Maximum/Minimum Value Application Problems

We can use a quadratic function to find the maximum or minimum value in certain scenarios. This is because a parabola facing up has a minimum value, and a parabola facing down has a maximum value.

Let's start with an example involving numbers.

[**Example 1**] One number is 8 more than another number. For example, the pair could be 1 and 9, 2 and 10, -3 and -11, etc. Find such a pair of numbers that their product is as small as possible.

[**Solution**] Assume one number is x, then the other number must be $x+8$. Let the function $f(x)=x(x+8)$ be the product of these two numbers. We need to find the minimum value of $f(x)$. First, we change $f(x)$ to standard form: $f(x)=x^2+8x$. Identify that $a=1, b=8, c=0$.

This parabola faces up, so $f(x)$ has a minimum value, which happens at its vertex.

To find the vertex of $f(x)$, we use the vertex formula. First, find $f(x)$'s axis:

$$x=-\frac{b}{2a}=-\frac{8}{2\cdot 1}=-4$$

Next, to find the vertex, plug $x=-4$ into $f(x)$, and we have:

$$f(-4)=(-4)^2+8(-4)=-16$$

$f(x)$'s vertex is $(-4,-16)$. This implies the minimum product is -16, when one number is -4, and the other number is $-4+8=4$.

Solution: The pair of numbers, -4 and 4, has the smallest product of all such pairs. Their product is -16.

It's important to understand the graph of $f(x)$ in Figure 1. Each point on the parabola represents the product of two numbers, where one number is 8 more than the other.

For example, the point $(-4,-16)$ represents the product of -16 when $x=-4$. The other number is $-4+8=4$, as it is 8 more than -4.

The point $(-5,-15)$ represents the product of -15 when $x=-5$ and the other number is $-5+8=3$.

The parabola's vertex represents the smallest product possible, -16.

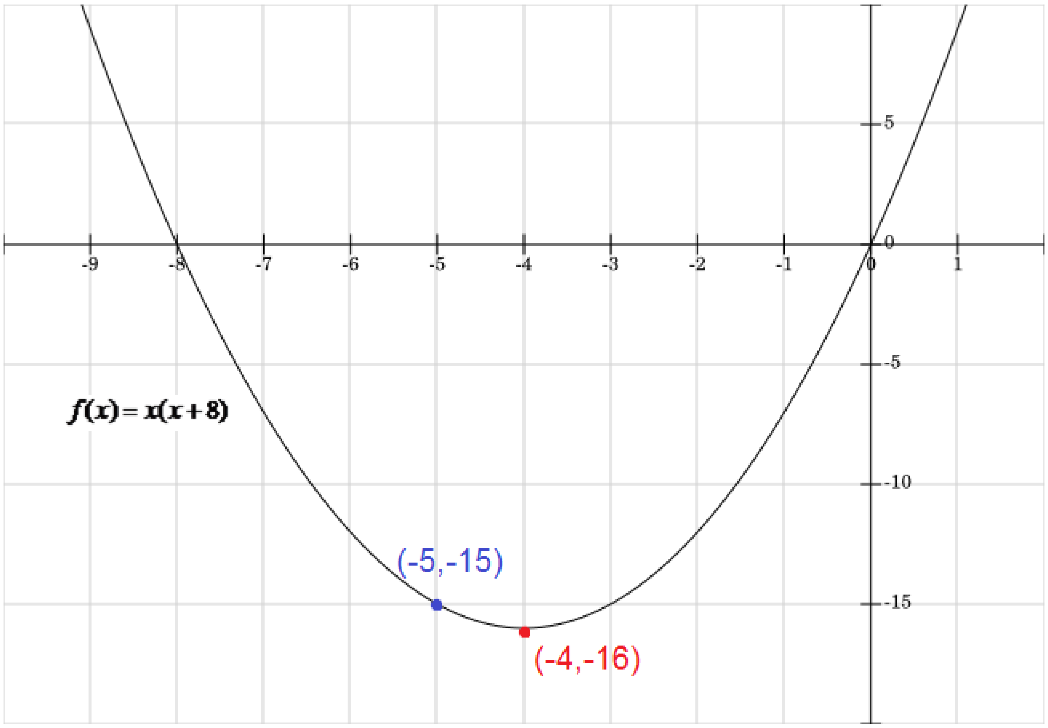

Figure 1: graph of f(x)=x(x+8)

Example 2 shows a classic type of maximum/minimum application problems involving a rectangle's area.

