Entropy:

1. A sample of 10.0 moles of a monatomic ideal gas, held at constant temperature (1000K), is expanded from 0.10 m^3 to 0.20 m^3. Calculate the entropy change during this process.

Ans. $\Delta S = \frac{Q}{T} = \frac{nRT\cdot\ln\left(\frac{V_f}{V_i}\right)}{T} = (10.0\ \text{mol})(8.314\frac{J}{mol\cdot K})\ln(2) = 57.6\frac{J}{K}$

2. A sample of 10.0 moles of a monatomic ideal gas, held at constant pressure (1.5 atm or $1.52\text{x}10^5$ Pa), is compressed from 0.10 m^3 to 0.05 m^3. Calculate the entropy change during this process.

Ans.

$$\Delta S = \int_{S_i}^{S_f} dS = \int_{Q_i}^{Q_f}\frac{dQ}{T} = \int_{T_i}^{T_f}\frac{\frac{3}{2}nRdT}{T} + \int_{V_i}^{V_f}\frac{pdV}{T} = \int_{T_i}^{T_f}\frac{\frac{3}{2}nRdT}{T} + \int_{V_i}^{V_f}\frac{pdV}{T} = \int_{T_i}^{T_f}\frac{\frac{5}{2}nRdT}{T} = \tfrac{5}{2}nR\ln\left(\frac{T_f}{T_i}\right)$$

$$\Delta S = \tfrac{5}{2}(10.0\ \text{mol})(8.314\tfrac{J}{mol\cdot K})\ln(\tfrac{1}{2}) = -144.\tfrac{J}{K}$$

3. A sample of 10.0 moles of a monatomic ideal gas, held at constant volume (1.0 m^3), is heated from 300 K to 400 K. Calculate the entropy change during this process.

Ans. $\Delta S = \int_{Q_i}^{Q_f}\frac{dQ}{T} = \int_{T_i}^{T_f}\frac{\frac{3}{2}nRdT}{T} = \tfrac{3}{2}nR\ln\left(\frac{T_f}{T_i}\right) = \tfrac{3}{2}(10.0\ \text{mol})(8.314\tfrac{J}{mol\cdot K})\ln(\tfrac{400K}{300K}) = 35.9\tfrac{J}{K}$

4. A sample of 1.5 moles of a thermally insulated (adiabatic) monatomic ideal gas is expanded from 0.10 m^3 to 0.20 m^3. During the expansion, the pressure of the gas decreases from $3.039\text{x}10^5$ Pa to $2.026\text{x}10^5$ Pa. Calculate the entropy change during this process.

Ans. $\Delta S = \int_{S_i}^{S_f} dS = \int_{Q_i}^{Q_f}\frac{dQ}{T} = 0\tfrac{J}{K}$

Heat Engines:

5. A Carnot heat engine consists of 2 isothermal and 2 adiabatic processes. Steps A$\rightarrow$B and C$\rightarrow$D are isothermic and steps B$\rightarrow$C and D$\rightarrow$A are adiabatic. The temperature at C$\rightarrow$D (T_H) is 1500 K and the temperature at A$\rightarrow$B (T_C) 300 K. For state A, P=2490 Pa & V = 3.0 m^3, and Carnot Engine performs $4.5\text{x}10^4$ J of work. Note: the working fluid is 3 moles of ideal diatomic gas.

A) Sketch the P-V diagram for this heat engine.

B) What is the efficiency of this heat engine?

Ans. $e = 1 - \dfrac{T_C}{T_H} = 0.80$

C) How much heat energy (Q_H) is absorbed during the isothermic phase B$\rightarrow$C?

Ans. $Q_H = \dfrac{W}{e} = \dfrac{4.5\times10^4 J}{0.80} = 5.63\times10^4 J$

D) How much heat energy (Q_C) is released during the isothermic phase D$\rightarrow$A?

Ans. $Q_C = -Q_H + W = -5.63\times10^4 J + 4.5\times10^4 J = -1.13\times10^4 J$

E) What is the pressure at each point A-D? Calculate using the ideal gas law.

