Locating Intercepts of Parabola

Earlier, we learned how to sketch the graph of a parabola $f(x) = a(x-h)^2 + k$:

1. Plot the vertex (h,k).
2. If $a > 0$, the parabola faces up; if $a < 0$, the parabola faces down.
3. If $|a| > 1$, make the parabola thinner than the standard parabola $y = x^2$; if $|a| < 1$, make the parabola wider than $y = x^2$; if $|a| = 1$, leave the parabola as wide as $y = x^2$.

In the last lesson, we learned how to locate the vertex even if the function is given in standard form $g(x) = ax^2 + bx + c$. However, there is still an issue: If $|a| \neq 1$, we don't know exactly how much thinner or wider the graph should become.

In this lesson, we will learn how to make our sketched parabolas more accurate by locating its *y*-intercept and *x*-intercept(s), if any.

Locating Parabola's Y-Intercept

Let's review how to find the *y*-intercept of $y = 2x + 3$. I hope you didn't simply memorized that the "3" is the *y*-value of the line's *y*-intercept.

Since a function's *y*-intercept crosses the *y*-axis, its *x* value must be 0—otherwise the point would not be on the *y*-axis. This is why, to find a function's y-intercept, we plug $x = 0$ into the equation and then solve for *y*. For $y = 2x + 3$, we have:

$$y = 2 \cdot 0 + 3 = 3$$

This is why the *y*-intercept of $y = 2x + 3$ is $(0,3)$.

We apply the same rule to find a quadratic function's *y*-intercept.

[**Example 1**] Find the *y*-intercept of $p(x) = 3x^2 - 2x + 10$ and $q(x) = 3(x-2)^2 + 10$.

[**Solution**] To find a function's *y*-intercept, we plug in $x = 0$. We have:

$p(0) = 3 \cdot 0^2 - 2 \cdot 0 + 10 = 10$, and

$q(0) = 3(0-2)^2 + 10 = 3 \cdot 2^2 + 10 = 22$.

Solution: The *y*-intercept of $p(x)$ is $(0,10)$, and the *y*-intercept of $q(x)$ is $(0,22)$.

Note that any function can have at most one *y*-intercept. If a function has two *y*-intercepts, say $(0,1)$ and $(0,2)$, it implies the function has two output values when the input is 0, and that would disqualify the relationship as a function.

For a parabola, it always has one and only one *y*-intercept.

Locating a Parabola's X-Intercept(s)

A parabola could have two, one or no *x*-intercepts. See the graph below:

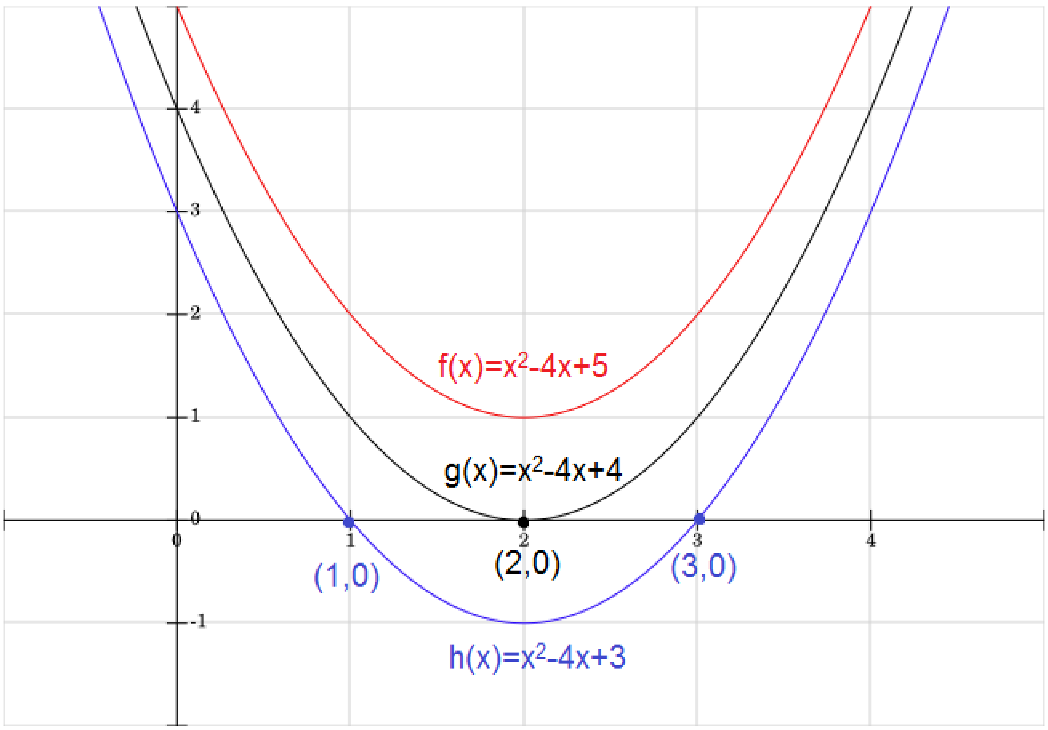

Figure 1: parabolas have different number of x-intercepts

In this graph, the parabola at the top has no *x*-intercepts; the one in the middle has one *x*-intercept: $(2,0)$; the one at the bottom has two *x*-intercepts: $(1,0)$ and $(3,0)$.

Since a function's *x*-intercept is always on the *x*-axis, its *y*-value must be 0. We can clearly see this in Figure 1. The strategy to find a function's *x*-intercept is to plug in $y = 0$ and then solve for x.

