5.2 Proportion

5.2.1 Motivation of Using Proportion

Let's start by doing a problem with rate.

Example 5.2.1 A car drove 150 miles in 6 hours. How long would it take the car to drive 250 miles?

Solution First, we use division to find the rate of change (speed in this case):

$$\frac{150 \text{ miles}}{6 \text{ hours}} = 25 \text{ miles/hour}$$

The car's speed is 25 miles per hour.
Next, we find how long it would take the car to drive 250 miles by fraction multiplication. We will use the rate $\frac{1 \text{ hour}}{25 \text{ miles}}$, and we have:

$$\begin{aligned}
&250 \text{ miles} \cdot \frac{1 \text{ hour}}{25 \text{ miles}} \\
&= \frac{250 \text{ miles}}{1} \cdot \frac{1 \text{ hour}}{25 \text{ miles}} \\
&= \frac{250}{1} \cdot \frac{1 \text{ hour}}{25} \\
&= \frac{250 \cdot 1}{1 \cdot 25} \text{ hours} \\
&= \frac{250}{25} \text{ hours} \\
&= 10 \text{ hours}
\end{aligned}$$

It takes the car 10 hours to drive 250 miles. ■

With proportion, solving this problem becomes much easier. The core of the solution is below (we will learn details later in this lesson):

$$\begin{aligned}
\frac{150 \text{ miles}}{6 \text{ hours}} &= \frac{250 \text{ miles}}{x \text{ hours}} \\
150x &= 250 \cdot 6 \\
150x &= 1500 \\
\frac{150x}{150} &= \frac{1500}{150} \\
x &= 10
\end{aligned}$$

We will learn how to set up proportion and then solve it. We start by learning how to solve equations like $150x = 1500$.

5.2.2 Solve Simple Equations

Think about this puzzle: 2 times which number gives 10? If we use x to represent the unknown number, we can write an equation:

$$2x = 10$$

We can omit the multiplication symbol between 2 and x, as $2x$ implies 2 times x.

We know the answer is 5. How do we solve this equation mathematically? Since division is the inverse operation of multiplication, we divide both sides of the equation with 2:

$$2x = 10$$
$$\frac{2x}{2} = \frac{10}{2}$$
$$x = 5$$

On the left side, from $\frac{2x}{2}$, since $2 \div 2 = 1$, we have $1x$. Since 1 times any number will not change that number's value (for example, $1 \cdot 3 = 3, 1 \cdot 4 = 4, \ldots$), $1x$ is the same as x.
Here are two more examples:

$$3x = 12 \qquad\qquad 15x = 45$$
$$\frac{3x}{3} = \frac{12}{3} \qquad\qquad \frac{15x}{15} = \frac{45}{15}$$
$$x = 4 \qquad\qquad x = 3$$

Basically, to solve an equation like $3x = 12$, we divide both sides of the equation by the number in front of x.

5.2.3 Cross-Multiplication

Let's observe a pattern:

$$\frac{1}{2} = \frac{3}{6} \rightarrow 1 \cdot 6 = 2 \cdot 3$$
$$\frac{1}{2} = \frac{4}{8} \rightarrow 1 \cdot 8 = 2 \cdot 4$$
$$\frac{3}{6} = \frac{4}{8} \rightarrow 3 \cdot 8 = 6 \cdot 4$$

We can see why this pattern is called "cross-multiplication". Now we can solve proportions. Let's look at a few examples:

$$\frac{x}{6} = \frac{2}{3} \qquad \frac{4}{x} = \frac{2}{3} \qquad \frac{2}{3} = \frac{x}{6} \qquad \frac{2}{3} = \frac{4}{x}$$
$$3x = 6 \cdot 2 \qquad 2x = 4 \cdot 3 \qquad 3x = 2 \cdot 6 \qquad 2x = 3 \cdot 4$$
$$3x = 12 \qquad 2x = 12 \qquad 3x = 12 \qquad 2x = 12$$
$$\frac{3x}{3} = \frac{12}{3} \qquad \frac{2x}{2} = \frac{12}{2} \qquad \frac{3x}{3} = \frac{12}{3} \qquad \frac{2x}{2} = \frac{12}{2}$$
$$x = 4 \qquad x = 6 \qquad x = 4 \qquad x = 6$$

5.2.4 Proportion Word Problems

It's important to organize information in a table when we write proportion equations. Let's look at a few examples.

Example 5.2.2 A car drove 150 miles in 6 hours. How long would it take the car to drive 250 miles?

Solution First, assume it would take the car x hours to drive 250 miles. Next, we will use a table to organize the given information:

	Situation 1	**Situation 2**
miles	150	250
hours	6	x

Now we can write a proportion equation and solve for x. It's critical to include units in the equation to make sure numbers are in the right places.

$$\begin{aligned} \frac{150 \text{ miles}}{6 \text{ hours}} &= \frac{250 \text{ miles}}{x \text{ hours}} \\ 150x &= 250 \cdot 6 \\ 150x &= 1500 \\ \frac{150x}{150} &= \frac{1500}{150} \\ x &= 10 \end{aligned}$$

It would take the car 10 hours to drive 250 miles. ■

In the example above, if we made a mistake by writing $\frac{150 \text{ miles}}{6 \text{ hours}} = \frac{x \text{ hours}}{250 \text{ miles}}$, it's easy to see the units don't match up, and thus the equation is incorrect. This is why it's important to include units in the equation.

Example 5.2.3 A restaurant's expense of labor cost to food cost is in the ratio of 8 : 3. In one month, if the restaurant spent \$600.00 in food cost, how much did it spend on labor cost?

Solution Assume the restaurant spent x dollars on labor in that month. We use a table to organize the given information:

	Ratio	**In that month**
labor cost in dollars	8	x
food cost in dollars	3	600

We write a proportion equation and solve for x.

$$\begin{aligned} \frac{8 \text{ labor cost in dollars}}{3 \text{ food cost in dollars}} &= \frac{x \text{ labor cost in dollars}}{600 \text{ food cost in dollars}} \\ 3x &= 8 \cdot 600 \\ 3x &= 4800 \\ \frac{3x}{3} &= \frac{4800}{3} \\ x &= 1600 \end{aligned}$$

In one month, if the restaurant spent \$600.00 in food cost, it spent \$1,600.00 on labor cost. ■