Solve Quadratic Equations with the Quadratic Formula

So far we can solve $x^2 = 9$ by the square root property.

We can solve $x^2 - x - 6 = 0$ by factoring.

However, we cannot solve equations like $5t^2 - 20t - 4 = 0$, where the polynomial on the left side cannot be factored. In this lesson, we will learn how to solve any quadratic equation by the Quadratic Formula.

[**Example 1**] A ball will be thrown straight up, from 4 meters above the ground, with a velocity of 20 meters/second. When will it hit the ground? Round your answer to the hundredth place.

By the law of gravity, we can model the height of this ball by the function $h(t) = -5t^2 + 20t + 4$, where $h(t)$ represents the height of the ball in meters, and t represents the number of seconds passed since the ball was thrown.

[**Solution**] When the ball hit the ground, it's height is 0 meter. So we plug $h(t) = 0$ into $h(t) = -5t^2 + 20t + 4$, and we have a quadratic equation to solve:

$$-5t^2 + 20t + 4 = 0$$

There is a trick to get rid of the leading negative sign in an equation: We can multiply all terms in the equation by -1, and we have:

$$\begin{aligned}
&-5t^2 + 20t + 4 = 0 \\
&(-1)\cdot(-5t^2) + (-1)\cdot 20t + (-1)\cdot 4 = (-1)\cdot 0 \\
&5t^2 - 20t - 4 = 0
\end{aligned}$$

Basically, all terms changed sign in this equation. Remember this trick.

After some trying, we cannot factor the polynomial on the left side. We will resort to the Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To use the formula, we identify $a = 5, b = -20, c = -4$ from the equation $5t^2 - 20t - 4 = 0$. Note that a is the number in front of t^2, b is the number in front of t, and c is the constant term in the equation.

Plug $a = 5, b = -20, c = -4$ into the Quadratic Formula, we have:

$$t=\frac{-(-20)\pm\sqrt{(-20)^2-4\cdot5\cdot(-4)}}{2\cdot5}$$
$$t=\frac{20\pm\sqrt{400-(-80)}}{10}$$
$$t=\frac{20\pm\sqrt{480}}{10}$$

It's important to use parentheses around negative numbers!

Since the question asks us to round the answer to the hundredth place, we will use a calculator from here, and we have:

$$t=\frac{20+\sqrt{480}}{10}\approx\frac{20+21.9089}{10}\approx4.19\text{, or}$$

$$t=\frac{20-\sqrt{480}}{10}\approx\frac{20-21.9089}{10}\approx-0.19$$

Since time cannot be negative, we ignore the second solution.

Solution: The ball will hit the ground approximately 4.19 seconds after it is thrown.

Whenever we study a free-falling object, most likely a quadratic function will be involved, and the Quadratic Formula will be used.

For quadratic equations which can be solved by the square root property or factoring, we can still use the Quadratic Formula, except it takes more time. See the next example:

[**Example 2**] Solve $x^2-x-6=0$ for x

[**Solution**] We could solve this by factoring:

$$(x-3)(x+2)=0$$
$$x-3=0 \quad \text{or} \quad x+2=0$$
$$x=3 \qquad \text{or} \quad x=-2$$

Next, we will show how to use Quadratic Formula to solve this equation. First, we identify $a=1,b=-1,c=-6$. Next, use the formula:

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-6)}}{2 \cdot 1} \\
x &= \frac{1 \pm \sqrt{1 - (-24)}}{2} \\
x &= \frac{1 \pm \sqrt{25}}{2} \\
x &= \frac{1 \pm 5}{2} \\
x &= \frac{1+5}{2} \quad \text{or} \quad x = \frac{1-5}{2} \\
x &= 3 \quad \text{or} \quad x = -2
\end{aligned}
$$

Sometimes we need to simplify square root, which can be tricky. See the next example.

[**Example 3**] Solve $4x + 3 - x^2 = 0$ for x

[**Solution**] When a quadratic equation is not ordered by degree, we need to order it:

$$
\begin{aligned}
&4x + 3 - x^2 = 0 \\
&-x^2 + 4x + 3 = 0
\end{aligned}
$$

Next, multiply each term by -1 to get rid of the leading negative sign:

$$
\begin{aligned}
&-x^2 + 4x + 3 = 0 \\
&(-1) \cdot (-x^2) + (-1) \cdot 4x + (-1) \cdot 3 = (-1) \cdot 0 \\
&x^2 - 4x - 3 = 0
\end{aligned}
$$

Now we try to factor the left side, but there is no way. We resort to the Quadratic Formula. Identify that $a = 1, b = -4, c = -3$, and we have:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1}$$

$$x = \frac{4 \pm \sqrt{16 - (-12)}}{2}$$

$$x = \frac{4 \pm \sqrt{28}}{2}$$

This is not the final answer yet, because $\sqrt{28}$ can be simplified:

$$x = \frac{4 \pm \sqrt{28}}{2}$$

$$x = \frac{4 \pm \sqrt{2 \cdot 2 \cdot 7}}{2}$$

$$x = \frac{4 \pm 2\sqrt{7}}{2}$$

We are still not done because this fraction can be simplified:

$$x = \frac{4 \pm 2\sqrt{7}}{2}$$

$$x = \frac{4}{2} \pm \frac{2\sqrt{7}}{2}$$

$$x = 2 \pm \sqrt{7}$$

The solution of $4x + 3 - x^2 = 0$ is $x = 2 + \sqrt{7}$ and $x = 2 - \sqrt{7}$, or simply $x = 2 \pm \sqrt{7}$.

Math content builds up. You might need to go back to review how to simplify square root and how to divide a polynomial by a monomial.

Sometimes there is no solution (or "no real solution"). See the next example.

[**Example 4**] Solve $x^2 + x + 2 = 0$ for x.

[**Solution**] There is no way to factor the left side. Identify $a = 1, b = 1, c = 2$, and use Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$$
$$x = \frac{-1 \pm \sqrt{-7}}{2}$$

Since we cannot square root a negative number, there is no solution for $x^2 + x + 2 = 0$, until we learn complex numbers later. It's more accurate to say "there is no real solution" for $x^2 + x + 2 = 0$.