Falling Object Application Problems

The earth pulls all objects at a constant acceleration. This physics law determines that all free falling objects' height can be modeled by a quadratic function.

If you decide to pursue a math/science career, you will learn in later math courses that parabolas are used in building bridges, making auto headlights, and many other scenarios.

You are studying quadratic functions not because mathematicians think that they are fun; it's because scientists of other subjects found that quadratic functions can be applied in many situations. This is true for almost all math contents you will study.

In this lesson, we will see how quadratic functions are used to model free falling objects. Here is the general formula for the height of a free falling object:

$$h(t) = -16t^2 + v_0 t + h_0$$

Let's look at each part of this formula:

t represents the number of seconds passed since the object's release.

$h(t)$ represents the height of the object in feet.

-16 is a constant determined by the earth's weight and Newton's Laws. It's unit is feet/second^2. In metric system, this number is -4.9 meters/second^2. In some problems, this number could be different, if we consider wind and other conditions affecting an object's flight.

v_0 represents the object's initial vertical speed. This number is positive if the object was thrown upward, and it is negative if the object was thrown downward.

h_0 represents the object's initial height. h_0's value is usually positive, unless the object is thrown from a spot below the sea level.

These should make more sense after you go through Example 1.

[**Example 1**] A ball was thrown straight upward at an initial speed of 64 feet/second. The point of release was very close to the ground. How high did the ball fly? After how many seconds did it fall to the ground?

[**Solution**] By the formula $h(t) = -16t^2 + v_0 t + h_0$, we have:

$$h(t) = -16t^2 + 64t,$$

where t represents the number of seconds passed since the object's release, and $h(t)$ represents the height of the object in feet. We assume $h_0 = 0$, because the point of release "was very close to the ground."

Identify that $a = -16, b = 64, c = 0$. Let's sketch the graph of $h(t)$ by the following information:

Axis: By the axis formula, the axis is $t = -\frac{b}{2a} = -\frac{64}{2\cdot(-16)} = 2$.

Vertex: Plug $t = 2$ into $h(t) = -16t^2 + 64t$, we have:

$$h(2) = -16\cdot 2^2 + 64\cdot 2 = 64$$

The vertex is $(2,64)$.

Up or down: Since $a = -16$, the parabola faces down, implying the vertex is its maximum value.

y-intercept: Plug $t = 0$ into $h(t) = -16t^2 + 64t$, and we have $h(0) = 0$. The y-intercept is $(0,0)$.

x-intercept(s): Plug $h(t) = 0$ into $h(t) = -16t^2 + 64t$ and solve for t, we have:

$$\begin{aligned} &h(t) = -16t^2 + 64t \\ &-16t^2 + 64t = 0 \\ &-16t(t-4) = 0 \\ &t = 0 \quad \text{or} \quad t - 4 = 0 \\ &t = 0 \quad \text{or} \quad t = 4 \end{aligned}$$

The parabola has two x-intercepts: $(0,0)$ and $(4,0)$. Note that we solve this quadratic equation by factoring. We ignored -16 when we used Zero Product Property, as -16 will not make the product 0.

With these information, we can sketch the graph of $h(t)$ now. See Figure 1 on the next page.

In the graph, the point $(0,0)$ means: When the ball was released, it was 0 feet above the ground.

The point $(2,64)$ means: 2 seconds after the ball was released, it reached its maximum height, 64 feet above the ground.

The point $(4,0)$ means: 4 seconds after the ball was released, it fell back to the ground.

Solution: The ball reached its highest point, 64 feet, two seconds after it was released.

The ball fell back to the ground 4 seconds after it was released.

