Phy 213: General Physics III

Chapter 25: Capacitors
Lecture Notes

Capacitors

1. For 2 charged conducting surfaces, with equal quantity but opposite sign charge, ΔV between the charged faces produces an electric field (E)
2. The electric potential (ΔV) between the faces is related to the electric field between them (and vice versa)

$$\Delta V = \int dV = -\int_{\text{face 1}}^{\text{face 2}} \vec{E}\cdot d\vec{s}$$

or

$$\vec{E} = -\frac{dV}{ds}\hat{i}_s = -\frac{dV}{dx}\hat{i} - \frac{dV}{dy}\hat{j} - \frac{dV}{dz}\hat{k}$$

3. The charge at each face is proportional to the potential difference $q \sim \Delta V \Rightarrow \frac{q}{\Delta V} = \text{constant} = C$
4. The proportionality constant (C) between q and ΔV is the capacitance (C).
5. The SI units of capacitance are C/V (called farads or F)

Capacitors as Differentiators

1. As the potential difference across a capacitor, C, increases by a small amount, dV, a corresponding amount of charge, dq, accumulates on each face: {where $\Delta V = V$ and $V_o = 0$ V}

$$dq = C\,dV \quad \{\text{note that } q(t) = C \cdot V(t)\}$$

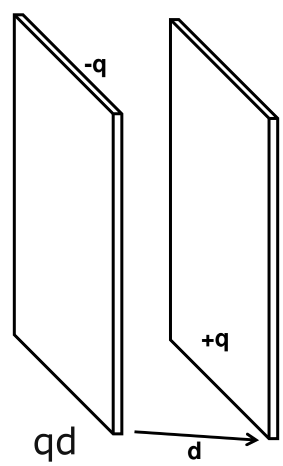
2. As dV occurs over a small time increment, dt, the rate of charge into/out of the capacitor can be determined by dividing through by dt:

$$\frac{dq}{dt} = C\frac{dV}{dt} = \text{rate of charge flow}$$

3. The quantity, dq/dt, is referred to as the electric current and the change in charge is the area under the dq/dt vs t graph

$$\Delta q = \int_{t_1}^{t_2} \left(\frac{dq}{dt}\right) dt = \text{the area under the } \frac{dq}{dt} \text{ vs t graph}$$

Parallel Plate Capacitors



Consider charged 2 parallel plates of equal area, evenly separated a distance d:

1. The capacitance is related to the accumulated charge q and potential difference between the plates:

$$C = \frac{q}{\Delta V} \quad \text{where} \quad \Delta V = -\int_0^d \vec{E} \cdot d\vec{s}$$

2. The electric field E between the plates is:

$$E = \frac{\sigma}{\varepsilon_o} = \frac{q}{A\varepsilon_o}$$

$$\Rightarrow \Delta V = V_+ - V_- = -\int_0^d \vec{E} \cdot d\vec{x} = \int_0^d E \cdot dx = \frac{qd}{A\varepsilon_o}$$

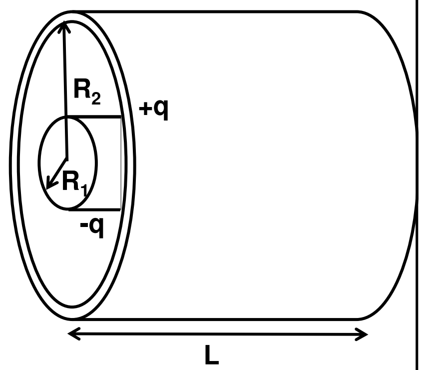
3. The capacitance is given by: $C = \dfrac{q}{\Delta V} = \dfrac{A\varepsilon_o}{d}$

Cylindrical Capacitors

For 2 concentric charged cylindrical plates of length L, with radii R_1 and R_2 (where $R_2 - R_1 << L$)

1. The E field between the plates:

$$\vec{E} = -\frac{\lambda}{2\pi\varepsilon_o r}\hat{i}_r = -\frac{q}{2\pi L\varepsilon_o r}\hat{i}_r$$



2. The potential difference between the plates is:

$$\Delta V = -\int_{R_1}^{R_2}\vec{E}\cdot d\vec{r} = \frac{q}{(2\pi\varepsilon_o)L}\int_{R_1}^{R_2}\frac{dr}{r}$$

$$\Delta V = V_2 - V_1 = \frac{q}{(2\pi\varepsilon_o)L}\ln\left(\frac{R_2}{R_1}\right)$$

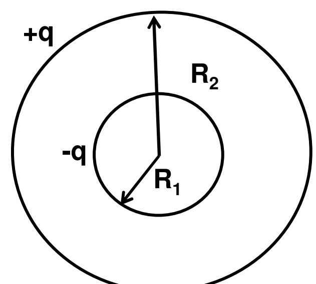
3. The capacitance is: $C = \frac{q}{\Delta V} = \frac{2\pi\varepsilon_o L}{\ln\left(\frac{R_2}{R_1}\right)}$

Spherical Capacitors

For 2 concentric charged cylindrical plates of length L, with radii R_1 and R_2 (where $R_2 - R_1 << L$)

