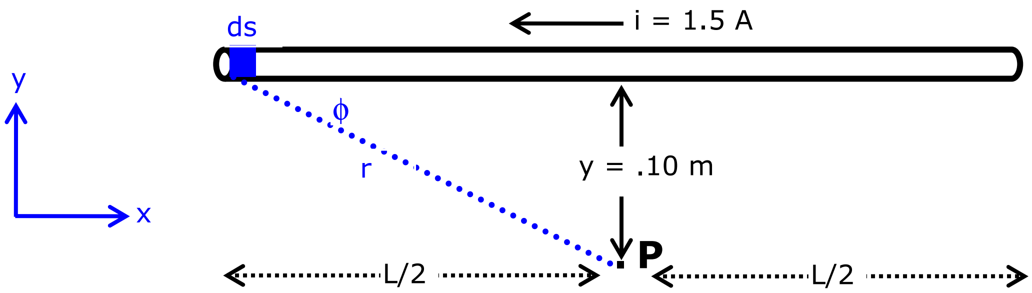
Biot-Savart Law

1. A straight wire horizontal wire, L=0.3 m, has a 1.5 A current running through it.



a. Using the Biot-Savart Law, derive the equation for the magnetic field vector produced by the wire at an arbitrary point P, below the wire and halfway between the ends.

$$d\vec{B}=\frac{\mu_o i}{4\pi}\frac{d\vec{s}\times\vec{r}}{r^3}$$

Ans.

$$\vec{B}=\int d\vec{B}=\frac{\mu_o i}{4\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{d\vec{s}\times\vec{r}}{r^3}=\frac{\mu_o i}{4\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dx}{r^2}\sin\phi\,\hat{k}$$

$$\vec{B}=\frac{\mu_o i y}{4\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dx}{\left(y^2+x^2\right)^{\frac{3}{2}}}\hat{k} = \frac{\mu_o i}{4\pi}\frac{x}{y\left(y^2+x^2\right)^{\frac{1}{2}}}\hat{k}\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}=\frac{\mu_o i}{4\pi}\frac{L}{y\left(y^2+\frac{L^2}{4}\right)^{\frac{1}{2}}}\hat{k}$$

b. Determine the magnitude and direction of the magnetic field vector at point P=0.1 m below the wire.

Ans.

$$\vec{B}=\frac{\mu_o}{4\pi}\frac{(1.5\text{A})(0.3\text{m})}{(0.1\text{m})\left((0.1\text{m})^2+\frac{(0.3\text{m})^2}{4}\right)^{\frac{1}{2}}}\hat{k} = 2.50\text{x}10^{-6}\text{T}\,\hat{k}$$

c. Determine the magnitude and direction of the magnetic field produced by the wire at a point 0.01 m to the left of point P.

Ans.

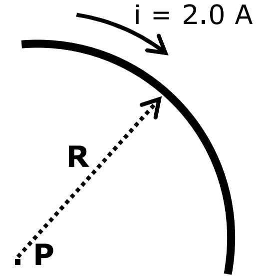
$$\vec{B}=\frac{\mu_o i}{4\pi}\frac{x}{y\left(y^2+x^2\right)^{\frac{1}{2}}}\hat{k}\Bigg|_{-\frac{L}{6}}^{\frac{5L}{6}}=\frac{\mu_o iL}{24\pi y}\left[\frac{5}{\left(y^2+\left(\frac{5L}{6}\right)^2\right)^{\frac{1}{2}}}+\frac{1}{\left(y^2+\left(\frac{L}{6}\right)^2\right)^{\frac{1}{2}}}\right]\hat{k}$$

$$\vec{B}=\frac{\mu_o(1.5\text{A})(0.3\text{m})}{24\pi(0.1\text{m})}\left[\frac{5}{\left((0.1\text{m})^2+\left(\frac{1.5\text{m}}{6}\right)^2\right)^{\frac{1}{2}}}+\frac{1}{\left((0.1\text{m})^2+\left(\frac{0.3\text{m}}{6}\right)^2\right)^{\frac{1}{2}}}\right]\hat{k} = 2.07\text{x}10^{-6}\text{T}\,\hat{k}$$

d. Determine the magnitude and direction of the magnetic field at point P for a wire of "infinite" length.

Ans. $$\vec{B}=\frac{\mu_o i}{4\pi}\frac{x}{y\left(y^2+x^2\right)^{\frac{1}{2}}}\hat{k}\Bigg|_{-\infty}^{\infty}=\frac{\mu_o i}{2\pi y}\hat{k} = 3.0\text{x}10^{-6}\text{T}\ \hat{k}$$

2. A quarter circle wire arc (90°), with radius of curvature R=0.1 m, has a 2.0 A current running through it.

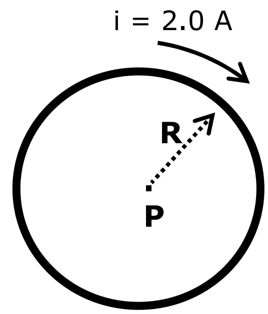


a. Using the Biot-Savart Law, determine the equation for magnetic field vector produced by the arc at point P, located at the center of curvature.