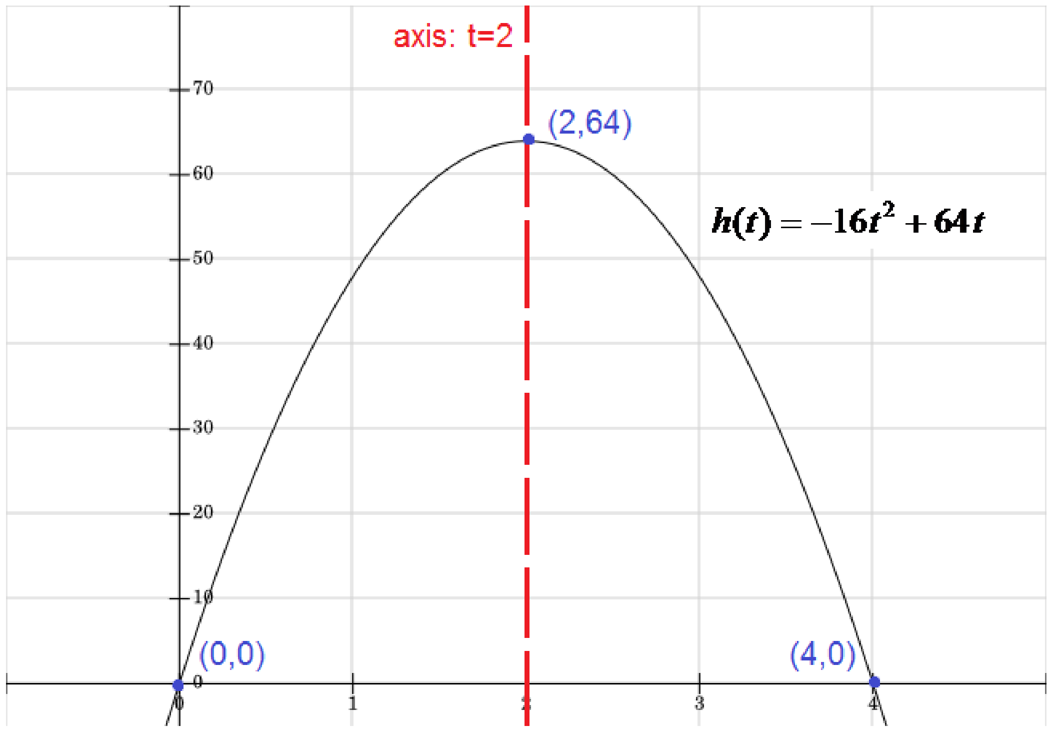

Figure 1: graph of h(t)=-16t^2+64t

Example 2 deals with the same situation as Example 1, except the questions are different.

[**Example 2**] A ball was thrown straight upward at an initial speed of 64 feet/second. The point of release was very close to the ground. Answer the following questions:

a) How high was the ball 3 seconds after it was released?

b) After how many seconds was the ball 28 feet above the ground?

[**Solution**] We are dealing with the same function $h(t) = -16t^2 + 64t$.

For Part a, we simply need to plug in $t = 3$, and we have $h(3) = -16 \cdot 3^2 + 64 \cdot 3 = 48$.

Solution for Part a: The ball was 48 feet high 3 seconds after it was released. We can see this point, $(3,48)$, in Figure 2.

Part b is more complicated. When the ball was 28 feet high, we have $h(t) = 28$. Plug this into the function and then solve for t, we have:

$$28 = -16t^2 + 64t$$
$$0 = -16t^2 + 64t - 28$$
$$0 = -4(4t^2 - 16t + 7)$$
$$0 = -4(2t-1)(2t-7)$$
$$2t - 1 = 0 \quad \text{or} \quad 2t - 7 = 0$$
$$t = 0.5 \quad \text{or} \quad t = 3.5$$

Note that this quadratic equation was solved by factoring. The Quadratic Formula would work, too.

Solution for Part b: The ball was 28 feet high twice during its flight: once 0.5 seconds after its release, another time 3.5 seconds after its release.

We can see these two points, $(0.5,28)$ and $(3.5,28)$, in Figure 2.

