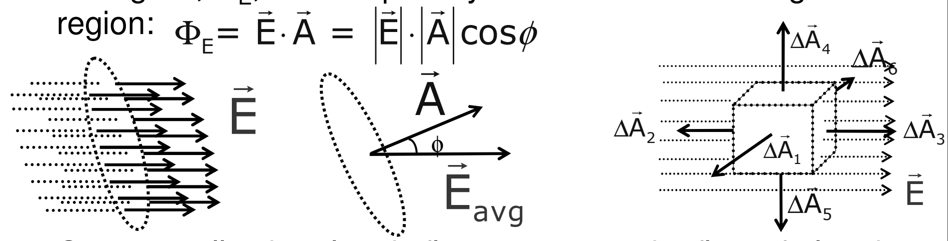
Phy 213: General Physics III

Chapter 23: Gauss' Law

Lecture Notes

Electric Flux

1. Consider a electric field passing through a flat region in space w/ area=A. The area vector ($\vec{A}$) with a magnitude of A and is directed normal to the surface. The **electric flux** through A, Φ_E, or the "quantity" of electric field through that region: $\Phi_E = \vec{E}\cdot\vec{A} = \left|\vec{E}\right|\cdot\left|\vec{A}\right|\cos\phi$



2. <u>Conceptually</u>, the electric flux represents the flow of electric field lines (normal) through a region of space
3. In general, the electric flux through a region of n areas, $\Delta A_{i,}$, is the sum of the fluxes through all of those regions:

$$\Phi_E = \Phi_1 + \Phi_2 + \ldots + \Phi_i = \sum_i \vec{E}_i \cdot \Delta\vec{A}_i$$

Gauss' Law

1. Gauss' Law is a fundamental "conservation" law of nature relating electric charge to electric flux
2. According to Gauss' Law, the total electric flux through any closed ("Gaussian") surface is equal to the enclosed charge ($Q_{enclosed}$) divided by the permittivity of free space (ε_o):
$$\Phi_E = \sum_i \vec{E}_i \cdot \Delta\vec{A}_i = \frac{q_{enclosed}}{\varepsilon_o}$$

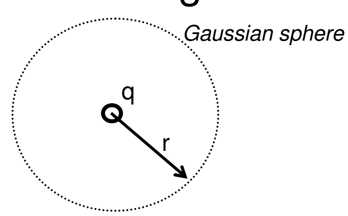
Or in general,

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

3. Gauss' Law can be used to determine the electric field (E) for many physical orientations (or distributions) of charge
4. Many consequences of Gauss' Law provide insights that are not necessarily obvious when applying Coulomb's Law

Gauss' Law for a Point Charge

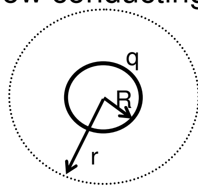
Consider a point charge, q



1. An appropriate "gaussian" surface for a point charge is a concentric sphere, radius r, since all E fields will be ∟ to the surface
2. The electric flux through the surface is:
$$\Phi_E = \int_{\substack{closed\\surface}} \vec{E} \cdot d\vec{A} = \vec{E} \cdot \int_{sphere} d\vec{A} = \vec{E} \cdot \vec{A} = E\left(4\pi r^2\right)$$
3. Applying Gauss' Law yields E:
$$\Phi_E = E\left(4\pi r^2\right) = \frac{q_{enclosed}}{\varepsilon_o} \Rightarrow E = \frac{q_{enclosed}}{4\pi\varepsilon_o r^2}$$

Spherical Symmetry: Charged Sphere

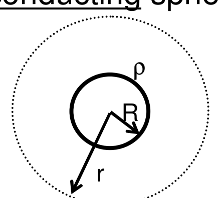
1. Spherical symmetry can be exploited for any spherical shape
2. Consider a hollow conducting sphere, radius R & charge q:



3. The flux through a "gaussian" sphere is: $\Phi_E = \vec{E} \cdot \vec{A} = \left|\vec{E}\right|\left(4\pi r^2\right)$
4. Applying Gauss' Law: $\Phi_E = \left|\vec{E}\right|\left(4\pi r^2\right) = \frac{q_{enclosed}}{\varepsilon_o} \Rightarrow \left|\vec{E}\right| = \frac{q_{enclosed}}{4\pi\varepsilon_o r^2}$
5. For r > R: $\left|\vec{E}\right| = \frac{q_{enclosed}}{4\pi\varepsilon_o r^2}$
6. For r < R: $\left|\vec{E}\right| = \frac{q_{enclosed}}{4\pi\varepsilon_o r^2} = 0 \ \{q_{enclosed} = 0C\}$ Inside a hollow conductor, E = 0

Spherical Symmetry: Charged Sphere (2)

Consider a solid <u>non-conducting</u> sphere with radius R & uniform charge density, ρ:

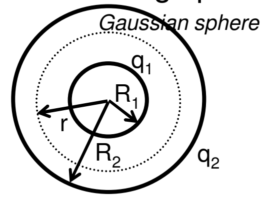


1. Applying Gauss' Law: $\Phi_E = \left|\vec{E}\right|\left(4\pi r^2\right) = \frac{q_{enclosed}}{\varepsilon_o} \Rightarrow \left|\vec{E}\right| = \frac{q_{enclosed}}{4\pi\varepsilon_o r^2}$
2. For r > R: $\left|\vec{E}\right| = \frac{q_{enclosed}}{4\pi\varepsilon_o r^2} = \frac{\rho\left(\frac{4}{3}\pi R^3\right)}{4\pi\varepsilon_o r^2} = \frac{\rho R^3}{3\varepsilon_o r^2}$
3. For r < R: $\left|\vec{E}\right| = \frac{q_{enclosed}}{4\pi\varepsilon_o r^2} = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{4\pi\varepsilon_o r^2} = \frac{\rho r}{3\varepsilon_o}$