[**Example 2**] You have 500 yards of fence to build a rectangular field. What are the dimensions of the field such that you will get the maximum area?

[**Solution**] Assume the rectangle's width is w yards, and its length is l yards. We cannot build a function with two variables. We have to use what's given to eliminate one variable.

Since the rectangle's perimeter is 500 yards, we have:

$$2l + 2w = 500$$

Now we can solve for l:

$$\begin{aligned}
&2l + 2w = 500 \\
&2l + 2w - 2w = 500 - 2w \\
&2l = 500 - 2w \\
&\frac{2l}{2} = \frac{500}{2} - \frac{2w}{2} \\
&l = 250 - w
\end{aligned}$$

Now we can build a function for the rectangle's area: $A(w)=w(250-w)$, where *w* is the rectangle's width in yards, and $A(w)$ is the rectangle's area in square yards.

Let's change $A(w)$ to standard form: $A(w)=w(250-w)=250w-w^2=-w^2+250w$. Identify that $a=-1, b=250, c=0$.

This parabola faces down, so it has a maximum value, which happens at its vertex.

We will use the vertex formula to find the vertex. First, the parabola's axis is:

$$w=-\frac{b}{2a}=-\frac{250}{2\cdot(-1)}=125$$

To find the vertex, we plug $w=125$ into $A(w)$, we have:

$$A(w)=w(250-w)=125(250-125)=15625$$

Solution: When the width is 125 yards, and then length is $250-125=125$ yards, the rectangle has the maximum area, which is 15625 square yards.

It's not a surprise that the maximum area happens when the rectangle is a square.

Again, it's important to be able to interpret the graph of $A(w)$ in Figure 2.

