

## Rotational Variables

- 1) Write out the expression for the average angular ( $\omega_{\text{avg}}$ ), in terms of the angular displacement ( $\theta$ ) and elapsed time ( $\Delta t$ ).
- 2) Write out the expression for the average angular acceleration ( $\alpha_{\text{avg}}$ ), in terms of the angular velocity ( $\omega$ ) and elapsed time ( $\Delta t$ ).
- 3) Write an expression for the average angular velocity ( $\omega_{\text{avg}}$ ) in terms of initial and final angular velocities ( $\omega_0$  and  $\omega$ , respectively), where the angular acceleration is constant.
- 4) Combine your answers to the above exercises and derive the rotational kinematic equations in terms of the angular variables ( $\theta, \omega, \alpha$ ). Assume the angular acceleration is constant.

## Rotational Motion

5) An ant is standing on a moving CD, 2 cm from the center. The CD is moving at 80 rpm.

a. When the CD makes a  $45.0^\circ$  revolution, what is the length of the ant's path?

Ans.  $\Delta\theta = 45^\circ \left( \frac{2\pi}{360^\circ} \right) = 0.785 \text{ rad} \Rightarrow s = \Delta\theta \cdot r = (0.785 \text{ rad})(2 \text{ cm}) = 1.57 \text{ cm}$

b. What is the ant's actual linear displacement ( $\Delta \vec{r}$ )?

$$\vec{r}_0 = (2 \text{ cm}) \hat{i} + (0 \text{ cm}) \hat{j}$$

$$\vec{r} = (r \cdot \cos 45^\circ) \hat{i} + (r \cdot \sin 45^\circ) \hat{j} = (1.41 \text{ cm}) \hat{i} + (1.41 \text{ cm}) \hat{j}$$

Ans.

$$\Delta \vec{r} = (1.41 \text{ cm} - 2 \text{ cm}) \hat{i} + (1.41 \text{ cm} - 0 \text{ cm}) \hat{j} = (-0.59 \text{ cm}) \hat{i} + (1.41 \text{ cm}) \hat{j}$$

$$|\Delta \vec{r}| = \sqrt{(-0.59 \text{ cm})^2 + (1.41 \text{ cm})^2} = 1.53 \text{ cm}$$

c. What is the rotational speed of the CD expressed in radians per sec?

Ans.  $\omega = \left( \frac{80 \text{ rev}}{1 \text{ min}} \right) \left( \frac{2\pi \text{ rad}}{1 \text{ rev}} \right) \left( \frac{1 \text{ min}}{60 \text{ s}} \right) = 8.38 \frac{\text{rad}}{\text{s}}$

d. The ant moves to a new position 4.0 cm from the center.

i. What is the angular velocity of the ant compared to its speed at  $r = 2.0 \text{ cm}$ ?

Ans. The angular velocity is the same,  $\omega = 80 \text{ rad/s}$ .

ii. What is the tangential speed of the ant compared to its speed at  $r = 2.0 \text{ cm}$ ?

Ans. Since  $r$  is doubled, the tangential speed  $v$  will also double  $\rightarrow v_{4 \text{ cm}} = 2 v_{2 \text{ cm}}$

e. The ant then moves to the center of the CD. What is the angular velocity and tangential speed of the ant as the CD rotates?

Ans. The angular velocity is the same,  $\omega = 80 \text{ rad/s}$ , but  $v = 0 \text{ m/s}$  (tangential speed)

### Moment of Inertia

6) Calculate the moment of inertia (in SI units) for the following objects:

a. A 2 kg mass attached to a thin 0.15 m long string (a simple pendulum).

$$\text{Ans. } I_{\text{pendulum}} = mr^2 = (2 \text{ kg})(0.15 \text{ m})^2 = 0.045 \text{ kg} \cdot \text{m}^2$$

b. A 0.5 kg hollow sphere with a 0.10 m diameter.