Ans. $$d\vec{B}=\frac{\mu_o i}{4\pi}\frac{d\vec{s}\times\vec{r}}{r^3}=-\frac{\mu_o i}{4\pi}\frac{R^2 d\phi}{R^3}\hat{k}=-\frac{\mu_o i}{4\pi}\frac{d\phi}{R}\hat{k}$$
$$\vec{B}=\int d\vec{B}=-\frac{\mu_o i}{4\pi R}\int_0^{\phi} d\phi\ \hat{k} = -\frac{\mu_o i}{4\pi R}\phi\ \hat{k}$$

b. Calculate the magnetic field vector produced by the arc at point P.

Ans. $$\vec{B}= -\frac{\mu_o i}{4\pi R}\phi\ \hat{k} = -3.1\text{x}10^{-6}\text{T}\ \hat{k}$$



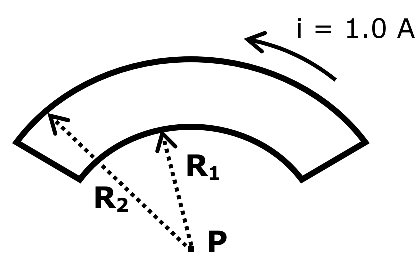
c. Derive the magnetic field vector at the center of a circular loop, of radius R.

Ans. $$\vec{B}=\int d\vec{B}=-\frac{\mu_o i}{4\pi R}\int_0^{2\pi} d\phi\ \hat{k} = -\frac{\mu_o i}{2R}\ \hat{k}$$

d. Determine the magnitude and direction of the magnetic field at the center of a circular loop, where R=0.1 m.

Ans. $$\vec{B}= -\frac{\mu_o i}{2R}\ \hat{k} = -1.26\text{x}10^{-7}\text{T}\ \hat{k}$$

3. Consider the current loop shown below, consisting of 2 concentric 90° arc segments of radii, R_1 and R_2, connected by 2 segments that are parallel to the center of curvature for the arcs. The current flowing through the loop is 1.0 A, counter-clockwise.



a. Using the Biot-Savart Law, determine the equations for the magnetic field vectors produced by each segment of the loop arc at point P, located at the center of curvature for both arc segments.

Ans. Each arc produces an oppositely directed B-field:
$$\vec{B}_1= -\frac{\mu_o\phi i}{4\pi R_1}\ \hat{k} \text{ and } \vec{B}_2= \frac{\mu_o\phi i}{4\pi R_2}\ \hat{k}$$

b. Calculate the total magnetic field vector at point P, where R_1=0.1 m and R_2= 0.15 m.

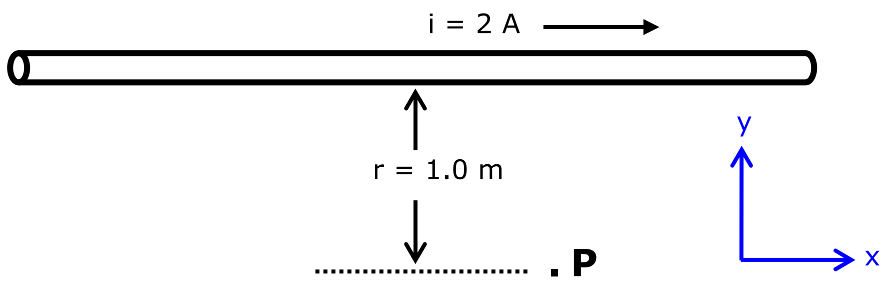
Ans. Each arc produces an oppositely directed B-field:

$$\vec{B} = \vec{B}_1+\vec{B}_2=\frac{\mu_o \phi i}{4\pi}\left(-\frac{1}{R_1}+\frac{1}{R_2}\right)\hat{k} = -5.25\text{x}10^{-7}\text{T}\ \hat{k}$$

c. Determine the net magnetic force vector exerted on the outer arc segment of the loop due to the other segments.

Ampere's Law

4. An infinitely long horizontal wire has a 2A current running through it.



a. Using Ampere's Law, determine the magnitude and direction of magnetic field produced by the wire at a point 1 m below the wire?

