ORCCA

Open Resources for Community College Algebra

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Portland Community College Faculty

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Acknowledgements

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This book uses WeBWorK to provide most of its exercises, which may be used for online homework. WeBWorK was created by Mike Gage and Arnie Pizer, and has benefited from over 25 years of contributions from open source developers. In 2013, Chris Hughes, Alex Jordan, and Carl Yao programmed most of the WeBWorK questions in this book with a PCC curriculum development grant.

The javascript library MathJax, created and maintained by David Cervone, Volker Sorge, Christian Lawson-Perfect, and Peter Krautzberger allows math content to render nicely on screen in the eBook. Additionally, MathJax makes web accessible mathematics possible.

The print edition (PDF) is built using the typesetting software LaTeX, created by Donald Knuth and enhanced by Leslie Lamport.

Each of these open technologies, along with many that we use but have not listed here, has been enhanced by many additional contributors spanning the past 40 years. We are grateful for all of these contributions.

To All

HTML, PDF, and print This book is available as an eBook, a free PDF, or printed and bound. All versions offer the same content and are synchronized such that cross-references match across versions. They can each be found at pcc.edu/orcca.

There are some differences between the eBook, PDF, and printed versions.

- The eBook is recommended, as it offers interactive elements and easier navigation than print. It requires no more than internet access and a modern web browser.
- A PDF version can be downloaded and then accessed without the internet. Some content is in color, but most of the colorized content from the eBook has been converted to black and white to ensure adequate contrast when printing in black and white. The exceptions are the graphs generated by WeBWorK.
- Printed and bound copies are available online. Up-to-date information about purchasing a copy should be available at pcc.edu/orcca. Contact the authors if you have trouble finding the latest version online. For each online sale, all royalties go to a PCC Foundation account, where roughly half will fund student scholarships, and half will fund continued maintenance of this book and other OER.

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Tables can be copied from the eBook version and pasted into applications like *MS Word*. However, mathematical content within tables will not always paste correctly without a little extra effort as described above.

Accessibility The HTML version is intended to meet or exceed web accessibility standards. If you encounter an accessibility issue, please report it.

- All graphs and images should have meaningful alt text that communicates what a sighted person would see, without necessarily giving away anything that is intended to be deduced from the image.
- All math content is rendered using MathJax. MathJax has a contextual menu that can be accessed in several ways, depending on what operating system and browser you are using. The most common way is to right-click or control-click on some piece of math content.
- In the MathJax contextual menu, you may set options for triggering a zoom effect on math content, and also by what factor the zoom will be. Also in the MathJax contextual menu, you can enable the Explorer, which allows for sophisticated navigation of the math content.
- A screen reader will generally have success verbalizing the math content from MathJax. With certain screen reader and browser combinations, you may need to set some configuration settings in the MathJax contextual menu.

Tablets and Smartphones PreTeXt documents like this book are “mobile-friendly.” When you view the HTML version, the display adapts to whatever screen size or window size you are using. A math teacher will always recommend that you do not study from the small screen on a phone, but if it’s necessary, the eBook gives you that option.

WeBWorK for Online Homework Most exercises are available in a ready-to-use collection of WeBWorK problem sets. Visit webwork.pcc.edu/webwork2/orcca-demonstration to see a demonstration WeBWorK course where guest login is enabled. Anyone interested in using these problem sets should contact the project leads.

Odd Answers The answers to the odd homework exercises at the end of each section are not contained in the PDF or print versions. As the eBook evolves, they may or may not be contained in an appendix there. In any case, odd answers are available *somewhere*. Check pcc.edu/orcca to see where.

Interactive and Static Examples Traditionally, a math textbook has examples throughout each section. This textbook uses two types of “example”:

Static These are labeled “Example.” Static examples may or may not be subdivided into a “statement” followed by a walk-through solution. This is basically what traditional examples from math textbooks do.

Active These are labeled “Checkpoint,” not to be confused with the exercises that come at the end of a section that might be assigned for homework, etc. In the HTML output, active examples have WeBWorK answer blanks where a reader could try submitting an answer. In the PDF output, active examples are almost indistinguishable from static examples, but there is a WeBWorK icon indicating that a reader could interact more actively using the eBook. Generally, a walk-through solution is provided immediately following the answer blank.

Some HTML readers will skip the opportunity to try an active example and go straight to its solution. Some readers will try an active example once and then move on to the solution. Some readers will tough it out for a period of time and resist reading the solution.

For readers of the PDF, it is expected that they would read the example and its solution just as they would read a static example.

A reader is *not* required to try submitting an answer to an active example before moving on. A reader *is* expected to read the solution to an active example, even if they succeed on their own at finding an answer.

Interspersed through a section there are usually several exercises that are intended as active reading exercises. A reader can work these examples and submit answers to WeBWorK to see if they are correct. The important thing is to keep the reader actively engaged instead of providing another static written example. In most cases, it is expected that a reader will read the solutions to these exercises just as they would be expected to read a more traditional static example.

Pedagogical Decisions

The authors and the greater PCC faculty have taken various stances on certain pedagogical and notational questions that arise in basic algebra instruction. We attempt to catalog these decisions here, although this list is certainly incomplete. If you find something in the book that runs contrary to these decisions, please let us know.

- Interleaving is our preferred approach, compared to a proficiency-based approach. To us, this means that once the book covers a topic, that topic will be appear in subsequent sections and chapters in indirect ways.
- Chapter 1 is mostly written as a *review*, and is not intended to teach all of these topics from first principles.
- We round decimal results to four significant digits, or possibly fewer leaving out trailing zeros. We do this to maintain consistency with the most common level of precision that WeBWorK uses to assess decimal answers. We *round*, not *truncate*. And we use the $\approx$ symbol. For example $\pi \approx 3.142$ and Portland's population is ≈ 609500.
- We offer *alternative* video lessons associated with each section, found in most sections in the eBook. We hope these videos provide readers with an alternative to whatever is in the reading, but there may be discrpancies here and there between the video content and reading content.
- We believe in always trying to open a topic with some level of application rather than abstract examples. From applications and practical questions, we move to motivate more abstract definitions and notation. This approach is perhaps absent in the first chapter, which is intended to be a review only. At first this may feel backwards to some instructors, with some "easier" examples (with no context) appearing after "more difficult" contextual examples.
- Linear inequalities are not strictly separated from linear equations. The section that teaches how to solve $2x + 3 = 8$ is immediately followed by the section teaching how to solve $2x + 3 < 8$.

  Our aim is to not treat inequalities as an add-on optional topic, but rather to show how intimately related they are to corresponding equations.
- When issues of "proper formatting" of student work arise, we value that the reader understand *why* such things help the reader to communicate outwardly. We believe that mathematics is about more than understanding a topic, but also about understanding it well enough to communicate results to others.

  For example we promote progression of equations like

  $$\begin{aligned} 1+1+1 &= 2+1 \\ &= 3 \end{aligned}$$

  instead of

  $$1+1+1 = 2+1 = 3.$$

  And we want students to *understand* that the former method makes their work easier for a reader to read. It is not simply a matter of "this is the standard and this is how it's done."

- When solving equations (or systems of linear equations), most examples should come with a check, intended to communicate to students that checking is part of the process. In Chapters 1–4, these checks will be complete simplifications using order of operations one step at a time. The later sections will often have more summary checks where either order of operations steps are skipped in groups, or we promote entering expressions into a calculator. Occasionally in later sections the checks will still have finer details, especially when there are issues like with negative numbers squared.
- Within a section, any first example of solving some equation (or system) should summarize with some variant of both "the solution is…" and "the solution set is…." Later examples can mix it up, but always offer at least one of these.
- There is a section on very basic arithmetic (five operations on natural numbers) in an appendix, not in the first chapter. This appendix is only available in the eBook.
- With applications of linear equations (as opposed to linear systems), we limit applications to situations where the setup will be in the form $x + f(x) = C$ and also certain rate problems where the setup will be in the form $5t + 4t = C$. There are other classes of application problem (mixing problems, interest problems, …) which can be handled with a system of two equations, and we reserve these until linear systems are covered.
- With simplifications of rational expressions in one variable, we always include domain restrictions that are lost in the simplification. For example, we would write $\frac{x(x+1)}{x+1} = x$, for $x \neq -1$. With multivariable rational expressions, we are content to ignore domain restrictions lost during simplification.

Entering WeBWorK Answers

This preface offers some guidance with syntax for WeBWorK answers. WeBWorK answer blanks appear in the active reading examples (called "checkpoints") in the HTML version of the book. If you are using WeBWorK for online homework, then you will also enter answers into WeBWorK answer blanks there.

Basic Arithemtic The five basic arithmetic operations are: addition, subtraction, multiplication, and raising to a power. The symbols for addition and subtraction are $+$ and $-$, and both of these are directly avialable on most keyboards as `+` and `-`.

On paper, multiplication is sometimes written using $\times$ and sometimes written using $\cdot$ (a centered dot). Since these symbols are not available on most keyboards, WeBWorK uses `*` instead, which is often `shift-8` on a full keyboard.

On paper, division is sometimes written using $\div$, sometimes written using a fraction layout like $\frac{4}{2}$, and sometimes written just using a slash, $/$. The slash is available on most full keyboards, near the question mark. WeBWorK uses `/` to indicate division.

On paper, raising to a power is written using a two-dimensional layout like 4^2. Since we don't have a way to directly type that with a simple keyboard, calculators and computers use the caret character, `^`, as in `4^2`. The character is usually `shift-6`.

Roots and Radicals On paper, a square root is represented with a radical symbol like $\sqrt{}$. Since a keyboard does not usually have this symbol, WeBWorK and many computer applications use `sqrt( )` instead. For example, to enter $\sqrt{17}$, type `sqrt(17)`.

Higher-index radicals are written on paper like $\sqrt[4]{12}$. Again we have no direct way to write this using most keyboards. In *some* WeBWorK problems it is possible to type something like `root(4, 12)` for the fourth root of twelve. However this is not enabled for all WeBWorK problems.

As an alternative that you may learn about in a later chapter, $\sqrt[4]{12}$ is mathematically equal to $12^{1/4}$, so it can be typed as `12^(1/4)`. Take note of the parentheses, which very much matter.

Common Hiccups with Grouping Symbols Suppose you wanted to enter $\frac{x+1}{2}$. You might type `x+1/2`, but this is not right. The computer will use the order of operations (see Section 1.4) and do your division first, dividing 1 by 2. So the computer will see $x + \frac{1}{2}$. To address this, you would need to use grouping symbols like parentheses, and type something like `(x+1)/2`.

Suppose you wanted to enter $6^{1/4}$, and you typed `6^1/4`. This is not right. The order of operations places a higher priority on exponentiation than division, so it calculates 6^1 first and then divides the result by 4. That is simply not the same as raising 6 to the $\frac{1}{4}$ power. Again the way to address this is to use grouping symbols, like `6^(1/4)`.

Entering Decimal Answers Often you will find a decimal answer with decimal places that go on and on. You are allowed to round, but not by too much. WeBWorK generally looks at how many *significant digits* you use, and generally expects you to use ***four or more*** correct significant digits.

"Significant digits" and "places past the decimal" are not the same thing. To count significant digits, read the number left to right and look for the first nonzero digit. Then count all the digits to the right including that first one.

The number 102.3 has four significant digits, but only one place past the decimal. This number could be a correct answer to a WeBWorK question. The number 0.0003 has one significant digit and four places past the decimal. This number might cause you trouble if you enter it, because maybe the "real" answer was 0.0003091, and rounding to 0.0003 was too much rounding.

Special Symbols There are a handful of special symbols that are easy to write on paper, but it's not clear how to type them. Here are WeBWorK's expectations.

Symbol	Name	How to Type
∞	infinity	`infinity` or `inf`
π	pi	`pi`
$\cup$	union	`U`
$\mathbb{R}$	the real numbers	`R`
$\vert$	such that	`|` (`shift-\`, where `\` is above the `enter` key)
$\leq$	less than or equal to	`<=`
$\geq$	greater than or equal to	`>=`
$\neq$	not equal to	`!=`

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CHAPTER 1

Basic Math Review

This chapter is *mostly* intended to *review* topics from a basic math course, especially Sections 1.1–1.4. These topics are covered differently than they would be covered for a student seeing them for the first time.

1.1 Arithmetic with Negative Numbers

Adding, subtracting, multiplying, dividing, and raising to powers each have peculiarities when using negative numbers. This section reviews arithmetic with signed (both positive and negative) numbers.

1.1.1 Signed Numbers

Is it valid to subtract a large number from a smaller one? It may be hard to imagine what it would mean physically to subtract 8 cars from your garage if you only have 1 car in there in the first place. Nevertheless, mathematics has found a way to give meaning to expressions like $1 - 8$ using **signed numbers**.

In daily life, the signed numbers we might see most often are temperatures. Most people on Earth use the Celsius scale; if you're not familiar with the Celsius temperature scale, think about these examples:

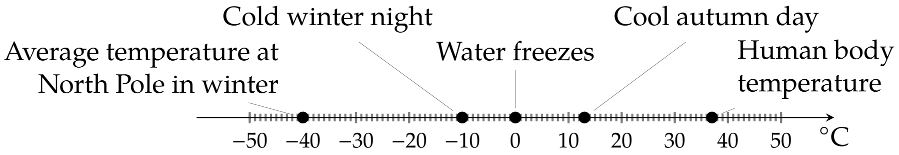


Figure 1.1.2: Number line with interesting Celsius temperatures

Figure 1.1.2 uses a **number line** to illustrate these positive and negative numbers. A number line is a useful device for visualizing how numbers relate to each other and combine with each other. Values to the right of 0 are called **positive** numbers and values to the left of 0 are called **negative numbers**.

Warning 1.1.3 Subtraction Sign versus Negative Sign. Unfortunately, the symbol we use for subtraction looks just like the symbol we use for marking a negative number. It will help to identify when a "minus" sign means "subtract" or means "negative." The key is to see if there is a number to its left, not counting anything farther left than an open parenthesis. Here are some examples.

- -13 has one negative sign and no subtraction sign.
- $20 - 13$ has no negative signs and one subtraction sign.
- $-20 - 13$ has a negative sign and then a subtraction sign.
- $(-20)(-13)$ has two negative signs and no subtraction sign.

Checkpoint 1.1.4. Identify “minus” signs.

In each expression, how many negative signs and subtraction signs are there?

a. $1 - 9$

has ☐ negative signs and ☐ subtraction signs.

b. $-12 + (-50)$

has ☐ negative signs and ☐ subtraction signs.

c. $\dfrac{-13 - (-15) - 17}{23 - 4}$

has ☐ negative signs and ☐ subtraction signs.

Explanation.

a. $1 - 9$ has zero negative signs and one subtraction sign.

b. $-12 + (-50)$ has two negative signs and zero subtraction signs.

c. $\dfrac{-13 - (-15) - 17}{23 - 4}$ has two negative signs and three subtraction signs.

1.1.2 Adding

An easy way to think about adding two numbers with the *same sign* is to simply (at first) ignore the signs, and add the numbers as if they were both positive. Then make sure your result is either positive or negative, depending on what the sign was of the two numbers you started with.

> **Example 1.1.5 Add Two Negative Numbers.** If you needed to add -18 and -7, note that both are negative. Maybe you have this expression in front of you:
>
> $$-18 + -7$$
>
> but that “plus minus” is awkward, and in this book you are more likely to have this expression:
>
> $$-18 + (-7)$$
>
> with extra parentheses. (How many subtraction signs do you see? How many negative signs?)
>
> Since *both* our terms are *negative*, we can add 18 and 7 to get 25 and immediately realize that our final result should be negative. So our result is -25:
>
> $$-18 + (-7) = -25$$

This approach works because adding numbers is like having two people tugging on a rope in one direction or the other, with strength indicated by each number. In Example 1.1.5 we have two people pulling to the left, one with strength 18, the other with strength 7. Their forces combine to pull *left* with strength 25, giving us our total of -25, as illustrated in Figure 1.1.6.

If we are adding two numbers that have *opposite* signs, then the two people tugging the rope are opposing each other. If either of them is using more strength, then the overall effect will be a net pull in that person’s direction. And the overall pull on the rope will be the *difference* of the two strengths. This is illustrated in Figure 1.1.7.

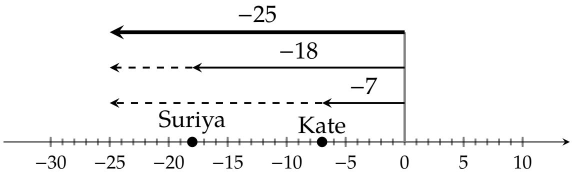


Figure 1.1.6: Working together

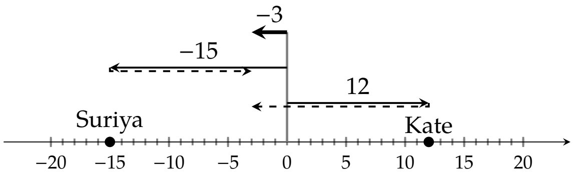


Figure 1.1.7: Working in opposition

Example 1.1.8 Adding One Number of Each Sign. Here are four examples of addition where one number is positive and the other is negative.

a. $-15 + 12$

We have one number of each sign, with sizes 15 and 12. Their difference is 3. But of the two numbers, the negative number dominated. So the result from adding these is -3.

b. $200 + (-100)$

We have one number of each sign, with sizes 200 and 100. Their difference is 100. But of the two numbers, the positive number dominated. So the result from adding these is 100.

c. $12.8 + (-20)$

We have one number of each sign, with sizes 12.8 and 20. Their difference is 7.2. But of the two numbers, the negative number dominated. So the result from adding these is -7.2.

d. $-87.3 + 87.3$

We have one number of each sign, both with size 87.3. The opposing forces cancel each other, leaving a result of 0.

Checkpoint 1.1.9. Take a moment to practice adding when at least one negative number is involved. The expectation is that readers can make these calculations here without a calculator.

a. Add $-1 + 9$.

b. Add $-12 + (-98)$.

c. Add $100 + (-123)$.

d. Find the sum $-2.1 + (-2.1)$.

e. Find the sum $-34.67 + 81.53$.

Explanation.

a. The two numbers have opposite sign, so we can think to subtract $9 - 1 = 8$. Of the two numbers we added, the positive is larger, so we stick with postive 8 as the answer.

b. The two numbers are both negative, so we can add $12 + 98 = 110$, and take the negative of that as the answer: -110.

c. The two numbers have opposite sign, so we can think to subtract $123 - 100 = 23$. Of the two numbers we added, the negative is larger, so we take the negative of 23 as the answer. That is, the answer is -23.

d. The two numbers are both negative, so we can add $2.1 + 2.1 = 4.2$, and take the negative of that as the answer: -4.2.

e. The two numbers have opposite sign, so we can think to subtract $81.53 - 34.67 = 46.86$. Of the two numbers we added, the positive is larger, so we stick with postive 46.86 as the answer.

1.1.3 Subtracting

Perhaps you can handle a subtraction such as $18 - 5$, where a small positive number is subtracted from a larger number. There are other instances of subtraction that might leave you scratching your head. In such situations, we recommend that you view each subtraction as *adding* the opposite number.

	Original	Adding the Opposite
Subtracting a larger positive number:	$12 - 30$	$12 + (-30)$
Subtracting from a negative number:	$-8.1 - 17$	$-8.1 + (-17)$
Subtracting a negative number:	$42 - (-23)$	$42 + 23$

The benefit is that perhaps you already mastered addition with positive and negative numbers, and this strategy that you convert subtraction to addition means you don't have all that much more to learn. These examples might be computed as follows:

$$\begin{aligned} 12 - 30 &= 12 + (-30) \\ &= -18 \end{aligned} \qquad \begin{aligned} -8.1 - 17 &= -8.1 + (-17) \\ &= -25.1 \end{aligned} \qquad \begin{aligned} 42 - (-23) &= 42 + 23 \\ &= 65 \end{aligned}$$

Checkpoint 1.1.10. Take a moment to practice subtracting when at least one negative number is involved. The expectation is that readers can make these calculations here without a calculator.

a. Subtract -1 from 9.

b. Subtract $32 - 50$.

c. Subtract $108 - (-108)$.

d. Find the difference $-5.9 - (-3.1)$.

e. Find the difference $-12.04 - 17.2$.

Explanation.

a. After writing this as $9 - (-1)$, we can rewrite it as $9 + 1$ and get 10.

b. Subtrcting in the oppsite order with the larger number first, $50 - 32 = 18$. But since we were asked to subtract the larger number *from* the smaller number, the answer is -18.

c. After writing this as $108 - (-108)$, we can rewrite it as $108 + 108$ and get 216.

d. After writing this as $-5.9 - (-3.1)$, we can rewrite it as $-5.9 + 3.1$. Now it is the *sum* of two numbers of opposite sign, so we can subtract $5.9 - 3.1$ to get 2.8. But we were adding numbers where the negative number was larger, so the final answer should be -2.8.

e. Since we are subtracting a positive number from a negative number, the result should be an even more negative number. We can add $12.04 + 17.2$ to get 29.24, but our final answer should be the opposite, -29.24.

1.1.4 Multiplying

Making sense of multiplication of negative numbers isn't quite so straightforward, but it's possible. Should the product of 3 and -7 be a positive number or a negative number? Remembering that we can view multiplication as repeated addition, we can see this result on a number line:

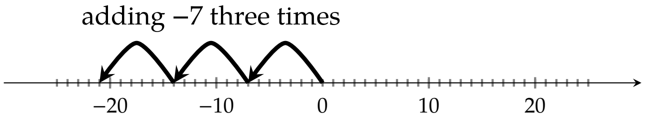


Figure 1.1.11: Viewing $3 \cdot (-7)$ as repeated addition

Figure 1.1.11 illustrates that $3 \cdot (-7) = -21$, and so it would seem that a positive number times a negative number will always give a negative result. (Note that it would not change things if the negative number came first in the product, since the order of multiplication doesn't affect the result.)

What about the product $-3 \cdot (-7)$, where both factors are negative? Should the product be positive or negative? If $3 \cdot (-7)$ can be seen as adding -7 three times as in Figure 1.1.12, then it isn't too crazy to interpret $-3 \cdot (-7)$ as *subtracting* -7 three times, as in Figure 1.1.12.

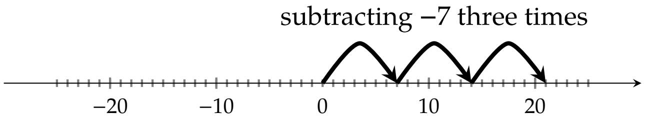


Figure 1.1.12: Viewing $-3 \cdot (-7)$ as repeated subtraction

This illustrates that $-3 \cdot (-7) = 21$, and it would seem that a negative number times a negative number always gives a positive result.

Positive and negative numbers are not the whole story. The number 0 is neither positive nor negative. What happens with multiplication by 0? You can choose to view $7 \cdot 0$ as adding the number 0 seven times. And you can choose to view $0 \cdot 7$ as adding the number 7 zero times. Either way, you really added nothing at all, which is the same as adding 0.

Fact 1.1.13 Multiplication by 0. *Multiplying any number by* 0 *results in* 0*.*

Checkpoint 1.1.14. Here are some practice exercises with multiplication and signed numbers. The expectation is that readers can make these calculations here without a calculator.

a. Multiply $-13 \cdot 2$.
b. Find the product of 30 and -50.
c. Compute $-12(-7)$.
d. Find the product $-285(0)$.

Explanation.

a. Since $13 \cdot 2 = 26$, and we are multiplying numbers of opposite signs, the answer is negative: -26.
b. Since $30 \cdot 50 = 1500$, and we are multiplying numbers of opposite signs, the answer is negative: -1500.
c. Since $12 \cdot 7 = 84$, and we are multiplying numbers of the same sign, the answer is positive: 84.
d. Any number multiplied by 0 is 0.

1.1.5 Powers

For early sections of this book the only exponents you will see will be the **natural numbers**: $\{1, 2, 3, \ldots\}$. But negative numbers can and will arise as the *base* of a power.

An exponent is a shorthand for how many times to multiply by the base. For example,

$$(-2)^5 \text{ means } \overbrace{(-2)\cdot(-2)\cdot(-2)\cdot(-2)\cdot(-2)}^{\text{5 instances}}$$

Will the result here be positive or negative? Since we can view $(-2)^5$ as repeated multiplication, and we now understand that multiplying two negatives gives a positive result, this expression can be thought of this way:

$$\underbrace{\underbrace{(-2)\cdot(-2)}_{\text{positive}}\cdot\underbrace{(-2)\cdot(-2)}_{\text{positive}}}_{\text{positive}}\cdot(-2)$$

and that lone last negative number will be responsible for making the final product negative.

More generally, if the base of a power is negative, then whether or not the result is positive or negative depends on if the exponent is even or odd. It depends on whether or not the factors can all be paired up to "cancel" negative signs, or if there will be a lone factor left by itself.

Once you understand whether the result is positive or negative, for a moment you may forget about signs. Continuing the example, you may calculate that $2^5 = 32$, and then since we know $(-2)^5$ is negative, you can report

$$(-2)^5 = -32$$

Warning 1.1.15 Negative Signs and Exponents. Expressions like -3^4 may not mean what you think they mean. What base do you see here? The correct answer is 3. The exponent 4 *only* applies to the 3, not to -3. So this expression, -3^4, is actually the same as $-\left(3^4\right)$, which is -81. Be careful not to treat -3^4 as having base -3. That would make it equivalent to $(-3)^4$, which is *positive* 81.

Checkpoint 1.1.16. Here is some practice with natural exponents on negative bases. The expectation is that readers can make these calculations here without a calculator.

a. Compute $(-8)^2$.

b. Calculate the power $(-1)^{203}$.

c. Find $(-3)^3$.

d. Calculate -5^2.

Explanation.

a. Since 8^2 is 64 and we are raising a negative number to an *even* power, the answer is positive: 64.

b. Since 1^{203} is 1 and we are raising a negative number to an *odd* power, the answer is negative: -1.

c. Since 3^3 is 27 and we are raising a negative number to an *odd* power, the answer is negative: -27.

d. Careful: here we are raising *positive* 5 to the second power to get 25 and *then* negating the result: -25. Since we don't see "$(-5)^2$," the answer is not positive 25.

1.1.6 Summary

Addition Add two negative numbers: add their positive counterparts and make the result negative.

Add a positive with a negative: find their difference using subtraction, and keep the sign of the dominant number.

Subtraction Any subtraction can be converted to addition of the opposite number. For all but the most basic subtractions, this is a useful strategy.

Multiplication Multiply two negative numbers: multiply their positive counterparts and make the result positive.

Multiply a positive with a negative: multiply their positive counterparts and make the result negative.

Multiply any number by 0: the result will be 0.

Division (not discussed in this section) Division by some number is the same as multiplication by its reciprocal. So the multiplication rules can be adopted.

Division of 0 by any nonzero number always results in 0.

Division of any number by 0 is always undefined.

Powers Raise a negative number to an even power: raise the positive counterpart to that power.

Raise a negative number to an odd power: raise the positive counterpart to that power, then make the result negative.

Expressions like -2^4 mean $-\left(2^4\right)$, not $(-2)^4$.

Exercises

Add the following.

1. a. $-8+(-1)=\boxed{}$
b. $-6+(-7)=\boxed{}$
c. $-1+(-7)=\boxed{}$

2. a. $-10+(-2)=\boxed{}$
b. $-5+(-3)=\boxed{}$
c. $-2+(-9)=\boxed{}$

3. a. $2+(-10)=\boxed{}$
b. $5+(-2)=\boxed{}$
c. $7+(-7)=\boxed{}$

4. a. $2+(-6)=\boxed{}$
b. $8+(-3)=\boxed{}$
c. $7+(-7)=\boxed{}$

5. a. $-8+3=\boxed{}$
b. $-4+10=\boxed{}$
c. $-4+4=\boxed{}$

6. a. $-10+3=\boxed{}$
b. $-1+7=\boxed{}$
c. $-4+4=\boxed{}$

7. a. $-41+(-31)=\boxed{}$
b. $-31+58=\boxed{}$
c. $64+(-27)=\boxed{}$

8. a. $-31+(-63)=\boxed{}$
b. $-84+33=\boxed{}$
c. $62+(-72)=\boxed{}$

Subtract the following.

9. a. 5 − 6 =
b. 8 − 4 =
c. 4 − 19 =

10. a. 5 − 9 =
b. 5 − 3 =
c. 4 − 14 =

11. a. −5 − 2 =
b. −8 − 3 =
c. −6 − 6 =

12. a. −4 − 1 =
b. −6 − 1 =
c. −6 − 6 =

13. a. −2 − (−8) =
b. −6 − (−2) =
c. −4 − (−4) =

14. a. −3 − (−6) =
b. −9 − (−2) =
c. −4 − (−4) =

Perform the given addition and subtraction.

15. a. −15 − 10 + (−9) =
b. 4 − (−15) + (−12) =

16. a. −13 − 7 + (−5) =
b. 1 − (−15) + (−17) =

17. a. −12 − 3 + (−1) =
b. 8 − (−15) + (−12) =

18. a. −11 − 10 + (−7) =
b. 5 − (−16) + (−17) =

Multiply the following.

19. a. (−8) · (−2) =
b. (−4) · 3 =
c. 7 · (−2) =
d. (−5) · 0 =

20. a. (−10) · (−3) =
b. (−7) · 7 =
c. 7 · (−6) =
d. (−4) · 0 =

21. a. (−3) · (−4) · (−3) =
b. 5 · (−8) · (−2) =
c. (−84) · (−52) · 0 =

22. a. (−3) · (−5) · (−5) =
b. 4 · (−8) · (−4) =
c. (−83) · (−70) · 0 =

23. a. (−2)(−1)(−2)(−1) =
b. (−3)(−3)(1)(−1) =

24. a. (−2)(−3)(−3)(−3) =
b. (−1)(−1)(−1)(−3) =

Evaluate the following.

25. a. $\frac{-30}{-5} =$ ☐

b. $\frac{36}{-6} =$ ☐

c. $\frac{-48}{6} =$ ☐

26. a. $\frac{-12}{-4} =$ ☐

b. $\frac{20}{-5} =$ ☐

c. $\frac{-24}{6} =$ ☐

27. a. $\frac{-3}{-1} =$ ☐

b. $\frac{9}{-1} =$ ☐

c. $\frac{110}{-110} =$ ☐

d. $\frac{-10}{-10} =$ ☐

e. $\frac{8}{0} =$ ☐

f. $\frac{0}{-4} =$ ☐

28. a. $\frac{-2}{-1} =$ ☐

b. $\frac{7}{-1} =$ ☐

c. $\frac{150}{-150} =$ ☐

d. $\frac{-13}{-13} =$ ☐

e. $\frac{8}{0} =$ ☐

f. $\frac{0}{-8} =$ ☐

29. a. $(-9)^2 =$ ☐

b. $-4^2 =$ ☐

30. a. $(-7)^2 =$ ☐

b. $-6^2 =$ ☐

31. a. $(-3)^3 =$ ☐

b. $-1^3 =$ ☐

32. a. $(-2)^3 =$ ☐

b. $-4^3 =$ ☐

33. a. $4^2 =$ ☐

b. $2^3 =$ ☐

c. $(-4)^2 =$ ☐

d. $(-3)^3 =$ ☐

34. a. $5^2 =$ ☐

b. $4^3 =$ ☐

c. $(-3)^2 =$ ☐

d. $(-5)^3 =$ ☐

35. a. $1^{10} =$ ☐

b. $(-1)^{11} =$ ☐

c. $(-1)^{12} =$ ☐

d. $0^{20} =$ ☐

36. a. $1^5 =$ ☐

b. $(-1)^{13} =$ ☐

c. $(-1)^{18} =$ ☐

d. $0^{18} =$ ☐

Simplify without using a calculator.

37. $-9.33 + (-21.7) = \boxed{}$

38. $-1.97 + (-81.3) = \boxed{}$

39. $6.6 - 1.62 = \boxed{}$

40. $6.3 - 3.32 = \boxed{}$

41. $-6.12 + 7.9 = \boxed{}$

42. $-3.82 + 7.6 = \boxed{}$

43. $-6.52 - (-8.3) = \boxed{}$

44. $-2.21 - (-8.9) = \boxed{}$

45. $75 - 6.91 = \boxed{}$

46. $82 - 1.71 = \boxed{}$

47. $-18 + 5.41 = \boxed{}$

48. $-25 + 8.11 = \boxed{}$

49. It's given that $32 \cdot 39 = 1248$. Use this fact to calculate the following without using a calculator:

$3.2(-0.039) = \boxed{}$

50. It's given that $48 \cdot 76 = 3648$. Use this fact to calculate the following without using a calculator:

$4.8(-7.6) = \boxed{}$

51. It's given that $55 \cdot 24 = 1320$. Use this fact to calculate the following without using a calculator:

$(-5.5)(-0.024) = \boxed{}$

52. It's given that $62 \cdot 51 = 3162$. Use this fact to calculate the following without using a calculator:

$(-6.2)(-5.1) = \boxed{}$

Applications

53. Consider the following situation in which you borrow money from your cousin:

- On June 1st, you borrowed 1400 dollars from your cousin.
- On July 1st, you borrowed 460 more dollars from your cousin.
- On August 1st, you paid back 690 dollars to your cousin.
- On September 1st, you borrowed another 960 dollars from your cousin.

How much money do you owe your cousin now?

54. Consider the following scenario in which you study your bank account.

- On Jan. 1, you had a balance of −450 dollars in your bank account.
- On Jan. 2, your bank charged 40 dollar overdraft fee.
- On Jan. 3, you deposited 870 dollars.
- On Jan. 10, you withdrew 650 dollars.

What is your balance on Jan. 11?

55. A mountain is 1100 feet *above* sea level. A trench is 360 feet *below* sea level. What is the difference in elevation between the mountain top and the bottom of the trench?

56. A mountain is 1200 feet *above* sea level. A trench is 420 feet *below* sea level. What is the difference in elevation between the mountain top and the bottom of the trench?

Challenge

57. Select the correct word to make each statement true.

a. A positive number minus a positive number is (☐ sometimes ☐ always ☐ never) negative.

b. A negative number plus a negative number is (☐ sometimes ☐ always ☐ never) negative.

c. A positive number minus a negative number is (☐ sometimes ☐ always ☐ never) positive.

d. A negative number multiplied by a negative number is (☐ sometimes ☐ always ☐ never) negative.

1.2 Fractions and Fraction Arithmetic

The word “fraction” comes from the Latin word *fractio*, which means “break into pieces.” For thousands of years, cultures from all over the world have used fractions to understand parts of a whole.

1.2.1 Visualizing Fractions

Parts of a Whole One approach to understanding fractions is to think of them as parts of a whole.

In Figure 1.2.2, we see 1 whole divided into 7 parts. Since 3 parts are shaded, we have an illustration of the fraction $\frac{3}{7}$. The **denominator** 7 tells us how many parts to cut up the whole; since we have 7 parts, they’re called “sevenths.” The **numerator** 3 tells us how many sevenths to consider.

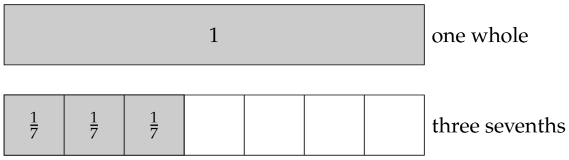


Figure 1.2.2: Representing $\frac{3}{7}$ as parts of a whole.

Checkpoint 1.2.3 A Fraction as Parts of a Whole. To visualize the fraction $\frac{14}{35}$, you might cut a rectangle into ____ equal parts, and then count up ____ of them.

Explanation. You could cut a rectangle into 35 equal pieces, and then 14 of them would represent $\frac{14}{35}$.

We can also locate fractions on number lines. When ticks are equally spread apart, as in Figure 1.2.4, each tick represents a fraction.

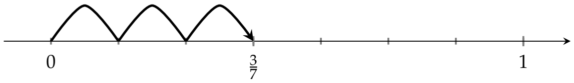
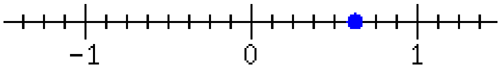


Figure 1.2.4: Representing $\frac{3}{7}$ on a number line.

Checkpoint 1.2.5 A Fraction on a Number Line. In the given number line, what fraction is marked?



Explanation. There are 8 subdivisions between 0 and 1, and the mark is at the fifth subdivision. So the mark is $\frac{5}{8}$ of the way from 0 to 1 and therefore represents the fraction $\frac{5}{8}$.

Division Fractions can also be understood through division.

For example, we can view the fraction $\frac{3}{7}$ as 3 divided into 7 equal parts, as in Figure 1.2.6. Just one of those parts represents $\frac{3}{7}$.

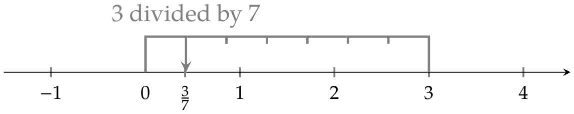


Figure 1.2.6: Representing $\frac{3}{7}$ on a number line.

Checkpoint 1.2.7 Seeing a Fraction as Division Arithmetic. The fraction $\frac{21}{40}$ can be thought of as dividing the whole number ☐ into ☐ equal-sized parts.

Explanation. Since $\frac{21}{40}$ means the same as $21 \div 40$, it can be thought of as dividing 21 into 40 equal parts.

1.2.2 Equivalent Fractions

It's common to have two fractions that represent the same amount. Consider $\frac{2}{5}$ and $\frac{6}{15}$ represented in various ways in Figures 1.2.8–1.2.10.

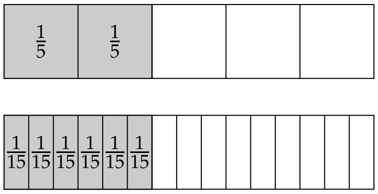


Figure 1.2.8: $\frac{2}{5}$ and $\frac{6}{15}$ as equal parts of a whole

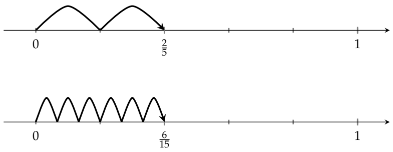


Figure 1.2.9: $\frac{2}{5}$ and $\frac{6}{15}$ as equal on a number line

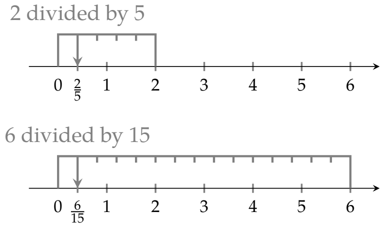


Figure 1.2.10: $\frac{2}{5}$ and $\frac{6}{15}$ as equal results from division

Those two fractions, $\frac{2}{5}$ and $\frac{6}{15}$ are equal, as those figures demonstrate. Also, because they each equal 0.4 as a decimal. If we must work with this number, the fraction that uses smaller numbers, $\frac{2}{5}$, is preferable. Working with smaller numbers decreases the likelihood of making a human arithmetic error. And it also increases the chances that you might make useful observations about the nature of that number.

So if you are handed a fraction like $\frac{6}{15}$, it is important to try to **reduce** it to "lowest terms." The most important skill you can have to help you do this is to know the multiplication table well. If you know it well, you know that $6 = 2 \cdot 3$ and $15 = 3 \cdot 5$, so you can break down the numerator and denominator that way. Both the numerator and denominator are divisible by 3, so they can be "factored out" and then as factors, cancel out.

$$\begin{aligned}\frac{6}{15} &= \frac{2 \cdot 3}{3 \cdot 5}\\ &= \frac{2 \cdot \cancel{3}}{\cancel{3} \cdot 5}\\ &= \frac{2 \cdot 1}{1 \cdot 5}\\ &= \frac{2}{5}\end{aligned}$$

Checkpoint 1.2.11. Reduce these fractions into lowest terms.

a. $\frac{14}{42} =$ ☐

b. $\frac{8}{30} =$ ☐

c. $\frac{70}{90} =$ ☐

Explanation.

a. With $\frac{14}{42}$, we have $\frac{2 \cdot 7}{2 \cdot 3 \cdot 7}$, which reduces to $\frac{1}{3}$.

b. With $\frac{8}{30}$, we have $\frac{2 \cdot 2 \cdot 2}{2 \cdot 3 \cdot 5}$, which reduces to $\frac{4}{15}$.

c. With $\frac{70}{90}$, we have $\frac{7 \cdot 10}{9 \cdot 10}$, which reduces to $\frac{7}{9}$.

Sometimes it is useful to do the opposite of reducing a fraction, and **build up** the fraction to use larger numbers.

Checkpoint 1.2.12. Sayid scored $\frac{21}{25}$ on a recent exam. Build up this fraction so that the denominator is 100, so that Sayid can understand what percent score he earned.

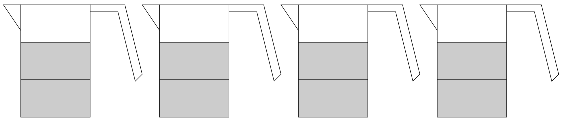
Explanation. To change the denominator from 25 to 100, it needs to be multiplied by 4. So we calculate

$$\begin{aligned}\frac{21}{25} &= \frac{21 \cdot 4}{25 \cdot 4}\\ &= \frac{84}{100}\end{aligned}$$

So the fraction $\frac{21}{25}$ is equivalent to $\frac{84}{100}$. (This means Sayid scored an 84%.)

1.2.3 Multiplying with Fractions

Example 1.2.13 Suppose a recipe calls for $\frac{2}{3}$ cup of milk, but we'd like to quadruple the recipe (make it four times as big). We'll need four times as much milk, and one way to measure this out is to fill a measuring cup to $\frac{2}{3}$ full, four times:



When you count up the shaded thirds, there are eight of them. So multiplying $\frac{2}{3}$ by the whole number 4, the result is $\frac{8}{3}$. Mathematically:

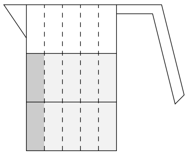
$$\begin{aligned}4 \cdot \frac{2}{3} &= \frac{4 \cdot 2}{3}\\ &= \frac{8}{3}\end{aligned}$$

Fact 1.2.14 Multiplying a Fraction and a Whole Number. *When you multiply a whole number by a fraction, you may just multiply the whole number by the numerator and leave the denominator alone. In other words, as long as d is not 0, then a whole number and a fraction multiply this way:*

$$a \cdot \frac{c}{d} = \frac{a \cdot c}{d}$$

Example 1.2.15 We could also use multiplication to decrease amounts. Suppose we needed to cut the recipe down to just one fifth. Instead of *four* of the $\frac{2}{3}$ cup milk, we need *one fifth* of the $\frac{2}{3}$ cup milk. So instead of multiplying by 4, we multiply by $\frac{1}{5}$. But how much is $\frac{1}{5}$ of $\frac{2}{3}$ cup?

If we cut the measuring cup into five equal vertical strips along with the three equal horizontal strips, then in total there are $3 \cdot 5 = 15$ subdivisions of the cup. Two of those sections represent $\frac{1}{5}$ of the $\frac{2}{3}$ cup.



> In the end, we have $\frac{2}{15}$ of a cup. The denominator 15 came from multiplying 5 and 3, the denominators of the fractions we had to multiply. The numerator 2 came from multiplying 1 and 2, the numerators of the fractions we had to multiply.
>
> $$\begin{aligned}\frac{1}{5}\cdot\frac{2}{3} &= \frac{1\cdot 2}{5\cdot 3}\\ &= \frac{2}{15}\end{aligned}$$

Fact 1.2.16 Multiplication with Fractions. *As long as b and d are not* 0*, then fractions multiply this way:*

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$$

Checkpoint 1.2.17. Simplify these fraction products.

a. $\frac{1}{3}\cdot\frac{10}{7} =$

b. $\frac{12}{3}\cdot\frac{15}{3} =$

c. $-\frac{14}{5}\cdot\frac{2}{3} =$

d. $\frac{70}{27}\cdot\frac{12}{-20} =$

Explanation.

a. Multiplying numerators gives 10, and multiplying denominators gives 21. The answer is $\frac{10}{21}$.

b. Before we multiply fractions, note that $\frac{12}{3}$ reduces to 4, and $\frac{15}{3}$ reduces to 5. So we just have $4\cdot 5 = 20$.

c. Multiplying numerators gives 28, and multiplying denominators gives 15. The answer is $\frac{28}{15}$.

d. Before we multiply fractions, note that $\frac{12}{-20}$ reduces to $\frac{-3}{5}$. So we have $\frac{70}{27}\cdot\frac{-3}{5}$. Both the numerator of the first fraction and denominator of the second fraction are divisible by 5, so it helps to reduce both fractions accordingly and get $\frac{14}{27}\cdot\frac{-3}{1}$. Both the denominator of the first fraction and numerator of the second fraction are divisible by 3, so it helps to reduce both fractions accordingly and get $\frac{14}{9}\cdot\frac{-1}{1}$. Now we are just multiplying $\frac{14}{9}$ by -1, so the result is $\frac{-14}{9}$.

1.2.4 Division with Fractions

How does division with fractions work? Are we able to compute/simplify each of these examples?

a. $3\div\frac{2}{7}$

b. $\frac{18}{19}\div 5$

c. $\frac{14}{3}\div\frac{8}{9}$

d. $\frac{\frac{2}{5}}{\frac{5}{2}}$

We know that when we divide something by 2, this is the same as multiplying it by $\frac{1}{2}$. Conversely, dividing a number or expression by $\frac{1}{2}$ is the same as multiplying by $\frac{2}{1}$, or just 2. The more general property is that when we divide a number or expression by $\frac{a}{b}$, this is equivalent to multiplying by the reciprocal $\frac{b}{a}$.

Fact 1.2.18 Division with Fractions. *As long as b, c and d are not* 0*, then division with fractions works this way:*

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$$

Example 1.2.19 With our examples from the beginning of this subsection:

a. $$\begin{aligned} 3 \div \frac{2}{7} &= 3 \cdot \frac{7}{2} \\ &= \frac{3}{1} \cdot \frac{7}{2} \\ &= \frac{21}{2} \end{aligned}$$

b. $$\begin{aligned} \frac{18}{19} \div 5 &= \frac{18}{19} \div \frac{5}{1} \\ &= \frac{18}{19} \cdot \frac{1}{5} \\ &= \frac{18}{95} \end{aligned}$$

c. $$\begin{aligned} \frac{14}{3} \div \frac{8}{9} &= \frac{14}{3} \cdot \frac{9}{8} \\ &= \frac{14}{1} \cdot \frac{3}{8} \\ &= \frac{7}{1} \cdot \frac{3}{4} \\ &= \frac{21}{4} \end{aligned}$$

d. $$\begin{aligned} \frac{\frac{2}{5}}{\frac{5}{2}} &= \frac{2}{5} \div \frac{5}{2} \\ &= \frac{2}{5} \cdot \frac{2}{5} \\ &= \frac{4}{25} \end{aligned}$$

Checkpoint 1.2.20. Simplify these fraction division expressions.

a. $\frac{1}{3} \div \frac{10}{7} = \boxed{}$ b. $\frac{12}{5} \div 5 = \boxed{}$ c. $-14 \div \frac{3}{2} = \boxed{}$ d. $\frac{70}{9} \div \frac{11}{-20} = \boxed{}$

Explanation.

a. $$\begin{aligned} \frac{1}{3} \div \frac{10}{7} &= \frac{1}{3} \cdot \frac{7}{10} \\ &= \frac{7}{30} \end{aligned}$$

b. $$\begin{aligned} \frac{12}{5} \div 5 &= \frac{12}{5} \cdot \frac{1}{5} \\ &= \frac{12}{25} \end{aligned}$$

c. $$\begin{aligned} -14 \div \frac{3}{2} &= -\frac{1}{14} \cdot \frac{2}{3} \\ &= -\frac{1}{7} \cdot \frac{1}{3} \\ &= -\frac{1}{21} \end{aligned}$$

d. $$\begin{aligned} \frac{70}{9} \div \frac{11}{-20} &= -\frac{70}{9} \cdot \frac{20}{11} \\ &= -\frac{1400}{99} \end{aligned}$$

1.2.5 Adding and Subtracting Fractions

With whole numbers and integers, operations of addition and subtraction are relatively straightforward. The situation is almost as straightforward with fractions *if the two fractions have the same denominator*. Consider

$$\frac{7}{2} + \frac{3}{2} = 7 \text{ halves} + 3 \text{ halves}$$

In the same way that 7 tacos and 3 tacos make 10 tacos, we have:

$$\begin{array}{ccccc} 7 \text{ halves} & + & 3 \text{ halves} & = & 10 \text{ halves} \\ \frac{7}{2} & + & \frac{3}{2} & = & \frac{10}{2} \\ & & & = & 5 \end{array}$$

Fact 1.2.21 Adding/Subtracting with Fractions Having the Same Denominator. *To add or subtract two fractions having the same denominator,* keep *that denominator, and add or subtract the numerators.*

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b} \qquad\qquad \frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$$

If it's possible, useful, or required of you, simplify the result by reducing to lowest terms.

Checkpoint 1.2.22. Add or subtract these fractions.

a. $\frac{1}{3}+\frac{10}{3}=$

b. $\frac{13}{6}-\frac{5}{6}=$

Explanation.

a. Since the denominators are both 3, we can add the numerators: $1+10=11$. The answer is $\frac{11}{3}$.

b. Since the denominators are both 6, we can subtract the numerators: $13-5=8$. The answer is $\frac{8}{6}$, but that reduces to $\frac{4}{3}$.

Whenever we'd like to combine fractional amounts that don't represent the same number of parts of a whole (that is, when the denominators are different), finding sums and differences is more complicated.

Example 1.2.23 Quarters and Dimes. Find the sum $\frac{3}{4}+\frac{2}{10}$. Does this seem intimidating? Consider this:

- $\frac{1}{4}$ of a dollar is a quarter, and so $\frac{3}{4}$ of a dollar is 75 cents.
- $\frac{1}{10}$ of a dollar is a dime, and so $\frac{2}{10}$ of a dollar is 20 cents.

So if you know what to look for, the expression $\frac{3}{4}+\frac{2}{10}$ is like adding 75 cents and 20 cents, which gives you 95 cents. As a fraction of one dollar, that is $\frac{95}{100}$. So we can report

$$\frac{3}{4}+\frac{2}{10}=\frac{95}{100}.$$

(Although we should probably reduce that last fraction to $\frac{19}{20}$.)

This example was not something you can apply to other fraction addition situations, because the denominators here worked especially well with money amounts. But there is something we can learn here. The fraction $\frac{3}{4}$ was equivalent to $\frac{75}{100}$, and the other fraction $\frac{2}{10}$ was equivalent to $\frac{20}{100}$. These *equivalent* fractions have the same denominator and are therefore "easy" to add. What we saw happen was:

$$\begin{aligned}\frac{3}{4}+\frac{2}{10}&=\frac{75}{100}+\frac{20}{100}\\&=\frac{95}{100}\end{aligned}$$

This realization gives us a strategy for adding (or subtracting) fractions.

Fact 1.2.24 Adding/Subtracting Fractions with Different Denominators. *To add (or subtract) generic fractions together, use their denominators to find a* ***common denominator****. This means some whole number that is a whole multiple of both of the original denominators. Then rewrite the two fractions as* equivalent *fractions that use this common denominator. Write the result* keeping *that denominator and adding (or subtracting) the numerators. Reduce the fraction if that is useful or required.*

Example 1.2.25 Let's add $\frac{2}{3} + \frac{2}{5}$. The denominators are 3 and 5, so the number 15 would be a good common denominator.

$$\begin{aligned}\frac{2}{3} + \frac{2}{5} &= \frac{2 \cdot 5}{3 \cdot 5} + \frac{2 \cdot 3}{5 \cdot 3} \\ &= \frac{10}{15} + \frac{6}{15} \\ &= \frac{16}{15}\end{aligned}$$

Checkpoint 1.2.26. A chef had $\frac{2}{3}$ cups of flour and needed to use $\frac{1}{8}$ cup to thicken a sauce. How much flour is left?

Explanation. We need to compute $\frac{2}{3} - \frac{1}{8}$. The denominators are 3 and 8. One common denominator is 24, so we move to rewrite each fraction using 24 as the denominator:

$$\begin{aligned}\frac{2}{3} - \frac{1}{8} &= \frac{2 \cdot 8}{3 \cdot 8} - \frac{1 \cdot 3}{8 \cdot 3} \\ &= \frac{16}{24} - \frac{3}{24} \\ &= \frac{13}{24}\end{aligned}$$

The numerical result is $\frac{13}{24}$, but a pure number does not answer this question. The amount of flour remaining is $\frac{13}{24}$ *cups*.

1.2.6 Mixed Numbers and Improper Fractions

A simple recipe for bread contains only a few ingredients:

$1\,{}^1\!/_2$	tablespoons yeast
$1\,{}^1\!/_2$	tablespoons kosher salt
$6\,{}^1\!/_2$	cups unbleached, all-purpose flour (more for dusting)

Each ingredient is listed as a **mixed number** that quickly communicates how many whole amounts and how many parts are needed. It's useful for quickly communicating a practical amount of something you are cooking with, measuring on a ruler, purchasing at the grocery store, etc. But it causes trouble in an algebra class. The number $1\,{}^1\!/_2$ means "one ***and*** one half." So really,

$$1\frac{1}{2} = 1 + \frac{1}{2}$$

The trouble is that with $1\,{}^1\!/_2$, you have two numbers written right next to each other. Normally with two math expressions written right next to each other, they should be *multiplied*, not *added*. But with a mixed number, they *should* be added.

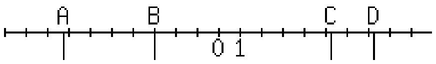
Fortunately we just reviewed how to add fractions. If we need to do any arithmetic with a mixed number like $1\,{}^{1}/_{2}$, we can treat it as $1 + \frac{1}{2}$ and simplify to get a "nice" fraction instead: $\frac{3}{2}$. A fraction like $\frac{3}{2}$ is called an **improper fraction** because it's actually larger than 1. And a "proper" fraction would be something small that is only *part* of a whole instead of *more* than a whole.

$$\begin{aligned} 1\frac{1}{2} &= 1 + \frac{1}{2} \\ &= \frac{1}{1} + \frac{1}{2} \\ &= \frac{2}{2} + \frac{1}{2} \\ &= \frac{3}{2} \end{aligned}$$

Exercises

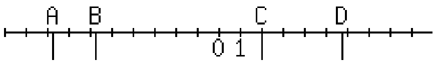
Review and Warmup

1. Which letter is $-\frac{29}{4}$ on the number line?



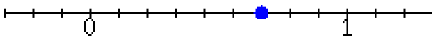
(☐ A ☐ B ☐ C ☐ D)

2. Which letter is $\frac{23}{4}$ on the number line?

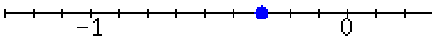


(☐ A ☐ B ☐ C ☐ D)

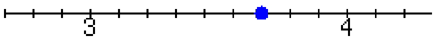
3. The dot in the graph can be represented by what fraction?



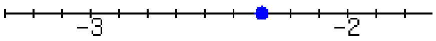
4. The dot in the graph can be represented by what fraction?



5. The dot in the graph can be represented by what fraction?



6. The dot in the graph can be represented by what fraction?



Reducing Fractions

7. Reduce the fraction $\frac{7}{70}$.

8. Reduce the fraction $\frac{14}{63}$.

9. Reduce the fraction $\frac{10}{50}$.

10. Reduce the fraction $\frac{20}{33}$.

11. Reduce the fraction $\frac{100}{70}$.

12. Reduce the fraction $\frac{42}{6}$.

Building Fractions

13. Find an equivalent fraction to $\frac{5}{7}$ with denominator 35.

14. Find an equivalent fraction to $\frac{3}{7}$ with denominator 14.

15. Find an equivalent fraction to $\frac{1}{17}$ with denominator 68.

16. Find an equivalent fraction to $\frac{13}{19}$ with denominator 38.

Multiplying/Dividing Fractions

17. Multiply: $\dfrac{1}{7} \cdot \dfrac{3}{8}$

18. Multiply: $\dfrac{3}{7} \cdot \dfrac{3}{8}$

19. Multiply: $\dfrac{6}{11} \cdot \dfrac{13}{6}$

20. Multiply: $\dfrac{7}{11} \cdot \dfrac{3}{14}$

21. Multiply: $3 \cdot \dfrac{3}{10}$

22. Multiply: $6 \cdot \dfrac{4}{5}$

23. Multiply: $-\dfrac{20}{7} \cdot \dfrac{7}{25}$

24. Multiply: $-\dfrac{18}{5} \cdot \dfrac{13}{28}$

25. Multiply: $28 \cdot \left(-\dfrac{9}{7}\right)$

26. Multiply: $6 \cdot \left(-\dfrac{1}{3}\right)$

27. Multiply: $\dfrac{7}{4} \cdot \dfrac{5}{49} \cdot \dfrac{6}{25}$

28. Multiply: $\dfrac{5}{9} \cdot \dfrac{14}{25} \cdot \dfrac{3}{4}$

29. Multiply: $\dfrac{14}{3} \cdot \dfrac{1}{4} \cdot 15$

30. Multiply: $\dfrac{7}{5} \cdot \dfrac{2}{49} \cdot 15$

31. Divide: $\dfrac{3}{7} \div \dfrac{7}{4}$

32. Divide: $\dfrac{4}{5} \div \dfrac{7}{4}$

33. Divide: $\dfrac{7}{10} \div \left(-\dfrac{5}{4}\right)$

34. Divide: $\dfrac{1}{20} \div \left(-\dfrac{5}{12}\right)$

35. Divide: $-\dfrac{3}{2} \div (-15)$

36. Divide: $-\dfrac{3}{8} \div (-9)$

37. Divide: $20 \div \dfrac{5}{3}$

38. Divide: $4 \div \dfrac{2}{3}$

39. Multiply: $3\frac{8}{9} \cdot 2\frac{1}{10}$

40. Multiply: $1\frac{5}{9} \cdot 6\frac{3}{4}$

Adding/Subtracting Fractions

41. Add: $\dfrac{2}{27} + \dfrac{1}{27}$

42. Add: $\dfrac{11}{40} + \dfrac{1}{40}$

43. Add: $\dfrac{5}{7} + \dfrac{11}{21}$

44. Add: $\frac{2}{9} + \frac{13}{18}$

45. Add: $\frac{1}{9} + \frac{13}{18}$

46. Add: $\frac{5}{6} + \frac{11}{30}$

47. Add: $\frac{3}{7} + \frac{1}{10}$

48. Add: $\frac{3}{8} + \frac{3}{7}$

49. Add: $\frac{1}{6} + \frac{1}{10}$

50. Add: $\frac{3}{10} + \frac{1}{6}$

51. Add: $\frac{5}{6} + \frac{9}{10}$

52. Add: $\frac{4}{5} + \frac{7}{10}$

53. Add: $-\frac{2}{5} + \frac{3}{5}$

54. Add: $-\frac{1}{5} + \frac{4}{5}$

55. Add: $-\frac{3}{7} + \frac{11}{14}$

56. Add: $-\frac{5}{9} + \frac{5}{54}$

57. Add: $-\frac{1}{8} + \frac{1}{7}$

58. Add: $-\frac{1}{8} + \frac{1}{5}$

59. Add: $2 + \frac{7}{8}$

60. Add: $4 + \frac{2}{5}$

61. Add: $\frac{3}{10} + \frac{1}{6} + \frac{1}{5}$

62. Add: $\frac{1}{3} + \frac{1}{8} + \frac{1}{6}$

63. Add: $\frac{2}{3} + \frac{1}{6} + \frac{3}{10}$

64. Add: $\frac{1}{3} + \frac{1}{6} + \frac{3}{5}$

65. Subtract: $\frac{16}{21} - \frac{4}{21}$

66. Subtract: $\frac{33}{32} - \frac{13}{32}$

67. Subtract: $\frac{4}{7} - \frac{33}{35}$

68. Subtract: $\frac{4}{9} - \frac{43}{45}$

69. Subtract: $\frac{1}{18} - \frac{5}{6}$

70. Subtract: $\frac{39}{40} - \frac{5}{8}$

71. Subtract: $-\frac{3}{10} - \frac{1}{6}$

72. Subtract: $-\frac{5}{6} - \frac{7}{10}$

73. Subtract: $-\frac{3}{10} - \left(-\frac{5}{6}\right)$

74. Subtract: $-\frac{5}{6} - \left(-\frac{3}{10}\right)$

75. Subtract: $-4 - \frac{15}{7}$

76. Subtract: $1 - \frac{10}{9}$

Applications

77. Michele walked $\frac{3}{10}$ of a mile in the morning, and then walked $\frac{3}{8}$ of a mile in the afternoon. How far did Michele walk altogether?

Michele walked a total of ☐ of a mile.

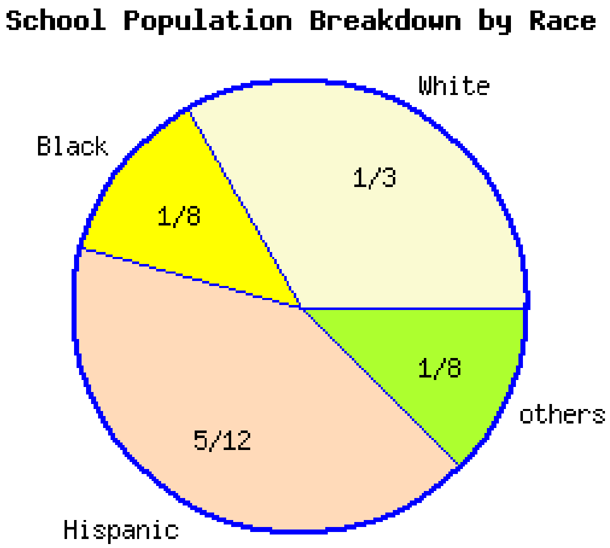
78. Kurt walked $\frac{1}{11}$ of a mile in the morning, and then walked $\frac{1}{9}$ of a mile in the afternoon. How far did Kurt walk altogether?

Kurt walked a total of ☐ of a mile.

79. Penelope and Kenji are sharing a pizza. Penelope ate $\frac{1}{10}$ of the pizza, and Kenji ate $\frac{3}{8}$ of the pizza. How much of the pizza was eaten in total?

They ate ☐ of the pizza.

80. The pie chart represents a school's student population.



Together, white and black students make up ☐ of the school's population.

81. A trail's total length is $\frac{19}{45}$ of a mile. It has two legs. The first leg is $\frac{2}{9}$ of a mile long. How long is the second leg?

The second leg is ☐ of a mile in length.

82. A trail's total length is $\frac{5}{18}$ of a mile. It has two legs. The first leg is $\frac{1}{9}$ of a mile long. How long is the second leg?

The second leg is ☐ of a mile in length.

83. Jessica is participating in a running event. In the first hour, she completed $\frac{1}{10}$ of the total distance. After another hour, in total she had completed $\frac{17}{70}$ of the total distance.

What fraction of the total distance did Jessica complete during the second hour?

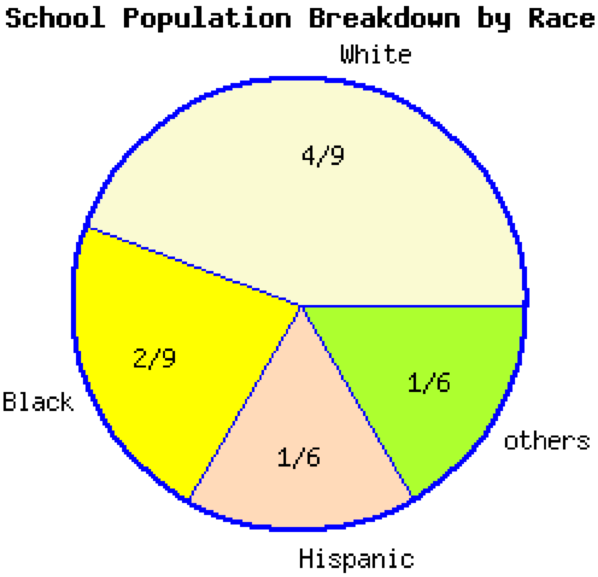
Jessica completed ☐ of the distance during the second hour.

84. Each page of a book is $6\frac{1}{2}$ inches in height, and consists of a header (a top margin), a footer (a bottom margin), and the middle part (the body). The header is $\frac{2}{9}$ of an inch thick and the middle part is $5\frac{8}{9}$ inches from top to bottom.

What is the thickness of the footer?

The footer is ☐ of an inch thick.

85. The pie chart represents a school's student population.



☐ more of the school is white students than black students.

86. Jessica and Carly are sharing a pizza. Jessica ate $\frac{2}{9}$ of the pizza, and Carly ate $\frac{1}{6}$ of the pizza. How much more pizza did Jessica eat than Carly?

Jessica ate ☐ more of the pizza than Carly ate.

87. Kayla and Blake are sharing a pizza. Kayla ate $\frac{2}{9}$ of the pizza, and Blake ate $\frac{1}{8}$ of the pizza. How much more pizza did Kayla eat than Blake?

Kayla ate ____ more of the pizza than Blake ate.

88. A school had a fund-raising event. The revenue came from three resources: ticket sales, auction sales, and donations. Ticket sales account for $\frac{1}{10}$ of the total revenue; auction sales account for $\frac{5}{7}$ of the total revenue. What fraction of the revenue came from donations?

____ of the revenue came from donations.

89. A few years back, a car was purchased for $10,800. Today it is worth $\frac{1}{3}$ of its original value. What is the car's current value?

The car's current value is ____.

90. A few years back, a car was purchased for $15,000. Today it is worth $\frac{1}{3}$ of its original value. What is the car's current value?

The car's current value is ____.

91. A town has 200 residents in total, of which $\frac{3}{4}$ are white/Caucasian Americans. How many white/Caucasian Americans reside in this town?

There are ____ white/Caucasian Americans residing in this town.

92. A company received a grant, and decided to spend $\frac{20}{21}$ of this grant in research and development next year. Out of the money set aside for research and development, $\frac{3}{5}$ will be used to buy new equipment. What fraction of the grant will be used to buy new equipment?

____ of the grant will be used to buy new equipment.

93. A food bank just received 28 kilograms of emergency food. Each family in need is to receive $\frac{2}{5}$ kilograms of food. How many families can be served with the 28 kilograms of food?

____ families can be served with the 28 kilograms of food.

94. A construction team maintains a 52-mile-long sewage pipe. Each day, the team can cover $\frac{4}{7}$ of a mile. How many days will it take the team to complete the maintenance of the entire sewage pipe?

It will take the team ____ days to complete maintaining the entire sewage pipe.

95. A child is stacking up tiles. Each tile's height is $\frac{3}{4}$ of a centimeter. How many layers of tiles are needed to reach 15 centimeters in total height?

To reach the total height of 15 centimeters, ____ layers of tiles are needed.

96. A restaurant made 450 cups of pudding for a festival.

Customers at the festival will be served $\frac{1}{10}$ of a cup of pudding per serving. How many customers can the restaurant serve at the festival with the 450 cups of pudding?

The restaurant can serve ____ customers at the festival with the 450 cups of pudding.

97. A 2×4 piece of lumber in your garage is $62\frac{17}{32}$ inches long. A second 2×4 is $42\frac{7}{16}$ inches long. If you lay them end to end, what will the total length be?

The total length will be ____ inches.

98. A 2×4 piece of lumber in your garage is $39\frac{3}{4}$ inches long. A second 2×4 is $31\frac{3}{4}$ inches long. If you lay them end to end, what will the total length be?

The total length will be ____ inches.

99. Each page of a book consists of a header, a footer and the middle part. The header is $\frac{6}{7}$ inches in height; the footer is $\frac{3}{14}$ inches in height; and the middle part is $3\frac{3}{7}$ inches in height.

What is the total height of each page in this book? Use mixed number in your answer if needed.

Each page in this book is ____ inches in height.

100. To pave the road on Ellis Street, the crew used $4\frac{1}{4}$ tons of cement on the first day, and used $5\frac{5}{6}$ tons on the second day. How many tons of cement were used in all?

____ tons of cement were used in all.

101. When driving on a high way, noticed a sign saying exit to Johnstown is $1\frac{3}{4}$ miles away, while exit to Jerrystown is $3\frac{1}{2}$ miles away. How far is Johnstown from Jerrystown?

Johnstown and Jerrystown are ____ miles apart.

102. A cake recipe needs $2\frac{1}{4}$ cups of flour. Using this recipe, to bake 9 cakes, how many cups of flour are needed?

To bake 9 cakes, ____ cups of flour are needed.

Sketching Fractions

103. Sketch a number line showing each fraction. (Be sure to carefully indicate the correct number of equal parts of the whole.)

(a) $\dfrac{2}{3}$ (b) $\dfrac{6}{8}$ (c) $\dfrac{5}{4}$ (d) $-\dfrac{4}{5}$

104. Sketch a number line showing each fraction. (Be sure to carefully indicate the correct number of equal parts of the whole.)

(a) $\dfrac{1}{6}$ (b) $\dfrac{3}{9}$ (c) $\dfrac{7}{6}$ (d) $-\dfrac{8}{5}$

105. Sketch a picture of the product $\frac{3}{5} \cdot \frac{1}{2}$, using a number line or rectangles.

106. Sketch a picture of the sum $\frac{2}{3} + \frac{1}{8}$, using a number line or rectangles.

Challenge

107. Given that $a \neq 0$, simplify $\dfrac{6}{a} + \dfrac{5}{a}$.

108. Given that $a \neq 0$, simplify $\dfrac{7}{a} + \dfrac{1}{2a}$.

109. Given that $a \neq 0$, simplify $\dfrac{8}{a} - \dfrac{8}{5a}$.

1.3 Absolute Value and Square Root

In this section, we will learn the basics of **absolute value** and **square root**. These are actions you can *do* to a given number, often changing the number into something else.

1.3.1 Introduction to Absolute Value

Definition 1.3.2. The **absolute value** of a number is the distance between that number and 0 on a number line. For the absolute value of x, we write $|x|$.

Let's look at $|2|$ and $|-2|$, the absolute value of 2 and the absolute value of -2.

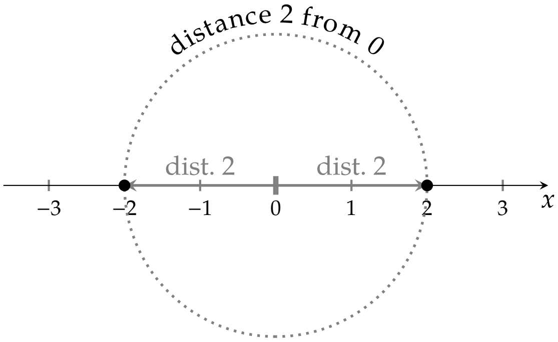


Figure 1.3.3: $|2|$ and $|-2|$

Since the distance between 2 and 0 on the number line is 2 units, the absolute value of 2 is 2. We write $|2| = 2$.

Since the distance between -2 and 0 on the number line is also 2 units, the absolute value of -2 is also 2. We write $|-2| = 2$.

Fact 1.3.4 Absolute Value. *Taking the absolute value of a number results in whatever the "positive version" of that number is. This is because the real meaning of absolute value is its* distance *from zero.*

Checkpoint 1.3.5 Calculating Absolute Value. Try calculating some absolute values.

a. $|57| =$ ☐

b. $|-43| =$ ☐

c. $\left|\frac{2}{-5}\right| =$ ☐

Explanation.

a. 57 is 57 units away from 0 on a number line, so $|57| = 57$. Another way to think about this is that the "positive version" of 57 is 57.

b. -43 is 43 units away from 0 on a number line, so $|-43| = 43$. Another way to think about this is that the "positive version" of -43 is 43.

c. $\frac{2}{-5}$ is $\frac{2}{5}$ units away from 0 on a number line, so $\left|\frac{2}{-5}\right| = \frac{2}{5}$. Another way to think about this is that the "positive version" of $\frac{2}{-5}$ is $\frac{2}{5}$.

Warning 1.3.6 Absolute Value Does Not Exactly "Make Everything Positive". Students may see an expression like $|2 - 5|$ and incorrectly think it is OK to "make everything positive" and write $2 + 5$. This is incorrect since $|2 - 5|$ works out to be 3, not 7, as we are actually taking the absolute value of -3 (the equivalent number inside the absolute value).

1.3.2 Square Root Facts

If you have learned your basic multiplication table, you know:

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Table 1.3.7: Multiplication table with squares

The numbers along the diagonal are special; they are known as **perfect squares**. And for working with square roots, it will be helpful if you can memorize these first few perfect square numbers.

“Taking a square root” is the opposite action of squaring a number. For example, when you square 3, the result is 9. So when you take the square root of 9, the result is 3. Just knowing that 9 comes about as 3^2 lets us realize that 3 is the square root of 9. This is why memorizing the perfect squares from the multiplication table can be so helpful.

The notation we use for taking a square root is the **radical**, $\sqrt{}$. For example, “the square root of 9” is denoted $\sqrt{9}$. And now we know enough to be able to write $\sqrt{9} = 3$.

Tossing in a few extra special square roots, it’s advisable to memorize the following:

$$\begin{array}{cccc}
\sqrt{0} = 0 & \sqrt{1} = 1 & \sqrt{4} = 2 & \sqrt{9} = 3 \\
\sqrt{16} = 4 & \sqrt{25} = 5 & \sqrt{36} = 6 & \sqrt{49} = 7 \\
\sqrt{64} = 8 & \sqrt{81} = 9 & \sqrt{100} = 10 & \sqrt{121} = 11 \\
\sqrt{144} = 12 & \sqrt{169} = 13 & \sqrt{196} = 14 & \sqrt{225} = 15
\end{array}$$

1.3.3 Calculating Square Roots with a Calculator

Most square roots are actually numbers with decimal places that go on forever. Take $\sqrt{5}$ as an example:

$$\sqrt{4} = 2 \qquad \sqrt{5} = ? \qquad \sqrt{9} = 3$$

Since 5 is between 4 and 9, then $\sqrt{5}$ must be somewhere between 2 and 3. There are no whole numbers between 2 and 3, so $\sqrt{5}$ must be some number with decimal places. If the decimal places eventually stopped, then squaring it would give you another number with decimal places that stop further out. But squaring it gives you 5 with no decimal places. So the only possibility is that $\sqrt{5}$ is a decimal between 2 and 3 that goes on forever. With a calculator, we can see:

$$\sqrt{5} \approx 2.236$$

Actually the decimal will not terminate, and that is why we used the ≈ symbol instead of an equals sign. To get 2.236 we rounded down slightly from the true value of $\sqrt{5}$. With a calculator, we can check that $2.236^2 = 4.999696$, a little shy of 5.

1.3.4 Square Roots of Fractions

We can calculate the square root of some fractions by hand, such as $\sqrt{\frac{1}{4}}$. The idea is the same: can you think of a number that you would square to get $\frac{1}{4}$? Being familiar with fraction multiplication, we know that $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ and so $\sqrt{\frac{1}{4}} = \frac{1}{2}$.

Checkpoint 1.3.8 Square Roots of Fractions. Try calculating some absolute values.

a. $\sqrt{\frac{1}{25}} = \boxed{}$

b. $\sqrt{\frac{4}{9}} = \boxed{}$

c. $\sqrt{\frac{81}{121}} = \boxed{}$

Explanation.

a. Since $\sqrt{1} = 1$ and $\sqrt{25} = 5$, then $\sqrt{\frac{1}{25}} = \frac{1}{5}$.

b. Since $\sqrt{4} = 2$ and $\sqrt{9} = 3$, then $\sqrt{\frac{4}{9}} = \frac{2}{3}$.

c. Since $\sqrt{81} = 9$ and $\sqrt{121} = 11$, then $\sqrt{\frac{81}{121}} = \frac{9}{11}$.

1.3.5 Square Root of Negative Numbers

Can we find the square root of a negative number, such as $\sqrt{-25}$? That would mean that there is some number out there that multiplies by itself to make -25. Would $\sqrt{-25}$ be positive or negative? Either way, once you square it (multiply it by itself) the result would be positive. So it couldn't possibly square to -25. So there is no square root of -25 or of any negative number for that matter.

If you are confronted with an expression like $\sqrt{-25}$, or any other square root of a negative number, you can state that "there is no real square root" or that the result "does not exist" (as a real number).

Imaginary Numbers. Mathematicians imagined a new type of number, neither positive nor negative, that would square to a negative result. But that is beyond the scope of this chapter.

Exercises

Review and Warmup

1. Evaluate the expressions.

a. 1^2
b. 3^2
c. 5^2
d. 7^2
e. 9^2
f. 11^2

2. Evaluate the expressions.

a. 2^2
b. 4^2
c. 6^2
d. 8^2
e. 10^2
f. 12^2

Absolute Value Evaluate the following.

3. $|-3| = \boxed{}$

4. $|0| = \boxed{}$

5. $|31.69| = \boxed{}$

6. $|-15.76| = \boxed{}$

7. $\left|\frac{21}{8}\right| = \boxed{}$

8. $\left|-\frac{17}{9}\right| = \boxed{}$

9. a. $|2| = \boxed{}$
b. $|-2| = \boxed{}$
c. $-|2| = \boxed{}$
d. $-|-2| = \boxed{}$

10. a. $|3| = \boxed{}$
b. $|-3| = \boxed{}$
c. $-|3| = \boxed{}$
d. $-|-3| = \boxed{}$

11. a. $|3| = \boxed{}$
b. $|-8| = \boxed{}$
c. $|0| = \boxed{}$
d. $|16 + (-6)| = \boxed{}$
e. $|-9 - (-1)| = \boxed{}$

12. a. $|4| = \boxed{}$
b. $|-2| = \boxed{}$
c. $|0| = \boxed{}$
d. $|20 + (-8)| = \boxed{}$
e. $|-9 - (-3)| = \boxed{}$

13. a. $-|3 - 7| = \boxed{}$
b. $|-3 - 7| = \boxed{}$
c. $-3|7 - 3| = \boxed{}$

14. a. $-|1 - 10| = \boxed{}$
b. $|-1 - 10| = \boxed{}$
c. $-3|10 - 1| = \boxed{}$

15. Which of the following are square numbers? There may be more than one correct answer.

☐ 81 ☐ 34 ☐ 9 ☐ 116 ☐ 55 ☐ 121

16. Which of the following are square numbers? There may be more than one correct answer.

☐ 15 ☐ 100 ☐ 64 ☐ 45 ☐ 32 ☐ 36

Square Roots Evaluate the following.

17. a. $\sqrt{144} = \boxed{}$
b. $\sqrt{4} = \boxed{}$
c. $\sqrt{1} = \boxed{}$

18. a. $\sqrt{1} = \boxed{}$
b. $\sqrt{100} = \boxed{}$
c. $\sqrt{36} = \boxed{}$

19. a. $\sqrt{\frac{4}{121}} = \boxed{}$
b. $\sqrt{-\frac{16}{9}} = \boxed{}$

20.

a. $\sqrt{\frac{16}{25}} =$

b. $\sqrt{-\frac{144}{49}} =$

21. Do not use a calculator.

a. $\sqrt{25} =$

b. $\sqrt{0.25} =$

c. $\sqrt{2500} =$

22. Do not use a calculator.

a. $\sqrt{49} =$

b. $\sqrt{0.49} =$

c. $\sqrt{4900} =$

23. Do not use a calculator.

a. $\sqrt{64} =$

b. $\sqrt{6400} =$

c. $\sqrt{640000} =$

24. Do not use a calculator.

a. $\sqrt{81} =$

b. $\sqrt{8100} =$

c. $\sqrt{810000} =$

25. Do not use a calculator.

a. $\sqrt{100} =$

b. $\sqrt{1} =$

c. $\sqrt{0.01} =$

26. Do not use a calculator.

a. $\sqrt{121} =$

b. $\sqrt{1.21} =$

c. $\sqrt{0.0121} =$

27. Use a calculator to approximate with a decimal.

$\sqrt{13} \approx$

28. Use a calculator to approximate with a decimal.

$\sqrt{17} \approx$

29. $\sqrt{\frac{9}{100}} =$.

30. $\sqrt{\frac{16}{121}} =$.

31. $-\sqrt{49} =$.

32. $-\sqrt{64} =$.

33. $\sqrt{-81} =$.

34. $\sqrt{-100} =$.

35. $\sqrt{-\frac{121}{144}} =$.

36. $\sqrt{-\frac{1}{49}} =$.

37. $-\sqrt{\frac{4}{25}} =$.

38. $-\sqrt{\frac{9}{121}} =$.

39.

a. $\sqrt{100} - \sqrt{36} =$

b. $\sqrt{100 - 36} =$

40.

a. $\sqrt{25} - \sqrt{9} =$

b. $\sqrt{25 - 9} =$

41. $\frac{5}{\sqrt{81}} =$.

42. $\frac{7}{\sqrt{36}} =$.

1.4 Order of Operations

Mathematical symbols are a means of communication, and it's important that when you write something, everyone else knows exactly what you intended. For example, if we say in English, "two times three squared," do we mean that:

- 2 is multiplied by 3, and then the result is squared?
- or that 2 is multiplied by the result of squaring 3?

English is allowed to have ambiguities like this. But mathematical language needs to be precise and mean the same thing to everyone reading it. For this reason, a standard **order of operations** has been adopted, which we review here.

1.4.1 Grouping Symbols

Consider the math expression $2 \cdot 3^2$. There are two mathematical operations here: a multiplication and an exponentiation. The result of this expression will change depending on which operation you decide to execute first: the multiplication or the exponentiation. If you multiply $2 \cdot 3$, and then square the result, you have 36. If you square 3, and then multiply 2 by the result, you have 18. If we want all people everywhere to interpret $2 \cdot 3^2$ in the same way, then only *one* of these can be correct.

The first tools that we have to tell readers what operations to execute first are grouping symbols, like parentheses and brackets. If you *intend* to execute the multiplication first, then writing

$$(2 \cdot 3)^2$$

clearly tells your reader to do that. And if you *intend* to execute the power first, then writing

$$2 \cdot \left(3^2\right)$$

clearly tells your reader to do that.