$$\text{Ans. } I_{\text{hollow sphere}} = \frac{2}{3}mr^2 = \frac{2}{3}(0.5 \text{ kg})(0.05 \text{ m})^2 = 0.0025 \text{ kg} \cdot \text{m}^2$$

c. Two 10 kg solid spheres ( $r=0.05 \text{ m}$ ) attached by a mass-less 0.5 m bar (simple dumbbell), rotated about the middle of the bar.

$$I_{\text{dumbbell}} = \frac{4}{5}mr^2 + 2mL^2$$

$$\text{Ans. } I_{\text{dumbbell}} = \frac{4}{5}(10 \text{ kg})(0.05 \text{ m})^2 + 2(10 \text{ kg})(0.30 \text{ m})^2$$

$$I_{\text{dumbbell}} = 1.82 \text{ kg} \cdot \text{m}^2$$

d. Two 10 kg solid spheres ( $r=0.05 \text{ m}$ ) attached by a mass-less 0.5 m bar (simple dumbbell), rotated about the center of one of the spheres.

$$I_{\text{dumbbell}} = \frac{2}{5}mr^2 + \left( \frac{2}{5}mr^2 + 2mL^2 \right)$$

$$\text{Ans. } I_{\text{dumbbell}} = \frac{4}{5}(10 \text{ kg})(0.05 \text{ m})^2 + (10 \text{ kg})(0.60 \text{ m})^2$$

$$I_{\text{dumbbell}} = 3.62 \text{ kg} \cdot \text{m}^2$$

e. A meter stick (0.15 kg) rotated about its center of mass.

$$I_{\text{meter stick}} = \frac{1}{12}mL^2$$

$$\text{Ans. } I_{\text{meter stick}} = \frac{1}{12}(0.15 \text{ kg})(1.0 \text{ m})^2$$

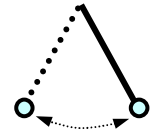
$$I_{\text{meter stick}} = 0.0125 \text{ kg} \cdot \text{m}^2$$

f. A meter stick (0.15 kg) rotated about one end.

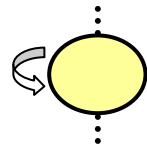
$$I_{\text{meter stick}} = \frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 = \frac{1}{3}mL^2$$

$$\text{Ans. } I_{\text{meter stick}} = \frac{1}{3}(0.15 \text{ kg})(1.0 \text{ m})^2$$

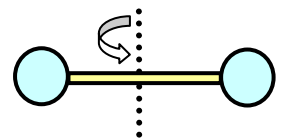
$$I_{\text{meter stick}} = 0.050 \text{ kg} \cdot \text{m}^2$$



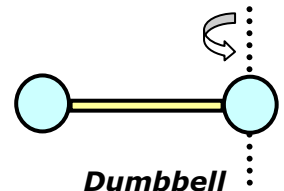
**Simple Pendulum**



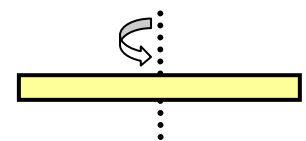
**Hollow Sphere**



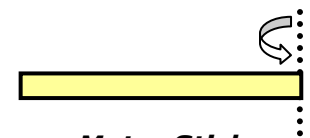
**Dumbbell**



**Dumbbell**



**Meter Stick**



**Meter Stick**

**Torque:**

7) Two people are sitting on a 20 kg teeter-totter ( $L = 7.0$  m), Person A (60 kg) and Person B (30 kg).

- a. Person A sits 1.5 m from the fulcrum. What is the torque exerted by Person A on the teeter-totter (assume the mass of the teeter-totter itself can be neglected)?

$$\vec{\tau}_A = \vec{r}_A \times m_A \vec{g} = |\vec{r}_A| |m_A g| \sin \theta \hat{k} =$$

Ans.  $\vec{\tau}_A = (-1.5 \text{ m})(60 \text{ kg})(-9.8 \frac{\text{m}}{\text{s}^2}) \hat{k} =$

$$\vec{\tau}_A = (882 \text{ N}\cdot\text{m}) \hat{k}$$

- b. What is the lever arm vector for Person B so that the teeter-totter balances? Draw a free body diagram of the 2-person teeter-totter system, include all forces & torques..

