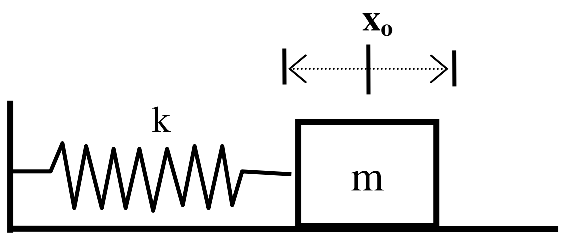
Simple Harmonic Motion:

1. A mass (0.5 kg) attached to an ideal spring (k=50 N/m), which is attached firmly to a wall at the other end, oscillates along a horizontal frictionless surface. The maximum displacement of the oscillating mass from its equilibrium position (x_m) is 0.2 m.



a. What is the angular frequency (ω), frequency (f) and period (T) of the oscillating mass?

Ans. $\omega = \sqrt{\frac{k}{m}} = 10\,\frac{\text{rad}}{\text{s}}$, $f = \frac{\omega}{2\pi} = 1.59\,\frac{\text{cycles}}{\text{s}}$ (or Hz), $T = \frac{1}{f} = \frac{1}{1.59\text{ Hz}} = 0.628\text{ s}$

b. Write out the equation for the position as a function of time, x(t), for this oscillating mass.

Ans. $x(t) = x_m\cos(\omega t + \phi) = (0.2\text{m})\cos\left[\left(10\,\frac{\text{rad}}{\text{s}}\right)t + \phi\right]$

c. Determine the velocity and acceleration equations for the oscillating mass.

Ans. $v(t) = v_m\sin(\omega t + \phi) = -\left(2\,\frac{\text{m}}{\text{s}}\right)\sin\left[\left(10\,\frac{\text{rad}}{\text{s}}\right)t + \phi\right]$

$a(t) = a_m\cos(\omega t + \phi) = -\left(20\,\frac{\text{m}}{\text{s}^2}\right)\cos\left[\left(10\,\frac{\text{rad}}{\text{s}}\right)t + \phi\right]$

d. What is the value of the maximum velocity and acceleration for the oscillating mass?

Ans. $|v_m| = \omega x_m = 2\,\frac{\text{m}}{\text{s}}$ and $|a_m| = \omega^2 x_m = 20\,\frac{\text{m}}{\text{s}^2}$

e. What is the velocity of the spring at the "rest" position of the spring? Hint: Apply conservation of energy and calculate the kinetic energy of the mass.

Ans. $E(t) = K(t) + U(t) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 = 1\text{ J}$

when $x(t) = x_o$, $E(t) = \frac{1}{2}mv^2 = 1\text{ J} \Rightarrow v = \sqrt{\frac{2(1\text{ J})}{m}} = 2\,\frac{\text{m}}{\text{s}}$

f. What is the potential energy of the mass at x_o? at $\pm\, x_m$?

Ans. $U(t) = \frac{1}{2}kx_o^2 = 0\text{ J}$ and $U(t) = \frac{1}{2}kx_m^2 = 1\text{ J}$

g. The above spring is then replaced with a different spring resulting in a period of oscillation of 0.10 s. What is the spring constant, k, for this new spring?

Ans. $T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow k = m\left(\frac{2\pi}{T}\right)^2 = 395\,\frac{\text{N}}{\text{m}}$

2. A cylinder (ρ_{cyl}=800 kg/m^3) floats on the surface of a calm body of fresh water. The cylinder has a radius of 0.10 m and a length of 0.8 m.

a. What is the volume of water displaced by the cylinder as it floats in the water surface?

Ans. According to Archimedes' Principle, $F_B = \rho_w g V_{disp} = mg \Rightarrow V_{disp} = \dfrac{m}{\rho_w} = \dfrac{\rho_{cyl}V_{cyl}}{\rho_w}$

The volume of the cylinder is: $V_{cyl} = A_{face}h = \pi r^2 h = 0.0251 \text{ m}^3$

The volume displaced is: $V_{disp} = \dfrac{\rho_{cyl}V_{cyl}}{\rho_w} = \dfrac{\left(800\frac{kg}{m^3}\right)\left(0.0251 \text{ m}^3\right)}{998\frac{kg}{m^3}} = 0.0201 \text{ m}^3$

b. How deep into the water does the cylinder extend?

Ans. $V_{disp} = A_{face}y_o \Rightarrow y_o = \dfrac{V_{disp}}{A_{face}} = \dfrac{0.0201 \text{ m}^3}{0.0314 \text{ m}^2} = 0.640 \text{ m},$

where y_o is the distance from water surface to the bottom surface of the cylinder.

3. The above cylinder is then pressed 0.05 m into the water, released and observed to oscillate on the water surface (ignore the production of water waves due to the cylinder's oscillation).

a. Apply Newton's 2nd Law to construct the force equation for the oscillating mass.

Ans. $F_{net} = ma = F_B - mg = \rho_w g V_{disp} - mg$

b. What is the angular frequency (ω), frequency (f) and period (T) of the oscillating cylinder?