To visualize the difference between $2 \cdot \left(3^2\right)$ or $(2 \cdot 3)^2$, consider these garden plots:

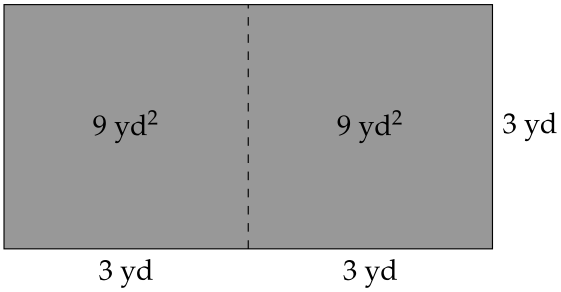


Figure 1.4.2: 3 yd is squared, then doubled: $2 \cdot \left(3^2\right)$

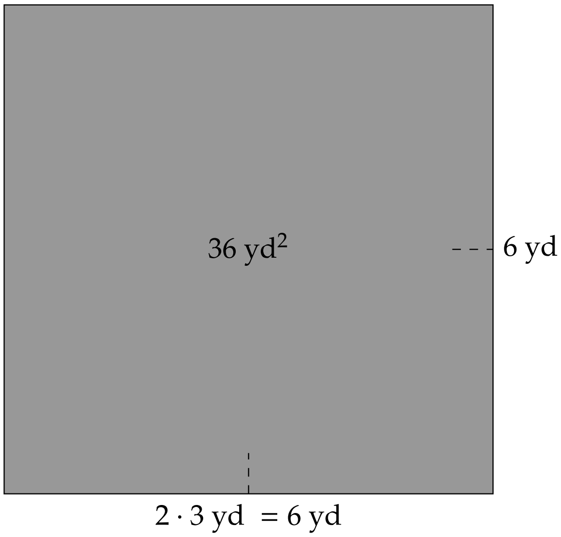


Figure 1.4.3: 3 yd is doubled, then squared: $(2 \cdot 3)^2$

If we calculate 3^2, we have the area of one of the small square garden plots on the left. If we then double that, we have $2 \cdot \left(3^2\right)$, the area of the left garden plot.

But if we calculate $(2 \cdot 3)^2$, then first we are doubling 3. So we are calculating the area of a square garden plot whose sides are twice as long. We end up with the area of the garden plot on the right.

The point is that these amounts are different.

Checkpoint 1.4.4. Calculate the value of $30-((2+3)\cdot 2)$, respecting the order that the grouping symbols are telling you to execute the arithmetic operations.

Explanation. The grouping symbols tell us what to work on first. In this exercise, we have grouping symbols within grouping symbols, so any operation in there (the addition) should be executed first:

$$\begin{aligned} 30-((2+3)\cdot 2) &= 30-(5\cdot 2)\\ &= 30-10\\ &= 20 \end{aligned}$$

1.4.2 Beyond Grouping Symbols

If all math expressions used grouping symbols for each and every arithmetic operation, we wouldn't need to say anything more here. In fact, some computer systems work that way, *requiring* the use of grouping symbols all the time. But it is much more common to permit math expressions with no grouping symbols at all, like $5+3\cdot 2$. Should the addition $5+3$ be executed first, or should the multiplication $3\cdot 2$? We need what's known formally as the **order of operations** to tell us what to do.

The **order of operations** is nothing more than an agreement that we all have made to prioritize the arithmetic operations in a certain order.

(P)arentheses and other grouping symbols Grouping symbols should always direct you to the highest priority arithmetic first.

(E)xponentiation After grouping symbols, exponentiation has the highest priority. Execute any exponentiation before other arithmetic operations.

(M)ultiplication, (D)ivision, and Negation After all exponentiation has been executed, start executing multiplications, divisions, and negations. These things all have equal priority. If there are more than one of them in your expression, the highest priority is the one that is leftmost (which comes first as you read it).

(A)ddition and (S)ubtraction After all other arithmetic has been executed, these are all that is left. Addition and subtraction have equal priority. If there are more than one of them in your expression, the highest priority is the one that is leftmost (which comes first as you read it).

List 1.4.5: Order of Operations

A common acronym to help you remember this order of operations is PEMDAS. There are a handful of mnemonic devices for remembering this ordering (such as Please Excuse My Dear Aunt Sally, People Eat

More Donuts After School, etc.).

We'll start with a few examples that only invoke a few operations each.

Example 1.4.6 Use the order of operations to simplify the following expressions.

a. $10 + 2 \cdot 3$. With this expression, we have the operations of addition and multiplication. The order of operations says the multiplication has higher priority, so execute that first:

$$\begin{aligned} 10 + 2 \cdot 3 &= 10 + \boxed{2 \cdot 3} \\ &= 10 + 6 \\ &= 16 \end{aligned}$$

b. $4 + 10 \div 2 - 1$. With this expression, we have addition, division, and subtraction. According to the order of operations, the first thing we need to do is divide. After that, we'll apply the addition and subtraction, working left to right:

$$\begin{aligned} 4 + 10 \div 2 - 1 &= 4 + \boxed{10 \div 2} - 1 \\ &= \boxed{4 + 5} - 1 \\ &= 9 - 1 \\ &= 8 \end{aligned}$$

c. $7 - 10 + 4$. This example *only* has subtraction and addition. While the acronym PEMDAS may mislead you to do addition before subtraction, remember that these operations have the same priority, and so we work left to right when executing them:

$$\begin{aligned} 7 - 10 + 4 &= \boxed{7 - 10} + 4 \\ &= -3 + 4 \\ &= 1 \end{aligned}$$

d. $20 \div 4 \cdot 7$. This expression has only division and multiplication. Again, remember that although PEMDAS shows "MD," the operations of multiplication and division have the same priority, so we'll apply them left to right:

$$\begin{aligned} 20 \div 4 \cdot 5 &= \boxed{20 \div 4} \cdot 5 \\ &= 5 \cdot 5 \\ &= 25 \end{aligned}$$

e. $(6 + 7)^2$. With this expression, we have addition inside a set of parentheses, and an exponent of 2 outside of that. We must compute the operation inside the parentheses first, and after that we'll apply the exponent:

$$\begin{aligned} (6 + 7)^2 &= (\boxed{6 + 7})^2 \\ &= 13^2 \\ &= 169 \end{aligned}$$

f. $4(2)^3$. This expression has multiplication and an exponent. There are parentheses too, but no operation inside them. Parentheses used in this manner make it clear that the 4 and 2 are separate numbers, not to be confused with 42. In other words, $4(2)^3$ and 42^3 mean very different things. Exponentiation has the higher priority, so we'll apply the exponent first, and then we'll multiply:

$$\begin{aligned} 4(2)^3 &= 4\boxed{(2)^3} \\ &= 4(8) \\ &= 32 \end{aligned}$$

Remark 1.4.7. There are many different ways that we write multiplication. We can use the symbols $\cdot$, $\times$, and $*$ to denote multiplication. We can also use parentheses to denote multiplication, as we've seen in Example 1.4.6, Item f. Once we start working with variables, there is even another way. No matter how multiplication is written, it does not change the priority that multiplication has in the order of operations.

Checkpoint 1.4.8 Practice with order of operations. Simplify this expression one step at a time, using the order of operations.

$5 - 3(7 - 4)^2 =$ ______

$=$ ______

$=$ ______

$=$ ______

Explanation.

$$\begin{aligned} 5 - 3(\boxed{7-4})^2 &= 5 - 3\boxed{(3)^2} \\ &= 5 - \boxed{3(9)} \\ &= 5 - 27 \\ &= -22 \end{aligned}$$

1.4.3 Absolute Value Bars, Radicals, and Fraction Bars are Grouping Symbols

When we first discussed grouping symbols, we only mentioned parentheses and brackets. Each of the following operations has an *implied* grouping symbol aside from parentheses and brackets.

Absolute Value Bars The absolute value bars, as in $|2 - 5|$, group the expression inside it just like a set of parentheses would.

Radicals The same is true of the radical symbol — everything inside the radical is grouped, as with $\sqrt{12-3}$.

Fraction Bars With a horizontal division bar, the numerator is treated as one group and the denominator as another, as with $\frac{2+3}{5-2}$.

We don't *need* parentheses for these three things since the absolute value bars, radical, and horizontal division bar each denote this grouping on their own. As far as priority in the order of operations goes, it's important to remember that these work just like our most familiar grouping symbols, parentheses.

With absolute value bars and radicals, these grouping symbols also *do* something to what's inside (but only *after* the operations inside have been executed). For example, $|-2| = 2$, and $\sqrt{9} = 3$.

Example 1.4.9 Use the order of operations to simplify the following expressions.

a. $4-3\left|5-7\right|$. For this expression, we'll treat the absolute value bars just like we treat parentheses. This implies we'll simplify what's inside the bars first, and then compute the absolute value. After that, we'll multiply and then finally subtract:

$$\begin{aligned}4-3\left|5-7\right| &= 4-3\left|\boxed{5-7}\right|\\ &= 4-3\boxed{\left|-2\right|}\\ &= 4-\boxed{3(2)}\\ &= 4-6\\ &= -2\end{aligned}$$

We may not do $4-3=1$ first, because 3 is connected to the absolute value bars by multiplication (although implicitly), which has a higher order than subtraction.

b. $8-\sqrt{5^2-8\cdot 2}$. This expression has an expression inside the radical of $5^2-8\cdot 2$. We'll treat this radical like we would a set of parentheses, and simplify that internal expression first. We'll then apply the square root, and then our last step will be to subtract that expression from 8:

$$\begin{aligned}8-\sqrt{5^2-8\cdot 2} &= 8-\sqrt{\boxed{5^2}-8\cdot 2}\\ &= 8-\sqrt{25-\boxed{8\cdot 2}}\\ &= 8-\sqrt{\boxed{25-16}}\\ &= 8-\boxed{\sqrt{9}}\\ &= 8-3\\ &= 5\end{aligned}$$

c. $\dfrac{2^4+3\cdot 6}{5-18\div 2}$. For this expression, the first thing we want to do is to recognize that the main fraction bar serves as a separator that groups the numerator and groups the denominator. Another way this expression could be written is $(2^4+3\cdot 6)\div(15-18\div 2)$. This implies we'll simplify the numerator and denominator separately according to the order of operations (since there are implicit parentheses around each of these). As a final step we'll simplify the resulting fraction (which is division).

$$\begin{aligned}\frac{2^4+3\cdot 6}{5-18\div 2} &= \frac{\boxed{2^4}+3\cdot 6}{5-\boxed{18\div 2}}\\ &= \frac{16+\boxed{3\cdot 6}}{\boxed{5-9}}\\ &= \frac{\boxed{16+18}}{-4}\\ &= \frac{34}{-4}\\ &= -\frac{17}{2}\end{aligned}$$

Checkpoint 1.4.10 More Practice with Order of Operations. Use the order of operations to evaluate

$$\frac{6+3\left|9-10\right|}{\sqrt{3+18\div 3}}.$$

Explanation. We start by identifying the innermost, highest priority operations:

$$\begin{aligned}
\frac{6+3\left|9-10\right|}{\sqrt{3+18\div 3}} &= \frac{6+3\left|\boxed{9-10}\right|}{\sqrt{3+\boxed{18\div 3}}} \\
&= \frac{6+3\boxed{\left|-1\right|}}{\sqrt{\boxed{3+6}}} \\
&= \frac{6+\boxed{3(1)}}{\boxed{\sqrt{9}}} \\
&= \frac{\boxed{6+3}}{3} \\
&= \frac{9}{3} = 3
\end{aligned}$$

1.4.4 Negation and Distinguishing $(-a)^m$ from $-a^m$

We noted in the order of operations that using the negative sign to negate a number has the same priority as multiplication and division. To understand why this is, observe that $-1\cdot 23=-23$, just for one example. So negating 23 gives the same result as multiplying 23 by -1. For this reason, negation has the same priority in the order of operations as multiplication. This can be a source of misunderstandings.

How would you write a math expression that takes the number -4 and squares it?

$$-4^2? \qquad (-4)^2? \qquad \text{it doesn't matter?}$$

It *does* matter. The second option, $(-4)^2$ is squaring the number -4. The parentheses emphasize this.

But the expression -4^2 is different. There are two actions in this expression: a negation and and exponentiation. According to the order of operations, the exponentiation has higher priority than the negation, so the exponent of 2 in -4^2 applies to the 4 *before* the negative sign (multiplication by -1) is taken into account.

$$\begin{aligned}
-4^2 &= -\boxed{4^2} \\
&= -16
\end{aligned}$$

and this is not the same as $(-4)^2$, which is *positive* 16.

Warning 1.4.11 Negative Numbers Raised to Powers. You may find yourself needing to raise a negative number to a power, and using a calculator to do the work for you. If you do not understand the issue described here, then you may get incorrect results.

- For example, entering `-4^2` into a calculator will result in -16, the negative of 4^2.
- But entering `(-4)^2` into a calculator will result in 16, the square of -4.

Go ahead and try entering these into your own calculator.

Checkpoint 1.4.12 Negating and Raising to Powers. Compute the following:

a. $-3^4 = \boxed{}$ and $(-3)^4 = \boxed{}$

b. $-4^3 = \boxed{}$ and $(-4)^3 = \boxed{}$

c. $-1.1^2 = \boxed{}$ and $(-1.1)^2 = \boxed{}$

Explanation. In each part, the first expression asks you to exponentiate and then negate the result. The second expression has a negative number raised to a power. So the answers are:

a. $-3^4 = -81$ and $(-3)^4 = 81$

b. $-4^3 = -64$ and $(-4)^3 = -64$

c. $-1.1^2 = -1.21$ and $(-1.1)^2 = 1.21$

Remark 1.4.13. You might observe in the previous example that there is no difference between -4^3 and $(-4)^3$. It's true that the results are the same, -64, but the two expressions still do say different things. With -4^3, you raise to a power first, then negate. With $(-4)^3$, you negate first, then raise to a power.

As was discussed in Subsection 1.1.5, if the base of a power is negative, then whether or not the result is positive or negative depends on if the exponent is even or odd. It depends on whether or not the factors can all be paired up to "cancel" negative signs, or if there will be a lone factor left by itself.

1.4.5 More Examples

Here are some example exercises that involve applying the order of operations to more complicated expressions. Try these exercises and read the steps given in each solution.

Example 1.4.14 Simplify $10 - 4(5-7)^3$.

Explanation. For the expression $10 - 4(5-7)^3$, we have simplify what's inside parentheses first, then we'll apply the exponent, then multiply, and finally subtract:

$$\begin{aligned}
10 - 4(5-7)^3 &= 10 - 4(\boxed{5-7})^3\\
&= 10 - 4\boxed{(-2)^3}\\
&= 10 - \boxed{4(-8)}\\
&= 10 - (-32)\\
&= 10 + 32\\
&= 42
\end{aligned}$$

Checkpoint 1.4.15. Simplify $24 \div (15 \div 3 + 1) + 2$.

Explanation. With the expression $24 \div (15 \div 3 + 1) + 2$, we'll simplify what's inside the parentheses according to the order of operations, and then take 24 divided by that expression as our last step:

$$\begin{aligned}24 \div (15 \div 3 + 1) + 2 &= 24 \div (\boxed{15 \div 3} + 1) + 2\\ &= 24 \div (\boxed{5 + 1}) + 2\\ &= \boxed{24 \div 6} + 2\\ &= 4 + 2\\ &= 6\end{aligned}$$

Example 1.4.16 Simplify $6 - (-8)^2 \div 4 + 1$.

Explanation. To simplify $6-(-8)^2 \div 4+1$, we'll first apply the exponent of 2 to -8, making sure to recall that $(-8)^2 = 64$. After this, we'll apply division. As a final step, we'll be have subtraction and addition, which we'll apply working left-to-right:

$$\begin{aligned}6 - (-8)^2 \div 4 + 1 &= 6 - \boxed{(-8)^2} \div 4 + 1\\ &= 6 - \boxed{(64) \div 4} + 1\\ &= \boxed{6 - 16} + 1\\ &= -10 + 1\\ &= -9\end{aligned}$$

Checkpoint 1.4.17. Simplify $(20 - 4^2) \div (4 - 6)^3$.

Explanation. The expression $(20 - 4^2) \div (4 - 6)^3$ has two sets of parentheses, so our first step will be to simplify what's inside each of those first according to the order of operations. Once we've done that, we'll apply the exponent and then finally divide:

$$\begin{aligned}(20 - 4^2) \div (4 - 6)^3 &= (20 - \boxed{4^2}) \div (4 - 6)^3\\ &= (\boxed{20 - 16}) \div (4 - 6)^3\\ &= 4 \div (\boxed{4 - 6})^3\\ &= 4 \div \boxed{(-2)^3}\\ &= 4 \div (-8)\\ &= \frac{4}{-8}\\ &= \frac{1}{-2}\\ &= -\frac{1}{2}\end{aligned}$$

Checkpoint 1.4.18. Simplify $\dfrac{2\,|9 - 15| + 1}{\sqrt{(-5)^2 + 12^2}}$.

Explanation. To simplify this expression, the first thing we want to recognize is the role of the main fraction bar, which groups the numerator and denominator. This implies we'll simplify the numerator and denominator separately according to the order of operations, and then reduce the fraction that results:

$$\begin{aligned}
\frac{2\,|9-15|+1}{\sqrt{(-5)^2+12^2}} &= \frac{2\left|\boxed{9-15}\right|+1}{\sqrt{\boxed{(-5)^2}+12^2}} \\
&= \frac{2\boxed{|-6|}+1}{\sqrt{25+\boxed{12^2}}} \\
&= \frac{\boxed{2(6)}+1}{\sqrt{\boxed{25+144}}} \\
&= \frac{\boxed{12+1}}{\boxed{\sqrt{169}}} \\
&= \frac{13}{13} \\
&= 1
\end{aligned}$$

Exercises

Review and Warmup

1. Multiply the following.

a. $(-9)\cdot(-1) =$

b. $(-4)\cdot 6 =$

c. $8\cdot(-6) =$

d. $(-4)\cdot 0 =$

2. Multiply the following.

a. $(-8)\cdot(-3) =$

b. $(-7)\cdot 4 =$

c. $8\cdot(-3) =$

d. $(-3)\cdot 0 =$

3. Multiply the following.

a. $(-1)\cdot(-4)\cdot(-3) =$

b. $2\cdot(-7)\cdot(-4) =$

c. $(-81)\cdot(-56)\cdot 0 =$

4. Multiply the following.

a. $(-1)\cdot(-5)\cdot(-5) =$

b. $7\cdot(-7)\cdot(-2) =$

c. $(-80)\cdot(-74)\cdot 0 =$

5. a. Compute -1^{96}.

b. Calculate the power $(-3)^4$.

c. Find $(-5)^3$.

d. Calculate -6^3.

6. a. Compute -2^3.

b. Calculate the power -8^2.

c. Find $(-7)^2$.

d. Calculate $(-3)^2$.

Order of Operations Skills Evaluate the following.

7. $4 + 6(3) = \boxed{}$

8. $7 + 3(4) = \boxed{}$

9. $5(3 + 2) = \boxed{}$

10. $4(4 + 4) = \boxed{}$

11. $(4 \cdot 2)^2 = \boxed{}$

12. $(4 \cdot 4)^2 = \boxed{}$

13. $5 \cdot 3^3 = \boxed{}$

14. $2 \cdot 2^2 = \boxed{}$

15. $(11 - 2) \cdot 4 = \boxed{}$

16. $(8 - 2) \cdot 2 = \boxed{}$

17. $11 - 3 \cdot 2 = \boxed{}$

18. $25 - 3 \cdot 5 = \boxed{}$

19. $6 + 3 \cdot 7 = \boxed{}$

20. $7 + 2 \cdot 9 = \boxed{}$

21. $4 - 4 \cdot 6 = \boxed{}$

22. $5 - 3 \cdot 8 = \boxed{}$

23. $1 - 2(-10) = \boxed{}$

24. $1 - 4(-8) = \boxed{}$

25. $-[2 - (1 - 5)^2] = \boxed{}$

26. $-[3 - (2 - 9)^2] = \boxed{}$

27. $7 - 5[7 - (5 + 3 \cdot 2)] = \boxed{}$

28. $7 - 3[1 - (2 + 3 \cdot 4)] = \boxed{}$

29. $4 + 4(139 - 5 \cdot 3^3) = \boxed{}$

30. $4 + 3(114 - 4 \cdot 3^3) = \boxed{}$

31. $-5[3 - (4 - 3 \cdot 3)^2] = \boxed{}$

32. $-2[9 - (7 - 4 \cdot 4)^2] = \boxed{}$

33. $9 - 5[2^2 - (4 - 1)] = \boxed{}$

34. $64 - 4[4^2 - (3 - 2)] = \boxed{}$

35. $(14 - 3)^2 + 5(14 - 3^2)\boxed{}$

36. $(12 - 3)^2 + 3(12 - 3^2)\boxed{}$

37. $(4 \cdot 2)^2 - 4 \cdot 2^2 = \boxed{}$

38. $(4 \cdot 5)^2 - 4 \cdot 5^2 = \boxed{}$

39. $9 \cdot 3^2 - 32 \div 4^2 \cdot 4 + 3 = \boxed{}$

40. $10 \cdot 2^2 - 75 \div 5^2 \cdot 6 + 3 = \boxed{}$

41. $4(7 - 2)^2 - 4(7 - 2^2) = \boxed{}$

42. $6(6 - 2)^2 - 6(6 - 2^2) = \boxed{}$

43. $\dfrac{1+3}{3-2} = \boxed{}$

44. $\dfrac{8+20}{7-3} = \boxed{}$

45. $\dfrac{8^2-3^2}{1+10} = \boxed{}$

46. $\dfrac{8^2-2^2}{5+5} = \boxed{}$

47. $\dfrac{27-(-4)^3}{10-17} = \boxed{}$

48. $\dfrac{27-(-2)^3}{5-10} = \boxed{}$

49. $\dfrac{(-2)\cdot(-1)-(-8)\cdot 3}{(-8)^2+(-66)} = \boxed{}$

50. $\dfrac{(-2)\cdot(-3)-(-8)\cdot 3}{(-8)^2+(-66)} = \boxed{}$

51. $-|1-4| = \boxed{}$

52. $-|2-10| = \boxed{}$

53. $2-7\,|3-5|+1 = \boxed{}$

54. $3-5\,|5-9|+1 = \boxed{}$

55. $-5^2-|8\cdot(-7)| = \boxed{}$

56. $-4^2-|6\cdot(-5)| = \boxed{}$

57. $8-2\left|-9+(2-5)^3\right| = \boxed{}$

58. $9-8\left|-5+(4-5)^3\right| = \boxed{}$

59. $\dfrac{\left|27+(-4)^3\right|}{-1} = \boxed{}$

60. $\dfrac{\left|1+(-2)^3\right|}{-1} = \boxed{}$

61. $\left|\dfrac{1+(-4)^3}{-3}\right| = \boxed{}$

62. $\left|\dfrac{1+(-2)^3}{-1}\right| = \boxed{}$

63. $\dfrac{-3\,|17-37|}{22-(-4)^2} = \boxed{}$

64. $\dfrac{-3\,|10-22|}{18-(-3)^2} = \boxed{}$

65. $\dfrac{8}{5}+6\cdot\dfrac{2}{5} = \boxed{}$

66. $\dfrac{8}{3}+10\cdot\dfrac{8}{15} = \boxed{}$

67. $\left(\dfrac{9}{8}-\dfrac{5}{24}\right)-5\left(\dfrac{5}{24}-\dfrac{9}{8}\right) = \boxed{}$

68. $\left(\dfrac{9}{4}-\dfrac{9}{8}\right)-5\left(\dfrac{9}{8}-\dfrac{9}{4}\right) = \boxed{}$

69. $\left|\dfrac{3}{2}-\dfrac{3}{10}\right|-5\left|\dfrac{3}{10}-\dfrac{3}{2}\right| = \boxed{}$

70. $\left|\dfrac{3}{8}-\dfrac{7}{32}\right|-5\left|\dfrac{7}{32}-\dfrac{3}{8}\right| = \boxed{}$

71. $\dfrac{2}{5}+2\left(\dfrac{3}{5}\right)^2 = \boxed{}$

72. $\dfrac{1}{5}+6\left(\dfrac{4}{5}\right)^2 = \boxed{}$

73. $\dfrac{3}{4}+\dfrac{5}{4}\div\dfrac{5}{2}-\dfrac{1}{2} = \boxed{}$

74. $\dfrac{4}{3}+\dfrac{2}{3}\div\dfrac{5}{4}-\dfrac{2}{5} = \boxed{}$

75. $5\sqrt{67+33} = \boxed{}$

76. $5\sqrt{9+27} = \boxed{}$

77. $5\sqrt{-27+6\cdot 6} = \boxed{}$

78. $2\sqrt{64+9\cdot 4} = \boxed{}$

79. $1-2\sqrt{34-9} = \boxed{}$

80. $8-3\sqrt{76-72} = \boxed{}$

81. $\sqrt{36}-2\sqrt{5+95} = \boxed{}$

82. $\sqrt{49} - 4\sqrt{9+7} = \boxed{}$

83. $\sqrt{40 + 3^2} = \boxed{}$

84. $\sqrt{-17 + 9^2} = \boxed{}$

85. $\sqrt{8^2 + 6^2} = \boxed{}$

86. $\sqrt{9^2 + 12^2} = \boxed{}$

87. $\dfrac{\sqrt{49} + 9}{\sqrt{49} - 9} = \boxed{}$

88. $\dfrac{\sqrt{9} + 5}{\sqrt{9} - 5} = \boxed{}$

89. $\dfrac{\sqrt{4 + 2 \cdot 6} + |-14 - 18|}{-4 - (-2)^3} = \boxed{}$

90. $\dfrac{\sqrt{94 + 2 \cdot 3} + |-17 - 45|}{-19 - (-3)^3} = \boxed{}$

91. $4[17 - 5(5 + 6)] = \boxed{}$

92. $4[15 - 7(3 + 6)] = \boxed{}$

93. $-8^2 - 7[2 - (2 - 3^3)] = \boxed{}$

94. $-10^2 - 5[6 - (7 - 3^3)] = \boxed{}$

Challenge

95. In this challenge, your job is to create expressions, using addition, subtraction, multiplication, and parentheses. You may use the numbers, $1, 2, 3$, and 4 in your expression, using each number only once. For example, you could make the expression: $1 + 2 \cdot 3 - 4$.

a. The greatest value that it is possible to create under these conditions is $\boxed{}$.

b. The least value that it is possible to create under these conditions is $\boxed{}$.

1.5 Set Notation and Types of Numbers

When we talk about *how many* or *how much* of something we have, it often makes sense to use different types of numbers. For example, if we are counting dogs in a shelter, the possibilities are only $0, 1, 2, \ldots$. (It would be difficult to have $\frac{1}{2}$ of a dog.) On the other hand if you were weighing a dog in pounds, it doesn't make sense to only allow yourself to work with whole numbers. The dog might weigh something like 28.35 pounds. These examples highlight how certain kinds of numbers are appropriate for certain situations. We'll classify various types of numbers in this section.

1.5.1 Set Notation

What is the mathematical difference between these three "lists?"

$$28, 31, 30 \qquad \{28, 31, 30\} \qquad (28, 31, 30)$$

To a mathematician, the last one, $(28, 31, 30)$ is an *ordered* triple. What matters is not merely the three numbers, but *also* the order in which they come. The ordered triple $(28, 31, 30)$ is not the same as $(30, 31, 28)$; they have the same numbers in them, but the order has changed. For some context, February has 28 days; *then* March has 31 days; *then* April has 30 days. The order of the three numbers is meaningful in that context.

With curly braces and $\{28, 31, 30\}$, a mathematician sees a collection of numbers and does not particularly care in which order they are written. Such a collection is called a **set**. All that matters is that these numbers are part of a collection. They've been *written* in some particular order because that's necessary to write them down. But you might as well have put the three numbers in a bag and shaken up the bag. For some context, maybe your favorite three NBA players have jersey numbers 30, 31, and 28, and you like them all equally well. It doesn't really matter what order you use to list them.

So we can say:

$$\{28, 31, 30\} = \{30, 31, 28\} \qquad\qquad (28, 31, 30) \neq (30, 31, 28)$$

What about just writing $28, 31, 30$? This list of three numbers is ambiguous. Without the curly braces or parentheses, it's unclear to a reader if the order is important. **Set notation** is the use of curly braces to surround a list/collection of numbers, and we will use set notation frequently in this section.

Checkpoint 1.5.2 Set Notation. Practice using (and not using) set notation.

According to Google, the three most common error codes from visiting a web site are `403`, `404`, and `500`.

a. Without knowing which error code is most common, express this set mathematically.

b. Error code `500` is the most common. Error code `403` is the least common of these three. And that leaves `404` in the middle. Express the error codes in a mathematical way that appreciates how frequently they happen, from most often to least often.

Explanation.

a. Since we only have to describe a collection of three numbers and their order doesn't matter, we can write {403,404,500}.

b. Now we must describe the same three numbers and we want readers to know that the order we are writing the numbers matters. We can write (500,404,403).

1.5.2 Different Number Sets

In the introduction, we mentioned how different sets of numbers are appropriate for different situations. Here are the basic sets of numbers that are used in basic algebra.

Natural Numbers When we count, we begin: $1, 2, 3, \ldots$ and continue on in that pattern. These numbers are known as **natural numbers**.

$\mathbb{N} = \{1, 2, 3, \ldots\}$

Whole Numbers If we include zero, then we have the set of **whole numbers**.

$\{0, 1, 2, 3, \ldots\}$ has no standard symbol, but some options are $\mathbb{N}_0$, $\mathbb{N} \cup \{0\}$, and $\mathbb{Z}_{\geq 0}$.

Integers If we include the negatives of whole numbers, then we have the set of **integers**.

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

A $\mathbb{Z}$ is used because one word in German for "numbers" is "Zahlen."

Rational Numbers A **rational number** is any number that *can* be written as a fraction of integers, where the denominator is nonzero. Alternatively, a **rational number** is any number that *can* be written with a decimal that terminates or that repeats.

$\mathbb{Q} = \left\{0, 1, -1, 2, \frac{1}{2}, -\frac{1}{2}, -2, 3, \frac{1}{3}, -\frac{1}{3}, -3, \frac{3}{2}, \frac{2}{3} \ldots\right\}$

$\mathbb{Q} = \left\{0, 1, -1, 2, 0.5, -0.5, -2, 3, 0.\overline{3}, -0.\overline{3}, -3, 1.5, 0.\overline{6} \ldots\right\}$

A $\mathbb{Q}$ is used because fractions are *q*uotients of integers.

Irrational Numbers Any number that *cannot* be written as a fraction of integers belongs to the set of **irrational numbers**. Another way to say this is that any number whose decimal places goes on forever without repeating is an **irrational number**. Some examples include $\pi \approx 3.1415926\ldots$, $\sqrt{15} \approx 3.87298\ldots$, $e \approx 2.71828\ldots$

There is no standard symbol for the set of irrational numbers.

Real Numbers Any number that can be marked somewhere on a number line is a **real number**. Real numbers might be the only numbers you are familiar with. For a number to *not* be real, you have to start considering things called *complex numbers*, which are not our concern right now.

The set of real numbers can be denoted with $\mathbb{R}$ for short.

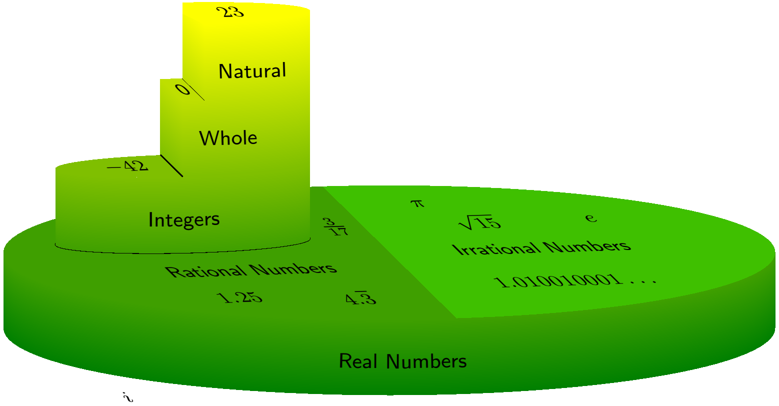


Figure 1.5.3: Types of Numbers

Warning 1.5.4 Rational Numbers in Other Forms. Any number that *can* be written as a ratio of integers is rational, even if it's not written that way at first. For example, these numbers might not look rational to you at first glance: -4, $\sqrt{9}$, 0π, and $\sqrt[3]{\sqrt{5}+2}-\sqrt[3]{\sqrt{5}-2}$. But they are all rational, because they can respectively be written as $\frac{-4}{1}$, $\frac{3}{1}$, $\frac{0}{1}$, and $\frac{1}{1}$.

Example 1.5.5 Determine If Numbers Are This Type or That Type. Determine which numbers from the set $\left\{-102, -7.25, 0, \frac{\pi}{4}, 2, \frac{10}{3}, \sqrt{19}, \sqrt{25}, 10.\overline{7}\right\}$ are natural numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers.

Explanation. All of these numbers are real numbers, because all of these numbers can be positioned on the real number line.

Each real number is either rational or irrational, and not both. -102, -7.25, 0, and 2 are rational because we can see directly that their decimal expressions terminate. $10.\overline{7}$ is also rational, because its decimal expression repeats. $\frac{10}{3}$ is rational because it is a ratio of integers. And last but not least, $\sqrt{25}$ is rational, because that's the same thing as 5.

This leaves only $\frac{\pi}{4}$ and $\sqrt{19}$ as irrational numbers. Their decimal expressions go on forever without entering a repetitive cycle.

Only -102, 0, 2, and $\sqrt{25}$ (which is really 5) are integers.

Of these, only 0, 2, and $\sqrt{25}$ are whole numbers, because whole numbers excludes the negative integers.

Of these, only 2 and $\sqrt{25}$ are natural numbers, because the natural numbers exclude 0.

Checkpoint 1.5.6.

a. Give an example of a whole number that is not an integer.
b. Give an example of an integer that is not a whole number.
c. Give an example of a rational number that is not an integer.
d. Give an example of a irrational number.
e. Give an example of a irrational number that is also an integer.

Explanation.

a. Since all whole numbers belong to integers, we cannot write any whole number which is not an integer. Type `DNE` (does not exist) for this question.
b. Any negative integer, like -1, is not a whole number, but is an integer.
c. Any terminating decimal, like 1.2, is a rational number, but is not an integer.
d. π is the easiest number to remember as an irrational number. Another constant worth knowing is $e \approx 2.718$. Finally, the square root of most integers are irrational, like $\sqrt{2}$ and $\sqrt{3}$.
e. All irrational numbers are non-repeating and non-terminating decimals. No irrational numbers are integers.

Checkpoint 1.5.7. In the introduction, we mentioned that the different types of numbers are appropriate in different situation. Which number set do you think is most appropriate in each of the following situations?

a. The number of people in a math class that play the ukulele.

   This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

b. The hypotenuse's length in a given right triangle.

   This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

c. The proportion of people in a math class that have a cat.

   This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

d. The number of people in the room with you who have the same birthday as you.

   This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

e. The total revenue (in dollars) generated for ticket sales at a Timbers soccer game.

   This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

Explanation.

a. The number of people who play the ukulele could be $0, 1, 2, \ldots$, so the whole numbers are the appropriate set.

b. A hypotenuse's length could be 1, 1.2, $\sqrt{2}$ (which is irrational), or any other positive number. So the real numbers are the appropriate set.

c. This proportion will be a ratio of integers, as both the total number of people in the class and the number of people who have a cat are integers. So the rational numbers are the appropriate set.

d. We know that the number of people must be a counting number, and since *you* are in the room with yourself, there is at least one person in that room with your birthday. So the natural numbers are the appropriate set.

e. The total revenue will be some number of dollars and cents, such as \$631,897.15, which is a terminating decimal and thus a rational number. So the rational numbers are the appropriate set.

1.5.3 Converting Repeating Decimals to Fractions

We have learned that a terminating decimal number is a rational number. It's easy to convert a terminating decimal number into a fraction of integers: you just need to multiply and divide by one of the numbers in the set $\{10, 100, 1000, \ldots\}$. For example, when we say the number 0.123 out loud, we say "one hundred and twenty-three thousandths." While that's a lot to say, it makes it obvious that this number can be written as a ratio:

$$0.123 = \frac{123}{1000}.$$

Similarly,

$$21.28 = \frac{2128}{100} = \frac{532 \cdot 4}{25 \cdot 4} = \frac{532}{25},$$

demonstrating how *any* terminating decimal can be written as a fraction.

Repeating decimals can also be written as a fraction. To understand how, use a calculator to find the decimal for, say, $\frac{73}{99}$ and $\frac{189}{999}$ You will find that

$$\frac{73}{99} = 0.73737373\ldots = 0.\overline{73} \qquad \frac{189}{999} = 0.189189189\ldots = 0.\overline{189}.$$

The pattern is that diving a number by a number from $\{9, 99, 999, \ldots\}$ with the same number of digits will create a repeating decimal that starts as "0." and then repeats the numerator. We can use this observation to reverse engineer some fractions from repeating decimals.

Checkpoint 1.5.8.

a. Write the rational number $0.772772772\ldots$ as a fraction.

b. Write the rational number $0.69696969\ldots$ as a fraction.

Explanation.

a. The *three*-digit number 772 repeats after the decimal. So we will make use of the *three*-digit denominator 999. And we have $\frac{772}{999}$.

b. The *two*-digit number 69 repeats after the decimal. So we will make use of the *two*-digit denominator 99. And we have $\frac{69}{99}$. But this fraction can be reduced to $\frac{23}{33}$.

Converting a repeating decimal to a fraction is not always quite this straightforward. There are complications if the number takes a few digits before it begins repeating. For your interest, here is one example on how to do that.

> **Example 1.5.9** Can we convert the repeating decimal $9.134343434\ldots = 9.1\overline{34}$ to a fraction? The trick is to separate its terminating part from its repeating part, like this:
>
> $$9.1 + 0.034343434\ldots.$$
>
> Now note that the terminating part is $\frac{91}{10}$, and the repeating part is almost like our earlier examples, except it has an extra 0 right after the decimal. So we have:
>
> $$\frac{91}{10} + \frac{1}{10} \cdot 0.34343434\ldots.$$
>
> With what we learned in the earlier examples and basic fraction arithmetic, we can continue:
>
> $$\begin{aligned} 9.134343434\ldots &= \frac{91}{10} + \frac{1}{10} \cdot 0.34343434\ldots \\ &= \frac{91}{10} + \frac{1}{10} \cdot \frac{34}{99} \\ &= \frac{91}{10} + \frac{34}{990} \\ &= \frac{91 \cdot 99}{10 \cdot 99} + \frac{34}{990} \\ &= \frac{9009}{990} + \frac{34}{990} = \frac{9043}{990} \end{aligned}$$
>
> Check that this is right by entering $\frac{9043}{990}$ into a calculator and seeing if it returns the decimal we started with, $9.134343434\ldots$.

Exercises

Review and Warmup

1. Write the decimal number as a fraction.

$0.55 =$

2. Write the decimal number as a fraction.

$0.65 =$

3. Write the decimal number as a fraction.

$7.65 =$

4. Write the decimal number as a fraction.

$8.25 =$

5. Write the decimal number as a fraction.

$0.988 =$

6. Write the decimal number as a fraction.

$0.152 =$

7. Write the fraction as a decimal number. Do not round your answers.

a. $\frac{3}{5} =$

b. $\frac{7}{16} =$

8. Write the fraction as a decimal number. Do not round your answers.

a. $\frac{1}{5} =$

b. $\frac{3}{20} =$

9. Write the mixed number as a decimal number. Do not round your answers.

a. $5\frac{3}{8} =$

b. $3\frac{1}{4} =$

10. Write the mixed number as a decimal number. Do not round your answers.

a. $4\frac{13}{16} =$

b. $9\frac{3}{4} =$

Set Notation

11. There are two numbers that you can square to get 36. Express this collection of two numbers using set notation.

12. There are four positive, even, one-digit numbers. Express this collection of four numbers using set notation.

13. There are six two-digit perfect square numbers. Express this collection of six numbers using set notation.

14. There is a set of three small positive integers where you can square all three numbers, then add the results, and get 61. Express this collection of three numbers using set notation.

Types of Numbers

15. Which of the following are whole numbers? There may be more than one correct answer.

☐ $-2.1\overline{97}$ ☐ $\sqrt{4}$ ☐ $\sqrt{3}$ ☐ 39
☐ -95159 ☐ -2 ☐ 5.101001000100001...
☐ 3.521

16. Which of the following are whole numbers? There may be more than one correct answer.

☐ $\frac{6}{31}$ ☐ 0 ☐ -2 ☐ $\sqrt{3}$ ☐ 4 ☐ -13126
☐ $-6.8\overline{71}$ ☐ $-4.101001000100001\ldots$

17. Which of the following are integers? There may be more than one correct answer.

☐ -6 ☐ 9.101001000100001... ☐ 69
☐ $\sqrt{16}$ ☐ 4.097 ☐ $-\frac{1}{28}$ ☐ π
☐ -4233

18. Which of the following are integers? There may be more than one correct answer.

☐ 17956 ☐ -2 ☐ 6.267 ☐ $-\frac{1}{8}$ ☐ -95340
☐ 35 ☐ π ☐ 6.101001000100001...

19. Which of the following are rational numbers? There may be more than one correct answer.

☐ 8.157 ☐ $5.3\overline{85}$ ☐ -6 ☐ 2.101001000100001...
☐ 0 ☐ $\sqrt{3}$ ☐ -86447 ☐ 99

20. Which of the following are rational numbers? There may be more than one correct answer.

☐ -0.957000000000001 ☐ 0 ☐ $-\frac{4}{69}$
☐ $-7.101001000100001\ldots$ ☐ -77554
☐ $\sqrt{36}$ ☐ $\sqrt{6}$ ☐ -2

21. Which of the following are irrational numbers? There may be more than one correct answer.

☐ -68660 ☐ $\frac{7}{96}$ ☐ -6 ☐ π ☐ 9.077
☐ 43001 ☐ 0 ☐ 7.101001000100001...

22. Which of the following are irrational numbers? There may be more than one correct answer.

☐ -59767 ☐ $-\frac{5}{17}$ ☐ 4.245 ☐ -2
☐ π ☐ 95 ☐ $\sqrt{3}$ ☐ 0

23. Which of the following are real numbers? There may be more than one correct answer.

☐ 60 ☐ 0 ☐ π ☐ -86770 ☐ $-\frac{6}{95}$
☐ $0.5\overline{70}$ ☐ -5 ☐ 5.101001000100001...

24. Which of the following are real numbers? There may be more than one correct answer.

☐ 8.303 ☐ $\frac{7}{13}$ ☐ 0 ☐ $\sqrt{4}$
☐ -2 ☐ $\sqrt{11}$ ☐ π ☐ -41981

25. Determine the validity of each statement by selecting True or False.

(a) The number $\frac{\pi}{2}$ is rational

(b) The number 98 is an integer, but not a whole number

(c) The number $-\frac{13}{102}$ is rational

(d) The number $\sqrt{\frac{49}{25}}$ is an integer, but not a whole number

(e) The number π is irrational

26. Determine the validity of each statement by selecting True or False.

(a) The number $\frac{3}{2}$ is rational, but not an integer

(b) The number $0.700700700700700\ldots$ is rational

(c) The number -11 is an integer that is also a natural number

(d) The number $\sqrt{2^2}$ is a real number, but not a rational number

(e) The number $\sqrt{\frac{4}{36}}$ is rational, but not an integer

27. In each situation, which number set do you think is most appropriate?

a. The number of dogs a student has owned throughout their lifetime.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

b. The difference between the projected annual expenditures and the actual annual expenditures for a given company.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

c. The length around swimming pool in the shape of a half circle with radius 10 ft.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

d. The proportion of students at a college who own a car.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

e. The width of a sheet of paper, in inches.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

f. The number of people eating in a non-empty restaurant.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

28.a. Give an example of a whole number that is not an integer.

b. Give an example of an integer that is not a whole number.

c. Give an example of a rational number that is not an integer.

d. Give an example of a irrational number.

e. Give an example of a irrational number that is also an integer.

Writing Decimals as Fractions

29. Write the rational number 6.35 as a fraction.

30. Write the rational number 77.162 as a fraction.

31. Write the rational number $0.\overline{78} = 0.7878\ldots$ as a fraction.

32. Write the rational number $0.\overline{955} = 0.955955\ldots$ as a fraction.

33. Write the rational number $4.4\overline{12} = 4.41212\ldots$ as a fraction.

34. Write the rational number $8.1\overline{238} = 8.1238238\ldots$ as a fraction.

Challenge

35. Imagine making up a number according to the following pattern. After the decimal point, write the natural numbers 1, 2, 3, 4, 5, etc. The decimal digits will extend infinitely according to my pattern.

$$0.12345\ldots$$

Is the number a rational number or an irrational number?

(☐ rational ☐ irrational)

1.6 Comparison Symbols and Notation for Intervals

As you know, 8 is larger than 3; that's a specific comparison between two numbers. We can also make a comparison between two less specific numbers, like if we say that average rent in Portland in 2016 is larger than it was in 2009. That makes a comparison using unspecified amounts. This section will go over the mathematical shorthand notation for making these kinds of comparisons.

In Oregon, only people who are 18 years old or older can vote in statewide elections.[1] Does that seem like a statement about the number 18? Maybe. But it's also a statement about numbers like 37 and 62: it says that people of these ages may vote as well. This section will also get into the mathematical notation for large collections of numbers like this.

1.6.1 Comparison Symbols

In everyday language you can say something like "8 is larger than 3." In mathematical writing, it's not convenient to write that out in English. Instead the symbol ">" has been adopted, and it's used like this:

$$8 > 3$$

and read out loud as "8 is greater than 3." The symbol ">" is called the **greater-than symbol.**

Checkpoint 1.6.2.

a. Use mathematical notation to write "11.5 is greater than 4.2."

b. Use mathematical notation to write "age is greater than 20."

Explanation.

a. $11.5 > 4.2$

b. We can just write the word age to represent age, and write $\text{age} > 20$. Or we could use an abbreviation like a for age, and write $a > 20$. Or, it is common to use x as a generic abbreviation, and we could write $x > 20$.

Remark 1.6.3. At some point in history, someone felt that > was a good symbol for "is greater than." In "$8 > 3$," the tall side of the symbol is with the larger of the two numbers, and the small pointed side is with the smaller of the two numbers.

We have to be careful when negative numbers are part of the comparison though. Is −8 larger or smaller than −3? In some sense −8 is larger, because if you owe someone 8 dollars, that's *more* than owing them 3 dollars. But that is not how the > symbol works. This symbol is meant to tell you which number is farther to the right on a number line. And if that's how it goes, then $-3 > -8$.

Alligator Jaws. Another visual was of thinking about the greater-than symbol ">" (and, as we will see later, the less-than symbol "<") is "the alligator wants to eat the larger number" as a way of remembering which direction to write the symbol.

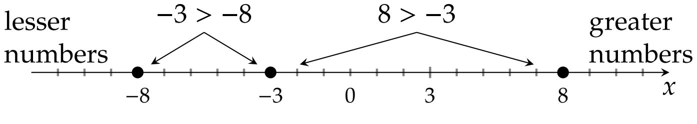


Figure 1.6.4: How the > symbol works.

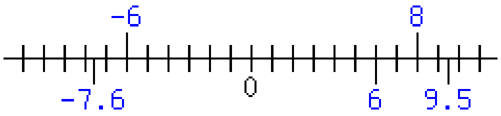
---

[1] Some other states like Washington allow 17-year-olds to vote in primary elections provided they will be 18 by the general election.

Checkpoint 1.6.5. Use the > symbol to arrange the following numbers in order from greatest to least. For example, your answer might look like `4>3>2>1>0`.

$$-7.6 \quad 6 \quad -6 \quad 9.5 \quad 8$$

Explanation. We can order these numbers by placing these numbers on a number line.

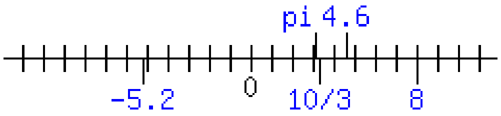


And so we see the answer is $9.5 > 8 > 6 > -6 > -7.6$.

Checkpoint 1.6.6. Use the > symbol to arrange the following numbers in order from greatest to least. For example, your answer might look like `4>3>2>1>0`.

$$-5.2 \quad \pi \quad \frac{10}{3} \quad 4.6 \quad 8$$

Explanation. We can order these numbers by placing these numbers on a number line. Knowing or computing their decimals helps with this.



And so we see the answer is $8 > 4.6 > 3.33333 > 3.14159 > -5.2$.

The greater-than symbol has a close relative, the **greater-than-or-equal-to symbol**, "≥." It means just like it sounds: the first number is either greater than, or equal to, the second number. These are all true statements:

$$8 \geq 3 \qquad 3 \geq -8 \qquad 3 \geq 3$$

but one of these three statements is false:

$$8 > 3 \qquad 3 > -8 \qquad 3 \overset{\text{no}}{>} 3$$

Remark 1.6.7. While it may not be that useful that we can write $3 \geq 3$, this symbol is quite useful when specific numbers aren't explicitly used on at least one side, like in these examples:

$$\text{(hourly pay rate)} \geq \text{(minimum wage)}$$
$$\text{(age of a voter)} \geq 18$$

Sometimes you want to emphasize that one number is *less than* another number instead of emphasizing which number is greater. To do this, we have symbols that are reversed from > and ≥. The symbol "<" is the **less-than symbol** and it's used like this:

$$3 < 8$$

and read out loud as “3 is less than 8.”

Table 1.6.8 gives the complete list of all six comparison symbols. Note that we’ve only discussed three in this section so far, but you already know the equals symbol and have likely also seen the symbol “≠,” which means “not equal to.”

Symbol	Means	Examples		
$=$	equals	$13 = 13$	$\frac{5}{4} = 1.25$	$5 \overset{\text{no}}{=} 6$
$>$	is greater than	$13 > 11$	$\pi > 3$	$9 \overset{\text{no}}{>} 9$
$\geq$	is greater than or equal to	$13 \geq 11$	$3 \geq 3$	$11.2 \overset{\text{no}}{\geq} 10.2$
$<$	is less than	$-3 < 8$	$\frac{1}{2} < \frac{2}{3}$	$2 \overset{\text{no}}{<} -2$
$\leq$	is less than or equal to	$-3 \leq 8$	$3 \leq 3$	$\frac{4}{5} \overset{\text{no}}{\leq} \frac{3}{5}$
$\neq$	is not equal to	$10 \neq 20$	$\frac{1}{2} \neq 1.2$	$\frac{3}{8} \overset{\text{no}}{\neq} 0.375$

Table 1.6.8: Comparison Symbols

1.6.2 Set-Builder and Interval Notation

If you say

$$\text{(age of a voter)} \geq 18$$

and have a particular voter in mind, what is that person’s age? There’s no way to know for sure. *Maybe* they are 18, but maybe they are older. It’s helpful to use a variable a to represent age (in years) and then to visualize the possibilities with a number line, as in Figure 1.6.9.

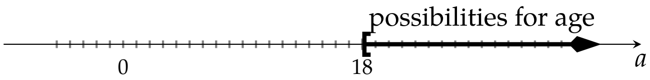


Figure 1.6.9: (age of a voter) ≥ 18

The shaded portion of the number line in Figure 1.6.9 is a mathematical **interval**. For now, that just means a collection of certain numbers. In this case, it’s all the numbers 18 and above.

The number line in Figure 1.6.9 is a *graphical* representation of a collection of certain numbers. We have two notations, set-builder notation and interval notation, that we also use to represent such collections of numbers.

Definition 1.6.10 Set-Builder Notation. Set-builder notation attempts to directly say the condition that numbers in the interval satisfy. In general, we write set-builder notation like:

$$\{x \mid \text{condition on } x\}$$

and read it out loud as “the set of all x such that ….” For example,

$$\{x \mid x \geq 18\}$$

is read out loud as “the set of all x such that x is greater than or equal to 18.” The breakdown is as follows.

$\{x \mid x \geq 18\}$	the set of
$\{x \mid x \geq 18\}$	all x
$\{x \mid x \geq 18\}$	such that
$\{x \mid x \geq 18\}$	x is greater than or equal to 18

Definition 1.6.11 Interval Notation. Interval notation represents a collection of numbers by only stating where the collection starts and stops, using parentheses and square brackets to show if the end values are included (or not). For example, in Figure 1.6.9, the interval starts at 18. To the right, the interval extends forever and has no end, so we use the ∞ symbol (meaning ”infinity”). This particular interval is denoted:

$$[18, \infty)$$

Why use “[” on one side and “)” on the other? The square bracket tells us that 18 *is* part of the interval and the round parenthesis tells us that ∞ is *not* part of the interval.[2]

In general there are four types of infinite intervals. Take note of the different uses of round parentheses and square brackets.

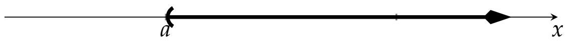


Figure 1.6.12: An **open, infinite** interval denoted by (a, ∞) means all numbers a or larger, *not* including a.

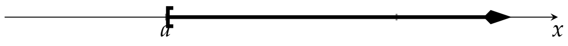


Figure 1.6.13: A **closed, infinite** interval denoted by $[a, \infty)$ means all numbers a or larger, *including* a.

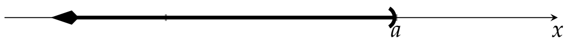


Figure 1.6.14: An **open, infinite** interval denoted by $(-\infty, a)$ means all numbers a or smaller, *not* including a.

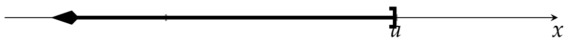
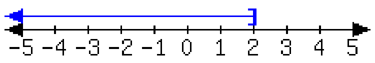
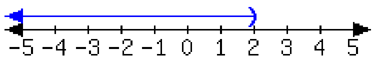
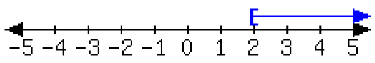


Figure 1.6.15: A **closed, infinite** interval denoted by $(-\infty, a]$ means all numbers a or smaller, *including* a.

Checkpoint 1.6.16 Interval and Set-Builder Notation from Number Lines. For each interval expressed in the number lines, give the interval notation and set-builder notation.







a. In set-builder notation:

In interval notation:

b. In set-builder notation:

In interval notation:

c. In set-builder notation:

In interval notation:

Explanation.

a. Since all numbers less than or equal to 2 are shaded, the set-builder notation is { x | x <= 2 }. The shaded interval “starts” at $-\infty$ and ends at 2 (including 2) so the interval notation is (-infinity,2].

b. Since all numbers less than to 2 are shaded, the set-builder notation is { x | x < 2 }. The shaded interval “starts” at $-\infty$ and ends at 2 (excluding 2) so the interval notation is (-infinity,2)

c. Since all numbers greater than or equal to 2 are shaded, the set-builder notation is { x | x >= 2 }. The shaded interval starts at 2 (including 2) and “ends” at ∞, so the interval notation is [2,infinity)

---

[2] And how could it be, since ∞ is not even a number?

Exercises

Review and Warmup

1. Write the decimal number as a fraction.

$0.85 =$ ☐

2. Write the decimal number as a fraction.

$0.95 =$ ☐

3. Write the decimal number as a fraction.

$1.95 =$ ☐

4. Write the decimal number as a fraction.

$2.65 =$ ☐

5. Write the decimal number as a fraction.

$0.332 =$ ☐

6. Write the decimal number as a fraction.

$0.494 =$ ☐

7. Write the fraction as a decimal number. Do not round your answers.

a. $\frac{13}{16} =$ ☐

b. $\frac{13}{25} =$ ☐

8. Write the fraction as a decimal number. Do not round your answers.

a. $\frac{5}{16} =$ ☐

b. $\frac{2}{5} =$ ☐

9. Write the mixed number as a decimal number. Do not round your answers.

a. $3\frac{17}{20} =$ ☐

b. $1\frac{18}{25} =$ ☐

10. Write the mixed number as a decimal number. Do not round your answers.

a. $8\frac{3}{25} =$ ☐

b. $3\frac{15}{16} =$ ☐

Ordering Numbers Use the > symbol to arrange the following numbers in order from greatest to least. For example, your answer might look like `4>3>2>1>0`.

11.

$$10 \quad -6 \quad 0 \quad 3 \quad 7$$

12.

$$-9 \quad 8 \quad 9 \quad -3 \quad 6$$

13.

$$-6.35 \quad 0.46 \quad -2.94 \quad -7.79 \quad 6.37$$

14.

$$-4.18 \quad -6.97 \quad 5.43 \quad 6.04 \quad -6.87$$

15.

$$-\frac{5}{4} \quad 7 \quad -7 \quad \frac{19}{7} \quad 6$$

16.

$$-\frac{19}{6} \quad -\frac{9}{5} \quad -\frac{11}{3} \quad -\frac{15}{2} \quad \frac{41}{8}$$

17. $3 \quad \frac{3}{7} \quad \frac{\pi}{2} \quad \frac{1}{2} \quad -8 \quad \sqrt{3}$

18. $\frac{2}{3} \quad 6 \quad \sqrt{3} \quad \frac{4}{7} \quad \pi \quad 5$

True/False

19. Decide if each comparison is true or false.

a. $2 \neq -3$ (☐ True ☐ False)
b. $2 = -4$ (☐ True ☐ False)
c. $4 \neq 4$ (☐ True ☐ False)
d. $-3 \leq 7$ (☐ True ☐ False)
e. $-3 < -3$ (☐ True ☐ False)
f. $6 = 6$ (☐ True ☐ False)

20. Decide if each comparison is true or false.

a. $-7 < -7$ (☐ True ☐ False)
b. $-3 < 4$ (☐ True ☐ False)
c. $8 \geq -4$ (☐ True ☐ False)
d. $-5 \geq -5$ (☐ True ☐ False)
e. $-4 \neq -4$ (☐ True ☐ False)
f. $-6 \neq 4$ (☐ True ☐ False)

21. Decide if each comparison is true or false.

a. $\frac{4}{2} \neq \frac{12}{6}$ (☐ True ☐ False)
b. $\frac{5}{5} > \frac{15}{15}$ (☐ True ☐ False)
c. $-\frac{43}{8} \neq \frac{5}{2}$ (☐ True ☐ False)
d. $-\frac{46}{5} = \frac{25}{9}$ (☐ True ☐ False)
e. $\frac{2}{7} \leq \frac{4}{14}$ (☐ True ☐ False)
f. $-\frac{45}{7} < \frac{11}{2}$ (☐ True ☐ False)

22. Decide if each comparison is true or false.

a. $-\frac{7}{6} = -\frac{21}{18}$ (☐ True ☐ False)
b. $\frac{13}{2} \geq -\frac{74}{8}$ (☐ True ☐ False)
c. $\frac{3}{8} < \frac{3}{8}$ (☐ True ☐ False)
d. $\frac{14}{3} = \frac{19}{2}$ (☐ True ☐ False)
e. $-\frac{5}{5} \neq -\frac{5}{5}$ (☐ True ☐ False)
f. $-\frac{9}{4} \geq -\frac{9}{4}$ (☐ True ☐ False)

Comparisons Choose <, >, or = to make a true statement.

23. $-\frac{7}{2}$ (☐ < ☐ > ☐ =) $-\frac{5}{4}$

24. $-\frac{6}{7}$ (☐ < ☐ > ☐ =) $-\frac{1}{2}$

25. $\frac{3}{5} + \frac{4}{3}$ (☐ < ☐ > ☐ =) $\frac{1}{2} \div \frac{5}{3}$

26. $\frac{4}{5} + \frac{3}{4}$ (☐ < ☐ > ☐ =) $\frac{2}{5} \div \frac{4}{3}$

27. $\frac{14}{13} \div \frac{14}{13}$ (☐ < ☐ > ☐ =) $\frac{8}{12} - \frac{2}{3}$

28. $\frac{17}{7} \div \frac{17}{7}$ (☐ < ☐ > ☐ =) $\frac{12}{10} - \frac{6}{5}$

29. $-6\frac{1}{3}$ (☐ < ☐ > ☐ =) -6

30. $-1\frac{2}{3}$ (☐ < ☐ > ☐ =) -1

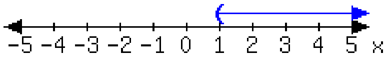
31. $-3\frac{1}{2}$ (☐ < ☐ > ☐ =) 3

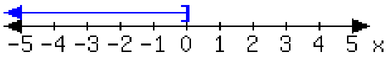
32. $-3\frac{2}{3}$ (☐ < ☐ > ☐ =) 1

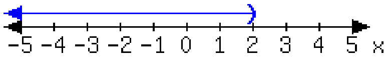
33. $\left|-\frac{3}{5}\right|$ (☐ < ☐ > ☐ =) $|0.6|$

34. $\left|-\frac{3}{8}\right|$ (☐ < ☐ > ☐ =) $|0.375|$

35. For each interval expressed in the number lines, give the interval notation and set-builder notation.







a. In set-builder notation:

In interval notation:

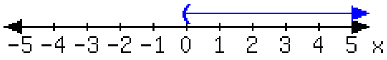
b. In set-builder notation:

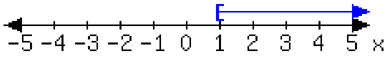
In interval notation:

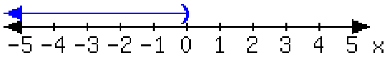
c. In set-builder notation:

In interval notation:

36. For each interval expressed in the number lines, give the interval notation and set-builder notation.







a. In set-builder notation:

In interval notation:

b. In set-builder notation:

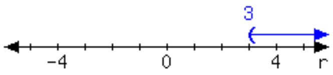
In interval notation:

c. In set-builder notation:

In interval notation:

Set-builder and Interval Notation

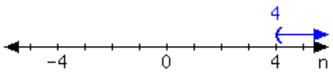
37. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

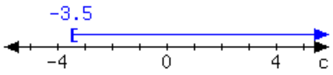
38. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

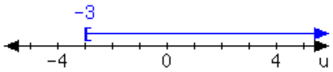
39. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

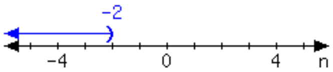
40. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

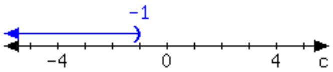
41. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

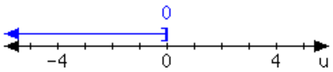
42. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

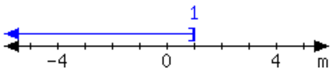
43. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

44. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

Convert to Interval Notation A set is written using set-builder notation. Write it using interval notation.

45. $\{x \mid x \leq 5\}$

46. $\{x \mid x \leq 7\}$

47. $\{x \mid x \geq 9\}$

48. $\{x \mid x \geq -9\}$

49. $\{x \mid x < -7\}$

50. $\{x \mid x < -5\}$

51. $\{x \mid x > -2\}$

52. $\{x \mid x > 10\}$

53. $\{x \mid 2 > x\}$

54. $\{x \mid 5 > x\}$

55. $\{x \mid 7 \geq x\}$

56. $\{x \mid 9 \geq x\}$

57. $\{x \mid -9 \leq x\}$

58. $\{x \mid -7 \leq x\}$

59. $\{x \mid -5 < x\}$

60. $\{x \mid -2 < x\}$

61. $\left\{x \mid \frac{5}{9} < x\right\}$

62. $\left\{x \mid \frac{7}{6} < x\right\}$

63. $\left\{x \mid x \leq -\frac{8}{3}\right\}$

64. $\left\{x \mid x \leq -\frac{9}{7}\right\}$

65. $\{x \mid x \leq 0\}$

66. $\{x \mid 0 < x\}$

1.7 Basic Math Chapter Review

1.7.1 Arithmetic with Negative Numbers

Adding Real Numbers with the Same Sign When adding two numbers with the same sign, we can ignore the signs, and simply add the numbers as if they were both positive.

> **Example 1.7.1**
>
> a. $5 + 2 = 7$
>
> b. $-5 + (-2) = -7$

Adding Real Numbers with Opposite Signs When adding two numbers with opposite signs, we find those two numbers' difference. The sum has the same sign as the number with the bigger value. If those two numbers have the same value, the sum is 0.

> **Example 1.7.2**
>
> a. $5 + (-2) = 3$
>
> b. $(-5) + 2 = -3$

Subtracting a Positive Number When subtracting a positive number, we can change the problem to adding the *opposite* number, and then apply the methods of adding numbers.

> **Example 1.7.3**
>
> a. $\begin{aligned} 5 - 2 &= 5 + (-2) \\ &= 3 \end{aligned}$
>
> b. $\begin{aligned} 2 - 5 &= 2 + (-5) \\ &= -3 \end{aligned}$
>
> c. $\begin{aligned} -5 - 2 &= -5 + (-2) \\ &= 3 \end{aligned}$

Subtracting a Negative Number When subtracting a negative number, we can change those two negative signs to a positive sign, and then apply the methods of adding numbers.

> **Example 1.7.4**
>
> a. $\begin{aligned} 5 - (-2) &= 5 + 2 \\ &= 7 \end{aligned}$
>
> b. $\begin{aligned} -5 - (-2) &= -5 + 2 \\ &= -3 \end{aligned}$
>
> c. $\begin{aligned} -2 - (-5) &= -2 + 5 \\ &= 3 \end{aligned}$

Multiplication and Division of Real Numbers When multiplying and dividing real numbers, each pair of negative signs cancel out each other (becoming a positive sign). If there is still one negative sign left, the result is negative; otherwise the result is positive.

> **Example 1.7.5**
>
> a. $(6)(-2) = -12$
>
> b. $(-6)(2) = -12$
>
> c. $(-6)(-2) = 12$
>
> d. $(-6)(-2)(-1) = -12$
>
> e. $(-6)(-2)(-1)(-1) = 12$
>
> f. $\frac{12}{-2} = -6$
>
> g. $\frac{-12}{2} = -6$
>
> h. $\frac{-12}{-2} = 6$

Powers When we raise a negative number to a certain power, apply the rules of multiplying real numbers: each pair of negative signs cancel out each other.

> **Example 1.7.6**
>
> a. $(-2)^2 = (-2)(-2) = 4$
>
> b. $(-2)^3 = (-2)(-2)(-2) = -8$
>
> c. $(-2)^4 = (-2)(-2)(-2)(-2) = 16$

Difference between $(-a)^n$ and $-a^n$ For the exponent expression 2^3, the number 2 is called the **base**, and the number 3 is called the **exponent**. The base of $(-a)^n$ is $-a$, while the base of $-a^n$ is a. This makes a difference in the result when the power is an even number.

> **Example 1.7.7**
>
> a. $(-4)^2 = (-4)(-4) = 16$
>
> b. $-4^2 = -(4)(4) = -16$
>
> c. $(-4)^3 = (-4)(-4)(-4) = -64$
>
> d. $-4^3 = -(4)(4)(4) = -64$

1.7.2 Fraction Arithmetic

> **Example 1.7.8 Multiplying Fractions.**
>
> When multiplying two fractions, we simply multiply the numerators and denominators. To avoid big numbers, we should reduce fractions before multiplying. If one number is an integer, we can write it as a fraction with a denominator of 1. For example, $2 = \frac{2}{1}$.
>
> $$\frac{1}{2} \cdot \frac{3}{4} = \frac{1 \cdot 3}{2 \cdot 4} = \frac{3}{8}$$

> **Example 1.7.9 Dividing Fractions.**
>
> When dividing two fractions, we "flip" the second number, and then do multiplication.
>
> $$\frac{1}{2} \div \frac{4}{3} = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$$

Example 1.7.10 Adding/Subtracting Fractions.

Before adding/subtracting fractions, we need to change each fraction's denominator to the same number, called the **common denominator**. Then, we add/subtract the numerators, and the denominator remains the same.

$$\begin{aligned}\frac{1}{2}-\frac{1}{3}&=\frac{1}{2}\cdot\frac{3}{3}-\frac{1}{3}\cdot\frac{2}{2}\\&=\frac{3}{6}-\frac{2}{6}\\&=\frac{1}{6}\end{aligned}$$

1.7.3 Absolute Value and Square Root

Example 1.7.11 Absolute Value.

The absolute value of a number is the distance from that number to 0 on the number line. An absolute value is always positive or 0.

a. $|2| = 2$

b. $\left|-\frac{1}{2}\right| = \frac{1}{2}$

c. $|0| = 0$

Example 1.7.12 Square Root.

The symbol $\sqrt{b}$ has meaning when $b \geq 0$. It means the positive number that can be squared to result in b.

a. $\sqrt{9} = 3$

b. $\sqrt{2} \approx 1.414$

c. $\sqrt{\frac{9}{16}} = \frac{3}{4}$

d. $\sqrt{-1}$ is undefined

1.7.4 Order of Operations

Example 1.7.13 Order of Operations.

When evaluating an expression with multiple operations, we must follow the order of operations:

1. (P)arentheses and other grouping symbols
2. (E)xponentiation
3. (M)ultiplication, (D)ivision, and Negation
4. (A)ddition and (S)ubtraction

$$\begin{aligned}4-2\left(3-(2-4)^2\right)&=4-2\left(3-(\boxed{2-4})^2\right)\\&=4-2\left(3-\boxed{(-2)^2}\right)\\&=4-2\left(\boxed{3-4}\right)\\&=4-\boxed{2\,(-1)}\\&=4-(-2)\\&=6\end{aligned}$$

1.7.5 Set Notation and Types of Numbers

A **set** is an unordered collection of items. Braces, $\{\}$, are used to show what items are in a set. For example, the set $\{1, 2, \pi\}$ is a set with three items that contains the numbers 1, 2, and π.

Types of Numbers Real numbers are categorized into the following sets: natural numbers, whole numbers, integers, rational numbers and irrational numbers.

> **Example 1.7.14** Here are some examples of numbers from each set of numbers:
>
> **Natural Numbers** The natural numbers are all counting numbers larger 1 and larger.
>
> $1, 251, 3462$
>
> **Whole Numbers** The whole numbers are all counting numbers larger 0 and larger.
>
> $0, 1, 42, 953$
>
> **Integers** The integers are all counting numbers both negative and positive.
>
> $-263, -10, 0, 1, 834$
>
> **Rational Numbers** The rational numbers are all possible fractions of integers.
>
> $\frac{1}{3}, -3, 1.1, 0, 0.\overline{73}$
>
> **Irrational Numbers** The irrational numbers are all numbers that cannot be written as a fraction of integers.
>
> $\pi, e, \sqrt{2}$

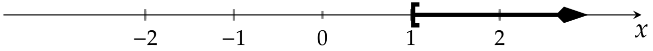
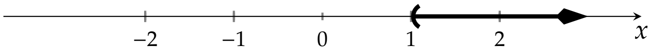
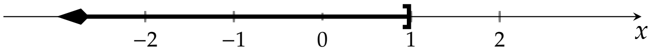
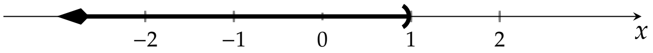
1.7.6 Comparison Symbols and Notation for Intervals

The following are symbols used to compare numbers.

Symbol	Meaning	Examples	
$=$	equals	$13 = 13$	$\frac{5}{4} = 1.25$
$>$	is greater than	$13 > 11$	$\pi > 3$
$\geq$	is greater than or equal to	$13 \geq 11$	$3 \geq 3$
$<$	is less than	$-3 < 8$	$\frac{1}{2} < \frac{2}{3}$
$\leq$	is less than or equal to	$-3 \leq 8$	$3 \leq 3$
$\neq$	is not equal to	$10 \neq 20$	$\frac{1}{2} \neq 1.2$

Table 1.7.15: Comparison Symbols

The following are some examples of set-builder notation and interval notation.

Graph	Set-builder Notation	Interval Notation
	$\{x \mid x \geq 1\}$	$[1, \infty)$
	$\{x \mid x > 1\}$	$(1, \infty)$
	$\{x \mid x \leq 1\}$	$(-\infty, 1]$
	$\{x \mid x < 1\}$	$(-\infty, 1)$

Exercises

1. Perform the given addition and subtraction.

a. $-19 - 8 + (-2) =$

b. $2 - (-19) + (-14) =$

2. Perform the given addition and subtraction.

a. $-18 - 5 + (-8) =$

b. $9 - (-19) + (-19) =$

3. Multiply the following.

a. $(-2) \cdot (-6) \cdot (-3) =$

b. $5 \cdot (-9) \cdot (-2) =$

c. $(-99) \cdot (-60) \cdot 0 =$

4. Multiply the following.

a. $(-2) \cdot (-4) \cdot (-5) =$

b. $3 \cdot (-9) \cdot (-5) =$

c. $(-98) \cdot (-77) \cdot 0 =$

5. Evaluate the following.

a. $\frac{-25}{-5} =$

b. $\frac{10}{-5} =$

c. $\frac{-35}{5} =$

6. Evaluate the following.

a. $\frac{-8}{-4} =$

b. $\frac{32}{-4} =$

c. $\frac{-15}{5} =$

7. Evaluate the following.

a. $(-1)^2 =$

b. $-4^2 =$

8. Evaluate the following.

a. $(-1)^2 =$

b. $-8^2 =$

9. Evaluate the following.

a. $(-4)^3 =$

b. $-1^3 =$

10. Evaluate the following.

a. $(-4)^3 =$

b. $-3^3 =$

11. Add: $-\frac{9}{10} + \frac{5}{6}$

12. Add: $-\frac{1}{6} + \frac{7}{10}$

13. Subtract: $-\frac{5}{6} - \left(-\frac{9}{10}\right)$

14. Subtract: $-\frac{1}{10} - \left(-\frac{5}{6}\right)$

15. Subtract: $2 - \frac{28}{9}$

16. Subtract: $4 - \dfrac{25}{6}$

17. Multiply: $-\dfrac{12}{13} \cdot \dfrac{7}{22}$

18. Multiply: $-\dfrac{2}{13} \cdot \dfrac{5}{26}$

19. Multiply: $-4 \cdot \dfrac{5}{6}$

20. Multiply: $-5 \cdot \dfrac{9}{20}$

21. Divide: $\dfrac{7}{15} \div \left(-\dfrac{5}{12}\right)$

22. Divide: $\dfrac{1}{9} \div \left(-\dfrac{5}{12}\right)$

23. Divide: $27 \div \dfrac{9}{4}$

24. Divide: $9 \div \dfrac{9}{4}$

25. Evaluate the following.

a. $-|3 - 10| = \boxed{\quad}$

b. $|-3 - 10| = \boxed{\quad}$

c. $-2|10 - 3| = \boxed{\quad}$

26. Evaluate the following.

a. $-|1 - 7| = \boxed{\quad}$

b. $|-1 - 7| = \boxed{\quad}$

c. $-2|7 - 1| = \boxed{\quad}$

27. Evaluate the following.

a. $\sqrt{1} = \boxed{\quad}$

b. $\sqrt{81} = \boxed{\quad}$

c. $\sqrt{100} = \boxed{\quad}$

28. Evaluate the following.

a. $\sqrt{4} = \boxed{\quad}$

b. $\sqrt{25} = \boxed{\quad}$

c. $\sqrt{9} = \boxed{\quad}$

29. Evaluate the following.

a. $\sqrt{\dfrac{16}{49}} = \boxed{\quad}$

b. $\sqrt{-\dfrac{25}{64}} = \boxed{\quad}$

30. Evaluate the following.

a. $\sqrt{\dfrac{25}{81}} = \boxed{\quad}$

b. $\sqrt{-\dfrac{144}{49}} = \boxed{\quad}$

31. Evaluate the following.

$-6^2 - 5[4 - (6 - 4^3)] = \boxed{\quad}$

32. Evaluate the following.

$-6^2 - 9[8 - (4 - 4^3)] = \boxed{\quad}$

33. Evaluate the following.

$\dfrac{27 - (-4)^3}{3 - 10} = \boxed{\quad}$

34. Evaluate the following.

$\dfrac{27 - (-2)^3}{7 - 12} = \boxed{\quad}$

35. Evaluate the following.

$10 - 8\left|-9 + (4 - 7)^3\right| = \boxed{\quad}$

36. Evaluate the following.

$1 - 6\left|-5 + (3 - 6)^3\right| = \boxed{\quad}$

Compare the following integers:

37. a. 2 (☐ < ☐ > ☐ =) −7

b. −2 (☐ < ☐ > ☐ =) −7

c. −7 (☐ < ☐ > ☐ =) 0

38. a. 3 (☐ < ☐ > ☐ =) −6

b. −1 (☐ < ☐ > ☐ =) −6

c. −6 (☐ < ☐ > ☐ =) 0

Determine the validity of each statement by selecting True or False.

39.
(a) The number $\sqrt{(-60)^2}$ is irrational
(b) The number $\sqrt{\frac{9}{16}}$ is an integer, but not a whole number
(c) The number $\sqrt{23}$ is rational
(d) The number 60 is an integer, but not a whole number
(e) The number 0 is a natural number

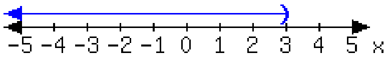
40.
(a) The number $\sqrt{\frac{25}{81}}$ is rational, but not an integer
(b) The number $\frac{19}{43}$ is rational, but not an integer
(c) The number $\sqrt{11}$ is a real number, but not an irrational number
(d) The number 0.14404004000400004... is rational
(e) The number $\sqrt{4}$ is a real number, but not a rational number

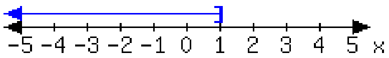
A set is written using set-builder notation. Write it using interval notation.

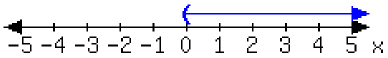
41. $\{x \mid x > 2\}$

42. $\{x \mid x > 4\}$

43. For each interval expressed in the number lines, give the interval notation and set-builder notation.





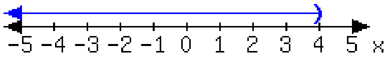


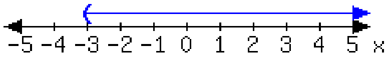
a. In set-builder notation: ☐
In interval notation: ☐

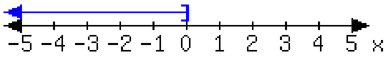
b. In set-builder notation: ☐
In interval notation: ☐

c. In set-builder notation: ☐
In interval notation: ☐

44. For each interval expressed in the number lines, give the interval notation and set-builder notation.







a. In set-builder notation: ☐
In interval notation: ☐

b. In set-builder notation: ☐
In interval notation: ☐

c. In set-builder notation: ☐
In interval notation: ☐

CHAPTER 2

Variables, Expressions, and Equations

2.1 Variables and Evaluating Expressions

To move past *arithmetic* to *algebra*, we begin working with **variables**. Any combination of numbers and variables using mathematical operations is called a mathematical **expression**. Some expressions are simple, and some are complicated. Some expressions are abstract, whereas some have context and meaning. One example of a simple mathematical expression with context is "$220 - a$," which has one variable, a, and is the expression for the maximum heart rate of a person who is a years old.

In this section, we'll focus on variables and expressions. In Section 2.2 we'll continue with a focus on geometry formulas. In the remainder of this chapter, we'll focus on mathematical **equations** and **inequalities** which are also very important in algebra.

2.1.1 Introduction to Variables

When we want to represent an unknown or changing numerical quantity, we use a **variable** to do so. For example, if you'd like to discuss the gas mileage of various cars, you could let the symbol "g" represent a car's gas mileage. The mileage might be 25 mpg, 30 mpg, or some other quantity. If we agree to use mpg for the units of measure, g might be a place holder for 25, 30, or some other number. Since we are using a variable and not a specific number, we can discuss gas mileage for Honda Civics at the same time we discuss gas mileage for Ford Explorers.

When variables stand for actual physical quantities, it's good to use letters that clearly correspond to the quantity they represent. For example, it's wise to use g to represent *g*as mileage. This helps the people who might read your work in the future to understand it better.

It is also important to identify what unit of measurement goes along with each variable you use, and clearly tell your reader this. For example, suppose you are working with $g = 25$. A car whose gas mileage is 25 mpg is very different from a car whose gas mileage is 25 kpg (kilometers per gallon). So it would be important to tell readers that g represents gas mileage *in miles per gallon*.

Checkpoint 2.1.2. Identify a variable you might use to represent each quantity. And identify what units would be most appropriate.

a. Let ☐ be the age of a student, measured in ☐.

b. Let ☐ be the amount of time passed since a driver left Portland, Oregon, bound for Boise, Idaho, measured in ☐.

c. Let ☐ be the area of a two-bedroom apartment, measured in ☐.

Explanation.

a. The unknown quantity is age, which we generally measure in years. So we could define this variable as:

   "Let a be the age of a student, measured in years."

b. The amount of time passed is the unknown quantity. Since this is a drive from Portland to Boise, it would make sense to measure this in hours. So we could define this variable as:

   "Let t be the amount of time passed since a driver left Portland, Oregon, bound for Boise, Idaho, measured in hours."

c. The unknown quantity is area. Apartment area is usually measured in square feet. So we'll define this variable as:

   "Let A be the area of a two-bedroom apartment, measured in ft^2."

Unless an algebra problem specifies which letter(s) to use, we may *choose* which letter(s) to use for our variable(s). However *without* any context to a problem, x, y, and z are the most common letters used as variables, and you may see these variables (especially x) a lot.

Also note that the units we use are often determined indirectly by other information given in an algebra problem. For example, if we're told that a car has used so many gallons of gas after traveling so many miles, then it suggests we should measure gas mileage in mpg.

2.1.2 Mathematical Expressions

A mathematical **expression** is any combination of variables and numbers using arithmetic operations. The following are all examples of mathematical expressions:

$$x+1 \qquad 2\ell+2w \qquad \frac{\sqrt{x}}{y+1} \qquad nRT$$

Note that this definition of "mathematical expression" does *not* include anything with signs like these in them: $=$, $<$, $\leq$, etc.

Example 2.1.3 The expression:

$$\frac{5}{9}(F-32)$$

can be used to convert from degrees Fahrenheit to degrees Celsius. To do this, we need a Fahrenheit temperature, F. Then we can **evaluate** the expression. This means replacing its variable(s) with specific numbers and calculating the result. In this case, we can replace F with a specific number.

Let's convert the temperature 89 °F to the Celsius scale by evaluating the expression.

$$\begin{aligned}\frac{5}{9}(F-32) &= \frac{5}{9}(89-32)\\ &= \frac{5}{9}(57)\\ &= \frac{285}{9} \approx 31.67\end{aligned}$$

This shows us that 89 °F is equivalent to approximately 31.67 °C.

Warning 2.1.4 Evaluating Versus Solving. The steps in Example 2.1.3 are not considered "solving" anything. "Solving" is a word you might be tempted to use, because in some sense the steps from Example 2.1.3

are "finding an answer." There is a special meaning in algebra for words like "solve" and "solution" that will come soon. Instead, when we substitute a value and compute the result, the proper vocabulary is "evaluating an expression."

Checkpoint 2.1.5. Try evaluating the temperature expression for yourself.