In the following examples, we will try to find the *x*-intercept(s) of those 3 functions in Figure 1 (pretend you don't know them yet).

[**Example 2**] Find the *x*-intercepts, if any, of $f(x) = x^2 - 4x + 5$.

[**Solution**] To find $f(x)$'s *x*-intercept, we plug $f(x) = 0$ into $f(x) = x^2 - 4x + 5$, and we have:

$$x^2 - 4x + 5 = 0$$

Since we cannot factor the left side, we will use the Quadratic Formula. Identify that $a = 1, b = -4, c = 5$, and we have:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} \\ x &= \frac{4 \pm \sqrt{-4}}{2} \end{aligned}$$

Since we cannot square root a negative number, there is no real solution for $x^2 - 4x + 5 = 0$, implying the function $f(x) = x^2 - 4x + 5$ doesn't have any *x*-intercept. Figure 1 verifies this result.

[**Example 3**] Find the *x*-intercepts, if any, of $g(x) = x^2 - 4x + 4$.

[**Solution**] To find $g(x)$'s *x*-intercept, we plug $g(x) = 0$ into $g(x) = x^2 - 4x + 4$, and we have:

$$x^2 - 4x + 4 = 0$$

The left side of this equation can be factored, so we have:

$$\begin{aligned} (x-2)^2 &= 0 \\ x - 2 &= 0 \\ x &= 2 \end{aligned}$$

There is one solution, $x = 2$, for $x^2 - 4x + 4 = 0$, implying the function $g(x) = x^2 - 4x + 4$ has one *x*-intercept: (2, 0). Figure 1 verifies this result.

[**Example 4**] Find the *x*-intercepts, if any, of $h(x) = x^2 - 4x + 3$.

[**Solution**] To find $h(x)$'s *x*-intercept, we plug $h(x) = 0$ into $h(x) = x^2 - 4x + 3$, and we have:

$$x^2 - 4x + 3 = 0$$

The left side of this equation can be factored, so we have:

$$(x-1)(x-3)=0$$
$$x-1=0 \quad \text{or} \quad x-3=0$$
$$x=1 \qquad \text{or} \quad x=3$$

There are two solutions, $x=1$ and $x=3$, for $x^2-4x+3=0$, implying the function $h(x)=x^2-4x+3$ has two *x*-intercept: (1, 0) and (3, 0). Figure 1 verifies this result.

Earlier, we learned that a quadratic equation's discriminant, $b^2–4ac$, determines the equation has 2, 1 or 0 solutions.

In this lesson, we just learned that a quadratic function can have 2, 1 or 0 *x*-intercept.

Is there a relationship between these two observations? Let's look at this table.

	Example 2	**Example 3**	**Example 4**
quadratic function	$f(x)=x^2-4x+5$	$g(x)=x^2-4x+4$	$h(x)=x^2-4x+3$
quadratic equation	$x^2-4x+5=0$	$x^2-4x+4=0$	$x^2-4x+3=0$
value of discriminant	$b^2–4ac=–4$	$b^2–4ac=0$	$b^2–4ac=4$
solutions	none	$x=2$	$x=1$ and $x=3$
x-intercepts	none	(2, 0)	(1, 0) and (3, 0)

Please take time to understand the following important relationships:

- When we solve a quadratic equation $ax^2+bx+c=0$, we are actually looking for the *x*-intercepts of the quadratic function $q(x)=ax^2+bx+c$.
- When $b^2–4ac>0$, the equation $ax^2+bx+c=0$ has two solutions, and the quadratic function $q(x)=ax^2+bx+c$ has two *x*-intercepts. This is the case for $h(x)$ in Figure 1.
- When $b^2–4ac=0$, the equation $ax^2+bx+c=0$ has one solution, and the quadratic function $q(x)=ax^2+bx+c$ has one *x*-intercept. The graph of $q(x)$ touches the *x*-axis at one point, like $g(x)$ in Figure 1.
- When $b^2–4ac<0$, the equation $ax^2+bx+c=0$ has no real solution, and the quadratic function $q(x)=ax^2+bx+c$ has not *x*-intercept. The whole graph of $q(x)$ is either above or below the *x*-axis, like $f(x)$ in Figure 1.

Studying mathematics is about understanding these relationships.

Let's look at one more example, where the parabola is given in vertex form.

[**Example 5**] Find the *x*-intercept(s) of $f(x)=2(x+3)^2-8$.

[**Solution**] To find $f(x)$'s *x*-intercepts, plug in $f(x)=0$, and we have:

$$f(x)=2(x+3)^2-8$$
$$0=2(x+3)^2-8$$

It's not wise to multiply out $(x+3)^2$ and then deal with a big mess. Instead, we will use the square root property:

$$0+8=2(x+3)^2-8+8$$
$$8=2(x+3)^2$$
$$\frac{8}{2}=\frac{2(x+3)^2}{2}$$
$$4=(x+3)^2$$

Since $4>0$, we can square root both sides and have:

$$\pm\sqrt{4}=x+3$$
$$\pm 2=x+3$$
$$x+3=2 \quad \text{or} \quad x+3=-2$$
$$x=-1 \quad \text{or} \quad x=-5$$

Solution: $f(x)=2(x+3)^2-8$'s *x*-intercepts are $(-1,0)$ and $(-5,0)$.

Look at the step $4=(x+3)^2$. In some other problem, at this step, you would see something like $-4=(x+3)^2$. Since $(x+3)^2$ cannot generate a negative number, at this step, we can claim that the parabola has no *x*-intercepts.