# MODULE 3

## 3.1 Fraction Definition and Equivalent Fractions

In this lesson, we will learn the definition of fractions, and how to change a fraction to an equivalent fraction.

### 3.1.1 Definition of Fraction

For a fraction like $\frac{2}{3}$, the number 2 is the *numerator*, and the number 3 is the *denominator*. When we look at a fraction like $\frac{2}{3}$, we naturally look at the numerator (2) first. Actually, we should look at the denominator (3) first. Here is how to interpret $\frac{2}{3}$:

1. We cut the whole evenly into 3 pieces.
2. We take 2 of those pieces.

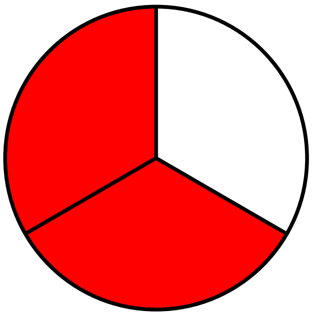

FIGURE 3.1: Red pieces in the pie represent $\frac{2}{3}$

There are more ways to represent fractions graphically. For example:

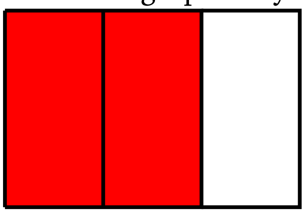

FIGURE 3.2: Red pieces in the rectangle represent $\frac{2}{3}$

Depending on the situations, we will use different shapes to represent fractions.

### 3.1.2 Equivalent Fractions

Let's look at the following two graphs:

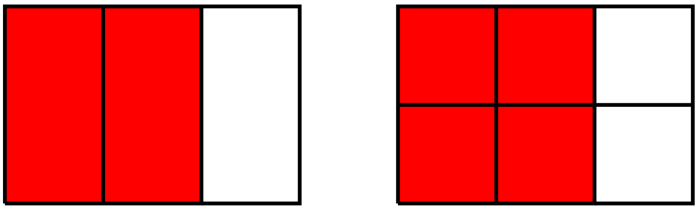

FIGURE 3.3: Compare $\frac{2}{3}$ and $\frac{4}{6}$

The graph on the left side represents $\frac{2}{3}$, and the graph on the right side represents $\frac{4}{6}$. We can see they actually cover the same area. In other words:

$$\frac{2}{3} = \frac{4}{6}$$

The difference is that the rectangle on the right side is cut into twice as many pieces as the rectangle on the left side. Let's look at this equation:

$$\frac{2}{3} = \frac{2 \cdot 2}{3 \cdot 2} = \frac{4}{6}$$

To change $\frac{2}{3}$ to $\frac{4}{6}$, we multiply 2 in both the numerator and denominator. This implies we cut the rectangle into twice as many pieces (6), and take twice as many pieces (4). The value of the fraction didn't change!

For $\frac{2}{3}$, if we cut the rectangle into 3 times as many pieces, and take 3 times as many pieces, the fraction's value would not change, either:

$$\frac{2}{3} = \frac{2 \cdot 3}{3 \cdot 3} = \frac{6}{9}$$

Let's look at a graphic representation of this equation:

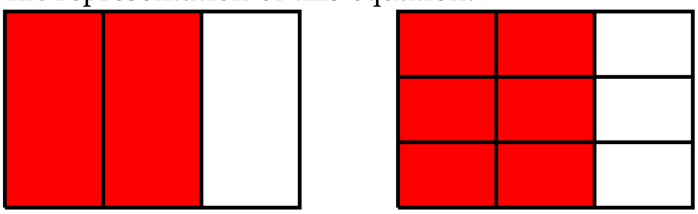

FIGURE 3.4: $\frac{2}{3} = \frac{6}{9}$

Here is the rule: If we multiply the same number in both the numerator and denominator of a fraction, the fraction's value doesn't change. For example, $\frac{2}{3} = \frac{2 \cdot 2}{3 \cdot 2} = \frac{4}{6}$, and $\frac{2}{3} = \frac{2 \cdot 3}{3 \cdot 3} = \frac{6}{9}$.