Use the expression $\frac{5}{9}(F-32)$ to evaluate some Celsius temperatures.

a. If a temperature is 50°F what is that temperature measured in Celsius?

b. If a temperature is −20°F what is that temperature measured in Celsius?

Explanation.

a.
$$\begin{aligned}\frac{5}{9}(F-32) &= \frac{5}{9}(50-32)\\ &= \frac{5}{9}(18)\\ &= \frac{5}{1}(2)\\ &= 10\end{aligned}$$

So 50°F is equavalent to 10°C.

b.
$$\begin{aligned}\frac{5}{9}(F-32) &= \frac{5}{9}(-20-32)\\ &= \frac{5}{9}(-52)\\ &= -\frac{260}{9}\\ &\approx -28.89\end{aligned}$$

So −20°F is equavalent to about −28.89°C.

Example 2.1.6 Target heart rate. According to the American Heart Association, a person's maximum heart rate, in beats per minute (bpm), is given by $220-a$, where a is their age in years.

a. Determine the maximum heart rate for someone who is 31 years old.

b. A person's *target* heart rate for moderate exercise is 50% to 70% of their maximum heart rate. If they want to reach 65% of their maximum heart rate during moderate exercise, we'd use the expression $0.65(220-a)$, where a is their age in years. Determine the target heart rate at this 65% level for someone who is 31 years old.

Explanation. Both of these parts ask us to evaluate an expression.

a. Since a is defined to be age in years, we will evaluate this expression by substituting a with 31:

$$\begin{aligned}220-a &= 220-31\\ &= 189\end{aligned}$$

This tells us that the maximum heart rate for someone who is 31 years old is 189 bpm.

b. We'll again substitute a with 31, but this time using the target heart rate expression:

$$\begin{aligned}0.65(220-a) &= 0.65(220-31)\\ &= 0.65(189)\\ &= 122.85\end{aligned}$$

This tells us that the target heart rate for someone who is 31 years old undertaking moderate exercise is 122.85 bpm.

Checkpoint 2.1.7. The target heart rate for moderate exercise is 50% to 70% of maximum heart rate. We can use the expression $\frac{p}{100}(220-a)$ to represent a person's target heart rate when their target rate is p% of their maximum heart rate, and they are a years old.

Determine the target heart rate at the 53% level for moderate exercise for someone who is 56 years old.

At the 53% level, the target heart rate for moderate exercise for someone who is 56 years old is $\boxed{}$ beats per minute.

Explanation.

$$\begin{aligned} \frac{p}{100}(220-a) &= \frac{53}{100}(220-56) \\ &= \frac{53}{100}(164) \\ &= \frac{53}{25}(41) \\ &= 86.92 \end{aligned}$$

At the 53% level, the target heart rate for moderate exercise for someone who is 56 years old is 86.92 beats per minute.

Checkpoint 2.1.8 Rising Rents. An expression estimating the average rent of a one-bedroom apartment in Portland, Oregon, from January, 2011 to October, 2016, is given by $10.173x + 974.78$, where x is the number of months since January, 2011.

a. According to this model, what was the average rent of a one-bedroom apartment in Portland in January, 2011?

b. According to this model, what was the average rent of a one-bedroom apartment in Portland in January, 2016?

Explanation.

a. This model uses x as the number of months after January, 2011. So in January, 2011, x is 0:

$$\begin{aligned} 10.173x + 974.48 &= 10.173(0) + 974.48 \\ &\approx 974.48 \end{aligned}$$

According to this model, the average monthly rent for a one-bedroom apartment in Portland, Oregon, in January, 2011, was \$974.78.

b. The date we are given is January, 2016, which is 5 years after January, 2011. Recall that x is the number of *months* since January, 2011. So we need to use $x = 60$:

$$\begin{aligned} 10.173x + 974.48 &= 10.173(60) + 974.48 \\ &\approx 1584.86 \end{aligned}$$

According to this model, the average monthly rent for a one-bedroom apartment in Portland, Oregon, was \$1584.86 in January 2016.

2.1.3 Evaluating Expressions with Exponents, Absolute Value, and Radicals

Mathematical expressions will often have exponents, absolute value bars, and radicals. This does not change the basic approach to evaluating them.

Example 2.1.9 Tsunami Speed. The speed of a tsunami (in meters per second) can be modeled by $\sqrt{9.8d}$, where d is the depth of the tsunami (in meters). Determine the speed of a tsunami that has a depth of 30 m to four significant digits.

Explanation. Using $d = 30$, we find:

$$\begin{aligned}\sqrt{9.8d} &= \sqrt{9.8(30)}\\ &= \sqrt{294}\\ &\approx \overbrace{17.14}^{\text{four}}6428\ldots\end{aligned}$$

The speed of tsunami with a depth of 30 m is about 17.15 $\frac{\text{m}}{\text{s}}$.

Up to now, we have been evaluating expressions, but we can evaluate formulas in the same way. A **formula** usually has a single variable that represents the output of an expression. For example, the expression for a person's maximum heart rate in beats per minute, $220 - a$, can be written as the formula, $H = 220 - a$. When we substitute a value for a we are evaluating the formula. Even though we have an equation, we are not solving yet. That will come soon.

Checkpoint 2.1.10 Tent Height. While camping, the height (in feet) inside a tent when you are d ft from the north side of the tent is given by the formula $h = -2\,|d-3| + 6$.

a. When you are 5 ft from the north side, the height will be ______.

b. When you are 2.5 ft from the north side, the height will be ______.

Explanation.

a. When $d = 5$, we have:

$$\begin{aligned}h &= -2\,|d-3| + 6\\ &= -2\,|5-3| + 6\\ &= -2\,|2| + 6\\ &= -2(2) + 6\\ &= -4 + 6\\ &= 2\end{aligned}$$

Thus when you are 5 ft from the north side, the height in the tent is 2 ft.

b. When $d = 2.5$, we have:

$$\begin{aligned}h &= -2\,|d-3| + 6\\ &= -2\,|2.5-3| + 6\\ &= -2\,|-0.5| + 6\\ &= -2(0.5) + 6\\ &= -1 + 6\\ &= 5\end{aligned}$$

Thus when you are 2.5 ft from the north side, the height in the tent is 5 ft.

Checkpoint 2.1.11 Mortgage Payments. If we borrow L dollars for a home mortgage loan at an annual interest rate r, and intend to pay off the loan after n months, then the amount we should pay each month M, in dollars, is given by the formula

$$M = \frac{rL\left(1+\frac{r}{12}\right)^n}{12\left(\left(1+\frac{r}{12}\right)^n - 1\right)}$$

If we borrow \$200,000 at an interest rate of 6% with the intent to pay off the loan in 30 years, what should our monthly payment be? (Using a calculator is appropriate here.)

Explanation. We must use $L = 200000$. Because the interest rate is a percentage, $r = 0.06$ (not 6). The variable n is supposed to be a number on months, but we will pay off the loan in 30 years. Therefore we take $n = 360$.

$$\begin{aligned} M = \frac{rL\left(1+\frac{r}{12}\right)^n}{12\left(\left(1+\frac{r}{12}\right)^n-1\right)} &= \frac{(0.06)(200000)\left(1+\frac{0.06}{12}\right)^{360}}{12\left(\left(1+\frac{0.06}{12}\right)^{360}-1\right)} \\ &= \frac{(0.06)(200000)(1+0.005)^{360}}{12\left((1+0.005)^{360}-1\right)} \\ &\approx \frac{(0.06)(200000)(6.022575\ldots)}{12\left(6.022575\ldots-1\right)} \\ &\approx \frac{(0.06)(200000)(6.022575\ldots)}{12(5.022575\ldots)} \\ &\approx \frac{72270.90\ldots}{60.2709\ldots} \\ &\approx 1199.10 \end{aligned}$$

Our monthly payment should be \$1,199.10.

Warning 2.1.12 Evaluating Expressions with Negative Numbers. When we substitute negative numbers into an expression, it's important to use parentheses around them or else it's easy to forget that a *negative* number is being raised to a power. Let's look at some examples.

Example 2.1.13 Evaluate x^2 if $x = -2$.

We substitute:

$$\begin{aligned} x^2 &= (-2)^2 \\ &= 4 \end{aligned}$$

If we don't use parentheses, we would have:

$$\begin{aligned} x^2 &= -2^2 && \text{incorrect!} \\ &= -4 \end{aligned}$$

The original expression takes x and squares it. With $-2^2 = -4$, the number -2 is not being squared. Since the exponent has higher priority than the negation, it's just the number 2 that is being squared. With $(-2)^2 = 4$ the number -2 *is* being squared, which is what we would want given the expression x^2.

So it is wise to always use some parentheses when substituting in any negative number.

Checkpoint 2.1.14. Evaluate and simplify the following expressions for $x = -5$ and $y = -2$:

a. $x^3y^2 =$ ☐

b. $(-2x)^3 =$ ☐

c. $-3x^2y =$ ☐

Explanation.

a. $$\begin{aligned} x^3y^2 &= (-5)^3(-2)^2 \\ &= (-125)(4) \\ &= -500 \end{aligned}$$

b. $$\begin{aligned} (-2x)^3 &= (-2(-5))^3 \\ &= (10)^3 \\ &= 1000 \end{aligned}$$

c. $$\begin{aligned} -3x^2y &= -3(-5)^2(-2) \\ &= -3(25)(-2) \\ &= 150 \end{aligned}$$

Exercises

Evaluating Expressions

1. Evaluate $x - 10$ for $x = 6$.

2. Evaluate $x + 4$ for $x = 9$.

3. Evaluate $-4 - x$ for $x = -10$.

4. Evaluate $10 - x$ for $x = -8$.

5. Evaluate $3x + 6$ for $x = -5$.

6. Evaluate $-5x - 6$ for $x = -3$.

7. Evaluate $-7c$ for $c = 9$.

8. Evaluate $-2B$ for $B = 2$.

9. Evaluate the expression r^2:

a. When $r = 3$, $r^2 = \boxed{}$

b. When $r = -5$, $r^2 = \boxed{}$

10. Evaluate the expression t^2:

a. When $t = 9$, $t^2 = \boxed{}$

b. When $t = -9$, $t^2 = \boxed{}$

11. Evaluate the expression t^3:

a. When $t = 4$, $t^3 = \boxed{}$

b. When $t = -3$, $t^3 = \boxed{}$

12. Evaluate the expression x^3:

a. When $x = 2$, $x^3 = \boxed{}$

b. When $x = -4$, $x^3 = \boxed{}$

13. Evaluate the following expressions.

a. Evaluate $5x^2$ when $x = 2$. $5x^2 = \boxed{}$

b. Evaluate $(5x)^2$ when $x = 2$. $(5x)^2 = \boxed{}$

14. Evaluate the following expressions.

a. Evaluate $3x^2$ when $x = 2$. $3x^2 = \boxed{}$

b. Evaluate $(3x)^2$ when $x = 2$. $(3x)^2 = \boxed{}$

15. Evaluate $-\left(y + 2\right)$ for $y = -7$.

16. Evaluate $-7\left(y + 9\right)$ for $y = 6$.

17. Evaluate $\dfrac{9r - 5}{3r}$ for $r = -1$.

18. Evaluate $\dfrac{3r - 7}{3r}$ for $r = -8$.

19. Evaluate $-9B + 9A$ for $B = -10$ and $A = -6$.

20. Evaluate $-2C - c$ for $C = -9$ and $c = 5$.

21. Evaluate $\dfrac{-5}{x} - \dfrac{4}{a}$ for $x = 7$ and $a = -3$.

22. Evaluate $\dfrac{-5}{y} - \dfrac{8}{B}$ for $y = 9$ and $B = -5$.

23. Evaluate $\dfrac{-7t+9c-3}{7t-2c}$ for $t=-4$ and $c=9$.

24. Evaluate $\dfrac{-2a-B+9}{-8a+7B}$ for $a=3$ and $B=-7$.

25. Evaluate the expression $\frac{1}{7}(x+2)^2-7$ when $x=-9$.

26. Evaluate the expression $\frac{1}{4}(x+3)^2-4$ when $x=-7$.

27. Evaluate the expression $\frac{1}{2}h(B+b)$ when $h=10,\ B=8,\ b=7$.

28. Evaluate the expression $\frac{1}{2}h(B+b)$ when $h=12,\ B=6,\ b=5$.

29. Evaluate the expression $-16t^2+64t+128$ when $t=3$.

30. Evaluate the expression $-16t^2+64t+128$ when $t=-5$.

31. Evaluate the following expressions.

a. Evaluate x^2r^3 when $x=-3$ and $r=-1$.

$x^2r^3=$ ☐

b. Evaluate x^3r^2 when $x=-3$ and $r=-1$.

$x^3r^2=$ ☐

32. Evaluate the following expressions.

a. Evaluate x^2y^3 when $x=-1$ and $y=-2$.

$x^2y^3=$ ☐

b. Evaluate x^3y^2 when $x=-1$ and $y=-2$.

$x^3y^2=$ ☐

33. Evaluate the following expressions.

a. Evaluate $(-3y)^2$ when $y=-1$.

$(-3y)^2=$ ☐

b. Evaluate $(-3y)^3$ when $y=-1$.

$(-3y)^3=$ ☐

34. Evaluate the following expressions.

a. Evaluate $(-y)^2$ when $y=-2$.

$(-y)^2=$ ☐

b. Evaluate $(-y)^3$ when $y=-2$.

$(-y)^3=$ ☐

35. Evaluate each algebraic expression for the given value(s):

$\dfrac{y^3+\sqrt{x-4}}{|2x-y|}$, for $x=104$ and $y=-4$: ☐

36. Evaluate each algebraic expression for the given value(s):

$\dfrac{y^3+\sqrt{x-4}}{|5x-y|}$, for $x=29$ and $y=7$: ☐

37. Evaluate each algebraic expression for the given value(s):

$\dfrac{\sqrt{x}}{y}-\dfrac{y}{x}$, for $x=81$ and $y=3$: ☐

38. Evaluate each algebraic expression for the given value(s):

$\dfrac{\sqrt{x}}{y}-\dfrac{y}{x}$, for $x=100$ and $y=-6$: ☐

39. Evaluate

$$\frac{y_2 - y_1}{x_2 - x_1}$$

for $x_1 = -20$, $x_2 = 13$, $y_1 = 19$, and $y_2 = -1$:

40. Evaluate

$$\frac{y_2 - y_1}{x_2 - x_1}$$

for $x_1 = -15$, $x_2 = -2$, $y_1 = -5$, and $y_2 = -12$:

41. Evaluate

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

for $x_1 = 8$, $x_2 = 4$, $y_1 = 8$, and $y_2 = 11$:

42. Evaluate

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

for $x_1 = -2$, $x_2 = 4$, $y_1 = -8$, and $y_2 = -16$:

43. Evaluate the algebraic expression $3a + b$ for $a = \frac{5}{7}$ and $b = \frac{4}{9}$.

44. Evaluate the algebraic expression $-8a + b$ for $a = \frac{6}{5}$ and $b = \frac{1}{2}$.

45. Evaluate each algebraic expression for the given value(s):

$\frac{5 + 5|y - x|}{x + 5y}$, for $x = 6$ and $y = -3$:

46. Evaluate each algebraic expression for the given value(s):

$\frac{4 + 5|y - x|}{x + 3y}$, for $x = 11$ and $y = -3$:

To convert a temperature measured in degrees Fahrenheit to degrees Celsius, there is a formula:

$$C = \frac{5}{9}(F - 32)$$

where C represents the temperature in degrees Celsius and F represents the temperature in degrees Fahrenheit.

47. If a temperature is 113°F, what is that temperature measured in Celsius?

48. If a temperature is 5°F, what is that temperature measured in Celsius?

49. If a temperature is 14°F, what is that temperature measured in Celsius?

50. If a temperature is 122°F, what is that temperature measured in Celsius?

A formula for converting meters into feet is

$$F = 3.28M$$

where M is a number of meters, and F is the corresponding number of feet.

51. Use the formula to find the number of feet that corresponds to fourteen meters.

feet corresponds to fourteen meters.

52. Use the formula to find the number of feet that corresponds to eight meters.

feet corresponds to eight meters.

The formula

$$y = \frac{1}{2} a t^2 + v_0 t + y_0$$

gives the vertical position of an object, at time t, thrown with an initial velocity v_0, from an initial position y_0 in a place where the acceleration of gravity is a. The acceleration of gravity on earth is $-9.8 \frac{\text{m}}{\text{s}^2}$. It is negative, because we consider the upward direction as positive in this situation, and gravity pulls down.

53. What is the height of a baseball thrown with an initial velocity of $v_0 = 78 \frac{\text{m}}{\text{s}}$, from an initial position of $y_0 = 97$ m, and at time $t = 14$ s?

Fourteen seconds after the baseball was thrown, it was $\boxed{}$ high in the air.

54. What is the height of a baseball thrown with an initial velocity of $v_0 = 84 \frac{\text{m}}{\text{s}}$, from an initial position of $y_0 = 80$ m, and at time $t = 5$ s?

Five seconds after the baseball was thrown, it was $\boxed{}$ high in the air.

The percentage of births in the U.S. delivered via C-section can be given by the following formula for the years since 1996:

$$p = 0.8(y - 1996) + 21$$

In this formula y is a year after 1996 and p is the percentage of births delivered via C-section for that year.

55. What percentage of births in the U.S. were delivered via C-section in the year 2010?

$\boxed{}$ of births in the U.S. were delivered via C-section in the year 2010.

56. What percentage of births in the U.S. were delivered via C-section in the year 2012?

$\boxed{}$ of births in the U.S. were delivered via C-section in the year 2012.

Target heart rate for moderate exercise is 50% to 70% of maximum heart rate. If we want to represent a certain percent of an individual's maximum heart rate, we'd use the formula

$$\text{rate} = p(220 - a)$$

where p is the percent, and a is age in years.

57. Determine the target heart rate at 51% level for someone who is 43 years old. Round your answer to an integer.

The target heart rate at 51% level for someone who is 43 years old is $\boxed{}$ beats per minute.

58. Determine the target heart rate at 53% level for someone who is 22 years old. Round your answer to an integer.

The target heart rate at 53% level for someone who is 22 years old is $\boxed{}$ beats per minute.

The diagonal length (D) of a rectangle with side lengths L and W is given by:

$$D = \sqrt{L^2 + W^2}$$

59. Determine the diagonal length of rectangles with $L = 6$ ft and $W = 8$ ft.

The diagonal length of rectangles with $L = 6$ ft and $W = 8$ ft is $\boxed{}$.

60. Determine the diagonal length of rectangles with $L = 9$ ft and $W = 12$ ft.

The diagonal length of rectangles with $L = 9$ ft and $W = 12$ ft is $\boxed{}$.

61. The height inside a camping tent when you are d feet from the edge of the tent is given by

$$h = -1.1|d - 5.4| + 6$$

where h stands for height in feet.

Determine the height when you are:

a. 6.5 ft from the edge.

The height inside a camping tent when you 6.5 ft from the edge of the tent is ☐

b. 4.4 ft from the edge.

The height inside a camping tent when you 4.4 ft from the edge of the tent is ☐

62. The height inside a camping tent when you are d feet from the edge of the tent is given by

$$h = -0.6|d - 5.8| + 5.5$$

where h stands for height in feet.

Determine the height when you are:

a. 9.7 ft from the edge.

The height inside a camping tent when you 9.7 ft from the edge of the tent is ☐

b. 2.9 ft from the edge.

The height inside a camping tent when you 2.9 ft from the edge of the tent is ☐

2.2 Geometry Formulas

Two- and three- dimensional shapes provide some formulas with variables that we can evaluate.

2.2.1 Evaluating Perimeter and Area Formulas

Rectangles The rectangle in Figure 2.2.2 has a length (as measured by the edges on the top and bottom) and a width (as measured by the edges on the left and right).

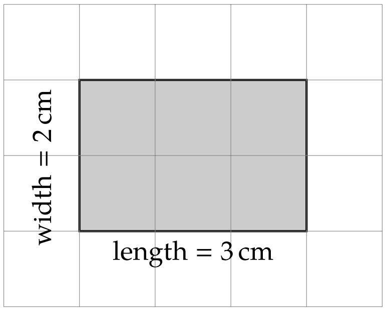


Figure 2.2.2: A Rectangle

Perimeter is the distance around the edge(s) of a two-dimensional shape. To calculate perimeter, start from a point on the shape (usually a corner), travel around the shape, and add up the total distance traveled. For the rectangle in the Figure 2.2.2, if we travel around it, the total distance would be:

$$\begin{aligned} \text{rectangle perimeter} &= 3\,\text{cm} + 2\,\text{cm} + 3\,\text{cm} + 2\,\text{cm} \\ &= 10\,\text{cm}. \end{aligned}$$

Another way to compute a rectangle's perimeter would be to start at one corner, add up the edge length half-way around, and then double that. So we could have calculated the perimeter this way:

$$\begin{aligned} \text{rectangle perimeter} &= 2(3\,\text{cm} + 2\,\text{cm}) \\ &= 2(5\,\text{cm}) \\ &= 10\,\text{cm}. \end{aligned}$$

There is nothing special about this rectangle having length 3 cm and width 2 cm. With a generic rectangle, it has some length we can represent with the variable ℓ and some width we can represent with the variable w. We can use P to represent its perimeter, and then the perimeter of the rectangle will be given by:

$$P = 2(\ell + w).$$

Area is the number of 1×1 squares that fit inside a two-dimensional shape (possibly after morphing them into non-square shapes). If the edges of the squares are, say, 1 cm long, then the area is measured in "square cm," written cm^2. In Figure 2.2.2, the rectangle has six 1 cm × 1 cm squares, so its area is 6 square centimeters.

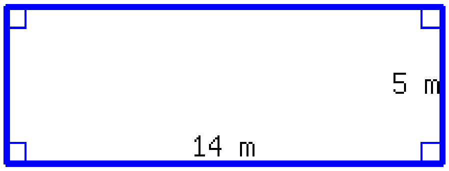
Note that we can find that area by multiplying the length and the width:

$$\begin{aligned} \text{rectangle area} &= (3\,\text{cm}) \cdot (2\,\text{cm}) \\ &= 6\,\text{cm}^2 \end{aligned}$$

Again, there is nothing special about this rectangle having length 3 cm and width 2 cm. With a generic rectangle, it has some length we can represent with the variable ℓ and some width we can represent with the variable w. We can represent its area with the variable A, and then the area of the rectangle will be given by:

$$A = \ell \cdot w.$$

Checkpoint 2.2.3. Find the perimeter and area of the rectangle.



Its perimeter is ☐ and its area is ☐.

Explanation. Using the perimeter and area formulas for a rectangle, we have:

$$\begin{aligned} P &= 2(\ell + w) & A &= \ell \cdot w \\ &= 2(14 + 10) & &= 14 \cdot 10 \\ &= 2(24) & &= 140 \\ &= 48 & & \end{aligned}$$

Since length and width were measured in meters, we find that the perimeter is 28 meters and the area is 140 square meters.

Triangles The perimeter of a general triangle has no special formula — all that is needed is to add the lengths of its three sides. The *area* of a triangle is a bit more interesting. In Figure 2.2.4, there are three triangles. From left to right, there is an acute triangle, a right triangle, and an obtuse triangle. Each triangle is drawn so that there is a "bottom" horizontal edge. This edge is referred to as the "base" of the triangle. With each triangle, a "height" that is perpendicular to the base is also illustrated.

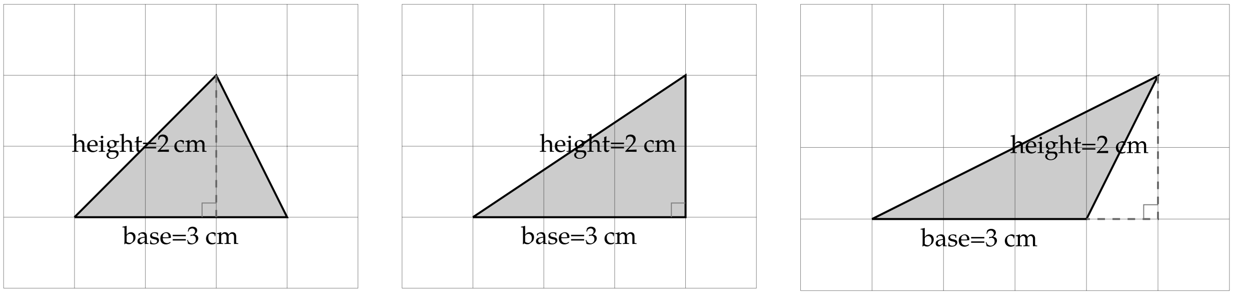


Figure 2.2.4: Triangles

Each of these triangles has the same base width, 3 cm, and the same height, 2 cm. Note that they each have the same area as well. Figure 2.2.5 illustrates how they each have an area of 3 cm^2.

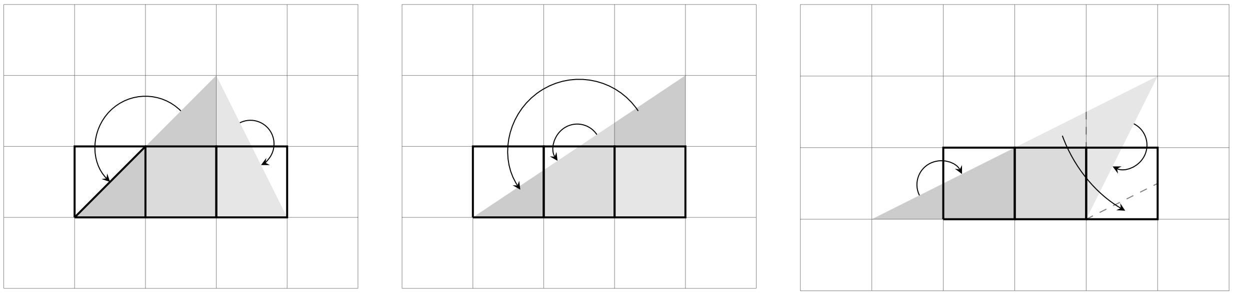
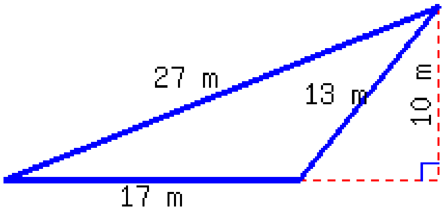


Figure 2.2.5: Triangles

As with the triangles in Figure 2.2.5, you can always rearrange little pieces of a triangle so that the resulting shape is a rectangle with the same base width, but with a height that's one-half of the triangle's height. With a generic rectangle, it has some base width we can represent with the variable b and some height we can represent with the variable h. We can represent its area with the variable A, and then the area of the triangle will be given by $A = b \cdot \left(\frac{1}{2}h\right)$, or more conventionally:

$$A = \frac{1}{2}bh.$$

Checkpoint 2.2.6. Find the perimeter and area of the triangle.



Its perimeter is ______ and its area is ______.

Explanation. For perimeter, we just add the three side lengths:

$$\begin{aligned} P &= 13 + 25 + 33 \\ &= 71 \end{aligned}$$

For area, we use the triangle area formula:

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(13)(22) \\ &= 13(11) \\ &= 143 \end{aligned}$$

Since length and width were measured in meters, we find that the perimeter is 71 meters and the area is 143 square meters.

Circles To find formulas for the perimeter and area of a circle, it helps to first know that there is a special number called π (spelled "pi" and pronounced like "pie") that appears in many places in mathematics. The decimal value of π is about $3.14159265\ldots$, and it helps to memorize some of these digits. It also helps to understand that π is a little larger than 3. There are many definitions for π that can explain where it comes from and how you can find all its decimal places, but here we are just going to accept that it is a special number, and it is roughly $3.14159265\ldots$.

The perimeter of a circle is the distance around its edge. For circles, the perimeter has a special name: the **circumference**. Imagine wrapping a string around the circle and cutting it so that it makes one complete loop. If we straighten out that piece of string, we have a length that is just as long as the circle's circumference.

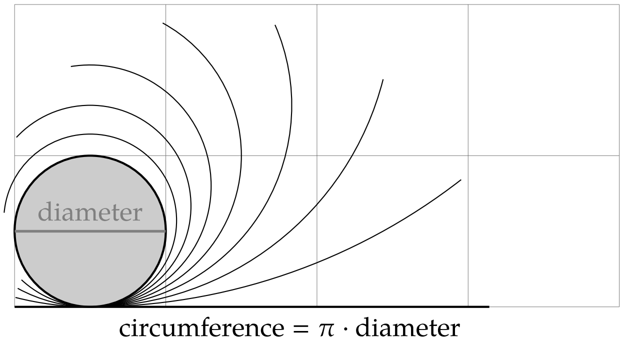


Figure 2.2.7: Circle Diameter and Circumference

As we can see in Figure 2.2.7, the circumference of a circle is a little more than three times as long as its diameter. (The diameter of a circle is the length of a straight line running from a point on the edge through the center to the opposite edge.) In fact, the circumference is actually exactly π times the length of the diameter. With a generic circle, it has some diameter we can represent with the variable d. We can represent its circumference with the variable c, and then the circumference of the circle will be given by:

$$c = \pi d.$$

Alternatively, we often prefer to work with a circle's **radius** instead of its diameter. The radius is the distance from any point on the circle's edge to its center. (Note that the radius is half the diameter.) From this perspective, we can see in Figure 2.2.8 that the circumference is a little more than 6 times the radius.

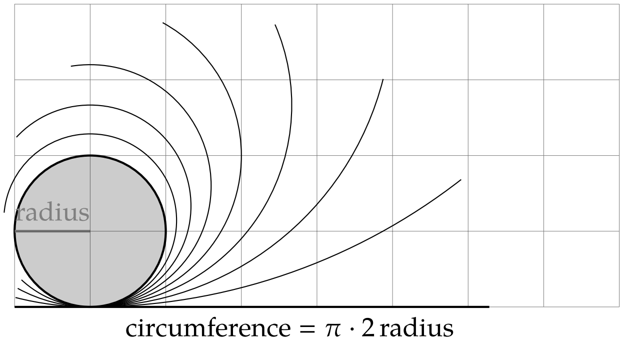


Figure 2.2.8: Circle Diameter and Circumference

This gives us another formula for a circle's circumerence that uses the variable r for its radius: $c = \pi \cdot 2r$.

Or more conventionally,

$$c = 2\pi r.$$

There is also a formula for the *area* of a circle based on its radius. Figure 2.2.9 shows how three squares can be cut up and rearranged to fit inside a circle. This shows how the area of a circle of radius r is just a little larger than $3r^2$. Since π is just a little larger than 3, could it be that the area of a circle is given by πr^2?

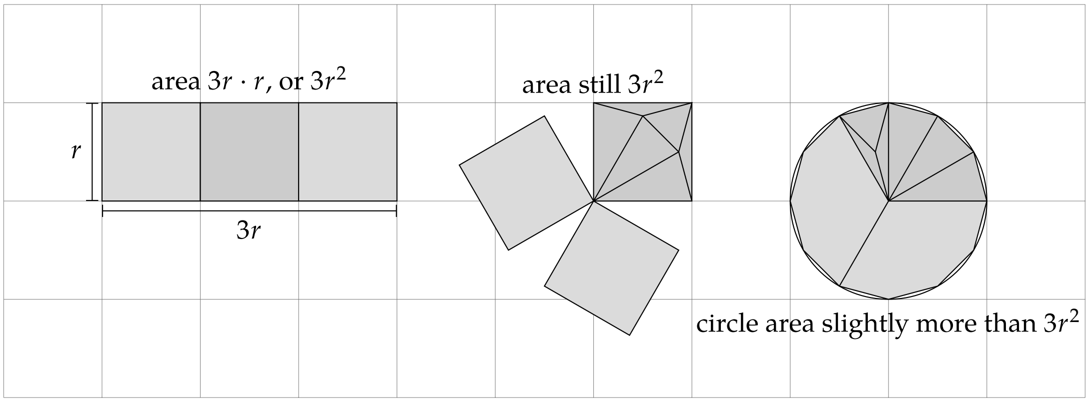

area $3r \cdot r$, or $3r^2$
r
$3r$
area still $3r^2$
circle area slightly more than $3r^2$

Figure 2.2.9: Circle area is slightly larger than $3r^2$.

One way to establish this formula is to imagine slicing up the circle into many pie slices as in Figure 2.2.10. Then you can rearrange the slices into a strange shape that is *almost* a rectangle with height equal to the radius of the original circle, and width equal to half the circumference of the original circle.

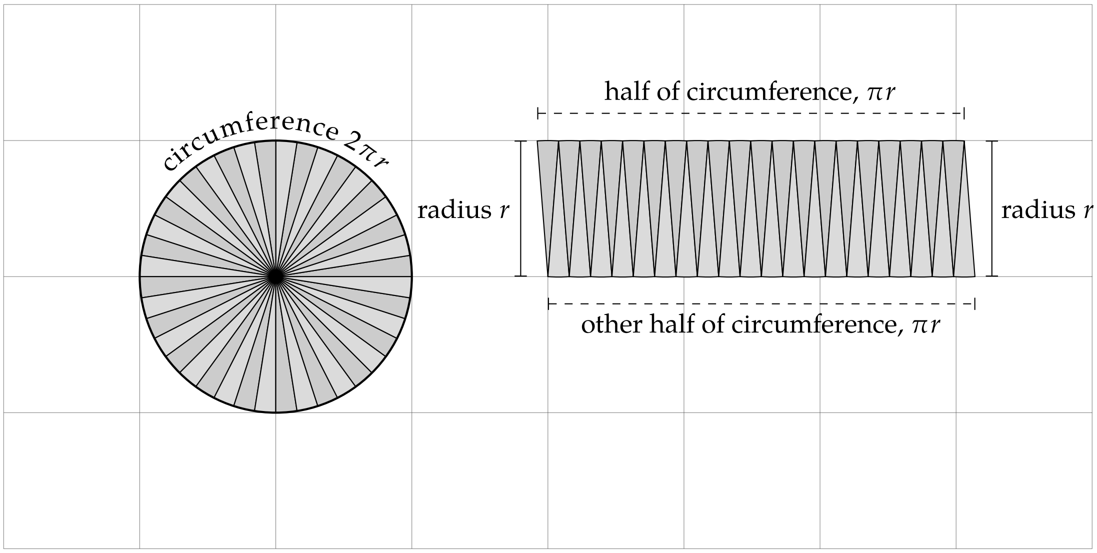

circumference $2\pi r$
radius r
half of circumference, πr
radius r
other half of circumference, πr

Figure 2.2.10: Reasoning the circle area formula.

Since the area of the circle is equal to the area of the almost-rectangular shape in Figure 2.2.10, we have the circle area formula:

$$A = \pi r^2.$$

Checkpoint 2.2.11. A circle's diameter is 6 m.

a. This circle's circumference, in terms of π, is ____.

b. This circle's circumference, rounded to the hundredth place, is ____.

c. This circle's area, in terms of π, is ____.

d. This circle's area, rounded to the hundredth place, is ____.

Explanation. We use r to represent radius and d to represent diameter. In this problem, it's given that the diameter is 6 m. A circle's radius is half as long as its diameter, so the radius is 3 m.

Throughout these computations, all quantities have units attached, but we only show them in the final step.

a. $$\begin{aligned} c &= \pi d \\ &= \pi \cdot 6 \\ &= 6\pi \text{ m} \end{aligned}$$

b. $$\begin{aligned} c &= \pi d \\ &\approx 3.1415926 \cdot 6 \\ &\approx 18.85 \text{ m} \end{aligned}$$

c. $$\begin{aligned} A &= \pi r^2 \\ &= \pi \cdot 3^2 \\ &= \pi \cdot 9 \\ &= 9\pi \text{ m}^2 \end{aligned}$$

d. $$\begin{aligned} A &= \pi r^2 \\ &\approx 3.1415926 \cdot 3^2 \\ &\approx 3.1415926 \cdot 9 \\ &\approx 28.27 \text{ m}^2 \end{aligned}$$

2.2.2 Volume

The **volume** of a three-dimensional object is the number of $1 \times 1 \times 1$ cubes that fit inside the object (possibly after morphing them into non-cube shapes). If the edges of the cubes are, say, 1 cm long, then the volume is measured in "cubic cm," written cm^3.

Rectangular Prisms The 3D shape in Figure 2.2.12 is called a rectangular prism.

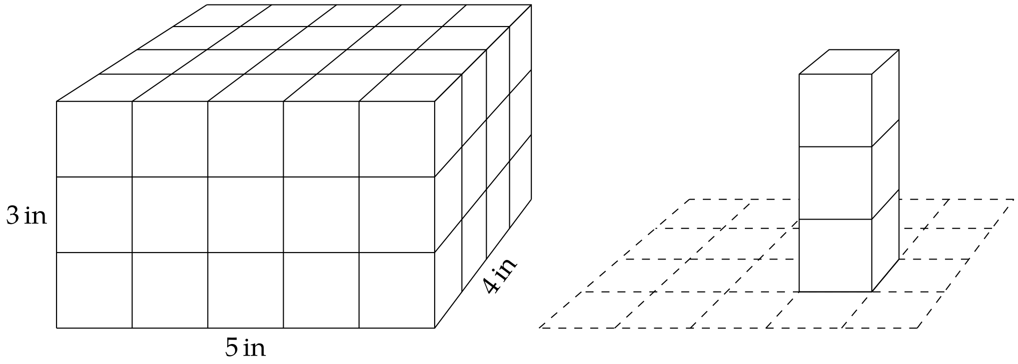


Figure 2.2.12: Volume of a Rectangular Prism

The rectangular prism in Figure 2.2.12 is composed of 1 in × 1 in × 1 in unit cubes, with each cube's volume being 1 cubic inch (or in^3). The shape's volume is the number of such unit cubes. The bottom face has

$5 \cdot 4 = 20$ unit squares. Since there are 3 layers of cubes, the shape has a total of $3 \cdot 20 = 60$ unit cubes. In other words, the shape's volume is $60\,\text{in}^3$ because it has sixty $1\,\text{in} \times 1\,\text{in} \times 1\,\text{in}$ cubes inside it.

We found the number of unit squares in the bottom face by multiplying $5 \cdot 4 = 20$. Then to find the volume, we multiplied by 3 because there are three layers of cubes. So one formula for a prism's volume is

$$V = wdh$$

where V stands for volume, w for width, d for depth, and h for height.

Checkpoint 2.2.13. A masonry brick is in the shape of a rectangular prism and is 8 inches wide, 3.5 inches deep, and 2.25 inches high. What is its volume?

Explanation. Using the formula for the volume of a rectangular prism:

$$\begin{aligned} V &= wdh \\ &= 8(3.5)(2.25) \\ &= 63 \end{aligned}$$

So the brick's volume is 63 cubic inches.

Cylinders A cylinder is not a prism, but it has some similarities. Instead of a square base, the base is a circle. Its volume can also be calculated in a similar way to how prism volume is calculated. Let's look at an example.

Example 2.2.14 Find the volume of a cylinder with a radius of 3 meters and a height of 2 meters.

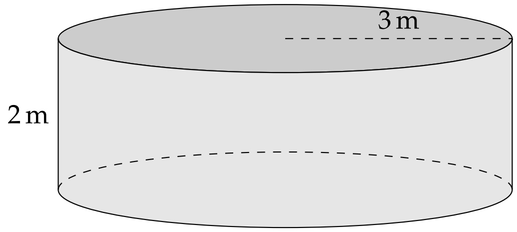


Figure 2.2.15: A Cylinder

Explanation. The base of the cylinder is a circle. We know the area of a circle is given by the formula $A = \pi r^2$, so the base area is $9\pi\,\text{m}^2$, or about $28.27\,\text{m}^2$. That means about 28.27 unit squares can fit into the base. One of them is drawn in Figure 2.2.16 along with two unit cubes above it.

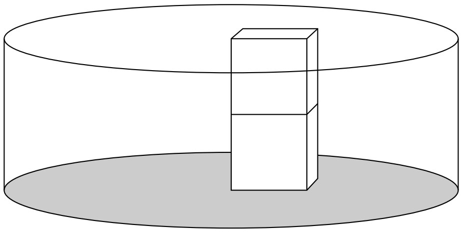


Figure 2.2.16: Finding Cylinder Volume

> For each unit square in the base circle, there are two unit cubes of volume. So the volume is the base area times the height: $9\pi\,\mathrm{m}^2 \cdot 2\,\mathrm{m}$, which equals $18\pi\,\mathrm{m}^3$. Approximating π with a decimal value, this is about $56.55\,\mathrm{m}^3$.

Example 2.2.14 demonstrates that the volume of a cylinder can be calculated with the formula

$$V = \pi r^2 h$$

where r is the radius and h is the height.

Checkpoint 2.2.17. A soda can is basically in the shape of a cylinder with radius 1.3 inches and height 4.8 inches. What is its volume?

Its exact volume in terms of π is: ______.

As a decimal approximation rounded to four significant digits, its volume is: ______.

Explanation. Using the formula for the volume of a cylinder:

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(1.3)^2(4.8) \\ &= 8.112\pi \\ &\approx 25.48 \end{aligned}$$

So the can's volume is 8.112π cubic inches, which is about 25.48 cubic inches.

Note that the volume formulas for a rectangular prism and a cylinder have something in common: both formulas first find the area of the base (which is a rectangle for a prism and a circle for a cylinder) and then multiply by the height. So there is another formula

$$V = Bh$$

that works for both shapes. Here, B stands for the base area (which is wd for a prism and πr^2 for a cylinder.)

2.2.3 Summary

You may be required to memorize the geometry formulas that have been cataloged in this section. Check that you have each of them memorized here.

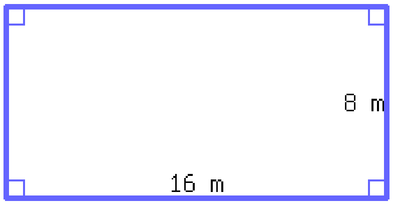
Checkpoint 2.2.18. Fill out the table with various formulas as they were given in this section.

Rectangle Perimeter ______
Rectangle Area ______
Triangle Area ______
Circle Circumference ______
Circle Area ______
Rectangular Prism Volume ______
Cylinder Volume ______
Volume of either Rectangular Prism or Cylinder ______

Exercises

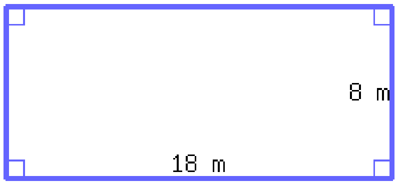
Perimeter and Area

1. Find the perimeter and area of the rectangle.



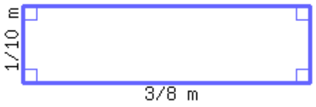
Its perimeter is and its area is .

2. Find the perimeter and area of the rectangle.

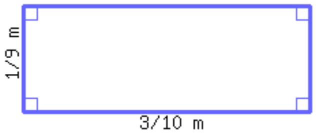


Its perimeter is and its area is .

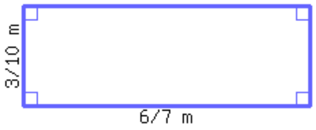
3. Find the perimeter of the rectangle below.



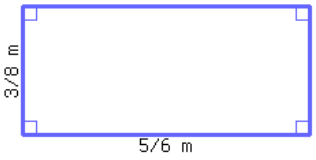
4. Find the perimeter of the rectangle below.



5. Find the area of the rectangle below.



6. Find the area of the rectangle below.



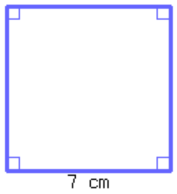
7. Find the perimeter and area of a rectangular table top with a length of 4 ft and a width of 27 in.

Its perimeter is and its area is .

8. Find the perimeter and area of a rectangular table top with a length of 4.2 ft and a width of 36 in.

Its perimeter is and its area is .

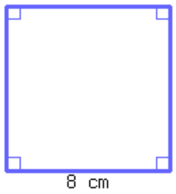
9. Find the perimeter and area of the square.



a. The square's perimeter is .

b. The square's area is .

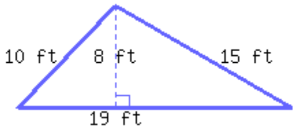
10. Find the perimeter and area of the square.



a. The square's perimeter is .

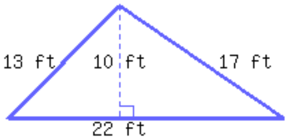
b. The square's area is .

11. Find the perimeter and area of the triangle.


10 ft
8 ft
15 ft
19 ft

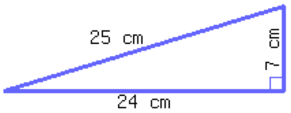
Its perimeter is
and its area is .

12. Find the perimeter and area of the triangle.


13 ft
10 ft
17 ft
22 ft

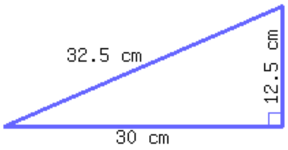
Its perimeter is
and its area is .

13. Find the perimeter and area of the right triangle.


25 cm
7 cm
24 cm

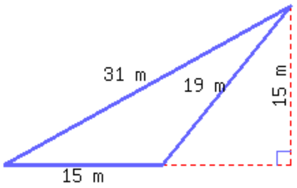
Its perimeter is
and its area is .

14. Find the perimeter and area of the right triangle.


32.5 cm
12.5 cm
30 cm

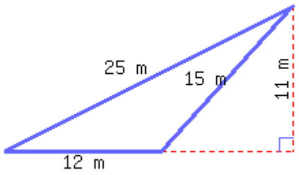
Its perimeter is
and its area is .

15. Find the perimeter and area of the triangle.


31 m
19 m
15 m
15 m

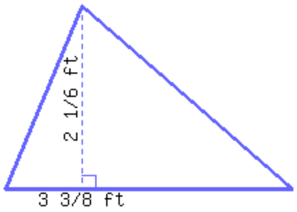
Its perimeter is
and its area is .

16. Find the perimeter and area of the triangle.

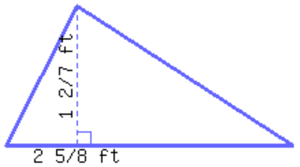

25 m
15 m
11 m
12 m

Its perimeter is
and its area is .

17. The area of the triangle below is
square feet.


2 1/6 ft
3 3/8 ft

18. The area of the triangle below is square feet.



19. Find the area of a triangular flag with a base of 2.6 m and a height of 60 cm.

Its area is .

20. Find the area of a triangular flag with a base of 2.9 m and a height of 130 cm.

Its area is .

21. A circle's radius is 6 m.

a. This circle's circumference, in terms of π, is .

b. This circle's circumference, rounded to the hundredth place, is .

c. This circle's area, in terms of π, is .

d. This circle's area, rounded to the hundredth place, is .

22. A circle's radius is 7 m.

a. This circle's circumference, in terms of π, is .

b. This circle's circumference, rounded to the hundredth place, is .

c. This circle's area, in terms of π, is .

d. This circle's area, rounded to the hundredth place, is .

23. A circle's diameter is 16 m.

a. This circle's circumference, in terms of π, is .

b. This circle's circumference, rounded to the hundredth place, is .

c. This circle's area, in terms of π, is .

d. This circle's area, rounded to the hundredth place, is .

24. A circle's diameter is 18 m.

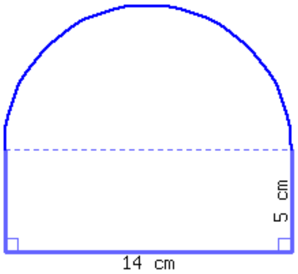
a. This circle's circumference, in terms of π, is .

b. This circle's circumference, rounded to the hundredth place, is .

c. This circle's area, in terms of π, is .

d. This circle's area, rounded to the hundredth place, is .

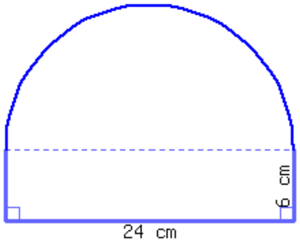
25. Find the perimeter and area of this shape, which is a semicircle on top of a rectangle.



Its perimeter is

and its area is .

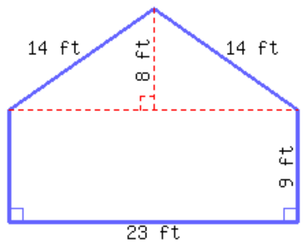
26. Find the perimeter and area of this shape, which is a semicircle on top of a rectangle.



Its perimeter is

and its area is .

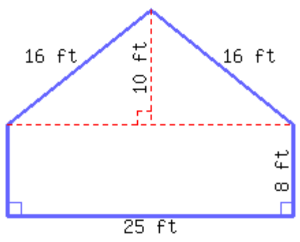
27. Find the perimeter and area of this polygon.


14 ft
14 ft
8 ft
9 ft
23 ft

Its perimeter is

and its area is .

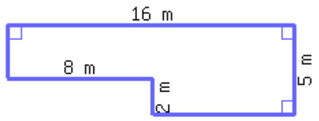
28. Find the perimeter and area of this polygon.


16 ft
16 ft
10 ft
8 ft
25 ft

Its perimeter is

and its area is .

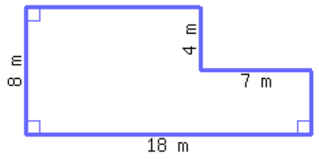
29. Find the perimeter and area of this shape.



Its perimeter is

and its area is .

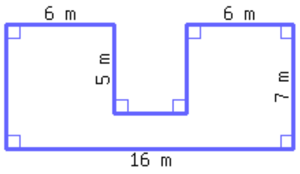
30. Find the perimeter and area of this shape.



Its perimeter is

and its area is .

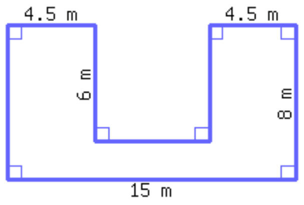
31. Find the perimeter and area of this polygon.


6 m
6 m
5 m
7 m
16 m

Its perimeter is

and its area is .

32. Find the perimeter and area of this polygon.


4.5 m
4.5 m
6 m
8 m
15 m

Its perimeter is

and its area is .

The formula

$$A = \frac{1}{2} r\, n\, s$$

gives the area of a regular polygon with side length s, number of sides n and, apothem r. (The *apothem* is the distance from the center of the polygon to one of its sides.)

33. What is the area of a regular pentagon with $s = 90$ in and $r = 82$ in?

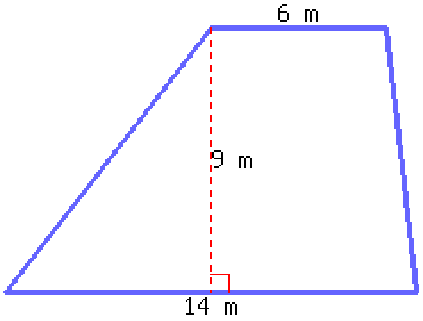
The area is .

34. What is the area of a regular 28-gon with $s = 99$ in and $r = 90$ in?

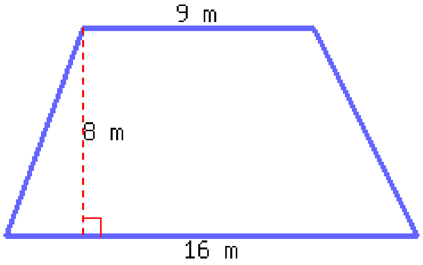
The area is .

A trapezoid's area can be calculated by the formula $A = \frac{1}{2}(b_1 + b_2)h$, where A stands for area, b_1 for the first base's length, b_2 for the second base's length, and h for height.

35. Find the area of the trapezoid below.

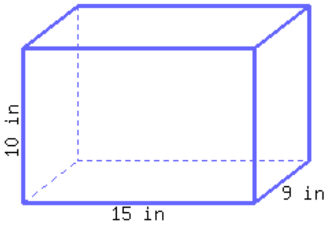


36. Find the area of the trapezoid below.

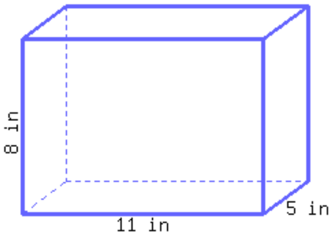


Volume

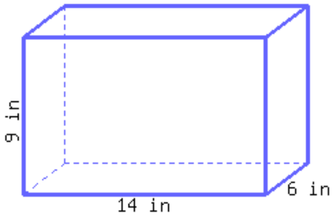
37. Find the volume of this rectangular prism.


10 in
15 in
9 in

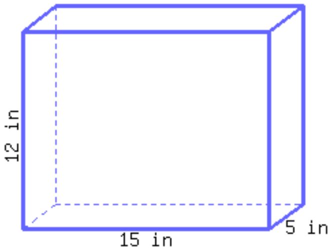
38. Find the volume of this rectangular prism.


8 in
11 in
5 in

39. Find the volume of this rectangular prism.


9 in
14 in
6 in

40. Find the volume of this rectangular prism.


12 in
15 in
5 in

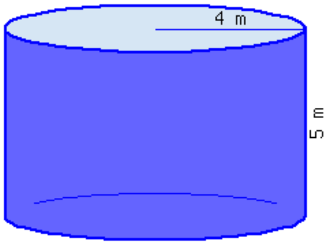
41. A cube's side length is 8 cm.

Its volume is ☐.

42. A cube's side length is 9 cm.

Its volume is ☐.

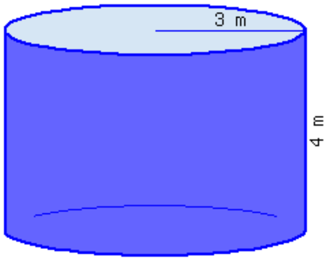
43. Find the volume of this cylinder.



a. This cylinder's volume, in terms of π, is ☐.

b. This cylinder's volume, rounded to the hundredth place, is ☐.

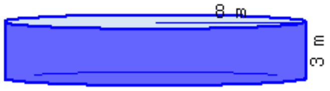
44. Find the volume of this cylinder.



a. This cylinder's volume, in terms of π, is ☐.

b. This cylinder's volume, rounded to the hundredth place, is ☐.

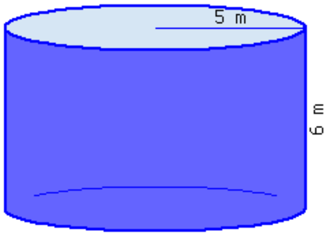
45. Find the volume of this cylinder.



a. This cylinder's volume, in terms of π, is ☐.

b. This cylinder's volume, rounded to the hundredth place, is ☐.

46. Find the volume of this cylinder.



a. This cylinder's volume, in terms of π, is ☐.

b. This cylinder's volume, rounded to the hundredth place, is ☐.

47. A cylinder's base's diameter is 10 ft, and its height is 5 ft.

a. This cylinder's volume, in terms of π, is ☐.

b. This cylinder's volume, rounded to the hundredth place, is ☐.

48. A cylinder's base's diameter is 4 ft, and its height is 6 ft.

a. This cylinder's volume, in terms of π, is ☐.

b. This cylinder's volume, rounded to the hundredth place, is ☐.

The formula $V = \frac{1}{3} \cdot s^2 \cdot h$ gives the volume of a right square pyramid.

49. What is the volume of a right square pyramid with $s = 66$ in and $h = 69$ in?

50. What is the volume of a right square pyramid with $s = 78$ in and $h = 34$ in?

2.3 Combining Like Terms

In the last section we worked with algebraic expressions. In order to simplify algebraic expressions, it is useful to identify which quantities we can combine.

2.3.1 Identifying Terms

In an algebraic expression, the **terms** are the quantities that are added. For example, the expression $3x + 2y$ has two terms, which are $3x$ and $2y$. Let's look at some more examples.

> **Example 2.3.2** List the terms in the expression $2l + 2w$.
>
> The expression has two terms that are being added, $2l$ and $2w$.

If there is any subtraction, we will rewrite the expression using addition. Here is an example of that.

> **Example 2.3.3** List the terms in the expression $-3x^2 + 5x - 4$.
>
> We can rewrite this expression as $-3x^2 + 5x + (-4)$ to see that the terms are $-3x^2$, $5x$, and -4. The last term is negative because subtracting is the same as adding the opposite.

> **Example 2.3.4** List the terms in the expression 3 cm + 2 cm + 3 cm + 2 cm.
>
> This expression has four terms: 3 cm, 2 cm, 3 cm, and 2 cm.

Checkpoint 2.3.5. List the terms in the expression $5x - 4x + 10z$.

Explanation. The terms are $5x$, $-4x$, and $10z$.

2.3.2 Combining Like Terms

In the examples above, you may have wanted to combine some of the terms. Look at the quantities below to see which ones you can add or subtract.

a. 5 in + 20 in
b. 16 ft − 4 ft^2
c. 2 🍎 + 5 🍎
d. 5 min + 50 ft
e. 5 🐶 − 2 🐱
f. 20 m − 6 m

The terms that we can combine are called **like terms**. We can combine terms with the same units, but we cannot combine units such as minutes and feet or cats and dogs. Here are the answers:

a. 5 in + 20 in = 25 in
b. 16 ft − 4 ft^2 cannot be simplified
c. 2 🍎 + 5 🍎 = 7 🍎
d. 5 min + 50 ft cannot be simplified
e. 5 🐶 − 2 🐱 cannot be simplified
f. 20 m − 6 m = 14 min

Now let's look at some examples that have variables in them.

Checkpoint 2.3.6. Which expressions have like terms that you can combine?

a. $10x + 3y$ (☐ can ☐ cannot) be combined.

b. $4x - 8x$ (☐ can ☐ cannot) be combined.

c. $9y - 4y$ (☐ can ☐ cannot) be combined.

d. $-6x + 17z$ (☐ can ☐ cannot) be combined.

e. $-3x - 7x$ (☐ can ☐ cannot) be combined.

f. $5t + 8t^2$ (☐ can ☐ cannot) be combined.

Explanation. The terms that we can combine have the same variable part, including any exponents.

a. $10x + 3y$ cannot be combined.

b. $4x - 8x = -4x$

c. $9y - 4y = 5y$

d. $-6x + 17z$ cannot be combined.

e. $-3x - 7x = -10x$

f. $5t + 8t^2$ cannot be combined.

> **Example 2.3.7** Simplify the expression $20x - 16x + 4y$, if possible, by combining like terms.
>
> This expression has two like terms, $20x$ and $-16x$, which we can combine.
>
> $$20x - 16x + 4y = 4x + 4y$$
>
> Note that we cannot combine $4x$ and $4y$ because x and y represent different quantities.

> **Example 2.3.8** Simplify the expression $100x + 100x^2$, if possible, by combining like terms.
>
> This expression cannot be simplified because the variable parts are not the same. We cannot add x and x^2 just like we cannot add feet, a measure of length, and square feet, a measure of area.

> **Example 2.3.9** Simplify the expression $-10r + 2s - 5t$, if possible, by combining like terms.
>
> This expression cannot be simplified because there are not any like terms.

> **Example 2.3.10** Simplify the expression $y + 5y$, if possible, by combining like terms.
>
> This expression can be thought of as $1y + 5y$. When we have a single y, the coefficient of 1 is not usually written. Now we have two like terms, $1y$ and $5y$. We will add those together:
>
> $$\begin{aligned} x + 5x &= 1x + 5x \\ &= 6x \end{aligned}$$

So far we have combined terms with whole numbers and integers, but we can also combine like terms when the coefficients are decimals (or fractions).

> **Example 2.3.11** Simplify the expression $x - 0.15x$, if possible, by combining like terms.
>
> Note that this expression can be rewritten as $1.00x - 0.15x$, and combined like this:
>
> $$\begin{aligned} x - 0.15x &= 1.00x - 0.15x \\ &= 0.85x \end{aligned}$$

Checkpoint 2.3.12. Simplify each expression, if possible, by combining like terms.

a. $4x - 7y + 10x$

b. $y - 8y + 2x^2$

c. $x + 0.25x$

d. $4x + 1.5y - 9z$

Explanation.

a. This expression has two like terms that can be combined to get $14x - 7y$.

b. In this expression we can combine the y terms to get $-7y + 2x^2$.

c. Rewrite this expression as $1.00x + 0.25x$ and simplify to get $1.25x$.

d. This expression cannot be simplified further because there are not any like terms.

Remark 2.3.13 The Difference Between Terms and Factors. We have learned that terms are quantities that are added, such as $3x$ and $-2x$ in $3x-2x$. These are different than **factors**, which are parts that are multiplied together. For example, the term $2x$ has two factors: 2 and x (with the multiplication symbol implied between them). The term $2\pi r$ has three factors: 2, π, and r.

Exercises

Review and Warmup

1. Add the following.

a. $2 + (-6) =$

b. $8 + (-3) =$

c. $9 + (-9) =$

2. Add the following.

a. $2 + (-7) =$

b. $10 + (-4) =$

c. $9 + (-9) =$

3. Add the following.

a. $-9 + 3 =$

b. $-1 + 7 =$

c. $-2 + 2 =$

4. Add the following.

a. $-6 + 3 =$

b. $-2 + 9 =$

c. $-2 + 2 =$

5. Subtract the following.

a. $4 - 8 =$

b. $6 - 2 =$

c. $6 - 12 =$

6. Subtract the following.

a. $4 - 7 =$

b. $8 - 1 =$

c. $6 - 17 =$

7. Subtract the following.

a. $-1 - 5 =$

b. $-10 - 5 =$

c. $-4 - 4 =$

8. Subtract the following.

a. $-1 - 3 =$

b. $-8 - 3 =$

c. $-4 - 4 =$

9. Subtract the following.

a. $-1 - (-6) =$

b. $-10 - (-1) =$

c. $-5 - (-5) =$

10. Subtract the following.

a. $-2-(-8)=$

b. $-6-(-2)=$

c. $-5-(-5)=$

Identifying Terms

11. Count the number of terms in each expression.

a. $8t^2-6t$

b. $4x-4-9y$

c. $-8z^2-z^2+2y^2+6t$

d. $-3s+2t+2z$

12. Count the number of terms in each expression.

a. $-7t-5x$

b. $-4s^2$

c. $2x^2-6x^2$

d. $-2s^2+8z+4t^2$

13. Count the number of terms in each expression.

a. $1.8t+0.7z^2+x^2$

b. $8.1s-8.9s+5.2+0.6y$

c. $3s^2+5.9z$

d. $0.1s$

14. Count the number of terms in each expression.

a. $-3.2t-6.6x^2+4.2$

b. $-8.5t+2.5s$

c. $-0.4x-6.2x-6.8+2s^2$

d. $-3.6x^2+7.6t+2.7+3.6s$

15. List the terms in each expression.

a. $-2t+6x+3x-2x^2$

b. $2t$

c. $-9s+4+6x$

d. $s^2-3z+2t^2$

16. List the terms in each expression.

a. $6t-1+4x+8t^2$

b. $-7s-3z^2+8y^2$

c. $-6s-x-3y+t^2$

d. $9z^2-8s$

17. List the terms in each expression.

a. $-5t-8t+4.4x-3.6y^2$

b. $8.3s^2-5.8z$

c. $6.7t-4.9s$

d. $3.9z$

18. List the terms in each expression.

a. $3.3t$

b. $3.4t^2+5.2x$

c. $-7.9s^2-4.4t^2+0.7-7.5x$

d. $-2.4t+4.5y+7.5t+0.8t$

19. List the terms in each expression.

a. $-1.8t^2+0.3y^2$

b. $7.8t+3.2t-3+8.9y$

c. $-6.6z^2$

d. $-2.2s^2-8.1x+5.2x$

20. List the terms in each expression.

a. $2.9t+6.9s^2$

b. $-3.7x+4.1y+1.1z$

c. $-6.6z-5.3z^2-4t$

d. $-3.9t$

Combining Like Terms Simplify each expression, if possible, by combining like terms.

21.
a. $7t^2 + 2s^2$
b. $8s^2 + 5z$
c. $8t^2 + t^2$
d. $-5y^2 - 5z$

22.
a. $-9z + 5z$
b. $7y^2 - 4s^2$
c. $-s^2 + 8x$
d. $-2z + 5t$

23.
a. $z + 8z^2 + 8z$
b. $-3z^2 + 7y^2$
c. $-9y - y - 7t^2 + s^2$
d. $3s^2 + 6y$

24.
a. $-5z + 3z^2 + 2z^2 - 3z^2$
b. $-2z^2 - 9 + 3y^2 + 9z^2$
c. $-9s - 8z$
d. $7z^2 + y^2$

25.
a. $-5z - 79z - 76 - 99z$
b. $-79x - 51t - 53s$
c. $-89x - 44x - 16z$
d. $22x + 38x^2 + 50x^2 + 90$

26.
a. $-21z + 14s - 18z - 56z$
b. $90z - 32$
c. $30y^2 - 61t^2 + 89t^2 + 94t^2$
d. $23y^2 + 53y^2 - 90y^2$

27.
a. $-3.8z - 2.5t$
b. $-5.2s - 6.7s - 5.4y - z^2$
c. $-1.5t - 4.5s^2 + 1.3t^2$
d. $8.8y + 2.3y$

28.
a. $1.3z - 0.4z$
b. $-5.7s^2 - 7.6s^2 + 4.5y^2$
c. $0.8y - 8.8 - 2.7t - 5.1t$
d. $5t^2 + 2.5t$

29.
a. $4z - \frac{5}{8}s$
b. $\frac{9}{5}y - \frac{2}{3}y$
c. $\frac{3}{4}y - 9y + y$
d. $\frac{4}{7}y + z^2 - z - \frac{5}{2}z^2$

30.
a. $\frac{5}{4}z + \frac{5}{9}x + \frac{4}{3}x$
b. $-\frac{4}{9}t^2 + \frac{7}{6}z^2 + 1 - \frac{6}{7}y^2$
c. $\frac{1}{3}y + \frac{8}{3}s^2 - \frac{8}{7}t$
d. $\frac{7}{3}x - \frac{3}{7}y$

31.
a. $\frac{8}{9}z^2 + \frac{4}{7}t^2 - \frac{7}{2}z$
b. $-\frac{2}{5}x^2 + \frac{8}{9}y^2 - \frac{7}{3}$
c. $6t^2 - \frac{2}{9}t^2$
d. $-\frac{7}{2}t^2 + \frac{5}{2}t$

32.
a. $-\frac{8}{3}z + \frac{4}{5}z + \frac{3}{4}z$
b. $-\frac{9}{4}y + y^2$
c. $\frac{4}{5}y + \frac{3}{2}t + z^2$
d. $\frac{9}{5}t + 6t + \frac{1}{3}t + 7x$

2.4 Equations and Inequalities as True/False Statements

This section introduces the concepts of algebraic **equations** and **inequalities**, and what it means for a number to be a **solution** to an equation or inequality.

2.4.1 Equations, Inequalities, and Solutions

An **equation** is two mathematical expressions with an equals sign between them. The two expressions can be relatively simple or more complicated:

A relatively simple equation:

$$x + 1 = 2$$

A more complicated equation:

$$\left(x^2 + y^2 - 1\right)^3 = x^2y^3$$

An **inequality** is quite similar, but the sign between the expressions is one of these: $<$, $\leq$, $>$, $\geq$, or $\neq$.

A relatively simple inequality:

$$x \geq 15$$

A more complicated inequality:

$$x^2 + y^2 < 1$$

A **linear equation** in one variable can be written in the form $ax + b = 0$, where a, b are real numbers, and $a \neq 0$. The variable doesn't have to be x. The variable cannot have an exponent other than 1 ($x = x^1$), and the variable cannot be inside a root symbol (square root, cube root, etc.) or in a denominator.

The following are some linear equations in one variable:

$$4 - y = 5 \qquad 4 - z = 5z \qquad 0 = \frac{1}{2}p$$

$$3 - 2(q + 2) = 10 \qquad \sqrt{2} \cdot r + 3 = 10 \qquad \frac{s}{2} + 3 = 5$$

(Note that r is outside the square root symbol.) We will see in later sections that all equations above can be converted into the form $ax + b = 0$.

The following are some non-linear equations:

$$1 + 2 = 3 \qquad \text{(There is no variable.)}$$

$$4 - 2y^2 = 5 \qquad \text{(The exponent of } y \text{ is not 1.)}$$

$$\sqrt{2r} + 3 = 10 \qquad (r \text{ is inside the square root.})$$

$$\frac{2}{s} + 3 = 5 \qquad (s \text{ is in a denominator.})$$

This chapter focuses on linear equations in *one* variable. We will study other types of equations in later chapters.

The simplest equations and inequalities have numbers and no variables. When this happens, the equation is either *true* or *false*. The following equations and inequalities are *true* statements:

$$2 = 2 \qquad -4 = -4 \qquad 2 > 1 \qquad -2 < -1 \qquad 3 \geq 3$$

The following equations and inequalities are *false* statements:

$$2 = 1 \qquad -4 = 4 \qquad 2 < 1 \qquad -2 \geq -1 \qquad 0 \neq 0$$

When equations and inequalities have variables, we can consider substituting values in for the variables. If replacing a variable with a number makes an equation or inequality *true*, then that number is called a **solution** to the equation.

> **Example 2.4.2 A Solution.** Consider the equation $y + 2 = 3$, which has only one variable, y. If we substitute in 1 for y and then simplify:
>
> $$\begin{aligned} y + 2 &= 3 \\ 1 + 2 &\overset{?}{=} 3 \\ 3 &\overset{\checkmark}{=} 3 \end{aligned}$$
>
> we get a true equation. So we say that 1 is a solution to $y + 2 = 3$. Notice that we used a question mark at first because we are unsure if the equation is true or false until the end.

If replacing a variable with a value makes a false equation or inequality, that number is not a solution.

> **Example 2.4.3 Not a Solution.** Consider the inequality $x + 4 > 5$, which has only one variable, x. If we substitute in 0 for x and then simplify:
>
> $$\begin{aligned} x + 4 &> 5 \\ 0 + 4 &\overset{?}{>} 5 \\ 4 &\overset{\text{no}}{>} 5 \end{aligned}$$
>
> we get a false equation. So we say that 0 is *not* a solution to $x + 4 > 5$.

2.4.2 Checking Possible Solutions

Given an equation or an inequality (with one variable), checking if some particular number is a solution is just a matter of replacing the value of the variable with the specified number and determining if the resulting equation/inequality is true or false. This may involve some amount of arithmetic simplification.

> **Example 2.4.4** Is 8 a solution to $x^2 - 5x = \sqrt{2x} + 20$?
>
> To find out, substitute in 8 for x and see what happens.
>
> $$\begin{aligned} x^2 - 5x &= \sqrt{2x} + 20 \\ 8^2 - 5(8) &\overset{?}{=} \sqrt{2(8)} + 20 \\ 64 - 5(8) &\overset{?}{=} \sqrt{16} + 20 \\ 64 - 40 &\overset{?}{=} 4 + 20 \\ 24 &\overset{\checkmark}{=} 24 \end{aligned}$$

So yes, 8 is a solution to $x^2 - 5x = \sqrt{2x} + 20$.

Example 2.4.5 Is -5 a solution to $\sqrt{169 - y^2} = y^2 - 2y$?

To find out, substitute in -5 for y and see what happens.

$$\begin{aligned}
\sqrt{169 - y^2} &= y^2 - 2y \\
\sqrt{169 - (-5)^2} &\overset{?}{=} (-5)^2 - 2(-5) \\
\sqrt{169 - 25} &\overset{?}{=} 25 - 2(-5) \\
\sqrt{144} &\overset{?}{=} 25 - (-10) \\
12 &\overset{\text{no}}{=} 35
\end{aligned}$$

So no, -5 is not a solution to $\sqrt{169 - y^2} = y^2 - 2y$.

But is -5 a solution to the *inequality* $\sqrt{169 - y^2} \leq y^2 - 2y$? Yes, because substituting -5 in for y would give you

$$12 \leq 35,$$

which is true.

Checkpoint 2.4.6. Is -3 a solution for x in the equation $2x - 3 = 5 - (4 + x)$? Evaluating the left and right sides gives:

$$\begin{array}{ccc}
2x - 3 & = & 5 - (4 + x) \\
\underline{\qquad} & \overset{?}{=} & \underline{\qquad}
\end{array}$$

So -3 (☐ is ☐ is not) a solution to $2x - 3 = 5 - (4 + x)$.

Explanation. We will substitute x with -3 in the equation and simplify each side of the equation to determine if the statement is true or false:

$$\begin{aligned}
2x - 3 &= 5 - (4 + x) \\
2(-3) - 3 &\overset{?}{=} 5 - (4 + (-3)) \\
-6 - 3 &\overset{?}{=} 5 - (1) \\
-9 &\overset{\times}{=} 4
\end{aligned}$$

Since $-9 = 4$ is not true, -3 is not a solution for x in the equation $2x - 3 = 5 - (4 + x)$.

Checkpoint 2.4.7. Is $\frac{1}{3}$ a solution for t in the equation $2t = 4\left(t - \frac{1}{2}\right)$? Evaluating the left and right sides gives:

$$\begin{array}{ccc}
2t & = & 4\left(t - \frac{1}{2}\right) \\
\underline{\qquad} & \overset{?}{=} & \underline{\qquad}
\end{array}$$

So $\frac{1}{3}$ (☐ is ☐ is not) a solution to $2t = 4\left(t - \frac{1}{2}\right)$.

Checkpoint 2.4.8. Is -2 a solution to $y^2 + y - 5 \leq y - 1$? Evaluating the left and right sides gives:

$$\begin{array}{ccc} y^2 + y - 5 & \leq & y - 1 \\ \underline{\qquad} & \overset{?}{\leq} & \underline{\qquad} \end{array}$$

So -3 (☐ is ☐ is not) a solution to $y^2 + y - 5 \leq y - 1$.

Checkpoint 2.4.9. Is 2 a solution to $\frac{z+3}{z-1} = \sqrt{18z}$? Evaluating the left and right sides gives:

$$\begin{array}{ccc} \dfrac{z+3}{z-1} & = & \sqrt{18z} \\ \underline{\qquad} & \overset{?}{=} & \underline{\qquad} \end{array}$$

So 2 (☐ is ☐ is not) a solution to $\frac{z+3}{z-1} = \sqrt{18z}$.

Checkpoint 2.4.10. Is -3 a solution to $x^2 + x + 1 \leq \frac{3x+2}{x+2}$? Evaluating the left and right sides gives:

$$\begin{array}{ccc} x^2 + x + 1 & \leq & \dfrac{3x+2}{x+2} \\ \underline{\qquad} & \overset{?}{\leq} & \underline{\qquad} \end{array}$$

So -3 (☐ is ☐ is not) a solution to $x^2 + x + 1 \leq \frac{3x+2}{x+2}$.

A cylinder's volume is related to its radius and its height by:

$$V = \pi r^2 h,$$

where V is the volume, r is the base's radius, and h is the height. If we know the volume is 96π cm^3 and the radius is 4 cm, then we have:

$$96\pi = 16\pi h$$

Is 4 cm the height of the cylinder? In other words, is 4 a solution to $96\pi = 16\pi h$? We will substitute h in the equation with 4 to check:

$$\begin{aligned} 96\pi &= 16\pi h \\ 96\pi &\overset{?}{=} 16\pi \cdot 4 \\ 96\pi &\overset{\text{no}}{=} 64\pi \end{aligned}$$

Since $96\pi = 64\pi$ is false, $h = 4$ does *not* satisfy the equation $96\pi = 16\pi h$.

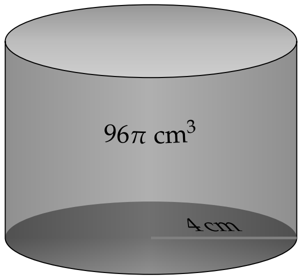


Figure 2.4.12: A cylinder

Example 2.4.11 Cylinder Volume. Next, we will try $h = 6$:

$$\begin{aligned}96\pi &= 16\pi h\\ 96\pi &\overset{?}{=} 16\pi \cdot 6\\ 96\pi &\overset{\checkmark}{=} 96\pi\end{aligned}$$

When $h = 6$, the equation $96\pi = 16\pi h$ is true. This tells us that 6 *is* a solution to $96\pi = 16\pi h$.

Remark 2.4.13. Note that we did not approximate π with 3.14 or any other approximation. We often leave π as π throughout our calculations. If we need to round, we do so as a final step.

Example 2.4.14 Jaylen has budgeted a maximum of \$300 for an appliance repair. The total cost of the repair can be modeled by $89 + 110(h - 0.25)$, where \$89 is the initial cost and \$110 is the hourly labor charge after the first quarter hour. Is 2 hours a solution for h in the inequality $89 + 110(h - 0.25) \leq 300$?

To determine if $h = 2$ satisfies the inequality, we will replace h with 2 and check if the statement is true:

$$\begin{aligned}89 + 110(h - 0.25) &\leq 300\\ 89 + 110(2 - 0.25) &\overset{?}{\leq} 300\\ 89 + 110(1.75) &\overset{?}{\leq} 300\\ 89 + 192.5 &\overset{?}{\leq} 300\\ 281.5 &\overset{\checkmark}{\leq} 300\end{aligned}$$

Thus, 2 hours is a solution for h in the inequality $89 + 110(h - 0.25) \leq 300$. In context, this means that Jaylen would stay within their \$300 budget if 2 hours of labor were performed.

Exercises

Review and Warmup

1. Evaluate $-9 - x$ for $x = -4$.
2. Evaluate $5 - x$ for $x = -2$.
3. Evaluate $-2x + 7$ for $x = 1$.
4. Evaluate $-10x - 5$ for $x = 3$.
5. Evaluate $-4(r + 2)$ for $r = 4$.
6. Evaluate $-10(t + 9)$ for $t = -3$.
7. Evaluate the expression $\frac{1}{3}(x + 4)^2 - 2$ when $x = -7$.
8. Evaluate the expression $\frac{1}{4}(x + 1)^2 - 7$ when $x = -5$.
9. Evaluate the expression $-16t^2+64t+128$ when $t = -3$.
10. Evaluate the expression $-16t^2+64t+128$ when $t = -5$.

Identifying Linear Equations and Inequalities

11. Are the equations below linear equations in one variable?

a. $\sqrt{9-5.5r}=7$ (☐ is ☐ is not) a linear equation in one variable.

b. $2-3r^2=18$ (☐ is ☐ is not) a linear equation in one variable.

c. $x-6q^2=27$ (☐ is ☐ is not) a linear equation in one variable.

d. $2\pi r=2\pi$ (☐ is ☐ is not) a linear equation in one variable.

e. $8.59y=-9$ (☐ is ☐ is not) a linear equation in one variable.

f. $12-3z=4$ (☐ is ☐ is not) a linear equation in one variable.

12. Are the equations below linear equations in one variable?

a. $-5-4y^2=-21$ (☐ is ☐ is not) a linear equation in one variable.

b. $\sqrt{1-4.7x}=2$ (☐ is ☐ is not) a linear equation in one variable.

c. $9q-y^2=1$ (☐ is ☐ is not) a linear equation in one variable.

d. $2\pi r=12\pi$ (☐ is ☐ is not) a linear equation in one variable.

e. $-r-11=3$ (☐ is ☐ is not) a linear equation in one variable.

f. $6.6y=6$ (☐ is ☐ is not) a linear equation in one variable.

13. Are the equations below linear equations in one variable?

a. $4.61q=14$ (☐ is ☐ is not) a linear equation in one variable.

b. $5qVx=-42$ (☐ is ☐ is not) a linear equation in one variable.

c. $V^2+y^2=-68$ (☐ is ☐ is not) a linear equation in one variable.

d. $\pi r^2=48\pi$ (☐ is ☐ is not) a linear equation in one variable.

e. $7-2r=17$ (☐ is ☐ is not) a linear equation in one variable.

f. $r\sqrt{12}=-84$ (☐ is ☐ is not) a linear equation in one variable.

14. Are the equations below linear equations in one variable?

a. $q\sqrt{24}=63$ (☐ is ☐ is not) a linear equation in one variable.

b. $\pi r^2=35\pi$ (☐ is ☐ is not) a linear equation in one variable.

c. $-2.96x=-22$ (☐ is ☐ is not) a linear equation in one variable.

d. $-8prz=52$ (☐ is ☐ is not) a linear equation in one variable.

e. $15x-4=28$ (☐ is ☐ is not) a linear equation in one variable.

f. $q^2+y^2=4$ (☐ is ☐ is not) a linear equation in one variable.

15. Are the inequalities below linear inequalities in one variable?

a. $4p^2-y>-51$ (☐ is ☐ is not) a linear inequality in one variable.

b. $-1>1-5p$ (☐ is ☐ is not) a linear inequality in one variable.

c. $-3z^2-7y^2<1$ (☐ is ☐ is not) a linear inequality in one variable.

16. Are the inequalities below linear inequalities in one variable?

a. $3x^2+6z^2>1$ (☐ is ☐ is not) a linear inequality in one variable.

b. $9V^2+6p>51$ (☐ is ☐ is not) a linear inequality in one variable.

c. $4\geq-5-3y$ (☐ is ☐ is not) a linear inequality in one variable.

17. Are the inequalities below linear inequalities in one variable?

a. $-5.8x > 96$ (☐ is ☐ is not) a linear inequality in one variable.

b. $-1 > -4878r - 2301p$ (☐ is ☐ is not) a linear inequality in one variable.

c. $\sqrt{9p} - 9 \leq -6$ (☐ is ☐ is not) a linear inequality in one variable.

18. Are the inequalities below linear inequalities in one variable?

a. $-6y < -43$ (☐ is ☐ is not) a linear inequality in one variable.

b. $\sqrt{9r} + 6 < -6$ (☐ is ☐ is not) a linear inequality in one variable.

c. $198 \leq 4182y - 9693x$ (☐ is ☐ is not) a linear inequality in one variable.

Checking a Solution for an Equation

19. Is -3 a solution for x in the equation $x + 7 = 3$? (☐ Yes ☐ No)

20. Is 2 a solution for x in the equation $x + 9 = 7$? (☐ Yes ☐ No)

21. Is -9 a solution for r in the equation $-3 - r = 5$? (☐ Yes ☐ No)

22. Is 6 a solution for r in the equation $5 - r = -1$? (☐ Yes ☐ No)

23. Is -1 a solution for r in the equation $-7r - 8 = -1$? (☐ Yes ☐ No)

24. Is -9 a solution for t in the equation $2t + 8 = -10$? (☐ Yes ☐ No)

25. Is 5 a solution for t in the equation $-10t + 2 = -9t - 3$? (☐ Yes ☐ No)

26. Is 2 a solution for x in the equation $-2x - 4 = -9x - 18$? (☐ Yes ☐ No)

27. Is -9 a solution for x in the equation $9(x - 11) = 20x$? (☐ Yes ☐ No)

28. Is 3 a solution for y in the equation $2(y + 14) = 9y$? (☐ Yes ☐ No)

29. Is -3 a solution for y in the equation $-3(y - 13) = 16(y + 6)$? (☐ Yes ☐ No)

30. Is -10 a solution for r in the equation $19(r + 1) = 9(r - 9)$? (☐ Yes ☐ No)

31. Is $\frac{7}{10}$ a solution for x in the equation $10x - 1 = -8$? (☐ Yes ☐ No)

32. Is $\frac{14}{5}$ a solution for x in the equation $5x - 4 = 9$? (☐ Yes ☐ No)

33. Is 4 a solution for t in the equation $-\frac{10}{3}t - \frac{3}{10} = -\frac{17}{15}$? (☐ Yes ☐ No)

34. Is $-\frac{9}{2}$ a solution for t in the equation $\frac{2}{5}t - \frac{9}{10} = -\frac{27}{10}$? (☐ Yes ☐ No)

Checking a Solution for an Inequality Decide whether each value is a solution to the given inequality.

