Phy 211: General Physics I

Chapter 10: Rotation
Lecture Notes

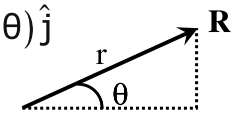
Rotational Motion & Angular Displacement

- When an object moves in a circular path (or rotates):
  - It remains a fixed distance (r) from the center of the circular path (or axis of rotation)
  - Since radial distance is fixed, position can be described by its angular position (θ)
- Angular position (θ) describes the position of an object along a circular path
  - Measured in radians (or degrees)
- **Angular displacement:** $\Delta\theta = \theta_f - \theta_i$
- **Angular velocity:** the rate at which angular position changes: $\vec{\omega} = \frac{d\vec{\theta}}{dt}$
- **Angular acceleration:** is the rate at which angular velocity changes: $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$

Relationship between rotational and linear variables for an object in circular motion

Position: $\vec{R} = \theta \cdot \vec{r}$

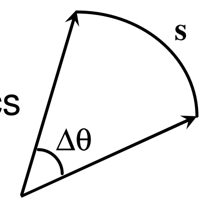
Where $\vec{x} = (r \cdot \sin\theta)\hat{i}$ and $\vec{y} = (r \cdot \cos\theta)\hat{j}$



Displacement (arc length): $s = \Delta\theta \cdot r$

Linear (tangential) speed: $v_T = \omega r$

Linear Acceleration: $a = \alpha r$



Equations of Rotational Kinematics

When $t_o = 0$ and α is constant:

1. $\omega - \omega_o = \alpha t$
2. $\theta - \theta_o = \frac{1}{2}(\omega_o + \omega)t$
3. $\theta - \theta_o = \omega_o + \frac{1}{2}\alpha t^2$
4. $\omega^2 - \omega_o^2 = 2\alpha\Delta\theta$

Centripetal & Tangential Acceleration

For an object moving in uniform circular motion, the magnitude of its centripetal acceleration is:

$$a_c = v^2/r$$

Since $v = \omega r \rightarrow v^2 = (\omega r)^2$ therefore:

$$a_c = \omega^2 r$$

When ω is not constant, the effect of angular acceleration must also included:

$$\vec{a} = (-a_c)\hat{i}_{\parallel} + (\alpha r)\hat{i}_{\perp}$$

Or

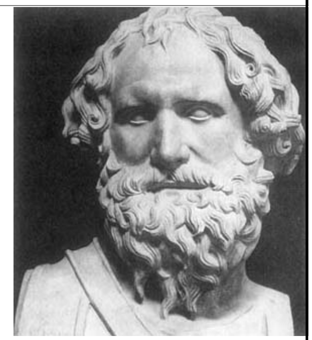
$$\vec{a} = \left(-\omega^2 r\right)\hat{i}_{\parallel} + (\alpha r)\hat{i}_{\perp}$$

Thus, an object's circular motion can be generalized in terms of both ω and α

<u>Notes:</u>

1. the 2 components of acceleration are perpendicular to each other
2. To determine the magnitude & direction of a, they must be treated as vectors

Archimedes (287-212 BC)

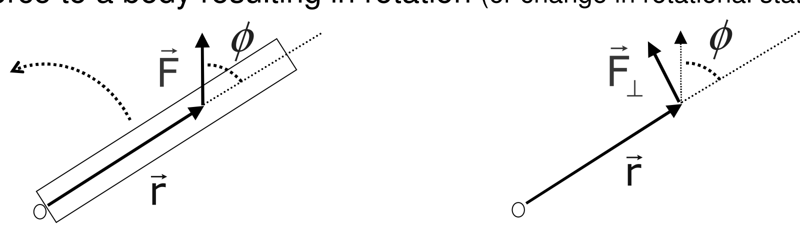


- Possibly the greatest mathematician in history
- Invented an early form of calculus
- Discovered the Principle of Buoyancy (now called Archimedes' Principle)
- Discovered the Principle of Leverage (Torque) and built several machines based on it.
- Famous quote:

  "Give me a point of support and I will move the Earth"

Rigid Objects & Torque

- Torque is a vector quantity that represents the application of force to a body resulting in rotation (or change in rotational state)



- **Definition:** $\vec{\tau} = \vec{r} \times \vec{F}$
- The SI units for torque are N·m (not to be confused with joules)
- Torque depends on:
  - The Lever arm (leverage), $\vec{r}$
  - The component of force perpendicular to lever arm, $\vec{F}_{\perp} = (F \cdot \sin\phi)\,\hat{i}_{\perp}$
  - The magnitude of torque: $|\vec{\tau}| = r \cdot F \cdot \sin\phi$

The Vector (Cross) Product

- Two vectors $(\vec{A} \text{ and } \vec{B})$ can be multiplied to produce a vector resultant, called the vector (or Cross) product.