Ans.

$$\text{Get } V_B \text{ from } Q_C\text{: } Q_C = W_{A\rightarrow B} = nRT\ln\left(\frac{V_B}{V_A}\right) = -1.13\times10^4 J$$

$$V_B = V_A e^{\frac{Q_C}{nRT}} = \left(3.0m^3\right)e^{\frac{-1.13\times10^4 J}{(3mol)(8.314\frac{J}{molK})(300K)}} = 0.663m^3$$

$$\Rightarrow P_B = \frac{(3mol)(8.314\frac{J}{mol\cdot K})(300K)}{0.663m^3} = 1.13\times10^5 Pa$$

$$\text{Get } P_C \text{ from } P_C V_C^\alpha = P_B V_B^\alpha\text{: } P_C\left(\frac{nRT_C}{P_C}\right)^\alpha = P_B\left(\frac{nRT_B}{P_B}\right)^\alpha \Rightarrow P_C = \left(P_B\left(\frac{T_B}{T_C P_B}\right)^\alpha\right)^{\frac{1}{1-\alpha}}$$

$$P_C = \left(1.13\times10^4 Pa\left(\frac{300K}{(1500K)(1.13\times10^4 Pa)}\right)^{\frac{7}{5}}\right)^{-\frac{5}{2}} = 3.16\times10^6 Pa$$

$$\text{Get } V_D \text{ from } Q_H\text{: } V_D = \left(\frac{nRT_H}{P_C}\right)e^{\frac{Q_H}{nRT}} = \left(\frac{nRT_H}{P_C}\right)e^{\frac{5.63\times10^4 J}{(3mol)(8.314\frac{J}{molK})(1500K)}} = 0.0531m^3$$

$$\Rightarrow P_B = \frac{(3mol)(8.314\frac{J}{mol\cdot K})(1500K)}{0.0531m^3} = 7.05\times10^5 Pa$$

F) How much work is performed by the gas during each step?

Ans.

$W_{A\rightarrow B} = Q_C = -1.13\times10^4 J$ and $W_{B\rightarrow C} = \frac{3}{2}nR\Delta T = -4.49\times10^4 J$

$W_{C\rightarrow D} = Q_H = 5.63\times10^4 J$ and $W_{D\rightarrow A} = \frac{3}{2}nR\Delta T = 4.49\times10^4 J$

H) What is the change in entropy ($\Delta S_{A\text{-}B}$) during A→B?

Ans. $\Delta S_{AB} = \frac{Q_C}{T_C} = \frac{-1.13\times10^4 J}{300K} = -37.7\,\frac{J}{K}$

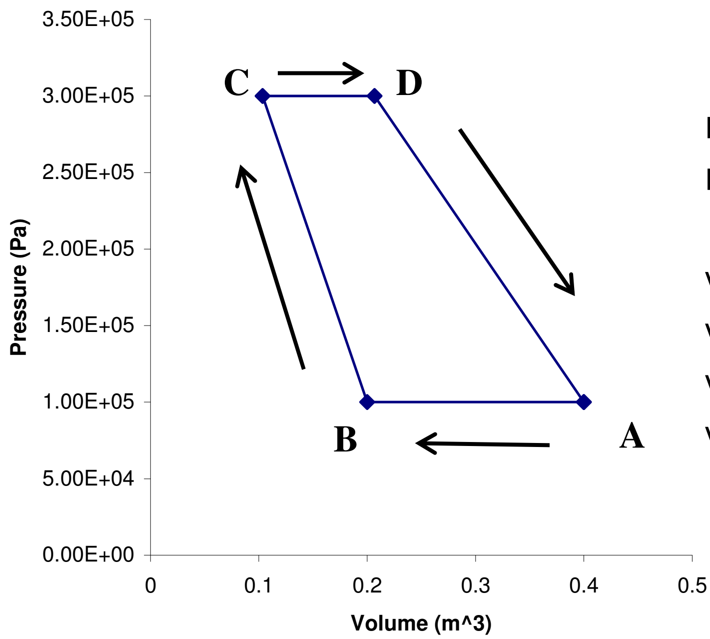
I) What is the change in entropy ($\Delta S_{C\text{-}D}$) during C→D?