35. $-2x + 9 > 7$

a. $x = -5$ (☐ is ☐ is not) a solution.

b. $x = 6$ (☐ is ☐ is not) a solution.

c. $x = 0$ (☐ is ☐ is not) a solution.

d. $x = 1$ (☐ is ☐ is not) a solution.

36. $2x - 5 > 1$

a. $x = 3$ (☐ is ☐ is not) a solution.

b. $x = 0$ (☐ is ☐ is not) a solution.

c. $x = 13$ (☐ is ☐ is not) a solution.

d. $x = 2$ (☐ is ☐ is not) a solution.

37. $3x - 8 \geq -5$

a. $x = 0$ (☐ is ☐ is not) a solution.

b. $x = 8$ (☐ is ☐ is not) a solution.

c. $x = 1$ (☐ is ☐ is not) a solution.

d. $x = -5$ (☐ is ☐ is not) a solution.

38. $-3x + 19 \geq 10$

a. $x = 2$ (☐ is ☐ is not) a solution.

b. $x = 0$ (☐ is ☐ is not) a solution.

c. $x = 3$ (☐ is ☐ is not) a solution.

d. $x = 13$ (☐ is ☐ is not) a solution.

39. $4x - 16 \leq 4$

a. $x = 0$ (☐ is ☐ is not) a solution.

b. $x = 10$ (☐ is ☐ is not) a solution.

c. $x = 5$ (☐ is ☐ is not) a solution.

d. $x = 3$ (☐ is ☐ is not) a solution.

40. $4x - 10 \leq -2$

a. $x = 1$ (☐ is ☐ is not) a solution.

b. $x = 2$ (☐ is ☐ is not) a solution.

c. $x = 12$ (☐ is ☐ is not) a solution.

d. $x = 0$ (☐ is ☐ is not) a solution.

Checking Solutions for Application Problems

41. A triangle's area is 99 square meters. Its height is 11 meters. Suppose we wanted to find how long is the triangle's base. A triangle's area formula is

$$A = \frac{1}{2}bh$$

where A stands for area, b for base and h for height. If we let b be the triangle's base, in meters, we can solve this problem using the equation:

$$99 = \frac{1}{2}(b)(11)$$

Check whether 36 is a solution for b of this equation. (☐ Yes ☐ No)

42. A triangle's area is 95 square meters. Its height is 10 meters. Suppose we wanted to find how long is the triangle's base. A triangle's area formula is

$$A = \frac{1}{2}bh$$

where A stands for area, b for base and h for height. If we let b be the triangle's base, in meters, we can solve this problem using the equation:

$$95 = \frac{1}{2}(b)(10)$$

Check whether 38 is a solution for b of this equation. (☐ Yes ☐ No)

43. When a plant was purchased, it was 2 inches tall. It grows 1 inches per day. How many days later will the plant be 17 inches tall?

Assume the plant will be 17 inches tall d days later. We can solve this problem using the equation:

$$1d + 2 = 17$$

Check whether 17 is a solution for d of this equation. (☐ Yes ☐ No)

44. When a plant was purchased, it was 1.2 inches tall. It grows 0.2 inches per day. How many days later will the plant be 5.2 inches tall?

Assume the plant will be 5.2 inches tall d days later. We can solve this problem using the equation:

$$0.2d + 1.2 = 5.2$$

Check whether 21 is a solution for d of this equation. (☐ Yes ☐ No)

45. A water tank has 163 gallons of water in it, and it is being drained at the rate of 7 gallons per minute. After how many minutes will there be 37 gallons of water left?

Assume the tank will have 37 gallons of water after m minutes. We can solve this problem using the equation:

$$163 - 7m = 37$$

Check whether 19 is a solution for m of this equation. (☐ Yes ☐ No)

46. A water tank has 171 gallons of water in it, and it is being drained at the rate of 9 gallons per minute. After how many minutes will there be 45 gallons of water left?

Assume the tank will have 45 gallons of water after m minutes. We can solve this problem using the equation:

$$171 - 9m = 45$$

Check whether 17 is a solution for m of this equation. (☐ Yes ☐ No)

47. A cylinder's volume is 126π cubic centimeters. Its height is 14 centimeters. Suppose we wanted to find how long is the cylinder's radius. A cylinder's volume formula is

$$V = \pi r^2 h$$

where V stands for volume, r for radius and h for height. Let r represent the cylinder's radius, in centimeters. We can solve this problem using the equation:

$$126\pi = \pi r^2(14)$$

Check whether 3 is a solution for r of this equation. (☐ Yes ☐ No)

48. A cylinder's volume is 960π cubic centimeters. Its height is 15 centimeters. Suppose we wanted to find how long is the cylinder's radius. A cylinder's volume formula is

$$V = \pi r^2 h$$

where V stands for volume, r for radius and h for height. Let r represent the cylinder's radius, in centimeters. We can solve this problem using the equation:

$$960\pi = \pi r^2(15)$$

Check whether 64 is a solution for r of this equation. (☐ Yes ☐ No)

49. A country's national debt was 140 million dollars in 2010. The debt increased at 50 million dollars per year. If this trend continues, when will the country's national debt increase to 640 million dollars?

Assume the country's national debt will become 640 million dollars y years after 2010. We can solve this problem using the equation:

$$50y + 140 = 640$$

Check whether 10 is a solution for y of this equation. (☐ Yes ☐ No)

50. A country's national debt was 100 million dollars in 2010. The debt increased at 60 million dollars per year. If this trend continues, when will the country's national debt increase to 1180 million dollars?

Assume the country's national debt will become 1180 million dollars y years after 2010. We can solve this problem using the equation:

$$60y + 100 = 1180$$

Check whether 20 is a solution for y of this equation. (☐ Yes ☐ No)

51. A school district has a reserve fund worth 41.1 million dollars. It plans to spend 2.9 million dollars per year. After how many years, will there be 15 million dollars left?

Assume there will be 15 million dollars left after y years. We can solve this problem using the equation:

$$41.1 - 2.9y = 15$$

Check whether 11 is a solution for y of this equation. (☐ Yes ☐ No)

52. A school district has a reserve fund worth 32 million dollars. It plans to spend 3 million dollars per year. After how many years, will there be 11 million dollars left?

Assume there will be 11 million dollars left after y years. We can solve this problem using the equation:

$$32 - 3y = 11$$

Check whether 8 is a solution for y of this equation. (☐ Yes ☐ No)

53. A rectangular frame's perimeter is 6 feet. If its length is 2 feet, suppose we want to find how long is its width. A rectangle's perimeter formula is

$$P = 2(l + w)$$

where P stands for perimeter, l for length and w for width. We can solve this problem using the equation:

$$6 = 2(2 + w)$$

Check whether 4 is a solution for w of this equation. (☐ Yes ☐ No)

54. A rectangular frame's perimeter is 7 feet. If its length is 2.1 feet, suppose we want to find how long is its width. A rectangle's perimeter formula is

$$P = 2(l + w)$$

where P stands for perimeter, l for length and w for width. We can solve this problem using the equation:

$$7 = 2(2.1 + w)$$

Check whether 1.4 is a solution for w of this equation. (☐ Yes ☐ No)

2.5 Solving One-Step Equations

We have learned how to check whether a specific number is a solution to an equation or inequality. In this section, we will begin learning how to *find* the solution(s) to basic equations ourselves.

2.5.1 Imagine Filling in the Blanks

Let's start with a very simple situation — so simple, that you might have success entirely in your head without writing much down. It's not exactly the algebra we want you to learn, but the example may serve as a good warm-up.

> **Example 2.5.2** A number plus 2 is 6. What is that number?
>
> You may be so familiar with basic arithmetic that you know the answer already. The *algebra* approach would be to start by translating "A number plus 2 is 6" into a math statement — in this case, an equation:
>
> $$x + 2 = 6$$
>
> where x is the number we are trying to find. In other words, what should be substituted in for x…
>
> $$x + 2 = 6$$
>
> … to make the equation true?
>
> Now, how do you determine what x is? One valid option is to just *imagine* what number you could put in place of x that would result in a true equation. Would 0 work? No, that would mean $0+2 \overset{\text{no}}{=} 6$. Would 17 work? No, that would mean $17 + 2 \overset{\text{no}}{=} 6$. Would 4 work? Yes, because $4 + 2 = 6$ is a true equation.
>
> So one solution to $x + 2 = 6$ is 4. No other numbers are going to be solutions, because when you add 2 to something smaller or larger than 4, the result is going to be smaller or larger than 6.

This approach might work for you to solve *very basic equations*, but in general equations are going to be too complicated to solve in your head this way. So we move on to more systematic approaches.

2.5.2 The Basic Principle of Algebra

2.5.2.1 Opposite Operations

Let's revisit Example 2.5.2, thinking it through differently.

> **Example 2.5.3** If a number plus 2 is 6, what is the number?
>
> One way to solve this riddle is to use the opposite operation. If a number *plus* 2 is 6, we should be able to *subtract* 2 from 6 and get that unknown number. So the unknown number is $6 - 2 = 4$.

Let's try this strategy with another riddle.

> **Example 2.5.4** If a number minus 2 is 6, what is the number? This time, we should be able to *add* 2 to 6 to get the unknown number. So the unknown number is $6 + 2 = 8$.

Does this strategy work with multiplication and division?

> **Example 2.5.5** If a number times 2 is 6, what is the number? This time, we should be able to *divide* 6 by 2 to get the unknown number. So the unknown number is $6 \div 2 = 3$.

> **Example 2.5.6** If a number divided by 2 equals 6, what is the number? This time, we can *multiply* 6 by 2, and the unknown number is $6 \cdot 2 = 12$.

These examples explore an important principle for solving an equation — applying an opposite arithmetic operation.

2.5.2.2 Balancing Equations Like a Scale

We can revisit Example 2.5.2 with yet another strategy. If a number plus 2 is 6, what is the number? As is common in algebra, we can use x to represent the unknown number. The question translates into the math equation

$$x + 2 = 6.$$

Try to envision the equals sign as the middle of a balanced scale. The left side has 2 one-pound objects and a block with unknown weight x lb. Together, the weight on the left is $x + 2$. The right side has 6 one-pound objects. Figure 2.5.7 shows the scale.

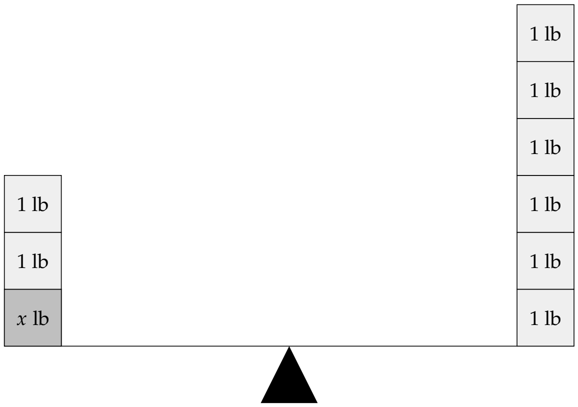


Figure 2.5.7: Scale to represent $x + 2 = 6$

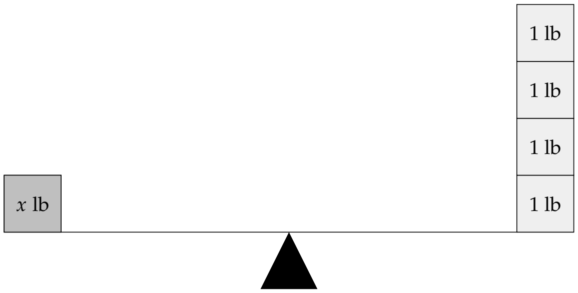


Figure 2.5.8: Scale to represent the solution to $x + 2 = 6$, after taking away 2 from each side

To find the weight of the unknown block, we can take away 2 one-pound blocks from *both* sides of the scale (to keep the scale balanced). Figure 2.5.8 shows the solution.

An equation is like a balanced scale, as the two sides of the equation are equal. In the same way that we can take away 2 lb from *both* sides of a balanced scale, we can subtract 2 on *both* sides of the equals sign. So instead of two pictures of balance scales, we can use algebra symbols and solve the equation $x + 2 = 6$ in the following manner:

$$\begin{aligned} x + 2 &= 6 && \text{a balanced scale} \\ x + 2 - 2 &= 6 - 2 && \text{remove the same quantity from each side} \\ x &= 4 && \text{still balanced; now it tells you the solution} \end{aligned}$$

It's important to note that each line shows what is called an **equivalent equation**. In other words, each equation shown is algebraically equivalent to the one above it and the one below it and will have exactly the same solution(s). The final equivalent equation $x = 4$ tells us that the **solution** to the equation is 4. The **solution set** to this equation is the set that lists every solution to the equation. For this example, the solution set is $\{4\}$.

We have learned we can add or subtract the same number on both sides of the equals sign, just like we can add or remove the same amount of weight on a balanced scale. Can we multiply and divide the same number on both sides of the equals sign?

Let's look at Example 2.5.5 again: If a number times 2 is 6, what is the number? Another balance scale can help visualize this.

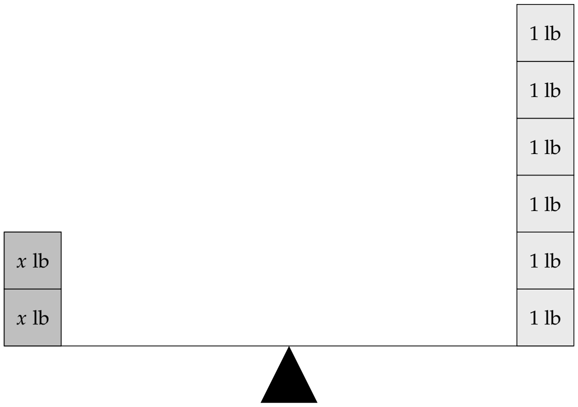


Figure 2.5.9: Scale to represent the equation $2x = 6$

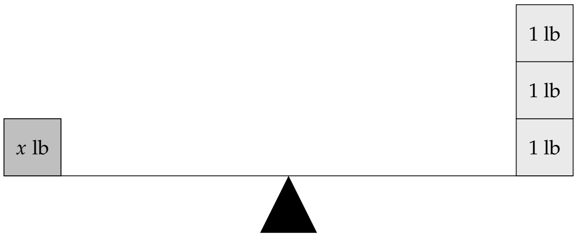


Figure 2.5.10: Scale to represent the equation the solution to $2x = 6$, after cutting each side in half

Currently, the scale is balanced. If we cut the weight in half on both sides, the scale should still be balanced.

We can see from the scale that $x = 3$ is correct. Removing half of the weight on both sides of the scale is like dividing both sides of an equation by 2:

$$\begin{aligned} 2x &= 6 \\ \frac{2x}{2} &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

The equivalent equation in this example is $x = 3$, which tells us that the solution to the equation is 3 and the solution set is $\{3\}$.

Remark 2.5.11. Note that when we divide each side of an equation by a number, we use the fraction line in place of the division symbol. The fact that $\frac{6}{2} = 6 \div 2$ is a reminder that the fraction line and division symbol have the same purpose. The division symbol is rarely used in later math courses.

Similarly, we could multiply both sides of an equation by 2, just like we can keep a scale balanced if we double the weight on each side. We will summarize these properties.

Fact 2.5.12 Properties of Equivalent Equations. *If there is an equation $A = B$, we can do the following to obtain an equivalent equation:*

- *add the same number to each side: $A + c = B + c$*

- *subtract the same number from each side:* $A - c = B - c$
- *multiply each side of the equation by the same* non-zero *number:* $A \cdot c = B \cdot c$
- *divide each side of the equation by the same* non-zero *number:* $\frac{A}{c} = \frac{B}{c}$

With practice, you will learn when it is helpful to use each of these properties.

2.5.3 Solving One-Step Equations and Stating Solution Sets

Notice that when we solved equations in Subsection 2.5.2.2, the final equation looked like $x =$ number, where the variable x is separated from other numbers and stands alone on one side of the equals sign. The goal of solving any equation is to *isolate the variable* in this same manner.

Putting together both strategies (applying the opposite operation and balancing equations like a scale) that we just explored, we will summarize how to solve a one-step linear equation.

Process 2.5.13 Steps to Solving Simple (One-Step) Linear Equations.

Apply *Apply the opposite operation to both sides of the equation. If a number was added to the variable, subtract that number, and vice versa. If a number was multiplied by the variable, divide by that number, and vice versa.*

Check *Check the solution. This means, verify that what you think is the solution actually solves the equation. For one reason, it's human to have made a simple arithmetic mistake, and by checking you will protect yourself from this. For another reason, there are situations where solving an equation will tell you that certain numbers are* possible *solutions, but they do not actually solve the original equation. Checking solutions will catch these situations.*

Summarize *State the solution set, or in the case of application problems, summarize the result using a complete sentence and appropriate units.*

Let's look at a few examples.

Example 2.5.14 Solve for y in the equation $7 + y = 3$.

Explanation.

To isolate y, we need to remove 7 from the left side. Since 7 is being *added* to y, we need to *subtract* 7 from each side of the equation.

$$\begin{aligned} 7 + y &= 3 \\ 7 + y - 7 &= 3 - 7 \\ y &= -4 \end{aligned}$$

We should always check the solution when we solve equations. For this problem, we will substitute y in the original equation with -4:

$$\begin{aligned} 7 + y &= 3 \\ 7 + (-4) &\overset{?}{=} 3 \\ 3 &\overset{\checkmark}{=} 3 \end{aligned}$$

The solution -4 is checked, so the solution set is $\{-4\}$.

Checkpoint 2.5.15. Solve for z in the equation $-7.3 + z = 5.1$.

The solution is ☐.

The solution set is ☐.

Explanation. To remove the -7.3 from the left side, we need to *add* 7.3 to each side of the equation. (If we make the mistake of subtracting 7.3 from each side, we would have $-7.3 + z - 7.3 = 5.1 - 7.3$, which simplifies to $z - 14.3 = -2.2$, which would not isolate z.)

$$\begin{aligned}
-7.3 + z &= 5.1 \\
-7.3 + z + 7.3 &= 5.1 + 7.3 \\
z &= 12.4
\end{aligned}$$

We will check the solution by substituting z in the original equation with 12.4:

$$\begin{aligned}
-7.3 + z &= 5.1 \\
-7.3 + (12.4) &\overset{?}{=} 5.1 \\
5.1 &\overset{\checkmark}{=} 5.1
\end{aligned}$$

The solution 12.4 is checked and the solution set is $\{12.4\}$.

Checkpoint 2.5.16. Solve for a in the equation $10 = -2a$.

The solution is .

The solution set is .

Explanation. To isolate the variable a, we need to divide each side by -2 (because a is being *multiplied* by -2). One common mistake is to add 2 to each side. This would not isolate a, but would instead leave us with the expression $-2a + 2$ on the right-hand side.

$$\begin{aligned}
10 &= -2a \\
\frac{10}{-2} &= \frac{-2a}{-2} \\
-5 &= a
\end{aligned}$$

We will check the solution by substituting a in the original equation with -5:

$$\begin{aligned}
10 &= -2a \\
10 &\overset{?}{=} -2(-5) \\
10 &\overset{\checkmark}{=} 10
\end{aligned}$$

The solution -5 is checked and the solution set is $\{-5\}$.

Note that in solving the equation in Checkpoint 2.5.16 we found that $-5 = a$, which is equivalent to $a = -5$. We did not write $a = -5$ as an extra step though, as $-5 = a$ identified the solution.

> **Example 2.5.17** The formula for a circle's circumference is $C = \pi d$, where C stands for circumference, d stands for diameter, and π is a constant with the value of $3.1415926\ldots$. If a circle's circumference is 12π ft, find this circle's diameter and radius.
>
> **Explanation**. The circumference is given as 12π feet. Approximating π with 3.14, this means the circumference is approximately 37.68 ft. It is nice to have a rough understanding of how long the circum-

ference is, but if we use 3.14 instead of π, we are using a slightly smaller number than π, and the result of any calculations we do would not be as accurate. This is why we will use the symbol π throughout solving this equation and round only at the end in the conclusion summary (if necessary).

We will substitute C in the formula with 12π and solve for d:

$$\begin{aligned} C &= \pi d \\ 12\pi &= \pi d \\ \frac{12\pi}{\pi} &= \frac{\pi d}{\pi} \\ 12 &= d \end{aligned}$$

So the circle's diameter is 12 ft. And since radius is half of diameter, the radius is 6 ft.

Example 2.5.18 Solve for b in $-b = 2$.

Explanation. Note that b is not yet isolated as there is a negative sign in front of it. One way to solve for b is to recognize that multiplying on both sides by -1 would clear away that negative sign:

$$\begin{aligned} -b &= 2 \\ -1 \cdot (-b) &= -1 \cdot (2) \\ b &= -2 \end{aligned}$$

We removed the negative sign from $-b$ using the fact that $-1 \cdot (-b) = b$. A second way to remove the negative sign -1 from $-b$ is to divide both sides by -1. If you view the original $-b$ as $-1 \cdot b$, then this approach resembles the solution from Checkpoint 2.5.16.

$$\begin{aligned} -b &= 2 \\ -1 \cdot b &= 2 \\ \frac{-1 \cdot b}{-1} &= \frac{2}{-1} \\ b &= -2 \end{aligned}$$

A third way to remove the original negative sign, is to recognize that the opposite operation of negation is negation. So negating both sides will work out too:

$$\begin{aligned} -b &= 2 \\ -(-b) &= -2 \\ b &= -2 \end{aligned}$$

We will check the solution by substituting b in the original equation with -2:

$$\begin{aligned} -b &= 2 \\ -(-2) &\stackrel{?}{=} 2 \end{aligned}$$

$$2 \overset{\checkmark}{=} 2$$

The solution -2 is checked and the solution set is $\{-2\}$.

2.5.4 Solving One-Step Equations Involving Fractions

When equations have fractions, solving them will make use of the same principles. You may need to use fraction arithmetic, and there may be special considerations that will make the calculations easier. So we have separated the following examples.

Example 2.5.19 Solve for g in $\frac{2}{3} + g = \frac{1}{2}$.

Explanation.

In Section 3.3, we will learn a skill to avoid fraction operations entirely in equations like this one. For now, let's solve the equation by using subtraction to isolate g:

$$\begin{aligned} \frac{2}{3} + g &= \frac{1}{2} \\ \frac{2}{3} + g - \frac{2}{3} &= \frac{1}{2} - \frac{2}{3} \\ g &= \frac{3}{6} - \frac{4}{6} \\ g &= -\frac{1}{6} \end{aligned}$$

We will check the solution by substituting g in the original equation with $-\frac{1}{6}$:

$$\begin{aligned} \frac{2}{3} + g &= \frac{1}{2} \\ \frac{2}{3} + \left(-\frac{1}{6}\right) &\overset{?}{=} \frac{1}{2} \\ \frac{4}{6} + \left(-\frac{1}{6}\right) &\overset{?}{=} \frac{1}{2} \\ \frac{3}{6} &\overset{?}{=} \frac{1}{2} \\ \frac{1}{2} &\overset{\checkmark}{=} \frac{1}{2} \end{aligned}$$

The solution $-\frac{1}{6}$ is checked and the solution set is $\left\{-\frac{1}{6}\right\}$.

Checkpoint 2.5.20. Solve for q in the equation $q - \frac{3}{7} = \frac{3}{2}$.

The solution is ☐.

The solution set is ☐.

Explanation. To remove the $\frac{3}{7}$ from the left side, we need to *add* $\frac{3}{7}$ to each side of the equation.

$$\begin{aligned} q - \frac{3}{7} &= \frac{3}{2} \\ q - \frac{3}{7} + \frac{3}{7} &= \frac{3}{2} + \frac{3}{7} \\ q &= \frac{27}{14} \end{aligned}$$

We will check the solution by substituting q in the original equation with $\frac{27}{14}$:

$$\begin{aligned} q-\frac{3}{7} &= \frac{3}{2} \\ \frac{27}{14}-\frac{3}{7} &\overset{?}{=} \frac{3}{2} \\ \frac{27}{14}-\frac{6}{14} &\overset{?}{=} \frac{3}{2} \\ \frac{21}{14} &\overset{?}{=} \frac{3}{2} \\ \frac{3}{2} &\overset{\checkmark}{=} \frac{3}{2} \end{aligned}$$

The solution $\frac{27}{14}$ is checked and the solution set is $\left\{\frac{27}{14}\right\}$.

Example 2.5.21 Solve for c in $\frac{c}{5} = 4$.

Explanation.

Note that the fraction line here implies division, so our variable c is being divided by 5. The opposite operation is to *multiply* by 5:

$$\begin{aligned} \frac{c}{5} &= 4 \\ 5\cdot\frac{c}{5} &= 5\cdot 4 \\ c &= 20 \end{aligned}$$

We will check the solution by substituting c in the original equation with 20:

$$\begin{aligned} \frac{c}{5} &= 4 \\ \frac{20}{5} &\overset{?}{=} 4 \\ 4 &\overset{\checkmark}{=} 4 \end{aligned}$$

The solution 20 is checked and the solution set is $\{20\}$.

Example 2.5.22 Solve for d in $-\frac{1}{3}d = 6$.

Explanation. It's true that in this example, the variable d is *multiplied* by $-\frac{1}{3}$. This means that *dividing* each side by $-\frac{1}{3}$ would be a valid strategy for solving this equation. However, dividing by a fraction could lead to human error, so consider this alternative strategy.

Another way to be rid of the $-\frac{1}{3}$ is to multiply by -3. Indeed, $-\frac{1}{3}d$ is the same as $\frac{d}{-3}$, and when we view the expression this way, d is being *divided* by -3. So multiplying by -3 would be the opposite operation.

$$\begin{aligned} -\frac{1}{3}d &= 6 \\ (-3)\cdot\left(-\frac{1}{3}d\right) &= (-3)\cdot 6 \\ d &= -18 \end{aligned}$$

If you choose to divide each side by $-\frac{1}{3}$, that will work out as well:

$$\begin{aligned}
-\frac{1}{3}d &= 6\\
\frac{-\frac{1}{3}d}{-\frac{1}{3}} &= \frac{6}{-\frac{1}{3}}\\
d &= \frac{6}{1}\cdot\frac{-3}{1}\\
d &= -18
\end{aligned}$$

This gives the same solution.

We will check the solution by substituting d in the original equation with -18:

$$\begin{aligned}
-\frac{1}{3}d &= 6\\
-\frac{1}{3}\cdot(-18) &\stackrel{?}{=} 6\\
6 &\stackrel{\checkmark}{=} 6
\end{aligned}$$

The solution -18 is checked and the solution set is $\{-18\}$.

Example 2.5.23 Solve for x in $\frac{3x}{4} = 10$.

Explanation. The variable x appears to have *two* operations that apply to it: first multiplication by 3, and then division by 4. But note that

$$\frac{3x}{4} = \frac{3}{4}\cdot\frac{x}{1} = \frac{3}{4}x.$$

If we view the left side this way, we can get away with solving the equation in one step, by multiplying on each side by the reciprocal of $\frac{3}{4}$.

$$\begin{aligned}
\frac{3x}{4} &= 10\\
\frac{3}{4}x &= 10\\
\frac{4}{3}\cdot\frac{3}{4}x &= \frac{4}{3}\cdot 10\\
x &= \frac{4}{3}\cdot\frac{10}{1}\\
x &= \frac{40}{3}
\end{aligned}$$

We will check the solution by substituting x in the original equation with $\frac{40}{3}$:

$$\begin{aligned}
\frac{3x}{4} &= 10\\
\frac{3\left(\frac{40}{3}\right)}{4} &\stackrel{?}{=} 10\\
\frac{40}{4} &\stackrel{?}{=} 10\\
10 &\stackrel{\checkmark}{=} 10
\end{aligned}$$

The solution $\frac{40}{3}$ is checked and the solution set is $\left\{\frac{40}{3}\right\}$.

Checkpoint 2.5.24. Solve for H in the equation $\frac{-7H}{12} = \frac{2}{3}$.

The solution is ☐.

The solution set is ☐.

Explanation. The left side is effectively the same things as $-\frac{7}{12}H$, so multiplying by $-\frac{12}{7}$ will isolate H.

$$\begin{aligned}
\frac{-7H}{12} &= \frac{2}{3}\\
-\frac{7}{12}H &= \frac{2}{3}\\
\left(-\frac{12}{7}\right)\cdot\left(-\frac{7}{12}H\right) &= \left(-\frac{12}{7}\right)\cdot\frac{2}{3}\\
H &= -\frac{4}{7}\cdot\frac{2}{1}\\
H &= -\frac{8}{7}
\end{aligned}$$

We will check the solution by substituting H in the original equation with $-\frac{8}{7}$:

$$\begin{aligned}
\frac{-7H}{12} &= \frac{2}{3}\\
\frac{-7\left(-\frac{8}{7}\right)}{12} &\stackrel{?}{=} \frac{2}{3}\\
\frac{8}{12} &\stackrel{?}{=} \frac{2}{3}\\
\frac{2}{3} &\stackrel{\checkmark}{=} \frac{2}{3}
\end{aligned}$$

The solution $-\frac{8}{7}$ is checked and the solution set is $\left\{-\frac{8}{7}\right\}$.

Exercises

Review and Warmup

1. Add the following.

a. $-8 + (-1) = \boxed{}$

b. $-7 + (-7) = \boxed{}$

c. $-2 + (-9) = \boxed{}$

2. Add the following.

a. $-8 + (-2) = \boxed{}$

b. $-5 + (-3) = \boxed{}$

c. $-2 + (-7) = \boxed{}$

3. Add the following.

a. $5 + (-9) = \boxed{}$

b. $5 + (-2) = \boxed{}$

c. $7 + (-7) = \boxed{}$

4. Add the following.

a. $1 + (-6) = \boxed{}$

b. $7 + (-3) = \boxed{}$

c. $6 + (-6) = \boxed{}$

5. Add the following.

a. $-8 + 1 = \boxed{}$

b. $-4 + 10 = \boxed{}$

c. $-5 + 5 = \boxed{}$

6. Add the following.

a. $-10 + 2 = \boxed{}$

b. $-1 + 6 = \boxed{}$

c. $-5 + 5 = \boxed{}$

7. Evaluate the following.

a. $\frac{-63}{-7} = \boxed{}$

b. $\frac{30}{-5} = \boxed{}$

c. $\frac{-35}{7} = \boxed{}$

8. Evaluate the following.

a. $\frac{-36}{-6} = \boxed{}$

b. $\frac{28}{-7} = \boxed{}$

c. $\frac{-63}{7} = \boxed{}$

9. Do the following multiplications:

a. $15 \cdot \frac{4}{5} = \boxed{}$

b. $20 \cdot \frac{4}{5} = \boxed{}$

c. $25 \cdot \frac{4}{5} = \boxed{}$

10. Do the following multiplications:

a. $36 \cdot \frac{4}{9} = \boxed{}$

b. $45 \cdot \frac{4}{9} = \boxed{}$

c. $54 \cdot \frac{4}{9} = \boxed{}$

11. Evaluate the following.

a. $\frac{-3}{-1} = \boxed{}$

b. $\frac{7}{-1} = \boxed{}$

c. $\frac{130}{-130} = \boxed{}$

d. $\frac{-16}{-16} = \boxed{}$

e. $\frac{11}{0} = \boxed{}$

f. $\frac{0}{-7} = \boxed{}$

12. Evaluate the following.

a. $\frac{-2}{-1} = \boxed{}$

b. $\frac{4}{-1} = \boxed{}$

c. $\frac{180}{-180} = \boxed{}$

d. $\frac{-19}{-19} = \boxed{}$

e. $\frac{11}{0} = \boxed{}$

f. $\frac{0}{-3} = \boxed{}$

Solving One-Step Equations with Addition/Subtraction Solve the equation.

13. $x + 7 = 10$

14. $x + 5 = 12$

15. $y + 1 = -9$

16. $y + 8 = 1$

17. $5 = y + 8$

18. $-2 = r + 6$

19. $-9 = r - 7$

20. $-14 = t - 9$

21. $t + 78 = 0$

22. $x + 50 = 0$

23. $x - 3 = 2$

24. $y - 10 = -2$

25. $-16 = y - 7$

26. $-9 = y - 4$

27. $r - (-5) = 7$

28. $r - (-7) = 10$

29. $-1 = t - (-5)$

30. $-8 = t - (-1)$

31. $3 + x = -4$

32. $1 + x = -9$

33. $3 = -3 + y$

34. $4 = -6 + y$

35. $y + \frac{3}{4} = \frac{3}{4}$

36. $r + \frac{7}{10} = \frac{9}{10}$

37. $-\frac{2}{7} + r = -\frac{5}{14}$

38. $-\frac{8}{3} + t = -\frac{5}{6}$

39. $\frac{2}{7} + q = -\frac{1}{8}$

40. $\frac{6}{5} + x = -\frac{1}{4}$

Solving One-Step Equations with Multiplication/Division Solve the equation.

41. $2x = 22$

42. $8y = 56$

43. $40 = -8y$

44. $33 = -3y$

45. $0 = 13B$

46. $0 = -22C$

47. $\frac{1}{10}n = 5$

48. $\frac{1}{7}q = 8$

49. $\frac{4}{13}x = 8$

50. $\frac{7}{12}r = 28$

51. $\frac{5}{2}t = 4$

52. $\frac{5}{3}b = 2$

53. $\frac{9}{8} = -\frac{c}{10}$

54. $\frac{3}{2} = -\frac{B}{6}$

55. $3C = -8$

56. $9n = -2$

57. $-35 = -30p$

58. $-25 = -10x$

59. $-\frac{r}{18} = \frac{4}{9}$

60. $-\frac{t}{30} = \frac{8}{5}$

61. $-\frac{b}{8} = -\frac{3}{2}$

62. $-\frac{c}{24} = -\frac{7}{8}$

63. $-\frac{3}{5} = \frac{3B}{10}$

64. $-\frac{7}{2} = \frac{7C}{9}$

65. $\frac{x}{72} = \frac{5}{9}$

66. $\frac{x}{50} = \frac{3}{10}$

67. $\frac{9}{2} = \frac{x}{18}$

68. $\frac{2}{3} = \frac{x}{21}$

Comparisons Solve the equation.

69.
a. $4y = 32$
b. $4 + r = 32$

70.
a. $8y = 16$
b. $8 + t = 16$

71.
a. $28 = -7y$
b. $28 = -7 + r$

72.
a. $18 = -3r$
b. $18 = -3 + x$

73.
a. $-r = 5$
b. $-t = -5$

74.
a. $-t = 13$
b. $-r = -13$

75.
a. $-\frac{1}{2}p = 7$
b. $-\frac{1}{2}r = -7$

76.
a. $-\frac{1}{6}x = 5$
b. $-\frac{1}{6}n = -5$

77.
a. $36 = -\frac{9}{2}y$
b. $-36 = -\frac{9}{2}A$

78.
a. $16 = -\frac{4}{9}t$
b. $-16 = -\frac{4}{9}r$

79.
a. $8y = 32$
b. $12t = 68$

80.
a. $4y = 16$
b. $21x = 51$

81.
a. $30 = -6r$
b. $64 = -12x$

82.
a. $40 = -10r$
b. $80 = -35t$

Geometry Application Problems

83. A circle's circumference is 18π mm.

a. This circle's diameter is ☐.

b. This circle's radius is ☐.

84. A circle's circumference is 20π mm.

a. This circle's diameter is ☐.

b. This circle's radius is ☐.

85. A circle's circumference is 30 cm. Find the following values. Round your answer to at least 2 decimal places.

a. This circle's diameter is ☐.

b. This circle's radius is ☐.

87. A circle's circumference is 8π mm.

a. This circle's diameter is ☐.

b. This circle's radius is ☐.

89. A circle's circumference is 39 cm. Find the following values. Round your answer to at least 2 decimal places.

a. This circle's diameter is ☐.

b. This circle's radius is ☐.

91. A cylinder's base's radius is 7 m, and its volume is 392π m^3.

This cylinder's height is ☐.

93. A rectangle's area is 570 mm^2. Its height is 19 mm.

Its base is ☐.

95. A rectangular prism's volume is 3822 ft^3. The prism's base is a rectangle. The rectangle's length is 21 ft and the rectangle's width is 13 ft.

This prism's height is ☐.

97. A triangle's area is 187.5 m^2. Its base is 25 m.

Its height is ☐.

86. A circle's circumference is 32 cm. Find the following values. Round your answer to at least 2 decimal places.

a. This circle's diameter is ☐.

b. This circle's radius is ☐.

88. A circle's circumference is 10π mm.

a. This circle's diameter is ☐.

b. This circle's radius is ☐.

90. A circle's circumference is 42 cm. Find the following values. Round your answer to at least 2 decimal places.

a. This circle's diameter is ☐.

b. This circle's radius is ☐.

92. A cylinder's base's radius is 4 m, and its volume is 144π m^3.

This cylinder's height is ☐.

94. A rectangle's area is 300 mm^2. Its height is 15 mm.

Its base is ☐.

96. A rectangular prism's volume is 10450 ft^3. The prism's base is a rectangle. The rectangle's length is 22 ft and the rectangle's width is 19 ft.

This prism's height is ☐.

98. A triangle's area is 162.5 m^2. Its base is 25 m.

Its height is ☐.

Challenge

99. Write a linear equation whose solution is $x = 5$. You may not write an equation whose left side is just "x" or whose right side is just "x."

There are infinitely many correct answers to this problem. *Be creative.* After finding an equation that works, see if you can come up with a different one that also works.

100. Fill in the blanks with the numbers 38 and 77 (using each number only once) to create an equation where x has the greatest possible value.

a. $\boxed{} + x = \boxed{}$

b. $\boxed{} = \boxed{} \cdot x$

2.6 Solving One-Step Inequalities

We have learned how to check whether a specific number is a solution to an equation or inequality. In this section, we will begin learning how to *find* the solution(s) to basic inequalities ourselves.

With one small complication, we can use very similar properties to Fact 2.5.12 when we solve inequalities (as opposed to equations).

Here are some numerical examples.

Add to both sides If $2 < 4$, then $2 + 1 \overset{\checkmark}{<} 4 + 1$.

Subtract from both sides If $2 < 4$, then $2 - 1 \overset{\checkmark}{<} 4 - 1$.

Multiply on both sides by a *positive* number If $2 < 4$, then $3 \cdot 2 \overset{\checkmark}{<} 3 \cdot 4$.

Divide on both sides by a *positive* number If $2 < 4$, then $\frac{2}{2} \overset{\checkmark}{<} \frac{4}{2}$.

However, something interesting happens when we multiply or divide by the same *negative* number on both sides of an inequality: the direction reverses! To understand why, consider Figure 2.6.2, where the numbers 2 and 4 are multiplied by the negative number −1.

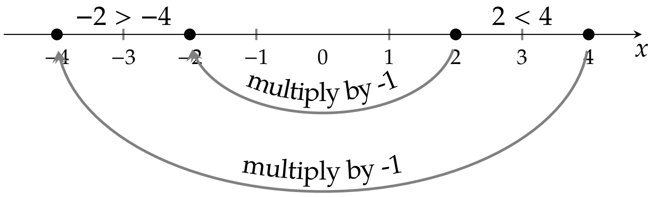


Figure 2.6.2: Multiplying two numbers by a negative number, and how their relationship changes

So even though $2 < 4$, if we multiply both sides by −1, we have $-2 \overset{\text{no}}{<} -4$. (The true inequality is $-2 > -4$.)

In general, we must apply the following property when solving an inequality.

Fact 2.6.3 Changing the Direction of the Inequality Sign. *When we multiply or divide each side of an inequality by the same* negative *number, the inequality sign must change direction.* Do not *change the inequality sign when multiplying/dividing by a positive number, or when adding/subtracting by any number.*

Example 2.6.4 Solve the inequality $-2x \geq 12$. State the solution set graphically, using interval notation, and using set-builder notation. (Interval notation and set-builder notation are discussed in Section 1.6.

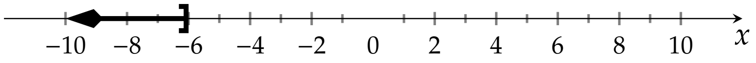
Explanation. To solve this inequality, we will divide each side by −2:

$$\begin{aligned} -2x &\geq 12 \\ \frac{-2x}{-2} &\leq \frac{12}{-2} && \text{Note the change in direction.} \\ x &\leq -6 \end{aligned}$$

Note that the inequality sign changed direction at the step where we divided each side of the inequality by a *negative* number.

When we solve a linear *equation*, there is usually exactly one solution. When we solve a linear *inequality*, there are usually infinitely many solutions. For this example, any number smaller than −6 or equal to −6 is a solution.

There are at least three ways to represent the solution set for the solution to an inequality: graphically, with set-builder notation, and with interval notation. Graphically, we represent the solution set as:



Using interval notation, we write the solution set as $(-\infty, -6]$. Using set-builder notation, we write the solution set as $\{x \mid x \leq -6\}$.

As with equations, we should check solutions to catch both human mistakes as well as for possible extraneous solutions (numbers which were *possible* solutions according to algebra, but which actually do not solve the inequality).

Since there are infinitely many solutions, it's impossible to literally check them all. We found that all values of x for which $x \leq -6$ are solutions. One approach is to check that −6 satisfies the inequality, and also that one number less than −6 (any number, your choice) is a solution.

$$\begin{aligned} -2x &\geq 12 \\ -2(-6) &\overset{?}{\geq} 12 \\ 12 &\overset{\checkmark}{\geq} 12 \end{aligned} \qquad\qquad \begin{aligned} -2x &\geq 12 \\ -2(-7) &\overset{?}{\geq} 12 \\ 14 &\overset{\checkmark}{\geq} 12 \end{aligned}$$

Thus both −6 and −7 are solutions. It's important to note this doesn't directly verify that *all* solutions to this inequality check. But it is evidence that our solution is correct, and it's valuable in that making these two checks would likely help us catch an error if we had made one. Consult your instructor to see if you're expected to check your answer in this manner.

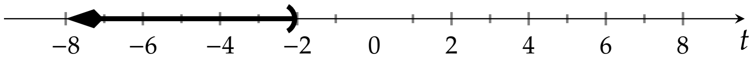
Example 2.6.5 Solve the inequality $t + 7 < 5$. State the solution set graphically, using interval notation, and using set-builder notation.

Explanation. To solve this inequality, we will subtract 7 from each side. There is not much difference between this process and solving the *equation* $t + 7 = 5$, because we are not going to multiply or divide by negative numbers.

$$\begin{aligned} t + 7 &< 5 \\ t + 7 - 7 &< 5 - 7 \\ t &< -2 \end{aligned}$$

Note again that the direction of the inequality did not change, since we did not multiply or divide each side of the inequality by a negative number at any point.

Graphically, we represent this solution set as:



Using interval notation, we write the solution set as $(-\infty, -2)$. Using set-builder notation, we write the solution set as $\{t \mid t < -2\}$.

We should check that -2 is *not* a solution, but that some number less than -2 *is* a solution.

$$\begin{aligned} t+7 &< 5 & \qquad t+7 &< 5 \\ -2+7 &\overset{?}{<} 5 & -10+7 &\overset{?}{<} 5 \\ 5 &\overset{\text{no}}{<} 5 & -3 &\overset{\checkmark}{<} 5 \end{aligned}$$

So our solution is reasonably checked.

Checkpoint 2.6.6. Solve the inequality $x - 5 > -4$. State the solution set using interval notation and using set-builder notation.

In interval notation, the solution set is ______.

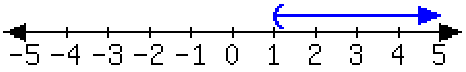
In set-builder notation, the solution set is ______.

Explanation. To solve this inequality, we will add 5 to each side.

$$\begin{aligned} x-5 &> -4 \\ x-5+5 &> -4+5 \\ x &> 1 \end{aligned}$$

Note again that the direction of the inequality did not change, since we did not multiply or divide each side of the inequality by a negative number at any point.

Graphically, we represent this solution set as:



Using interval notation, we write the solution set as $(1, \infty)$. Using set-builder notation, we write the solution set as $\{x \mid x > 1\}$.

We should check that 1 is *not* a solution, but that some number greater than 1 *is* a solution.

$$\begin{aligned} x-5 &> -4 & \qquad x-5 &> -4 \\ 1-5 &\overset{?}{<} -4 & 10-5 &\overset{?}{<} -4 \\ -4 &\overset{\text{no}}{<} -4 & 5 &\overset{\checkmark}{<} -4 \end{aligned}$$

So our solution is reasonably checked.

Checkpoint 2.6.7. Solve the inequality $-\frac{1}{2}z \geq -1.74$. State the solution set using interval notation and using set-builder notation.

In interval notation, the solution set is ______.

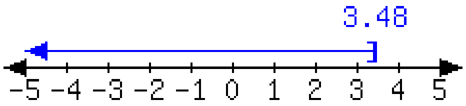
In set-builder notation, the solution set is ______.

Explanation. To solve this inequality, we will multiply by -2 to each side.

$$\begin{aligned} -\frac{1}{2}z &\geq -1.74 \\ (-2)\left(-\frac{1}{2}z\right) &\leq (-2)(-1.74) \\ z &\leq 3.48 \end{aligned}$$

In this exercise, we *did* multiply by a negative number and so the direction of the inequality sign changed.

Graphically, we represent this solution set as:



Using interval notation, we write the solution set as $(-\infty, 3.48]$. Using set-builder notation, we write the solution set as $\{z \mid z \leq 3.48\}$.

We should check that 3.48 *is* a solution, and also that some number less than 3.48 is a solution.

$$\begin{aligned} -\frac{1}{2}z &\geq -1.74 & -\frac{1}{2}z &\geq -1.74 \\ -\frac{1}{2}(3.48) &\overset{?}{\geq} -1.74 & -\frac{1}{2}(0) &\overset{?}{\geq} -1.74 \\ -1.74 &\overset{\checkmark}{\geq} -1.74 & 0 &\overset{\checkmark}{\geq} -1.74 \end{aligned}$$

So our solution is reasonably checked.

Exercises

Review and Warmup

1. Add the following.

a. $-10 + (-2) =$

b. $-5 + (-3) =$

c. $-3 + (-10) =$

2. Add the following.

a. $-10 + (-3) =$

b. $-7 + (-4) =$

c. $-3 + (-8) =$

3. Add the following.

a. $2 + (-7) =$

b. $8 + (-3) =$

c. $3 + (-3) =$

4. Add the following.

a. $2 + (-8) =$

b. $10 + (-4) =$

c. $3 + (-3) =$

5. Add the following.

a. $-10 + 3 =$

b. $-1 + 7 =$

c. $-8 + 8 =$

6. Add the following.

a. $-7 + 3 =$

b. $-2 + 9 =$

c. $-8 + 8 =$

7. Evaluate the following.

a. $\frac{-20}{-4} =$

b. $\frac{28}{-7} =$

c. $\frac{-12}{6} =$

8. Evaluate the following.

a. $\frac{-6}{-3} =$

b. $\frac{50}{-5} =$

c. $\frac{-42}{6} =$

9. Do the following multiplications:

a. $16 \cdot \frac{5}{8} =$

b. $24 \cdot \frac{5}{8} =$

c. $32 \cdot \frac{5}{8} =$

10. Do the following multiplications:

a. $15 \cdot \frac{2}{5} =$

b. $20 \cdot \frac{2}{5} =$

c. $25 \cdot \frac{2}{5} =$

11. Evaluate the following.

a. $\frac{-9}{-1} =$

b. $\frac{10}{-1} =$

c. $\frac{190}{-190} =$

d. $\frac{-19}{-19} =$

e. $\frac{5}{0} =$

f. $\frac{0}{-10} =$

12. Evaluate the following.

a. $\frac{-8}{-1} =$

b. $\frac{8}{-1} =$

c. $\frac{130}{-130} =$

d. $\frac{-11}{-11} =$

e. $\frac{5}{0} =$

f. $\frac{0}{-5} =$

Solving One-Step Inequalities using Addition/Subtraction Solve this inequality.

13. $x + 2 > 7$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

14. $x + 3 > 10$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

15. $x - 3 \leq 8$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

16. $x - 4 \leq 7$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

17. $4 \leq x + 10$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

18. $5 \leq x + 8$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

19. $1 > x - 7$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

20. $1 > x - 10$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

Solving One-Step Inequalities using Multiplication/Division Solve this inequality.

21. $2x \leq 6$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

22. $3x \leq 6$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

23. $9x > 5$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

24. $1x > 6$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

25. $-4x \geq 8$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

26. $-5x \geq 20$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

27. $15 \geq -5x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

28. $4 \geq -2x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

29. $3 < -x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

30. $4 < -x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

31. $-x \leq 5$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

32. $-x \leq 6$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

33. $\frac{6}{5}x > 6$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

34. $\frac{7}{10}x > 14$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

35. $-\frac{8}{7}x \leq 32$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

36. $-\frac{9}{4}x \leq 18$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

37. $-3 < \frac{1}{10}x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

38. $-2 < \frac{2}{7}x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

39. $-9 < -\frac{3}{4}x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

40. $-8 < -\frac{4}{9}x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

41. $3x > -9$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

42. $4x > -8$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

43. $-16 < -4x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

44. $-15 < -5x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

45. $\frac{3}{10} \geq \frac{x}{20}$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

46. $\frac{9}{2} \geq \frac{x}{8}$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

47. $-\frac{z}{12} < -\frac{5}{2}$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

48. $-\frac{z}{12} < -\frac{3}{4}$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

Challenge

49. Choose the correct inequality or equal sign to make the relation true.

a. Let x and y be integers, such that $x < y$.

Then $x - y$ (☐ < ☐ > ☐ =) $y - x$.

b. Let x and y be integers, such that $1 < x < y$.

Then xy (☐ < ☐ > ☐ =) $x + y$.

c. Let x and y be rational numbers, such that $0 < x < y < 1$.

Then xy (☐ < ☐ > ☐ =) $x + y$.

d. Let x and y be integers, such that $x < y$.

Then $x + 2y$ (☐ < ☐ > ☐ =) $2x + y$.

2.7 Percentages

Percent-related problems arise in everyday life. This section reviews some basic calculations that can be made with percentages.

In many situations when translating from English to math, the word "of" translates as multiplication. Also the word "is" (and many similar words related to "to be") translates to an equals sign. For example:

One third of thirty is ten.

$$\frac{1}{3} \cdot 30 = 10$$

Here is another example, this time involving a percentage. We know that "2 is 50% of 4," so we can say:

$$\begin{aligned}&2 \text{ is } 50\% \text{ of } 4\\ &2 = 0.5 \cdot 4\end{aligned}$$

Example 2.7.2 Translate each statement involving percents below into an equation. Define any variables used. (Solving these equations is an exercise).

a. How much is 30% of $24.00?

b. $7.20 is what percent of $24.00?

c. $7.20 is 30% of how much money?

Explanation. Each question can be translated from English into a math equation by reading it slowly and looking for the right signals.

a. The word "is" means about the same thing as the equals sign. "How much" is a question phrase, and we can let x be the unknown amount (in dollars). The word "of" translates to multiplication, as discussed earlier. So we have:

$$\overset{\substack{\text{how much}\\ |}}{x} \;\overset{\substack{\text{is}\\ |}}{=}\; \overset{\substack{30\%\\ |}}{0.30} \;\overset{\substack{\text{of}\\ |}}{\cdot}\; \overset{\substack{\$24\\ |}}{24}$$

b. Let P be the unknown value. We have:

$$\overset{\substack{\$7.20\\ |}}{7.2} \;\overset{\substack{\text{is}\\ |}}{=}\; \overset{\substack{\text{what percent}\\ |}}{P} \;\overset{\substack{\text{of}\\ |}}{\cdot}\; \overset{\substack{\$24\\ |}}{24}$$

With this setup, P is going to be a decimal value (0.30) that you would translate into a percentage (30%).

c. Let x be the unknown amount (in dollars). We have:

$$\overset{\substack{\$7.20\\ |}}{7.2} \;\overset{\substack{\text{is}\\ |}}{=}\; \overset{\substack{30\%\\ |}}{0.30} \;\overset{\substack{\text{of}\\ |}}{\cdot}\; \overset{\substack{\text{how much}\\ |}}{x}$$

Checkpoint 2.7.3. Solve each equation from Example 2.7.2.

a. How much is 30% of \$24.00?

$$x = 0.3 \cdot 24$$

x is $\boxed{}$.

b. \$7.20 is what percent of \$24.00?

$$7.2 = P \cdot 24$$

P is $\boxed{}$.

c. \$7.20 is 30% of how much money?

$$7.2 = 0.3 \cdot x$$

x is $\boxed{}$.

Explanation.

a. $$\begin{aligned} x &= 0.3 \cdot 24 \\ x &= 8 \end{aligned}$$

b. $$\begin{aligned} 7.2 &= P \cdot 24 \\ \frac{7.2}{24} &= \frac{P \cdot 24}{24} \\ 0.3 &= P \end{aligned}$$

c. $$\begin{aligned} 7.2 &= 0.3 \cdot x \\ \frac{7.2}{0.3} &= \frac{0.3 \cdot x}{0.3} \\ 24 &= x \end{aligned}$$

2.7.1 Setting up and Solving Percent Equations

An important skill for solving percent-related problems is to boil down a complicated word problem into a simple form like “2 is 50% of 4.” Let’s look at some further examples.

Example 2.7.4

In Fall 2016, Portland Community College had 89,900 enrolled students. According to Figure 2.7.5, how many black students were enrolled at PCC in Fall 2016?

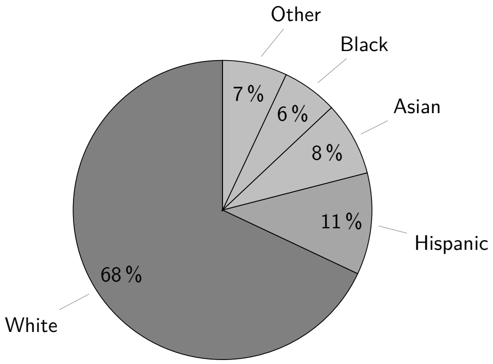


Figure 2.7.5: Racial breakdown of PCC students in Fall 2016

Explanation. After reading this word problem and the chart, we can translate the problem into “what is 6% of 89,900?” Let x be the number of black students enrolled at PCC in Fall 2016. We can set up and solve the equation:

$$\begin{aligned} \overset{\text{what}}{\overset{|}{x}} \; &\overset{\text{is}}{\overset{|}{=}} \overset{6\%}{\overset{|}{0.06}} \overset{\text{of}}{\overset{|}{\cdot}} \overset{89{,}900}{\overset{|}{89900}} \\ x &= 5394 \end{aligned}$$

There was not much “solving” to do, since the variable we wanted to isolate was already isolated.

As of Fall 2016, Portland Community College had 5394 black students. Note: this is not likely to be perfectly accurate, because the numbers we started with (89,900 enrolled students and 6%) appear to be rounded.

Example 2.7.6

The bar graph in Figure 2.7.7 displays how many students are in each class at a local high school. According to the bar graph, what percentage of the school's student population is freshman?

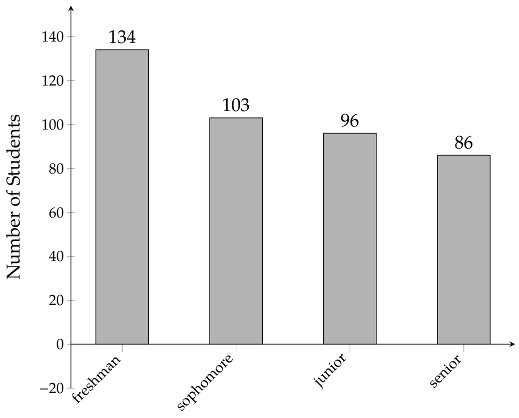


Figure 2.7.7: Number of students at a high school by class

Explanation. The school's total number of students is:

$$134 + 103 + 96 + 86 = 419$$

With that calculated, we can translate the main question:

> "What percentage of the school's student population is freshman?"

into:

> "What percent of 419 is 134?"

Using P to represent the unknown quantity, we write and solve the equation:

$$\begin{aligned}
\overset{\substack{\text{what percent}\\ |}}{P} \overset{\substack{\text{of}\\ |}}{\cdot} \overset{\substack{419\\ |}}{419} \overset{\substack{\text{is}\\ |}}{=} \overset{\substack{134\\ |}}{134}& \\
\frac{P \cdot 419}{419} &= \frac{134}{419} \\
P &\approx 0.3198 \\
P &\approx 31.98\%
\end{aligned}$$

Approximately 31.98% of the school's student population is freshman.

Remark 2.7.8. When solving equations that do *not* have context we state the solution set. However, when solving an equation or inequality that arises in an application problem (such as the context of the high school in Example 2.7.6), it makes more sense to summarize our result with a sentence, using the context of the application. This allows us to communicate the full result, including appropriate units.

Example 2.7.9 Carlos just received his monthly paycheck. His gross pay (the amount before taxes and related things are deducted) was \$2,346.19, and his total tax and other deductions was \$350.21. The rest was deposited directly into his checking account. What percent of his gross pay went into his checking account?

Explanation. Train yourself to read the word problem and not try to pick out numbers to substitute into formulas. You may find it helps to read the problem over to yourself three or more times before you attempt to solve it. There are *three* dollar amounts to discuss in this problem, and many students fall into a trap of using the wrong values in the wrong places. There is the gross pay, the amount that was deducted, and the amount that was deposited. Only two of these have been explicitly written down. We need to use subtraction to find the dollar amount that was deposited:

$$2346.19 - 350.21 = 1995.98$$

Now, we can translate the main question:

> "What percent of his gross pay went into his checking account?"

into:

> "What percent of \$2346.19 is \$1995.98?"

Using P to represent the unknown quantity, we write and solve the equation:

$$\begin{aligned}
\overset{\substack{\text{what percent}\\|}}{P} \;\overset{\substack{\text{of}\\|}}{\cdot}\; \overset{\substack{\$2346.19\\|}}{2346.19} &\overset{\substack{\text{is}\\|}}{=} \overset{\substack{\$1995.98\\|}}{1995.98}\\
\frac{P \cdot 2346.19}{2346.19} &= \frac{1995.98}{2346.19}\\
P &\approx 0.8507\\
P &\approx 85.07\%
\end{aligned}$$

Approximately 85.07% of his gross pay went into his checking account.

Checkpoint 2.7.10. Alexis sells cars for a living, and earns 28% of the dealership's sales profit as commission. In a certain month, she plans to earn \$2200 in commissions. How much total sales profit does she need to bring in for the dealership?

Alexis needs to bring in ______ in sales profit.

Explanation. Be careful that you do not calculate 28% of \$2200. That might be what a student would do who doesn't thoroughly read the question. If you have ever trained yourself to quickly find numbers in word problems and substitute them into formulas, you must *unlearn* this. The issue is that \$2200 is not the dealership's sales profit, and if you mistakenly multiply $0.28 \cdot 2200 = 616$, then \$616 makes no sense as an answer to this question. How could Alexis bring in only \$616 of sales profit, and be rewarded with \$2200 in commission?

We can translate the problem into "\$2200 is 28% of what?" Letting x be the sales profit for the dealership (in dollars), we can write and solve the equation:

$$\begin{aligned}
\overset{\text{\$2200}}{\overset{|}{2200}} &\overset{\text{is}}{\overset{|}{=}} \overset{\text{28\%}}{\overset{|}{0.28}} \overset{\text{of}}{\overset{|}{\cdot}} \overset{\text{what}}{\overset{|}{x}}\\
\frac{2200}{0.28} &= \frac{0.28x}{0.28}\\
7857.14 &\approx x\\
x &\approx 7857.14
\end{aligned}$$

To earn \$2200 in commission, Alexis needs to bring in approximately \$7857.14 of sales profit for the dealership.

Example 2.7.11

According to e-Literate, the average cost of a new college textbook has been increasing. Find the percentage of increase from 2009 to 2013.

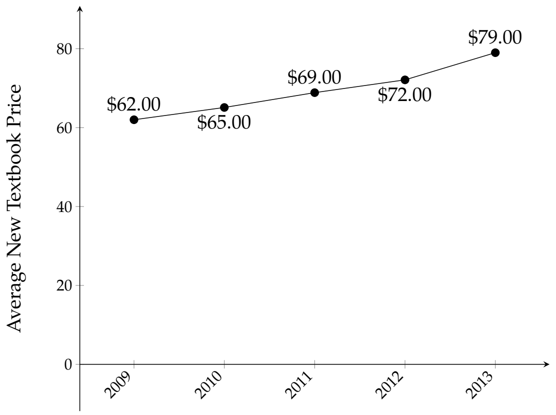


Figure 2.7.12: Average New Textbook Price from 2009 to 2013

Explanation. The actual amount of increase from 2009 to 2013 was 79 − 62 = 17, dollars. We need to answer the question “\$17 is what percent of \$62?” Note that we are comparing the 17 to 62, not to 79. In these situations where one amount is the earlier amount, the earlier original amount is the one that represents 100%. Let P represent the percent of increase. We can set up and solve the equation:

$$\begin{aligned}
\overset{\text{\$17}}{\overset{|}{17}} &\overset{\text{is}}{\overset{|}{=}} \overset{\text{what percent}}{\overset{|}{P}} \overset{\text{of}}{\overset{|}{\cdot}} \overset{\text{\$62}}{\overset{|}{62}}\\
17 &= 62P\\
\frac{17}{62} &= \frac{62P}{62}\\
0.2742 &\approx P
\end{aligned}$$

From 2009 to 2013, the average cost of a new textbook increased by approximately 27.42%.

Checkpoint 2.7.13. Last month, a full tank of gas for a car you drive cost you \$40.00. You hear on the news that gas prices have risen by 12%. By how much, in dollars, has the cost of a full tank gone up?

A full tank of gas now costs ☐ more than it did last month.

Explanation. Let x represent the amount of increase. We can set up and solve the equation:

$$\begin{aligned} \overset{\text{12\%}}{\overset{|}{0.12}} \cdot \overset{\text{of}}{\overset{|}{}} \overset{\text{old cost}}{\overset{|}{40}} &\overset{\text{is}}{\overset{|}{=}} \overset{\text{how much}}{\overset{|}{x}} \\ 4.8 &= x \end{aligned}$$

A full tank now costs $4.80 more than it did last month.

Example 2.7.14 Enrollment at your neighborhood's elementary school two years ago was 417 children. After a 15% increase last year and a 15% decrease this year, what's the new enrollment?

Explanation. It is tempting to think that increasing by 15% and then decreasing by 15% would bring the enrollment right back to where it started. But the 15% decrease applies to the enrollment *after* it had already increased. So that 15% decrease is going to translate to *more* students lost than were gained.

Using 100% as corresponding to the enrollment from two years ago, the enrollment last year was 100% + 15% = 115% of that. But then using 100% as corresponding to the enrollment from last year, the enrollment this year was 100% − 15% = 85% of that. So we can set up and solve the equation

$$\begin{aligned} \overset{\text{this year's enrollment}}{\overset{|}{x}} &\overset{\text{is}}{\overset{|}{=}} \overset{\text{85\%}}{\overset{|}{0.85}} \cdot \overset{\text{of}}{\overset{|}{}} \overset{\text{115\%}}{\overset{|}{1.15}} \cdot \overset{\text{of}}{\overset{|}{}} \overset{\text{enrollment two years ago}}{\overset{|}{417}} \\ x &= 0.85 \cdot 1.15 \cdot 417 \\ x &= 407.6175 \end{aligned}$$

We would round and report that enrollment is now 408 students. (The percentage rise and fall of 15% were probably rounded in the first place, which is why we did not end up with a whole number.)

Exercises

Review and Warmup

1. Change the following percentages into decimals:

17% = ☐

54% = ☐

2. Change the following percentages into decimals:

18% = ☐

51% = ☐

3. Convert the following decimals into percentages:

0.29 = ☐

0.67 = ☐

4. Convert the following decimals into percentages:

0.21 = ☐

0.64 = ☐

5. Change the following percentages into decimals:

3% = ☐

30% = ☐

100% = ☐

300% = ☐

6. Change the following percentages into decimals:

4% = ☐

40% = ☐

100% = ☐

400% = ☐

7. Convert the following decimals into percentages:

0.05 = ☐

0.5 = ☐

5 = ☐

1 = ☐

8. Convert the following decimals into percentages:

0.06 = ☐

0.6 = ☐

6 = ☐

1 = ☐

9. Convert the following decimals into percentages:

6.67 = ☐

0.667 = ☐

0.0667 = ☐

10. Convert the following decimals into percentages:

7.22 = ☐

0.722 = ☐

0.0722 = ☐

11. Change the following percentages into decimals:

895% = ☐

89.5% = ☐

8.95% = ☐

12. Change the following percentages into decimals:

959% = ☐

95.9% = ☐

9.59% = ☐

Basic Percentage Calculation

13. 3% of 200 is ☐.

14. 8% of 300 is ☐.

15. 60% of 400 is ☐.

16. 30% of 490 is ☐.

17. 780% of 590 is ☐.

18. 530% of 690 is ☐.

19. Answer with a percent.

176 is ☐ of 220.

20. Answer with a percent.

720 is ☐ of 800.

21. Answer with a percent.

142.1 is ☐ of 49.

22. Answer with a percent.

21.6 is ☐ of 18.

23. Answer with a percent.

12 is about ☐ of 47.

24. Answer with a percent.

8 is about ☐ of 44.

25. 14% of ☐ is 68.6.

26. 72% of ☐ is 424.8.

27. 4% of ☐ is 27.6.

28. 9% of ☐ is 71.1.

29. 420% of ☐ is 3738.

30. 250% of ☐ is 2450.

Applications

31. A town has 1400 registered residents. Among them, 39% were Democrats, 38% were Republicans. The rest were Independents. How many registered Independents live in this town?

There are registered Independent residents in this town.

32. A town has 1800 registered residents. Among them, 36% were Democrats, 27% were Republicans. The rest were Independents. How many registered Independents live in this town?

There are registered Independent residents in this town.

33. Martha is paying a dinner bill of $29.00. Martha plans to pay 12% in tips. How much tip will Martha pay?

Martha will pay in tip.

34. Evan is paying a dinner bill of $33.00. Evan plans to pay 20% in tips. How much tip will Evan pay?

Evan will pay in tip.

35. Ashley is paying a dinner bill of $36.00. Ashley plans to pay 16% in tips. How much in total (including bill and tip) will Ashley pay?

Ashley will pay in total (including bill and tip).

36. Rita is paying a dinner bill of $40.00. Rita plans to pay 12% in tips. How much in total (including bill and tip) will Rita pay?

Rita will pay in total (including bill and tip).

37. A watch's wholesale price was $440.00. The retailer marked up the price by 40%. What's the watch's new price (markup price)?

The watch's markup price is .

38. A watch's wholesale price was $460.00. The retailer marked up the price by 30%. What's the watch's new price (markup price)?

The watch's markup price is .

39. In the past few seasons' basketball games, Timothy attempted 170 free throws, and made 153 of them. What percent of free throws did Timothy make?

Timothy made of free throws in the past few seasons.

40. In the past few seasons' basketball games, Nina attempted 430 free throws, and made 86 of them. What percent of free throws did Nina make?

Nina made of free throws in the past few seasons.

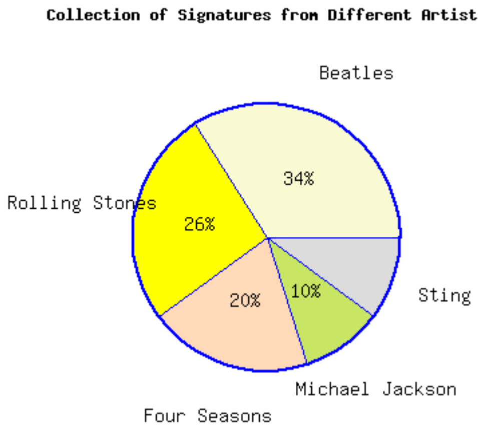
41. A painting is on sale at $360.00. Its original price was $450.00. What percentage is this off its original price?

The painting was off its original price.

42. A painting is on sale at $450.00. Its original price was $500.00. What percentage is this off its original price?

The painting was off its original price.

43. The pie chart represents a collector's collection of signatures from various artists.



If the collector has a total of 1050 signatures, there are ☐ signatures by Sting.

45. In the last election, 34% of a county's residents, or 12240 people, turned out to vote. How many residents live in this county?

This county has ☐ residents.

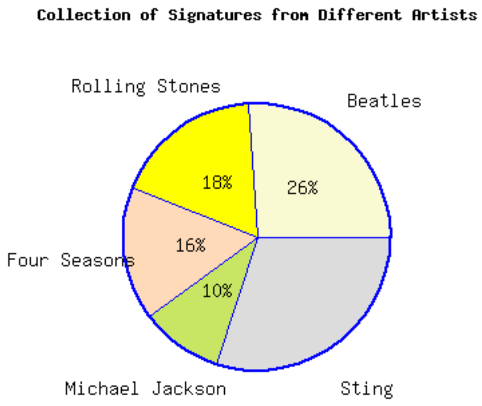
47. 70.68 grams of pure alcohol was used to produce a bottle of 18.6% alcohol solution. What is the weight of the solution in grams?

The alcohol solution weighs ☐.

49. Connor paid a dinner and left 17%, or \$3.74, in tips. How much was the original bill (without counting the tip)?

The original bill (not including the tip) was ☐.

44. The pie chart represents a collector's collection of signatures from various artists.



If the collector has a total of 1250 signatures, there are ☐ signatures by Sting.

46. In the last election, 59% of a county's residents, or 23836 people, turned out to vote. How many residents live in this county?

This county has ☐ residents.

48. 43.6 grams of pure alcohol was used to produce a bottle of 10.9% alcohol solution. What is the weight of the solution in grams?

The alcohol solution weighs ☐.

50. Sydney paid a dinner and left 13%, or \$3.38, in tips. How much was the original bill (without counting the tip)?

The original bill (not including the tip) was ☐.

51. Rebecca sells cars for a living. Each month, she earns $2,000.00 of base pay, plus a certain percentage of commission from her sales.

One month, Rebecca made $52,500.00 in sales, and earned a total of $3,186.50 in that month (including base pay and commission). What percent commission did Rebecca earn?

Rebecca earned in commission.

53. The following is a nutrition fact label from a certain macaroni and cheese box.

Nutrition Facts	
Serving Size 1 cup	
Servings Per Container 2	
Amount Per Serving	
Calories 300	Calories from Fat 110
	% Daily Value
Total Fat 7.8 g	12%
Saturated Fat 2 g	15%
Trans Fat 2 g	
Cholesterol 30 mg	10%
Sodium 400 mg	20%
Total Carbohydrate 30 g	11%
Dietary Fiber 0 g	0%
Sugars 5 g	
Protein 5 g	
Vitamin A 2 mg	3%
Vitamin C 2 mg	2.5%
Calcium 2 mg	20%
Iron 3 mg	4%

The highlighted row means each serving of macaroni and cheese in this box contains 7.8 g of fat, which is 12% of an average person's daily intake of fat. What's the recommended daily intake of fat for an average person?

The recommended daily intake of fat for an average person is .

55. Sydney earned $278.07 of interest from a mutual fund, which was 0.69% of his total investment. How much money did Sydney invest into this mutual fund?

Sydney invested in this mutual fund.

52. Ryan sells cars for a living. Each month, he earns $2,000.00 of base pay, plus a certain percentage of commission from his sales.

One month, Ryan made $57,000.00 in sales, and earned a total of $4,644.80 in that month (including base pay and commission). What percent commission did Ryan earn?

Ryan earned in commission.

54. The following is a nutrition fact label from a certain macaroni and cheese box.

Nutrition Facts	
Serving Size 1 cup	
Servings Per Container 2	
Amount Per Serving	
Calories 300	Calories from Fat 110
	% Daily Value
Total Fat 14 g	20%
Saturated Fat 2 g	15%
Trans Fat 2 g	
Cholesterol 30 mg	10%
Sodium 400 mg	20%
Total Carbohydrate 30 g	11%
Dietary Fiber 0 g	0%
Sugars 5 g	
Protein 5 g	
Vitamin A 2 mg	3%
Vitamin C 2 mg	2.5%
Calcium 2 mg	20%
Iron 3 mg	4%

The highlighted row means each serving of macaroni and cheese in this box contains 14 g of fat, which is 20% of an average person's daily intake of fat. What's the recommended daily intake of fat for an average person?

The recommended daily intake of fat for an average person is .

56. Shane earned $102.81 of interest from a mutual fund, which was 0.23% of his total investment. How much money did Shane invest into this mutual fund?

Shane invested in this mutual fund.

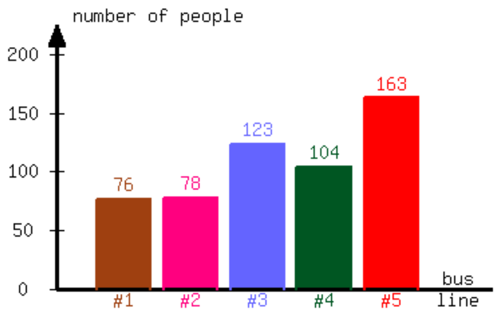
57. A town has 5000 registered residents. Among them, there are 1950 Democrats and 1750 Republicans. The rest are Independents. What percentage of registered voters in this town are Independents?

In this town, ☐ of all registered voters are Independents.

58. A town has 1300 registered residents. Among them, there are 468 Democrats and 299 Republicans. The rest are Independents. What percentage of registered voters in this town are Independents?

In this town, ☐ of all registered voters are Independents.

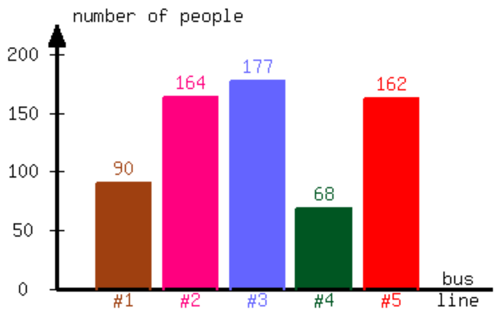
59. A community college conducted a survey about the number of students riding each bus line available. The following bar graph is the result of the survey.



What percent of students ride Bus #1?

Approximately ☐ of students ride Bus #1.

60. A community college conducted a survey about the number of students riding each bus line available. The following bar graph is the result of the survey.



What percent of students ride Bus #1?

Approximately ☐ of students ride Bus #1.

Percent Increase/Decrease

61. The population of cats in a shelter decreased from 100 to 75. What is the percentage decrease of the shelter's cat population?

The percentage decrease is ☐.

62. The population of cats in a shelter decreased from 120 to 108. What is the percentage decrease of the shelter's cat population?

The percentage decrease is ☐.

63. The population of cats in a shelter increased from 55 to 73. What is the percentage increase of the shelter's cat population?

The percentage increase is approximately ☐.

64. The population of cats in a shelter increased from 63 to 75. What is the percentage increase of the shelter's cat population?

The percentage increase is approximately ☐.

65. Last year, a small town's population was 760. This year, the population decreased to 753. What is the percentage decrease?

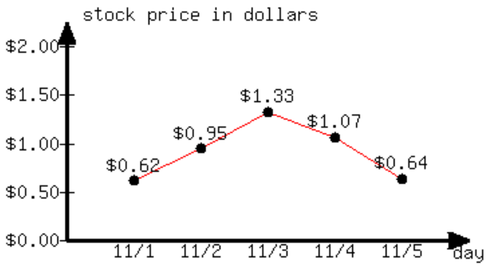
The percentage decrease of the town's population was approximately .

67. Your salary used to be \$39,000 per year.

You had to take a 2% pay cut. After the cut, your salary was per year.

Then, you earned a 2% raise. After the raise, your salary was per year.

69. This line graph shows a certain stock's price change over a few days.



From 11/1 to 11/5, what is the stock price's percentage change?

From 11/1 to 11/5, the stock price's percentage change was approximately .

71. A house was bought two years ago at the price of \$110,000. Each year, the house's value decreased by 5%. What's the house's value this year?

The house's value this year is .

66. Last year, a small town's population was 800. This year, the population decreased to 798. What is the percentage decrease?

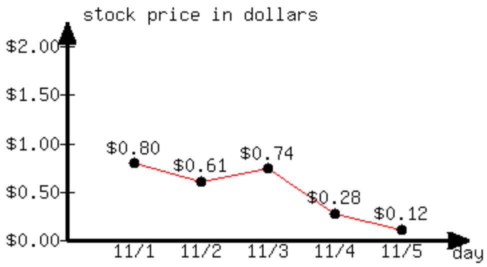
The percentage decrease of the town's population was approximately .

68. Your salary used to be \$31,000 per year.

You had to take a 2% pay cut. After the cut, your salary was per year.

Then, you earned a 2% raise. After the raise, your salary was per year.

70. This line graph shows a certain stock's price change over a few days.



From 11/1 to 11/5, what is the stock price's percentage change?

From 11/1 to 11/5, the stock price's percentage change was approximately .

72. A house was bought two years ago at the price of \$380,000. Each year, the house's value decreased by 6%. What's the house's value this year?

The house's value this year is .

2.8 Modeling with Equations and Inequalities

One purpose of learning math is to be able to model real-life situations and then use the model to ask and answer questions about the situation. In this lesson, we will examine the basics of modeling to set up an equation (or inequality).

2.8.1 Setting Up Equations for Rate Models

To set up an equation modeling a real world scenario, the first thing we need to do is identify what variable we will use. The variable we use will be determined by whatever is unknown in our problem statement. Once we've identified and defined our variable, we'll use the numerical information provided to set up our equation.

Example 2.8.2 A savings account starts with \$500. Each month, an automatic deposit of \$150 is made. Write an equation that represents the number of months it will take for the balance to reach \$1,700.

Explanation.

To determine this equation, we might start by making a table in order to identify a general pattern for the total amount in the account after m months.

Using this pattern, we can determine that an equation showing the unknown number of months, m, when the total savings equals \$1700 would look like this:

$$500 + 150m = 1700$$

Months Since Saving Started	Total Amount Saved (in Dollars)
0	500
1	$500 + 150 = 650$
2	$500 + 150(2) = 800$
3	$500 + 150(3) = 950$
4	$500 + 150(4) = 1100$
$\vdots$	$\vdots$
m	$500 + 150m$

Table 2.8.3: Amount in Savings Account

Remark 2.8.4. To determine the solution to the equation in Example 2.8.2, we could continue the pattern in Table 2.8.3:

We can see that the value of m that makes the equation true is 8 as $500 + 150(8) = 1700$. Thus it would take 8 months for an account starting with \$500 to reach \$1,700 if \$150 is saved each month.

Months Since Saving Started	Total Amount Saved (in Dollars)
5	$500 + 150(5) = 1250$
6	$500 + 150(6) = 1400$
7	$500 + 150(7) = 1550$
8	$500 + 150(8) = 1700$

Table 2.8.5: Amount in Savings Account

Here we are able to determine the solution by creating a table and using inputs that were whole numbers. Often the solution will not be something we can find this way. We will need to solve the equation using algebra, as we'll learn how to do in later sections. For this section, we'll only focus on setting up the equation.

Example 2.8.6 A bathtub contains 2.5 ft^3 of water. More water is being poured in at a rate of 1.75 ft^3 per minute. Write an equation representing when the amount of water in the bathtub will reach 6.25 ft^3.

Explanation.

Since this problem refers to *when* the amount of water will reach a certain amount, we immediately know that the unknown quantity is time. As the volume of water in the tub is measured in ft^3 per minute, we know that time needs to be measured in minutes. We'll define t to be the number of minutes that water is poured into the tub. To determine this equation, we'll start by making a table of values:

Minutes Water Has Been Poured	Total Amount of Water (in ft^3)
0	2.5
1	$2.5 + 1.75 = 4.25$
2	$2.5 + 1.75(2) = 6$
3	$2.5 + 1.75(3) = 7.75$
⋮	⋮
t	$2.5 + 1.75t$

Table 2.8.7: Amount of Water in the Bathtub

Using this pattern, we can determine that the equation representing when the amount will be 6.25 ft^3 is:

$$2.5 + 1.75t = 6.25$$

2.8.2 Setting Up Equations for Percent Problems

Section 2.7 reviewed some basics of working with percentages, and even solved some one-step equations that were set up using percentages. Here we look at some scenarios where there is an equation to set up based on percentages, but it is slightly more involved than a one-step equation.

Example 2.8.8 Jakobi's annual salary as a nurse in Portland, Oregon, is $73,290. His salary increased by 4% from last year. Write a linear equation modeling this scenario, where the unknown value is Jakobi's salary last year.

Explanation. We need to know Jakobi's salary last year. So we'll introduce s, defined to be Jakobi's salary last year (in dollars). To set up the equation, we need to think about how he arrived at this year's salary. To get to this year's salary, his employer took last year's salary and added 4% to it. Conceptually, this means we have:

$$\text{(last year's salary)} + \text{(4\% of last year's salary)} = \text{(this year's salary)}$$

We'll represent 4% of last year's salary with $0.04s$ since 0.04 is the decimal representation of 4%. This means that the equation we set up is:

$$s + 0.04s = 73290$$

Checkpoint 2.8.9. Kirima offered to pay the bill and tip at a restaurant where she and her freinds had dinner. In total she paid $150, which made the tip come out to a little more than 19%. We'd like to know what was the bill before tip. Set up an equation for this situation.