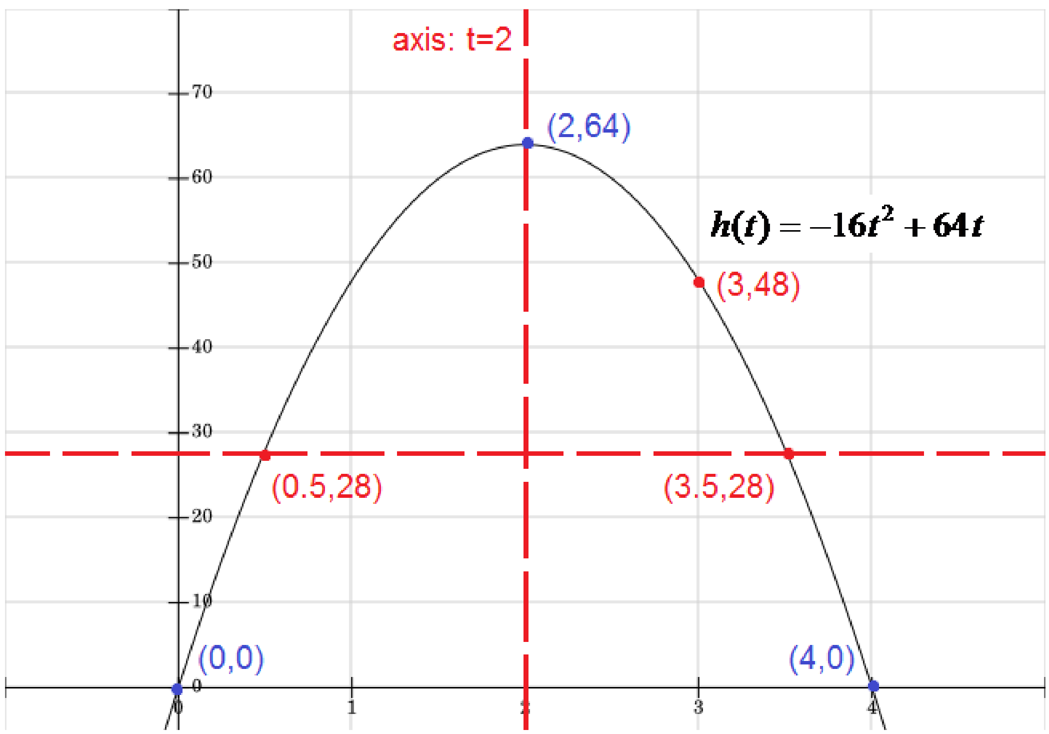

Figure 2: graph of h(t) and horizontal line y=28

In Figure 2, the horizontal line $y = 28$ is graphed. It intersects with the parabola at two points: $(0.5,28)$ and $(3.5,28)$. They represent those two instances when the ball was 28 feet high.

The next example has an initial height.

[**Example 3**] Fireworks were shot straight up on top of a 100-foot hill. Their initial speed was 120 feet per second. Answer the following questions:

a) What's the maximum height these fireworks reached?

b) How many seconds did these fireworks fly in the air before they hit the ground at the foot of the hill?

c) 7 seconds after fireworks were released, how high were they in the air?

d) After fireworks were released, after how many seconds were they 200 feet in the air?

[**Solution**] First of all, in real life, people don't shoot fireworks straight up; otherwise they would land on the canons. Instead, they were shot up at a certain angle. However, when angles are involved, trigonometry must be used to solve these problems. At this level of math, let's ignore the angle and only consider the vertical speed.

In this problem, the height function is $h(t) = -16t^2 + 120t + 100$, where t represents the number of seconds after release, and $h(t)$ represents the height of fireworks in feet.

The axis of this parabola is $t = -\frac{b}{2a} = -\frac{120}{2 \cdot (-16)} = 3.75$.

To find the vertex of this parabola, plug in $t = 3.75$, and we have:
$h(3.75) = -16 \cdot 3.75^2 + 120 \cdot 3.75 + 100 = 325$. The parabola's vertex is $(3.75, 325)$.

Solution of Part a: The maximum height of those fireworks was 325 feet.

When those fireworks land at the foot of the hill, $h(t) = 0$. Plug in $h(t) = 0$ and solve for t, we have:

$$h(t) = -16t^2 + 120t + 100$$
$$-16t^2 + 120t + 100 = 0$$
$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$t = \frac{-120 \pm \sqrt{120^2 - 4(-16)100}}{2 \cdot (-16)}$$
$$t \approx -0.76 \quad \text{or} \quad t \approx 8.26$$

Solution of Part b: Those fireworks flew approximately 8.26 seconds in the air before they hit the ground at the foot of the hill.

To find how high those fireworks were after flying 7 seconds, we plug $t = 7$ into $h(t)$, and we have $h(7) = -16 \cdot 7^2 + 120 \cdot 7 + 100 = 156$.

Solution of Part c: 7 seconds after those fireworks were shot up, they were 156 feet high.

When those fireworks were 200 feet in the air, we have $h(t) = 200$. Plug $h(t) = 200$ into the equation, we have:

$$
\begin{aligned}
&h(t) = -16t^2 + 120t + 100 \\
&200 = -16t^2 + 120t + 100 \\
&0 = -16t^2 + 120t - 100 \\
&t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&t = \frac{-120 \pm \sqrt{120^2 - 4(-16)(-100)}}{2 \cdot (-16)} \\
&t \approx 0.95 \quad \text{or} \quad t \approx 6.55
\end{aligned}
$$

Solution of Part d: Those fireworks were 200 feet in the air in two instances: 0.95 seconds and 6.55 seconds after they were released.