Ans. $\Delta S_{CD} = \frac{Q_H}{T_H} = \frac{5.63\times10^4 J}{1500K} = 37.7\,\frac{J}{K}$

J) What is the net change in entropy ($\Delta S_{universe}$) between the steps in H and I?

Ans. $\Delta S_{universe} = \Delta S_{AB} + \Delta S_{AB} = 0\,\frac{J}{K}$

6. Consider the following heat engine, where steps A→B and C→D are isobaric and steps B→C and D→A are adiabatic (yes, the adiabatic steps should have a slight curvature). Note: the working fluid in this engine is a monatomic ideal gas (10 moles).



$P_A=P_B=1.0\text{x}10^5$ Pa

$P_C=P_D=3.0\text{x}10^5$ Pa

$V_A=0.40m^3$

$V_B=0.20m^3$

$V_C=0.103m^3$

$V_D=0.207m^3$

A) What is the temperature at each point A-D?

Ans. $T=\frac{PV}{nR} \Rightarrow T_A=481$ K, $T_B=241$ K, $T_C=372$ K, and $T_D=747$ K

B) How much work is performed <u>by the gas</u> during each step?

Ans. $W_{A\to B} = p\Delta V = -2.00\times10^4 J$ and $W_{B\to C} = \frac{3}{2}nR\Delta T = -1.63\times10^4 J$

$W_{C\to D} = p\Delta V = 3.12\times10^4 J$ and $W_{D\to A} = \frac{3}{2}nR\Delta T = 3.3\times10^4 J$

C) What is the net work during the complete cycle?

Ans. $W_{Net}=W_{A\to B}+W_{B\to C}+W_{C\to D}+W_{D\to A}=-2.00\times10^4 J-1.63\times10^4 J+3.12\times10^4 J+3.3\times10^4 J$

$W_{Net}=2.79\times10^4 J$

D) How much heat (Q_H) is absorbed by the gas?

Ans. $Q_{in} = Q_{CD} = nc_p\Delta T = 7.80\times10^4 J$

E) How much heat (Q_C) is discarded by the gas?

Ans. $Q_{out} = Q_{CD} = nc_p\Delta T = -5.00\times10^4 J$

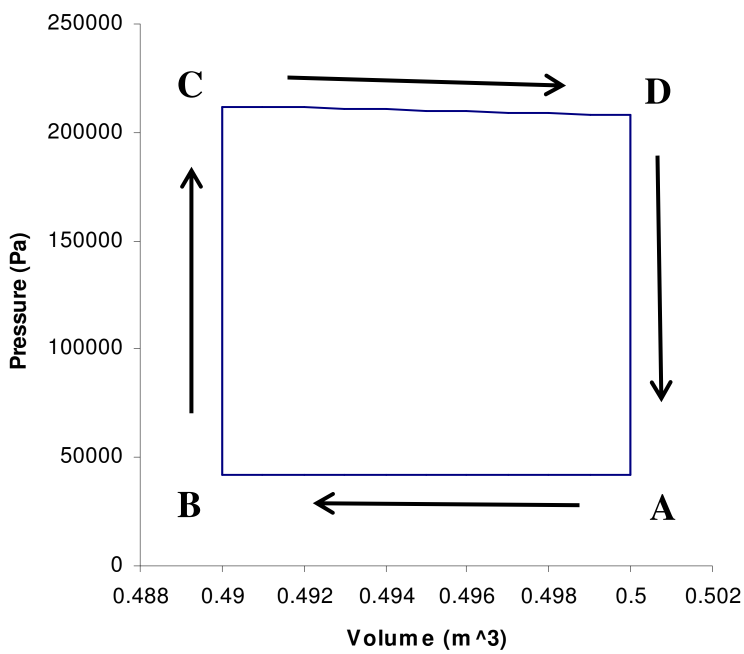
F) What is the efficiency of the heat engine?