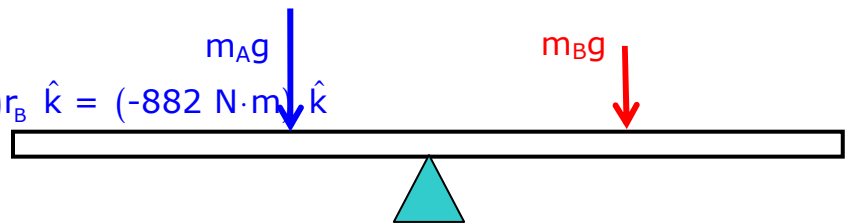
Ans.

$$\vec{\tau}_{\text{net}} = \vec{\tau}_A + \vec{\tau}_B = 0$$

$$\vec{\tau}_B = -\vec{\tau}_A = (-882 \text{ N}\cdot\text{m}) \hat{k}$$

$$\vec{\tau}_B = (-m_B g) r_B \hat{k} = (30 \text{ kg})(-9.8 \frac{\text{m}}{\text{s}^2}) r_B \hat{k} = (-882 \text{ N}\cdot\text{m}) \hat{k}$$

$$\vec{r}_B = \frac{(-882 \text{ N}\cdot\text{m})}{(30 \text{ kg})(-9.8 \frac{\text{m}}{\text{s}^2})} \hat{i} = (3 \text{ m}) \hat{i}$$



- c. Re-calculate all of the torques using a fulcrum located at one end of the balanced teeter-totter.

- d. Express the lever arm vector and how far from the fulcrum Person B is positioned when the teeter-totter is balanced at a  $20^\circ$  angle (clockwise), where Person A is on the left side.

Ans. {assume Person A is located at same position on the teeter-totter as above}

$$\vec{\tau}_A = \vec{r}_A \times m_A \vec{g} = |\vec{r}_A| |m_A g| \sin 20^\circ \hat{k} =$$

$$\vec{\tau}_A = (-1.5 \text{ m})(60 \text{ kg})(-9.8 \frac{\text{m}}{\text{s}^2}) \sin 20^\circ \hat{k} =$$

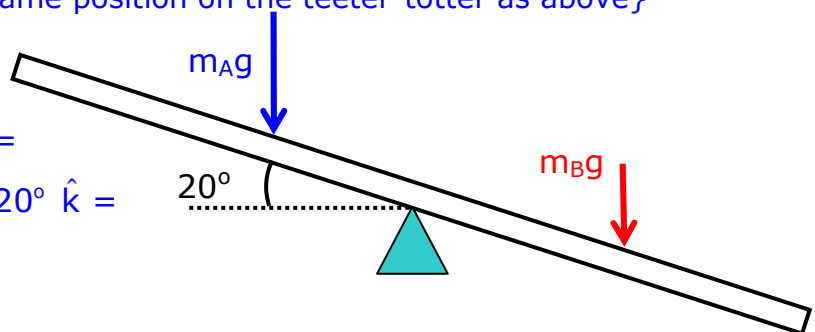
$$\vec{\tau}_A = (829 \text{ N}\cdot\text{m}) \hat{k}$$

$$\vec{\tau}_{\text{net}} = \vec{\tau}_A + \vec{\tau}_B = 0$$

$$\vec{\tau}_B = -\vec{\tau}_A = (-829 \text{ N}\cdot\text{m}) \hat{k}$$

$$\vec{\tau}_B = (-m_B g) r_B \sin 70^\circ \hat{k} = (30 \text{ kg})(-9.8 \frac{\text{m}}{\text{s}^2})(\sin 70^\circ) r_B \hat{k} = (-829 \text{ N}\cdot\text{m}) \hat{k}$$

$$\vec{r}_B = \frac{(-829 \text{ N}\cdot\text{m})}{(30 \text{ kg})(-9.8 \frac{\text{m}}{\text{s}^2})(\sin 70^\circ)} \hat{i} = (3 \text{ m}) \hat{i}$$