Inside a uniformly charged insulator, E ≠ 0

Spherical Symmetry: Charged Spheres (3)

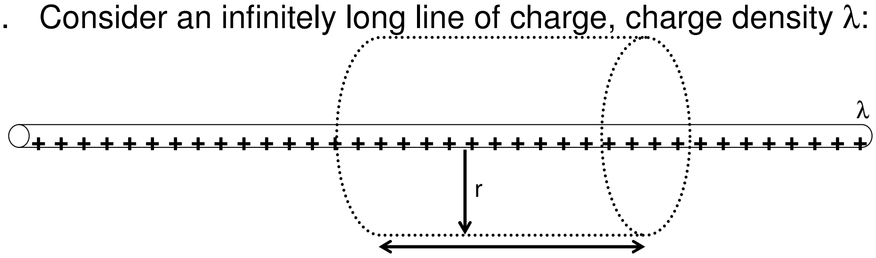
Consider 2 concentric hollow conducting spheres with radii R_1 & R_1 and charges q_1 & q_2:



1. Applying Gauss' Law: $\Phi_E = \left|\vec{E}\right|\left(4\pi r^2\right) = \frac{q_{enclosed}}{\varepsilon_o} \Rightarrow \left|\vec{E}\right| = \frac{q_{enclosed}}{4\pi\varepsilon_o r^2}$
2. For $r < R_1$: $\left|\vec{E}\right| = \frac{q_{enclosed}}{4\pi\varepsilon_o r^2} = 0$
3. For $R_2 > r > R_1$: $\left|\vec{E}\right| = \frac{q_{enclosed}}{4\pi\varepsilon_o r^2} = \frac{q_1}{4\pi\varepsilon_o r^2}$
4. For $r > R_2$: $\left|\vec{E}\right| = \frac{q_{enclosed}}{4\pi\varepsilon_o r^2} = \frac{(q_1+q_2)}{4\pi\varepsilon_o r^2}$ *{opposite charge subtracts}*

Application: Line of Charge

1. Cylindrical symmetry is useful for evaluating a "line of charge" as well as charged cylindrical shapes
2. Consider an infinitely long line of charge, charge density λ:



3. The flux through each of the 3 surfaces is:

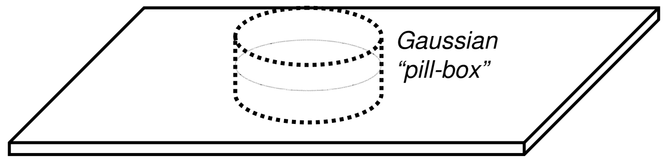
$$\Phi_E = \Phi_1 + \Phi_2 + \Phi_3 = \Phi_3 \quad \text{since } \Phi_1 = \Phi_2 = 0$$

$$\Phi_E = E(2\pi r)x = \frac{q_{enclosed}}{\varepsilon_o} = \frac{\lambda x}{\varepsilon_o}$$

4. Applying Gauss' Law: $E = \frac{\lambda}{2\pi\varepsilon_o r}$

Planar Symmetry: A Charged Sheet

1. A Gaussian "pill-box" is an appropriate Gaussian surface for evaluating a charged flat sheet
2. Consider a uniformly charged flat sheet with a surface charge density, σ:



3. The flux through the pill-box surface is:

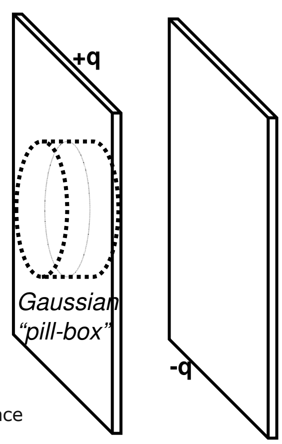
$$\Phi_E = \Phi_{side} + \Phi_{face1} + \Phi_{face2} = 2\Phi_{face} \qquad \{\Phi_{face} = \Phi_{face1} = \Phi_{face2}\}$$

4. The electric field:

$$\Phi_E = 2\left|\vec{E}\right|\left(\pi r^2\right) = \frac{q_{enclosed}}{\varepsilon_o} \Rightarrow \left|\vec{E}\right| = \frac{q_{enclosed}}{2\pi\varepsilon_o r^2} = \frac{\sigma\pi r^2}{2\pi\varepsilon_o r^2} = \frac{\sigma}{2\varepsilon_o}$$

Application: 2 Parallel Charged Plates

1. Two oppositely charged conducting plates, with charge density σ:
2. Surface charges draw toward each other on the inner face of each plate
3. To determine the electric field within the plates, apply a Gaussian "pill-box" to one of the plates



4. The flux through the pill-box surface is: $\Phi_E = \Phi_{side} + \Phi_{inner\ face} + \Phi_{outer\ face} = \Phi_{outer\ face}$

$$\{where\ \Phi_{inner\ face} = \Phi_{side} = 0\}$$

5. The electric field is:

$$\Phi_E = \left|\vec{E}\right|\left(\pi r^2\right) = \frac{q_{enclosed}}{\varepsilon_o} \Rightarrow \left|\vec{E}\right| = \frac{q_{enclosed}}{\pi\varepsilon_o r^2} = \frac{\sigma\pi r^2}{\pi\varepsilon_o r^2} = \frac{\sigma}{\varepsilon_o}$$

Consequences of Gauss' Law

1. Electric flux is a conserved quantity for an enclosed electric charge
2. The electric field inside a charged solid conductor is zero
3. The electric field inside a charged hollow conductor or non-conductor is zero
4. Geometrical symmetries can make the calculation of an electric field less cumbersome even for complicated charge distributions