Ans. from above, $a = \dfrac{d^2y}{dt^2} = \dfrac{-\rho_w g A_{face}\Delta y(t) - mg}{m} = \dfrac{-\rho_w g A_{face}\left(y(t)-y_o\right) - mg}{m}$

Since,

$$\rho_w g A_{face} y_o = mg$$

$$\Rightarrow a(t) = \frac{d^2y(t)}{dt^2} = -\left(\frac{\rho_w g A_{face}}{m}\right)y(t) = -\left(\frac{\rho_w g A_{face}}{\rho_{cyl}A_{face}h}\right)y(t) = -\left(\frac{\rho_w g}{\rho_{cyl}h}\right)y(t)$$

Therefore, $\omega = \sqrt{\dfrac{\rho_w g}{\rho_{cyl}h}} = 3.91\frac{rad}{s}$, $f = \dfrac{\omega}{2\pi} = 0.622 \text{ Hz}$, $T = \dfrac{1}{f} = 1.61 \text{ s}$

c. Determine the displacement equation for the cylinder.

Ans. $y(t) = y_m\cos(\omega t + \phi) = (-0.05\text{m})\cos\left[\left(3.91\frac{rad}{s}\right)t + \phi\right] = (0.05\text{m})\cos\left[\left(3.91\frac{rad}{s}\right)t \pm \pi\right]$

d. Determine the velocity and acceleration equations for the oscillating mass.

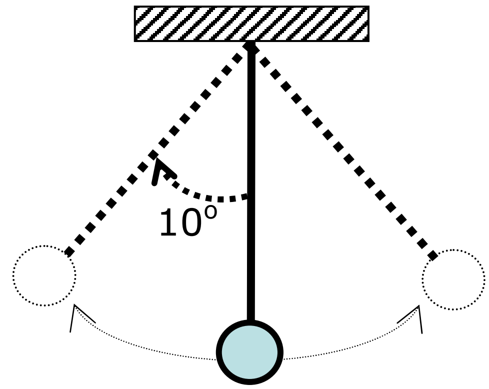
Ans. $v(t) = v_m\sin(\omega t + \phi) = -\left(0.196\frac{m}{s}\right)\sin\left[\left(3.91\frac{rad}{s}\right)t \pm \pi\right]$

$a(t) = a_m\cos(\omega t + \phi) = -\left(0.764\frac{m}{s^2}\right)\cos\left[\left(3.91\frac{rad}{s}\right)t \pm \pi\right]$

e. What is the value of the maximum velocity and acceleration for the oscillating mass?

Ans. $\left|v_m\right| = \omega y_m = 0.196\frac{m}{s}$ and $\left|a_m\right| = \omega^2 y_m = 0.764\frac{m}{s^2}$

Simple Pendulum:



4. A simple pendulum (mass = 2 kg and r = 0.5 m) is held in an elevated 10^o position then released, resulting in oscillating motion. Assume the small angle approximation, $\sin\theta \approx \theta$ (in radians), for this problem.

a. Determine the torque equation, $\tau(t)$, for the oscillating pendulum.

Ans. $$\tau_{net} = I\frac{d^2\theta(t)}{dt^2} = -mgr\sin\theta(t) \approx -mgr\theta(t)$$

b. What is the angular frequency (ω), frequency (f) and period (T) for the pendulum?

Ans. from above, $$\frac{d^2\theta}{dt^2} = -\left(\frac{mgr}{I}\right)\theta = -\left(\frac{mgr}{mr^2}\right)\theta = -\left(\frac{g}{r}\right)\theta$$

Therefore, $$\omega = \sqrt{\frac{g}{r}} = 4.43\,\tfrac{rad}{s},\quad f = \frac{\omega}{2\pi} = 0.705\text{ Hz},\quad T = \frac{1}{f} = 1.42\text{ s}$$

c. If the pendulum length, r, were increased to 1.5 m, what is the frequency (f) and period (T) for the pendulum?

Ans. $$\omega = \sqrt{\frac{g}{r}} = 2.56\,\tfrac{rad}{s},\quad f = \frac{\omega}{2\pi} = 0.407\text{ Hz},\quad T = \frac{1}{f} = 2.46\text{ s}$$

d. What is the period (T) for the (original) pendulum on the moon (g_{moon}= 1.63 m/s^2)?

Ans. $$\omega = \sqrt{\frac{g_{moon}}{r}} = 1.81\,\tfrac{rad}{s},\quad f = \frac{\omega}{2\pi} = 0.287\text{ Hz},\quad T = \frac{1}{f} = 3.48\text{ s}$$

Physical Pendulum

5. A hollow thin-walled ring (mass = 0.1 kg and r = 0.1 m) is hung by a small support nail and given a light tap, resulting in oscillating motion. Assume the small angle approximation, $\sin\theta \approx \theta$ (in radians) and there is no slippage between the cylinder and the nail.

a. Determine the torque equation, $\tau(t)$, for the oscillating ring.

Ans. $$\tau_{net} = I\frac{d^2\theta(t)}{dt^2} = -mgr\sin\theta(t) \approx -mgr\theta(t)$$

b. What is the angular frequency (ω), frequency (f) and period (T) for the pendulum?

Ans. from above, $$\frac{d^2\theta}{dt^2} = -\left(\frac{mgr}{I}\right)\theta = -\left(\frac{mgr}{mr^2 + mr^2}\right)\theta = -\left(\frac{g}{2r}\right)\theta$$

Therefore, $$\omega = \sqrt{\frac{g}{2r}} = 7.00\,\tfrac{rad}{s},\quad f = 1.11\text{ Hz},\quad T = 0.898\text{ s}$$

c. Determine the effective length of this physical pendulum (i.e. the center of oscillation).

Ans. since $$\omega^2 = \frac{g}{r} \Rightarrow r = \frac{g}{\omega^2} = 0.2\text{ m}$$