Explanation. A common mistake is to translate a question like this into "what is 19% of $150?" as a way to calculate the tip amount, and then subtract that from $150. But that is not how tipping works. The tip percentage is applied to the original bill, not the final total. If we let x represent the original bill, then:

$$\overset{\text{bill}}{\overset{|}{x}} \;\overset{\text{plus}}{\overset{|}{+}}\; \overset{19\%}{\overset{|}{0.19}} \;\overset{\text{of}}{\overset{|}{\cdot}}\; \overset{\text{bill}}{\overset{|}{x}} \;\overset{\text{is}}{\overset{|}{=}}\; \overset{\$150}{\overset{|}{150}}$$

Example 2.8.10 The price of a refrigerator after a 15% discount is \$612. Write a linear equation modeling this scenario, where the original price of the refrigerator (before the discount was applied) is the unknown quantity.

Explanation. We'll let c be the original price of the refrigerator. To obtain the discounted price, we take the original price and subtract 15% of that amount. Conceptually, this looks like:

$$(\text{original price}) - (15\% \text{ of the original price}) = (\text{discounted price})$$

Since the amount of the discount is 15% of the original price, we'll represent this with $0.15c$. The equation we set up is then:

$$c - 0.15c = 612$$

Checkpoint 2.8.11. A shirt is on sale at 20% off. The current price is \$51.00. Write an equation based on this scenario where the variable represents the shirt's original price.

Explanation. Let x represent the original price of the shirt. Since 20% is removed to bring the cost to \$51, we can set up the equation:

$$\overset{\text{original}}{\overset{|}{x}} \;\overset{\text{minus}}{\overset{|}{-}}\; \overset{20\%}{\overset{|}{0.20}} \;\overset{\text{of}}{\overset{|}{\cdot}}\; \overset{\text{original}}{\overset{|}{x}} \;\overset{\text{is}}{\overset{|}{=}}\; \overset{\$51}{\overset{|}{51}}$$

2.8.3 Setting Up Equations for Geometry Problems

With geometry problems and algebra, there is often the possibility to draw some picture to help understand the scenario better. Additionally it is often necessary to rely on some formula from geometry, such as the formulas from Subsection 2.2.1.

Example 2.8.12 An Olympic-size swimming pool is rectangular and 50 m in length. We don't know its width, but we do know that it required 150 m of painter's tape to outline the edge of the pool during recent renovations. Use this information to set up an equation that models the width of the pool.

Explanation.

Since the pool's shape is a rectangle, it helps to sketch a rectangle representing the pool as in Figure 2.8.13. Since we know its length is 50 m, it is a good idea to label that in the sketch. The width is our unknown quantity, so we can use w as a variable to represent the pool's width in meters and label that too.

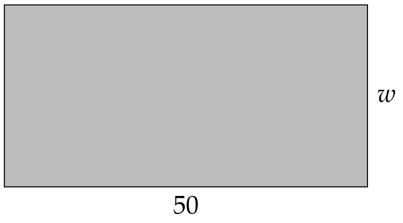
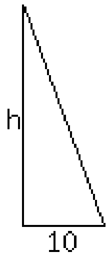


Figure 2.8.13: An Olympic-size pool

> Since it required 150 m of painter's tape to outline the pool, we know the perimeter of the pool is 150 m. This suggests using the perimeter formula for a rectangle: $P = 2(\ell + w)$. (This formula was discussed in Subsection 2.2.1).
>
> With this formula, we can substitute 150 in for P and 50 in for ℓ:
>
> $$150 = 2(50 + w)$$
>
> and this equation models the width of the pool.

Checkpoint 2.8.14. One sail on a sail boat is approximately shaped like a triangle. If the base length is 10 feet and the total sail area is 125 square feet, we can wonder how tall is the sail. Set up an equation to model the sail's height.

Explanation. Since the sail's shape is (approximately) a triangle, it helps to sketch a triangle representing the sail. Since we know its base width is 10 feet, it is a good idea to label that in the sketch. The heigth is our unknown quantity, so we can use h as a variable to represent the sail's height in feet and label that too.



Since the total area is known to be 125 square feet, this suggests using the area formula for a triangle: $A = \frac{1}{2}bh$.

With this formula, we can substitute 125 in for A and 100 in for b:

$$125 = \frac{1}{2}(10)h$$

and this equation models the height of the pool.

2.8.4 Setting Up Inequalities for Models

In general, we'll model using inequalities when we want to determine a maximum or minimum value. To identify that an inequality is needed instead of an equality, we'll look for phrases like *at least, at most, at a minimum* or *at a maximum*.

> **Example 2.8.15** The car share company car2go has a one-time registration fee of \$5 and charges \$14.99 per hour for use of their vehicles. Hana wants to use car2go and has a maximum budget of \$300. Write a linear inequality representing this scenario, where the unknown quantity is the number of hours she uses their vehicles.
>
> **Explanation**. We'll let h be the number of hours that Hana uses car2go. We need the initial cost and the cost from the hourly charge to be less than or equal to \$300, which we set up as:
>
> $$5 + 14.99h \le 300$$

Example 2.8.16 When an oil tank is decommissioned, it is drained of its remaining oil and then re-filled with an inert material, such as sand. A cylindrical oil tank has a volume of 275 gal and is being filled with sand at a rate of 700 gal per hour. Write a linear inequality representing this scenario, where the time it takes for the tank to overflow with sand is the unknown quantity.

Explanation. The unknown in this scenario is time, so we'll define t to be the number of hours that sand is poured into the tank. (Note that we chose hours based on the rate at which the sand is being poured.) We'll represent the amount of sand poured in as $700t$ as each hour an additional 700 gal are added. Given that we want to know when this amount exceeds 275 gal, we set this equation up as:

$$700t > 275$$

2.8.5 Translating Phrases into Mathematical Expressions and Equations/Inequalities

The following table shows how to translate common phrases into mathematical expressions:

English Phrases	Math Expressions
the sum of 2 and a number	$x + 2$ or $2 + x$
2 more than a number	$x + 2$ or $2 + x$
a number increased by 2	$x + 2$ or $2 + x$
a number and 2 together	$x + 2$ or $2 + x$
the difference between a number and 2	$x - 2$
the difference of 2 and a number	$2 - x$
2 less than a number	$x - 2$ (*not* $2 - x$)
a number decreased by 2	$x - 2$
2 decreased by a number	$2 - x$
2 subtracted from a number	$x - 2$
a number subtracted from 2	$2 - x$
the product of 2 and a number	$2x$
twice a number	$2x$
a number times 2	$x \cdot 2$ or $2x$
two thirds of a number	$\frac{2}{3}x$
25% of a number	$0.25x$
the quotient of a number and 2	$x/2$
the quotient of 2 and a number	$2/x$
the ratio of a number and 2	$x/2$
the ratio of 2 and a number	$2/x$

Table 2.8.17: Translating English Phrases into Math Expressions

We can extend this to setting up equations and inequalities. Let's look at some examples. The key is to break a complicated phrase or sentence into smaller parts, identifying key vocabulary such as "is," "of," "greater than," "at most," etc.

English Sentences	Math Equations and Inequalities
The sum of 2 and a number is 6.	$x+2=6$
2 less than a number is at least 6.	$x-2\geq 6$
Twice a number is at most 6.	$2x\leq 6$
6 is the quotient of a number and 2.	$6=\frac{x}{2}$
4 less than twice a number is greater than 10.	$2x-4>10$
Twice the difference between 4 and a number is 10.	$2(4-x)=10$
The product of 2 and the sum of 3 and a number is less than 10.	$2(x+3)<10$
The product of 2 and a number, subtracted from 5, yields 8.	$5-2x=8$
Two thirds of a number subtracted from 10 is 2.	$10-\frac{2}{3}x=2$
25% of the sum of 7 and a number is 2.	$0.25(x+7)=2$

Table 2.8.18: Translating English Sentences into Math Equations

Exercises

Review and Warmup

1. Identify a variable you might use to represent each quantity. And identify what units would be most appropriate.

a. Let ☐ be the area of a house, measured in ☐.

b. Let ☐ be the age of a dog, measured in ☐.

c. Let ☐ be the amount of time passed since a driver left Seattle, Washington, bound for Portland, Oregon, measured in ☐.

2. Identify a variable you might use to represent each quantity. And identify what units would be most appropriate.

a. Let ☐ be the age of a person, measured in ☐.

b. Let ☐ be the distance traveled by a driver that left Portland, Oregon, bound for Boise, Idaho, measured in ☐.

c. Let ☐ be the surface area of the walls of a room, measured in ☐.

Modeling with Linear Equations

3. Chris's annual salary as a radiography technician is \$39,858.00. His salary increased by 2.2% from last year. What was his salary last year?

Assume his salary last year was s dollars. Write an equation to model this scenario. There is no need to solve it.

5. A bicycle for sale costs \$194.04, which includes 7.8% sales tax. What was the cost before sales tax?

Assume the bicycle's price before sales tax is p dollars. Write an equation to model this scenario. There is no need to solve it.

7. The price of a washing machine after 10% discount is \$216.00. What was the original price of the washing machine (before the discount was applied)?

Assume the washing machine's price before the discount is p dollars. Write an equation to model this scenario. There is no need to solve it.

9. The price of a restaurant bill, including an 15% gratuity charge, was \$115.00. What was the price of the bill before gratuity was added?

Assume the bill without gratuity is b dollars. Write an equation to model this scenario. There is no need to solve it.

11. In May 2016, the median rent price for a one-bedroom apartment in a city was reported to be \$908.10 per month. This was reported to be an increase of 0.9% over the previous month. Based on this reporting, what was the median price of a one-bedroom apartment in April 2016?

Assume the median price of a one-bedroom apartment in April 2016 was p dollars. Write an equation to model this scenario. There is no need to solve it.

4. Sherial's annual salary as a radiography technician is \$42,630.00. Her salary increased by 1.5% from last year. What was her salary last year?

Assume her salary last year was s dollars. Write an equation to model this scenario. There is no need to solve it.

6. A bicycle for sale costs \$224.07, which includes 6.7% sales tax. What was the cost before sales tax?

Assume the bicycle's price before sales tax is p dollars. Write an equation to model this scenario. There is no need to solve it.

8. The price of a washing machine after 30% discount is \$189.00. What was the original price of the washing machine (before the discount was applied)?

Assume the washing machine's price before the discount is p dollars. Write an equation to model this scenario. There is no need to solve it.

10. The price of a restaurant bill, including an 11% gratuity charge, was \$11.10. What was the price of the bill before gratuity was added?

Assume the bill without gratuity is b dollars. Write an equation to model this scenario. There is no need to solve it.

12. In May 2016, the median rent price for a one-bedroom apartment in a city was reported to be \$1,006.00 per month. This was reported to be an increase of 0.6% over the previous month. Based on this reporting, what was the median price of a one-bedroom apartment in April 2016?

Assume the median price of a one-bedroom apartment in April 2016 was p dollars. Write an equation to model this scenario. There is no need to solve it.

13. Izabelle is driving an average of 42 miles per hour, and she is 58.8 miles away from home. After how many hours will she reach his home?

Assume Izabelle will reach home after h hours. Write an equation to model this scenario. There is no need to solve it.

14. Blake is driving an average of 46 miles per hour, and he is 156.4 miles away from home. After how many hours will he reach his home?

Assume Blake will reach home after h hours. Write an equation to model this scenario. There is no need to solve it.

15. Uhaul charges an initial fee of \$32.65 and then \$0.68 per mile to rent a 15-foot truck for a day. If the total bill is \$116.29, how many miles were driven?

Assume m miles were driven. Write an equation to model this scenario. There is no need to solve it.

16. Uhaul charges an initial fee of \$34.85 and then \$0.53 per mile to rent a 15-foot truck for a day. If the total bill is \$132.90, how many miles were driven?

Assume m miles were driven. Write an equation to model this scenario. There is no need to solve it.

17. A cat litter box has a rectangular base that is 24 inches by 24 inches. What will the height of the cat litter be if 4 cubic feet of cat litter is poured? (Hint: 1 $\text{ft}^3 = 1728\ \text{in}^3$)

Assume h inches will be the height of the cat litter if 4 cubic feet of cat litter is poured. Write an equation to model this scenario. There is no need to solve it.

18. A cat litter box has a rectangular base that is 24 inches by 18 inches. What will the height of the cat litter be if 6 cubic feet of cat litter is poured? (Hint: 1 $\text{ft}^3 = 1728\ \text{in}^3$)

Assume h inches will be the height of the cat litter if 6 cubic feet of cat litter is poured. Write an equation to model this scenario. There is no need to solve it.

Modeling with Linear Inequalities

19. A truck that hauls water is capable of carrying a maximum of 1500 lb. Water weighs $8.3454\,\frac{\text{lb}}{\text{gal}}$, and the plastic tank on the truck that holds water weighs 53 lb. Assume the truck can carry a maximum of g gallons of water. Write an *inequality* to model this scenario. There is no need to solve it.

20. A truck that hauls water is capable of carrying a maximum of 2600 lb. Water weighs $8.3454\,\frac{\text{lb}}{\text{gal}}$, and the plastic tank on the truck that holds water weighs 59 lb. Assume the truck can carry a maximum of g gallons of water. Write an *inequality* to model this scenario. There is no need to solve it.

21. Grant's maximum lung capacity is 5.2 liters. If his lungs are full and he exhales at a rate of 0.8 liters per second, write an *inequality* that models when he will still have at least 0.4 liters of air left in his lungs. There is no need to solve it.

22. Izabelle's maximum lung capacity is 5.6 liters. If her lungs are full and she exhales at a rate of 0.8 liters per second, write an *inequality* that models when she will still have at least 2.8 liters of air left in his lungs. There is no need to solve it.

23. A swimming pool is being filled with water from a garden hose at a rate of 8 gallons per minute. If the pool already contains 60 gallons of water and can hold up to 276 gallons, set up an *inequality* modeling how much time can pass without the pool overflowing. There is no need to solve it.

24. A swimming pool is being filled with water from a garden hose at a rate of 6 gallons per minute. If the pool already contains 70 gallons of water and can hold up to 154 gallons, set up an *inequality* modeling how much time can pass without the pool overflowing. There is no need to solve it.

25. An engineer is designing a cylindrical springform pan (the kind of pan a cheesecake is baked in). The pan needs to be able to hold a volume at least 398 cubic inches and have a diameter of 13 inches. Write an *inequality* modeling possible height of the pan. There is no need to solve it.

26. An engineer is designing a cylindrical springform pan (the kind of pan a cheesecake is baked in). The pan needs to be able to hold a volume at least 338 cubic inches and have a diameter of 14 inches. Write an *inequality* modeling possible height of the pan. There is no need to solve it.

Translating English Phrases into Math Expressions and Equations Translate the following phrase or sentence into a math expression or equation (whichever is appropriate).

27. three more than a number

28. ten less than a number

29. the sum of a number and six

30. the difference between a number and three

31. the difference between nine and a number

32. the difference between six and a number

33. two subtracted from a number

34. nine added to a number

35. five decreased by a number

36. two increased by a number

37. a number decreased by eight

38. a number increased by five

39. two times a number, increased by five

40. eight times a number, decreased by ten

41. five less than four times a number

42. one less than eight times a number

43. eight more than the quotient of three and a number

44. four less than the quotient of seven and a number

45. Two times a number is sixteen.

46. Seven times a number is twenty-eight.

47. The sum of fifty-six and a number is seventy-three.

48. The difference between thirty-three and a number is twenty-eight.

49. The quotient of a number and seventeen is fourteen seventeenths.

50. The quotient of a number and twenty-five is one twenty-fifth.

51. The quotient of twenty-six and a number is thirteen twenty-fifths.

52. The quotient of eighteen and a number is nine twenty-thirds.

53. The sum of three times a number and eleven is fifty-six.

54. The sum of eight times a number and twenty-three is thirty-one.

55. One less than five times a number yields fifty-nine.

56. Two less than three times a number yields ninety-one.

57. The product of seven and a number, increased by eight, yields ninety-nine.

58. The product of five and a number, added to four, yields 234.

59. The product of three and a number increased by seven, yields 123.

60. The product of seven and a number added to three, yields 168.

61. one sixth of a number

62. one half of a number

63. twenty-three thirty-eighths of a number

64. thirteen forty-firsts of a number

65. a number decreased by one eleventh of itself

66. a number decreased by seven thirtieths of itself

67. A number increased by two ninths is one ninth of that number.

68. A number decreased by one sixth is one eighth of that number.

69. One more than the product of three elevenths and a number yields three tenths of that number.

70. Five more than the product of one fifth and a number gives two sevenths of that number.

Challenge

71. Last year, Joan received a 2.5% raise. This year, she received a 4% raise. Her current wage is $11.46 an hour. What was her wage before the two raises?

2.9 Introduction to Exponent Rules

In this section, we will look at some rules or properties we use when simplifying expressions that involve multiplication and exponents.

2.9.1 Exponent Basics

Before we discuss any exponent rules, we need to quickly remind ourselves of some important concepts and vocabulary.

When working with expressions with exponents, we have the following vocabulary:

$$\text{base}^{\text{exponent}} = \text{power}$$

For example, when we calculate $8^2 = 64$, the **base** is 8, the **exponent** is 2, and the expression 8^2 is called the 2nd **power** of 8.

The other foundational concept is that if an exponent is a positive integer, the power can be rewritten as repeated multiplication of the base. For example, the 4th power of 3 can be written as 4 factors of 3 like so:

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3$$

2.9.2 Exponent Rules

Product Rule If we write out $3^5 \cdot 3^2$ without using exponents, we'd have:

$$3^5 \cdot 3^2 = (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3)$$

If we then count how many 3s are being multiplied together, we find we have $5 + 2 = 7$, a total of seven 3s.

$$\begin{aligned} 3^5 \cdot 3^2 &= 3^{5+2} \\ &= 3^7 \end{aligned}$$

> **Example 2.9.2** Simplify $x^2 \cdot x^3$.
>
> To simplify $x^2 \cdot x^3$, we write this out in its expanded form, as a product of x's, we have
>
> $$\begin{aligned} x^2 \cdot x^3 &= (x \cdot x)(x \cdot x \cdot x) \\ &= x \cdot x \cdot x \cdot x \cdot x \\ &= x^5 \end{aligned}$$
>
> Note that we obtained the exponent of 5 by adding 2 and 3.

This is our first rule, the **Product Rule**: when multiplying two expressions that have the same base, we can simplify the product by adding the exponents.

$$x^m \cdot x^n = x^{m+n} \tag{2.9.1}$$

Checkpoint 2.9.3. Use the properties of exponents to simplify the expression.

$x^{19} \cdot x^{13}$

Explanation. We *add* the exponents as follows:

$$\begin{aligned} x^{19} \cdot x^{13} &= x^{19+13} \\ &= x^{32} \end{aligned}$$

Power to a Power Rule The second rule is an extension of the first rule. If we write out $\left(3^5\right)^2$ without using exponents, we'd have 3^5 multiplied by itself:

$$\begin{aligned} \left(3^5\right)^2 &= \left(3^5\right) \cdot \left(3^5\right) \\ &= (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \end{aligned}$$

If we again count how many 3s are being multiplied, we have a total of two groups each with five 3s. So we'd have $2 \cdot 5 = 10$ instances of a 3.

$$\begin{aligned} \left(3^5\right)^2 &= 3^{2 \cdot 5} \\ &= 3^{10} \end{aligned}$$

Example 2.9.4 Simplify $\left(x^2\right)^3$.

To simplify $\left(x^2\right)^3$, we write this out in its expanded form, as a product of x's, we have

$$\begin{aligned} \left(x^2\right)^3 &= \left(x^2\right) \cdot \left(x^2\right) \cdot \left(x^2\right) \\ &= (x \cdot x) \cdot (x \cdot x) \cdot (x \cdot x) \\ &= x^6 \end{aligned}$$

Note that we obtained the exponent of 6 by multiplying 2 and 3.

We have our second rule, the **Power to a Power Rule**: when a base is raised to an exponent and that expression is raised to another exponent, we multiply the exponents.

$$(x^m)^n = x^{m \cdot n}$$

Checkpoint 2.9.5. Use the properties of exponents to simplify the expression.

$\left(r^2\right)^5$

Explanation. We *multiply* the exponents as follows:

$$\begin{aligned} \left(r^2\right)^5 &= r^{2 \cdot 5} \\ &= r^{10} \end{aligned}$$

Product to a Power Rule The third exponent rule deals with having multiplication inside a set of parentheses and an exponent outside the parentheses. If we write out $(3t)^5$ without using an exponent, we'd have $3t$ multiplied by itself five times:

$$(3t)^5 = (3t)(3t)(3t)(3t)(3t)$$

Keeping in mind that there is multiplication between every 3 and t and multiplication between all of the parentheses, we can reorder and regroup the factors:

$$\begin{aligned}(3t)^5 &= (3\cdot t)\cdot(3\cdot t)\cdot(3\cdot t)\cdot(3\cdot t)\cdot(3\cdot t)\\ &= (3\cdot 3\cdot 3\cdot 3\cdot 3)\cdot(t\cdot t\cdot t\cdot t\cdot t)\\ &= 3^5t^5\end{aligned}$$

We essentially applied the outer exponent to each factor inside the parentheses.

> **Example 2.9.6** Simplify $(xy)^5$.
>
> To simplify $(xy)^5$, we write this out in its expanded form, as a product of x's and y's, we have
>
> $$\begin{aligned}(xy)^5 &= (x\cdot y)\cdot(x\cdot y)\cdot(x\cdot y)\cdot(x\cdot y)\cdot(x\cdot y)\\ &= (x\cdot x\cdot x\cdot x\cdot x)\cdot(y\cdot y\cdot y\cdot y\cdot y)\\ &= x^5y^5\end{aligned}$$
>
> Note that the exponent on xy can simply be applied to both x and y.

This is our third rule, the **Product to a Power Rule**: when a product is raised to an exponent, we can apply the exponent to each factor in the product.

$$\left(x\cdot y\right)^n = x^n\cdot y^n$$

Checkpoint 2.9.7. Use the properties of exponents to simplify the expression.

$(5x)^2$

Explanation. We *multiply* the exponents and apply the rule $(ab)^m = a^m\cdot b^m$ as follows:

$$\begin{aligned}(5x)^2 &= (5)^2x^2\\ &= 25x^2\end{aligned}$$

If a and b are real numbers, and n and m are positive integers, then we have the following rules:

Product Rule $a^n\cdot a^m = a^{n+m}$

Power to a Power Rule $(a^n)^m = a^{n\cdot m}$

Product to a Power Rule $(ab)^n = a^n\cdot b^n$

List 2.9.8: Summary of the Rules of Exponents for Multiplication

Many examples we'll come across will make use of more than one exponent rule. In deciding which exponent rule to work with first, it's important to remember that the order of operations still applies.

> **Example 2.9.9** Simplify the following expressions.
>
> a. $\left(3^7r^5\right)^4$
>
> b. $\left(t^3\right)^2\cdot\left(t^4\right)^5$

Explanation.

a. Since we cannot simplify anything inside the parentheses, we'll begin simplifying this expression using the Product to a Power Rule. We'll apply the outer exponent of 4 to each factor inside the parentheses. Then we'll use the Power to a Power Rule to finish out simplification process:

$$\begin{aligned}\left(3^7 r^5\right)^4 &= \left(3^7\right)^4 \cdot \left(r^5\right)^4\\ &= 3^{7\cdot 4} \cdot r^{5\cdot 4}\\ &= 3^{28} r^{20}\end{aligned}$$

b. According to the order of operations, we should first simplify any exponents before carrying out any multiplication. Therefore, we'll begin simplifying this by applying the Power to a Power Rule and then finish using the Product Rule:

$$\begin{aligned}\left(t^3\right)^2 \cdot \left(t^4\right)^5 &= t^{3\cdot 2} \cdot t^{4\cdot 5}\\ &= t^6 \cdot t^{20}\\ &= t^{6+20}\\ &= t^{26}\end{aligned}$$

Remark 2.9.10. We cannot simplify an expression like $x^2 y^3$ using the Product Rule, as the factors x^2 and y^3 do not have the same base.

Exercises

Review and Warmup Evaluate the following.

1.
a. $3^2 =$
b. $4^3 =$
c. $(-4)^2 =$
d. $(-5)^3 =$

2.
a. $3^2 =$
b. $2^3 =$
c. $(-2)^2 =$
d. $(-3)^3 =$

3.
a. $1^7 =$
b. $(-1)^{15} =$
c. $(-1)^{14} =$
d. $0^{19} =$

4.
a. $1^8 =$
b. $(-1)^{13} =$
c. $(-1)^{16} =$
d. $0^{17} =$

5.
a. $(-3)^2 =$
b. $-10^2 =$

6.
a. $(-1)^2 =$
b. $-2^2 =$

7.
a. $(-1)^3 =$
b. $-2^3 =$

8.
a. $(-4)^3 =$
b. $-1^3 =$

Exponent Rules Use the properties of exponents to simplify the expression.

9. $3 \cdot 3^9$

10. $4 \cdot 4^6$

11. $5^2 \cdot 5^7$

12. $5^8 \cdot 5^2$

13. $r^{13} \cdot r^9$

14. $t^{15} \cdot t^3$

15. $y^{17} \cdot y^{15} \cdot y^8$

16. $x^{19} \cdot x^8 \cdot x^{16}$

17. $\left(2^2\right)^3$

18. $\left(4^8\right)^7$

19. $\left(x^4\right)^5$

20. $\left(y^6\right)^{12}$

21. $(4t)^3$

22. $(3y)^3$

23. $(5rx)^4$

24. $(4xt)^4$

25. $\left(4r^{12}\right)^2$

26. $\left(5y^2\right)^4$

27. $(5r^4) \cdot (10r^{12})$

28. $(9t^7) \cdot (-4t^5)$

29. $\left(-\dfrac{x^9}{3}\right) \cdot \left(\dfrac{x^{18}}{5}\right)$

30. $\left(-\dfrac{x^{11}}{6}\right) \cdot \left(-\dfrac{x^{11}}{4}\right)$

31. $\left(-10t^8\right)^2$

32. $\left(-7r^9\right)^3$

Use the properties of exponents to simplify the expression.

33. $(-3y^{17}) \cdot (-9y^{10}) \cdot (-y^9)$

34. $(-2r^{19}) \cdot (-5r^4) \cdot (4r^4)$

35. a. $\left(-2x^2\right)^2 = \boxed{}$

b. $-\left(2x^2\right)^2 = \boxed{}$

36. a. $\left(-6r^3\right)^2 = \boxed{}$

b. $-\left(6r^3\right)^2 = \boxed{}$

Challenge

37. a. Let $x^7 \cdot x^a = x^{17}$. Let's say that a is a natural number. How many possibilities are there for a?

b. Let $x^b \cdot x^c = x^{75}$. Let's say that b and c are natural numbers. How many possibilities are there for b?

c. Let $x^d \cdot x^e = x^{800}$. Let's say that d and e are natural numbers. How many possibilities are there for d?

38. Choose the correct inequality or equal sign to make the relation true.

2^{500} (☐ < ☐ > ☐ =) 5^{200}

2.10 Simplifying Expressions

We know that if we have two apples and add three more, then our result is the same as if we'd had three apples and added two more. In this section, we'll formally define and extend these basic properties we know about numbers to variable expressions.

2.10.1 Identities and Inverses

We will start with some definitions. The number 0 is called the **additive identity**. If the sum of two numbers is the additive identity, 0, these two numbers are called **additive inverses**. For example, 2 is the additive inverse of -2, and the additive inverse of -2 is 2.

Similarly, the number 1 is called the **multiplicative identity**. If the product of two numbers is the multiplicative identity, 1, these two numbers are called **multiplicative inverses**. For example, 2 is the multiplicative inverse of $\frac{1}{2}$, and the multiplicative inverse of $-\frac{2}{3}$ is $-\frac{3}{2}$. The multiplicative inverse is also called **reciprocal**.

2.10.2 Introduction to Algebraic Properties

Commutative Property When we compute the area of a rectangle, we generally multiply the length by the width. Does the result change if we multiply the width by the length?

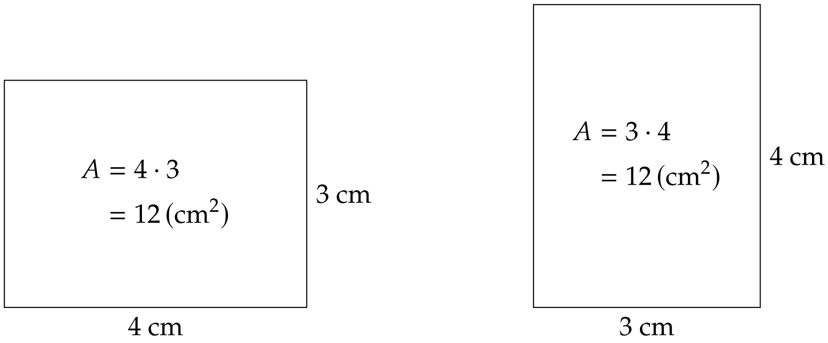


Figure 2.10.2: Horizontal and Vertical Rectangles

We can see $3 \cdot 2 = 2 \cdot 3$. If we denote the length of a rectangle with ℓ and the width with w, this implies $\ell w = w\ell$. This is referred to as the **commutative property of multiplication**. The commutative property also applies to addition, as in $1+2 = 2+1$, where it is called the **commutative property of addition**. However, there is no commutative property of subtraction or division, as $2 - 1 \neq 1 - 2$, and $\frac{4}{2} \neq \frac{2}{4}$.

Associative Property Let's extend that example to a rectangular prism with width $w = 4\,\text{cm}$, depth $d = 3\,\text{cm}$, and height $h = 2\,\text{cm}$. To compute the volume of this solid, we multiply the width, depth and height, which we write as wdh.

In the following figure, on the left side, we multiply the length and width first, and then multiply the height; on the right side, we multiply the width and height first, and then multiply the length. Let's compare the products.

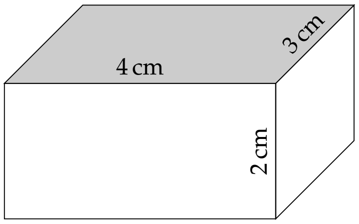


Figure 2.10.3: $(4 \cdot 3) \cdot 2 = 24$

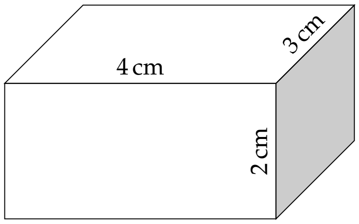


Figure 2.10.4: $4 \cdot (3 \cdot 2) = 24$

We can see $(wd)h = w(dh)$. This is known as the **associative property of multiplication**. The associative property also applies to addition, as in $(1 + 2) + 3 = 1 + (2 + 3)$, which is called the **associative property of addition**. However, there is no associative property of subtraction, as $(3 - 2) - 1 \neq 3 - (2 - 1)$.

Distributive Property The final property we'll explore is called the **distributive property**, which involves both multiplication and addition. To conceptualize this property, let's consider what happens if we buy 3 boxes that each contain one apple and one pear. This will have the same total cost as if we'd bought 3 apples and 3 pears. We write this algebraically:

$$3(a + p) = 3a + 3p.$$

Visually, we can see that it's just a means of re-grouping: 3(🍎 + 🍐) = 3(🍎) + 3(🍐).

2.10.3 Summary of Algebraic Properties

Let a, b, and c represent real numbers, variables, or algebraic expressions. Then the following properties hold:

Commutative Property of Multiplication $a \cdot b = b \cdot a$

Associative Property of Multiplication $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

Commutative Property of Addition $a + b = b + a$

Associative Property of Addition $a + (b + c) = (a + b) + c$

Distributive Property $a(b + c) = ab + ac$

List 2.10.5: Algebraic Properties

Let's practice these properties in the following exercises.

Checkpoint 2.10.6.

a. Use the commutative property of multiplication to write an equivalent expression to $53m$.
b. Use the associative property of multiplication to write an equivalent expression to $3(5n)$.
c. Use the commutative property of addition to write an equivalent expression to $q + 84$.

d. Use the associative property of addition to write an equivalent expression to $x + (20 + c)$.

e. Use the distributive property to write an equivalent expression to $3(r + 7)$ that has no grouping symbols.

Explanation.

a. To use the commutative property of multiplication, we change the order in which two factors are multiplied:

$$\begin{aligned} &53m \\ &= m \cdot 53. \end{aligned}$$

b. To use the associative property of multiplication, we leave factors written in their original order, but change the grouping symbols so that a different multiplication has higher priority:

$$\begin{aligned} &3(5n) \\ &= (3 \cdot 5)n. \end{aligned}$$

You may further simplify by carrying out the multiplication between the two numbers:

$$\begin{aligned} &3(5n) \\ &= (3 \cdot 5)n \\ &= 15n. \end{aligned}$$

c. To use the commutative property of addition, we change the order in which two terms are added:

$$\begin{aligned} &q + 84 \\ &= 84 + q. \end{aligned}$$

d. To use the associative property of addition, we leave terms written in their original order, but change the grouping symbols so that a different addition has higher priority:

$$\begin{aligned} &x + (20 + c) \\ &= (x + 20) + c. \end{aligned}$$

e. To use the distributive property, we multiply the number outside the parentheses, 3, with each term inside the parentheses:

$$\begin{aligned} &3(r + 7) \\ &= 3 \cdot r + 3 \cdot 7 \\ &= 3r + 21. \end{aligned}$$

2.10.4 Applying the Commutative, Associative, and Distributive Properties

Like Terms One of the main ways that we will use the commutative, associative, and distributive properties is to simplify expressions. In order to do this, we need to recognize **like terms**, as discussed in Section 2.3. We combine like terms when we take an expression lilke $2a + 3a$ and write the result as $5a$. The formal process actually involves using the distributive property:

$$\begin{aligned} 2a + 3a &= (2+3)a \\ &= 5a \end{aligned}$$

In practice, however, it's more helpful to think of this as having 2 of an object and then an additional 3 of that same object. In total, we then have 5 of that object.

Example 2.10.7 Where possible, simplify the following expressions by combining like terms.

a. $6c + 12c - 5c$

b. $-5q^2 - 3q^2$

c. $x - 5y + 4x$

d. $2x - 3y + 4z$

Explanation.

a. All three terms are like terms, so they may combined. We combine them two at a time:

$$\begin{aligned} 6c + 12c - 5c &= 18c - 5c \\ &= 13c \end{aligned}$$

b. The two terms $-5q^2$ and $-3q^2$ are like terms, so we may combine them:

$$-5q^2 - 3q^2 = -8q^2$$

c. The two terms x and $4x$ are like terms, while the other term is different. Using the associative and commutative properties of addition in the first step allows us to place the two like terms next to each other, and then combine them:

$$\begin{aligned} x - 5y + 4x &= x + 4x + (-5y) \\ &= 1x + 4x + (-5y) \\ &= 5x - 5y \end{aligned}$$

Note the expression x is the same as $1x$. Usually we don't write the "1" as it is implied. However, it's helpful when combining like terms to remember that $x = 1x$. (Similarly, $-x$ is equal to $-1x$, which can be helpful when combining $-x$ with like terms.)

d. The expression $2x - 3y + 4z$ cannot be simplified as there are no like terms.

Adding Expressions When we add an expression like $4x - 5$ to an expression like $3x - 7$, we write them as follows:

$$(4x - 5) + (3x - 7)$$

In order to remove the given sets of parentheses and apply the commutative property of addition, we will rewrite the subtraction operation as "adding the opposite":

$$4x + (-5) + 3x + (-7)$$

At this point we can apply the commutative property of addition and then combine like terms. Here's how the entire problem will look:

$$\begin{aligned} (4x - 5) + (3x - 7) &= 4x + (-5) + 3x + (-7) \\ &= 4x + 3x + (-5) + (-7) \end{aligned}$$

$$\begin{aligned} &= 7x + (-12) \\ &= 7x - 12 \end{aligned}$$

Remark 2.10.8. Once we become more comfortable simplifying such expressions, we will simply write this kind of simplification in one step:

$$(4x - 5) + (3x - 7) = 7x - 12$$

Example 2.10.9 Use the associative, commutative, and distributive properties to simplify the following expressions as much as possible.

a. $(2x + 3) + (4x + 5)$

b. $(-5x + 3) + (4x - 7)$

Explanation.

a. We will remove parentheses, and then combine like terms:

$$\begin{aligned} (2x + 3) + (4x + 5) &= 2x + 3 + 4x + 5 \\ &= 2x + 4x + 3 + 5 \\ &= 6x + 8 \end{aligned}$$

b. We will remove parentheses, and then combine like terms:

$$\begin{aligned} (-5x + 3) + (4x - 7) &= -5x + 3 + 4x + (-7) \\ &= -x + (-4) \\ &= -x - 4 \end{aligned}$$

Applying the Distributive Property with Negative Coefficients Applying the distributive property in an expression such as $2(3x+4)$ is fairly straightforward, in that this becomes $2(3x)+2(4)$ which we then simplify to $6x + 8$. Applying the distributive property is a little trickier when subtraction or a negative constant is involved, for example, with the expression $2(3x - 4)$. Recalling that subtraction is defined as "adding the opposite," we can change the subtraction of positive 4 to the addition of negative 4:

$$2\big(3x + (-4)\big)$$

Now when we distribute, we obtain:

$$2(3x) + 2(-4)$$

As a final step, we see that this simplifies to:

$$6x - 8$$

Remark 2.10.10. We can also extend the distributive property to use subtraction, and state that $a(b - c) = ab - ac$. With this property, we would simplify $2(3x - 4)$ more efficiently:

$$\begin{aligned} 2(3x - 4) &= 2(3x) - 2(4) \\ &= 6x - 8 \end{aligned}$$

In general, we will use this approach.

Example 2.10.11 Apply the distributive property to each expression and simplify it as much as possible.

a. $-3(5x+7)$

b. $2(-4x-1)$

Explanation.

a. We will distribute -3 to the $5x$ and 7:

$$\begin{aligned} -3(5x+7) &= -3(5x)+(-3)(7) \\ &= -15x-21 \end{aligned}$$

b. We will distribute 2 to the $-4x$ and -1:

$$\begin{aligned} 2(-4x-1) &= 2(-4x)-2(1) \\ &= -8x-2 \end{aligned}$$

Checkpoint 2.10.12. Use the distributive property to write an equivalent expression to $-4(x-2)$ that has no grouping symbols.

Explanation. To use the distributive property, we multiply the number outside the parentheses, -4, with each term inside the parentheses:

$$\begin{aligned} -4(x-2) &= -4 \cdot x - 4(-2) \\ &= -4x+8 \end{aligned}$$

Subtracting Expressions To subtract one expression from another expression, such as $(5x+9)-(3x+2)$, we will again rely on the fact that subtraction is defined as "adding the opposite." To add the *opposite* of an expression, we will technically distribute a constant factor of -1 and simplify from there:

$$\begin{aligned} (5x+9)-(3x+2) &= (5x+9)+(-1)(3x+2) \\ &= 5x+9+(-1)(3x)+(-1)(2) \\ &= 5x+9+(-3x)+(-2) \\ &= 2x+7 \end{aligned}$$

Remark 2.10.13. The above example demonstrates *how* we apply the distributive property in order to subtract two expressions. But in practice, it can be pretty cumbersome. A shorter (and often clearer) approach is to instead subtract every term in the expression we are subtracting, which is shown like this:

$$\begin{aligned} (5x+9)-(3x+2) &= 5x+9-3x-2 \\ &= 2x+7 \end{aligned}$$

In general, we'll use this approach.

Example 2.10.14 Use the associative, commutative, and distributive properties to simplify the following expressions as much as possible.

a. $(-6x+4)-(3x-7)$

b. $(-2x-5)-(-4x-6)$

Explanation.

a. We will remove parentheses using the distributive property, and then combine like terms:

$$(-6x+4)-(3x-7) = -6x+4-3x-(-7)$$

$$\begin{aligned} &= -6x + 4 - 3x + 7 \\ &= -9x + 11 \end{aligned}$$

b. We will remove parentheses using the distributive property, and then combine like terms:

$$\begin{aligned} (-2x - 5) - (-4x - 6) &= -2x - 5 - (-4x) - (-6) \\ &= -2x - 5 + 4x + 6 \\ &= 2x + 1 \end{aligned}$$

2.10.5 The Role of the Order of Operations in Applying the Commutative, Associative, and Distributive Properties

When simplifying an expression such as $3 + 4(5x + 7)$, we need to respect the order of operations. Since the terms inside the parentheses are not like terms, there is nothing to simplify there. The next highest priority operation is multiplying the 4 by $(5x + 7)$. This must be done *before* anything happens with the adding of that 3. We cannot say $3 + 4(5x + 7) = 7(5x + 7)$, because that would mean we treated the addition as having higher priority than the multiplication.

So to simplify $3 + 4(5x + 7)$, we will first examine the multiplication of 4 with $(5x + 7)$, and here we may apply the distributive property. After that, we will use the commutative and associative properties:

$$\begin{aligned} 3 + 4(5x + 7) &= 3 + 4(5x) + 4(7) \\ &= 3 + 20x + 28 \\ &= 20x + 3 + 28 \\ &= 20x + 31 \end{aligned}$$

Example 2.10.15 Simplify the following expressions using the commutative, associative, and distributive properties.

a. $4 - (3x - 9)$

b. $5x + 9(-2x + 3)$

c. $5(x - 9) + 4(x + 4)$

Explanation.

a. We will remove parentheses using the distributive property, and then combine like terms:

$$\begin{aligned} 4 - (3x - 9) &= 4 - 3x - (-9) \\ &= 4 - 3x + 9 \\ &= -3x + 13 \end{aligned}$$

b. We will remove parentheses using the distributive property, and then combine like terms:

$$\begin{aligned} 5x + 9(-2x + 3) &= 5x + 9(-2x) + 9(3) \\ &= 5x - 18x + 27 \\ &= -13x + 27 \end{aligned}$$

c. We will remove parentheses using the distributive property, and then combine like terms:

$$\begin{aligned} 5(x-9)+4(x+4) &= 5x-45+4x+16 \\ &= 9x-29 \end{aligned}$$

Checkpoint 2.10.16. Use the distributive property to simplify $3-9(9-8r)$ completely.

Explanation. We first use distributive property to get rid of parentheses, and then combine like terms:

$$\begin{aligned} 3-9(9-8r) &= 3+(-9)(9-8r) \\ &= 3+(-9)(9)+(-9)(-8r) \\ &= 3-81+72r \\ &= -78+72r \\ &= 72r-78 \end{aligned}$$

Note that either of the last two expressions are acceptable final answers.

2.10.6 Rules of Exponents and Simplifying

In Section 2.9, we introduced three exponent rules. We continue to use these rules when simplifying expressions. Sometimes though, students incorrectly apply "rules" of exponents where they have misremembered the actual rule. Let's summarize what we can and cannot do.

When we add/subtract two expressions, we can only combine *like* terms. For example:

- $3x-x=2x$
- $t^2+t^2=2t^2$
- q^2+q cannot be combined.

However, we can multiply two expressions regardless of whether or not they are like terms. For example:

- $x\cdot x=x^2$
- $t^2\cdot t^3=t^5$
- $(q^2)(q)=q^3$

Consider:

- When we combine like terms that have a variable, the exponent doesn't change, as in $x^2+x^2=2x^2$.
- When we multiply powers of a variable that use the same variable, the exponent *will* change, as in $(x^2)(x^2)=x^4$.
- We *cannot* combine "unlike terms," as something like x^2+x is as simplified as it can be.
- We *can* multiply powers with different exponents, as in $(x^2)(x)=x^3$.

The next few examples test your understanding of these concepts.

Example 2.10.17 Simplify the following expressions using the rules of exponents and the distributive property.

a. $3x^2+2x+x^2$

b. $(3x^2)(2x)(x^2)$

c. $2x(3x+4)$

d. $x^3-3x^2(5x-2)$

Explanation.

a. We will combine like terms $3x^2$ and x^2:

$$3x^2 + 2x + x^2 = 4x^2 + 2x$$

b. We will apply the Product Rule:

$$(3x^2)(2x)(x^2) = 6x^5$$

c. To simplify $2x(3x + 4)$, we want to first distribute $2x$, and then we can apply the Product Rule:

$$\begin{aligned} 2x(3x + 4) &= 2x(3x) + 2x(4) \\ &= 6x^2 + 8x \end{aligned}$$

d. We will use the distributive property first, apply the Product Rule, and combine like terms:

$$\begin{aligned} x^3 - 3x^2(5x - 2) &= x^3 - 3x^2(5x) - (-3x^2)(2) \\ &= x^3 - 15x^3 + 6x^2 \\ &= -14x^3 + 6x^2 \end{aligned}$$

Exercises

Review and Warmup

1. Count the number of terms in each expression.

a. $-6x - 7 + 3t - 8z^2$

b. $-6t + 1$

c. $4s^2 - 6y - 6s^2$

d. $2s - 2z - 5y^2 + 7s$

2. Count the number of terms in each expression.

a. $-4x^2 + z + y - 9t$

b. $8x^2 - 6t + 2s^2 + 5t$

c. $-7x^2$

d. $-8z^2 + 5y - 5$

3. List the terms in each expression.

a. $-2x$

b. $-8.5s^2 - 1.2s + 8t^2$

c. $3.8t + 6.4 - 0.3x$

d. $-2.3y - 7.8$

4. List the terms in each expression.

a. $-0.4x^2 - 0.3$

b. $-3.2t + 6.8t$

c. $-0.2y + 0.2s - 2.6x + 8.3s$

d. $-3.5y^2$

5. List the terms in each expression.

a. $1.2x + 7.9t$

b. $-5.2y + 5.6 - 4.3s^2 - 6$

c. $-4.8y + 0.9z$

d. $8.4y - 0.1z - 8.5t$

6. List the terms in each expression.

a. $2.8x^2 - 2x^2$

b. $-7.2s + 4.3t^2 - 8.3z$

c. $5.8s - 7.1 - 6.5x + 7.4z$

d. $8.4x^2 - z$

7. Simplify each expression, if possible, by combining like terms.

a. $5x + 6y^2 + 9y$

b. $3s + 6s$

c. $-z + 5z^2$

d. $x^2 + 4x^2$

8. Simplify each expression, if possible, by combining like terms.

a. $6x^2 - 4s^2 + 7s^2$

b. $2s^2 + 2y^2 - 7 + 7y$

c. $-3y^2 + 6z + 8z$

d. $-4s - 2x - 8s^2 + 2s^2$

These exercises involve the concepts of like terms and the commutative, associative, and distributive properties.

9. The additive inverse of 4 is

10. The additive inverse of 7 is

11. The multiplicative inverse of 9 is

12. The multiplicative inverse of −10 is

13. Use the associative property of addition to write an equivalent expression to $r + (38 + q)$.

14. Use the associative property of addition to write an equivalent expression to $t + (2 + n)$.

15. Use the associative property of addition to write an equivalent expression to $10 + (9 + b)$.

16. Use the associative property of addition to write an equivalent expression to $3 + (17 + c)$.

17. Use the associative property of multiplication to write an equivalent expression to $4(5m)$.

18. Use the associative property of multiplication to write an equivalent expression to $8(3b)$.

19. Use the commutative property of addition to write an equivalent expression to $n + 8$.

20. Use the commutative property of addition to write an equivalent expression to $p + 73$.

21. Use the commutative property of addition to write an equivalent expression to $10x + 38$.

22. Use the commutative property of addition to write an equivalent expression to $5y + 3$.

23. Use the commutative property of addition to write an equivalent expression to $8(t + 69)$.

24. Use the commutative property of addition to write an equivalent expression to $3(b + 34)$.

25. Use the commutative property of multiplication to write an equivalent expression to $99c$.

26. Use the commutative property of multiplication to write an equivalent expression to $92x$.

27. Use the commutative property of multiplication to write an equivalent expression to $43 + 7a$.

28. Use the commutative property of multiplication to write an equivalent expression to $95 + 9n$.

29. Use the commutative property of multiplication to write an equivalent expression to $4(p + 60)$.

30. Use the commutative property of multiplication to write an equivalent expression to $8(x+25)$.

31. Use the distributive property to write an equivalent expression to $10(y+2)$ that has no grouping symbols.

32. Use the distributive property to write an equivalent expression to $6(t+6)$ that has no grouping symbols.

33. Use the distributive property to write an equivalent expression to $-9(b+9)$ that has no grouping symbols.

34. Use the distributive property to write an equivalent expression to $-3(c-3)$ that has no grouping symbols.

35. Use the distributive property to write an equivalent expression to $-(m-2)$ that has no grouping symbols.

36. Use the distributive property to write an equivalent expression to $-(r-7)$ that has no grouping symbols.

37. Use the distributive property to simplify $9+7(9+8n)$ completely.

38. Use the distributive property to simplify $6+2(7+7p)$ completely.

39. Use the distributive property to simplify $3-7(-4+3x)$ completely.

40. Use the distributive property to simplify $8-3(-10+3y)$ completely.

41. Use the distributive property to simplify $5-(-5+5t)$ completely.

42. Use the distributive property to simplify $2-(4+2a)$ completely.

43. Use the distributive property to simplify $8-(-6c-8)$ completely.

44. Use the distributive property to simplify $6-(2b+9)$ completely.

45. Use the distributive property to simplify $\frac{10}{7}(2-5x)$ completely.

46. Use the distributive property to simplify $\frac{8}{5}(-3+n)$ completely.

47. Use the distributive property to simplify $\frac{5}{8}\left(-9+\frac{3}{2}p\right)$ completely.

48. Use the distributive property to simplify $\frac{10}{3}\left(7+\frac{3}{4}x\right)$ completely.

49. The expression $y+t+c$ would be ambiguous if we did not have a left-to-right reading convention. Use grouping symbols to emphasize the order that these additions should be carried out.

Use the associative property of addition to write an equivalent (but different) algebraic expression.

50. The expression $t+a+x$ would be ambiguous if we did not have a left-to-right reading convention. Use grouping symbols to emphasize the order that these additions should be carried out.

Use the associative property of addition to write an equivalent (but different) algebraic expression.

51. A student has (correctly) simplified an algebraic expression in the following steps. Between each pair of steps, identify the algebraic property that justifies moving from one step to the next.

$$9(a+5)+9a$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$=(9a+45)+9a$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$=(45+9a)+9a$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$=45+(9a+9a)$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$=45+(9+9)\,a$$

$$=45+18a$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$=18a+45$$

52. A student has (correctly) simplified an algebraic expression in the following steps. Between each pair of steps, identify the algebraic property that justifies moving from one step to the next.

$$6(c+8)+7c$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$=(6c+48)+7c$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$=(48+6c)+7c$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$= 48 + (6c + 7c)$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$= 48 + (6 + 7)\,c$$

$$= 48 + 13c$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$= 13c + 48$$

53. The number of students enrolled in math courses at Portland Community College has grown over the years. The formulas

$$M = 0.47x + 3.7 \quad W = 0.38x + 4.3 \quad N = 0.02x + 0.1$$

describe the numbers (of thousands) of men, women, and gender-non-binary students enrolled in math courses at PCC x years after 2005. (Note this is an exercise using randomized numbers, not actual data.) Give a simplified formula for the total number T of thousands of students at PCC taking math classes x years after 2005. Be sure to give the entire formula, starting with `T=`.

54. The number of students enrolled in math courses at Portland Community College has grown over the years. The formulas

$$M = 0.51x + 5.7 \quad W = 0.51x + 3.4 \quad N = 0.02x + 0.2$$

describe the numbers (of thousands) of men, women, and gender-non-binary students enrolled in math courses at PCC x years after 2005. (Note this is an exercise using randomized numbers, not actual data.) Give a simplified formula for the total number T of thousands of students at PCC taking math classes x years after 2005. Be sure to give the entire formula, starting with `T=`.

Multiply the polynomials.

55. $6x\,(x - 6) = \boxed{}$

56. $8x\,(x + 4) = \boxed{}$

57. $2x\,(-10x - 5) = \boxed{}$

58. $3x\,(-5x + 5) = \boxed{}$

59. $-6x^2\,(x + 4) = \boxed{}$

60. $-3x^2\,(x - 8) = \boxed{}$

61. $6y^2\left(-3y^2 + 4y\right) = \boxed{}$

62. $-3r^2\left(-10r^2 + 8r\right) = \boxed{}$

Simplify the following expressions if possible.

63. a. $m^2 + m^2 =$
b. $(m^2)(m^2) =$
c. $m^2 + m^3 =$
d. $(m^2)(m^3) =$

64. a. $n^2 + n^2 =$
b. $(n^2)(n^2) =$
c. $n^2 + n^4 =$
d. $(n^2)(n^4) =$

65. a. $p + p =$
b. $(p)(p) =$
c. $p + p^2 =$
d. $(p)(p^2) =$

66. a. $x^3 + x^3 =$
b. $(x^3)(x^3) =$
c. $x^3 + x^4 =$
d. $(x^3)(x^4) =$

67. a. $-y - 4y =$
b. $(-y)(-4y) =$
c. $-y - 2y^4 =$
d. $(-y)(-2y^4) =$

68. a. $-4t^3 - 2t^3 =$
b. $(-4t^3)(-2t^3) =$
c. $-4t^3 - 2t =$
d. $(-4t^3)(-2t) =$

69. a. $2a^2 + 4a^2 =$
b. $(2a^2)(4a^2) =$
c. $2a^2 + 4a =$
d. $(2a^2)(4a) =$

70. a. $-c - c =$
b. $(-c)(-c) =$
c. $-c + 2c^3 =$
d. $(-c)(2c^3) =$

71. a. $2q - q^2 - 2q =$
b. $(2q)(-q^2)(-2q) =$

72. a. $-3r^3 - 2r^4 + 2r^3 =$
b. $(-3r^3)(-2r^4)(2r^3) =$

Simplify the following expression.

73. $2n^2\left(5n^4\right)^2 =$

74. $5p^5\left(-4p^2\right)^2 =$

75. $2x^2c^2\left(4xc^2\right)^2 =$

76. $-3y^5t^4\left(-2yt^5\right)^4 =$

77. $(-5t^3)(-4t^4) + (t^4)(-5t^3) =$

78. $(3a^5)(-3a^5) - (4a^4)(2a^6) =$

79. $(-2c^5)\left(-2c^4\right)^3 - (c)(4c^8) =$

80. $(3p^4)\left(p^5\right)^4 - (2p^2)(p^7) =$

81. $(4q^4)\left(-2q^4\right)^2 + \left(q^2\right)^5(5q^2) =$

82. $(3m^5)\left(2m^4\right)^4 + \left(m^2\right)^2(3m^{17}) =$

83. $\left(3p^3\right)^2 n^9 + 3\left(p^2n^3\right)^3 =$

84. $\left(3x^4\right)^3 c^{12} + 4\left(x^3c^3\right)^4 =$

85. Use the distributive property to write an equivalent expression to $-9y(7y+2)$ that has no grouping symbols.

86. Use the distributive property to write an equivalent expression to $-4t(4t-10)$ that has no grouping symbols.

87. Use the distributive property to write an equivalent expression to $-7a^2(a-1)$ that has no grouping symbols.

88. Use the distributive property to write an equivalent expression to $-10c^4(c+8)$ that has no grouping symbols.

89. Use the distributive property to simplify $4+6m(2+3m)$ completely.

90. Use the distributive property to simplify $8+3b(10+7b)$ completely.

91. Use the distributive property to simplify $2m-9m(8+10m^2)$ completely.

92. Use the distributive property to simplify $8p-6p(3+10p^3)$ completely.

93. Use the distributive property to simplify $5x^3-2x^3(-3+9x^2)$ completely.

94. Use the distributive property to simplify $10y^3-7y^3(-9+9y^3)$ completely.

95. Fully simplify $3(7x-8)+6(3x-7)$.

96. Fully simplify $4(3x+2)-6(7x-1)$.

97. Fully simplify $-4(9x-6)+7(-x-4)$.

98. Fully simplify $5(6x+1)-8(4x-6)$.

Challenge

99. Fill in the blanks with algebraic expressions that make the equation true. You may not use 0 or 1 in any of the blank spaces.Here is an example: $? + ? = 8x$.

One possible answer is: $3x + 5x = 8x$.

There are infinitely many correct answers to this problem. I encourage you to be creative. After finding a correct answer, see if you can come up with a different answer that is also correct.

a. $\boxed{} + \boxed{} = -13x$

b. $\boxed{} + \boxed{} = -13x^{30}$

c. $\boxed{} \cdot \boxed{} \cdot \boxed{} = 2x^{80}$

2.11 Variables, Expressions, and Equations Chapter Review

2.11.1 Variables and Evaluating Expressions

In Section 2.1 we covered the definitions of variables and expressions, and how to evaluate an expression with a particular number. We learned the formulas for perimeter and area of rectangles, triangles, and circles.

Evaluating Expressions When we evaluate an expression's value, we substitute each variable with its given value.

> **Example 2.11.1** Evaluate the value of $\frac{5}{9}(F-32)$ if $F = 212$.
>
> $$\begin{aligned} \frac{5}{9}(F-32) &= \frac{5}{9}(212-32) \\ &= \frac{5}{9}(180) \\ &= 100 \end{aligned}$$

Substituting a Negative Number When we substitute a variable with a negative number, it's important to use parentheses around the number.

> **Example 2.11.2** Evaluate the following expressions if $x = -3$.
>
> a. $\begin{aligned} x^2 &= (-3)^2 \\ &= 9 \end{aligned}$
>
> b. $\begin{aligned} x^3 &= (-3)^3 \\ &= (-3)(-3)(-3) \\ &= -27 \end{aligned}$
>
> c. $\begin{aligned} -x^2 &= -(-3)^2 \\ &= -9 \end{aligned}$
>
> d. $\begin{aligned} -x^3 &= -(-3)^3 \\ &= -(-27) \\ &= 27 \end{aligned}$

2.11.2 Geometry Formulas

In Section 2.2 we established the following formulas.

Perimeter of a Rectangle $P = 2(\ell + w)$

Area of a Rectangle $A = \ell w$

Area of a Triangle $A = \frac{1}{2}bh$

Circumference of a Circle $c = 2\pi r$

Area of a Circle $A = \pi r^2$

Volume of a Rectangular Prism $V = wdh$

Volume of a Cylinder $V = \pi r^2 h$

Volume of a Rectangular Prism or Cylinder $V = Bh$

2.11.3 Combining Like Terms

In Section 2.3 we covered the definitions of a term and how to combine like terms.

Example 2.11.3 List the terms in the expression $5x - 3y + \frac{2w}{3}$.

Explanation. The expression has three terms that are being added, $5x$, $-3y$ and $\frac{2w}{3}$.

Example 2.11.4 Simplify the expression $5x - 3x^2 + 2x + 5x^2$, if possible, by combining like terms.

Explanation. This expression has four terms: $5x$, $-3x^2$, $2x$, and $5x^2$. Both $5x$ and $2x$ are like terms; also $-3x^2$ and $5x^2$ are like terms. When we combine like terms, we get:

$$5x - 3x^2 + 2x + 5x^2 = 7x + 2x^2$$

Note that we cannot combine $7x$ and $2x^2$ because x and x^2 represent different quantities.

2.11.4 Equations and Inequalities as True/False Statements

In Section 2.4 we covered the definitions of an equation and an inequality, as well as how to verify if a particular number is a solution to them.

Checking Possible Solutions Given an equation or an inequality (with one variable), checking if some particular number is a solution is just a matter of replacing the value of the variable with the specified number and determining if the resulting equation/inequality is true or false. This may involve some amount of arithmetic simplification.

Example 2.11.5 Is -5 a solution to $2(x + 3) - 2 = 4 - x$?

Explanation. To find out, substitute in -5 for x and see what happens.

$$\begin{aligned}
2(x+3)-2 &= 4-x\\
2((-5)+3)-2 &\overset{?}{=} 4-(-5)\\
2(-2)-2 &\overset{?}{=} 9\\
-4-2 &\overset{?}{=} 9\\
-6 &\overset{\text{no}}{=} 9
\end{aligned}$$

So no, -5 is not a solution to $2(x + 3) - 2 = 4 - x$.

2.11.5 Solving One-Step Equations

In Section 2.5 we covered to to add, subtract, multiply, or divide on both sides of an equation to isolate the variable, summarized in Fact 2.5.12. We also learned how to state our answer, either as a solution or a solution set. Last we discussed how to solve equations with fractions.

Solving One-Step Equations When we solve linear equations, we use Properties of Equivalent Equations and follow an algorithm to solve a linear equation.

Example 2.11.6 Solve for g in $\frac{1}{2} = \frac{2}{3} + g$.

Explanation. We will subtract $\frac{2}{3}$ on both sides of the equation:

$$\begin{aligned} \frac{1}{2} &= \frac{2}{3} + g \\ \frac{1}{2} - \frac{2}{3} &= \frac{2}{3} + g - \frac{2}{3} \\ \frac{3}{6} - \frac{4}{6} &= g \\ -\frac{1}{6} &= g \end{aligned}$$

We will check the solution by substituting g in the original equation with $-\frac{1}{6}$:

$$\begin{aligned} \frac{1}{2} &= \frac{2}{3} + g \\ \frac{1}{2} &\stackrel{?}{=} \frac{2}{3} + \left(-\frac{1}{6}\right) \\ \frac{1}{2} &\stackrel{?}{=} \frac{4}{6} + \left(-\frac{1}{6}\right) \\ \frac{1}{2} &\stackrel{?}{=} \frac{3}{6} \\ \frac{1}{2} &\stackrel{\checkmark}{=} \frac{1}{2} \end{aligned}$$

The solution $-\frac{1}{6}$ is checked and the solution set is $\left\{-\frac{1}{6}\right\}$.

2.11.6 Solving One-Step Inequalities

In Section 2.6 we covered how solving inequalities is very much like how we solve equations, except that if we multiply or divide by a negative we switch the inequality sign.

Solving One-Step Inequalities When we solve linear inequalities, we also use Properties of Equivalent Equations with one small complication: When we multiply or divide by the same *negative* number on both sides of an inequality, the direction reverses!

Example 2.11.7 Solve the inequality $-2x \geq 12$. State the solution set with both interval notation and set-builder notation.

Explanation. To solve this inequality, we will divide each side by -2:

$$\begin{aligned} -2x &\geq 12 \\ \frac{-2x}{-2} &\leq \frac{12}{-2} && \text{Note the change in direction.} \\ x &\leq -6 \end{aligned}$$

- The inequality's solution set in interval notation is $(-\infty, -6]$.

- The inequality's solution set in set-builder notation is $\{x \mid x \leq -6\}$.

2.11.7 Percentages

In Section 2.7 we covered how to translate sentences with percentages into equations that we can solve.

Solving One-Step Equations Involving Percentages An important skill for solving percent-related problems is to boil down a complicated word problem into a simple form like "2 is 50% of 4."

Example 2.11.8 What percent of 2346.19 is 1995.98?

Using P to represent the unknown quantity, we write and solve the equation:

$$\begin{aligned} \overbrace{P}^{\text{what percent}} \quad \overbrace{\cdot}^{\text{of}} \quad \overbrace{2346.19}^{\text{\$2346.19}} \quad &\overbrace{=}^{\text{is}} \quad \overbrace{1995.98}^{\text{\$1995.98}} \\ \frac{P \cdot 2346.19}{2346.19} &= \frac{1995.98}{2346.19} \\ P &= 0.85073\ldots \\ P &\approx 85.07\% \end{aligned}$$

In summary, 1995.98 is approximately 85.07% of 2346.19.

2.11.8 Modeling with Equations and Inequalities

In Section 2.8 we covered how to translate phrases into mathematics, and how to set up equations and inequalities for application models.

Modeling with Equations and Inequalities To set up an equation modeling a real world scenario, the first thing we need to do is identify what variable we will use. The variable we use will be determined by whatever is unknown in our problem statement. Once we've identified and defined our variable, we'll use the numerical information provided in the equation to set up our equation.

Example 2.11.9 A bathtub contains $2.5\,\text{ft}^3$ of water. More water is being poured in at a rate of $1.75\,\text{ft}^3$ per minute. When will the amount of water in the bathtub reach $6.25\,\text{ft}^3$?

Explanation. Since the question being asked in this problem starts with "when," we immediately know that the unknown is time. As the volume of water in the tub is measured in ft^3 per minute, we know that time needs to be measured in minutes. We'll defined t to be the number of minutes that water is poured into the tub. Since each minute there are $1.75\,\text{ft}^3$ of water added, we will add the expression $1.75t$ to 2.5 to obtain the total amount of water. Thus the equation we set up is:

$$2.5 + 1.75t = 6.25$$

2.11.9 Introduction to Exponent Rules

In Section 2.9 we covered the rules of exponents for multiplication.

Example 2.11.10 Simplify the following expressions using the rules of exponents:

a. $-2t^3 \cdot 4t^5$

b. $5\left(v^4\right)^2$

c. $-(3u)^2$

d. $(-3z)^2$

Explanation.

a. $-2t^3 \cdot 4t^5 = -8t^8$

b. $5\left(v^4\right)^2 = 5v^8$

c. $-(3u)^2 = -9u^4$

d. $(-3z)^2 = 9z^4$

2.11.10 Simplifying Expressions

In Section 2.10 we covered the definitions of the identities and inverses, and the various algebraic properties. We then learned about the order of operations.

Example 2.11.11 Use the associative, commutative, and distributive properties to simplify the expression $5x + 9(-2x + 3)$ as much as possible.

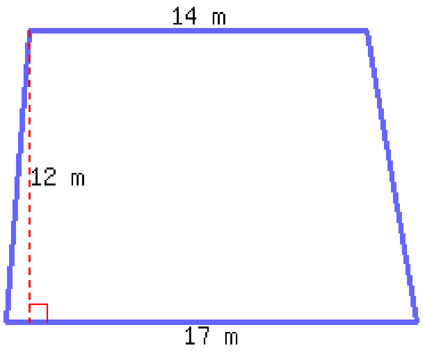
Explanation. We will remove parentheses by the distributive property, and then combine like terms:

$$\begin{aligned}5x + 9(-2x + 3) &= 5x + 9(-2x + 3)\\ &= 5x + 9(-2x) + 9(3)\\ &= 5x - 18x + 27\\ &= -13x + 27\end{aligned}$$

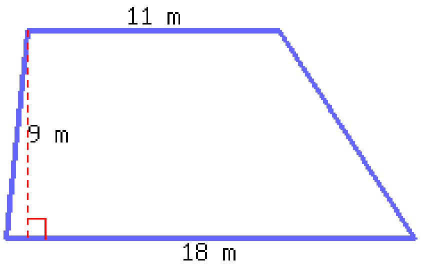
Exercises

A trapezoid's area can be calculated by the formula $A = \frac{1}{2}(b_1 + b_2)h$, where A stands for area, b_1 for the first base's length, b_2 for the second base's length, and h for height.

1. Find the area of the trapezoid below.



2. Find the area of the trapezoid below.



To convert a temperature measured in degrees Fahrenheit to degrees Celsius, there is a formula:

$$C = \frac{5}{9}(F - 32)$$

where C represents the temperature in degrees Celsius and F represents the temperature in degrees Fahrenheit.

3. If a temperature is 122°F, what is that temperature measured in Celsius?

4. If a temperature is 14°F, what is that temperature measured in Celsius?

5. Evaluate the expression x^2:

a. When $x = 6$, $x^2 = \boxed{}$

b. When $x = -6$, $x^2 = \boxed{}$

6. Evaluate the expression y^2:

a. When $y = 3$, $y^2 = \boxed{}$

b. When $y = -9$, $y^2 = \boxed{}$

7. Evaluate the expression y^3:

a. When $y = 5$, $y^3 = \boxed{}$

b. When $y = -3$, $y^3 = \boxed{}$

8. Evaluate the expression r^3:

a. When $r = 3$, $r^3 = \boxed{}$

b. When $r = -5$, $r^3 = \boxed{}$

9. List the terms in each expression.

a. $4t + 2z + 6$

b. $7z^2$

c. $9t + y$

d. $2t + 7t$

10. List the terms in each expression.

a. $8t^2 + 6 + 2x^2 - t^2$

b. $5x^2 - 6y^2 + 7y$

c. $-5z^2 + 3t^2 - 7y^2$

d. $-2y^2 + 4 - 3s + 2t$

11. Simplify each expression, if possible, by combining like terms.

a. $8t - t + 3t + 9t$

b. $-8z^2 + 5z^2 + 6z^2$

c. $3z - 3z$

d. $-3x^2 - 2 - 7x$

12. Simplify each expression, if possible, by combining like terms.

a. $5t^2 - 8y^2 + 4 + 2t$

b. $7t - 2t^2$

c. $-7z^2 - 2z^2 + 8x^2$

d. $-6s + x$

13. Simplify each expression, if possible, by combining like terms.

a. $\frac{4}{3}t + \frac{4}{9}$

b. $-\frac{1}{7}y^2 + y^2 - \frac{4}{3}y^2 + \frac{2}{7}s^2$

c. $-\frac{3}{8}y + \frac{3}{2}z^2 - 3z^2 + 2x$

d. $-\frac{2}{3}t - s$

14. Simplify each expression, if possible, by combining like terms.

a. $-\frac{1}{6}t - \frac{9}{4}t$

b. $-\frac{3}{8}s - z - \frac{2}{5}s$

c. $\frac{1}{3}y^2 - \frac{9}{5}y^2$

d. $-\frac{2}{9}y - \frac{1}{3}y^2 + \frac{1}{4}y^2 + 9y$

15. Is -2 a solution for x in the equation $2x + 2 = 2 - (5 + x)$? Evaluating the left and right sides gives:

$$\begin{array}{ccc} 2x + 2 & = & 2 - (5 + x) \\ \underline{\qquad} & \stackrel{?}{=} & \underline{\qquad} \end{array}$$

So -2 (☐ is ☐ is not) a solution to $2x + 2 = 2 - (5 + x)$.

16. Is -1 a solution for x in the equation $4x - 4 = -3 - (4 + x)$? Evaluating the left and right sides gives:

$$\begin{array}{ccc} 4x - 4 & = & -3 - (4 + x) \\ \underline{\qquad} & \stackrel{?}{=} & \underline{\qquad} \end{array}$$

So -1 (☐ is ☐ is not) a solution to $4x - 4 = -3 - (4 + x)$.

17. Is 1 a solution for x in the inequality $-4x^2 + 5x \leq 2x - 7$? Evaluating the left and right sides gives:

$$\begin{array}{ccc} -4x^2 + 5x & \leq & 2x - 7 \\ \underline{\qquad} & \stackrel{?}{\leq} & \underline{\qquad} \end{array}$$

So 1 (☐ is ☐ is not) a solution to $-4x^2 + 5x \leq 2x - 7$.

18. Is 2 a solution for x in the inequality $-2x^2 + 5x \leq 2x - 12$? Evaluating the left and right sides gives:

$$\begin{array}{ccc} -2x^2 + 5x & \leq & 2x - 12 \\ \underline{\qquad} & \stackrel{?}{\leq} & \underline{\qquad} \end{array}$$

So 2 (☐ is ☐ is not) a solution to $-2x^2 + 5x \leq 2x - 12$.

Solve the equation.

19. $t + 7 = 2$

20. $t + 4 = 1$

21. $-10 = t - 6$

22. $-9 = x - 7$

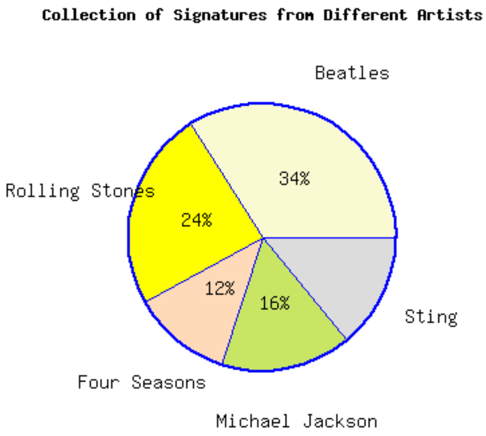
23. $96 = -8x$

24. $24 = -3y$

25. $\frac{5}{13}c = 25$

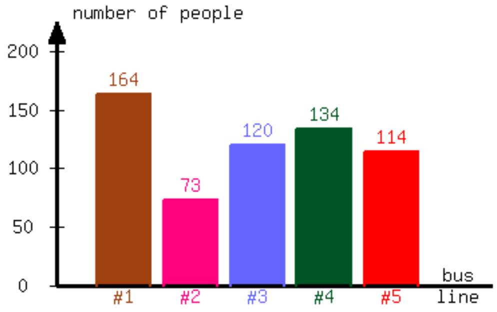
26. $\frac{4}{7}A = 12$

27. The pie chart represents a collector's collection of signatures from various artists.



If the collector has a total of 1450 signatures, there are signatures by Sting.

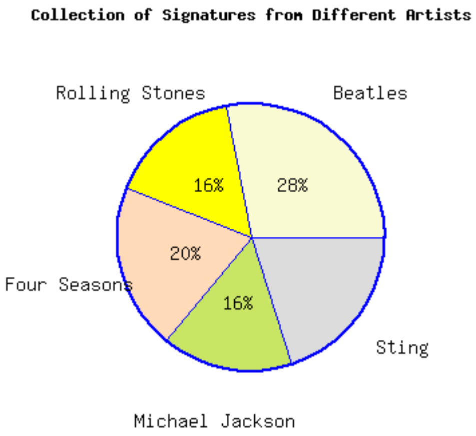
29. A community college conducted a survey about the number of students riding each bus line available. The following bar graph is the result of the survey.



What percent of students ride Bus #1?

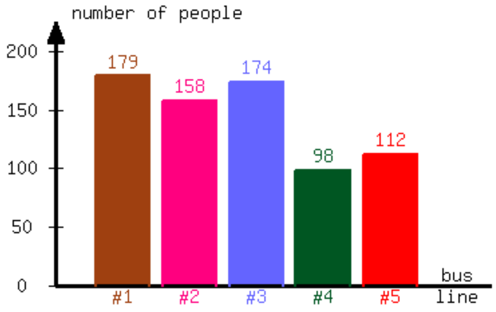
Approximately of students ride Bus #1.

28. The pie chart represents a collector's collection of signatures from various artists.



If the collector has a total of 1650 signatures, there are signatures by Sting.

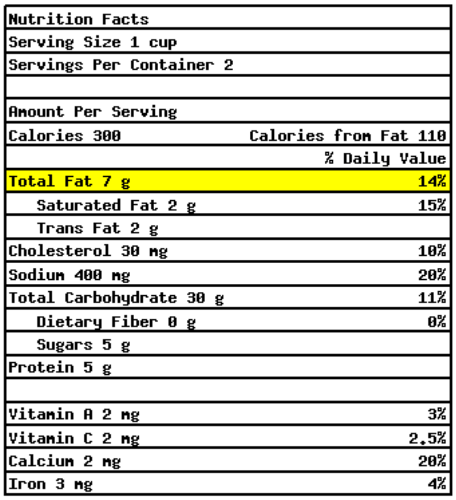
30. A community college conducted a survey about the number of students riding each bus line available. The following bar graph is the result of the survey.



What percent of students ride Bus #1?

Approximately of students ride Bus #1.

31. The following is a nutrition fact label from a certain macaroni and cheese box.

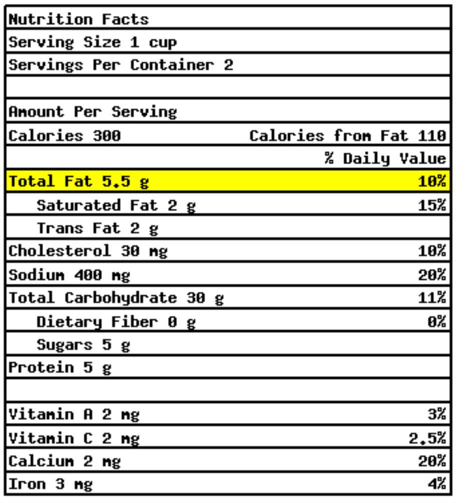


Nutrition Facts	
Serving Size 1 cup	
Servings Per Container 2	
Amount Per Serving	
Calories 300	Calories from Fat 110
	% Daily Value
Total Fat 7 g	14%
Saturated Fat 2 g	15%
Trans Fat 2 g	
Cholesterol 30 mg	10%
Sodium 400 mg	20%
Total Carbohydrate 30 g	11%
Dietary Fiber 0 g	0%
Sugars 5 g	
Protein 5 g	
Vitamin A 2 mg	3%
Vitamin C 2 mg	2.5%
Calcium 2 mg	20%
Iron 3 mg	4%

The highlighted row means each serving of macaroni and cheese in this box contains 7 g of fat, which is 14% of an average person's daily intake of fat. What's the recommended daily intake of fat for an average person?

The recommended daily intake of fat for an average person is $\boxed{}$.

32. The following is a nutrition fact label from a certain macaroni and cheese box.



Nutrition Facts	
Serving Size 1 cup	
Servings Per Container 2	
Amount Per Serving	
Calories 300	Calories from Fat 110
	% Daily Value
Total Fat 5.5 g	10%
Saturated Fat 2 g	15%
Trans Fat 2 g	
Cholesterol 30 mg	10%
Sodium 400 mg	20%
Total Carbohydrate 30 g	11%
Dietary Fiber 0 g	0%
Sugars 5 g	
Protein 5 g	
Vitamin A 2 mg	3%
Vitamin C 2 mg	2.5%
Calcium 2 mg	20%
Iron 3 mg	4%

The highlighted row means each serving of macaroni and cheese in this box contains 5.5 g of fat, which is 10% of an average person's daily intake of fat. What's the recommended daily intake of fat for an average person?

The recommended daily intake of fat for an average person is $\boxed{}$.

33. Jerry used to make 13 dollars per hour. After he earned his Bachelor's degree, his pay rate increased to 48 dollars per hour. What is the percentage increase in Jerry's salary?

The percentage increase is $\boxed{}$.

34. Eileen used to make 14 dollars per hour. After she earned her Bachelor's degree, her pay rate increased to 49 dollars per hour. What is the percentage increase in Eileen's salary?

The percentage increase is $\boxed{}$.

35. After a 10% increase, a town has 550 people. What was the population before the increase?

Before the increase, the town's population was $\boxed{}$.

36. After a 70% increase, a town has 1020 people. What was the population before the increase?

Before the increase, the town's population was $\boxed{}$.

37. A bicycle for sale costs \$254.88, which includes 6.2% sales tax. What was the cost before sales tax?

Assume the bicycle's price before sales tax is p dollars. Write an equation to model this scenario. There is no need to solve it.

38. A bicycle for sale costs \$283.77, which includes 5.1% sales tax. What was the cost before sales tax?

Assume the bicycle's price before sales tax is p dollars. Write an equation to model this scenario. There is no need to solve it.

39. The property taxes on a 2100-square-foot house are \$4,179.00 per year. Assuming these taxes are proportional, what are the property taxes on a 1700-square-foot house?

Assume property taxes on a 1700-square-foot house is t dollars. Write an equation to model this scenario. There is no need to solve it.

40. The property taxes on a 1600-square-foot house are \$1,600.00 per year. Assuming these taxes are proportional, what are the property taxes on a 2000-square-foot house?

Assume property taxes on a 2000-square-foot house is t dollars. Write an equation to model this scenario. There is no need to solve it.

41. A swimming pool is being filled with water from a garden hose at a rate of 5 gallons per minute. If the pool already contains 30 gallons of water and can hold 135 gallons, after how long will the pool overflow?

Assume m minutes later, the pool would overflow. Write an equation to model this scenario. There is no need to solve it.

42. A swimming pool is being filled with water from a garden hose at a rate of 8 gallons per minute. If the pool already contains 40 gallons of water and can hold 280 gallons, after how long will the pool overflow?

Assume m minutes later, the pool would overflow. Write an equation to model this scenario. There is no need to solve it.

43. Use the commutative property of addition to write an equivalent expression to $5b + 31$.

44. Use the commutative property of addition to write an equivalent expression to $6q + 79$.

45. Use the associative property of multiplication to write an equivalent expression to $3(4r)$.

46. Use the associative property of multiplication to write an equivalent expression to $4(7m)$.

47. Use the distributive property to write an equivalent expression to $10(p + 2)$ that has no grouping symbols.

48. Use the distributive property to write an equivalent expression to $7(q + 6)$ that has no grouping symbols.

49. Use the distributive property to simplify $4 + 9(2 + 4y)$ completely.

50. Use the distributive property to simplify $9 + 4(9 + 3r)$ completely.

51. Use the distributive property to simplify $6 - 4(1 - 6a)$ completely.

52. Use the distributive property to simplify $3 - 9(-5 - 6b)$ completely.

53. Use the properties of exponents to simplify the expression.

$$r^{12} \cdot r^{17}$$

54. Use the properties of exponents to simplify the expression.

$$t^{14} \cdot t^{11}$$

55. Use the properties of exponents to simplify the expression.

$\left(y^{10}\right)^3$

56. Use the properties of exponents to simplify the expression.

$\left(t^{11}\right)^{10}$

57. Use the properties of exponents to simplify the expression.

$(3x)^4$

58. Use the properties of exponents to simplify the expression.

$(2r)^2$

59. Use the properties of exponents to simplify the expression.

$(-2t^5) \cdot (4t^{16})$

60. Use the properties of exponents to simplify the expression.

$(-5y^7) \cdot (3y^9)$

61. Use the properties of exponents to simplify the expression.

a. $\left(-2b^3\right)^6 = \boxed{}$

b. $-\left(2b^3\right)^6 = \boxed{}$

62. Use the properties of exponents to simplify the expression.

a. $\left(-2a^3\right)^2 = \boxed{}$

b. $-\left(2a^3\right)^2 = \boxed{}$

63. Simplify the following expression.

$\left(3r^3\right)^4\left(r^2\right)^2 = \boxed{}$

64. Simplify the following expression.

$\left(5t^2\right)^2\left(t^5\right)^2 = \boxed{}$

65. Simplify the following expressions if possible.

a. $p^2 + 2p^2 = \boxed{}$

b. $(p^2)(2p^2) = \boxed{}$

c. $p^2 - 4p^3 = \boxed{}$

d. $(p^2)(-4p^3) = \boxed{}$

66. Simplify the following expressions if possible.

a. $-2q + 4q = \boxed{}$

b. $(-2q)(4q) = \boxed{}$

c. $-2q - q^4 = \boxed{}$

d. $(-2q)(-q^4) = \boxed{}$

Multiply the polynomials.

67. $-10x^2\left(3x^2 + 5x\right) = \boxed{}$

68. $7x^2\left(9x^2 + 10x\right) = \boxed{}$

CHAPTER 3

Linear Equations and Inequalities

3.1 Solving Multistep Linear Equations

We have learned how to solve one-step equations in Section. In this section, we will learn how to solve multistep equations.

3.1.1 Solving Two-Step Equations

Example 3.1.2 A water tank can hold 140 gallons of water, but it has only 5 gallons of water. A tap was turned on, pouring 15 gallons of water into the tank every minute. After how many minutes will the tank be full? Let's find a pattern first.