**Example 3.1.1** Write a few equivalent fractions of $\frac{4}{5}$.

**Solution**

$$\frac{4}{5} = \frac{4 \cdot 2}{5 \cdot 2} = \frac{8}{10}$$
$$\frac{4}{5} = \frac{4 \cdot 3}{5 \cdot 3} = \frac{12}{15}$$
$$\frac{4}{5} = \frac{4 \cdot 4}{5 \cdot 4} = \frac{16}{20}$$
$$\frac{4}{5} = \frac{4 \cdot 5}{5 \cdot 5} = \frac{20}{25}$$
$$\ldots$$

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**Example 3.1.2** Change $\frac{2}{3}$ to an equivalent fraction with the denominator of 12.

**Solution** The question asks:

$$\frac{2}{3} = \frac{?}{12}$$

To change the denominator from 3 to 12, we need to do $3 \cdot 4 = 12$. If we multiply the denominator with 4, we must do the same to the numerator. We have:

$$\frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 4} = \frac{8}{12}$$

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Similarly, we can reduce fractions by dividing the same number in both the numerator and denominator:

$$\frac{4}{6} = \frac{4 \div 2}{6 \div 2} = \frac{2}{3}$$
$$\frac{6}{9} = \frac{6 \div 3}{9 \div 3} = \frac{2}{3}$$
$$\frac{20}{25} = \frac{20 \div 5}{25 \div 5} = \frac{4}{5}$$
$$\ldots$$

How do we know which number to divide when we reduce fractions? Think about prime numbers: $2, 3, 5, 7, 11 \ldots$. Try prime numbers one by one.

**Example 3.1.3** Reduce the fraction $\frac{30}{36}$.

**Solution**

$$\begin{aligned}
&\frac{24}{36} \\
&= \frac{24 \div 2}{36 \div 2} && \text{2 goes into both 24 and 36.} \\
&= \frac{12}{18} \\
&= \frac{12 \div 2}{18 \div 2} && \text{2 goes into both 12 and 18.} \\
&= \frac{6}{9} \\
&= \frac{6 \div 3}{9 \div 3} && \text{3 goes into both 6 and 9.} \\
&= \frac{2}{3} && \text{No more prime numbers go into both 2 and 3.}
\end{aligned}$$

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You only need to memorize the first 5 prime numbers: $2, 3, 5, 7, 11$. Rarely would an instructor expect you to see the prime number 13 goes into two numbers. Actually, to see whether 7 goes into a number, usually we resort to a calculator.

If we can reduce a fraction, we must do so! We don't allow reducible fractions like $\frac{2}{4}$ and $\frac{3}{9}$ as the final answer of a problem.

There is an easier way to reduce $\frac{24}{36}$:

$$\frac{24}{36} = \frac{24 \div 12}{36 \div 12} = \frac{2}{3}$$

If you can see 12 goes into both 24 and 36, you can reduce $\frac{24}{36}$ in one step. However, if you cannot see it, simply go through the list of prime numbers one by one, $2, 3, 5, 7, 11...$, you can still reduce $\frac{24}{36}$ to $\frac{2}{3}$.

**Example 3.1.4** Reduce the fraction $\frac{42}{126}$.

**Solution**

$$\begin{aligned} &\frac{42}{126} \\ &= \frac{42 \div 2}{126 \div 2} && \text{2 goes into both 42 and 126.} \\ &= \frac{21}{63} \\ &= \frac{21 \div 3}{63 \div 3} && \text{3 goes into both 21 and 63.} \\ &= \frac{7}{21} \\ &= \frac{7 \div 7}{21 \div 7} && \text{7 goes into both 7 and 21.} \\ &= \frac{1}{3} \end{aligned}$$

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Here are a few special cases in fraction reduction. Remember: The fraction line has the same function as the division symbol.

$$\frac{10}{10} = 10 \div 10 = 1$$
$$\frac{10}{1} = 10 \div 1 = 10$$
$$\frac{10}{0} = 10 \div 0 = \text{undefined}$$
$$\frac{0}{10} = 0 \div 10 = 0$$

We can easily change a fraction to a decimal with a simple division: $\frac{1}{2} = 1 \div 2 = 0.5$. We will cover this in the decimal chapter.

