Phy 213: General Physics III

Chapter 25: Capacitors
Lecture Notes

Capacitors

1. For 2 charged conducting surfaces, with equal quantity but opposite sign charge, ∆V between the charged faces produces an electric field (E).
2. The electric potential (∆V) between the faces is related to the electric field between them (and vice versa)
   \[ \Delta V = \int dV = - \int_{\text{face 2}}^{\text{face 1}} \vec{E} \cdot d\vec{s} \]
   or
   \[ \vec{E} = - \frac{dV}{ds} \hat{s} = - \frac{dV}{dx} \hat{i} - \frac{dV}{dy} \hat{j} - \frac{dV}{dz} \hat{k} \]
3. The charge at each face is proportional to the potential difference
   \[ q \sim \Delta V \Rightarrow \frac{q}{\Delta V} = \text{constant} = C \]
4. The proportionality constant (C) between q and ∆V is the capacitance (C).
5. The SI units of capacitance are C/V (called farads or F)
Capacitors as Differentiators

1. As the potential difference across a capacitor, $C$, increases by a small amount, $dV$, a corresponding amount of charge, $dq$, accumulates on each face: \[ \Delta V = V \text{ and } V_0 = 0 \text{ V} \]
   \[ dq = C dV \{ \text{note that } q(t) = C \cdot V(t) \} \]
2. As $dV$ occurs over a small time increment, $dt$, the rate of charge into/out of the capacitor can be determined by dividing through by $dt$:
   \[ \frac{dq}{dt} = C \frac{dV}{dt} = \text{rate of charge flow} \]
3. The quantity, $dq/dt$, is referred to as the electric current and the change in charge is the area under the $dq/dt$ vs $t$ graph:
   \[ \Delta q = \int_{t_i}^{t_f} \left( \frac{dq}{dt} \right) dt = \text{the area under the } \frac{dq}{dt} \text{ vs } t \text{ graph} \]

Parallel Plate Capacitors

Consider charged 2 parallel plates of equal area, evenly separated a distance $d$:

1. The capacitance is related to the accumulated charge $q$ and potential difference between the plates:
   \[ C = \frac{q}{1. \Delta V} \text{ where } \Delta V = - \int_{0}^{d} \bar{E} \cdot d\vec{s} \]
2. The electric field $E$ between the plates is:
   \[ E = \frac{\sigma}{\varepsilon_0} = \frac{q}{A \varepsilon_0} \]
   \[ \Rightarrow \Delta V = V_+ - V_- = -\int_{0}^{d} \bar{E} \cdot d\vec{x} = \int_{0}^{d} E \cdot dx = \frac{qd}{A \varepsilon_0} \]
3. The capacitance is given by:
   \[ C = \frac{q}{\Delta V} = \frac{A \varepsilon_0}{d} \]
Cylindrical Capacitors

For 2 concentric charged cylindrical plates of length L, with radii \( R_1 \) and \( R_2 \) (where \( R_2 - R_1 \ll L \))

1. The E field between the plates:
\[
\vec{E} = -\frac{\lambda}{2\pi\varepsilon_0 r} \hat{\l}\quad \text{or} \quad\vec{E} = -\frac{q}{2\pi L \varepsilon_0 r} \hat{\l}
\]

2. The potential difference between the plates is:
\[
\Delta V = \int_{R_1}^{R_2} \vec{E} \cdot d\vec{r} = \frac{q}{(2\pi\varepsilon_0)L} \int_{R_1}^{R_2} \frac{dr}{r} = \frac{q}{(2\pi\varepsilon_0)L} \ln\left(\frac{R_2}{R_1}\right)
\]

3. The capacitance is:
\[
C = \frac{q}{\Delta V} = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{R_2}{R_1}\right)}
\]

Spherical Capacitors

For 2 concentric charged cylindrical plates of length L, with radii \( R_1 \) and \( R_2 \) (where \( R_2 - R_1 \ll L \))

1. The E field between the plates:
\[
\vec{E} = -\frac{q}{4\pi\varepsilon_0 r^2} \hat{\l}
\]

2. The potential difference between the plates is:
\[
\Delta V = \int_{R_1}^{R_2} \vec{E} \cdot d\vec{r} = \frac{q}{4\pi\varepsilon_0} \int_{R_1}^{R_2} \frac{dr}{r^2} = \frac{q}{4\pi\varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2}\right)
\]

3. The capacitance is:
\[
C = \frac{q}{\Delta V} = \frac{4\pi\varepsilon_0}{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)} = 4\pi\varepsilon_0 \left(\frac{R_1 R_2}{R_2 - R_1}\right)
\]
Energy Stored in a Capacitor

1. It takes work to charge a capacitor, that work is stored as electrical potential energy:

\[ \Delta U = U_2 - U_1 = -W = \int_{V_1}^{V_2} qdV = C \int_{V_1}^{V_2} dV = \frac{1}{2} CV_2^2 - \frac{1}{2} CV_1^2 \]

2. Therefore, the potential energy stored in a capacitor can be defined as:

\[ U = \frac{1}{2} CV^2 \]

3. The energy density, energy per unit volume, is given by:

\[ u = \frac{U}{V} = \frac{CV^2}{2V} \]

where \( V \) is the volume between the plates

4. For a parallel plate capacitor with plate area \( A_{\text{plate}} \) and separation \( d \):

\[ u = \frac{U}{V} = \frac{CV^2}{2A_{\text{plate}}d} = \frac{\varepsilon_0 V^2}{2d^2} \]

Capacitors in Parallel

When 2 or more capacitors are connected in parallel, attached so that they share the same potential difference across their plates, the combined (equivalent) capacitance is additive:

\[ C_{\text{eq}} = C_1 + C_2 + \ldots + C_N = \sum_{i=1}^{N} C_i \]

\[ \frac{q_{\text{tot}}}{V} = C_1 + C_2 + \ldots + C_N = C_{\text{eq}} \]
Capacitors in Series

When 2 or more capacitors are connected in series, attached end to end, the combined (equivalent) capacitance is given by:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N} = \sum_{i=1}^{N} \frac{1}{C_i}$$

**Justification:** The total $V$ across all of the capacitors is the sum of the potential across each individual capacitor

$$V = V_1 + V_2 + \ldots + V_N = \frac{q}{C_1} + \frac{q}{C_2} + \ldots + \frac{q}{C_N}$$

$$V = q \left( \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N} \right) \Rightarrow V = \frac{1}{C_{eq}} \cdot \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}$$

Capacitors & Dielectrics

1. In general, the charge capacity of a capacitor depends on the material (or dielectric) or lack thereof between the plates.
2. In general, for any type of capacitor, the constant, $\varepsilon_0$, can be replaced by an effective permittivity constant, $\varepsilon = \kappa \varepsilon_0$.
   a. For vacuum or air-filled capacitor: $\kappa \approx 1$
   b. For other materials: $\kappa > 1$
3. For a parallel plate capacitor:
   $$C = \frac{\varepsilon A}{d} = \frac{\kappa \varepsilon_0 A}{d}$$
4. For a cylindrical capacitor:
   $$C = \frac{2\pi \varepsilon L}{\ln\left(\frac{R_2}{R_1}\right)} = \frac{2\pi \kappa \varepsilon_0 L}{\ln\left(\frac{R_2}{R_1}\right)}$$
5. For a spherical capacitor:
   $$C = 4\pi \varepsilon \left(\frac{R_1 R_2}{R_2 - R_1}\right) = 4\pi \kappa \varepsilon_0 \left(\frac{R_1 R_2}{R_2 - R_1}\right)$$