Minutes since Tap Was Turned on	Amount of Water in the Tank (in Gallons)
0	5
1	$15 \cdot 1 + 5 = 20$
2	$15 \cdot 2 + 5 = 35$
3	$15 \cdot 3 + 5 = 50$
4	$15 \cdot 4 + 5 = 65$
$\vdots$	$\vdots$
m	$15m + 5$

Table 3.1.3: Amount of Water in the Tank

We can see that after m minutes, the tank has $15m + 5$ gallons of water. This makes sense since the tap pours $15m$ gallons of water into the tank in m minutes and it had 5 gallons to start with. To find when the tank will be full (with 140 gallons of water), we can write the equation

$$15m + 5 = 140$$

First, we need to isolate the variable term, $15m$, in the equation. In other words, we need to remove 5 from the left side of the equals sign. We can do this by subtracting 5 from both sides of the equation. Once the variable term is isolated, we can eliminate the coefficient and solve for m. The full process is:

$$\begin{aligned} 15m + 5 &= 140 \\ 15m + 5 - 5 &= 140 - 5 \\ 15m &= 135 \\ \frac{15m}{15} &= \frac{135}{15} \\ m &= 9 \end{aligned}$$

Next, we need to substitute m with 9 in the equation $15m + 5 = 140$ to check the solution:

$$\begin{aligned}15m + 5 &= 140\\ 15(9) + 5 &\overset{?}{=} 140\\ 135 + 5 &\overset{\checkmark}{=} 140\end{aligned}$$

The solution 9 is checked. In summary, the tank will be full after 9 minutes.

In solving the two-step equation in Example 3.1.2, we first isolated the variable expression $15m$ and then eliminated the coefficient of 15 by dividing each side of the equation by 15. These two steps will be at the heart of our approach to solving linear equations. For more complicated equations, we may need to simplify some of the expressions first. Below is a general approach to solving linear equations that we will use as we solve more and more complicated equations.

Simplify Simplify the expressions on each side of the equation by distributing and combining like terms.

Isolate Use addition or subtraction to separate the variable terms and constant terms (numbers) so that they are on different sides of the equation.

Eliminate Use multiplication or division to eliminate the variable term's coefficient.

Check Check the solution. Substitute values into the original equation and use the order of operations to simplify both sides. It's important to use the order of operations alone rather than properties like the distributive law. Otherwise you might repeat the same arithmetic errors made while solving and fail to catch an incorrect solution.

Summarize State the solution set or (in the case of an application problem) summarize the result in a complete sentence using appropriate units.

List 3.1.4: Steps to Solve Linear Equations

Let's look at some more examples.

Example 3.1.5 Solve for y in the equation $7 - 3y = -8$.

Explanation. To solve, we will first separate the variable terms and constant terms into different sides of the equation. Then we will eliminate the variable term's coefficient.

$$\begin{aligned}7 - 3y &= -8\\ 7 - 3y - 7 &= -8 - 7\\ -3y &= -15\\ \frac{-3y}{-3} &= \frac{-15}{-3}\\ y &= 5\end{aligned}$$

Checking the solution $y = 5$:

$$\begin{aligned}
7 - 3y &= -8\\
7 - 3(5) &\overset{?}{=} -8\\
7 - 15 &\overset{\checkmark}{=} -8
\end{aligned}$$

Therefore the solution to the equation $7 - 3y = -8$ is 5 and the solution set is $\{5\}$.

3.1.2 Solving Multistep Linear Equations

Example 3.1.6 Ahmed has saved \$2,500.00 in his savings account and is going to start saving \$550.00 per month. Julia has saved \$4,600.00 in her savings account and is going to start saving \$250.00 per month. If this situation continues, how many months later would Ahmed catch up with Julia in savings?

Ahmed saves \$550.00 per month, so he can save $550m$ dollars in m months. With the \$2,500.00 he started with, after m months he has $550m+2500$ dollars. Similarly, after m months, Julia has $250m+4600$ dollars. To find when those two accounts will have the same amount of money, we write the equation

$$550m + 2500 = 250m + 4600.$$

$$\begin{aligned}
550m + 2500 &= 250m + 4600\\
550m + 2500 - 2500 &= 250m + 4600 - 2500\\
550m &= 250m + 2100\\
550m - 250m &= 250m + 2100 - 250m\\
300m &= 2100\\
\frac{300m}{300} &= \frac{2100}{300}\\
m &= 7
\end{aligned}$$

Checking the solution 7:

$$\begin{aligned}
550m + 2500 &= 250m + 4600\\
550(7) + 2500 &\overset{?}{=} 250(7) + 4600\\
3850 + 2500 &\overset{?}{=} 1750 + 4600\\
6350 &\overset{\checkmark}{=} 6350
\end{aligned}$$

In summary, Ahmed will catch up with Julia's savings in 7 months.

Example 3.1.7 Solve for x in $5 - 2x = 5x - 9$.

Explanation.

$$\begin{aligned}
5 - 2x &= 5x - 9\\
5 - 2x - 5 &= 5m - 9 - 5\\
-2x &= 5x - 14\\
-2x - 5x &= 5x - 14 - 5x\\
-7x &= -14\\
\frac{-7x}{-7} &= \frac{-14}{-7}\\
x &= 2
\end{aligned}$$

> Checking the solution 2:
>
> $$\begin{aligned} 5-2x &= 5x-9 \\ 5-2(2) &\overset{?}{=} 5(2)-9 \\ 5-4 &\overset{?}{=} 10-9 \\ 1 &\overset{\checkmark}{=} 1 \end{aligned}$$
>
> Therefore the solution is 2 and the solution set is $\{2\}$.

In Example 3.1.7, we could have moved variable terms to the right side of the equals sign, and number terms to the left side. We chose not to. There's no reason we *couldn't* have moved variable terms to the right side though. Let's compare:

$$\begin{aligned} 5-2x &= 5x-9 \\ 5-2x+9 &= 5x-9+9 \\ 14-2x &= 5x \\ 14-2x+2x &= 5x+2x \\ 14 &= 7x \\ \frac{14}{7} &= \frac{7x}{7} \\ 2 &= x \end{aligned}$$

Lastly, we could save a step by moving variable terms and number terms in one step:

$$\begin{aligned} 5-2x &= 5x-9 \\ 5-2x+2x+9 &\quad 5x-9+2x+9 \\ 14 &= 7x \\ \frac{14}{7} &= \frac{7x}{7} \\ 2 &= x \end{aligned}$$

Remark 3.1.8. This textbook will move variable terms and number terms separately throughout this chapter. Check with your instructor for their expectations.

Checkpoint 3.1.9. Solve the equation.

$7a+3=a+45$

Explanation. The first step is to subtract terms in order to separate the variable and non-variable terms.

$$\begin{aligned} 7a+3 &= a+45 \\ 7a+3-\mathbf{a}-\mathbf{3} &= a+45-\mathbf{a}-\mathbf{3} \\ 6a &= 42 \\ \frac{6a}{6} &= \frac{42}{6} \\ a &= 7 \end{aligned}$$

The solution to this equation is 7. To stress that this is a value assigned to a, some report $a=7$. We can also say that the solution set is $\{7\}$, or that $a \in \{7\}$. If we substitute 7 in for a in the original equation $7a + 3 = a + 45$, the equation will be true. Please check this on your own; it is an important habit.

The next example requires combining like terms.

Example 3.1.10 Solve for n in $n - 9 + 3n = n - 3n$.

Explanation.

To start solving this equation, we'll need to combine like terms. After this, we can put all terms containing n on one side of the equation and finish solving for n.

$$\begin{aligned} n - 9 + 3n &= n - 3n \\ 4n - 9 &= -2n \\ 4n - 9 - 4n &= -2n - 4n \\ -9 &= -6n \\ \frac{-9}{-6} &= \frac{-6n}{-6} \\ n &= \frac{3}{2} \end{aligned}$$

Checking the solution $\frac{3}{2}$:

$$\begin{aligned} n - 9 + 3n &= n - 3n \\ \frac{3}{2} - 9 + 3\left(\frac{3}{2}\right) &\overset{?}{=} \frac{3}{2} - 3\left(\frac{3}{2}\right) \\ \frac{3}{2} - 9 + \frac{9}{2} &\overset{?}{=} \frac{3}{2} - \frac{9}{2} \\ \frac{12}{2} - 9 &\overset{?}{=} -\frac{6}{2} \\ 6 - 9 &\overset{?}{=} -3 \\ -3 &\overset{?}{=} -3 \end{aligned}$$

The solution to the equation $n - 9 + 3n = n - 3n$ is $\frac{3}{2}$ and the solution set is $\left\{\frac{3}{2}\right\}$.

Checkpoint 3.1.11. Solve the equation.

$-1 + 7 = -8b - b - 3$

Explanation. The first step is simply to combine like terms.

$$\begin{aligned} -1 + 7 &= -8b - b - 3 \\ 6 &= -9b - 3 \\ 6 + \mathbf{3} &= -9b - 3 + \mathbf{3} \\ 9 &= -9b \\ \frac{9}{-9} &= \frac{-9b}{-9} \\ -1 &= b \\ b &= -1 \end{aligned}$$

The solution to this equation is -1. To stress that this is a value assigned to b, some report $b = -1$. We can also say that the solution set is $\{-1\}$, or that $b \in \{-1\}$. If we substitute -1 in for b in the original equation $-1 + 7 = -8b - b - 3$, the equation will be true. Please check this on your own; it is an important habit.

Example 3.1.12 Azul is designing a rectangular garden and they have 40 meters of wood for the border. Their garden's length must be 4 meters less than three times the width, and its perimeter must be 40 meters. Find the garden's length and width.

Explanation. Reminder: A rectangle's perimeter formula is $P = 2(L+W)$, where P stands for perimeter,

L stands for length and W stands for width.

Let Azul's garden width be W meters. We can then represent the length as $3W - 4$ meters since we are told that it is 4 meters less than three times the width. It's given that the perimeter is 40 meters. Substituting those values into the formula, we have:

$$\begin{aligned} P &= 2(L + W) \\ 40 &= 2(3W - 4 + W) \\ 40 &= 2(4W - 4) && \text{Like terms were combined.} \end{aligned}$$

The next step to solve this equation is to remove the parentheses by distribution.

$$\begin{aligned} 40 &= 2(4W - 4) \\ 40 &= 8W - 8 \\ 40 + 8 &= 8W - 8 + 8 \\ 48 &= 8W \\ \frac{48}{8} &= \frac{8W}{8} \\ 6 &= W. \end{aligned}$$

Checking the solution $W = 6$:

$$\begin{aligned} 40 &= 2(4W - 4) \\ 40 &\stackrel{?}{=} 2(4(6) - 4) \\ 40 &\stackrel{\checkmark}{=} 2(20). \end{aligned}$$

To determine the length, recall that this was represented by $3W - 4$, which is:

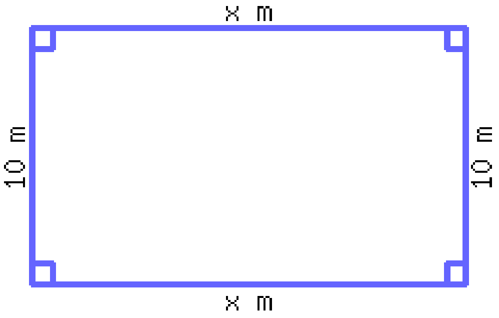
$$\begin{aligned} 3W - 4 &= 3(6) - 4 \\ &= 14. \end{aligned}$$

Thus, the width of Azul's garden is 6 meters and the length is 14 meters.

Checkpoint 3.1.13. A rectangle's perimeter is 52 m. Its width is 10 m. Use an equation to solve for the rectangle's length.

Its length is ☐.

Explanation. When we deal with a geometric figure, it's always a good idea to sketch it to help us think. Let the length be x meters.



The perimeter is given as 52 m. Adding up the rectangle's 4 sides gives the perimeter. The equation is:

$$\begin{aligned}
x+x+10+10&=52\\
2x+20&=52\\
2x+20\mathbf{-20}&=52\mathbf{-20}\\
2x&=32\\
\frac{2x}{2}&=\frac{32}{2}\\
x&=16
\end{aligned}$$

So the rectangle's length is 16 m. Don't forget the unit m.

We should be careful when we distribute a negative sign into the parentheses, like in the next example.

Example 3.1.14 Solve for a in $4-(3-a)=-2-2(2a+1)$.

Explanation. To solve this equation, we will simplify each side of the equation, manipulate it so that all variable terms are on one side and all constant terms are on the other, and then solve for a:

$$\begin{aligned}
4-(3-a)&=-2-2(2a+1)\\
4-3+a&=-2-4a-2\\
1+a&=-4-4a\\
1+a+4a&=-4-4a+4a\\
1+5a&=-4\\
1+5a-1&=-4-1\\
5a&=-5\\
\frac{5a}{5}&=\frac{-5}{5}\\
a&=-1
\end{aligned}$$

Checking the solution −1:

$$\begin{aligned}
4-(3-a)&=-2-2(2a+1)\\
4-(3-(-1))&\stackrel{?}{=}-2-2(2(-1)+1)\\
4-(4)&\stackrel{?}{=}-2-2(-1)\\
0&\stackrel{\checkmark}{=}0
\end{aligned}$$

Therefore the solution to the equation is −1 and the solution set is $\{-1\}$.

3.1.3 Differentiating between Simplifying Expressions, Evaluating Expressions and Solving Equations

Let's look at the following similar, yet different examples.

Example 3.1.15 Simplify the expression $10-3(x+2)$.

Explanation.

$$\begin{aligned}
10-3(x+2)&=10-3x-6\\
&=-3x+4
\end{aligned}$$

An equivalent result is $4 - 3x$. Note that our final result is an *expression*.

Example 3.1.16 Evaluate the expression $10 - 3(x + 2)$ when $x = 2$ and when $x = 3$.

Explanation. We will substitute $x = 2$ into the expression:

$$\begin{aligned} 10 - 3(x + 2) &= 10 - 3(2 + 2) \\ &= 10 - 3(4) \\ &= 10 - 12 \\ &= -2 \end{aligned}$$

When $x = 2$, $10 - 3(x + 2) = -2$.

Similarly, we will substitute $x = 3$ into the expression:

$$\begin{aligned} 10 - 3(x + 2) &= 10 - 3(3 + 2) \\ &= 10 - 3(5) \\ &= 10 - 15 = -5 \end{aligned}$$

When $x = 3$, $10 - 3(x + 2) = -5$.

Note that the final results here are *values of the original expression*.

Example 3.1.17 Solve the equation $10 - 3(x + 2) = x - 16$.

Explanation.

$$\begin{aligned} 10 - 3(x + 2) &= x - 16 \\ 10 - 3x - 6 &= x - 16 \\ -3x + 4 &= x - 16 \\ -3x + 4 - 4 &= x - 16 - 4 \\ -3x &= x - 20 \\ -3x - x &= x - 20 - x \\ -4x &= -20 \\ \frac{-4x}{-4} &= \frac{-20}{-4} \\ x &= 5 \end{aligned}$$

Checking the solution $x = 5$:

$$\begin{aligned} 10 - 3(x + 2) &= x - 16 \\ 10 - 3(5 + 2) &\stackrel{?}{=} 5 - 16 \\ 10 - 3(7) &\stackrel{?}{=} -11 \\ 10 - 21 &\stackrel{\checkmark}{=} -11 \end{aligned}$$

We have checked that $x = 5$ is a solution of the equation $10 - 3(x + 2) = x - 16$.

Note that the final results here are *solutions* to the equations.

- An expression like $10 - 3(x + 2)$ can be simplified to $-3x + 4$ (as in Example 3.1.15), but we cannot solve for x in an expression.
- As x takes different values, an expression has different values. In Example 3.1.16, when $x = 2$, $10 - 3(x + 2) = -2$; but when $x = 3$, $10 - 3(x + 2) = -5$.
- An equation connects two expressions with an equals sign. In Example 3.1.17, $10 - 3(x + 2) = x - 16$ has the expression $10 - 3(x + 2)$ on the left side of equals sign, and the expression $x - 16$ on the right side.
- When we solve the equation $10 - 3(x + 2) = x - 16$, we are looking for a number which makes those two expressions have the same value. In Example 3.1.17, we found the solution to be $x = 5$, which makes both $10 - 3(x + 2) = -11$ and $x - 16 = -11$, as shown in the checking part.

List 3.1.18: A summary the differences among simplifying expressions, evaluating expressions and solving equations:

Exercises

Warmup and Review Solve the equation.

1. $r + 3 = -3$
2. $r + 9 = 6$
3. $t - 6 = -2$
4. $t - 2 = 6$

5. $44 = -4x$
6. $42 = -7x$
7. $\frac{10}{3}a = 2$
8. $\frac{5}{9}b = 9$

Solving Two-Step Equations Solve the equation.

9. $6A + 4 = 58$
10. $2B + 2 = 8$
11. $8m - 6 = 18$
12. $5n - 5 = -50$

13. $-9 = 2q + 3$
14. $-23 = 8y + 1$
15. $-26 = 5r - 6$
16. $6 = 2a - 4$

17. $-5b + 3 = 48$
18. $-8A + 1 = 49$
19. $-2B - 8 = -28$
20. $-5m - 5 = 5$

21. $17 = -n + 9$
22. $5 = -q + 3$
23. $7y + 35 = 0$
24. $4r + 40 = 0$

Application Problems for Solving Two-Step Equations

25. A gym charges members $25 for a registration fee, and then $38 per month. You became a member some time ago, and now you have paid a total of $557 to the gym. How many months have passed since you joined the gym?

months have passed since you joined the gym.

26. Your cell phone company charges a $18 monthly fee, plus $0.15 per minute of talk time. One month your cell phone bill was $90. How many minutes did you spend talking on the phone that month?

You spent talking on the phone that month.

27. A school purchased a batch of T-shirts from a company. The company charged $7 per T-shirt, and gave the school a $60 rebate. If the school had a net expense of $2,460 from the purchase, how many T-shirts did the school buy?

The school purchased T-shirts.

28. Joshua hired a face-painter for a birthday party. The painter charged a flat fee of $80, and then charged $5.50 per person. In the end, Joshua paid a total of $217.50. How many people used the face-painter's service?

people used the face-painter's service.

29. A certain country has 676.8 million acres of forest. Every year, the country loses 7.52 million acres of forest mainly due to deforestation for farming purposes. If this situation continues at this pace, how many years later will the country have only 368.48 million acres of forest left? (Use an equation to solve this problem.)

After years, this country would have 368.48 million acres of forest left.

30. Heather has $87 in her piggy bank. She plans to purchase some Pokemon cards, which costs $1.55 each. She plans to save $62.20 to purchase another toy. At most how many Pokemon cards can he purchase?

Write an equation to solve this problem.

Heather can purchase at most Pokemon cards.

Solving Equations with Variable Terms on Both Sides Solve the equation.

31. $9q + 10 = q + 50$

32. $8x + 5 = x + 26$

33. $-6r + 9 = -r - 1$

34. $-8a + 3 = -a - 39$

35. $5 - 7b = 6b + 96$

36. $2 - 2A = 6A + 82$

37. $5B + 7 = 9B + 10$

38. $4m + 4 = 2m + 3$

39.
a. $7n + 3 = 3n + 39$
b. $3x + 3 = 7x - 29$

40.
a. $9q + 10 = 3q + 34$
b. $3C + 10 = 9C - 44$

Application Problems for Solving Equations with Variable Terms on Both Sides Use a linear equation to solve the word problem.

41. Two trees are 6 feet and 11.5 feet tall. The shorter tree grows 2.5 feet per year; the taller tree grows 2 feet per year. How many years later would the shorter tree catch up with the taller tree?

It would take the shorter tree ☐ years to catch up with the taller tree.

42. Massage Heaven and Massage You are competitors. Massage Heaven has 3400 registered customers, and it gets approximately 900 newly registered customers every month. Massage You has 10600 registered customers, and it gets approximately 450 newly registered customers every month. How many months would it take Massage Heaven to catch up with Massage You in the number of registered customers?

These two companies would have approximately the same number of registered customers ☐ months later.

43. Two truck rental companies have different rates. V-Haul has a base charge of \$60.00, plus \$0.60 per mile. W-Haul has a base charge of \$52.60, plus \$0.65 per mile. For how many miles would these two companies charge the same amount?

If a driver drives ☐ miles, those two companies would charge the same amount of money.

44. Massage Heaven and Massage You are competitors. Massage Heaven has 9200 registered customers, but it is losing approximately 400 registered customers every month. Massage You has 1200 registered customers, and it gets approximately 400 newly registered customers every month. How many months would it take Massage Heaven to catch up with Massage You in the number of registered customers?

These two companies would have approximately the same number of registered customers ☐ months later.

45. Tammy has \$85.00 in her piggy bank, and she spends \$4.00 every day.

Laurie has \$8.00 in her piggy bank, and she saves \$1.50 every day.

If they continue to spend and save money this way, how many days later would they have the same amount of money in their piggy banks?

days later, Tammy and Laurie will have the same amount of money in their piggy banks.

46. Lindsay has \$95.00 in her piggy bank, and she spends \$4.00 every day.

Derick has \$18.00 in his piggy bank, and he saves \$3.00 every day.

If they continue to spend and save money this way, how many days later would they have the same amount of money in their piggy banks?

days later, Lindsay and Derick will have the same amount of money in their piggy banks.

Solving Linear Equations with Like Terms Solve the equation.

47. $4m + 7m + 2 = 112$

48. $9n + 2n + 2 = 90$

49. $6q + 6 + 4 = 40$

50. $3x + 10 + 3 = 22$

51. $-2 + 4 = -3r - r - 30$

52. $-6 + 8 = -5t - t - 46$

53. $2y + 3 - 7y = 38$

54. $5r + 7 - 7r = 27$

55. $-6r + 10 + r = -20$

56. $-3r + 5 + r = 5$

57. $45 = -8n - 9 - n$

58. $-58 = -5q - 4 - q$

59. $2 - x - x = -7 + 3$

60. $8 - r - r = -2 + 14$

61. $3 - 2t - 8 = -5$

62. $2 - 9b - 10 = -8$

63. $A - 8 - 5A = -6 - 8A + 26$

64. $B - 4 - 8B = -4 - 2B + 25$

65. $-10m + 2m = 10 - 2m - 40$

66. $-9n + 4n = 8 - 2n - 11$

67. $4q + 10 = -5q + 10 - 2q$

68. $10x + 5 = -3x + 5 - 2x$

69. $-8 + 9 = 7r - 6 - 10r + 4 + 2r$

70. $-2 + (-1) = 4t - 6 - 7t + 2 + 2t$

Application Problems for Solving Linear Equations with Like Terms

71. A 138-meter rope is cut into two segments. The longer segment is 28 meters longer than the shorter segment. Write and solve a linear equation to find the length of each segment. Include units.

The segments are and long.

72. In a doctor's office, the receptionist's annual salary is \$142,000 less than that of the doctor. Together, the doctor and the receptionist make \$208,000 per year. Find each person's annual income.

The receptionist's annual income is . The doctor's annual income is .

73. Phil and Penelope went picking strawberries. Phil picked 116 fewer strawberries than Penelope did. Together, they picked 214 strawberries. How many strawberries did Penelope pick?

Penelope picked strawberries.

74. Virginia and Ross collect stamps. Ross collected 27 fewer than five times the number of Virginia's stamps. Altogether, they collected 1005 stamps. How many stamps did Virginia and Ross collect?

Virginia collected stamps. Ross collected stamps.

75. Diane and Tracei sold girl scout cookies. Diane's sales were \$37 more than three times of Tracei's. Altogether, their sales were \$437. How much did each girl sell?

Diane's sales were . Tracei's sales were .

76. A hockey team played a total of 191 games last season. The number of games they won was 11 more than five times of the number of games they lost.

Write and solve an equation to answer the following questions.

The team lost games. The team won games.

77. After a 55% increase, a town has 155 people. What was the population before the increase?

Before the increase, the town's population was .

78. After a 35% increase, a town has 270 people. What was the population before the increase?

Before the increase, the town's population was .

Solving Linear Equations Involving Distribution Solve the equation.

79. $2(t+2)=24$

80. $8(b+9)=112$

81. $5(c-6)=5$

82. $2(B-3)=-18$

83. $24=-8(m+7)$

84. $-30=-5(n+1)$

85. $12=-2(q-5)$

86. $128=-8(x-9)$

87. $-(r-4)=8$

88. $-(t-8)=-2$

89. $-14=-(7-b)$

90. $-2=-(3-c)$

91. $10(10B-8)=520$

92. $7(5C-8)=-56$

93. $2=-2(9-2n)$

94. $110=-5(3-5q)$

95. $3+9(x+8)=111$

96. $1+6(r+7)=31$

97. $5-8(t+7)=13$

98. $3-10(b+7)=-137$

99. $22=2-4(c-7)$

100. $97=9-8(B-7)$

101. $3-6(C-7)=105$

102. $1-8(n-7)=17$

103. $3=9-(5-q)$

104. $-1=8-(3-x)$

105. $1-(r+10)=-18$

106. $4-(t+7)=-6$

107.
a. $5+(b+4)=12$
b. $5-(b+4)=12$

108.
a. $2+(c+1)=-6$
b. $2-(c+1)=-6$

Solve the equation.

109. $4(B+5)-10(B-7)=90$

110. $3(C+10)-8(C-2)=46$

111. $5+8(n-5)=-28-(7-2n)$

112. $4+9(p-10)=-84-(2-2p)$

113. $7(x-2)-x=67-3(7+3x)$

114. $10(r-6)-r=-90-3(2+3r)$

115. $7(-10t+10)=14(-2-6t)$

116. $3(-10b+6)=6(-9-6b)$

117. $23+6(6-4c)=-4(c-13)+7$

118. $12+4(3-3B)=-4(B-4)+8$

Application Problems for Solving Linear Equations Involving Distribution

119. A rectangle's perimeter is 78 cm. Its base is 27 cm.

Its height is ☐.

120. A rectangle's perimeter is 58 m. Its width is 12 m. Use an equation to solve for the rectangle's length.

Its length is ☐.

121. A rectangle's perimeter is 116 in. Its length is 8 in longer than its width. Use an equation to find the rectangle's length and width.

Its width is ☐.

Its length is ☐.

122. A rectangle's perimeter is 120 cm. Its length is 2 times as long as its width. Use an equation to find the rectangle's length and width.

It's width is ☐.

Its length is ☐.

123. A rectangle's perimeter is 106 ft. Its length is 2 ft shorter than four times its width. Use an equation to find the rectangle's length and width.

Its width is ☐.

Its length is ☐.

124. A rectangle's perimeter is 184 ft. Its length is 4 ft longer than three times its width. Use an equation to find the rectangle's length and width.

Its width is ☐.

Its length is ☐.

Comparisons

125. Solve the equation.

a. $-b + 7 = 7$

b. $-y + 7 = -7$

c. $-r - 7 = 7$

d. $-a - 7 = -7$

126. Solve the equation.

a. $-c + 4 = 4$

b. $-m + 4 = -4$

c. $-B - 4 = 4$

d. $-y - 4 = -4$

127. a. Solve the following linear equation:

$r - 2 = 8$

b. Evaluate the following expression when $r = 10$:

$r - 2 =$ ☐

128. a. Solve the following linear equation:

$r - 8 = -3$

b. Evaluate the following expression when $r = 5$:

$r - 8 =$ ☐

129. a. Solve the following linear equation:

$4(t+6)-4=36$

b. Evaluate the following expression when $t=4$:

$4(t+6)-4=$

c. Simplify the following expression:

$4(t+6)-4=$

130. a. Solve the following linear equation:

$3(t-5)+9=6$

b. Evaluate the following expression when $t=4$:

$3(t-5)+9=$

c. Simplify the following expression:

$3(t-5)+9=$

131. Choose True or False for the following questions about the difference between expressions and equations.

a. We can evaluate $-10x-10$ when $x=1$ (☐ True ☐ False)

b. $-10x-10=-10x-10$ is an equation. (☐ True ☐ False)

c. We can evaluate $-10x-10=-10x-10$ when $x=1$ (☐ True ☐ False)

d. We can check whether $x=1$ is a solution of $-10x-10$. (☐ True ☐ False)

e. $-10x-10$ is an expression. (☐ True ☐ False)

f. We can check whether $x=1$ is a solution of $-10x-10=-10x-10$. (☐ True ☐ False)

g. $-10x-10=-10x-10$ is an expression. (☐ True ☐ False)

h. $-10x-10$ is an equation. (☐ True ☐ False)

132. Choose True or False for the following questions about the difference between expressions and equations.

a. $-7x+4$ is an expression. (☐ True ☐ False)

b. $4x-7$ is an equation. (☐ True ☐ False)

c. $-7x+4=4x-7$ is an equation. (☐ True ☐ False)

d. We can check whether $x=1$ is a solution of $-7x+4$. (☐ True ☐ False)

e. $-7x+4=4x-7$ is an expression. (☐ True ☐ False)

f. We can evaluate $-7x+4=4x-7$ when $x=1$ (☐ True ☐ False)

g. We can evaluate $-7x+4$ when $x=1$ (☐ True ☐ False)

h. We can check whether $x=1$ is a solution of $-7x+4=4x-7$. (☐ True ☐ False)

Challenge

133. Think of a number. Add four to your number. Now double that. Then add six. Then halve it. Finally, subtract 7. What is the result? Do you always get the same result, regardless of what number you start with? How does this work? Explain using algebra.

134. Write a linear equation whose solution is $x=-9$.

Note that you may not write an equation whose left side is just "x" or whose right side is just "x."

There are infinitely many correct answers to this problem. Be creative. After finding an equation that works, see if you can come up with a different one that also works.

3.2 Solving Multistep Linear Inequalities

We have learned how to solve one-step inequalities in Section. In this section, we will learn how to solve multistep inequalities.

3.2.1 Solving Multistep Inequalities

When solving a linear inequality, we follow the same steps in List 3.1.4. The only difference in our steps to solving is that when we multiply or divide by a negative number on both sides of an inequality, the direction of the inequality symbol must switch. We will look at some examples.

Simplify Simplify the expressions on each side of the inequality by distributing and combining like terms.

Isolate Use addition or subtraction to isolate the variable terms and constant terms (numbers) so that they are on different sides of the inequality symbol.

Eliminate Use multiplication or division to eliminate the variable term's coefficient. If each side of the inequality is multiplied or divided by a negative number, switch the direction of the inequality symbol.

Check When specified, verify the infinite solution set by checking multiple solutions.

Summarize State the solution set or (in the case of an application problem) summarize the result in a complete sentence using appropriate units.

List 3.2.2: Steps to Solve Linear Inequalities

Example 3.2.3 Solve for t in the inequality $-3t + 5 \geq 11$. Write the solution set in both set-builder notation and interval notation.

Explanation.

$$\begin{aligned}
-3t + 5 &\geq 11\\
-3t + 5 - 5 &\geq 11 - 5\\
-3t &\geq 6\\
\frac{-3t}{-3} &\leq \frac{6}{-3}\\
t &\leq -2
\end{aligned}$$

Note that when we divided both sides of the inequality by -3, we had to switch the direction of the inequality symbol.

The solution set in set-builder notation is $\{t \mid t \leq -2\}$.

The solution set in interval notation is $(-\infty, -2]$.

Remark 3.2.4. Since the inequality solved in Example 3.2.3 has infinitely many solutions, it's difficult to check. We found that all values of t for which $t \leq -2$ are solutions, so one approach is to check if -2 is a solution and additionally if one other number less than -2 is a solution.

Here, we'll check that -2 satisfies this inequality:

$$\begin{aligned} -3t + 5 &\geq 11 \\ -3(-2) + 5 &\overset{?}{\geq} 11 \\ 6 + 5 &\overset{?}{\geq} 11 \\ 11 &\overset{\checkmark}{\geq} 11 \end{aligned}$$

Next, we can check another number smaller than -2, such as -5:

$$\begin{aligned} -3t + 5 &\geq 11 \\ -3(-5) + 5 &\overset{?}{\geq} 11 \\ 15 + 5 &\overset{?}{\geq} 11 \\ 20 &\overset{\checkmark}{\geq} 11 \end{aligned}$$

Thus both -2 and -5 are solutions. It's important to note that this doesn't directly verify that *all* solutions to this inequality check. It's valuable though in that it would likely help us catch an error if we had made one. Consult your instructor to see if you're expected to check your answer in this manner.

Example 3.2.5 Solve for z in the inequality $(6z + 5) - (2z - 3) < -12$. Write the solution set in both set-builder notation and interval notation.

Explanation.

$$\begin{aligned} (6z + 5) - (2z - 3) &< -12 \\ 6z + 5 - 2z + 3 &< -12 \\ 4z + 8 &< -12 \\ 4z + 8 - 8 &< -12 - 8 \\ 4z &< -20 \\ \frac{4z}{4} &< \frac{-20}{4} \\ z &< -5 \end{aligned}$$

Note that we divided both sides of the inequality by 4 and since this is a positive number we *did not* need to switch the direction of the inequality symbol.

The solution set in set-builder notation is $\{z \mid z < -5\}$.

The solution set in interval notation is $(-\infty, -5)$.

Example 3.2.6 Solve for x in $-2-2(2x+1) > 4-(3-x)$. Write the solution set in both set-builder notation and interval notation.

Explanation.

$$\begin{aligned} -2 - 2(2x + 1) &> 4 - (3 - x) \\ -2 - 4x - 2 &> 4 - 3 + x \\ -4x - 4 &> x + 1 \\ -4x - 4 - x &> x + 1 - x \end{aligned}$$

$$\begin{aligned} -5x - 4 &> 1 \\ -5x - 4 + 4 &> 1 + 4 \\ -5x &> 5 \\ \frac{-5x}{-5} &< \frac{5}{-5} \\ x &< -1 \end{aligned}$$

Note that when we divided both sides of the inequality by -5, we had to switch the direction of the inequality symbol.

The solution set in set-builder notation is $\{x \mid x < -1\}$.

The solution set in interval notation is $(-\infty, -1)$.

Example 3.2.7 When a stopwatch started, the pressure inside a gas container was 4.2 atm (standard atmospheric pressure). As the container was heated, the pressure increased by 0.7 atm per minute. The maximum pressure the container can handle was 21.7 atm. Heating must be stopped once the pressure reaches 21.7 atm. In what time interval was the container safe?

Explanation. The pressure increases by 0.7 atm per minute, so it increases by $0.7m$ after m minutes. Counting in the original pressure of 4.2 atm, pressure in the container can be modeled by $0.7m + 4.2$, where m is the number of minutes since the stop watch started.

The container is safe when the pressure is 21.7 atm or lower. We can write and solve this inequality:

$$\begin{aligned} 0.7m + 4.2 &\leq 21.7 \\ 0.7m + 4.2 - 4.2 &\leq 21.7 - 4.2 \\ 0.7m &\leq 17.5 \\ \frac{0.7m}{0.7} &\leq \frac{17.5}{0.7} \\ m &\leq 25 \end{aligned}$$

In summary, the container was safe as long as $m \leq 25$. Assuming that m also must be greater than or equal to zero, this means $0 \leq m \leq 25$. We can write this as the time interval as $[0, 25]$. Thus the container was safe between 0 minutes and 25 minutes.

Exercises

Review and Warmup

1. Solve this inequality.

$x + 3 > 9$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

2. Solve this inequality.

$x + 3 > 7$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

3. Solve this inequality.

$4 > x - 10$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

4. Solve this inequality.

$5 > x - 8$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

5. Solve this inequality.

$5x \leq 10$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

6. Solve this inequality.

$2x \leq 8$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

7. Solve this inequality.

$6 \geq -2x$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

8. Solve this inequality.

$6 \geq -3x$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

9. Solve this inequality.

$\frac{4}{9}x > 12$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

10. Solve this inequality.

$\frac{5}{6}x > 5$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

11. A swimming pool is being filled with water from a garden hose at a rate of 5 gallons per minute. If the pool already contains 60 gallons of water and can hold 160 gallons, after how long will the pool overflow?

Assume m minutes later, the pool would overflow. Write an equation to model this scenario. There is no need to solve it.

12. An engineer is designing a cylindrical springform pan. The pan needs to be able to hold a volume of 305 cubic inches and have a diameter of 12 inches. What's the minimum height it can have? (Hint: The formula for the volume of a cylinder is $V = \pi r^2 h$).

Assume the pan's minimum height is h inches. Write an equation to model this scenario. There is no need to solve it.

Solving Multistep Linear Inequalities Solve this inequality.

13. $9x + 6 > 42$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

14. $10x + 3 > 83$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

15. $4 \geq 3x - 5$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

16. $25 \geq 4x - 3$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

17. $41 \leq 1 - 4x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

18. $33 \leq 8 - 5x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

19. $-6x - 4 < -58$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

20. $-7x - 1 < -29$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

21. $4 \geq -8x + 4$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

22. $3 \geq -9x + 3$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

23. $-5 > 5 - x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

24. $-8 > 1 - x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

25. $3(x + 4) \geq 24$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

26. $4(x + 8) \geq 72$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

27. $7t + 7 < 1t + 37$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

28. $8t + 4 < 3t + 19$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

29. $-9z + 6 \leq -z - 58$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

30. $-8z + 7 \leq -z - 28$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

31. $a - 9 - 3a > -9 - 4a + 12$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

32. $a - 9 - 9a > -10 - 10a + 7$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

33. $-8p + 6 - 8p \geq 2p + 6$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

34. $-2p + 3 - 6p \ge 3p + 3$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

35. $64 < -4(p - 9)$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

36. $-20 < -5(p - 5)$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

37. $-(x - 2) \ge 8$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

38. $-(x - 8) \ge 13$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

39. $47 \le 7 - 5(z - 1)$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

40. $20 \le 8 - 2(z - 8)$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

41. $5 - (y + 9) < 6$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

42. $1 - (y + 7) < -4$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

43. $1 + 7(x - 9) < -15 - (7 - 3x)$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

44. $2 + 9(x - 5) < -20 - (7 - 5x)$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

Applications

45. You are riding in a taxi and can only pay with cash. You have to pay a flat fee of \$25, and then pay \$2.80 per mile. You have a total of \$193 in your pocket.

Let x be the number of miles the taxi will drive you. You want to know how many miles you can afford. Write an inequality to represent this situation in terms of how many miles you can afford:

Solve this inequality. At most how many miles can you afford?

You can afford at most miles.

Use interval notation to express the number of miles you can afford.

46. You are riding in a taxi and can only pay with cash. You have to pay a flat fee of \$30, and then pay \$3.50 per mile. You have a total of \$135 in your pocket.

Let x be the number of miles the taxi will drive you. You want to know how many miles you can afford. Write an inequality to represent this situation in terms of how many miles you can afford:

Solve this inequality. At most how many miles can you afford?

You can afford at most miles.

Use interval notation to express the number of miles you can afford.

47. A car rental company offers the following two plans for renting a car:

Plan A: \$31 per day and 16 cents per mile

Plan B: \$49 per day with free unlimited mileage

How many miles must one drive in order to justify choosing Plan B?

One must drive more than miles to justify choosing Plan B. In other words, it's more economical to use plan B if your number of miles driven will be in the interval (answer with interval notation).

48. A car rental company offers the following two plans for renting a car:

Plan A: \$29 per day and 17 cents per mile

Plan B: \$52 per day with free unlimited mileage

How many miles must one drive in order to justify choosing Plan B?

One must drive more than miles to justify choosing Plan B. In other words, it's more economical to use plan B if your number of miles driven will be in the interval (answer with interval notation).

49. You are offered two different sales jobs. The first company offers a straight commission of 9% of the sales. The second company offers a salary of \$300 per week *plus* 4% of the sales. How much would you have to sell in a week in order for the straight commission offer to be at least as good?

You'd have to sell more than worth of goods for the straight commission to be better for you. In other words, the dollar amount of goods sold would have to be in the interval (answer using interval notation).

50. You are offered two different sales jobs. The first company offers a straight commission of 8% of the sales. The second company offers a salary of \$430 per week *plus* 4% of the sales. How much would you have to sell in a week in order for the straight commission offer to be at least as good?

You'd have to sell more than ☐ worth of goods for the straight commission to be better for you. In other words, the dollar amount of goods sold would have to be in the interval ☐ (answer using interval notation).

3.3 Linear Equations and Inequalities with Fractions

In this section, we will learn how to solve linear equations and inequalities with fractions.

3.3.1 Introduction

So far, in our last step of solving for a variable we have divided each side of the equation by a constant, as in:

$$\begin{aligned} 2x &= 10 \\ \frac{2x}{2} &= \frac{10}{2} \\ x &= 5 \end{aligned}$$

If we have a coefficient that is a fraction, we *could* proceed in exactly the same manner:

$$\begin{aligned} \frac{1}{2}x &= 10 \\ \frac{\frac{1}{2}x}{\frac{1}{2}} &= \frac{10}{\frac{1}{2}} \\ x &= 10 \cdot \frac{2}{1} = 20 \end{aligned}$$

What if our equation or inequality was more complicated though, for example $\frac{1}{4}x + \frac{2}{3} = \frac{1}{6}$? We would have to first do a lot of fraction arithmetic in order to then divide each side by the coefficient of x. An alternate approach is to instead *multiply* each side of the equation by a chosen constant that eliminates the denominator. In the equation $\frac{1}{2}x = 10$, we could simply multiply each side of the equation by 2, which would eliminate the denominator of 2:

$$\begin{aligned} \frac{1}{2}x &= 10 \\ 2 \cdot \left(\frac{1}{2}x\right) &= 2 \cdot 10 \\ x &= 20 \end{aligned}$$

For more complicated equations, we will multiply each side of the equation by the least common denominator (LCD) of all fractions contained in the equation.

3.3.2 Eliminating Denominators

Deshawn planted a sapling in his yard that was 4-feet tall. The tree will grow $\frac{2}{3}$ of a foot every year. How many years will it take for his tree to be 10 feet tall?

Since the tree grows $\frac{2}{3}$ of a foot every year, we can use a table to help write a formula modeling the tree's growth:

Years Passed	Tree's Height (ft)
0	4
1	$4 + \frac{2}{3}$
2	$4 + \frac{2}{3} \cdot 2$
$\vdots$	$\vdots$
y	$4 + \frac{2}{3}y$

Example 3.3.2 From this, we've determined that y years since the tree was planted, the tree's height will be $4 + \frac{2}{3}y$ feet.

To find when Deshawn's tree will be 10 feet tall, we write and solve this equation:

$$\begin{aligned}4+\frac{2}{3}y&=10\\ 3\cdot\left(4+\frac{2}{3}y\right)&=3\cdot 10\\ 3\cdot 4+3\cdot\frac{2}{3}y&=30\\ 12+2y&=30\\ 2y&=18\\ y&=9\end{aligned}$$

Now we will check the solution 9 in the equation $4+\frac{2}{3}y=10$:

$$\begin{aligned}4+\frac{2}{3}y&=10\\ 4+\frac{2}{3}(9)&\stackrel{?}{=}10\\ 4+6&\stackrel{\checkmark}{=}10\end{aligned}$$

In summary, it will take 9 years for Deshawn's tree to reach 10 feet tall.

Let's look at a few more examples.

Example 3.3.3 Solve for x in $\frac{1}{4}x+\frac{2}{3}=\frac{1}{6}$.

Explanation. To solve this equation, we first need to identify the LCD of all fractions in the equation. On the left side we have $\frac{1}{4}$ and $\frac{2}{3}$. On the right side we have $\frac{1}{6}$. The LCD of $3, 4$, and 6 is 12, so we will multiply each side of the equation by 12 in order to eliminate *all* of the denominators:

$$\begin{aligned}\frac{1}{4}x+\frac{2}{3}&=\frac{1}{6}\\ 12\cdot\left(\frac{1}{4}x+\frac{2}{3}\right)&=12\cdot\frac{1}{6}\\ 12\cdot\left(\frac{1}{4}x\right)+12\cdot\left(\frac{2}{3}\right)&=12\cdot\frac{1}{6}\\ 3x+8&=2\\ 3x&=-6\\ \frac{3x}{3}&=\frac{-6}{3}\\ x&=-2\end{aligned}$$

Checking the solution -2:

$$\begin{aligned}\frac{1}{4}x+\frac{2}{3}&=\frac{1}{6}\\ \frac{1}{4}(-2)+\frac{2}{3}&\stackrel{?}{=}\frac{1}{6}\\ -\frac{2}{4}+\frac{2}{3}&\stackrel{?}{=}\frac{1}{6}\\ -\frac{6}{12}+\frac{8}{12}&\stackrel{?}{=}\frac{1}{6}\\ \frac{2}{12}&\stackrel{\checkmark}{=}\frac{1}{6}\end{aligned}$$

The solution is therefore -2 and the solution set is $\{-2\}$.

Example 3.3.4 Solve for z in $-\frac{2}{5}z-\frac{3}{2}=-\frac{1}{2}z+\frac{4}{5}$.

Explanation.

The first thing we need to do is identify the LCD of all denominators in this equation. Since the denominators are 2 and 5, the LCD is 10. So as our first step, we will multiply each side of the equation by 10 in order to eliminate all denominators:

$$\begin{aligned}
-\frac{2}{5}z-\frac{3}{2} &= -\frac{1}{2}z+\frac{4}{5}\\
10\cdot\left(-\frac{2}{5}z-\frac{3}{2}\right) &= 10\cdot\left(-\frac{1}{2}z+\frac{4}{5}\right)\\
10\left(-\frac{2}{5}z\right)-10\left(\frac{3}{2}\right) &= 10\left(-\frac{1}{2}z\right)+10\left(\frac{4}{5}\right)\\
-4z-15 &= -5z+8\\
z-15 &= 8\\
z &= 23
\end{aligned}$$

Checking the solution 23:

$$\begin{aligned}
-\frac{2}{5}z-\frac{3}{2} &= -\frac{1}{2}z+\frac{4}{5}\\
-\frac{2}{5}(23)-\frac{3}{2} &\stackrel{?}{=} -\frac{1}{2}(23)+\frac{4}{5}\\
-\frac{46}{5}-\frac{3}{2} &\stackrel{?}{=} -\frac{23}{2}+\frac{4}{5}\\
-\frac{46}{5}\cdot\frac{2}{2}-\frac{3}{2}\cdot\frac{5}{5} &\stackrel{?}{=} -\frac{23}{2}\cdot\frac{5}{5}+\frac{4}{5}\cdot\frac{2}{2}\\
-\frac{92}{10}-\frac{15}{10} &\stackrel{?}{=} -\frac{115}{10}+\frac{8}{10}\\
-\frac{107}{10} &\stackrel{\checkmark}{=} -\frac{107}{10}
\end{aligned}$$

Thus the solution is 23 and so the solution set is $\{23\}$.

Example 3.3.5 Solve for a in the equation $\frac{2}{3}(a+1)+5=\frac{1}{3}$.

Explanation.

$$\begin{aligned}
\frac{2}{3}(a+1)+5 &= \frac{1}{3}\\
3\cdot\left(\frac{2}{3}(a+1)+5\right) &= 3\cdot\frac{1}{3}\\
3\cdot\frac{2}{3}(a+1)+3\cdot 5 &= 1\\
2(a+1)+15 &= 1\\
2a+2+15 &= 1\\
2a+17 &= 1\\
2a &= -16\\
a &= -8
\end{aligned}$$

Check the solution -8 in the equation $\frac{2}{3}(a+1)+5=\frac{1}{3}$, we find that:

$$\begin{aligned}
\frac{2}{3}(a+1)+5 &= \frac{1}{3}\\
\frac{2}{3}(-8+1)+5 &\stackrel{?}{=} \frac{1}{3}\\
\frac{2}{3}(-7)+5 &\stackrel{?}{=} \frac{1}{3}\\
-\frac{14}{3}+\frac{15}{3} &\stackrel{\checkmark}{=} \frac{1}{3}
\end{aligned}$$

The solution is therefore -8 and the solution set is $\{-8\}$.

Example 3.3.6 Solve for b in the equation $\frac{2b+1}{3}=\frac{2}{5}$.

Explanation.

$$\begin{aligned}
\frac{2b+1}{3} &= \frac{2}{5}\\
15\cdot\frac{2b+1}{3} &= 15\cdot\frac{2}{5}\\
5(2b+1) &= 6\\
10b+5 &= 6\\
10b &= 1\\
b &= \frac{1}{10}
\end{aligned}$$

Checking the solution $\frac{1}{10}$:

$$\begin{aligned}
\frac{2b+1}{3} &= \frac{2}{5}\\
\frac{2\left(\frac{1}{10}\right)+1}{3} &\overset{?}{=} \frac{2}{5}\\
\frac{\frac{1}{5}+1}{3} &\overset{?}{=} \frac{2}{5}\\
\frac{\frac{1}{5}+\frac{5}{5}}{3} &\overset{?}{=} \frac{2}{5}\\
\frac{\frac{6}{5}}{3} &\overset{?}{=} \frac{2}{5}\\
\frac{6}{5}\cdot\frac{1}{3} &\overset{\checkmark}{=} \frac{2}{5}
\end{aligned}$$

The solution is $\frac{1}{10}$ and the solution set is $\left\{\frac{1}{10}\right\}$.

Remark 3.3.7. You might know about solving Example 3.3.6 with a technique called **cross-multiplication**. Cross-multiplication is a specialized application of the process of clearing the denominators from an equation. This process will be discussed in Section 3.5.

Example 3.3.8 In a science lab, a container had 21 ounces of water at 9:00 A.M.. Water has been evaporating at the rate of 3 ounces every 5 minutes. When will there be 8 ounces of water left?

Explanation. Since the container has been losing 3 oz of water every 5 minutes, it loses $\frac{3}{5}$ oz every minute. In m minutes since 9:00 A.M., the container would lose $\frac{3}{5}m$ oz of water. Since the container had 21 oz of water at the beginning, the amount of water in the container can be modeled by $21-\frac{3}{5}m$ (in oz).

To find when there would be 8 oz of water left, we write and solve this equation:

$$\begin{aligned}
21-\frac{3}{5}m &= 8\\
5\cdot\left(21-\frac{3}{5}x\right) &= 5\cdot 8\\
5\cdot 21-5\cdot\frac{3}{5}x &= 40\\
105-3m &= 40\\
105-3m-105 &= 40-105\\
-3m &= -65\\
\frac{-3m}{-3} &= \frac{-65}{-3}\\
m &= \frac{65}{3}
\end{aligned}$$

Checking the solution $\frac{65}{3}$:

$$\begin{aligned}
21-\frac{3}{5}m &= 8\\
21-\frac{3}{5}\left(\frac{65}{3}\right) &\overset{?}{=} 8\\
21-13 &\overset{\checkmark}{=} 8
\end{aligned}$$

> Therefore the solution is $\frac{65}{3}$. As a mixed number, this is $21\frac{2}{3}$. In context, this means that 21 minutes and 40 seconds after 9:00 A.M., at 9:21:40 A.M., the container will have 8 ounces of water left.

Checkpoint 3.3.9. Solve the equation.

$$21 = \frac{x}{5} + \frac{x}{2}$$

Explanation. To clear fractions in an equation, we multiply each term by a common denominator. For this problem, a common denominator is 10.

$$\begin{aligned} 21 &= \frac{x}{5} + \frac{x}{2} \\ 10 \cdot 21 &= 10 \cdot \frac{x}{5} + 10 \cdot \frac{x}{2} \\ 210 &= 2x + 5x \\ 210 &= 7x \\ \frac{210}{7} &= \frac{7x}{7} \\ 30 &= x \\ x &= 30 \end{aligned}$$

The solution to this equation is 30. To stress that this is a value assigned to x, some report $x = 30$. We can also say that the solution set is $\{30\}$, or that $x \in \{30\}$. If we substitute 30 in for x in the original equation $21 = \frac{x}{5} + \frac{x}{2}$, the equation will be true. Please check this on your own; it is an important habit.

3.3.3 Solving Inequalities with Fractions

We can also solve linear inequalities involving fractions by multiplying each side of the inequality by the LCD of all fractions within the inequality. Remember that with inequalities, everything works exactly the same except that the inequality sign reverses direction whenever we multiply each side of the inequality by a negative number.

> **Example 3.3.10** Solve for x in the inequality $\frac{3}{4}x - 2 > \frac{4}{5}x$. Write the solution set in both set-builder notation and interval notation.
>
> **Explanation**.
>
> $$\begin{aligned} \frac{3}{4}x - 2 &> \frac{4}{5}x \\ 20 \cdot \left(\frac{3}{4}x - 2\right) &> 20 \cdot \frac{4}{5}x \\ 20 \cdot \frac{3}{4}x - 20 \cdot 2 &> 16x \\ 15x - 40 &> 16x \\ 15x - 40 - 15x &> 16x - 15x \\ -40 &> x \\ x &< -40 \end{aligned}$$
>
> The solution set in set-builder notation is $\{x \mid x < -40\}$. Note that it's equivalent to write $\{x \mid -40 > x\}$,

but it's easier to understand if we write x first in an inequality.

The solution set in interval notation is $(-\infty, -40)$.

Example 3.3.11 Solve for y in the inequality $\frac{4}{7} - \frac{4}{3}y \leq \frac{2}{3}$. Write the solution set in both set-builder notation and interval notation.

Explanation.

$$\begin{aligned}
\frac{4}{7} - \frac{4}{3}y &\leq \frac{2}{3} \\
21 \cdot \left(\frac{4}{7} - \frac{4}{3}y\right) &\leq 21 \cdot \left(\frac{2}{3}\right) \\
21\left(\frac{4}{7}\right) - 21\left(\frac{4}{3}y\right) &\leq 21\left(\frac{2}{3}\right) \\
12 - 28y &\leq 14 \\
-28y &\leq 2 \\
\frac{-28y}{-28} &\geq \frac{2}{-28} \\
y &\geq -\frac{1}{14}
\end{aligned}$$

Note that when we divided each side of the inequality by -28, the inequality symbol reversed direction.

The solution set in set-builder notation is $\left\{y \mid y \geq -\frac{1}{14}\right\}$.

The solution set in interval notation is $\left[-\frac{1}{14}, \infty\right)$.

Example 3.3.12 In a certain class, a student's grade is calculated by the average of their scores on 3 tests. Aidan scored 78% and 54% on the first two tests. If he wants to earn at least a grade of C (70%), what's the lowest score he needs to earn on the third exam?

Explanation. Assume Aidan will score x% on the third test. To make his average test score greater than or equal to 70%, we write and solve this inequality:

$$\begin{aligned}
\frac{78 + 54 + x}{3} &\geq 70 \\
\frac{132 + x}{3} &\geq 70 \\
3 \cdot \frac{132 + x}{3} &\geq 3 \cdot 70 \\
132 + x &\geq 210 \\
x &\geq 78
\end{aligned}$$

To earn at least a C grade, Aidan needs to score at least 78% on the third test.

Exercises

Review and Warmup

1. Multiply: $7 \cdot \frac{1}{10}$

2. Multiply: $2 \cdot \frac{2}{7}$

3. Multiply: $9 \cdot \left(-\frac{4}{3}\right)$

4. Multiply: $45 \cdot \left(-\frac{5}{9}\right)$

5. Do the following multiplications:

a. $14 \cdot \frac{4}{7} = \boxed{}$

b. $21 \cdot \frac{4}{7} = \boxed{}$

c. $28 \cdot \frac{4}{7} = \boxed{}$

6. Do the following multiplications:

a. $20 \cdot \frac{4}{5} = \boxed{}$

b. $25 \cdot \frac{4}{5} = \boxed{}$

c. $30 \cdot \frac{4}{5} = \boxed{}$

Solving Linear Equations with Fractions Solve the equation.

7. $\frac{n}{9} + 88 = 5n$

8. $\frac{p}{6} + 92 = 4p$

9. $\frac{x}{3} + 3 = 9$

10. $\frac{y}{9} + 10 = 13$

11. $5 - \frac{t}{8} = 0$

12. $1 - \frac{a}{3} = -3$

13. $2 = 8 - \frac{2c}{7}$

14. $-26 = 4 - \frac{10A}{3}$

15. $3C = \frac{5C}{2} + 4$

16. $3m = \frac{9m}{8} + 45$

17. $51 = \frac{2}{5}p + 3p$

18. $150 = \frac{8}{7}x + 6x$

19. $81 - \frac{3}{8}y = 3y$

20. $45 - \frac{7}{4}t = 2t$

21. $6a = \frac{10}{9}a + 10$

22. $3c = \frac{8}{5}c + 7$

23. $\frac{9}{4} - 5A = 4$

24. $\frac{3}{10} - 2C = 3$

25. $\frac{7}{6} - \frac{1}{6}m = 9$

26. $\frac{3}{4} - \frac{1}{4}p = 6$

27. $\frac{4x}{9} - 5 = -\frac{73}{9}$

28. $\frac{8y}{7} - 10 = -\frac{102}{7}$

29. $\frac{4}{5} + \frac{8}{5}t = 3t$

30. $\frac{8}{9} + \frac{2}{3}a = 3a$

31. $\frac{3c}{5} - \frac{8}{5} = -\frac{1}{5}c$

32. $\frac{2A}{7} - \frac{25}{7} = -\frac{3}{7}A$

33. $\frac{8C}{9} + \frac{1}{8} = C$

34. $\frac{2m}{5} + \frac{1}{4} = m$

35. $\frac{2p}{3} - 57 = -\frac{5}{2}p$

36. $\frac{2q}{7} - 19 = -\frac{1}{6}q$

37. $-\frac{9}{8}y + 81 = \frac{9y}{16}$

38. $-\frac{1}{2}t + 5 = \frac{3t}{4}$

39. $\frac{5a}{8} - 5a = \frac{9}{16}$

40. $\frac{9c}{4} - 5c = \frac{7}{8}$

41. $\frac{5A}{2} + \frac{8}{5} = \frac{3}{8}A$

42. $\frac{9C}{4} + \frac{8}{3} = \frac{5}{6}C$

43. $\frac{4}{3}m = \frac{3}{2} + \frac{3m}{7}$

44. $\frac{4}{7}p = \frac{4}{5} + \frac{5p}{3}$

45. $\frac{9}{8} = \frac{q}{24}$

46. $\frac{5}{4} = \frac{y}{24}$

47. $-\frac{t}{45} = \frac{8}{9}$

48. $-\frac{a}{28} = \frac{2}{7}$

49. $-\frac{c}{8} = -\frac{7}{4}$

50. $-\frac{A}{60} = -\frac{3}{10}$

51. $-\frac{7}{6} = \frac{8C}{9}$

52. $-\frac{8}{3} = \frac{8m}{5}$

53. $\frac{3}{10} = \frac{p+9}{50}$

54. $\frac{7}{6} = \frac{q+8}{36}$

55. $\frac{3}{2} = \frac{y-9}{7}$

56. $\frac{7}{8} = \frac{r-9}{3}$

57. $\frac{a-10}{4} = \frac{a+10}{6}$

58. $\frac{c-4}{2} = \frac{c+7}{4}$

59. $\frac{A+8}{6} - \frac{A-4}{12} = \frac{13}{6}$

60. $\frac{C+2}{4} - \frac{C-2}{8} = \frac{11}{8}$

61. $\frac{m}{3} - 15 = \frac{m}{8}$

62. $\frac{p}{7} - 4 = \frac{p}{9}$

63. $\frac{q}{4} - 3 = \frac{q}{7} + 6$

64. $\frac{y}{2} - 4 = \frac{y}{9} + 3$

65. $\frac{4}{5}r + \frac{3}{5} = \frac{2}{5}r + \frac{1}{2}$

66. $\frac{3}{5}a + \frac{7}{5} = \frac{8}{5}a + \frac{8}{5}$

Solve the equation.

67. $\frac{3c+9}{4} - \frac{4-c}{8} = \frac{2}{3}$

68. $\frac{10A+9}{4} - \frac{2-A}{8} = \frac{5}{7}$

69. $6 = \frac{C}{5} + \frac{C}{10}$

70. $26 = \frac{m}{9} + \frac{m}{4}$

71. $p + \frac{2}{7} = -\frac{4}{5}p - 1$

72. $-2q + \frac{9}{4} = -\frac{3}{10}q - \frac{3}{7}$

Solve the equation.

73. $\frac{9}{7}y + \frac{3}{5} = -\frac{8}{9}y + 1$

74. $-3r + \frac{2}{3} = -\frac{1}{3}r - 2$

75.

a. $-\frac{a}{6} + 4 = -1$

b. $\frac{-r}{6} + 4 = -1$

c. $\frac{c}{-6} + 4 = -1$

d. $\frac{-q}{-6} + 4 = -1$

76.

a. $-\frac{b}{3} + 8 = 4$

b. $\frac{-n}{3} + 8 = 4$

c. $\frac{C}{-3} + 8 = 4$

d. $\frac{-A}{-3} + 8 = 4$

Applications

77. Nina is jogging in a straight line. She got a head start of 4 meters from the starting line, and she ran 4 meters every 7 seconds. After how many seconds will Nina be 16 meters away from the starting line?

Nina will be 16 meters away from the starting line ☐ seconds since she started running.

78. Charlotte is jogging in a straight line. She started at a place 44 meters from the starting line, and ran toward the starting line at the speed of 4 meters every 5 seconds. After how many seconds will Charlotte be 36 meters away from the starting line?

Charlotte will be 36 meters away from the starting line ☐ seconds since she started running.

79. Nina had only $5.00 in her piggy bank, and she decided to start saving more. She saves $5.00 every 9 days. After how many days will she have $30.00 in the piggy bank?

Nina will save $30.00 in her piggy bank after ☐ days.

80. Cody has saved $45.00 in his piggy bank, and he decided to start spending them. He spends $5.00 every 8 days. After how many days will he have $25.00 left in the piggy bank?

Cody will have $25.00 left in his piggy bank after ☐ days.

Solving Inequalities with Fractions Solve this inequality.

81. $\frac{x}{10} + 98 \geq 5x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

82. $\frac{x}{2} + 25 \geq 3x$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

83. $\frac{3}{2} - 5y < 5$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

84. $\frac{3}{2} - 2y < 4$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

85. $-\frac{1}{4}t > \frac{6}{5}t - 58$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

86. $-\frac{5}{2}t > \frac{4}{5}t - 66$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

87. $\frac{3}{8} \geq \frac{x}{32}$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

88. $\frac{7}{8} \geq \frac{x}{48}$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

89. $-\frac{z}{30} < -\frac{3}{10}$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

90. $-\frac{z}{50} < -\frac{9}{10}$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

91. $\frac{x}{8} - 5 \leq \frac{x}{3}$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

92. $\frac{x}{6} - 2 \leq \frac{x}{3}$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

93. $\frac{y-10}{4} \geq \frac{y+10}{2}$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

94. $\frac{y-6}{6} \geq \frac{y+4}{4}$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

Solve this inequality.

95. $\frac{3}{2} < \frac{x+3}{4} - \frac{x-8}{8}$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

96. $\frac{13}{8} < \frac{x+9}{4} - \frac{x-2}{8}$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

Applications

97. Your grade in a class is determined by the average of three test scores. You scored 74 and 88 on the first two tests. To earn at least 83 for this course, how much do you have to score on the third test?

Let x be the score you will earn on the third test. Write an inequality to represent this situation.

☐ ☐ ☐

Solve this inequality. What is the minimum that you have to earn on the third test in order to earn a 83 for the course?

You cannot score over 100 on the third test. Use interval notation to represent the range of scores you can earn on the third test in order to earn at least 83 for this course.

98. Your grade in a class is determined by the average of three test scores. You scored 75 and 86 on the first two tests. To earn at least 76 for this course, how much do you have to score on the third test?

Let x be the score you will earn on the third test. Write an inequality to represent this situation.

Solve this inequality. What is the minimum that you have to earn on the third test in order to earn a 76 for the course?

You cannot score over 100 on the third test. Use interval notation to represent the range of scores you can earn on the third test in order to earn at least 76 for this course.

3.4 Isolating a Linear Variable

In this section, we will learn how to solve linear equations and inequalities with more than one variable.

3.4.1 Solving for a Variable

The formula of calculating a rectangle's area is $A = \ell w$, where ℓ stands for the rectangle's length, and w stands for width. When a rectangle's length and width are given, we can easily calculate its area.

What if a rectangle's area and length are given, and we need to calculate its width?

If a rectangle's area is given as $12\,\mathrm{m}^2$, and its length is given as $4\,\mathrm{m}$, we could find its width this way:

$$\begin{aligned} A &= \ell w \\ 12 &= 4w \\ \frac{12}{4} &= \frac{4w}{4} \\ 3 &= w \\ w &= 3 \end{aligned}$$

If we need to do this many times, we would love to have an easier way, without solving an equation each time. We will solve for w in the formula $A = \ell w$:

$$\begin{aligned} A &= \ell w \\ \frac{A}{\ell} &= \frac{\ell w}{\ell} \\ \frac{A}{\ell} &= w \\ w &= \frac{A}{\ell} \end{aligned}$$

Now if we want to find the width when $\ell = 4$ is given, we can simply replace ℓ with 4 and simplify.

We solved for w in the formula $A = \ell w$ once, and we could use the new formula $w = \frac{A}{\ell}$ again and again saving us a lot of time down the road. Let's look at a few examples.

Remark 3.4.2. Note that in solving for A, we divided each side of the equation by ℓ. The operations that we apply, and the order in which we do them, are determined by the operations in the original equation. In the original equation $A = \ell w$, we saw that w was *multiplied* by ℓ, and so we knew that in order to "undo" that operation, we would need to *divide* each side by ℓ. We will see this process of "un-doing" the operations throughout this section.

> **Example 3.4.3** Solve for R in $P = R - C$. (This is the relationship between profit, revenue, and cost.)
>
> To solve for R, we first want to note that C is *subtracted* from R. To "undo" this, we will need to *add* C to each side of the equation:
>
> $$\begin{aligned} P &= \overset{\downarrow}{R} - C \\ P + C &= \overset{\downarrow}{R} - C + C \\ P + C &= \overset{\downarrow}{R} \\ R &= P + C \end{aligned}$$

Example 3.4.4 Solve for x in $y = mx + b$. (This is a line's equation in slope-intercept form.)

In the equation $y = mx + b$, we see that x is multiplied by m and then b is added to that. Our first step will be to isolate mx, which we'll do by subtracting b from each side of the equation:

$$\begin{aligned}
y &= m\overset{\downarrow}{x} + b \\
y - b &= m\overset{\downarrow}{x} + b - b \\
y - b &= m\overset{\downarrow}{x}
\end{aligned}$$

Now that we have mx on its own, we'll note that x is multiplied by m. To "undo" this, we'll need to divide each side of the equation by m:

$$\begin{aligned}
\frac{y - b}{m} &= \frac{m\overset{\downarrow}{x}}{m} \\
\frac{y - b}{m} &= \overset{\downarrow}{x} \\
x &= \frac{y - b}{m}
\end{aligned}$$

Warning 3.4.5. It's important to note in Example 3.4.4 that each *side* was divided by m. We can't simply divide y by m, as the equation would no longer be equivalent.

Example 3.4.6 Solve for b in $A = \frac{1}{2}bh$. (This is the area formula for a triangle.)

To solve for b, we need to determine what operations need to be "undone." The expression $\frac{1}{2}bh$ has multiplication between $\frac{1}{2}$ and b and h. As a first step, we will multiply each side of the equation by 2 in order to eliminate the denominator of 2:

$$\begin{aligned}
A &= \frac{1}{2}\overset{\downarrow}{b}h \\
2 \cdot A &= 2 \cdot \frac{1}{2}\overset{\downarrow}{b}h \\
2A &= \overset{\downarrow}{b}h
\end{aligned}$$

As a last step, we will "undo" the multiplication between b and h by dividing each side by h:

$$\begin{aligned}
\frac{2A}{h} &= \frac{\overset{\downarrow}{b}h}{h} \\
\frac{2A}{h} &= \overset{\downarrow}{b} \\
b &= \frac{2A}{h}
\end{aligned}$$

Example 3.4.7 Solve for y in $2x + 5y = 10$. (This is a linear equation in standard form.)

To solve for y, we will first have to solve for $5y$ by subtracting $2x$ from each side of the equation. After that, we'll be able to divide each side by 5 to finish solving for y:

$$\begin{aligned} 2x + 5\overset{\downarrow}{y} &= 10 \\ 2x + 5\overset{\downarrow}{y} - 2x &= 10 - 2x \\ 5\overset{\downarrow}{y} &= 10 - 2x \\ \frac{5\overset{\downarrow}{y}}{5} &= \frac{10 - 2x}{5} \\ y &= \frac{10 - 2x}{5} \end{aligned}$$

Remark 3.4.8. As we will learn in later sections, the result in Example 3.4.7 can also be written as $y = \frac{10}{5} - \frac{2x}{5}$ which can then be written as $y = 2 - \frac{2}{5}x$.

Example 3.4.9 Solve for F in $C = \frac{5}{9}(F - 32)$. (This represents the relationship between temperature in degrees Celsius and degrees Fahrenheit.)

To solve for F, we first need to see that it is contained inside a set of parentheses. To get the expression $F - 32$ on its own, we'll need to eliminate the $\frac{5}{9}$ outside those parentheses. One way we can "undo" this multiplication is by dividing each side by $\frac{5}{9}$. As we learned in Section 3.3 though, a better approach is to instead multiply each side by the reciprocal of $\frac{9}{5}$:

$$\begin{aligned} C &= \frac{5}{9}(\overset{\downarrow}{F} - 32) \\ \frac{9}{5} \cdot C &= \frac{9}{5} \cdot \frac{5}{9}(\overset{\downarrow}{F} - 32) \\ \frac{9}{5}C &= \overset{\downarrow}{F} - 32 \end{aligned}$$

Now that we have $F - 32$, we simply need to add 32 to each side to finish solving for F:

$$\begin{aligned} \frac{9}{5}C + 32 &= \overset{\downarrow}{F} - 32 + 32 \\ \frac{9}{5}C + 32 &= \overset{\downarrow}{F} \\ F &= \frac{9}{5}C + 32 \end{aligned}$$

Exercises

Review and Warmup Solve the equation.

1. $9q + 2 = 56$

2. $6y + 1 = 55$

3. $-9r - 8 = 37$

4. $-3a - 5 = -17$

5. $-6b + 3 = -b - 12$

6. $-2A + 6 = -A - 4$

7. $56 = -8(B - 10)$

8. $35 = -5(m - 4)$

Solving for a Variable

9. a. Solve this linear equation for t.

$t + 1 = 9$

b. Solve this linear equation for x.

$x + m = y$

11. a. Solve this linear equation for x.

$x - 5 = -3$

b. Solve this linear equation for y.

$y - B = -3$

13. a. Solve this linear equation for y.

$-y + 1 = -5$

b. Solve this linear equation for r.

$-r + c = q$

15. a. Solve this linear equation for r.

$7r = 56$

b. Solve this linear equation for t.

$ct = x$

17. a. Solve this linear equation for r.

$\frac{r}{7} = 10$

b. Solve this linear equation for y.

$\frac{y}{p} = x$

19. a. Solve this linear equation for t.

$7t + 5 = 12$

b. Solve this linear equation for r.

$qr + a = x$

10. a. Solve this linear equation for t.

$t + 1 = 5$

b. Solve this linear equation for r.

$r + n = y$

12. a. Solve this linear equation for x.

$x - 5 = -3$

b. Solve this linear equation for y.

$y - c = -3$

14. a. Solve this linear equation for y.

$-y + 7 = 3$

b. Solve this linear equation for t.

$-t + C = b$

16. a. Solve this linear equation for r.

$3r = 12$

b. Solve this linear equation for x.

$yx = n$

18. a. Solve this linear equation for t.

$\frac{t}{3} = 2$

b. Solve this linear equation for r.

$\frac{r}{B} = x$

20. a. Solve this linear equation for x.

$6x + 4 = 64$

b. Solve this linear equation for y.

$qy + r = n$

21. a. Solve this linear equation for x.

$xt = m$

b. Solve this linear equation for t.

$xt = m$

23. a. Solve this linear equation for y.

$y + x = B$

b. Solve this linear equation for x.

$y + x = B$

25. a. Solve this linear equation for B.

$cy + B = C$

b. Solve this linear equation for c.

$cy + B = C$

27. a. Solve this linear equation for n.

$x = tn + p$

b. Solve this linear equation for t.

$x = tn + p$

29. Solve this linear equation for x.

$$y = mx - b$$

31. a. Solve this equation for b:

$12 = \frac{1}{2}b \cdot 4$

b. Solve this equation for b:

$A = \frac{1}{2}b \cdot h$

33. Solve this linear equation for r.

$$C = 2\pi r$$

22. a. Solve this linear equation for y.

$yr = c$

b. Solve this linear equation for r.

$yr = c$

24. a. Solve this linear equation for r.

$r + t = x$

b. Solve this linear equation for t.

$r + t = x$

26. a. Solve this linear equation for m.

$rt + m = b$

b. Solve this linear equation for r.

$rt + m = b$

28. a. Solve this linear equation for q.

$r = cq + n$

b. Solve this linear equation for c.

$r = cq + n$

30. Solve this linear equation for x.

$$y = -mx + b$$

32. a. Solve this equation for b:

$9 = \frac{1}{2}b \cdot 6$

b. Solve this equation for b:

$A = \frac{1}{2}b \cdot h$

34. Solve this linear equation for h.

$$V = \pi r^2 h$$

35. Solve these linear equations for r.

a. $\frac{r}{5} + 9 = 12$

b. $\frac{r}{x} + 9 = B$

36. Solve these linear equations for t.

a. $\frac{t}{5} + 7 = 8$

b. $\frac{t}{r} + 7 = x$

37. Solve this linear equation for t.

$$\frac{t}{y} + c = a$$

38. Solve this linear equation for x.

$$\frac{x}{t} + p = m$$

39. Solve this linear equation for x.

$$\frac{x}{9} + r = a$$

40. Solve this linear equation for y.

$$\frac{y}{8} + r = A$$

41. Solve this linear equation for b.

$$t = y - \frac{8b}{q}$$

42. Solve this linear equation for A.

$$C = q - \frac{2A}{B}$$

43. Solve this linear equation for x.

$$Ax + By = C$$

44. Solve this linear equation for y.

$$Ax + By = C$$

Solve the linear equation for y.

45.

$$25x + 5y = -75$$

46.

$$30x - 5y = -65$$

47.

$$4x + 2y = 6$$

48.

$$18x - 2y = 16$$

49.

$$4x - y = 14$$

50.

$$2x - y = -12$$

51.

$$-4x - 6y = -24$$

52.

$$-7x - 6y = -18$$

53.

$$2x + 7y = 2$$

54.

$$6x - 8y = 2$$

55.

$$-87x - 87y = 38$$

56.

$$24y - 46x = 25$$

3.5 Ratios and Proportions

3.5.1 Introduction

A **ratio** is a means of comparing two quantities using division. One common example is a unit price. For example, if a box of cereal costs \$3.99 and weighs 21 ounces then we can write this ratio as:

$$\frac{\$3.99}{21\,\text{oz}}$$

If we want to know the unit price (that is, how much each individual ounce costs), then we can divide \$3.99 by 21 ounces and obtain \$0.19 per ounce. These two ratios, $\frac{\$3.99}{21\,\text{oz}}$ and $0.19\frac{\$}{\text{oz}}$ are equivalent, and the equation showing that they are equal is a **proportion**. In this case, we could write the following proportion:

$$\frac{\$3.99}{21\,\text{oz}} = \frac{\$0.19}{1\,\text{oz}}$$

In this section, we will extend this concept and write proportions where one quantity is unknown and solve for that unknown.

Remark 3.5.2. Sometimes ratios are stated using a colon instead of a fraction. For example, the ratio $\frac{2}{1}$ can be written as $2 : 1$.

Example 3.5.3 Suppose we want to know the total cost for a box of cereal that weighs 18 ounces, assuming it costs the same per ounce as the 21-ounce box. Letting C be this unknown cost (in dollars), we could set up the following proportion:

$$\begin{aligned}\frac{\text{cost in dollars}}{\text{weight in oz}} &= \frac{\text{cost in dollars}}{\text{weight in oz}}\\ \frac{\$3.99}{21\,\text{oz}} &= \frac{\$C}{18\,\text{oz}}\end{aligned}$$

To solve this proportion, we will first note that it will be easier to solve without units:

$$\frac{3.99}{21} = \frac{C}{18}$$

Next we want to recognize that each side contains a fraction. Our usual approach for solving this type of equation is to multiply each side by the least common denominator (LCD). In this case, the LCD of 21 and 18 is 126. As with many other proportions we solve, it is often easier to just multiply each side by the common denominator of $18 \cdot 21$, which we know will make each denominator cancel:

$$\begin{aligned}\frac{3.99}{21} &= \frac{C}{18}\\ 18 \cdot 21 \cdot \frac{3.99}{21} &= \frac{C}{18} \cdot 18 \cdot 21\\ 18 \cdot \cancel{21}\frac{3.99}{\cancel{21}} &= \frac{C}{\cancel{18}} \cdot \cancel{18} \cdot 21\\ 71.82 &= 21C\\ \frac{71.82}{21} &= \frac{21C}{21}\\ C &= 3.42\end{aligned}$$

> So assuming the cost is proportional to the cost of the 21-ounce box, the cost for an 18-ounce box of cereal would be \$3.42.

3.5.2 Solving Proportions

Solving proportions uses the process of clearing denominators that we covered in Section 3.3. Because a proportion is exactly one fraction equal to another, we can simplify the process of clearing the denominators simply by multiplying both sides of the equation by both denominators. In other words, we don't specifically need the LCD to clear the denominators.

> **Example 3.5.4** Solve $\frac{x}{8} = \frac{15}{12}$ for x.
>
> Instead of finding the LCD of the two fractions, we'll simply multiply both sides of the equation by 8 and by 12. This will still have the effect of canceling the denominators on both sides of the equation.
>
> $$\begin{aligned} \frac{x}{8} &= \frac{15}{12} \\ 12 \cdot 8 \cdot \frac{x}{8} &= \frac{15}{12} \cdot 12 \cdot 8 \\ 12 \cdot \cancel{8} \cdot \frac{x}{\cancel{8}} &= \frac{15}{\cancel{12}} \cdot \cancel{12} \cdot 8 \\ 12 \cdot x &= 15 \cdot 8 \\ 12x &= 120 \\ \frac{12x}{12} &= \frac{120}{12} \\ x &= 10 \end{aligned}$$
>
> Our work indicates 10 is the solution. We can check this as we would for any equation, by substituting 10 for x and verifying we obtain a true statement:
>
> $$\begin{aligned} \frac{10}{8} &\stackrel{?}{=} \frac{15}{12} \\ \frac{5}{4} &\stackrel{\checkmark}{=} \frac{5}{4} \end{aligned}$$
>
> Since both fractions reduce to $\frac{5}{4}$, we know the solution to the equation $\frac{x}{8} = \frac{15}{12}$ is 10 and the solution set is $\{10\}$.

When solving proportions, we can use the name **cross-multiplication** to describe the process of what just occurred. Say we have a proportion

$$\frac{a}{b} = \frac{c}{d}$$

To remove fractions, we multiply both sides with the common denominator, bd, and we have:

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} \\ bd \cdot \frac{a}{b} &= \frac{c}{d} \cdot bd \\ \cancel{b}d \cdot \frac{a}{\cancel{b}} &= \frac{c}{\cancel{d}} \cdot b\cancel{d} \\ ad &= bc \end{aligned}$$

Since a and d are diagonally across the equals sign from each other in $\frac{a}{b} = \frac{c}{d}$, as are b and c, we call this approach **cross-multiplication**.

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

If we understand cross-multiplication, we are able to rewrite a proportion $\frac{a}{b} = \frac{c}{d}$ in an equivalent form that does not have any fractions, $ad = bc$, as our first step of work. If we had used this skill in Example 3.5.4, we would have had:

$$\begin{aligned}\frac{x}{8} &= \frac{15}{12}\\ 12\cdot x &= 15\cdot 8\\ 12x &= 120\end{aligned}$$

Notice this is the same equation we had in the fifth line of our work in solving Example 3.5.4, but we obtained it without having to contemplate what we need to multiply by to clear the fractions.

We are able to use cross-multiplication when solving proportions, but it is extremely important to note that cross-multiplication only works when we are solving a proportion, an equation that has one ratio or fraction equal to another ratio or fraction. If an equation has anything more than one ratio or fraction on a single side of an equation, we cannot use cross-multiplication. For example, we cannot use cross-multiplication to solve $\frac{3}{4}x - \frac{2}{5} = \frac{9}{4}$, unless we first manipulate the equation to have exactly one fraction and nothing else on each side of the equation.

It is also important to be aware of the fact that cross-multiplication is a special version of our general process of clearing fractions: multiplying both sides of an equation by a common denominator of all the fractions in an equation.

Example 3.5.5 Solve $\frac{t}{5} = \frac{t+2}{3}$ for t.

Explanation. Again this equation is a proportion, so we are able to multiply both sides of the equation by both denominators to clear the fractions:

$$\begin{aligned}\frac{t}{5} &= \frac{t+2}{3}\\ 5\cdot 3\cdot\frac{t}{5} &= \frac{t+2}{3}\cdot 5\cdot 3\\ \cancel{5}\cdot 3\cdot\frac{t}{\cancel{5}} &= \frac{t+2}{\cancel{3}}\cdot 5\cdot\cancel{3}\\ 3\cdot t &= 5\cdot(t+2)\end{aligned}$$

It is critical that we include the parentheses around $t+2$, so that we are multiplying 5 against the entire numerator.

$$\begin{aligned}3t &= 5(t+2)\\ 3t &= 5t+10\\ 3t-5t &= 5t+10-5t\\ -2t &= 10\\ \frac{-2t}{-2} &= \frac{10}{-2}\\ t &= -5\end{aligned}$$

We should check that this value -5 is actually the solution of the equation:

$$\begin{aligned}\frac{-5}{5} &\overset{?}{=} \frac{-5+2}{3}\\ -1 &\overset{?}{=} \frac{-3}{3}\\ -1 &\overset{\checkmark}{=} -1\end{aligned}$$

Since we have verified that -5 is the solution for $\frac{t}{5} = \frac{t+2}{3}$, we know that the solution set is $\{-5\}$.

Example 3.5.6 Solve $\frac{r+7}{8} = -\frac{9}{4}$ for r.

Explanation. This proportion is a bit different in the fact that one fraction is negative. The key to working with a negative fraction is to attach the negative sign to either the numerator or denominator, but not both:

$$\frac{-9}{4} = -\frac{9}{4} \quad \text{and} \quad \frac{9}{-4} = -\frac{9}{4}, \quad \text{but} \quad \frac{-9}{-4} = +\frac{9}{4}$$

Since we're trying to eliminate the fractions, it will likely make the work a bit easier to attach the negative to the numerator.