Ans. $e = \dfrac{W}{Q_H} = \dfrac{2.79\times10^4 J}{7.80\times10^4 J} = 0.35$

G) What is the Carnot efficiency for the engine?

Ans. $e = 1 - \dfrac{T_C}{T_H} = 1 - \dfrac{241K}{747K} = 0.68$

7. Consider the following heat engine (the Stirling Engine). Steps A→B and C→D are isothermic and steps B→C and D→A are isochoric. The temperature at C→D (T_H) is 2500 K and the temperature at A→B (T_C) 500 K. Note: the working fluid in this engine is an ideal monatomic gas (10 moles).



A) What is the pressure at each point A-D? Calculate using the ideal gas law.

Ans.

$$P = \frac{nRT}{V}$$
$$P_A = 8.31\times10^4 Pa$$
$$P_B = 8.48\times10^4 Pa$$
$$P_C = 4.24\times10^5 Pa$$
$$P_D = 4.16\times10^5 Pa$$

B) How much work is performed <u>by the gas</u> during each step?

$$W_{AB} = nRT\cdot\ln\left(\frac{V_B}{V_A}\right) = (10.0\text{ mol})(8.314\tfrac{J}{mol\cdot K})(500K)\ln\left(\frac{0.49}{0.50}\right) = -8.40\times10^2 J$$

Ans. $$W_{CD} = nRT\cdot\ln\left(\frac{V_B}{V_A}\right) = (10.0\text{ mol})(8.314\tfrac{J}{mol\cdot K})(2500K)\ln\left(\frac{0.50}{0.49}\right) = 4.20\times10^3 J$$

$$W_{BC} = W_{DA} = 0\ J$$

C) What is the net work during the complete cycle?

Ans. $W_{net} = W_{AB} + W_{CD} = -8.40\times10^2 J + 4.20\times10^3 J = 3.36\times10^3 J$

D) How much heat ($Q_{in} = Q_{BC} + Q_H$) is absorbed by the gas?

Ans. $Q_{in} = Q_{CD} + Q_{DA} = 4.20\times10^3 J + 2.49\times10^5 J = 2.53\times10^5 J$

E) How much heat ($Q_{out} = Q_{DA} + Q_C$) is discarded by the gas?

Ans. $Q_{out} = Q_{AB} + Q_{BC} = -8.40\times10^2\text{J} - 2.49\times10^5\text{J} = -2.50\times10^5\text{J}$

F) What is the efficiency of this Stirling engine?

Ans. $e = \dfrac{W}{Q_{in}} = \dfrac{3.36\times10^3\text{J}}{2.53\times10^5\text{J}} = 0.013$

G) What is the Carnot efficiency for this engine?

Ans. $e = 1 - \dfrac{T_C}{T_H} = 1 - \dfrac{500\text{K}}{2500\text{K}} = 0.80$

H) What is the change in entropy ($\Delta S_{B\text{-}D}$) during B$\rightarrow$D?

Ans.
$$\Delta S = \Delta S_{BC} + \Delta S_{CD} = \tfrac{3}{2}nR\ln\left(\frac{T_H}{T_C}\right) + \frac{Q_{CD}}{T_H}$$
$$\Delta S = \tfrac{3}{2}(10.0\text{ mol})(8.314\tfrac{\text{J}}{\text{mol}\cdot\text{K}})\ln\left(\tfrac{2500\text{K}}{500\text{K}}\right) + \frac{4.20\times10^3\text{J}}{2500\text{K}} = 202.4\tfrac{\text{J}}{\text{K}}$$

I) What is the change in entropy ($\Delta S_{D\text{-}B}$) during D$\rightarrow$B?