Definition: (3-D using vector components)

$$\vec{A}\times\vec{B}=\left(A_y B_z - A_z B_y\right)\hat{i}+\left(A_z B_x - A_x B_z\right)\hat{j}+\left(A_x B_y - A_y B_x\right)\hat{k}$$

- The magnitude of the vector product:

$$\left|\vec{A}\times\vec{B}\right|=\left|\vec{A}\right|\left|\vec{B}\right|\sin\phi$$

where ϕ is the angle between vectors **A** and **B**

- Useful properties of vector products: $\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$

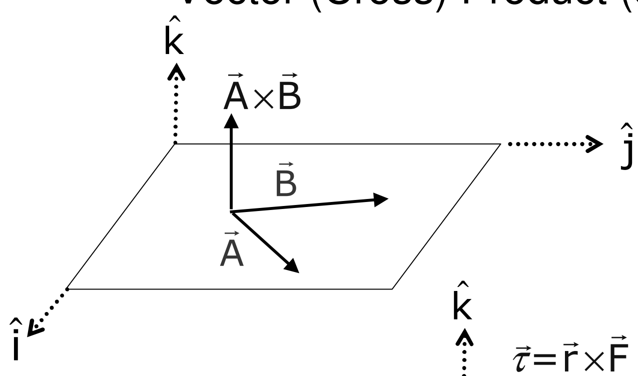
$$\vec{A}\times\vec{A}=0$$

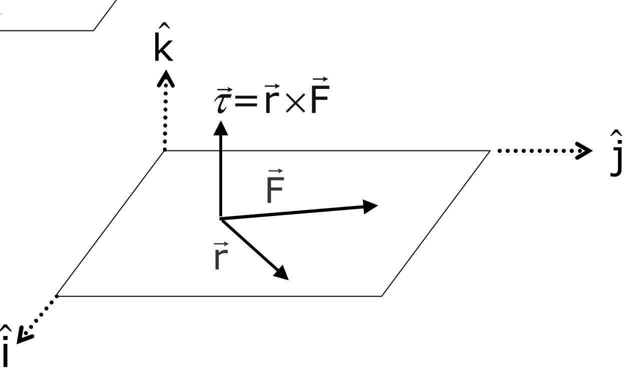
$$\hat{i}\times\hat{j} = \hat{k}$$

Notes:

1. *The vector product of vectors A & B is perpendicular to both A & B.*
2. *If vectors A & B lie within an x-y plane, the vector product points in the z-direction:* $\vec{A}\times\vec{B}=\left(A_x B_y - A_y B_x\right)\hat{k}$

Vector (Cross) Product (cont.)





Newton's 2nd Law

(for Rotational Motion)

When there is a net torque exerted on a rigid body its state of rotation ($\vec{\omega}$) will change depending on:

1. Amount of net torque, $\vec{\tau}_{Net}$
2. Rotational inertia of object, **I**

$$\vec{\alpha} = \frac{\vec{\tau}_{Net}}{I} = \frac{\vec{\tau}_1 + \vec{\tau}_2 + \ldots}{I} = \frac{1}{I}\sum_{i=1}^{n}\vec{\tau}_i$$

Note: No net torque means $\alpha = 0$ and ω = constant}

Alternatively, the net torque exerted on a body is equal to the product of the rotational inertia and the angular acceleration:

$$\vec{\tau}_{Net} = \vec{\tau}_1 + \vec{\tau}_2 + \ldots = \sum_{i=1}^{n}\vec{\tau}_i = I\vec{\alpha}$$

This is Newton's 2nd Law (for rotation)!!