We'll work with the equation in the form $\frac{r+7}{8} = \frac{-9}{4}$

$$\begin{aligned}
\frac{r+7}{8} &= \frac{-9}{4}\\
8 \cdot 4 \cdot \frac{r+7}{8} &= \frac{-9}{4} \cdot 8 \cdot 4\\
\cancel{8} \cdot 4 \cdot \frac{r+7}{\cancel{8}} &= \frac{-9}{\cancel{4}} \cdot 8 \cdot \cancel{4}\\
4 \cdot (r+7) &= 8 \cdot (-9)\\
4r + 28 &= -72\\
4r + 28 - 28 &= -72 - 28\\
4r &= -100\\
\frac{4r}{4} &= \frac{-100}{4}\\
r &= -25
\end{aligned}$$

We should check that this value −25 is actually the solution of the equation:

$$\begin{aligned}
\frac{-25+7}{8} &\stackrel{?}{=} -\frac{9}{4}\\
\frac{-18}{8} &\stackrel{?}{=} -\frac{9}{4}\\
-\frac{9}{4} &\stackrel{\checkmark}{=} -\frac{9}{4}
\end{aligned}$$

Since we have verified that −25 is the solution for $\frac{r+7}{8} = -\frac{9}{4}$, we know that the solution set is $\{-25\}$.

Example 3.5.7 Solve $\frac{x}{15} = \frac{40}{25}$ for x.

Explanation. To solve this proportion, begin by multiplying both sides by both denominators.

$$\begin{aligned}
\frac{x}{15} &= \frac{40}{25}\\
15 \cdot 25 \cdot \frac{x}{15} &= \frac{40}{25} \cdot 15 \cdot 25\\
\cancel{15} \cdot 25 \cdot \frac{x}{\cancel{15}} &= \frac{40}{\cancel{25}} \cdot 15 \cdot \cancel{25}\\
25 \cdot x &= 40 \cdot 15\\
25x &= 600\\
\frac{25x}{25} &= \frac{600}{25}\\
x &= 24
\end{aligned}$$

You can easily verify that this value 24 is actually the solution of the equation:

$$\begin{aligned}\frac{24}{15} &\overset{?}{=} \frac{40}{25}\\ \frac{8}{5} &\overset{\checkmark}{=} \frac{8}{5}\end{aligned}$$

Since we have verified that 24 is the solution for $\frac{x}{15} = \frac{40}{25}$, we know that the solution set is $\{24\}$.

Example 3.5.8 Solve $\frac{x-4}{6} = \frac{x+3}{4}$ for x.

Explanation. To solve this proportion, begin by multiplying both sides by both denominators.

$$\begin{aligned}\frac{x-4}{6} &= \frac{x+3}{4}\\ 6\cdot 4\cdot\frac{x-4}{6} &= \frac{x+3}{4}\cdot 6\cdot 4\\ \cancel{6}\cdot 4\cdot\frac{x-4}{\cancel{6}} &= \frac{x+3}{\cancel{4}}\cdot 6\cdot\cancel{4}\\ 4\cdot(x-4) &= (x+3)\cdot 6\\ 4x-16 &= 6x+18\\ 4x-16+16 &= 6x+18+16\\ 4x &= 6x+34\\ 4x-6x &= 6x+34-6x\\ -2x &= 34\\ \frac{-2x}{-2} &= \frac{34}{-2}\\ x &= -17\end{aligned}$$

We can check that this value is correct by substituting it back into the original equation:

$$\begin{aligned}\frac{x-4}{6} &= \frac{x+3}{4}\\ \frac{-17-4}{6} &\overset{?}{=} \frac{-17+3}{4}\\ \frac{-21}{6} &\overset{?}{=} \frac{-14}{4}\\ \frac{-7}{2} &\overset{\checkmark}{=} \frac{-7}{2}\end{aligned}$$

Since we have verified that -17 is the solution for $\frac{x-4}{6} = \frac{x+3}{4}$, we know that the solution set is $\{-17\}$.

3.5.3 Proportionality in Similar Triangles

One really useful example of ratios and proportions involves similar triangles. Two triangles are considered **similar** if they have the same angles and their side lengths are proportional, as shown in Figure 3.5.9:

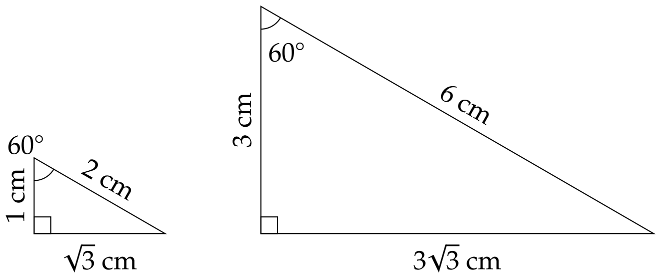


Figure 3.5.9: Similar Triangles

In the first triangle in Figure 3.5.9, the ratio of the left side length to the hypotenuse length is $\frac{1\,\text{cm}}{2\,\text{cm}}$; in the second triangle, the ratio of the left side length to the hypotenuse length is $\frac{3\,\text{cm}}{6\,\text{cm}}$. Since both reduce to $\frac{1}{2}$, we can write the following proportion:

$$\frac{1\,\text{cm}}{2\,\text{cm}} = \frac{3\,\text{cm}}{6\,\text{cm}}$$

If we extend this concept, we can use it to solve for an unknown side length. Consider the two similar triangles in the next example.

Example 3.5.10

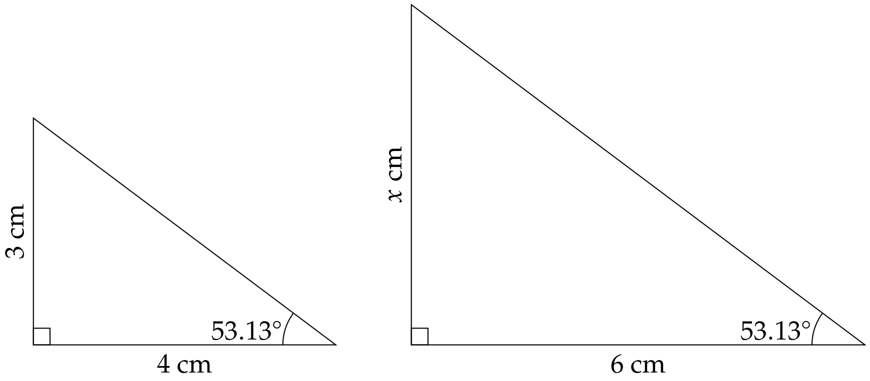


Figure 3.5.11: Similar Triangles

Since the two triangles are similar, we know that their side length should be proportional. To determine the unknown length, we can set up a proportion and solve for x:

$$\begin{aligned}
\frac{\text{bigger triangle's left side length in cm}}{\text{bigger triangle's bottom side length in cm}} &= \frac{\text{smaller triangle's left side length in cm}}{\text{smaller triangle's bottom side length in cm}} \\
\frac{x\,\text{cm}}{6\,\text{cm}} &= \frac{3\,\text{cm}}{4\,\text{cm}} \\
\frac{x}{6} &= \frac{3}{4} \\
12 \cdot \frac{x}{6} &= 12 \cdot \frac{3}{4} \qquad \text{(12 is the least common denominator)} \\
2x &= 9
\end{aligned}$$

$$\begin{aligned}\frac{2x}{2} &= \frac{9}{2}\\ x &= \frac{9}{2} \text{ or } 4.5\end{aligned}$$

The unknown side length is then 4.5 cm.

Remark 3.5.12. Looking at the triangles in Figure 3.5.9, you may notice that there are many different proportions you could set up, such as:

$$\begin{aligned}\frac{2\,\text{cm}}{1\,\text{cm}} &= \frac{6\,\text{cm}}{3\,\text{cm}}\\ \frac{2\,\text{cm}}{6\,\text{cm}} &= \frac{1\,\text{cm}}{3\,\text{cm}}\\ \frac{6\,\text{cm}}{2\,\text{cm}} &= \frac{3\,\text{cm}}{1\,\text{cm}}\\ \frac{3\sqrt{3}\,\text{cm}}{\sqrt{3}\,\text{cm}} &= \frac{3\,\text{cm}}{1\,\text{cm}}\end{aligned}$$

This is often the case when we set up ratios and proportions.

If we take a second look at Figure 3.5.11, there are also several other proportions we could have used to find the value of x.

$$\frac{\text{bigger triangle's left side length}}{\text{smaller triangle's left side length}} = \frac{\text{bigger triangle's bottom side length}}{\text{smaller triangle's bottom side length}}$$

$$\frac{\text{smaller triangle's bottom side length}}{\text{bigger triangle's bottom side length}} = \frac{\text{smaller triangle's left side length}}{\text{bigger triangle's left side length}}$$

$$\frac{\text{bigger triangle's bottom side length}}{\text{smaller triangle's bottom side length}} = \frac{\text{bigger triangle's left side length}}{\text{smaller triangle's left side length}}$$

Written as algebraic proportions, these three equations would, respectively, be

$$\frac{x\,\text{cm}}{3\,\text{cm}} = \frac{6\,\text{cm}}{4\,\text{cm}}, \qquad \frac{4\,\text{cm}}{6\,\text{cm}} = \frac{3\,\text{cm}}{x\,\text{cm}}, \qquad \frac{6\,\text{cm}}{4\,\text{cm}} = \frac{x\,\text{cm}}{3\,\text{cm}}$$

While these are only a few of the possibilities, if we clear the denominators from any properly designed proportion, every one is equivalent to $x = 4.5$.

3.5.4 Creating and Solving Proportions

Proportions can be used to solve many real-life applications. The key to using proportions to solve such applications is to first set up a ratio where all values are known. We then set up a second ratio that will be proportional to the first, but has one value in the ratio unknown. Let's look at a few examples.

Example 3.5.13 Property taxes for a residential property are proportional to the assessed value of the property. Assume that a certain property in a given neighborhood is assessed at \$234,100 and its annual property taxes are \$2,518.92. What are the annual property taxes for a house that is assessed at \$287,500?

Explanation. Let T be the annual property taxes (in dollars) for a property assessed at \$287,500. We can write and solve this proportion:

$$\begin{aligned}\frac{\text{tax}}{\text{property value}} &= \frac{\text{tax}}{\text{property value}}\\ \frac{2518.92}{234100} &= \frac{T}{287500}\end{aligned}$$

The least common denominator of this proportion is rather large, so we will instead multiply each side by 234100 and 287500 and simplify from there:

$$\begin{aligned}\frac{2518.92}{234100} &= \frac{T}{287500}\\ 234100 \cdot 287500 \cdot \frac{2518.92}{234100} &= \frac{T}{287500} \cdot 234100 \cdot 287500\\ 287500 \cdot 2518.92 &= T \cdot 234100\\ \frac{287500 \cdot 2518.92}{234100} &= \frac{234100T}{234100}\\ T &\approx 3093.50\end{aligned}$$

The property taxes for a property assessed at $\$287,500$ are $\$3,093.50$.

Example 3.5.14 Tagging fish is a means of estimating the size of the population of fish in a body of water (such as a lake). A sample of fish is taken, tagged, and then redistributed into the lake. When another sample is taken, the proportion of fish that are tagged out of that sample are assumed to be proportional to the total number of fish tagged out of the entire population of fish in the lake.

$$\frac{\text{number of tagged fish in sample}}{\text{number of fish in sample}} = \frac{\text{number of tagged fish total}}{\text{number of fish total}}$$

Assume that 90 fish are caught and tagged. Once they are redistributed, a sample of 200 fish is taken. Of these, 7 are tagged. Estimate how many fish total are in the lake.

Explanation. Let n be the number of fish in the lake. We can set up a proportion for this scenario:

$$\frac{7}{200} = \frac{90}{n}$$

To solve for n, which is in a denominator, we'll need to multiply each side by both 200 and n:

$$\begin{aligned}\frac{7}{200} &= \frac{90}{n}\\ 200 \cdot n \cdot \frac{7}{200} &= \frac{90}{n} \cdot 200 \cdot n\\ \cancel{200} \cdot n \cdot \frac{7}{\cancel{200}} &= \frac{90}{\cancel{n}} \cdot 200 \cdot \cancel{n}\\ 7n &= 1800\\ \frac{7n}{7} &= \frac{1800}{7}\\ n &\approx 2471.4286\end{aligned}$$

According to this sample, we can estimate that there are about $2,471$ fish in the lake.

Example 3.5.15 Infant Tylenol contains 160 mg of acetaminophen in each 5 mL of liquid medicine. If Bao's baby is prescribed 60 mg of acetaminophen, how many milliliters of liquid medicine should he give them?

Explanation. Assume Bao should give q milliliters of liquid medicine, and we can set up the following proportion:

$$\begin{aligned}
\frac{\text{amount of liquid medicine in mL}}{\text{amount of acetaminophen in mg}} &= \frac{\text{amount of liquid medicine in mL}}{\text{amount of acetaminophen in mg}}\\
\frac{5\,\text{mL}}{160\,\text{mg}} &= \frac{q\,\text{mL}}{60\,\text{mg}}\\
\frac{5}{160} &= \frac{q}{60}\\
160\cdot 60\cdot\frac{5}{160} &= \frac{q}{60}\cdot 160\cdot 60\\
60\cdot 5 &= q\cdot 160\\
300 &= 160q\\
\frac{300}{160} &= \frac{160q}{160}\\
q &= 1.875
\end{aligned}$$

So to give 60 mg of acetaminophen to his baby, Bao should give 1.875 mL of liquid medicine.

Example 3.5.16 Sarah is an architect and she's making a scale model of a building. The actual building will be 30 ft tall. In the model, the height of the building will be 2 in. How tall should she make the model of a person who is 5 ft 6 in tall so that the model is to scale?

Explanation. Let h be the height of the person in Sarah's model, which we'll measure in inches. We'll create a proportion that compares the building and person's heights in the model to their heights in real life:

$$\begin{aligned}
\frac{\text{height of model building in inches}}{\text{height of actual building in feet}} &= \frac{\text{height of model person in inches}}{\text{height of actual person in feet}}\\
\frac{2\,\text{in}}{30\,\text{ft}} &= \frac{h\,\text{in}}{5\,\text{ft}\,6\text{in}}
\end{aligned}$$

Before we can just eliminate the units, we'll need to convert 5 ft 6 in to feet:

$$\frac{2\,\text{in}}{30\,\text{ft}} = \frac{h\,\text{in}}{5.5\,\text{ft}}$$

Now we can remove the units and continue solving:

$$\begin{aligned}
\frac{2}{30} &= \frac{h}{5.5}\\
30\cdot 5.5\cdot\frac{2}{30} &= \frac{h}{5.5}\cdot 30\cdot 5.5\\
5.5\cdot 2 &= h\cdot 30\\
11 &= 30h
\end{aligned}$$

$$\begin{aligned}\frac{11}{30} &= \frac{30h}{30}\\ \frac{11}{30} &= h\\ h &\approx 0.3667\end{aligned}$$

Sarah should make the model of a person who is 5 ft 6 in tall be $\frac{11}{30}$ inches (about 0.3667 inches) tall.

Exercises

Review and Warmup

1. Reduce the fraction $\frac{3}{15}$.
2. Reduce the fraction $\frac{2}{10}$.
3. Reduce the fraction $\frac{9}{21}$.
4. Reduce the fraction $\frac{8}{14}$.
5. Reduce the fraction $\frac{60}{105}$.
6. Reduce the fraction $\frac{84}{147}$.
7. Reduce the fraction $\frac{252}{105}$.
8. Reduce the fraction $\frac{420}{245}$.

Setting Up Ratios and Proportions

9. Ibuprofen for infants comes in a liquid form and contains 30 milligrams of ibuprofen for each 0.75 milliliters of liquid. If a child is to receive a dose of 50 milligrams of ibuprofen, how many milliliters of liquid should they be given?

   Assume l milliliters of liquid should be given. Write an equation to model this scenario. There is no need to solve it.

10. Ibuprofen for infants comes in a liquid form and contains 35 milligrams of ibuprofen for each 0.875 milliliters of liquid. If a child is to receive a dose of 45 milligrams of ibuprofen, how many milliliters of liquid should they be given?

    Assume l milliliters of liquid should be given. Write an equation to model this scenario. There is no need to solve it.

11. The property taxes on a 2400-square-foot house are \$2,904.00 per year. Assuming these taxes are proportional, what are the property taxes on a 1200-square-foot house?

    Assume property taxes on a 1200-square-foot house is t dollars. Write an equation to model this scenario. There is no need to solve it.

12. The property taxes on a 1900-square-foot house are \$2,527.00 per year. Assuming these taxes are proportional, what are the property taxes on a 2500-square-foot house?

    Assume property taxes on a 2500-square-foot house is t dollars. Write an equation to model this scenario. There is no need to solve it.

Solving Proportions

13. Solve $\frac{x}{48} = \frac{15}{40}$ for x.

14. Solve $\frac{x}{63} = \frac{10}{35}$ for x.

15. Solve $\frac{10}{x} = \frac{35}{63}$ for x.

16. Solve $\frac{12}{x} = \frac{16}{24}$ for x.

17. Solve $\frac{x}{6} = \frac{x-15}{9}$ for x.

18. Solve $\frac{x}{5} = \frac{x+24}{9}$ for x.

19. Solve $\frac{x}{7} = \frac{x-3}{6}$ for x.

20. Solve $\frac{x}{7} = \frac{x-20}{11}$ for x.

21. Solve $\frac{x+3}{5} = \frac{x-5}{9}$ for x.

22. Solve $\frac{x-10}{5} = \frac{x+16}{7}$ for x.

23. Solve $\frac{x-16}{6} = \frac{x-12}{14}$ for x.

24. Solve $\frac{x-8}{7} = \frac{x-8}{11}$ for x.

25. Solve $\frac{x}{24} = -\frac{45}{27}$ for x.

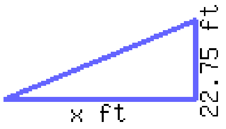
26. Solve $\frac{x}{21} = -\frac{18}{14}$ for x.

27. Solve $\frac{x+2}{42} = -\frac{24}{18}$ for x.

28. Solve $\frac{x-2}{6} = -\frac{45}{10}$ for x.

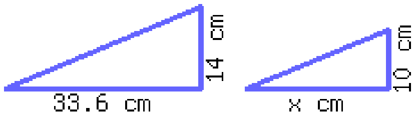
Applications

29. The following two triangles are similar to each other. Find the length of the missing side.



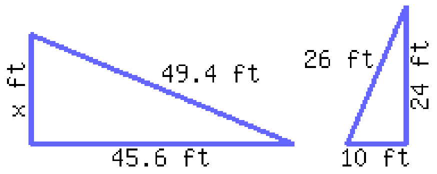
The missing side's length is

30. The following two triangles are similar to each other. Find the length of the missing side.



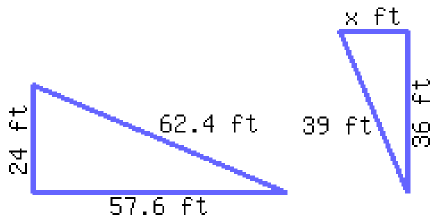
The missing side's length is

31. The following two triangles are similar to each other. Find the length of the missing side.



The missing side's length is

32. The following two triangles are similar to each other. Find the length of the missing side.



The missing side's length is

33. According to a salad recipe, each serving requires 2 teaspoons of vegetable oil and 12 teaspoons of vinegar. If 12 teaspoons of vegetable oil were used, how many teaspoons of vinegar should be used?

If 12 teaspoons of vegetable oil were used, teaspoons of vinegar should be used.

34. According to a salad recipe, each serving requires 5 teaspoons of vegetable oil and 35 teaspoons of vinegar. If 119 teaspoons of vinegar were used, how many teaspoons of vegetable oil should be used?

If 119 teaspoons of vinegar were used, teaspoons of vegetable oil should be used.

35. Laurie makes $105 every six hours she works. How much will she make if she works twenty-two hours this week?

If Laurie works twenty-two hours this week, she will make .

36. Corey makes $81 every six hours he works. How much will he make if he works twenty-six hours this week?

If Corey works twenty-six hours this week, he will make .

37. A mutual fund consists of 23% stock and 77% bond. In other words, for each 23 dollars of stock, there are 77 dollars of bond. For a mutual fund with $2,850.00 of stock, how many dollars of bond are there?

For a mutual fund with $2,850.00 of stock, there are approximately of bond.

38. A mutual fund consists of 32% stock and 68% bond. In other words, for each 32 dollars of stock, there are 68 dollars of bond. For a mutual fund with $2,510.00 of bond, how many dollars of stock are there?

For a mutual fund with $2,510.00 of bond, there are approximately of stock.

39. Farshad jogs every day. Last month, he jogged 14.5 hours for a total of 17.4 miles. At this speed, if Farshad runs 35 hours, how far can he run?

At this speed, Farshad can run in 35 hours.

40. Scot jogs every day. Last month, he jogged 5.5 hours for a total of 9.9 miles. At this speed, how long would it take Scot to run 90 miles?

At this speed, Scot can run 90 mi in .

41. Blake purchased 3.7 pounds of apples at the total cost of $16.28. If he purchases 9.4 pounds of apples at this store, how much would it cost?

It would cost to purchase 9.4 pounds of apples.

42. Jay purchased 5.4 pounds of apples at the total cost of $22.14. If the price doesn't change, how many pounds of apples can Jay purchase with $40.18?

With $40.18, Jay can purchase of apples.

43. Timothy collected a total of 2261 stamps over the past 17 years. At this rate, how many stamps would he collect in 23 years?

At this rate, Timothy would collect stamps in 23 years.

44. Tiffany collected a total of 2016 stamps over the past 14 years. At this rate, how many years would it take she to collect 3888 stamps?

At this rate, Tiffany can collect 3888 stamps in years.

45. In a city, the owner of a house valued at 380 thousand dollars needs to pay $425.60 in property tax. At this tax rate, how much property tax should the owner pay if a house is valued at 890 thousand dollars?

The owner of a 890-thousand-dollar house should pay ______ in property tax.

46. In a city, the owner of a house valued at 300 thousand dollars needs to pay $525.00 in property tax. At this tax rate, if the owner of a house paid $1,540.00 of property tax, how much is the house worth?

If the owner of a house paid $1,540.00 of property tax, the house is worth ______ thousand dollars.

47. To try to determine the health of the Rocky Mountain elk population in the Wenaha Wildlife Area, the Oregon Department of Fish and Wildlife caught, tagged, and released 39 Rocky Mountain elk. A week later, they returned and observed 42 Rocky Mountain elk, 9 of which had tags. Approximately how many Rocky Mountain elk are in the Wenaha Wildlife Area?

There are approximately ______ elk in the wildlife area.

48. To try to determine the health of the black-tailed deer population in the Jewell Meadow Wildlife Area, the Oregon Department of Fish and Wildlife caught, tagged, and released 28 black-tailed deer. A week later, they returned and observed 63 black-tailed deer, 18 of which had tags. Approximately how many black-tailed deer are in the Jewell Meadow Wildlife Area?

There are approximately ______ deer in the wildlife area.

49. A restaurant used 1105.2 lb of vegetable oil in 36 days. At this rate, how many pounds of vegetable oil will be used in 50 days?

The restaurant will use ______ of vegetable oil in 50 days.

50. A restaurant used 777.6 lb of vegetable oil in 24 days. At this rate, 1458 lb of oil will last how many days?

The restaurant will use 1458 lb of vegetable oil in ______ days.

Challenge

51. The ratio of girls to boys in a preschool is 6 to 7. If there are 104 kids in the school, how many girls are there in the preschool?

3.6 Special Solution Sets

Most of the time, a linear equation's final equivalent equation looks like $x = 3$, and the solution set is written to show that there is only one solution: $\{3\}$. Similarly, a linear inequality's final equivalent equation looks like $x < 5$, and the solution set is represented with either $(-\infty, 5)$ in interval notation or $\{x|x < 5\}$ in set-builder notation. It's possible that both linear equations and inequalities have all real numbers as possible solutions, and it's possible that no real numbers are solutions to each. In this section, we will explore these special solution sets.

3.6.1 Special Solution Sets

Recall that for the equation $x + 2 = 5$, there is only one number which will make the equation true: 3. This means that our solution is 3, and we write the **solution set** as $\{3\}$. We say the equation's solution set has one **element**, 3.

We'll now explore equations that have all real numbers as possible solutions or no real numbers as possible solutions.

> **Example 3.6.2** Solve for x in $3x = 3x + 4$.
>
> To solve this equation, we need to move all terms containing x to one side of the equals sign:
>
> $$\begin{aligned} 3x &= 3x + 4 \\ 3x - 3x &= 3x + 4 - 3x \\ 0 &= 4 \end{aligned}$$
>
> Notice that x is no longer present in the equation. What value can we substitute into x to make $0 = 4$ true? Nothing! We say this equation has no solution. Or, the equation has an empty solution set. We can write this as $\emptyset$, which is the symbol for the empty set.
>
> The equation $0 = 4$ is known a **false statement** since it is false no matter what x is. It indicates there is no solution to the original equation.

> **Example 3.6.3** Solve for x in $2x + 1 = 2x + 1$.
>
> We will move all terms containing x to one side of the equals sign:
>
> $$\begin{aligned} 2x + 1 &= 2x + 1 \\ 2x + 1 - 2x &= 2x + 1 - 2x \\ 1 &= 1 \end{aligned}$$
>
> At this point, x is no longer contained in the equation. What value can we substitute into x to make $1 = 1$ true? Any number! This means that all real numbers are possible solutions to the equation $2x + 1 = 2x + 1$. We say this equation's solution set contains *all real numbers*. We can write this set using set-builder notation as $\{x \mid x \text{ is a real number}\}$ or using interval notation as $(-\infty, \infty)$.
>
> The equation $1 = 1$ is known as an **identity** since it is true no matter what x is. It indicates that all real numbers are solutions to the original linear equation.

Remark 3.6.4. What would have happened if we had continued solving after we obtained $1 = 1$ in Example 3.6.3?

$$\begin{aligned}1&=1\\1-1&=1-1\\0&=0\end{aligned}$$

As we can see, all we found was another identity — a different equation that is true for all values of x.

Warning 3.6.5. Note that there is a very important difference between ending with an equivalent equation of $0=0$ and $x=0$. The first holds true for all real numbers, and the solution set is $\{x \mid x \text{ is a real number}\}$. The second has only one solution: 0. We write that solution set to show that only the number zero is the solution: $\{0\}$.

Example 3.6.6 Solve for t in the inequality $4t+5>4t+2$.

To solve for t, we will first subtract $4t$ from each side to get all terms containing t on one side:

$$\begin{aligned}4t+5&>4t+2\\4t+5-4t&>4t+2-4t\\5&>2\end{aligned}$$

Notice that again, the variable t is no longer contained in the inequality. We then need to consider which values of t make the inequality true. The answer is *all values*, so our solution set is all real numbers, which we can write as $\{t \mid t \text{ is a real number}\}$.

Example 3.6.7 Solve for x in the inequality $-5x+1\leq -5x$.

To solve for x, we will first add $5x$ to each side to get all terms containing x on one side:

$$\begin{aligned}-5x+1&\leq -5x\\-5x+1+5x&\leq -5x+5x\\1&\leq 0\end{aligned}$$

Once more, the variable x is absent. So we can ask ourselves, "For which values of x is $1\leq 0$ true?" The answer is *none*, and so there is no solution to this inequality. We can write the solution set using $\emptyset$.

Remark 3.6.8. Again consider what would have happened if we had continued solving after we obtained $1\leq 0$ in Example 3.6.7.

$$\begin{aligned}1&\leq 0\\1-1&\leq 0-1\\0&\leq -1\end{aligned}$$

As we can see, all we found was another false statement—a different equation that is not true for any real number.

Let's summarize the two special cases when solving linear equations and inequalities:

All Real Numbers When the equivalent equation or inequality is an *identity* such as $2 = 2$ or $0 < 2$, all real numbers are solutions. We write this solution set as either $(-\infty, \infty)$ or $\{x \mid x \text{ is a real number}\}$.

No Solution When the equivalent equation or inequality is a *false statement* such as $0 = 2$ or $0 > 2$, no real number is a solution. We write this solution set as either $\{\ \}$ or $\emptyset$ or write the words "no solution exists."

List 3.6.9: Special Solution Sets for Equations and Inequalities

3.6.2 Solving Equations and Inequalities with Special Solution Sets

Example 3.6.10 Solve for a in $\frac{2}{3}(a+1) - \frac{5}{6} = \frac{2}{3}a$.

To solve this equation for a, we'll want to recall the technique of multiplying each side of the equation by the LCD of all fractions. Here, this means that we will multiply each side by 6 as our first step. After that, we'll be able to simplify each side of the equation and continue solving for a:

$$\begin{aligned}
\frac{2}{3}(a+1) - \frac{5}{6} &= \frac{2}{3}a \\
6 \cdot \left(\frac{2}{3}(a+1) - \frac{5}{6}\right) &= 6 \cdot \frac{2}{3}a \\
6 \cdot \frac{2}{3}(a+1) - 6 \cdot \frac{5}{6} &= 6 \cdot \frac{2}{3}a \\
4(a+1) - 5 &= 4a \\
4a + 4 - 5 &= 4a \\
4a - 1 &= 4a \\
4a - 1 - 4a &= 4a - 4a \\
-1 &= 0
\end{aligned}$$

The statement $-1 = 0$ is false, so the equation has no solution. We can write the empty set as: $\emptyset$.

Example 3.6.11 Solve for x in the equation $3(x+2) - 8 = (5x+4) - 2(x+1)$.

To solve for x, we will first need to simplify the left side and right side of the equation as much as possible by distributing and combining like terms:

$$\begin{aligned}
3(x+2) - 8 &= (5x+4) - 2(x+1) \\
3x + 6 - 8 &= 5x + 4 - 2x - 2 \\
3x - 2 &= 3x + 2
\end{aligned}$$

From here, we'll want to subtract $3x$ from each side:

$$\begin{aligned}
3x - 2 - 3x &= 3x + 2 - 3x \\
-2 &= 2
\end{aligned}$$

As the equation $-2 = 2$ is not true for any value of x, there is no solution to this equation. We write the solution set as: $\emptyset$.

Example 3.6.12 Solve for z in the inequality $\frac{3z}{5} + \frac{1}{2} \leq \left(\frac{z}{10} + \frac{3}{4}\right) + \left(\frac{z}{2} - \frac{1}{4}\right)$.

To solve for z, we will first need to multiply each side of the inequality by the LCD, which is 40. After that, we'll finish solving by putting all terms containing a variable on one side of the inequality:

$$\begin{aligned}
\frac{3z}{5} + \frac{1}{2} &\leq \left(\frac{z}{10} + \frac{3}{4}\right) + \left(\frac{z}{2} - \frac{1}{4}\right)\\
40 \cdot \left(\frac{3z}{5} + \frac{1}{2}\right) &\leq 40 \cdot \left(\left(\frac{z}{10} + \frac{3}{4}\right) + \left(\frac{z}{2} - \frac{1}{4}\right)\right)\\
40 \cdot \left(\frac{3z}{5}\right) + 40 \cdot \left(\frac{1}{2}\right) &\leq 40 \cdot \left(\frac{z}{10} + \frac{3}{4}\right) + 40 \cdot \left(\frac{z}{2} - \frac{1}{4}\right)\\
40 \cdot \left(\frac{3z}{5}\right) + 40 \cdot \left(\frac{1}{2}\right) &\leq 40 \cdot \left(\frac{z}{10}\right) + 40 \cdot \left(\frac{3}{4}\right) + 40 \cdot \left(\frac{z}{2}\right) - 40 \cdot \left(\frac{1}{4}\right)\\
24z + 20 &\leq 4z + 30 + 20z - 10\\
24z + 20 &\leq 24z + 20\\
24z + 20 - 24z &\leq 24z + 20 - 24z\\
20 &\leq 20
\end{aligned}$$

As the equation $20 \leq 20$ is true for all values of z, all real numbers are solutions to this inequality. Thus the solution set is $\{z \mid z \text{ is a real number}\}$.

Exercises

Review and Warmup Solve the equation.

1. $7n + 4 = 18$
2. $4q + 3 = 27$
3. $-2x - 2 = -18$
4. $-5r - 9 = 11$
5. $-4t + 8 = -t - 10$
6. $-10b + 3 = -b - 24$
7. $96 = -6(c - 6)$
8. $15 = -3(B - 10)$

Solving Equations with Special Solution Sets Solve the equation.

9. $10C = 10C + 4$
10. $6n = 6n + 8$
11. $2p + 2 = 2p + 2$
12. $10x + 6 = 10x + 6$
13. $6r - 2 - 7r = -4 - r + 2$
14. $3t - 6 - 4t = -7 - t + 1$
15. $-9 - 10b + 8 = -b + 13 - 9b$
16. $-7 - 6c + 4 = -c + 12 - 5c$
17. $2(B - 9) = 2(B - 1)$

18. $8(C-5)=8(C-4)$

19.
a. $7n+7=4n+7$
b. $7n+7=7n+7$
c. $7n+7=7n+11$

20.
a. $5p+10=2p+10$
b. $5p+10=5p+10$
c. $5p+10=5p+13$

Solve the equation.

21. $4(4-2x)-(6x-8)=23-2(8+7x)$

22. $3(8-10r)-(6r-3)=19-2(9+18r)$

23. $19-6(4+5t)=-31t-(5-t)$

24. $30-5(7+4b)=-21b-(5-b)$

Solving Inequalities with Special Solution Sets Solve this inequality. Answer using interval notation.

25. $6x>6x+3$

26. $6x>6x+9$

27. $-8x\leq -8x-5$

28. $-10x\leq -10x-8$

29. $-8+10x+18\geq 10x+10$

30. $-2+2x+8\geq 2x+6$

31. $-6+2x+9<2x+3$

32. $-10+4x+19<4x+9$

33. $-7-8z+3>-z+16-7z$

34. $-7-5z+6>-z+10-4z$

35. $6(k-9)\leq 6(k-1)$

36. $8(k-7)\leq 8(k-3)$

37. $10x\leq 10x+2$

38. $10x\leq 10x+7$

Solve this inequality. Answer using interval notation.

39. $2(4-10m)-(2m-4)>8-2(8+11m)$

40. $2(1-4m)-(10m-4)>10-2(3+9m)$

Challenge

41. Fill in the right side of the equation to create a linear equation with the properties listed.

a. Create a linear equation with *infinitely many solutions*.

$6(x+4)=$ ☐

b. Create a linear equation with the solution $x=2$.

$6(x+4)=$ ☐

3.7 Linear Equations and Inequalities Chapter Review

3.7.1 Solving Multistep Linear Equations

In Section 3.1 we covered the steps to solve a linear equation and the differences among simplifying expressions, evaluating expressions and solving equations.

Example 3.7.1 Solve for a in $4-(3-a)=-2-2(2a+1)$.

Explanation. To solve this equation, we will simplify each side of the equation, manipulate it so that all variable terms are on one side and all constant terms are on the other, and then solve for a:

$$\begin{aligned}4-(3-a)&=-2-2(2a+1)\\4-3+a&=-2-4a-2\\1+a&=-4-4a\\1+a+4a&=-4-4a+4a\\1+5a&=-4\\1+5a-1&=-4-1\\5a&=-5\\\frac{5a}{5}&=\frac{-5}{5}\\a&=-1\end{aligned}$$

Checking the solution -1 in the original equation, we get:

$$\begin{aligned}4-(3-a)&=-2-2(2a+1)\\4-(3-(-1))&\stackrel{?}{=}-2-2(2(-1)+1)\\4-(4)&\stackrel{?}{=}-2-2(-1)\\0&\stackrel{\checkmark}{=}0\end{aligned}$$

Therefore the solution to the equation is -1 and the solution set is $\{-1\}$.

3.7.2 Solving Multistep Linear Inequalities

In Section 3.2 we covered how solving inequalities is very much like how we solve equations, except that if we multiply or divide by a negative we switch the inequality sign.

Example 3.7.2 Solve for x in $-2-2(2x+1)>4-(3-x)$. Write the solution set in both set-builder notation and interval notation.

Explanation.

$$\begin{aligned}-2-2(2x+1)&>4-(3-x)\\-2-4x-2&>4-3+x\\-4x-4&>x+1\end{aligned}$$

$$\begin{aligned}
-4x-4-x &> x+1-x\\
-5x-4 &> 1\\
-5x-4+4 &> 1+4\\
-5x &> 5\\
\frac{-5x}{-5} &< \frac{5}{-5}\\
x &< -1
\end{aligned}$$

Note that when we divided both sides of the inequality by -5, we had to switch the direction of the inequality symbol.

The solution set in set-builder notation is $\{x \mid x < -1\}$. The solution set in interval notation is $(-\infty, -1)$.

3.7.3 Linear Equations and Inequalities with Fractions

In Section 3.3 we covered how to eliminate denominators in an equation with the LCD to help solve the equation.

Example 3.7.3 Solve for x in $\frac{1}{4}x + \frac{2}{3} = \frac{1}{6}$.

Explanation.

We'll solve by multiplying each side of the equation by 12:

$$\begin{aligned}
\frac{1}{4}x+\frac{2}{3} &= \frac{1}{6}\\
12\cdot\left(\frac{1}{4}x+\frac{2}{3}\right) &= 12\cdot\frac{1}{6}\\
12\cdot\left(\frac{1}{4}x\right)+12\cdot\left(\frac{2}{3}\right) &= 12\cdot\frac{1}{6}\\
3x+8 &= 2\\
3x &= -6\\
\frac{3x}{3} &= \frac{-6}{3}\\
x &= -2
\end{aligned}$$

Checking the solution:

$$\begin{aligned}
\frac{1}{4}x+\frac{2}{3} &= \frac{1}{6}\\
\frac{1}{4}(-2)+\frac{2}{3} &\overset{?}{=} \frac{1}{6}\\
-\frac{2}{4}+\frac{2}{3} &\overset{?}{=} \frac{1}{6}\\
-\frac{6}{12}+\frac{8}{12} &\overset{?}{=} \frac{1}{6}\\
\frac{2}{12} &\overset{?}{=} \frac{1}{6}\\
\frac{1}{6} &\overset{\checkmark}{=} \frac{1}{6}
\end{aligned}$$

The solution is therefore -2. We write the solution set s $\{-2\}$.

3.7.4 Isolating a Linear Variable

In Section 3.4 we covered how to solve an equation when there are multiple variables in the equation.

Example 3.7.4 Solve for x in $y = mx + b$.

Explanation.

$$\begin{aligned} y &= mx + b \\ y - b &= mx + b - b \\ y - b &= mx \\ \frac{y-b}{m} &= \frac{mx}{m} \\ \frac{y-b}{m} &= x \end{aligned}$$

3.7.5 Ratios and Proportions

In Section 3.5 we covered the definitions of a ratio and a proportion and how to solve a proportion. We learned about cross multiplication, did problems about similar triangles, and used proportions to solve word problems.

Example 3.7.5 Solve $\frac{6-x}{5} = \frac{x}{4}$ for x.

Explanation. To solve this proportion, begin by multiplying both sides by both denominators.

$$\begin{aligned} \frac{6-x}{5} &= \frac{x}{4} \\ 5 \cdot 4 \cdot \frac{6-x}{5} &= \frac{x}{4} \cdot 5 \cdot 4 \\ \cancel{5} \cdot 4 \cdot \frac{6-x}{\cancel{5}} &= \frac{x}{\cancel{4}} \cdot 5 \cdot \cancel{4} \\ 4 \cdot (6 - x) &= x \cdot 5 \\ 24 - 4x &= 5x \\ 24 - 4x + 4x &= 5x + 4x \\ 24 &= 9x \\ \frac{24}{9} &= \frac{9x}{9} \\ \frac{24}{9} &= x \end{aligned}$$

So, the solution set is $\left\{\frac{24}{9}\right\}$.

Example 3.7.6 Property taxes for a residential property are proportional to the assessed value of the property. Assume that a certain property in a given neighborhood is assessed at \$234,100 and its annual property taxes are \$2,518.92. What are the annual property taxes for a house that is assessed at \$287,500?

Explanation. Let T be the annual property taxes (in dollars) for a property assessed at \$287,500. We can write and solve this proportion:

$$\begin{aligned} \frac{\text{tax}}{\text{property value}} &= \frac{\text{tax}}{\text{property value}} \\ \frac{2518.92}{234100} &= \frac{T}{287500} \end{aligned}$$

$$\begin{aligned}
234100 \cdot 287500 \cdot \frac{2518.92}{234100} &= \frac{T}{287500} \cdot 234100 \cdot 287500 \\
287500 \cdot 2518.92 &= T \cdot 234100 \\
\frac{287500 \cdot 2518.92}{234100} &= \frac{234100T}{234100} \\
T &\approx 3093.50
\end{aligned}$$

The property taxes for a property assessed at \$287,500 are \$3,093.50.

Example 3.7.7

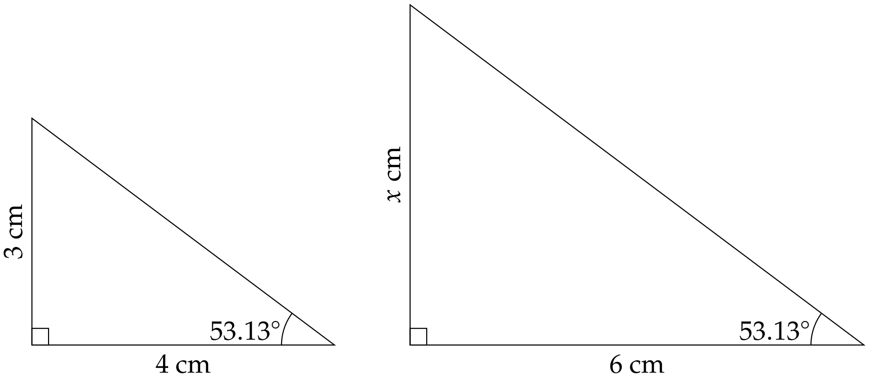


Figure 3.7.8: Similar Triangles

Since the two triangles are similar, we know that their side length should be proportional. To determine the unknown length, we can set up a proportion and solve for x:

$$\begin{aligned}
\frac{\text{bigger triangle's left side length in cm}}{\text{bigger triangle's bottom side length in cm}} &= \frac{\text{smaller triangle's left side length in cm}}{\text{smaller triangle's bottom side length in cm}} \\
\frac{x\,\text{cm}}{6\,\text{cm}} &= \frac{3\,\text{cm}}{4\,\text{cm}} \\
\frac{x}{6} &= \frac{3}{4} \\
12 \cdot \frac{x}{6} &= 12 \cdot \frac{3}{4} \qquad \text{(12 is the least common denominator)} \\
2x &= 9 \\
\frac{2x}{2} &= \frac{9}{2} \\
x &= \frac{9}{2} \text{ or } 4.5
\end{aligned}$$

The unknown side length is then 4.5 cm.

3.7.6 Special Solution Sets

In Section 3.6 we covered linear equations that have no solutions and also linear equations that have infinitely many solutions. When solving linear inequalities, it's also possible that no solution exists or that all real numbers are solutions.

Example 3.7.9

a. Solve for x in the equation $3x = 3x + 4$.

b. Solve for t in the inequality $4t + 5 > 4t + 2$.

Explanation.

a. To solve this equation, we need to move all terms containing x to one side of the equals sign:

$$\begin{aligned} 3x &= 3x + 4 \\ 3x - 3x &= 3x + 4 - 3x \\ 0 &= 4 \end{aligned}$$

This equation has no solution. We write the solution set as $\emptyset$, which is the symbol for the empty set.

b. To solve for t, we will first subtract $4t$ from each side to get all terms containing t on one side:

$$\begin{aligned} 4t + 5 &> 4t + 2 \\ 4t + 5 - 4t &> 4t + 2 - 4t \\ 5 &> 2 \end{aligned}$$

All values of the variable t make the inequality true. The solution set is all real numbers, which we can write as $\{t \mid t \text{ is a real number}\}$ in set notation, or $(-\infty, \infty)$ in interval notation.

Exercises

1. a. Solve the following linear equation:

$3(y - 7) - 4 = -13$

b. Evaluate the following expression when $y = 4$:

$3(y - 7) - 4 =$ ☐

c. Simplify the following expression:

$3(y - 7) - 4 =$ ☐

2. a. Solve the following linear equation:

$2(y + 1) - 9 = 1$

b. Evaluate the following expression when $y = 4$:

$2(y + 1) - 9 =$ ☐

c. Simplify the following expression:

$2(y + 1) - 9 =$ ☐

3. Solve the equation.

$-55 = -8B - 10 - B$

4. Solve the equation.

$1 = -5C - 5 - C$

5. Solve the equation.

$5 + 10(n - 4) = -73 - (2 - 2n)$

6. Solve the equation.

$4 + 7(p - 10) = -84 - (7 - 2p)$

7. Solve the equation.

$-6 - 8x + 4 = -x + 4 - 7x$

8. Solve the equation.

$-10 - 5y + 4 = -y + 0 - 4y$

9. Solve the equation.

$$13 = \frac{t}{7} + \frac{t}{6}$$

10. Solve the equation.

$$9 = \frac{a}{5} + \frac{a}{10}$$

11. Solve the equation.

$$\frac{c-4}{6} = \frac{c+6}{8}$$

12. Solve the equation.

$$\frac{B-8}{4} = \frac{B+3}{6}$$

13. Solve this inequality.

$$4 - (y+7) < 4$$

In set-builder notation, the solution set is $\boxed{}$.

In interval notation, the solution set is $\boxed{}$.

14. Solve this inequality.

$$4 - (y+10) < -7$$

In set-builder notation, the solution set is $\boxed{}$.

In interval notation, the solution set is $\boxed{}$.

15. Solve this inequality. Answer using interval notation.

$$10(k-8) \le 10(k-4)$$

16. Solve this inequality. Answer using interval notation.

$$2(k-6) \le 2(k-2)$$

17. Solve this inequality.

$$1 + 10(x-9) < -23 - (2-2x)$$

In set-builder notation, the solution set is $\boxed{}$.

In interval notation, the solution set is $\boxed{}$.

18. Solve this inequality.

$$2 + 8(x-4) < -21 - (1-4x)$$

In set-builder notation, the solution set is $\boxed{}$.

In interval notation, the solution set is $\boxed{}$.

19. Solve this inequality.

$$-\frac{1}{4}t > \frac{2}{5}t - 13$$

In set-builder notation, the solution set is $\boxed{}$.

In interval notation, the solution set is $\boxed{}$.

20. Solve this inequality.

$$-\frac{1}{2}t > \frac{6}{5}t - 17$$

In set-builder notation, the solution set is $\boxed{}$.

In interval notation, the solution set is $\boxed{}$.

21. Solve this linear equation for x.

$$Ax + By = C$$

22. Solve this linear equation for y.

$$Ax + By = C$$

23. Solve this linear equation for n.

$$r = a - \frac{2n}{m}$$

24. Solve this linear equation for p.

$$q = m - \frac{2p}{b}$$

25. Carly has $70 in her piggy bank. She plans to purchase some Pokemon cards, which costs $1.15 each. She plans to save $52.75 to purchase another toy. At most how many Pokemon cards can he purchase?

Write an equation to solve this problem.

Carly can purchase at most ☐ Pokemon cards.

26. Maygen has $72 in her piggy bank. She plans to purchase some Pokemon cards, which costs $2.55 each. She plans to save $43.95 to purchase another toy. At most how many Pokemon cards can he purchase?

Write an equation to solve this problem.

Maygen can purchase at most ☐ Pokemon cards.

27. Use a linear equation to solve the word problem.

Evan has $85.00 in his piggy bank, and he spends $2.50 every day.

Bobbi has $31.00 in her piggy bank, and she saves $2.00 every day.

If they continue to spend and save money this way, how many days later would they have the same amount of money in their piggy banks?

☐ days later, Evan and Bobbi will have the same amount of money in their piggy banks.

28. Use a linear equation to solve the word problem.

Will has $100.00 in his piggy bank, and he spends $4.00 every day.

Ross has $34.00 in his piggy bank, and he saves $2.00 every day.

If they continue to spend and save money this way, how many days later would they have the same amount of money in their piggy banks?

☐ days later, Will and Ross will have the same amount of money in their piggy banks.

29. A hockey team played a total of 167 games last season. The number of games they won was 17 more than five times of the number of games they lost.

Write and solve an equation to answer the following questions.

The team lost ☐ games. The team won ☐ games.

30. A hockey team played a total of 117 games last season. The number of games they won was 13 more than three times of the number of games they lost.

Write and solve an equation to answer the following questions.

The team lost ☐ games. The team won ☐ games.

31. A rectangle's perimeter is 278 ft. Its length is 4 ft longer than four times its width. Use an equation to find the rectangle's length and width.

Its width is ☐.

Its length is ☐.

32. A rectangle's perimeter is 226 ft. Its length is 1 ft longer than three times its width. Use an equation to find the rectangle's length and width.

Its width is .

Its length is .

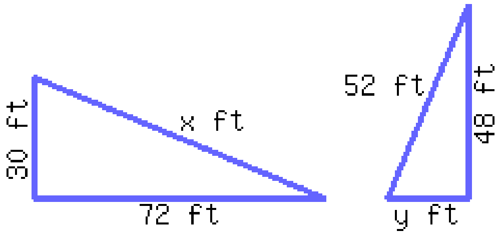
33. Briana has saved $45.00 in her piggy bank, and she decided to start spending them. She spends $5.00 every 7 days. After how many days will she have $30.00 left in the piggy bank?

Briana will have $30.00 left in her piggy bank after days.

34. Maygen has saved $49.00 in her piggy bank, and she decided to start spending them. She spends $5.00 every 6 days. After how many days will she have $29.00 left in the piggy bank?

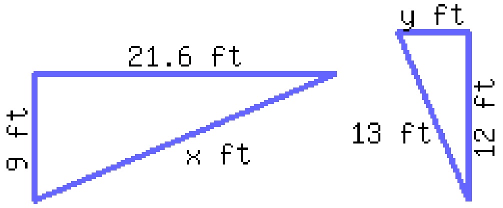
Maygen will have $29.00 left in her piggy bank after days.

35. The following two triangles are similar to each other. Find the length of the missing side.



The length of the side labeled x is and the length of the side labeled y is .

36. The following two triangles are similar to each other. Find the length of the missing side.



The length of the side labeled x is and the length of the side labeled y is .

37. A restaurant used 639.4 lb of vegetable oil in 23 days. At this rate, 1306.6 lb of oil will last how many days?

The restaurant will use 1306.6 lb of vegetable oil in days.

38. A restaurant used 914.5 lb of vegetable oil in 31 days. At this rate, 1209.5 lb of oil will last how many days?

The restaurant will use 1209.5 lb of vegetable oil in days.

39. Use a linear equation to solve the word problem.

Massage Heaven and Massage You are competitors. Massage Heaven has 6500 registered customers, and it gets approximately 550 newly registered customers every month. Massage You has 8500 registered customers, and it gets approximately 450 newly registered customers every month. How many months would it take Massage Heaven to catch up with Massage You in the number of registered customers?

These two companies would have approximately the same number of registered customers months later.

40. Use a linear equation to solve the word problem.

Two truck rental companies have different rates. V-Haul has a base charge of \$70.00, plus \$0.70 per mile. W-Haul has a base charge of \$60.40, plus \$0.80 per mile. For how many miles would these two companies charge the same amount?

If a driver drives miles, those two companies would charge the same amount of money.

41. A rectangle's perimeter is 134 ft. Its length is 5 ft shorter than three times its width. Use an equation to find the rectangle's length and width.

Its width is .

Its length is .

42. A rectangle's perimeter is 178 ft. Its length is 2 ft longer than two times its width. Use an equation to find the rectangle's length and width.

Its width is .

Its length is .

CHAPTER 4

Graphing Lines

4.1 Cartesian Coordinates

When we model relationships with graphs, we use the **Cartesian coordinate system**. This section covers the basic vocabulary and ideas that come with the Cartesian coordinate system.

The Cartesian coordinate system identifies the location of every point in a plane. Basically, the system gives every point in a plane its own "address" in relation to a starting point. We'll use a street grid as an analogy. Here is a map with Carl's home at the center. The map also shows some nearby businesses. Assume each unit in the grid represents one city block.

René Descartes. Several conventions used in mathematics are attributed to (or at least named after) René Descartes[a]. The Cartesian coordinate system is one of these.

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[a] en.wikipedia.org/wiki/René_Descartes

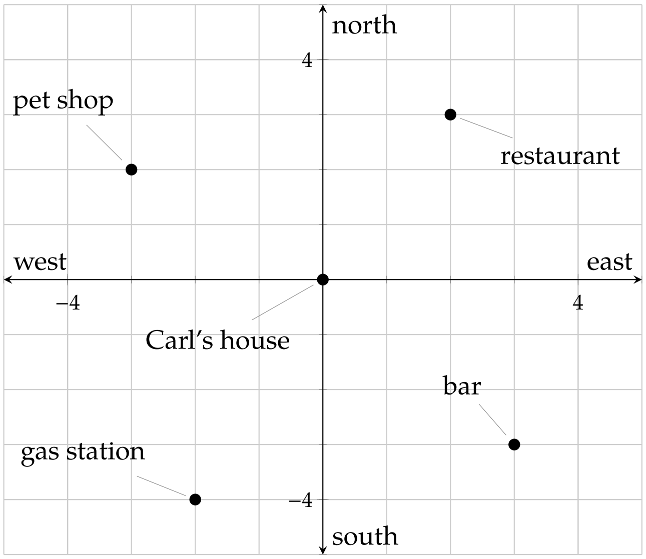

Figure 4.1.2: Carl's neighborhood

If Carl has an out-of-town guest who asks him how to get to the restaurant, Carl could say:

> "First go 2 blocks east, then go 3 blocks north."

Carl uses two numbers to locate the restaurant. In the Cartesian coordinate system, these numbers are called **coordinates** and they are written as the **ordered pair** $(2, 3)$. The first coordinate, 2, represents distance traveled from Carl's house to the east (or to the right horizontally on the graph). The second coordinate, 3, represents distance to the north (up vertically on the graph).

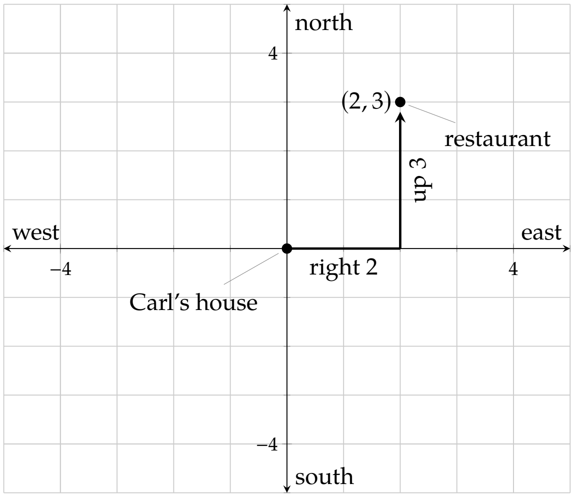

Figure 4.1.3: Carl's path to the restaurant

Alternatively, to travel from Carl's home to the pet shop, he would go 3 blocks west, and then 2 blocks north.

In the Cartesian coordinate system, the *positive* directions are to the *right* horizontally and *up* vertically. The *negative* directions are to the *left* horizontally and *down* vertically. So the pet shop's Cartesian coordinates are $(-3, 2)$.

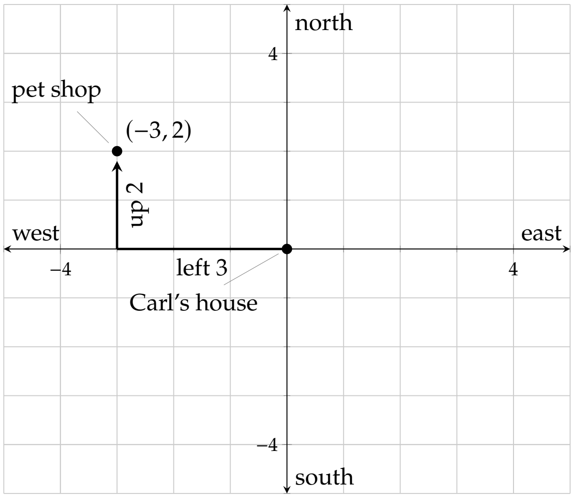

Figure 4.1.4: Carl's path to the pet shop

Remark 4.1.5. It's important to know that the order of Cartesian coordinates is (horizontal, vertical). This idea of communicating horizontal information *before* vertical information is consistent throughout most of mathematics.

Checkpoint 4.1.6. Use Figure 4.1.2 to answer the following questions.

a. What are the coordinates of the bar? ☐

b. What are the coordinates of the gas station? ☐

c. What are the coordinates of Carl's house? ☐

Traditionally, the variable x represents numbers on the horizontal axis, so it is called the **x-axis**. The variable y represents numbers on the vertical axis, so it is called the **y-axis**. The axes meet at the point $(0, 0)$, which is called the **origin**. Every point in the plane is represented by an **ordered pair**, (x, y).

In a Cartesian coordinate system, the map of Carl's neighborhood would look like this:

Notation Issue: Coordinates or Interval?. Unfortunately, the notation for an ordered pair looks exactly like interval notation for an open interval. *Context* will help you understand if $(2, 3)$ indicates the point 2 units right of the origin and 3 units up, or if $(2, 3)$ indicates the interval of all real numbers between 2 and 3.

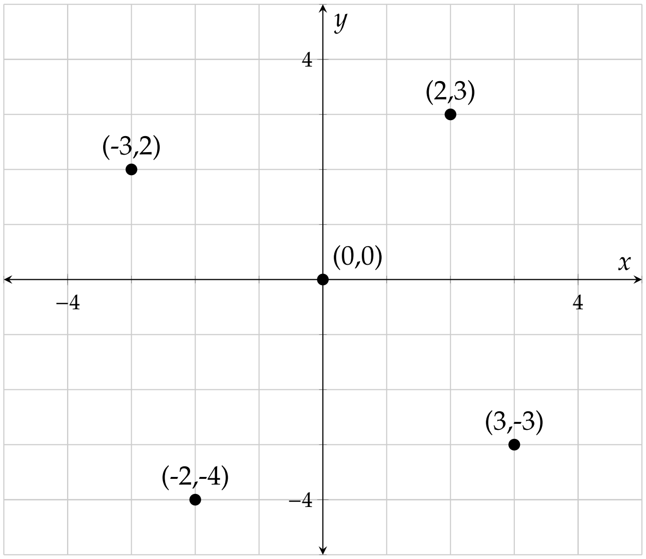

Figure 4.1.7: Carl's Neighborhood in a Cartesian Coordinate System

Definition 4.1.8 Cartesian Coordinate System. A Cartesian coordinate system[1] is a coordinate system that specifies each point uniquely in a plane by a pair of numerical coordinates, which are the signed (positive/negative) distances to the point from two fixed perpendicular directed lines, measured in the same unit of length. Those two reference lines are called the **horizontal axis** and **vertical axis**, and the point where they meet is the **origin**. The horizontal and vertical axes are often called the **x-axis** and **y-axis**.

The plane based on the x-axis and y-axis is called a **coordinate plane**. The ordered pair used to locate a point is called the point's **coordinates**, which consists of an **x-coordinate** and a **y-coordinate**. For example, for the point $(1, 2)$, its x-coordinate is 1, and its y-coordinate is 2. The origin has coordinates $(0, 0)$.

A Cartesian coordinate system is divided into four **quadrants**, as shown in Figure 4.1.9. The quadrants are traditionally labeled with Roman numerals.

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[1] en.wikipedia.org/wiki/Cartesian_coordinate_system

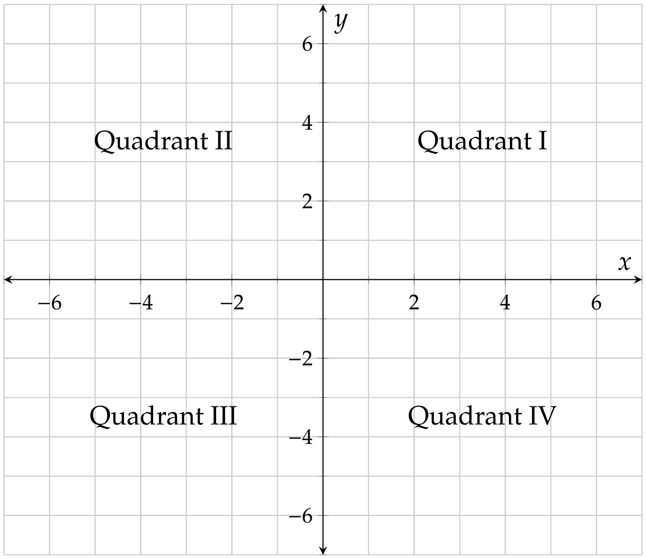
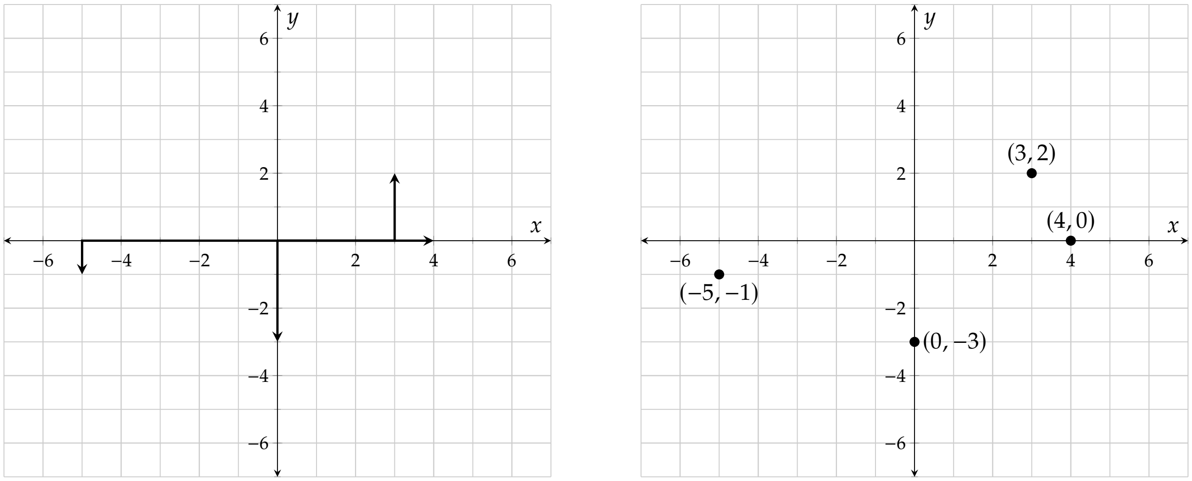

y
6
4
2
x
−6
−4
−2
2
4
6
−2
−4
−6
Quadrant II
Quadrant I
Quadrant III
Quadrant IV

Figure 4.1.9: A Cartesian grid with four quadrants marked

Example 4.1.10 On paper, sketch a Cartesian coordinate system with units, and then plot the following points: $(3, 2), (-5, -1), (0, -3), (4, 0)$.

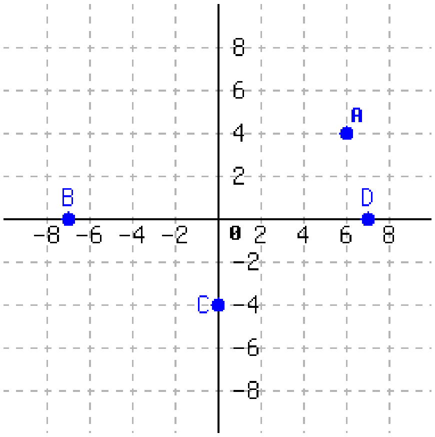
Explanation.



Exercises

Identifying Coordinates Locate each point in the graph:

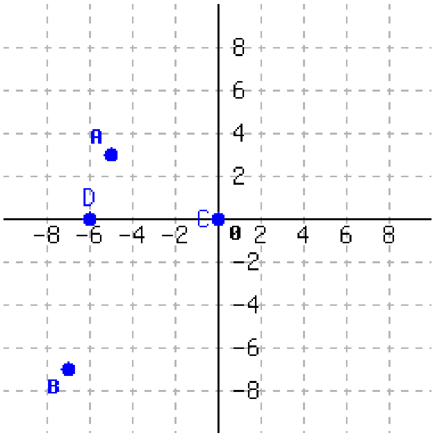
1.



Write each point's position as an ordered pair, like $(1, 2)$.

$A =$ ______ $B =$ ______
$C =$ ______ $D =$ ______

2.



Write each point's position as an ordered pair, like $(1, 2)$.

$A =$ ______ $B =$ ______
$C =$ ______ $D =$ ______

Creating Sketches of Graphs

3. Sketch the points $(8, 2)$, $(5, 5)$, $(-3, 0)$, and $(2, -6)$ on a Cartesian plane.

4. Sketch the points $(1, -4)$, $(-3, 5)$, $(0, 4)$, and $(-2, -6)$ on a Cartesian plane.

5. Sketch the points $(208, -50)$, $(97, 112)$, $(-29, 103)$, and $(-80, -172)$ on a Cartesian plane.

6. Sketch the points $(110, 38)$, $(-205, 52)$, $(-52, 125)$, and $(-172, -80)$ on a Cartesian plane.

7. Sketch the points $(5.5, 2.7)$, $(-7.3, 2.75)$, $\left(-\frac{10}{3}, \frac{1}{2}\right)$, and $\left(-\frac{28}{5}, -\frac{29}{4}\right)$ on a Cartesian plane.

8. Sketch the points $(1.9, -3.3)$, $(-5.2, -8.11)$, $\left(\frac{7}{11}, \frac{15}{2}\right)$, and $\left(-\frac{16}{3}, \frac{19}{5}\right)$ on a Cartesian plane.

9. Sketch a Cartesian plane and shade the quadrants where the x-coordinate is negative.

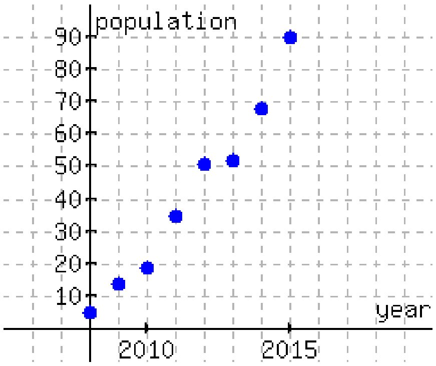
10. Sketch a Cartesian plane and shade the quadrants where the y-coordinate is positive.

11. Sketch a Cartesian plane and shade the quadrants where the x-coordinate has the same sign as the y-coordinate.

12. Sketch a Cartesian plane and shade the quadrants where the x-coordinate and the y-coordinate have opposite signs.

Cartesian Plots in Context

13. This graph gives the minimum estimates of the wolf population in Washington from 2008 through 2015.
(Source: `http://wdfw.wa.gov/publications/01793/wdfw01793.pdf`)

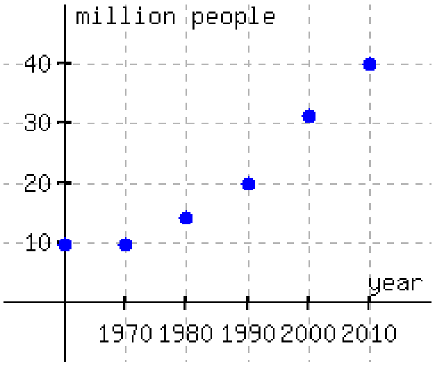


What are the Cartesian coordinates for the point representing the year 2010?

Between 2010 and 2011, the wolf population grew by wolves.

List at least three ordered pairs in the graph.

14. Here is a graph of the foreign-born US population (in millions) during Census years 1960 to 2010.
(Source: `http://www.pewhispanic.org/2015/09/28/chapter-5-u-s-foreign-born-population-trends/`.)



What are the Cartesian coordinates for the point representing the year 1970?

Between 1970 and 1990, the US population that is foreign-born increased by million people.

List at least three ordered pairs in the graph.

Regions in the Cartesian Plane

15. The point $(1, -10)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

The point $(4, 2)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

The point $(-6, 8)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

The point $(-10, -2)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

16. The point $(4, 4)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

The point $(-7, -10)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

The point $(4, -9)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

The point $(-9, 10)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

17. Assume the point (x, y) if in Quadrant II, locate the following points:

The point $(-x, y)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

The point $(x, -y)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

The point $(-x, -y)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

18. Assume the point (x, y) if in Quadrant IV, locate the following points:

The point $(-x, y)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

The point $(x, -y)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

The point $(-x, -y)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

19. Answer the following questions on the coordinate system:

For the point (x, y), if $x > 0$ and $y > 0$, then the point is in/on (☐ Quadrant I ☐ Quadrant II ☐ Quadrant III ☐ Quadrant IV ☐ the x-axis ☐ the y-axis) .

For the point (x, y), if $x > 0$ and $y < 0$, then the point is in/on (☐ Quadrant I ☐ Quadrant II ☐ Quadrant III ☐ Quadrant IV ☐ the x-axis ☐ the y-axis) .

For the point (x, y), if $x < 0$ and $y < 0$, then the point is in/on (☐ Quadrant I ☐ Quadrant II ☐ Quadrant III ☐ Quadrant IV ☐ the x-axis ☐ the y-axis) .

For the point (x, y), if $x < 0$ and $y > 0$, then the point is in/on (☐ Quadrant I ☐ Quadrant II ☐ Quadrant III ☐ Quadrant IV ☐ the x-axis ☐ the y-axis) .

For the point (x, y), if $y = 0$, then the point is in/on (☐ Quadrant I ☐ Quadrant II ☐ Quadrant III ☐ Quadrant IV ☐ the x-axis ☐ the y-axis) .

For the point (x, y), if $x = 0$, then the point is in/on (☐ Quadrant I ☐ Quadrant II ☐ Quadrant III ☐ Quadrant IV ☐ the x-axis ☐ the y-axis) .

Plotting Points and Choosing a Scale

20. What would be the difficulty with trying to plot (12, 4), (13, 5), and (310, 208) all on the same graph?

21. The points (3, 5), (5, 6), (7, 7), and (9, 8) all lie on a straight line. What can go wrong if you make a plot of a Cartesion plane with these points marked, and you don't have tick marks that are evenly spaced apart?

4.2 Graphing Equations

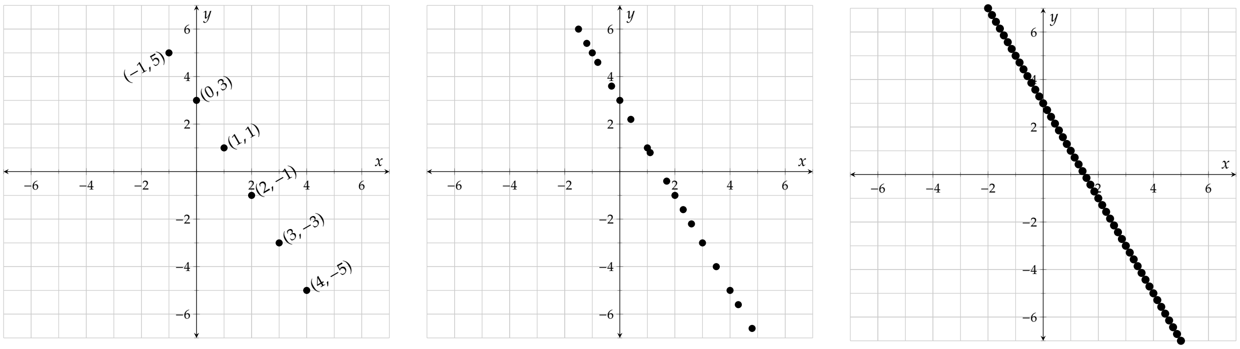
We have graphed *points* in a coordinate system, and now we will graph *lines* and *curves*.

A **graph** of an equation is a picture of that equation's solution set. For example, the graph of $y = -2x + 3$ is shown in Figure 4.1c. The graph plots the ordered pairs whose coordinates make $y = -2x + 3$ true. Table 4.2.2 shows a few points that make the equation true.

$y = -2x + 3$	(x, y)
$5 \overset{\checkmark}{=} -2(-1) + 3$	$(-1, 5)$
$3 \overset{\checkmark}{=} -2(0) + 3$	$(0, 3)$
$1 \overset{\checkmark}{=} -2(1) + 3$	$(1, 1)$
$-1 \overset{\checkmark}{=} -2(2) + 3$	$(2, -1)$
$-3 \overset{\checkmark}{=} -2(3) + 3$	$(3, -3)$
$-5 \overset{\checkmark}{=} -2(4) + 3$	$(4, -5)$

Table 4.2.2

Table 4.2.2 tells us that the points $(-1, 5)$, $(0, 3)$, $(1, 1)$, $(2, -1)$, $(3, -3)$, and $(4, -5)$ are all solutions to the equation $y = -2x + 3$, and so they should all be shaded as part of that equation's graph. You can see them in Figure 4.1a. But there are many more points that make the equation true. More points are plotted in Figure 4.1b. Even more points are plotted in Figure 4.1c — so many, that together the points look like a straight line.



(a) A few points…

(b) More points…

(c) So many points it looks like a straight line

Figure 4.2.2: Graphs of the Equation $y = -2x + 3$

Remark 4.2.3. The graph of an equation shades all the points (x, y) that make the equation true once the x- and y-values are substituted in. Typically, there are *so many* points shaded, that the final graph appears to be a continuous line or curve that you could draw with one stroke of a pen.

Checkpoint 4.2.4. The point $(4, -5)$ is on the graph in Figure 4.2.3.(c). What happens when you substitute these values into the equation $y = -2x + 3$?

$$\begin{aligned} y &= -2x + 3 \\ \_\_ &= \_\_\_\_\_\_\_\_\_\_\_\_ \end{aligned}$$

This equation is (☐ true ☐ false) .

Checkpoint 4.2.5. Decide whether $(5,-2)$ and $(-10,-7)$ are on the graph of the equation $y=-\frac{3}{5}x+1$.

At $(5,-2)$:

$$\begin{aligned} y &= -\tfrac{3}{5}x+1 \\ \underline{\hphantom{xx}} &= \underline{\hphantom{xxxxxxxxxxxxxxxxxxxx}} \end{aligned}$$

This equation is (☐ true ☐ false) and $(5,-2)$ is (☐ part of ☐ not part of) the graph of $y=-\frac{3}{5}x+1$.

At $(-10,-7)$:

$$\begin{aligned} y &= -\tfrac{3}{5}x+1 \\ \underline{\hphantom{xx}} &= \underline{\hphantom{xxxxxxxxxxxxxxxxxxxx}} \end{aligned}$$

This equation is (☐ true ☐ false) and $(-10,-7)$ is (☐ part of ☐ not part of) the graph of $y=-\frac{3}{5}x+1$.

Explanation. If the point $(5,-2)$ is on $y=-\frac{3}{5}x+1$, once we substitute $x=5$ and $y=-2$ into the line's equation, the equation should be true. Let's try:

$$\begin{aligned} y &= -\frac{3}{5}x+1 \\ -2 &\overset{?}{=} -\frac{3}{5}(5)+1 \\ -2 &\overset{\checkmark}{=} -3+1 \end{aligned}$$

Because this last equation is true, we can definitively say that $(5,-2)$ is on the graph of $y=-\frac{3}{5}x+1$.

However if we substitute $x=-10$ and $y=-7$ into the equation, it leads to $-7=7$, which is false. This definitively tells us that $(-10,-7)$ is not on the graph.

So to make our own graph of an equation with two variables x and y, we can choose some reasonable x-values, then calculate the corresponding y-values, and then plot the (x,y)-pairs as points. For many (not-so-complicated) algebraic equations, connecting those points with a smooth curve will produce an excellent graph.

Example 4.2.6 Let's plot a graph for the equation $y=-2x+5$. We use a table to organize our work:

x	$y=-2x+5$	Point
-2		
-1		
0		
1		
2		

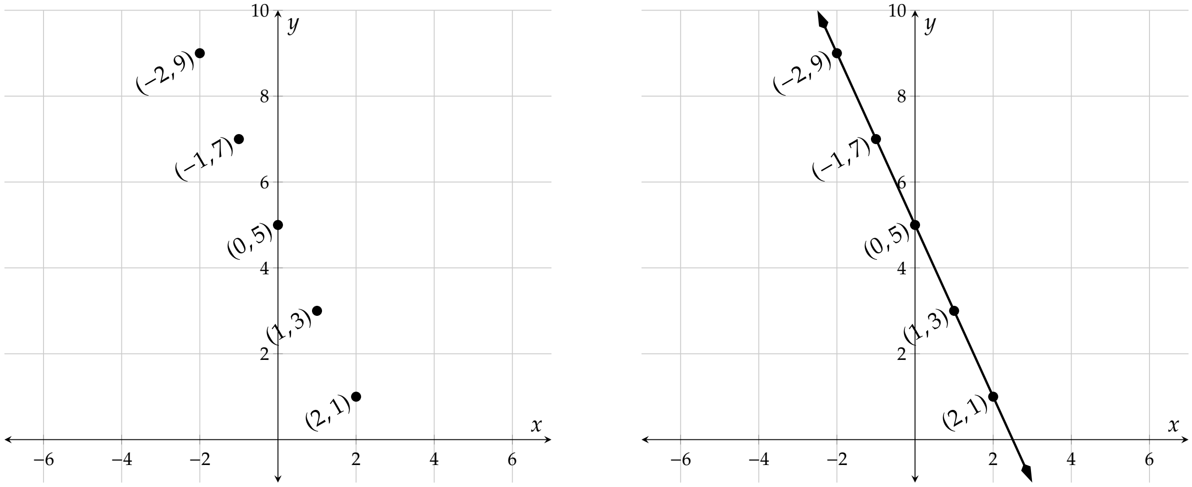
(a) Set up the table

x	$y=-2x+5$	Point
-2	$-2(-2)+5=9$	$(-2,9)$
-1	$-2(-1)+5=7$	$(-1,7)$
0	$-2(0)+5=5$	$(0,5)$
1	$-2(1)+5=3$	$(1,3)$
2	$-2(2)+5=1$	$(2,1)$

(b) Complete the table

Figure 4.2.6: Making a table for $y=-2x+5$

We use points from the table to graph the equation. First, plot each point carefully. Then, connect the points with a smooth curve. Here, the curve is a straight line. Lastly, we can communicate that the graph extends further by sketching arrows on both ends of the line.



(a) Use points from the table

(b) Connect the points in whatever pattern is apparent

Figure 4.2.6: Graphing the Equation $y = -2x + 5$

Remark 4.2.7. Note that our choice of x-values is arbitrary. As long as we determine the coordinates of enough points to indicate the behavior of the graph, we may choose whichever x-values we like. For simpler calculations, people often start with the integers from -2 to 2. However sometimes the equation has context that suggests using other x-values, as in the next examples.

Example 4.2.8 The gas tank in Sofia's car holds 14 gal of fuel. Over the course of a long road trip, her car uses fuel at an average rate of $0.032\,\frac{\text{gal}}{\text{mi}}$. If Sofia fills the tank at the beginning of a long trip, then the amount of fuel remaining in the tank, y, after driving x miles is given by the equation $y = 14 - 0.032x$. Make a suitable table of values and graph this equation.

Explanation. Choosing x-values from -2 to 2, as in our previous example, wouldn't make sense here. Sofia cannot drive a negative number of miles, and any long road trip is longer than 2 miles. So in this context, choose x-values that reflect the number of miles Sofia might drive in a day.

x	$y = 14 - 0.032x$	Point
20	13.36	$(20, 13.36)$
50	12.4	$(50, 12.4)$
80	11.44	$(80, 11.44)$
100	10.8	$(100, 10.8)$
200	7.6	$(200, 7.6)$

Table 4.2.9: Make the table

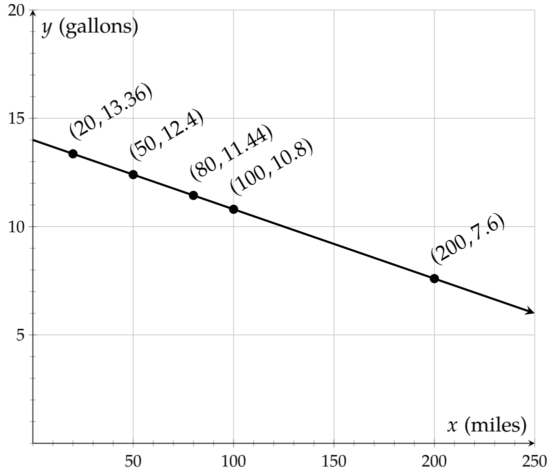


Figure 4.2.10: Make the graph

In the graph from Example 4.2.8, notice how both axes indicate units that help describe the meaning of each variable. Whenever a graph has real-world context, be sure to label both axes clearly with both variable name (like x) and units.

Example 4.2.11 Plot a graph for the equation $y = \frac{4}{3}x - 4$.

Explanation. This equation doesn't have any context to help us choose x-values for a table. We could use x-values like -2, -1, and so on. But note the fraction in the equation. If we use an x-value like -2, we will have to multiply by the fraction $\frac{4}{3}$ which will leave us still holding a fraction. And then we will have to subtract 4 from that fraction. Since we know that everyone can make mistakes with that kind of arithmetic, maybe we can avoid it with a more wise selection of x-values.