### Kinetic Energy & Rotation:

8) Determine the rotational kinetic energy for the following objects, rotated about the axis described above.

a. A mass attached to a string and rotating at 25 rad/sec.

$$\text{Ans. } K = \frac{1}{2}I\omega^2 = \frac{1}{2}(0.045 \text{ kg}\cdot\text{m}^2)(25 \frac{\text{rad}}{\text{s}})^2 = 14.06 \text{ J}$$

b. A hollow sphere rotating about its center at 15 rad/sec.

$$\text{Ans. } K = \frac{1}{2}I\omega^2 = \frac{1}{2}(0.0025 \text{ kg}\cdot\text{m}^2)(15 \frac{\text{rad}}{\text{s}})^2 = 0.281 \text{ J}$$

c. Two 10 kg solid spheres ( $r=0.05 \text{ m}$ ) attached by a massless 0.5 m bar (simple dumbbell), rotated about the middle of the bar at 15 rad/sec.

$$\text{Ans. } K = \frac{1}{2}I\omega^2 = \frac{1}{2}(1.82 \text{ kg}\cdot\text{m}^2)(15 \frac{\text{rad}}{\text{s}})^2 = 204.8 \text{ J}$$

d. Two 10 kg solid spheres ( $r=0.05 \text{ m}$ ) attached by a massless 0.5 m bar (simple dumbbell), rotated about the center of one of the spheres at 15 rad/sec.

$$\text{Ans. } K = \frac{1}{2}I\omega^2 = \frac{1}{2}(3.62 \text{ kg}\cdot\text{m}^2)(15 \frac{\text{rad}}{\text{s}})^2 = 407.3 \text{ J}$$

e. A meter stick (0.15 kg) rotated about its center of mass, at 5 rpm.

$$\text{Ans. } K = \frac{1}{2}I\omega^2 = \frac{1}{2}(0.0125 \text{ kg}\cdot\text{m}^2)(0.524 \frac{\text{rad}}{\text{s}})^2 = 0.00171 \text{ J}$$

f. A meter stick (0.15 kg) rotated about one end, at 5 rpm.

$$\text{Ans. } K = \frac{1}{2}I\omega^2 = \frac{1}{2}(0.0375 \text{ kg}\cdot\text{m}^2)(0.524 \frac{\text{rad}}{\text{s}})^2 = 0.00514 \text{ J}$$

9) Determine the rotational kinetic energy of a 0.15 kg golf ball, radius of 2.0 cm.

a. Treat the golf ball as a solid sphere that is rotating at 1000 rpm and simultaneously traveling at 10 m/s, what is the rotational kinetic energy of the ball?

$$I = \frac{2}{5}mr^2 = \frac{2}{5}(0.15 \text{ kg})(.020 \text{ m})^2 = 2.4 \times 10^{-5} \text{ kg}\cdot\text{m}^2$$

$$\text{Ans. } \omega = \left(\frac{1000 \text{ rev}}{1 \text{ min}}\right)\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 1.05 \frac{\text{rad}}{\text{s}}$$