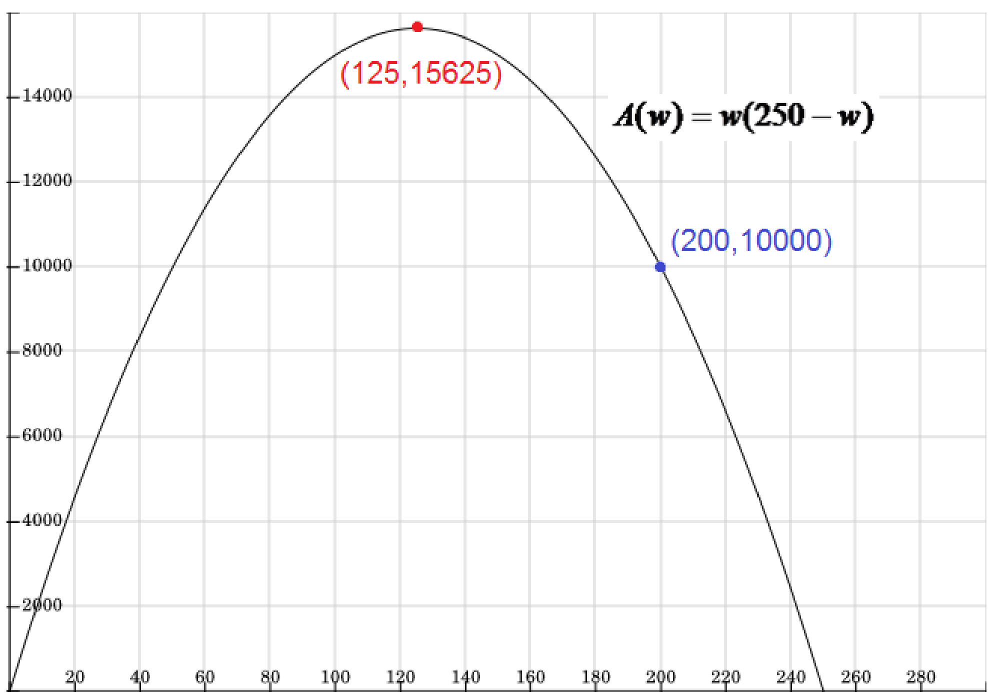

Figure 2: graph of A(w)=w(250-w)

In Figure 2, each point on the parabola represents the area of the rectangle of some dimensions.

The point $(200,10000)$ means the rectangle's area is 10000 square yards when the width is 200 yards. Since the perimeter is 500 yards, it implies the length is 50 yards in this situation.

The point $(125,15625)$ means the rectangle's area is 15625 square yards when the width is 125 yards. Since this point is the parabola's vertex, 15625 square yards is the maximum area for the rectangle.

Let's make the rectangle area problem more interesting.

[**Example 3**] You have 600 yards of fence. You will build two rectangular pens right next to each other, sharing the fence in the middle. These two pens will have the same dimensions. What dimensions will achieve the maximum area?

[**Solution**] Let's sketch a diagram first. Assume each pen's width is *w* yards, and each pen's length is *l* yards.

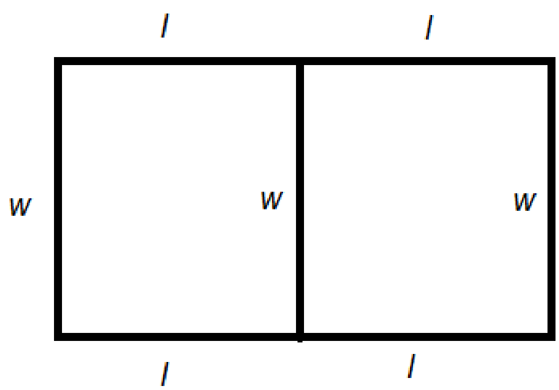

Figure 3: two pens built next to each other

Since you have a total of 600 yards of fence, we have:

$$3w + 4l = 600$$

As in Example 2, let's solve for l:

$$\begin{aligned}
&3w + 4l - 3w = 600 - 3w \\
&4l = -3w + 600 \\
&\frac{4l}{4} = \frac{-3w}{4} + \frac{600}{4} \\
&l = -\frac{3}{4}w + 150
\end{aligned}$$

Now we can build a function for the area of ONE pen: $A(w) = w(-\frac{3}{4}w + 150)$.

Change the function into standard form, we have: $A(w) = -\frac{3}{4}w^2 + 150w$.

Identify that $a = -\frac{3}{4}, b = 150, c = 0$.

This parabola faces down, so it has a maximum value, which happens at its vertex.

The axis is $w = -\frac{b}{2a} = -\frac{150}{2 \cdot (-\frac{3}{4})} = -\frac{150 \cdot 4}{2 \cdot (-\frac{3}{4}) \cdot 4} = -\frac{600}{-6} = 100$.

To find the vertex, we plug $w = 100$ into $A(w) = -\frac{3}{4}w^2 + 150w$, and we have:

$$A(100) = -\frac{3}{4}(100)^2 + 150 \cdot 100 = 7500.$$

The vertex is $(100, 7500)$, implying when $w = 100$ yards, each pen has the maximum area 7500 square yards. When $w = 100$, the length is $l = -\frac{3}{4}w + 150 = -\frac{3}{4} \cdot 100 + 150 = 75$ yards.

Solution: The maximum area is 7500 square yards. This happens when $w = 100$ yards and $l = 75$ yards.

The graph of this function is very similar to Figure 2. Please sketch and interpret the vertex of the parabola. This part is left as an exercise.