Ans. Apply a circular "Amperean loop" centered on the wire:

$$\oint \vec{B}\cdot d\vec{\ell} = \left|\vec{B}\right|(2\pi r) = \mu_o i_{enc}$$

$$\left|\vec{B}\right| = \frac{\mu_o i_{enc}}{2\pi r} = -4.01\text{x}10^{-7}\text{T} \Rightarrow \vec{B} = -4.01\text{x}10^{-7}\ \text{T}\ \hat{k}$$

b. An electron travels with a velocity: $\vec{v} = 2\text{x}10^{3}\,\frac{m}{s}\hat{j}$ at point P. Determine the magnetic force vector exerted on the electron at point P.

Ans. $\vec{F}_B = e\vec{v}\times\vec{B} = evB\ \hat{i} = 1.28\text{x}10^{-22}\text{N}\ \hat{i}$

c. A second wire oriented parallel to wire 1 above, with a current of 1 A (in the same direction), is then placed 1.0 meter directly below the first wire. What is the magnetic field vector exactly halfway between the wires?

Ans. The 2 B-fields are additive:

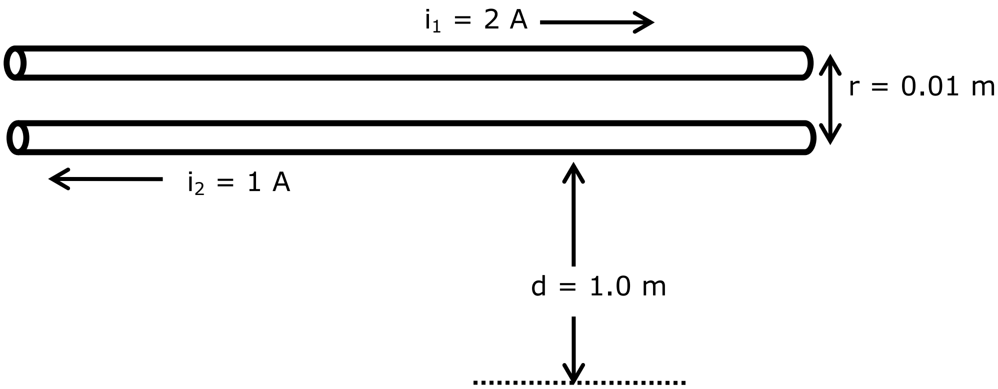
$$\vec{B} = \vec{B}_1+\vec{B}_2 = -\frac{\mu_o}{2\pi r}(i_1+i_2)\hat{k} = -1.20\text{x}10^{-6}\text{T}\ \hat{k}$$

d. What is the magnetic force acting on a 1 m segment of the 2nd wire?

Ans. $\vec{F}_B = i_2\vec{L}_2\times\vec{B}_1 = -i_2L_2B_1\hat{j} = -4.01\text{x}10^{-7}\text{N}\ \hat{j}$

e. Derive the equation for the magnetic field vector between the wires in the plane of the wires.

5. Two infinitely long horizontal wires separated by 0.01 m have a current running through them (in opposite directions).



a. Using Ampere's Law, determine the magnitude and direction of magnetic field produced by both wires at a point 1 m below the lowest wire.

Ans. Since r<<d, apply a circular "Amperean loop" centered halfway between the wires:

$$\oint \vec{B}\cdot d\vec{\ell} = \left|\vec{B}\right|(2\pi r) = \mu_o i_{enc} = \mu_o (i_1 - i_2)$$

$$\left|\vec{B}\right| = \frac{\mu_o (i_1 - i_2)}{2\pi r} = 2.01\text{x}10^{-7}\text{T} \Rightarrow \vec{B} = -2.01\text{x}10^{-7}\text{T}\ \hat{k}$$

b. Determine the magnitude and direction of the magnetic field produced by both wires at a point halfway between the two wires in (b).

Ans. Apply a circular "Amperean loop" around each wire separately:

$$\left|\vec{B}_1\right|(2\pi r) = \mu_o i_1 \Rightarrow \left|\vec{B}_1\right| = 8.02\text{x}10^{-5}\text{T}$$

$$\left|\vec{B}_2\right|(2\pi r) = \mu_o i_2 \Rightarrow \left|\vec{B}_2\right| = 4.01\text{x}10^{-5}\text{T}$$

$$\vec{B}_{net} = \vec{B}_1 + \vec{B}_2 = -\left(\left|\vec{B}_1\right| + \left|\vec{B}_2\right|\right)\hat{k} = -1.203\text{x}10^{-4}\text{T}\ \hat{k}$$

c. What is the magnetic force vector exerted by wire 1 on wire 2?

Ans. $$\vec{F}_B = i_2\vec{L}_2 \times \vec{B}_1 = -i_2 L_2\left(\frac{\mu_o i_1}{2\pi r}\right)\hat{j} \Rightarrow \frac{\vec{F}_B}{L_2} = -6.37\text{x}10^{-6}\ \tfrac{\text{N}}{\text{m}}\ \hat{j}$$

d. If the separation between the wires were 1 m, determine the magnitude and direction of magnetic field produced by both wires at a point 1 m below the lowest wire.