Ans.
$$\Delta S = \Delta S_{AB} + \Delta S_{BC} = \frac{Q_{AB}}{T_C} + \tfrac{3}{2}nR\ln\left(\frac{T_H}{T_C}\right)$$
$$\Delta S = \frac{-8.40\times10^2\text{J}}{500\text{K}} + \tfrac{3}{2}(10.0\text{ mol})(8.314\tfrac{\text{J}}{\text{mol}\cdot\text{K}})\ln\left(\tfrac{500\text{K}}{2500\text{K}}\right) = -202.4\tfrac{\text{J}}{\text{K}}$$

J) What is the net change in entropy ($\Delta S_{universe}$) between the steps in H and I?

Ans. $\Delta S_{system} = \Delta S_{universe} = 0\tfrac{\text{J}}{\text{K}}$

Statistical Interpretation of Entropy:

8. Consider a system of 2 identical coins, with equal probability of "heads" or "tails", respectively.

A) How many possible unique configurations are possible in this system?

Ans. $W_{tot} = X^N - 1 = 2^2 - 1 = 3$

B) What is the multiplicity of configurations (W) that both coins will and on "tails"?

Ans. $W = \dfrac{N!}{n_1!n_2!} = \dfrac{2!}{0!2!} = 1,\ n_1=\text{"heads"}\ \&\ n_2=\text{"tails"}$

C) What is the entropy of the state when they are tossed and both land on "tails"?

Ans. $S = k\cdot\ln W = \left(1.38\times10^{-23}\tfrac{\text{J}}{\text{K}}\right)\ln 1 = 0$

D) What does your answer in (C) imply about that state of the 2 dice system? Be as specific as possible.

Ans. All "tails" represents a perfectly ordered state and this corresponds to S=0, the lowest possible entropy state.

9. Consider a system of 5 identical dice, with equal probability of landing on any side, respectively.

A) How many possible unique configurations are possible in this system?

Ans. W_{tot} = 6 (yahtzee) + 30 (full house) + 30 (4-of-a-kind) + 120 (3-of-a-kind) + 120 (2 pairs) and 360 (1 pair) = 667 unique combinations (states)

B) What is the multiplicity of configurations (W) that all dice will land on the same value (i.e. a Yahtzee)?

Ans. for any particular Yahtzee value: $W = \dfrac{N!}{n_1!n_2!n_3!n_4!n_5!n_6!} = \dfrac{5!}{5!0!0!0!0!0!} = 1$

C) What is the entropy of a "Yahtzee" state (all the same value) for the 5 dice system?

Ans. $S = k \cdot \ln W = \left(1.38\times10^{-23}\,\tfrac{J}{K}\right) \ln 1 = 0$

10. In information theory, entropy (measured in bits) is used to define a lower boundary on the necessary number of numerical bit digits required to uniquely specify all possible states of an information system (e.g. unique letters of an alphabet).

A) What is the entropy of a 36 character alphabet letter (incl. 10 numbers) system? Assume that all letters and numbers are equally probable.

Ans: $S_{\text{in bits}}=-(1.443)\sum_{i=1}^{N} p_i(\ln p_i)=-(1.443)\sum_{i=1}^{36}\left(\frac{1}{36}\right)\ln\left(\frac{1}{36}\right)=(1.443)\ln(36)=5.2 \text{ bits}$

A minimum of 6 bits is necessary to uniquely express this character system.

B) What is the entropy of this 36 character alphabet letter system, when numbers are 5 times more probable that letters?

Ans: Note, the probability of each character is: $26p_L + 10\,P_\# = 26p_L + 10\cdot5p_L = 1$

Therefore: $p_L = 1/76$ & $p_\# = 5/76$.

$$S_{\text{in bits}}=-(1.443)\left(\sum_{i=1}^{10} p_\#(\ln p_\#) + \sum_{i=1}^{26} p_L(\ln p_L)\right)$$

$$S_{\text{in bits}}=-(1.443)\left(\left(\frac{50}{76}\right)\ln\left(\frac{5}{76}\right) + \left(\frac{26}{76}\right)\ln\left(\frac{1}{76}\right)\right)= 4.7 \text{ bits}$$

A minimum of 5 bits is necessary to uniquely express this character system.