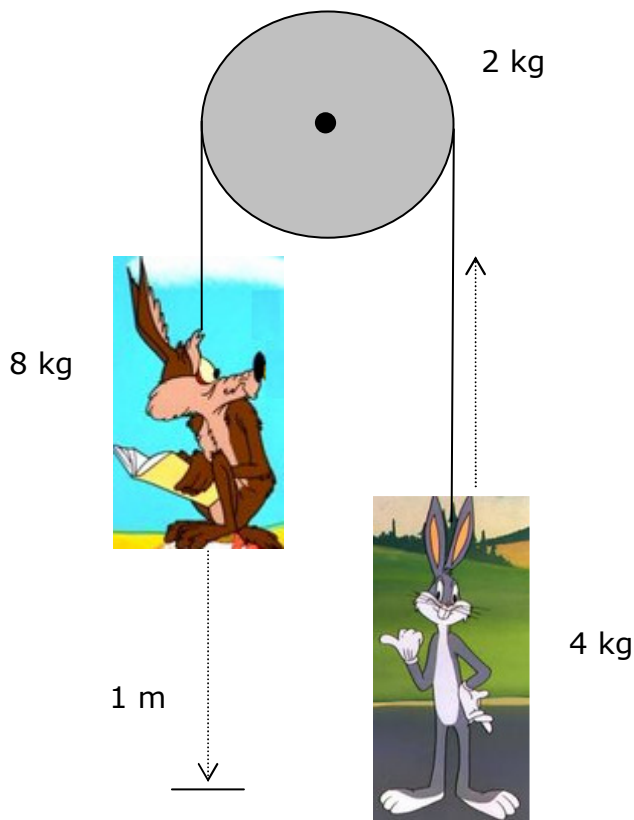
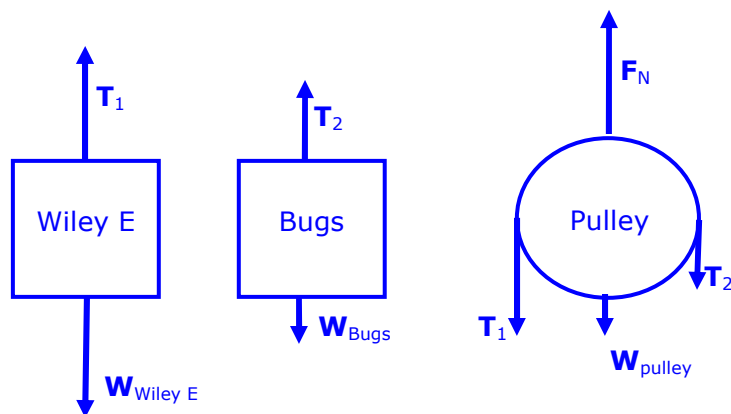
$$K_{\text{rot}} = \frac{1}{2}I\omega^2 = \frac{1}{2}(2.4 \times 10^{-5} \text{ kg}\cdot\text{m}^2)(1.05 \frac{\text{rad}}{\text{s}})^2 = 1.3 \times 10^{-5} \text{ J}$$

b. What is the total kinetic energy of the ball?

$$\text{Ans. } K_{\text{tot}} = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(0.15 \text{ kg})(40 \frac{\text{m}}{\text{s}})^2 + 1.3 \times 10^{-5} \text{ J} \approx 120 \text{ J}$$

10) Consider 2 attached masses (Bugs and Wiley E.) suspended by a simple (solid cylindrical) pulley (2 kg). Bugs has a mass of 4 kg and Wiley E. has a mass of 8 kg. The radius of the pulley is 0.1 m. Starting from rest, Wiley E. drops 1 m and Bugs ascends the same amount.

a. Draw a force vector diagram for each object (including the pulley). *Note: the tension forces on each end of the rope are not equal.*



b. Apply Newton's 2<sup>nd</sup> Law to Wiley E. and Bugs. Use it to solve for the tension forces,  $T_1$  and  $T_2$ , in terms of  $g$ ,  $m_{\text{Wiley}}$  and  $m_{\text{Bugs}}$ .