1. The E field between the plates:

$$\vec{E} = -\frac{q}{4\pi\varepsilon_o r^2}\hat{i}_r$$



2. The potential difference between the plates is:

$$\Delta V = -\int_{R_1}^{R_2}\vec{E}\cdot d\vec{r} = \frac{q}{4\pi\varepsilon_o}\int_{R_1}^{R_2}\frac{dr}{r^2}$$

$$\Delta V = V_2 - V_1 = \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

3. The capacitance is:

$$C = \frac{q}{\Delta V} = \frac{4\pi\varepsilon_o}{\left(\frac{1}{R_1}-\frac{1}{R_2}\right)} = 4\pi\varepsilon_o\left(\frac{R_1R_2}{R_2-R_1}\right)$$

Energy Stored in a Capacitor

1. It takes work to charge a capacitor, that work is stored as electrical potential energy:

$$\Delta U = U_2 - U_1 = -W = \int_{V_1}^{V_2} q\,dV = C\int_{V_1}^{V_2} V\,dV = \tfrac{1}{2}CV_2^2 - \tfrac{1}{2}CV_1^2$$

2. Therefore, the potential energy stored in a capacitor can be defined as:

$$U = \tfrac{1}{2}CV^2$$

3. The energy density, energy per unit volume, is given by:

$$u = \frac{U}{\mathcal{V}} = \frac{CV^2}{2\mathcal{V}}$$

   where $\mathcal{V}$ is the volume between the plates

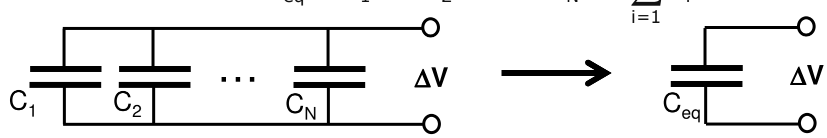
4. For a parallel plate capacitor with plate area A_{plate} and separation d:

$$u = \frac{U}{\mathcal{V}} = \frac{CV^2}{2A_{plate}d} = \frac{\varepsilon_o V^2}{2d^2}$$

Capacitors in Parallel

When 2 or more capacitors are connected in parallel, attached so that they share the same potential difference across their plates, the combined (equivalent) capacitance is additive:

$$C_{eq} = C_1 + C_2 + \ldots C_N = \sum_{i=1}^{N} C_i$$



Justification: Each capacitor has certain amount of q at a given V and q_{tot} is given by:

$$q_{tot} = q_1 + q_2 + \ldots + q_N = C_1V + C_2V + \ldots + C_NV = (C_1 + C_2 + \ldots + C_N)V$$

or

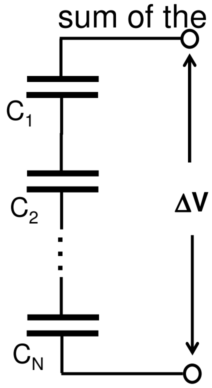
$$\frac{q_{tot}}{V} = \mathbf{C_1 + C_2 + \ldots + C_N = C_{eq}}$$

Capacitors in Series

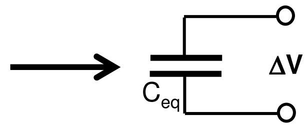
When 2 or more capacitors are connected in series, attached end to end, the combined (equivalent) capacitance is given by:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots \frac{1}{C_N} = \sum_{i=1}^{N} \frac{1}{C_i}$$

Justification: The total V across all of the capacitors is the sum of the potential across each individual capacitor



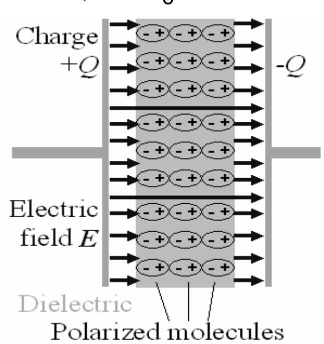
$$V = V_1 + V_2 + \ldots + V_N = \frac{q}{C_1} + \frac{q}{C_2} + \ldots \frac{q}{C_N}$$



$$V = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \ldots \frac{1}{C_N}\right) \Rightarrow \frac{V}{q} = \boldsymbol{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots \frac{1}{C_N}}$$

Capacitors & Dielectrics

1. In general, the charge capacity of a capacitor depends on the material (or dielectric) or lack thereof between the plates
2. In general, for any type of capacitor, the constant, ε_o can be replaced by an effective permittivity constant, $\varepsilon = \kappa\varepsilon_o$
   a. For vacuum or air-filled capacitor: $\kappa \sim 1$
   b. For other materials: $\kappa > 1$



3. For a parallel plate capacitor:

$$C = \frac{\varepsilon A}{d} = \frac{\kappa\varepsilon_o A}{d}$$

4. For a cylindrical capacitor:

$$C = \frac{2\pi\varepsilon L}{\ln\left(\frac{R_2}{R_1}\right)} = \frac{2\pi\kappa\varepsilon_o L}{\ln\left(\frac{R_2}{R_1}\right)}$$

5. For a spherical capacitor:

$$C = 4\pi\varepsilon\left(\frac{R_1 R_2}{R_2 - R_1}\right) = 4\pi\kappa\varepsilon_o\left(\frac{R_1 R_2}{R_2 - R_1}\right)$$