Introduction to Square Root

We know $10^2 = 100$. The inverse operation of "square" is "square root": $\sqrt{100} = 10$.

Similarly, since $10^3 = 1000$, the cube root of 1000 is 10: $\sqrt[3]{1000} = 10$.

We call numbers like $\sqrt{2}$ and $\sqrt[3]{3}$ "radicals." In this lesson, we will only study square root.

Legend about $\sqrt{2}$

If the number inside the square root is not a square number, the result is an irrational number, meaning a non-repeating decimal which never ends. For example:

$$\sqrt{2} = 1.414213...$$

There is a legend behind irrational numbers (Note the word "legend"). In ancient Greece, there was a mathematician community called Pythagoreans. They believed that all numbers were rational, meaning all numbers can all be presented by a fraction in the form of $\frac{a}{b}$, where *a* and *b* are integers. For example, $2 = \frac{2}{1}, -3.5 = \frac{-7}{2}, 0.\overline{3} = \frac{1}{3}, ...$

Hippasus was a member of this community. He draw a unit square (each side measuring 1 unit), and then drew the diagonal. Now we know, by Pythagorean Theorem, that the diagonal measures $\sqrt{2}$ units. Hippasus proved that $\sqrt{2}$ cannot be presented by $\frac{a}{b}$, implying $\sqrt{2}$ is not a rational number. This broke the hearts of Pythagoreans in the community. Hippasus was thrown into the sea for violating the beauty of the world...

Estimating Square Root Values

Let's estimate the value of $\sqrt{5}$ and $\sqrt{6}$. Notice that:

$$\sqrt{4} = 2$$
$$\sqrt{5} = ?$$
$$\sqrt{6} = ?$$
$$\sqrt{9} = 3$$

By this pattern, we can tell the value of both $\sqrt{5}$ and $\sqrt{6}$ are in the range of (2, 3). This estimation is good enough. If we need more accurate values, we will use a calculator.

[**Example 1**] The value of $\sqrt{80}$ is between which two integers?

[**Solution**] Since $\sqrt{64} = 8$ and $\sqrt{81} = 9$, we can tell $\sqrt{80}$ is in the range of (8, 9). Actually, its value is very close to 9.

Square Root of Fractions and Decimals

Don't be scared by fractions and decimals inside a square root. Look at these examples:

$$\sqrt{\frac{1}{4}} = \frac{1}{2} \text{ because } (\frac{1}{2})^2 = \frac{1}{4}$$

$$\sqrt{\frac{9}{4}} = \frac{3}{2} \text{ because } (\frac{3}{2})^2 = \frac{9}{4}$$

$$\sqrt{0.01} = 0.1 \text{ because } 0.1^2 = 0.01$$

Square Root of Negative Numbers

$\sqrt{4} = 2$ because $2^2 = 4$. What's the value of $\sqrt{-4}$?

If $\sqrt{-4} = x$, then $x^2 = -4$. The square of which number is -4 ? It's not -2, because $(-2)^2 = 4$.

We cannot find such a number whose square is -4. As a result, $\sqrt{-4}$ **does not exist**.

Note the difference in the following two problems:

$$-\sqrt{4} = -2 \text{, but } \sqrt{-4} \text{ does not exist}$$

At the same time, $\sqrt{0} = 0$ because $0^2 = 0$.

Square Root and Square

Square root and square are inverse operations, implying they would cancel out each other. Observe these two patterns:

$$(\sqrt{4})^2 = 2^2 = 4 \qquad\qquad \sqrt{4^2} = \sqrt{16} = 4$$

$$(\sqrt{9})^2 = 3^2 = 9 \qquad \text{and} \qquad \sqrt{9^2} = \sqrt{81} = 9$$

$$\cdots \qquad\qquad \cdots$$

$$(\sqrt{x})^2 = x \qquad\qquad \sqrt{x^2} = x$$

Note that *x* must be **positive** for these formulas to work.