Ans. Apply a circular "Amperean loop" around each wire separately:

$$\left|\vec{B}_1\right|(4\pi d) = \mu_o i_1 \Rightarrow \left|\vec{B}_1\right| = \frac{\mu_o i_1}{4\pi d}$$

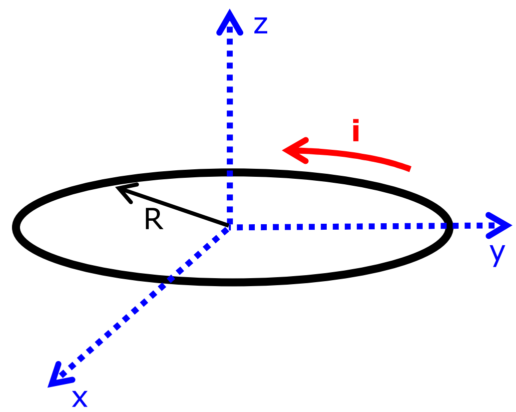
$$\left|\vec{B}_2\right|(2\pi d) = \mu_o i_2 \Rightarrow \left|\vec{B}_2\right| = \frac{\mu_o i_2}{2\pi d}$$

$$\vec{B}_{net} = \vec{B}_1 + \vec{B}_2 = \left(\left|\vec{B}_1\right| - \left|\vec{B}_2\right|\right)\hat{k} = \left(\frac{\mu_o i_1}{4\pi d} - \frac{\mu_o i_2}{2\pi d}\right)\hat{k} = 0\hat{k}$$

e. The current in wire 2 is increased to 2A. What is the magnetic force vector exerted by wire 2 on wire 1?

Ans. $\vec{F}_B = i_1\vec{L}_1\times\vec{B}_2 = i_1L_1\left(\frac{\mu_o i_2}{2\pi r}\right)\hat{j} \Rightarrow \frac{\vec{F}_B}{L_1} = 1.27\text{x}10^{-5}\,\frac{N}{m}\,\hat{j}$

6. Consider a current carrying coil (i=0.5 A, counter-clockwise as viewed looking down), where the number of coils is 20 and the radius is 5.0 cm.



a. Calculate the magnitude of the magnetic moment for the coil.

Ans. $\vec{\mu}_o = NiA\,\hat{j} = 7.85\text{x}10^{-2}A\cdot m^2\,\hat{j}$

b. Derive the magnetic field B along the central axis (i.e. in the z direction) of the coil.

Ans.
$$d\vec{B}=\frac{\mu_o i}{4\pi}\frac{d\vec{s}\times\vec{r}}{r^3}=\frac{\mu_o i}{4\pi}\frac{r^2d\theta}{r^3}\sin\phi\,\hat{k} = \frac{\mu_o i}{4\pi}\frac{r^2d\theta}{r^3}\left(\frac{R}{r}\right)\hat{k}$$
$$\vec{B}=\int d\vec{B}=-\frac{N\mu_o iR}{4\pi\left(z^2+R^2\right)}\int_0^{2\pi}d\theta\,\hat{k} = \frac{N\mu_o iR}{2\left(z^2+R^2\right)}\hat{k}$$

c. What is the magnetic field at z=20 cm above the coil?

Ans. $\vec{B}=\frac{N\mu_o iR}{2\left(z^2+R^2\right)}\hat{k} = 7.87\text{x}10^{-7}T\,\hat{k}$

d. What is the magnetic field vector at z=40 cm above the coil?

Ans. $\vec{B}=\frac{N\mu_o iR}{2\left(z^2+R^2\right)}\hat{k} = 1.96\text{x}10^{-7}T\,\hat{k}$

e. What is the magnetic field vector at z=80 cm above the coil?

Ans. $\vec{B}=\frac{N\mu_o iR}{2\left(z^2+R^2\right)}\hat{k} = 4.90\text{x}10^{-8}T\,\hat{k}$

f. At what z distance does the magnetic field become effectively independent of the radius, i.e. the magnetic field can be calculated to within 3% when the radius is ignored?

Ans.

$$\frac{\left|\vec{B}_{R=0}\right|}{\left|\vec{B}_{R\neq0}\right|} = \frac{\dfrac{N\mu_o iR}{2z^2}}{\dfrac{N\mu_o iR}{2\left(z^2+R^2\right)}} = \frac{\dfrac{1}{z^2}}{\dfrac{1}{\left(z^2+R^2\right)}} = \frac{z^2+R^2}{z^2} \geq 0.97$$

$$z^2 \geq \frac{0.97}{1-0.97}R^2 \Rightarrow z^2 \geq \sqrt{\frac{0.97}{1-0.97}}R = 5.9R \text{ (or 28.4 cm)}$$