### 3.1.3 Compare Fractions

It's easy to understand that $\frac{2}{3} > \frac{1}{3}$, and $\frac{3}{10} < \frac{7}{10}$. How would we compare fractions when the denominators are different? Like $\frac{5}{6}$ and $\frac{7}{8}$?

Now that we learned how to change a fraction to its equivalent, we can change the denominators to the same number and then compare them.

**Example 3.1.5** Compare $\frac{5}{6}$ and $\frac{7}{8}$.

**Solution** Since $6 \cdot 8 = 48$, we know both 6 and 8 go into 48, so we will change both denominators to 48:

$$\frac{5}{6} = \frac{5 \cdot 8}{6 \cdot 8} = \frac{40}{48} \text{ and } \frac{7}{8} = \frac{7 \cdot 6}{8 \cdot 6} = \frac{42}{48}$$

**Conclusion:** Now we can tell $\frac{5}{6} < \frac{7}{8}$.

If you can see both 6 and 8 go into 24, you can avoid dealing with bigger numbers, but the result would be the same. In that case, you would do:

$$\frac{5}{6} = \frac{5 \cdot 4}{6 \cdot 4} = \frac{20}{24} \text{ and } \frac{7}{8} = \frac{7 \cdot 3}{8 \cdot 3} = \frac{21}{24}$$

The conclusion stays the same: $\frac{7}{8}$ is bigger. ■

### 3.1.4 Fractions on Number Line

The following figures show how to locate fractions on the number line. These figures are pretty self-explanatory. We need to count each unit (from 0 to 1) is cut into how many segments.

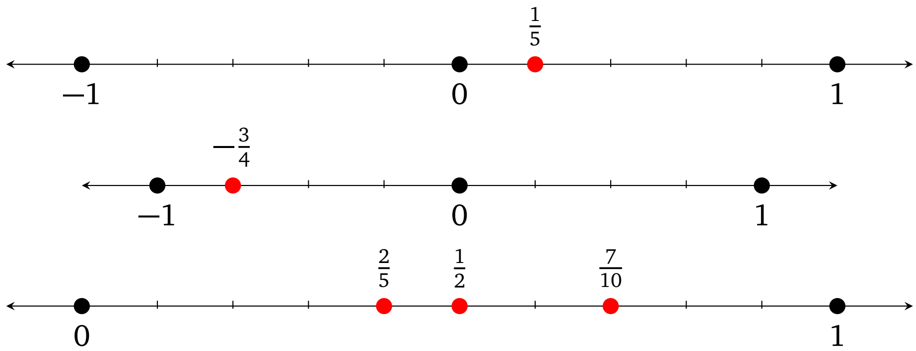


FIGURE 3.5: Locate fractions on number line

On the last number line, note that the segment from 0 to 1 is cut evenly into 10 pieces, implying each piece represents $\frac{1}{10}$. The first red dot is 4 pieces away from 0, which represents $\frac{4}{10}$. We must reduce the fraction: $\frac{4}{10} = \frac{2}{5}$.
Similarly, the second red dot represents $\frac{5}{10}$. We must again reduce the fraction: $\frac{5}{10} = \frac{1}{2}$.
The third red dot represents $\frac{7}{10}$. We cannot reduce this fraction.

### 3.1.5 Summary

Let's review what we learned in this lesson:

- To understand a fraction like $\frac{2}{3}$, we look at the denominator 3 first, and then look at the numerator 2. For $\frac{2}{3}$, we cut the whole evenly into 3 pieces, and then take 2 of those pieces.
- If we multiply the same number in both the numerator and denominator of a fraction, the fraction's value doesn't change. For example,

$$\frac{2}{3} = \frac{2 \cdot 2}{3 \cdot 2} = \frac{4}{6}$$

- If we divide the same number in both the numerator and denominator of a fraction, the fraction's value doesn't change. For example,

$$\frac{4}{6} = \frac{4 \div 2}{6 \div 2} = \frac{2}{3}$$

- When we compare two fractions with different denominators, we can change the denominators to the same number, and then compare the numerators, like in example 3.1.5.