Ans.

$$(\text{Wiley E}): \quad \vec{F}_{\text{Net Wiley}} = (m_{\text{Wiley}}g - T_1) \hat{j} = m_{\text{Wiley}}\vec{a}$$

$$(\text{Bugs}): \quad \vec{F}_{\text{Net Bugs}} = (T_2 - m_{\text{Bugs}}g) \hat{j} = m_{\text{Bugs}}\vec{a}$$

$$(\text{Pulley}): \quad \vec{F}_p = (F_N - T_1 - T_2 - m_p g) \hat{j} = 0$$

c. Apply Newton's 2<sup>nd</sup> Law (torque) to the pulley.

$$\vec{\tau}_{\text{Net}} = \vec{r}_p \times \vec{T}_1 + \vec{r}_p \times \vec{T}_2 = (r_p T_1 - r_p T_2) \hat{k} = I \vec{\alpha}$$

Ans.

$$T_1 - T_2 = \frac{I|\vec{\alpha}|}{r_p} = \frac{\left(\frac{m_p r_p^2}{2}\right)\left(\frac{a}{r_p}\right)}{r_p} = \frac{1}{2}m_p a$$

d. Use the expressions for the tension forces in (b) and the torque equation obtained in (c) to solve for the acceleration of the system (a).

$$\text{Ans. } a = \left[ \frac{(m_{\text{Wiley}} - m_{\text{Bugs}})}{(m_{\text{Wiley}} + m_{\text{Bugs}} + \frac{1}{2}m_p)} \right] g = 3.02 \frac{\text{m}}{\text{s}^2}$$

e. What is the magnitude of the tension force at each end of the rope,  $T_1$  and  $T_2$ , respectively?

$$\text{Ans. } T_1 = m_{\text{Wiley}}(g - a) = 54.24 \text{ N and } T_2 = m_{\text{Bugs}}(g + a) = 51.28 \text{ N}$$

f. What is KE for Wiley E. and Bugs, respectively, at the end of the 1 m displacement?

Ans.  $K_{\text{Wiley}} = \frac{1}{2} m_{\text{Wiley}} v^2 = m_{\text{Wiley}} a \Delta r_{\text{Wiley}} = 24.1 \text{ J}$   
 $K_{\text{Bugs}} = \frac{1}{2} m_{\text{Bugs}} v^2 = m_{\text{Bugs}} a \Delta r_{\text{Bugs}} = 12.0 \text{ J}$

g. What is the rotational KE of the pulley at the end of the 1 m displacement?

Ans.  $v = \sqrt{\frac{2K_{\text{Wiley}}}{m_{\text{Wiley}}}} = \sqrt{\frac{2(24.16 \text{ J})}{8 \text{ kg}}} = 2.46 \frac{\text{m}}{\text{s}}$   
 $K_{\text{rot}} = \frac{1}{2} I \omega^2 = \frac{1}{2} \left( \frac{1}{2} m_p r_p^2 \right) \left( \frac{v}{r_p} \right)^2 = \frac{1}{4} m_p v^2 = 3.0 \text{ J}$

h. Calculate the total KE and mechanical energy of the system at the end of the 1 m displacement. Compare this value with the total KE for the same system where the pulley is mass-less.

Ans.  $K_{\text{system}} = K_{\text{Wiley}} + K_{\text{Bugs}} + K_p = 24.1 \text{ J} + 12.0 \text{ J} + 3.0 \text{ J} = \mathbf{39.1 \text{ J}}$

For the same system w/ mass-less pulley:

$$a = \left[ \frac{(m_{\text{Wiley}} - m_{\text{Bugs}})}{(m_{\text{Wiley}} + m_{\text{Bugs}})} \right] g = 3.26 \frac{\text{m}}{\text{s}^2}$$

$$K_{\text{system}} = \Delta K_{\text{Wiley}} + \Delta K_{\text{Bugs}} = W_{\text{Wiley}} + W_{\text{Bugs}} = (m_{\text{Wiley}} + m_{\text{Bugs}}) a \Delta r = \mathbf{39.1 \text{ J}}$$