Number of Solutions of Quadratic Equation

In earlier lessons, we saw that sometimes a quadratic equation has two solutions, sometimes one, and sometimes none. How can we tell? Let's look at 3 examples, and then find a pattern.

[**Example 1**] Solve $x^2 - 2x = 0$ for x.

[**Solution**] It's easier to solve this equation by factoring. However, to find a pattern, we will instead solve this equation by Quadratic Formula. Identify that $a = 1, b = -2, c = 0$, and we have:

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 0}}{2 \cdot 1} \\
x &= \frac{2 \pm \sqrt{4}}{2} \\
x &= \frac{2 \pm 2}{2} \\
x &= \frac{2+2}{2} \quad \text{or} \quad x = \frac{2-2}{2} \\
x &= 2 \qquad\quad \text{or} \quad x = 0
\end{aligned}
$$

The equation $x^2 - 2x = 0$ has two solutions: $x = 2$ and $x = 0$.

[**Example 2**] Solve $x^2 - 2x + 1 = 0$ for x.

[**Solution**] Identify that $a = 1, b = -2, c = 1$, and we have:

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \\
x &= \frac{2 \pm \sqrt{0}}{2} \\
x &= \frac{2 \pm 0}{2} \\
x &= \frac{2+0}{2} \quad \text{or} \quad x = \frac{2-0}{2} \\
x &= 1 \qquad\quad \text{or} \quad x = 1
\end{aligned}
$$

Actually, it's silly to write $x=1$ or $x=1$ in the last line. The equation $x^2-2x+1=0$ has one solution: $x=1$.

[**Example 3**] Solve $x^2-2x+2=0$ for x.

[**Solution**] Identify that $a=1, b=-2, c=2$, and we have:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4\cdot1\cdot2}}{2\cdot1}$$
$$x=\frac{2\pm\sqrt{-4}}{2}$$

Since we cannot square root a negative number, the equation $x^2-2x+2=0$ has no real solutions. Actually, it has complex solutions, but we can save that for a later math course.

Look at these 3 examples. Which part in the Quadratic Formula determines the number of solutions?

It is the part inside the square root: b^2-4ac. Notice that:

- In Example 1, $b^2-4ac=4$, and the equation has two solutions.
- In Example 2, $b^2-4ac=0$, and the equation has one solution.
- In Example 3, $b^2-4ac=-4$, and the equation has no real solution.

The part b^2-4ac is so important that we give it a special name: **discriminant**. Let's summarize the pattern: For a quadratic equation $ax^2+bx+c=0$,

- if $b^2-4ac>0$, there are two solutions;
- if $b^2-4ac=0$, there is one solution;
- if $b^2-4ac<0$, there is no real solution, or, it has complex solutions.

[**Example 4**] Tell how many solutions the equation $9x^2-24x+16=0$ has.

[**Solution**] Identify that $a=9, b=-24, c=16$.

Since $b^2-4ac=(-24)^2-4\cdot9\cdot16=576-576=0$, the equation has 1 solution.