Rigid Objects in Equilibrium

A rigid body is in equilibrium when:

- No net force → no change in state of motion
  (v = constant)

$$\vec{F}_{Net} = \vec{F}_x + \vec{F}_y + \vec{F}_z = 0$$

- No net torque → no change in state of rotation
  (ω = constant)

$$\vec{\tau}_{Net} = 0$$

Both of the above conditions must be met for an object to be in mechanical equilibrium!

Note: an object can be both moving and rotating and still be in mechanical equilibrium

Moment of Inertia

The resistance of a rigid body to changes in its state of rotational motion ($\vec{\omega}$) is called the moment of inertia (or rotational inertia)

The Moment of Inertia (I) depends on:

1. Mass of the object, m
2. The axis of rotation
3. Distribution (position) of mass about the axis of rotation

Definition *(continuous mass distribution)***:**

$$I = \int r^2 dm \quad \text{or} \quad \rho\int r^2 dV \quad \left\{\text{where } \rho = \frac{dm}{dV} = \frac{m}{V}\right\}$$

For a discrete distribution of mass:

$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots m_n r_n^2 = \sum_{i=1}^{n} m_i r_i^2$$

Where:

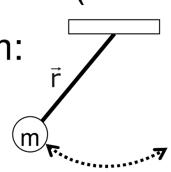
m_i *is the mass of a small segment of the object*

r_i *is the distance of the mass mi from the axis of rotation*

The SI units for I are kg·m²

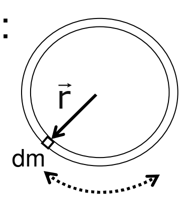
Moment of Inertia (common examples)

1. A simple pendulum:



$$I_{pendulum} = mr^2$$

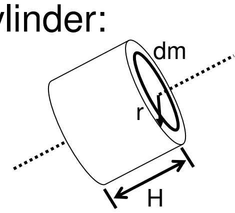
2. A thin ring:



$$I_{ring} = \int r^2 dm = \frac{m}{2\pi}\int_0^{2\pi} r^2 d\theta$$

$$I_{ring} = \frac{m}{2\pi} r^2 \theta \Big|_0^{2\pi} = mr^2$$

3. A solid cylinder:



$$I_{cylinder} = \int r^2 dm = \frac{2\pi m H}{\pi r^2 H}\int_0^{r} r^3 dr$$

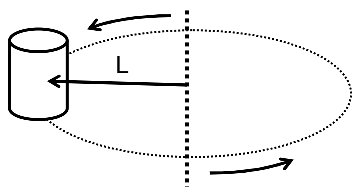
$$I_{cylinder} = \left(\frac{2m}{r^2}\right)\frac{r^4}{4}\Big|_0^{r} = \frac{mr^2}{2}$$

The Parallel Axis Theorem

- The Parallel Axis Theorem is used to determine the moment of inertia for a body rotated about an axis a distance, l, from the center of mass:

$$I = I_{cm} + mL^2$$

Example: A 0.2 kg cylinder (r=0.1 m) rotated about an axis located 0.5 m from its center:



$$I = I_{cm} + mL^2$$

$$I = \frac{mr^2}{2} + mL^2$$

$$I = (0.002 + 0.050)\ kg\cdot m^2$$

$$I = 0.052\ kg\cdot m^2$$

Work & Rotational Kinetic Energy

When torque is applied to an object and a rotation is produced, the torque does work:

$$W = \int_{\theta_i}^{\theta_f} \vec{\tau}\cdot d\vec{\theta}$$

When there is net torque:

$$W_{Net} = \int_{\theta_i}^{\theta_f} \vec{\tau}_{Net}\cdot d\vec{\theta} = \int_{\theta_i}^{\theta_f} I\vec{\alpha}\cdot d\vec{\theta} = I\vec{\alpha}\cdot\Delta\vec{\theta}$$

Since $\omega^2 - \omega_o^2 = 2\alpha\Delta\theta$

$$W_{Net} = I\vec{\alpha}\cdot\Delta\vec{\theta} = I\left(\frac{\omega^2-\omega_o^2}{2}\right) = \tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_o^2$$

The rotational kinetic energy (K_{rot}) for an object is defined as:

$$K_{rot} = \tfrac{1}{2}I\omega^2 \Rightarrow W_{Net} = \tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_o^2 = K_{rot} - K_{rot_o} = \Delta K_{rot}$$

This is the Work-Energy Theorem (for rotation)!!