If we use onnly multiples of 3 for the x-values, then multiplying by $\frac{4}{3}$ will leave us with an integer, which will be easy to subtract 4 from. So we decide to use -6, -3, 0, 3, and 6 for x.

x	$y = \frac{4}{3}x - 4$	Point
-6		
-3		
0		
3		
6		

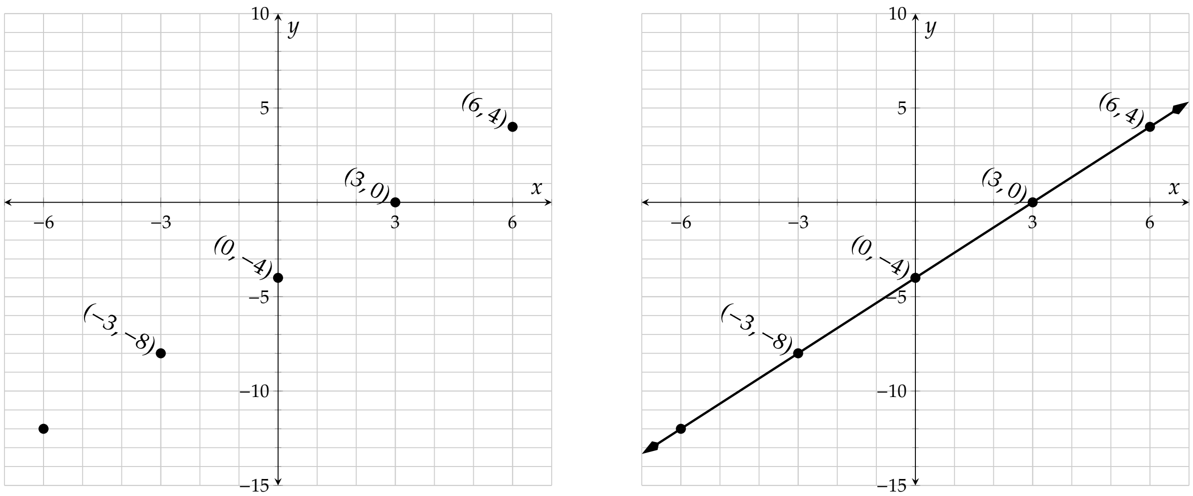
(a) Set up the table

x	$y = \frac{4}{3}x - 4$	Point
-6	$\frac{4}{3}(-6) - 2 = -12$	$(-6, -12)$
-3	$\frac{4}{3}(-3) - 2 = -8$	$(-3, -8)$
0	$\frac{4}{3}(0) - 2 = -4$	$(0, -4)$
3	$\frac{4}{3}(3) - 2 = 0$	$(3, 0)$
6	$\frac{4}{3}(6) - 2 = 4$	$(6, 4)$

(b) Complete the table

Figure 4.2.11: Making a table for $y = \frac{4}{3}x - 4$

We use points from the table to graph the equation. First, plot each point carefully. Then, connect the points with a smooth curve. Here, the curve is a straight line. Lastly, we can communicate that the graph extends further by sketching arrows on both ends of the line.



(a) Use points from the table

(b) Connect the points in whatever pattern is apparent

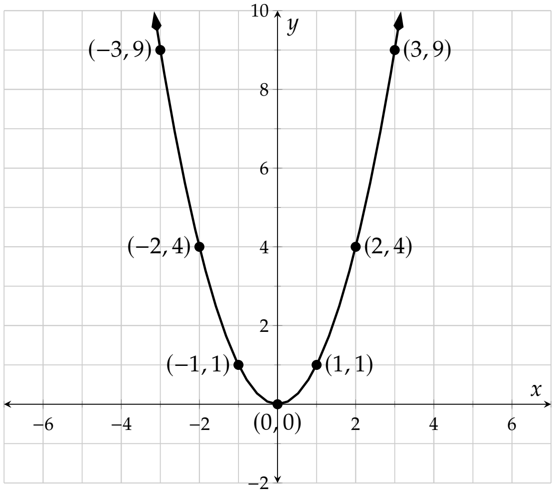
Figure 4.2.11: Graphing the Equation $y = \frac{4}{3}x - 4$

Not all equations make a straight line once they are plotted.

Example 4.2.12 Build a table and graph the equation $y = x^2$. Use x-values from −3 to 3.

Explanation.

x	$y = x^2$	Point
-3	$(-3)^2 = 9$	$(-3, 9)$
-2	$(-2)^2 = 4$	$(-2, 4)$
-1	$(-1)^2 = 1$	$(-1, 1)$
0	$(0)^2 = 0$	$(0, 0)$
1	$(1)^2 = 1$	$(0, 1)$
2	$(2)^2 = 4$	$(2, 4)$
3	$(3)^2 = 9$	$(3, 9)$



In this example, the points do not fall on a straight line. Many algebraic equations have graphs that are non-linear, where the points do not fall on a straight line. Since each x-value corresponds to a single y-value (the square of x) we connected the points with a smooth curve, sketching from left to right.

Exercises

Testing Points as Solutions Consider the equation

1. $y = 8x + 7$

   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

   ☐ $(10, 90)$ ☐ $(-4, -25)$
   ☐ $(0, 10)$ ☐ $(-5, -33)$

2. $y = 9x + 3$

   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

   ☐ $(0, 4)$ ☐ $(-5, -42)$
   ☐ $(-3, -24)$ ☐ $(4, 41)$

3. $y = -2x - 2$

   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

   ☐ $(8, -18)$ ☐ $(-6, 13)$
   ☐ $(-2, 2)$ ☐ $(0, -2)$

4. $y = -10x - 5$

   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

   ☐ $(3, -35)$ ☐ $(0, -5)$
   ☐ $(-6, 56)$ ☐ $(-4, 35)$

5. $y = \frac{2}{3}x - 3$

   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

   ☐ $(-15, -13)$ ☐ $(0, 0)$
   ☐ $(6, 1)$ ☐ $(-9, -6)$

6. $y = \frac{2}{3}x - 5$

   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

   ☐ $(3, -3)$ ☐ $(-15, -15)$
   ☐ $(0, 0)$ ☐ $(-9, -10)$

7. $y = -\frac{3}{4}x - 3$

   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

   ☐ $(-12, 6)$ ☐ $(20, -15)$
   ☐ $(0, -3)$ ☐ $(-16, 14)$

8. $y = -\frac{3}{4}x - 5$

   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

   ☐ $(12, -11)$ ☐ $(0, -5)$
   ☐ $(-20, 10)$ ☐ $(-16, 9)$

Tables for Equations Make a table for the equation.

9. The first row is an example.

x	$y = -x + 6$	Points
-3	9	$(-3, 9)$
-2		
-1		
0		
1		
2		

10. The first row is an example.

x	$y = -x + 7$	Points
-3	10	$(-3, 10)$
-2		
-1		
0		
1		
2		

11. The first row is an example.

x	$y = 5x + 1$	Points
-3	-14	$(-3, -14)$
-2	______	____________
-1	______	____________
0	______	____________
1	______	____________
2	______	____________

12. The first row is an example.

x	$y = 6x + 8$	Points
-3	-10	$(-3, -10)$
-2	______	____________
-1	______	____________
0	______	____________
1	______	____________
2	______	____________

13. The first row is an example.

x	$y = -5x + 4$	Points
-3	19	$(-3, 19)$
-2	__________	____________________
-1	__________	____________________
0	__________	____________________
1	__________	____________________
2	__________	____________________

14. The first row is an example.

x	$y = -5x + 1$	Points
-3	16	$(-3, 16)$
-2	__________	____________________
-1	__________	____________________
0	__________	____________________
1	__________	____________________
2	__________	____________________

15. The first row is an example.

x	$y = \frac{3}{8}x + 5$	Points
-24	-4	$(-24, -4)$
-16	______	____________
-8	______	____________
0	______	____________
8	______	____________
16	______	____________

16. The first row is an example.

x	$y = \frac{3}{4}x - 7$	Points
-12	-16	$(-12, -16)$
-8	______	____________
-4	______	____________
0	______	____________
4	______	____________
8	______	____________

17. The first row is an example.

x	$y = -\frac{5}{2}x + 2$	Points
-6	17	$(-6, 17)$
-4	______	____________
-2	______	____________
0	______	____________
2	______	____________
4	______	____________

18. The first row is an example.

x	$y = -\frac{5}{8}x + 10$	Points
-24	25	$(-24, 25)$
-16	______	____________
-8	______	____________
0	______	____________
8	______	____________
16	______	____________

19.

x	$y = 8x$

20.

x	$y = 12x$

21.

x	$y = 8x + 2$

22.

x	$y = 10x - 4$

23.

x	$y = \frac{19}{2}x - 8$

24.

x	$y = \frac{3}{4}x - 6$

25.

x	$y = -\frac{11}{19}x - 4$

26.

x	$y = \frac{15}{8}x - 1$

Cartesian Plots in Context

27. A certain water heater will cost you \$900 to buy and have installed. This water heater claims that its operating expense (money spent on electricity or gas) will be about \$31 per month. According to this information, the equation $y = 900 + 31x$ models the total cost of the water heater after x months, where y is in dollars. Make a table of at least five values and plot a graph of this equation.

28. You bought a new Toyota Corolla for \$18,600 with a zero interest loan over a five-year period. That means you'll have to pay \$310 each month for the next five years (sixty months) to pay it off. According to this information, the equation $y = 18600 - 310x$ models the loan balance after x months, where y is in dollars. Make a table of at least five values and plot a graph of this equation. Make sure to include a data point representing when you will have paid off the loan.

29. The pressure inside a full propane tank will rise and fall if the ambient temperature rises and falls. The equation $P = 0.1963(T + 459.67)$ models this relationship, where the temperature T is measured in °F and the pressure and the pressure P is measured in $\frac{\text{lb}}{\text{in}^2}$. Make a table of at least five values and plot a graph of this equation. Make sure to use T-values that make sense in context.

30. A beloved coworker is retiring and you want to give her a gift of week-long vacation rental at the coast that costs \$1400 for the week. You might end up paying for it yourself, but you ask around to see if the other 29 office coworkers want to split the cost evenly. The equation $y = \frac{1400}{x}$ models this situation, where x people contribute to the gift, and y is the dollar amount everyone contributes. Make a table of at least five values and plot a graph of this equation. Make sure to use x-values that make sense in context.

Graphs of Equations

31. Create a table of ordered pairs and then make a plot of the equation $y = 2x + 3$.

32. Create a table of ordered pairs and then make a plot of the equation $y = 3x + 5$.

33. Create a table of ordered pairs and then make a plot of the equation $y = -4x + 1$.

34. Create a table of ordered pairs and then make a plot of the equation $y = -x - 4$.

35. Create a table of ordered pairs and then make a plot of the equation $y = \frac{5}{2}x$.

36. Create a table of ordered pairs and then make a plot of the equation $y = \frac{4}{3}x$.

37. Create a table of ordered pairs and then make a plot of the equation $y = -\frac{2}{5}x - 3$.

38. Create a table of ordered pairs and then make a plot of the equation $y = -\frac{3}{4}x + 2$.

39. Create a table of ordered pairs and then make a plot of the equation $y = x^2 + 1$.

40. Create a table of ordered pairs and then make a plot of the equation $y = (x-2)^2$. Use x-values from 0 to 4.

41. Create a table of ordered pairs and then make a plot of the equation $y = -3x^2$.

42. Create a table of ordered pairs and then make a plot of the equation $y = -x^2 - 2x - 3$.

4.3 Exploring Two-Variable Data and Rate of Change

This section is about examining data that has been plotted on a Cartesian coordinate system, and then making observations. In some cases, we'll be able to turn those observations into useful mathematical calculations.

4.3.1 Modeling data with two variables

Using mathematics, we can analyze real data from the world around us. We can use what we discover to better understand the world, and sometimes to make predictions. Here's an example of data about the economic situation in the US:

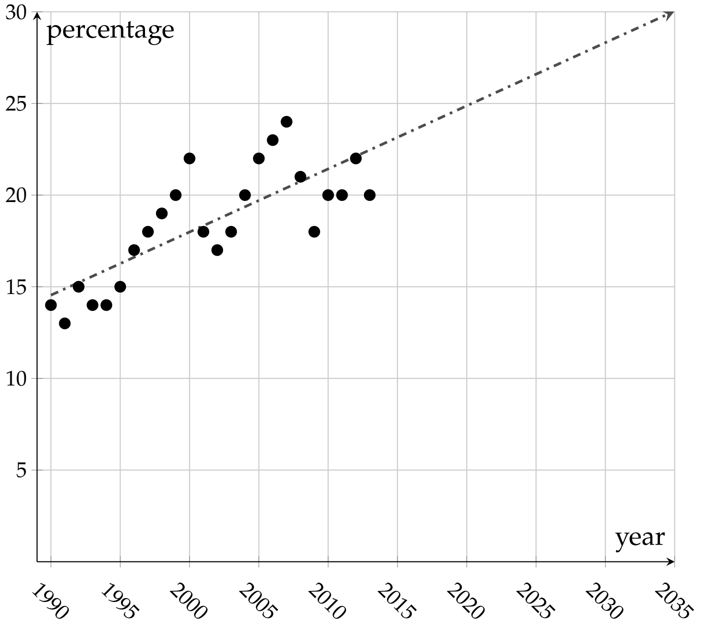


Figure 4.3.2: Share of all income held by the top 1 %, United States, 1990–2013 (www.epi.org)

If this trend continues, what percentage of all income will the top 1 % have in the year 2030? If we model data in the chart with the trend line, we can estimate the value to be 28.6 %. This is one way math is used in real life.

Does that trend line have an equation like those we looked at in Section 4.2? Is it even correct to look at this data set and decide that a straight line is a good model? These are some of the questions we want to consider as we begin this section. The answers will evolve through the next several sections.

4.3.2 Patterns in Tables

Example 4.3.3 Find a pattern in each table. What is the missing entry in each table? Can you describe each pattern in words and/or mathematics?

black	white
big	small
short	tall
few	

USA	Washington
UK	London
France	Paris
Mexico	

1	2
2	4
3	6
5	

Figure 4.3.4: Patterns in 3 tables

Explanation.

black	white
big	small
short	tall
few	*many*

USA	Washington
UK	London
France	Paris
Mexico	*Mexico City*

1	2
2	4
3	6
5	*10*

Figure 4.3.5: Patterns in 3 tables

First table Each word on the right has the opposite meaning of the word to its left.

Second table Each city on the right is the capital of the country to its left.

Third table Each number on the right is double the number to its left.

We can view each table as assigning each input in the left column a corresponding output in the right column. In the first table, for example, when the input "big" is on the left, the output "small" is on the right. The first table's function is to output a word with the opposite meaning of each input word. (This is not a numerical example.)

The third table *is* numerical. And its function is to take a number as input, and give twice that number as its output. Mathematically, we can describe the pattern as "$y = 2x$," where x represents the input, and y represents the output. Labeling the table mathematically, we have Table 4.3.6.

x (input)	y (output)
1	2
2	4
3	6
5	10
10	20
Pattern: $y = 2x$	

Table 4.3.6: Table with a mathematical pattern

The equation $y = 2x$ summarizes the pattern in the table. For each of the following tables, find an equation that describes the pattern you see. Numerical pattern recognition may or may not come naturally for you. Either way, pattern recognition is an important mathematical skill that anyone can develop. Solutions for these exercises provide some ideas for recognizing patterns.

Checkpoint 4.3.7. Write an equation in the form $y = \ldots$ suggested by the pattern in the table.

x	y
0	10
1	11
2	12
3	13

Explanation. Looking for a similar relationship in each row is one approach to pattern recognition. Here, the y-value in each row is 10 greater than its corresponding x-value. So the equation $y = x + 10$ describes the pattern. Of course, there are more complicated patterns to explore, as we'll see in the next exercise.

Checkpoint 4.3.8. Write an equation in the form $y = \ldots$ suggested by the pattern in the table.

x	y
0	-1
1	2
2	5
3	8

Explanation. The relationship between x and y in each row is not as clear here. Another popular approach for finding patterns: in each column, consider how the values change from one row to the next. From row to row, the x-value increases by 1. Also, the y-value increases by 3 from row to row.

	x	y	
	0	-1	
$+1 \longrightarrow$	1	2	$\longleftarrow +3$
$+1 \longrightarrow$	2	5	$\longleftarrow +3$
$+1 \longrightarrow$	3	8	$\longleftarrow +3$

Since row-to-row change is always 1 for x and is always 3 for y, the rate of change from one row to another row is always the same: 3 units of y for every 1 unit of x.

We know that the output for $x = 0$ is $y = -1$. And our observation about the constant rate of change tells us that if we increase the input by x units from 0, the output should increase by $\overbrace{3 + 3 + \cdots + 3}^{x \text{ times}}$, which is $3x$. So the output would be $-1 + 3x$. So the equation is $y = 3x - 1$.

Checkpoint 4.3.9. Write an equation in the form $y = \ldots$ suggested by the pattern in the table.

x	y
0	0
1	1
2	4
3	9

Explanation. Looking for a relationship in each row here, we see that each y-value is the square of the corresponding x-value. So the equation is $y = x^2$.

What if we had tried the approach we used in the previous exercise, comparing change from row to row in each column?

$$\begin{array}{rcccl} & x & y & & \\ & 0 & 0 & & \\ +1 \rightarrow & 1 & 1 & \leftarrow & +1 \\ +1 \rightarrow & 2 & 4 & \leftarrow & +3 \\ +1 \rightarrow & 3 & 9 & \leftarrow & +5 \end{array}$$

Here, the rate of change is *not* constant from one row to the next. While the x-values are increasing by 1 from row to row, the y-values increase more and more from row to row. Notice that there is a pattern there as well? Mathematicians are fascinated by relationships that produce more complicated patterns. (We'll study more complicated patterns later.)

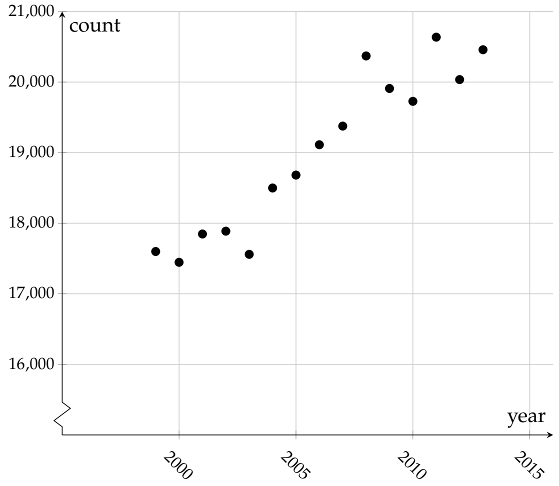
4.3.3 Rate of Change

For an hourly wage-earner, the amount of money they earn depends on how many hours they work. If a worker earns \$15 per hour, then 10 hours of work corresponds to \$150 of pay. Working *one* additional hour will change 10 hours to 11 hours; and this will cause the \$150 in pay to rise by *fifteen* dollars to \$165 in pay. Any time we compare how one amount changes (dollars earned) as a consequence of another amount changing (hours worked), we are talking about a **rate of change**.

Given a table of two-variable data, between any two rows we can compute a **rate of change**.

Example 4.3.10 The following data, given in both table and graphed form, gives the counts of invasive cancer diagnoses in Oregon over a period of time. (wonder.cdc.gov)

Year	Invasive Cancer Incidents
1999	17,599
2000	17,446
2001	17,847
2002	17,887
2003	17,559
2004	18,499
2005	18,682
2006	19,112
2007	19,376
2008	20,370
2009	19,909
2010	19,727
2011	20,636
2012	20,035
2013	20,458



What is the **rate of change** in Oregon invasive cancer diagnoses between 2000 and 2010? The total (net) change in diagnoses over that timespan is

$$19727 - 17446 = 2281.$$

Since 10 years passed (which you can calculate as 2010 − 2000), the rate of change is 2281 diagnoses per 10 years, or

$$\frac{2281\,\text{diagnoses}}{10\,\text{year}} = 228.1\,\frac{\text{diagnoses}}{\text{year}}.$$

We read that last quantity as "228.1 diagnoses per year." This rate of change means that between the years 2000 and 2010, there were 228.1 more diagnoses each year, on average. (Notice that there was no single year in that span when diagnoses increased by 228.1.)

Let's practice calculating rates of change over different timespans:

Checkpoint 4.3.11. Use the data in Example 4.3.10 to find the rate of change in Oregon invasive cancer diagnoses between 1999 and 2002. Just give the numerical value; the units are provided.

$\boxed{}\ \frac{\text{diagnoses}}{\text{year}}$

And what was the rate of change between 2003 and 2011?

$\boxed{}\ \frac{\text{diagnoses}}{\text{year}}$

Explanation. To find the rate of change between 1999 and 2002, calculate

$$\frac{17887 - 17599}{2002 - 1999} = 96.$$

To find the rate of change between 2003 and 2011, calculate

$$\frac{20636 - 17559}{2011 - 2003} = 384.625.$$

We are ready to give a formal definition for **rate of change**. Considering our work from Example 4.3.10 and Checkpoint 4.3.11, we settle on:

Definition 4.3.12 Rate of Change. If (x_1, y_1) and (x_2, y_2) are two data points from a set of two-variable data, then the **rate of change** between them is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

(The Greek letter delta, Δ, is used to represent "change in" since it is the first letter of the Greek word for "difference.")

In Example 4.3.10 and Checkpoint 4.3.11 we found three rates of change. Figure 4.3.13 highlights the three pairs of points that were used to make these calculations.

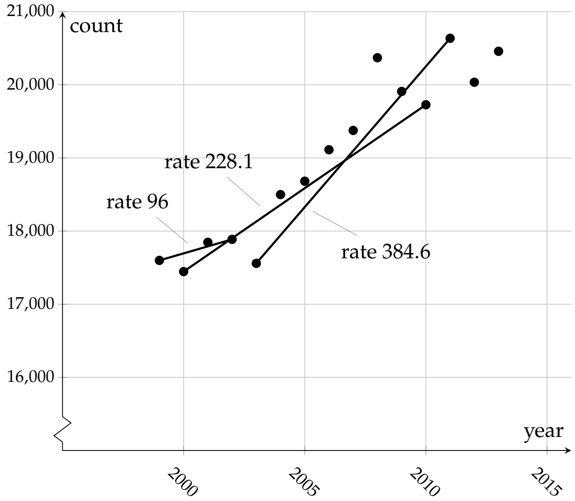


Figure 4.3.13

Note how the larger the numerical rate of change between two points, the steeper the line is that connects them. This is such an important observation, we'll put it in an official remark.

Remark 4.3.14. The rate of change between two data points is intimately related to the steepness of the line segment that connects those points.

1. The steeper the line, the larger the rate of change, and vice versa.
2. If one rate of change between two data points equals another rate of change between two different data points, then the corresponding line segments will have the same steepness.
3. When a line segment between two data points slants down from left to right, the rate of change between those points will be negative.

In the solution to Checkpoint 4.3.8, the key observation was that the **rate of change** from one row to the next was constant: 3 units of increase in y for every 1 unit of increase in x. Graphing this pattern in Figure 4.3.15, we see that every line segment here has the same steepness, so the entire graph is a line.

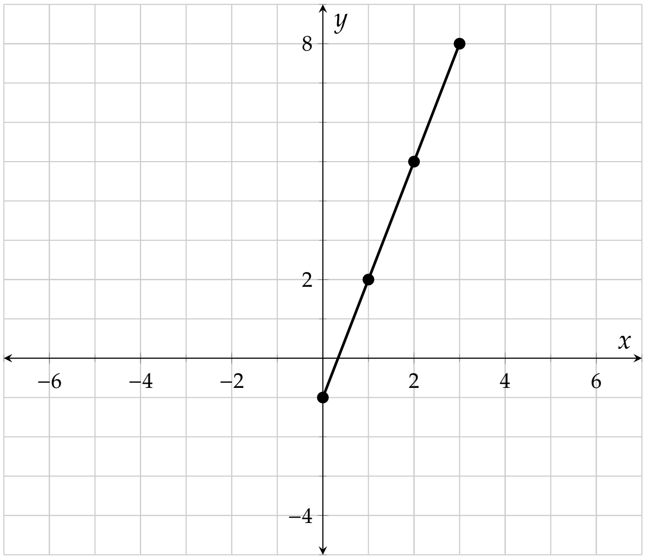


Figure 4.3.15

Whenever the rate of change is constant no matter which two (x, y)-pairs (or data pairs) are chosen from a data set, then you can conclude the graph will be a straight line *even without making the graph*. We call this kind of relationship a **linear** relationship. We'll study linear relationships in more detail throughout this chapter. Right now in this section, we feel it is important to simply identify if data has a linear relationship or not.

Checkpoint 4.3.16. Is there a linear relationship in the table?

x	y
−8	3.1
−5	2.1
−2	1.1
1	0.1

(☐ The relationship is linear ☐ The relationship is not linear)

Explanation. From one x-value to the next, the change is always 3. From one y-value to the next, the change is alwasy −1. So the rate of change is always $\frac{-1}{3} = -\frac{1}{3}$. Since the rate of change is constant, the data have a linear relationship.

Checkpoint 4.3.17. Is there a linear relationship in the table?

x	y
11	208
13	210
15	214
17	220

(☐ The relationship is linear ☐ The relationship is not linear)

Explanation. The rate of change between the first two points is $\frac{210-208}{13-11} = 1$. The rate of change between the last two points is $\frac{220-214}{17-15} = 3$. This is one way to demonstrate that the rate of change differs for different pairs of points, so this pattern is not linear.

Checkpoint 4.3.18. Is there a linear relationship in the table?

x	y
3	−2
6	−8
8	−12
12	−20

(☐ The relationship is linear ☐ The relationship is not linear)

Explanation. The changes in x from one row to the next are +3,+2, and +8. That's not a consistent pattern, but we need to consider rates of change between points. The rate of change between the first two points is $\frac{-8-(-2)}{6-3} = -2$. The rate of change between the next two points is $\frac{-12-(-8)}{8-6} = -2$. And the rate of change between the last two points is $\frac{-20-(-12)}{12-8} = -2$. So the rate of change, −2, is constant regardless of which pairs we choose. That means these pairs describe a linear relationship.

Let's return to the data that we opened the section with, in Figure 4.3.2. Is that data linear? Well, yes and no. To be completely honest, it's not linear. It's easy to pick out pairs of points where the steepness changes from one pair to the next. In other words, the points do not all fall into a single line.

However if we stand back, there does seem to be an overall upward trend that is captured by the line someone has drawn over the data. Points *on this line* do have a linear pattern. Let's estimate the rate of change between some points on this line. We are free to use any points to do this, so let's make this calculation easier by choosing points we can clearly identify on the graph: $(1991, 15)$ and $(2020, 25)$.

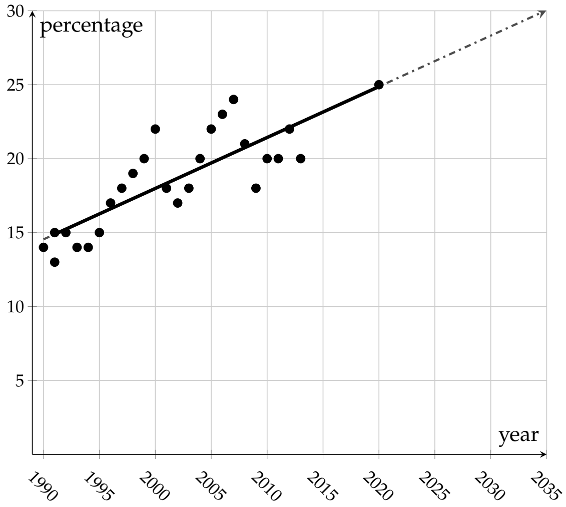


Figure 4.3.19: Share of all income held by the top 1 %, United States, 1990–2013 (www.epi.org)

The rate of change between those two points is

$$\frac{25-15}{2020-1991} = \frac{10}{29} \approx 0.3448.$$

So we might say that *on average* the rate of change expressed by this data is $0.3448\,\frac{\%}{\text{yr}}$.

Exercises

Finding Patterns Write an equation in the form $y = \ldots$ suggested by the pattern in the table.

1.

x	y
−2	−6
−1	−3
0	0
1	3
2	6

2.

x	y
3	12
4	16
5	20
6	24
7	28

3.

x	y
5	11
6	12
7	13
8	14
9	15

4.

x	y
6	10
7	11
8	12
9	13
10	14

5.

x	y
15	23
13	21
6	14
4	12
1	9

6.

x	y
17	16
6	5
14	13
19	18
1	0

7.

x	y
25	5
1	1
4	2
16	4
9	3

8.

x	y
−5	5
−2	2
−3	3
−2	2
−4	4

9.

x	y
2	4
3	9
4	16
5	25
6	36

10.

x	y
7	49
9	81
11	121
13	169
15	225

11.

x	y
43	$\frac{1}{43}$
62	$\frac{1}{62}$
84	$\frac{1}{84}$
58	$\frac{1}{58}$
1	1

12.

x	y
54	$\frac{1}{54}$
27	$\frac{1}{27}$
25	$\frac{1}{25}$
33	$\frac{1}{33}$
99	$\frac{1}{99}$

Linear Relationships Does the following table show that x and y have a linear relationship? (☐ yes ☐ no)

13.

x	y
0	93
1	100
2	107
3	114
4	121
5	128

14.

x	y
0	62
1	70
2	78
3	86
4	94
5	102

15.

x	y
5	51
6	49
7	47
8	45
9	43
10	41

16.

x	y
10	74
11	72
12	70
13	68
14	66
15	64

17.

x	y
3	19
4	27
5	43
6	75
7	139
8	267

18.

x	y
8	260
9	516
10	1028
11	2052
12	4100
13	8196

19.

x	y
0	17
1	18
2	25
3	44
4	81
5	142

20.

x	y
1	11
2	18
3	37
4	74
5	135
6	226

21.

x	y
−10	35.92
−9	36.32
−8	36.72
−7	37.12
−6	37.52
−5	37.92

22.

x	y
−2	82.57
−1	84.08
0	85.59
1	87.1
2	88.61
3	90.12

23.

x	y
5	85
10	125
12	141
16	173
17	181
18	189

24.

x	y
1	18
5	50
7	66
13	114
16	138
19	162

Calculating Rate of Change

25. This table gives population estimates for Portland, Oregon from 1990 through 2014.

Year	Population	Year	Population
1990	487849	2003	539546
1991	491064	2004	533120
1992	493754	2005	534112
1993	497432	2006	538091
1994	497659	2007	546747
1995	498396	2008	556442
1996	501646	2009	566143
1997	503205	2010	585261
1998	502945	2011	593859
1999	503637	2012	602954
2000	529922	2013	609520
2001	535185	2014	619360
2002	538803		

Find the rate of change in Portland population between 2005 and 2006. Just give the numerical value; the units are provided.

☐ $\frac{\text{people}}{\text{year}}$

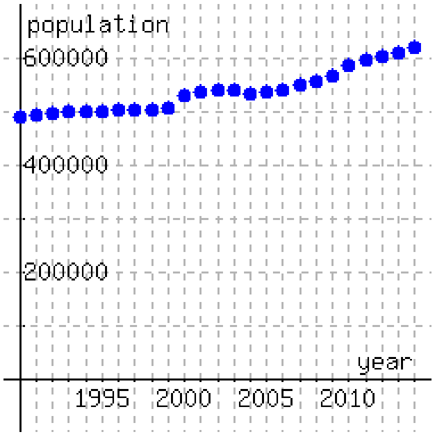
And what was the rate of change between 2008 and 2014?

☐ $\frac{\text{people}}{\text{year}}$

List all the years where there is a negative rate of change between that year and the next year.

26. This table and graph gives population estimates for Portland, Oregon from 1990 through 2014.

Year	Population	Year	Population
1990	487849	2003	539546
1991	491064	2004	533120
1992	493754	2005	534112
1993	497432	2006	538091
1994	497659	2007	546747
1995	498396	2008	556442
1996	501646	2009	566143
1997	503205	2010	585261
1998	502945	2011	593859
1999	503637	2012	602954
2000	529922	2013	609520
2001	535185	2014	619360
2002	538803		



Between what two years that are two years apart was the rate of change highest?

What was that rate of change? Just give the numerical value; the units are provided.

☐ $\frac{\text{people}}{\text{year}}$

4.4 Slope

In Section 4.3, we observed that a constant rate of change between points produces a linear relationship, whose graph is a straight line. Such a constant rate of change has a special name, **slope**, and we'll explore slope in more depth here.

4.4.1 What is slope?

When the **rate of change** from point to point never changes, those points must fall on a straight line, as in Figure 4.4.2, and there is a **linear relationship** between the variables x and y.

Rather than say "constant rate of change" in every such situation, mathematicians call that common rate of change **slope.**

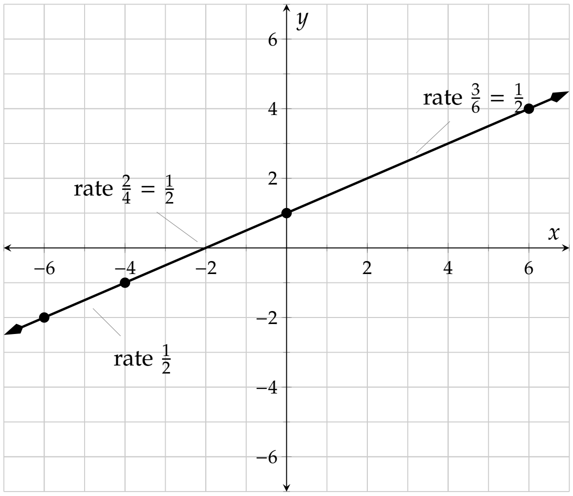


Figure 4.4.2: Between successive points, the rate of change is always 1/2.

Definition 4.4.3 Slope. When x and y are two variables where the rate of change between any two points is always the same, we call this common rate of change the **slope**. Since having a constant rate of change means the graph will be a straight line, its also called the **slope of the line**.

Considering the definition for Definition 4.3.12, this means that when x and y are two variables where the rate of change between any two points is always the same, then you can calculate slope, m, by finding two distinct data points (x_1, y_1) and (x_2, y_2), and calculating

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{4.4.1}$$

A slope is a rate of change. So if there are units for the horizontal and vertical variables, then there will be units for the slope. The slope will be measured in $\frac{\text{vertical units}}{\text{horizontal units}}$.

If the slope is nonzero, we say that there is a **linear relationship** between x and y. When the slope is 0, we say that y is **constant** with respect to x.

Here are some scenarios with different slopes. Note that a slope is more meaningful with units.

Slope m. Why is the letter m commonly used as the symbol for "slope?" Some believe that it comes from the French word "monter" which means "to climb."

- If a tree grows 2.5 feet every year, its rate of change in height is the same from year to year. So the height and time have a linear relationship where the **slope** is 2.5 $\frac{\text{ft}}{\text{yr}}$.

- If a company loses 2 million dollars every year, its rate of change in reserve funds is the same from year to year. So the company's reserve funds and time have a linear relationship where the **slope** is -2 million dollars per year.
- If Sakura is an adult who has stopped growing, her rate of change in height is the same from year to year—it's zero. So the **slope** is $0\,\frac{\text{in}}{\text{yr}}$. Sakura's height is **constant** with respect to time. In a statistics course, you would say that height and time don't have a relationship at all, in the sense that information about Sakura's height tells you nothing about her age.

Remark 4.4.4. A useful phrase for remembering the definition of **slope** is "rise over run." Here, "rise" refers to "change in y," Δy, and "run" refers to "change in x," Δx. Be careful though. As we have learned, the horizontal direction comes *first* in mathematics, followed by the vertical direction. The phrase "rise over run" reverses this. (It's a bit awkward to say, but the phrase "run under rise" puts the horizontal change first.)

Example 4.4.5 Yara's Savings. On Dec. 31, Yara had only \$50 in her savings account. For the the new year, she resolved to deposit \$20 into her savings account each week, without withdrawing any money from the account.

Yara keeps her resolution, and her account balance increases steadily by \$20 each week. That's a constant rate of change, so her account balance and time have a linear relationship with slope $20\,{}^{\text{dollars}}/_{\text{wk}}$.

We can model the balance, y, in Yara's savings account after x weeks with an equation. Since Yara started with \$50 and adds \$20 each week, the account balance y after x weeks is

$$y = 50 + 20x \tag{4.4.2}$$

where y is a dollar amount. Notice that the slope, $20\,{}^{\text{dollars}}/_{\text{wk}}$, serves as the multiplier for x weeks.

We can also consider Yara's savings using a table.

	x, weeks since Dec. 31	y, savings account balance (dollars)	
	0	50	
x increases by $1\longrightarrow$	1	70	$\longleftarrow y$ increases by 20
x increases by $1\longrightarrow$	2	90	$\longleftarrow y$ increases by 20
x increases by $2\longrightarrow$	4	130	$\longleftarrow y$ increases by 40
x increases by $3\longrightarrow$	7	190	$\longleftarrow y$ increases by 60
x increases by $5\longrightarrow$	12	290	$\longleftarrow y$ increases by 100

Table 4.4.6: Yara's savings

In first few rows of the table, we see that when the number of weeks x increases by 1, the balance y increases by 20. The row-to-row rate of change is $\frac{20}{1} = 20$, the slope. In any table for a linear relationship, whenever x increases by 1 unit, y will increase by the slope.

In further rows, notice that as row-to-row change in x increases, row-to-row change in y increases proportionally to preserve the constant rate of change. Looking at the change in the last two rows of the table, we see x increases by 5 and y increases by 100, which gives a rate of change of $\frac{100}{5} = 20$, the value of the slope again.

We can "see" the rates of change between consecutive rows of the table on a graph of Yara's savings by including **slope triangles**.

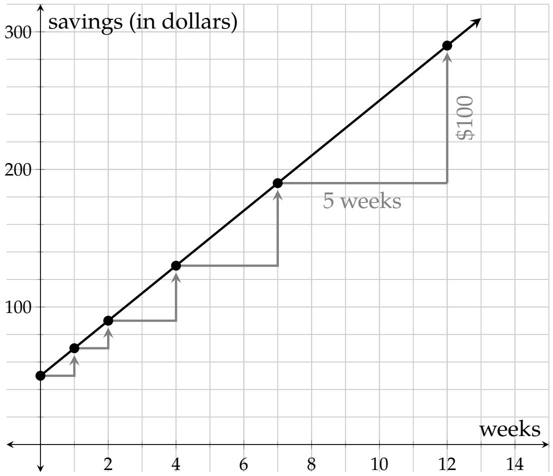


Figure 4.4.7: Yara's savings

The large, labeled slope triangle indicates that when 5 weeks pass, Yara saves \$100. This is the rate of change between the last two rows of the table, $\frac{100}{5} = 20\,{}^{\text{dollars}}/_{\text{wk}}$.

The smaller slope triangles indicate, from left to right, the rates of change $\frac{20}{1}$, $\frac{20}{1}$, $\frac{40}{2}$, and $\frac{60}{3}$ respectively. All of these rates simplify to the slope, $20\,{}^{\text{dollars}}/_{\text{wk}}$.

Every slope triangle on the graph of Yara's savings has the same shape (geometrically, they are called similar triangles) since the ratio of vertical change to horizontal change is always $20\,{}^{\text{dollars}}/_{\text{wk}}$. On any graph of any line, we can draw a slope triangle and compute slope as "rise over run."

Of course, we could draw a slope triangle on the other side of the line:

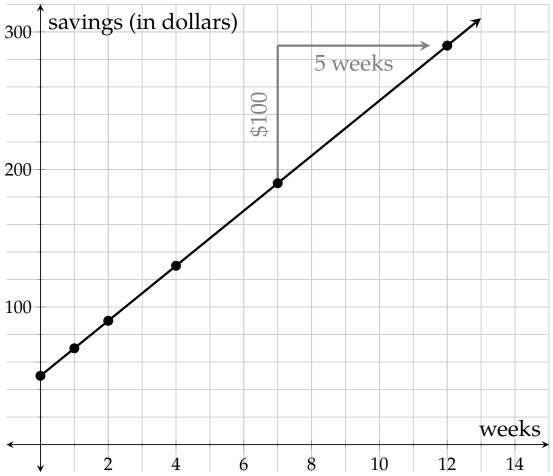


Figure 4.4.8: Yara's savings

This slope triangle works just as well for identifying "rise" and "run," but it focuses on vertical change before horizontal change. For consistency with mathematical conventions, we will generally draw slope triangles showing horizontal change followed by vertical change, as in Figure 4.4.7.

Example 4.4.9 The following graph of a line models the amount of gas, in gallons, in Kiran's gas tank as they drive their car. Find the line's slope, and interpret its meaning in this context.

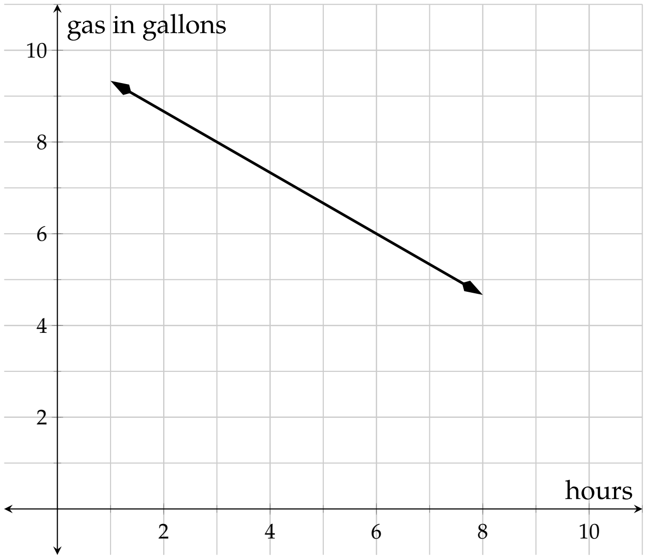


Figure 4.4.10: Amount of gas in Kiran's gas tank

Explanation. To find a line's slope using its graph, we first identify two points on it, and then draw a slope triangle. Naturally, we would want to choose two points whose x- and y-coordinates are easy to identify exactly based on the graph. We will pick the two points where $x = 3$ and $x = 6$, because they are right on the grid lines:

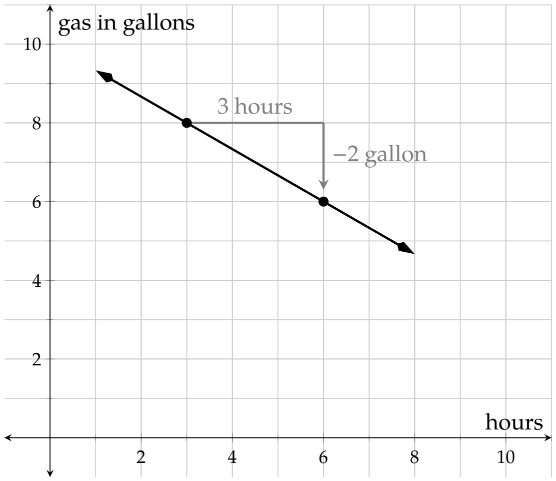


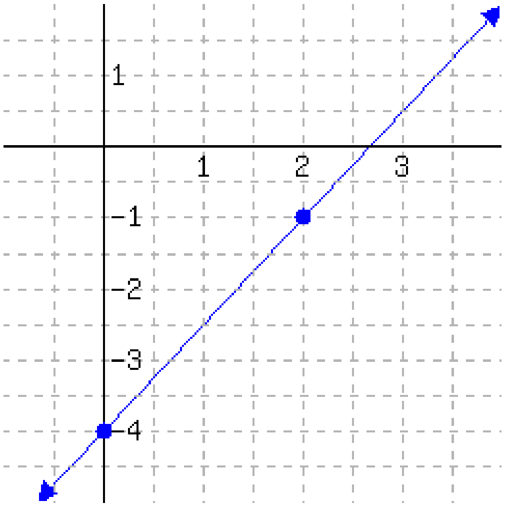
Figure 4.4.11: A Good Slope Triangle

Notice that the *change* in y is negative, because the amount of gas is decreasing. Since we chose points with integer coordinates, we can easily calculate the slope:

$$\text{slope} = \frac{-2}{3} = -\frac{2}{3}.$$

With units, the slope is $-\frac{2}{3}\ \frac{\text{gal}}{\text{h}}$. In the given context, this slope implies gas in the tank is *decreasing* at the rate of $\frac{2}{3}\ \frac{\text{gal}}{\text{h}}$. Since this slope is written as a fraction, there is another way to understand it: the gas in Kiran's tank is decreasing by 2 gallons every 3 hours.

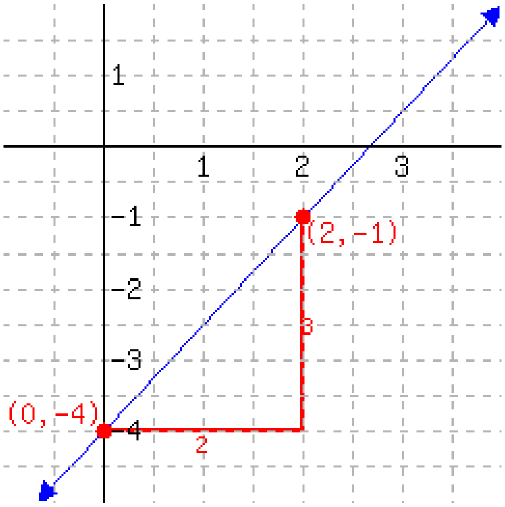
Checkpoint 4.4.12. Below is a line's graph.



The slope of this line is ______.

Explanation. To find the slope of a line from its graph, we first need to identify two points that the line passes through. It is wise to choose points with integer coordinates. For this problem, we choose $(0, -4)$ and $(2, -1)$.

Next, we sketch a slope triangle and find the *rise* and *run*. In the sketch below, the rise is 3 and the run is 2.



$$\begin{aligned}\text{slope} &= \frac{\text{rise}}{\text{run}}\\ &= \frac{3}{2}\end{aligned}$$

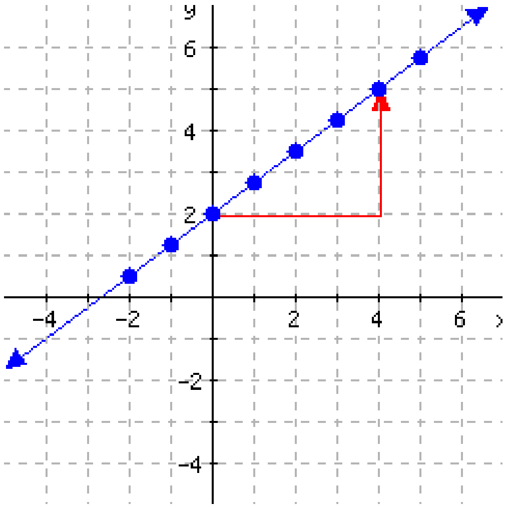
This line's slope is $\frac{3}{2}$.

Checkpoint 4.4.13. Make a table and plot the equation $y = \frac{3}{4}x + 2$, which makes a straight line. Use the plot to determine the slope of this line.

Explanation. First, we choose some x-values to make a table, and compute the corresponding y-values.

x	$y = \frac{3}{4}x + 2$	Point
-2	$\frac{3}{4}(-2) + 2 = 0.5$	$(-2, 0.5)$
-1	$\frac{3}{4}(-1) + 2 = 1.25$	$(-1, 1.25)$
0	$\frac{3}{4}(0) + 2 = 2$	$(0, 2)$
1	$\frac{3}{4}(1) + 2 = 2.75$	$(1, 2.75)$
2	$\frac{3}{4}(2) + 2 = 3.5$	$(1, 3.5)$
3	$\frac{3}{4}(3) + 2 = 4.25$	$(1, 4.25)$
4	$\frac{3}{4}(4) + 2 = 5$	$(1, 5)$
5	$\frac{3}{4}(5) + 2 = 5.75$	$(2, 5.75)$

This table lets us plot the graph and identify a slope triangle that is easy to work with.



Since the slope triangle runs 4 units and then rises 3 units, the slope is $\frac{3}{4}$.

4.4.2 Comparing Slopes

It's useful to understand what it means for different slopes to appear on the same coordinate system.

> **Example 4.4.14** Effie, Ivan and Cleo are in a foot race. Figure 4.4.15 models the distance each has traveled in the first few seconds. Each runner takes a second to accelerate up to their running speed, but then runs at a constant speed. So they are then traveling with a constant rate of change, and the straight line portions of their graphs have a slope. Find each line's slope, and interpret its meaning in this context. What comparisons can you make with these runners?

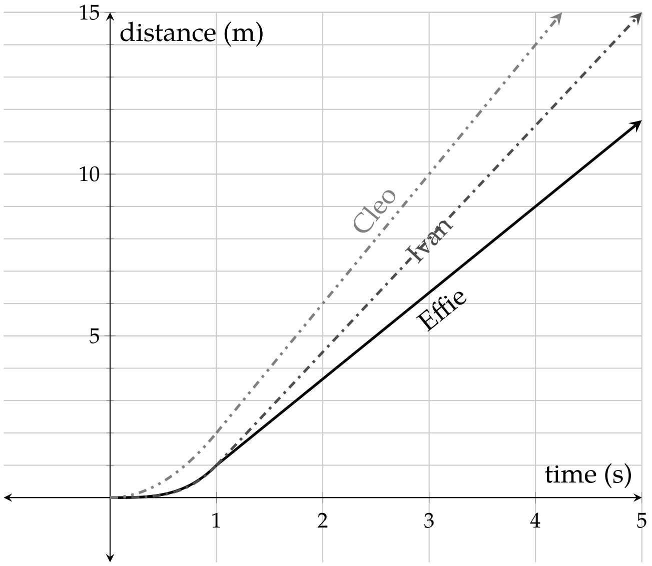


Figure 4.4.15: A three-way foot race

We will draw slope triangles to find each line's slope.

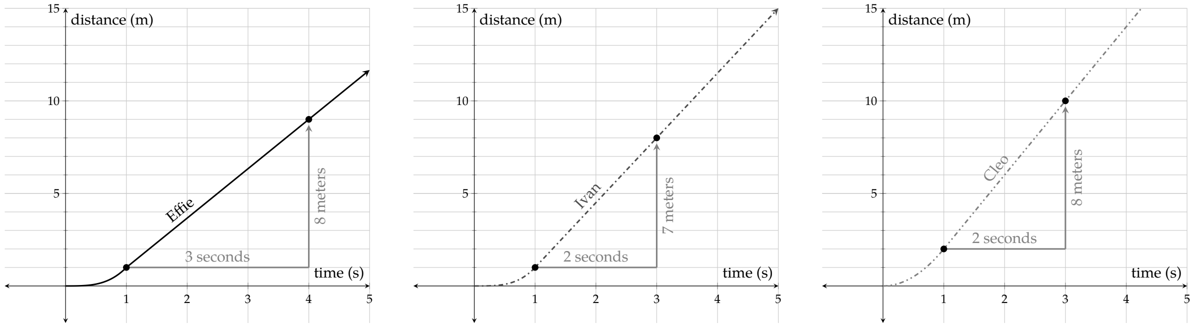


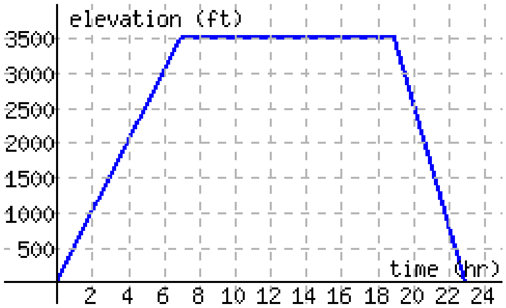
Figure 4.4.16: Find the Slope of Each Line

Using Formula (4.4.1), we have:

- Effie's slope is $\frac{8}{3} \approx 2.666$ meters per second
- Ivan's slope is $\frac{7}{2} = 3.5$ meters per second
- Cleo's slope is $\frac{8}{2} = 4$ meters per second

In a time-distance graph, the slope of a line represents speed. The slopes in these examples and the running speeds of these runners are both measured in $\frac{\text{m}}{\text{s}}$. Another important relationship we can see is that, the more sharply a line is slanted, the bigger the slope is. This should make sense because for each passing second, the faster person travels longer, making a slope triangle's height taller. This means that, numerically, we can tell that Cleo is the fastest runner (and Effie is the slowest) just by comparing the slopes $4 > 3.5 > 2.666$.

Checkpoint 4.4.17 Jogging on Mt. Hood. Kato is training for a race up the slope of Mt. Hood, from Sandy to Government Camp, and then back. The graph below models his elevation from his starting point as time passes. Find the slopes of the three line segments, and interpret their meanings in this context.



a. The first segment has slope ______.

b. The second segment has slope ______.

c. The third segment has slope ______.

Explanation. The first segment started at $(0, 0)$ and stopped at $(7, 3500)$. This implies, Kato started at the starting point, traveled 7 hours and reached a point 3500 feet higher in elevation from the starting point. The slope of the line is

$$\frac{\Delta y}{\Delta x} = \frac{3500}{7} = 500$$

and with units, that is 500 ft/hr. In context, Kato was running, gaining 500 feet in elevation per hour.

The third segment started from $(19, 3500)$ and stopped at $(23, 0)$. This implies, Kato started this part of his trip from a spot 3500 feet higher in elevation from the starting point, traveled for 4 hours and returned to the starting elevation. The slope of the line is

$$\frac{\Delta y}{\Delta x} = \frac{-3500}{4} = -875$$

and with units, that is -875 ft/hr. In context, Kato was running, dropping in elevation by 875 feet per hour.

What happened in the second segment, which started at $(7, 3500)$ and ended at $(19, 3500)$? This implies he started this portion 3500 feet higher in elevation from the starting point, and didn't change elevation for 19 hours. The slope of the line is

$$\frac{\Delta y}{\Delta x} = \frac{0}{19} = 0$$

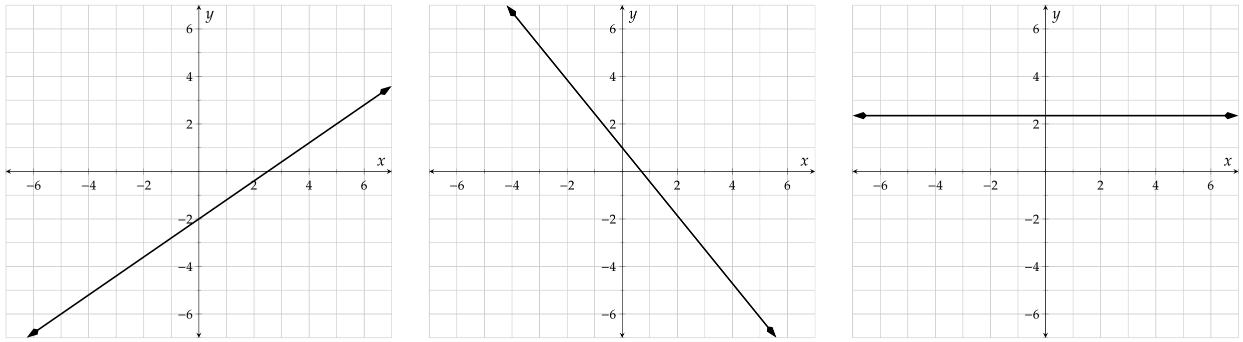
and with units, that is 0 ft/hr. In context, Kato was running but neither gaining nor losing elevation.

Some important properties are demonstrated in Exercise 4.4.17.

Fact 4.4.18 The Relationship Between Slope and Increase/Decrease. *In a linear relationship, as the x-value increases (in other words as you read its graph from left to right):*

- *if the y-values increase (in other words, the line goes upward), its slope is positive.*
- *if the y-values decrease (in other words, the line goes downward), its slope is negative.*
- *if the y-values don't change (in other words, the line is flat, or horizontal), its slope is* 0.

These properties are summarized graphically in Figure 4.4.18.



(a) A line with positive slope **(b)** A line with negative slope **(c)** A line with a slope of zero

Figure 4.4.18

4.4.3 Finding Slope by Two Given Points

Several times in this section we computed a slope by drawing a slope triangle. That's not really necessary if you have coordinates for two points that a line passes through. In fact, sometimes it's impractical to draw a slope triangle.[1] Here we will stress how to find a line's slope without drawing a slope triangle.

Example 4.4.19 Your neighbor planted a sapling from Portland Nursery in his front yard. Ever since, for several years now, it has been growing at a constant rate. By the end of the third year, the tree was 15 ft tall; by the end of the sixth year, the tree was 27 ft tall. What's the tree's rate of growth (i.e.the slope)?

We *could* sketch a graph for this scenario, and include a slope triangle. If we did that, it would look like:

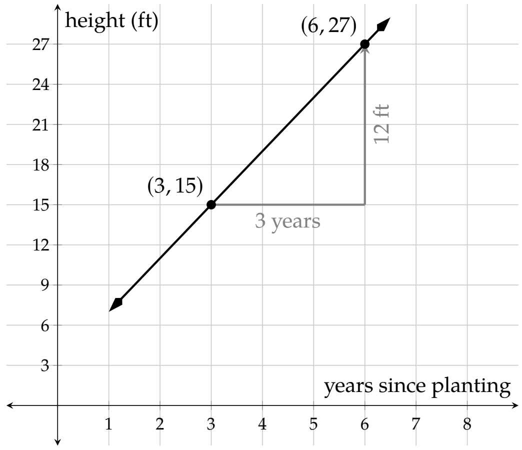


Figure 4.4.20: Height of a Tree

By the slope triangle and Equation (4.4.1) we have:

$$\begin{aligned}\text{slope} = m &= \frac{\Delta y}{\Delta x}\\ &= \frac{12}{3}\\ &= 4\end{aligned}$$

So the tree is growing at a rate of $4\,\frac{\text{ft}}{\text{yr}}$.

---

[1] For instance if you only have specific information about two points that are too close together to draw a triangle, or if you cannot clearly see precise coordinates where you might start and stop your slope triangle.

But hold on. Did we really *need* this picture? The “rise” of 12 came from a subtraction of two y-values: $27-15$. And the “run” of 3 came from a subtraction of two x-values: $6-3$.

Here is a picture-free approach. We know that after 3 yr, the height is 15 ft. As an ordered pair, that information gives us the point $(3,15)$ which we can label as $(\overset{x_1}{3},\overset{y_1}{15})$. Similarly, the background information tells us to consider $(6,27)$, which we label as $(\overset{x_2}{6},\overset{y_2}{27})$. Here, x_1 and y_1 represent the first point’s x-value and y-value, and x_2 and y_2 represent the second point’s x-value and y-value.

Now we can write an alternative to Equation (4.4.1):

$$\text{slope} = m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{4.4.3}$$

This is known as the **slope formula**. The following graph will help you understand why this formula works. Basically, we are still using a slope triangle to calculate the slope.

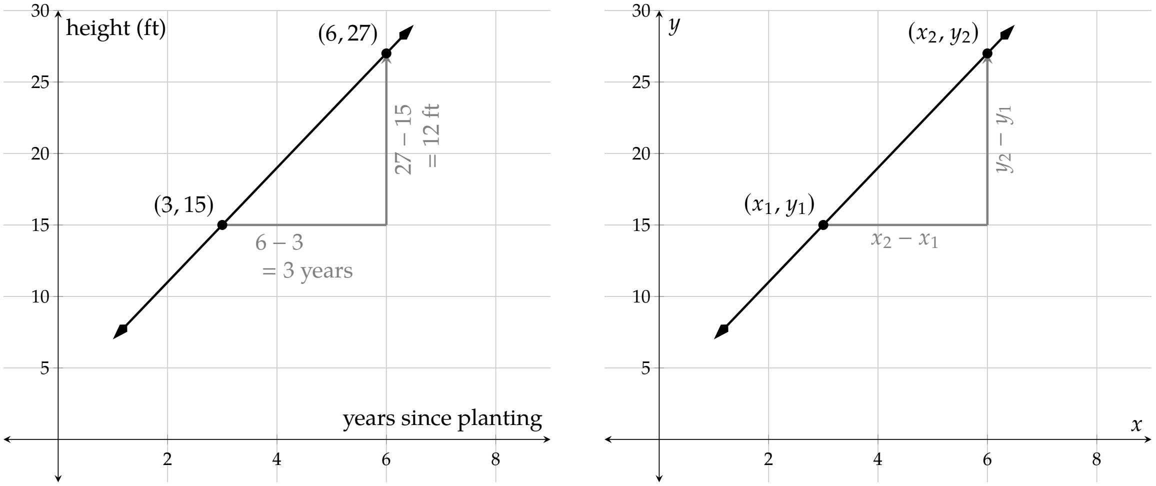


Figure 4.4.21: Understanding the slope formula

It’s important to use subscript instead of superscript in the slope equation, because y^2 means to take the number y and square it. Whereas y_2 tells you that there are at least two y-values in the conversation, and y_2 is the second of them.

The beauty of the slope formula (4.4.3) is that to find a line’s slope, we don’t need to draw a slope triangle any more. Let’s look at an example.

Example 4.4.22 A line passes the points $(-5,25)$ and $(4,-2)$. Find this line’s slope.

Explanation. If you are new to this formula, it’s important to label each number before using the formula. The two given points are:

$$(\overset{x_1}{-5},\overset{y_1}{25}) \text{ and } (\overset{x_2}{4},\overset{y_2}{-2})$$

Now apply the slope formula (4.4.3):

$$\begin{aligned}\text{slope} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{-2 - 25}{4 - (-5)}\\ &= \frac{-27}{9}\\ &= -3\end{aligned}$$

Note that we used parentheses when substituting in x_1 and y_1. This is a good habit to protect yourself from making errors with subtraction and double negatives.

Checkpoint 4.4.23. A line passes through the points $(-6, 26)$ and $(6, -16)$. Find this line's slope.

Explanation. To find a line's slope, we can use the slope formula:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

First, we mark which number corresponds to which variable in the formula:

$$(-6, 26) \longrightarrow (x_1, y_1)$$

$$(6, -16) \longrightarrow (x_2, y_2)$$

Now we substitute these numbers into the corresponding variables in the slope formula:

$$\begin{aligned}\text{slope} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{-16 - 26}{6 - (-6)}\\ &= \frac{-42}{12}\\ &= -\frac{7}{2}\end{aligned}$$

So the line's slope is $-\frac{7}{2}$.

Exercises

Review and Warmup

1. Reduce the fraction $\frac{5}{40}$.

2. Reduce the fraction $\frac{3}{27}$.

3. Reduce the fraction $\frac{15}{18}$.

4. Reduce the fraction $\frac{15}{27}$.

5. Reduce the fraction $\frac{35}{210}$.

6. Reduce the fraction $\frac{42}{189}$.

7. Reduce the fraction $\frac{135}{75}$.

8. Reduce the fraction $\frac{100}{30}$.

9. Reduce the fraction $\frac{245}{35}$.

10. Reduce the fraction $\frac{280}{35}$.

Slope and Points

11. A line passes through the points $(2, 1)$ and $(7, 21)$. Find this line's slope.

12. A line passes through the points $(4, 27)$ and $(6, 37)$. Find this line's slope.

13. A line passes through the points $(1, -1)$ and $(9, -9)$. Find this line's slope.

14. A line passes through the points $(3, -22)$ and $(8, -47)$. Find this line's slope.

15. A line passes through the points $(-4, 2)$ and $(-8, -2)$. Find this line's slope.

16. A line passes through the points $(-3, -7)$ and $(-5, -11)$. Find this line's slope.

17. A line passes through the points $(-1, -7)$ and $(3, -11)$. Find this line's slope.

18. A line passes through the points $(-3, -14)$ and $(3, 4)$. Find this line's slope.

19. A line passes through the points $(-2, 2)$ and $(-8, 14)$. Find this line's slope.

20. A line passes through the points $(-4, -1)$ and $(-5, 1)$. Find this line's slope.

21. A line passes through the points $(14, 16)$ and $(-7, -8)$. Find this line's slope.

22. A line passes through the points $(5, 17)$ and $(-10, -7)$. Find this line's slope.

23. A line passes through the points $(-2, 0)$ and $(4, -9)$. Find this line's slope.

24. A line passes through the points $(-16, 10)$ and $(8, 1)$. Find this line's slope.

25. A line passes through the points $(2, -4)$ and $(-5, -4)$. Find this line's slope.

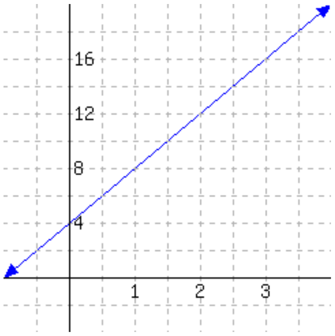
26. A line passes through the points $(5, -2)$ and $(-3, -2)$. Find this line's slope.

27. A line passes through the points $(1, -2)$ and $(1, 1)$. Find this line's slope.

28. A line passes through the points $(3, -4)$ and $(3, 3)$. Find this line's slope.

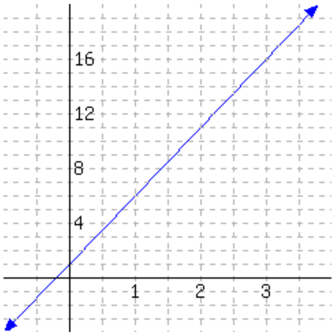
Slope and Graphs

29. Below is a line's graph.



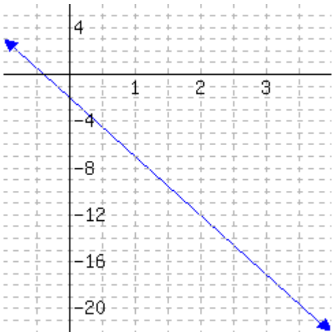
The slope of this line is ☐.

30. Below is a line's graph.



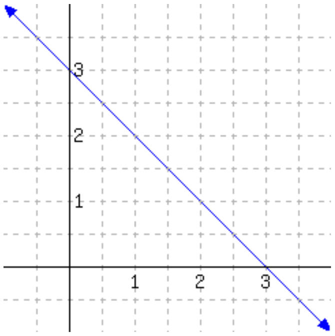
The slope of this line is ☐.

31. Below is a line's graph.



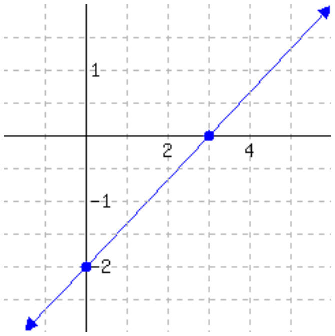
The slope of this line is ☐.

32. Below is a line's graph.



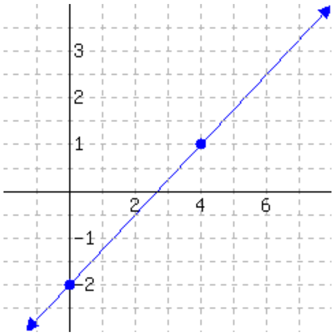
The slope of this line is ☐.

33. Below is a line's graph.



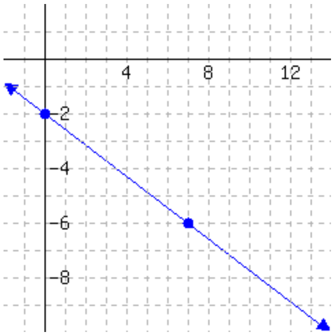
The slope of this line is ☐.

34. Below is a line's graph.



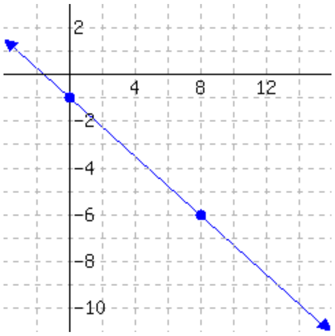
The slope of this line is ☐.

35. Below is a line's graph.



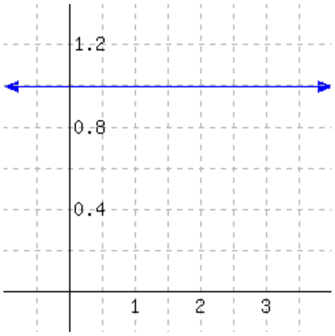
The slope of this line is ☐.

36. Below is a line's graph.



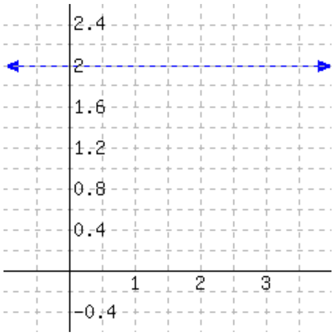
The slope of this line is ☐.

37. Below is a line's graph.



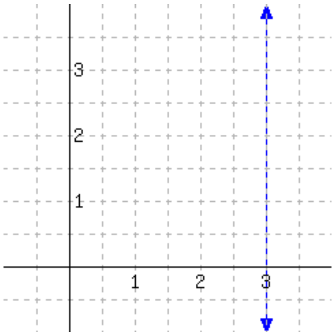
The slope of this line is ☐.

38. Below is a line's graph.



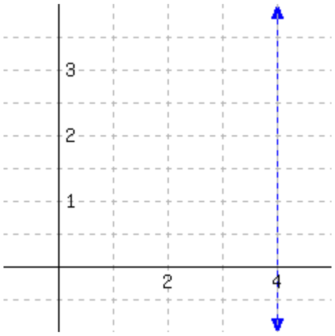
The slope of this line is ☐.

39. Below is a line's graph.



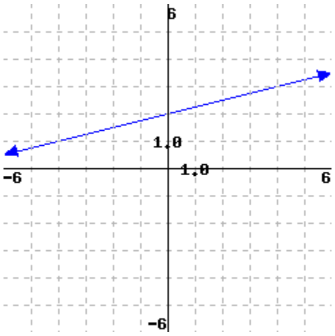
The slope of this line is ☐.

40. Below is a line's graph.



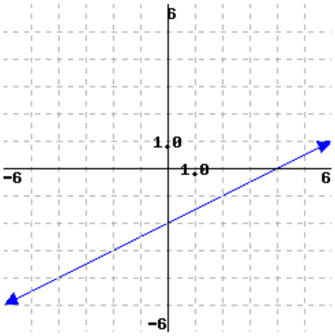
The slope of this line is ☐.

41. A line's graph is shown below.



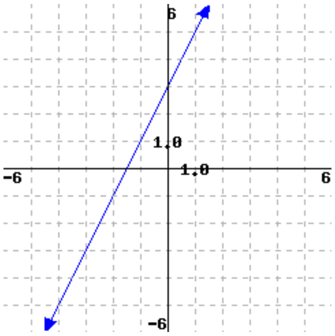
The slope is ☐.

42. A line's graph is shown below.



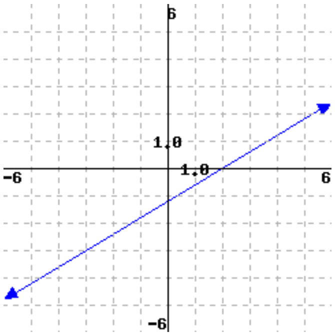
The slope is ☐.

43. A line's graph is shown below.



The slope is ☐.

44. A line's graph is shown below.



The slope is ☐.

Slope in Context

45. By your cell phone contract, you pay a monthly fee plus some money for each minute you use the phone during the month. In one month, you spent 260 minutes on the phone, and paid \$25.70. In another month, you spent 390 minutes on the phone, and paid \$31.55. What is the rate (in dollars per minute) that the phone company is charging you? That is, what is the slope of the line if you plotted the bill versus the number of minutes spent on the phone?

The rate is per minute.

46. By your cell phone contract, you pay a monthly fee plus some money for each minute you use the phone during the month. In one month, you spent 300 minutes on the phone, and paid \$24.50. In another month, you spent 360 minutes on the phone, and paid \$27.20. What is the rate (in dollars per minute) that the phone company is charging you? That is, what is the slope of the line if you plotted the bill versus the number of minutes spent on the phone?

The rate is per minute.

47. A company set aside a certain amount of money in the year 2000. The company spent exactly the same amount from that fund each year on perks for its employees. In 2003, there was still \$743,000 left in the fund. In 2005, there was \$659,000 left. What is the rate (in dollars per year) at which this company is spending from this fund?

The company is spending per year on perks for its employees.

48. A company set aside a certain amount of money in the year 2000. The company spent exactly the same amount from that fund each year on perks for its employees. In 2004, there was still \$546,000 left in the fund. In 2005, there was \$500,000 left. What is the rate (in dollars per year) at which this company is spending from this fund?

The company is spending per year on perks for its employees.

49. A biologist has been observing a tree's height. Eleven months into the observation, the tree was 22.2 feet tall. Eleven months into the observation, the tree was 24.6 feet tall. What is the rate at which the tree is growing? In other words, what is the slope if you plotted heigth versus time?

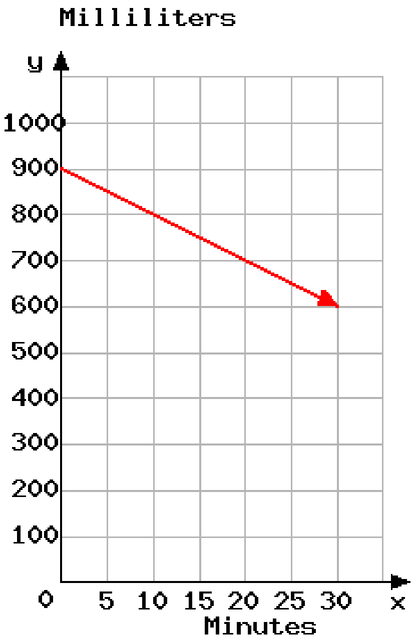
50. A biologist has been observing a tree's height. Thirteen months into the observation, the tree was 17.53 feet tall. Thirteen months into the observation, the tree was 18.08 feet tall. What is the rate at which the tree is growing? In other words, what is the slope if you plotted heigth versus time?

51. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way. Five minutes since the experiment started, the gas had a mass of 133.2 grams. Twelve minutes since the experiment started, the gas had a mass of 108 grams. At what rate is the gas leaking?

52. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way. Seven minutes since the experiment started, the gas had a mass of 338.4 grams. Nineteen minutes since the experiment started, the gas had a mass of 225.6 grams. At what rate is the gas leaking?

53. A liquid solution is slowly leaking from a container. This graph shows the

milliters of solution y remaining in the container after x minutes.

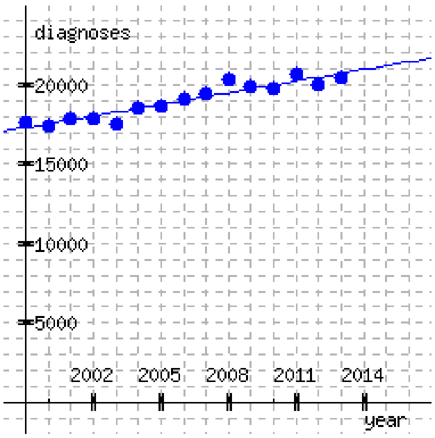


a. The y coordinate of the line is ☐.

b. The slope of the line is ☐.

c. Use the graph and your answer to part b to predict the number of minutes it will take for the container to empty if the solution continues leaking at the same rate. That time is ☐ minutes.

54. The graph plots the number of invasive cancer diagnoses in Oregon over time, and a trend-line has been drawn.



Estimate the slope of the trend-line. Just give the numerical value; the units are provided.

☐ $\frac{\text{diagnoses}}{\text{year}}$

Challenge

55. True or False: A slope of $\frac{2}{5}$ is steeper than a slope of $\frac{1}{4}$. (☐ true ☐ false)

56. True or False: A slope of $\frac{1}{8}$ is steeper than a slope of $\frac{2}{5}$. (☐ true ☐ false)

4.5 Slope-Intercept Form

In this section, we will explore one of the "standard" ways to write the equation of a line. It's known as **slope-intercept form**.

4.5.1 Slope-Intercept Definition

Recall Example 4.4.5, where Yara had \$50 in her savings account when the year began, and decided to deposit \$20 each week without withdrawing any money. In that example, we model using x to represent how many weeks have passed. After x weeks, Yara has added $20x$ dollars. And since she started with \$50, she has

$$y = 20x + 50$$

in her account after x weeks. In this example, there is a constant rate of change of 20 dollars per week, so we call that the **slope** as discussed in Section 4.4. We also saw in Figure 4.4.7 that plotting Yara's balance over time gives us a straight-line graph.

The graph of Yara's savings has some things in common with almost every straight-line graph. There is a **slope**, and there is a place where the line crosses the y-axis. Figure 4.5.2 illustrates this in the abstract.

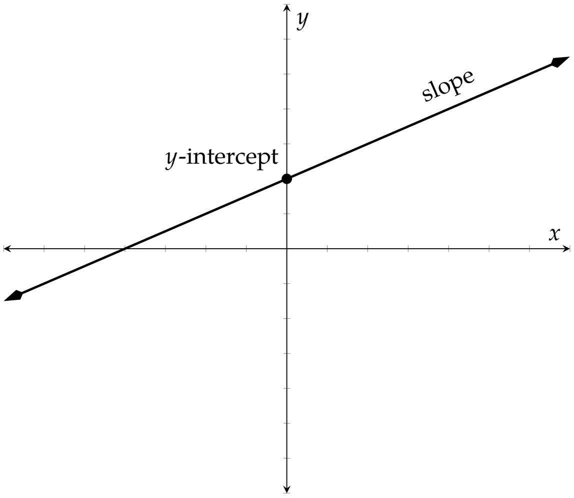


Figure 4.5.2: Generic line with slope and y-intercept

What else is there?. Can you think of a type of straight line that does not have a notion of slope? Or that does not cross the y-axis somewhere?

We already have an accepted symbol, m, for the slope of a line. The **y-intercept** is a *point* on the y-axis where the line crosses. Since it's on the y-axis, the x-coordinate of this point is 0. It is standard to call the y-intercept $(0, b)$ where b represents the position of the y-intercept on the y-axis.

Checkpoint 4.5.3. Use Figure 4.4.7 to answer this question.

What was the value of b in the plot of Yara's savings?

What is the y-intercept?

Explanation. The line crosses the y-axis at $(0, 50)$, so the value of b is 50. And the y-intercept is $(0, 50)$

One way to write the equation for Yara's savings was

$$y = 20x + 50,$$

where both $m = 20$ and $b = 50$ are immediately visible in the equation. Now we are ready to generalize this.

Definition 4.5.4 Slope-Intercept Form. When x and y have a linear relationship where m is the slope and $(0, b)$ is the y-intercept, one equation for this relationship is

$$y = mx + b \tag{4.5.1}$$

and this equation is called the **slope-intercept form** of the line. It is called this because the slope and y-intercept are immediately discernible from the numbers in the equation.

Checkpoint 4.5.5. What are the slope and y-intercept for each of the following line equations?

Equation	Slope	y-intercept
$y = 3.1x + 1.78$	__________	__________
$y = -17x + 112$	__________	__________
$y = \frac{3}{7}x - \frac{2}{3}$	__________	__________
$y = 13 - 8x$	__________	__________
$y = 1 - \frac{2x}{3}$	__________	__________
$y = 2x$	__________	__________
$y = 3$	__________	__________

Explanation. In the first three equations, simply read the slope m according to slope-intercept form. The slopes are 3.1, -17, and $\frac{3}{7}$.

The fourth equation was written with the terms not in the slope-intercept form order. It could be written $y = -8x + 13$, and then it is clear that its slope is -8. In any case, the slope is the coefficient of x.

The fifth equation is also written with the terms not in the slope-intercept form order. Changing the order of the terms, it could be written $y = -\frac{2x}{3} + 1$, but this still does not match the pattern of slope-intercept form. Considering how fraction multiplication works, $\frac{2x}{3} = \frac{2}{3} \cdot \frac{x}{1} = \frac{2}{3}x$. So we can write this equation as $y = -\frac{2}{3}x + 1$, and we see the slope is $-\frac{2}{3}$.

The last two equations could be written $y = 2x + 0$ and $y = 0x + 3$, allowing us to read their slopes as 2 and 0.

For the y-intercepts, remember that we are expected to answer using an ordered pair $(0, b)$, not just a single number b. We can simply read that the first two y-intercepts are $(0, 1.78)$ and $(0, 112)$.

The third equation does not exactly match the slope-intercept form, until you view it as $y = \frac{3}{7}x + \left(-\frac{2}{3}\right)$, and then you can see that its y-intercept is $-\frac{2}{3}$.

With the fourth equation, after rewriting it as $y = -8x + 13$, we can see that its y-intercept is $(0, 13)$.

We already explored rewriting the fifth equation as $y = -\frac{2}{3}x + 1$, where we can see that its y-intercept is $(0, 1)$.

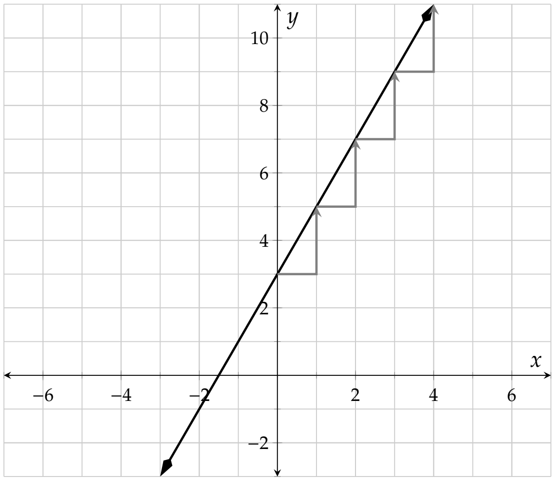
The last two equations could be written $y = 2x + 0$ and $y = 0x + 3$, allowing us to read their y-intercepts as $(0, 0)$ and $(0, 3)$.

Alternatively, we know that y-intercepts happen where $x = 0$, and substituting $x = 0$ into each equation gives you the y-value of the y-intercept.

Remark 4.5.6. The number b is the y-value when $x = 0$. Therefore it is common to refer to b as the **initial value** or **starting value** of a linear relationship.

Example 4.5.7 With a simple equation like $y = 2x + 3$, we can see that this is a line whose slope is 2 and which has initial value 3. So starting at $y = 3$ when $x = 0$ (that is, on the y-axis), each time we increase the x-value by 1, the y-value increases by 2. With these basic observations, we can quickly produce a table and/or a graph.

	x	y	
start on y-axis $\longrightarrow$	0	3	initial $\longleftarrow$ value
increase by 1 $\longrightarrow$	1	5	increase $\longleftarrow$ by 2
increase by 1 $\longrightarrow$	2	7	increase $\longleftarrow$ by 2
increase by 1 $\longrightarrow$	3	9	increase $\longleftarrow$ by 2
increase by 1 $\longrightarrow$	4	11	increase $\longleftarrow$ by 2



Example 4.5.8 Decide whether data in the table has a linear relationship. If so, write the linear equation in slope-intercept form (4.5.1).

x-values	y-values
0	−4
2	2
5	11
9	23

Explanation. To assess whether the relationship is linear, we have to recall from Section 4.3 that we should examine rates of change between data points. Note that the changes in y-values are not consistent. However, the rates of change are calculated as follows:

- When x increases by 2, y increases by 6. The first rate of change is $\frac{6}{2} = 3$.
- When x increases by 3, y increases by 9. The second rate of change is $\frac{9}{3} = 3$.
- When x increases by 4, y increases by 12. The third rate of change is $\frac{12}{4} = 3$.

Since the rates of change are all the same, 3, the relationship is linear and the slope m is 3.

According to the table, when $x = 0$, $y = -4$. So the starting value, b, is -4.

So in slope-intercept form, the line's equation is $y = 3x - 4$.

Checkpoint 4.5.9. Decide whether data in the table has a linear relationship. If so, write the linear equation in slope-intercept form. This may not be as easy as the previous example. Read the solution for a full explanation.

x-values	y-values
3	−2
6	−8
8	−12
11	−18

The data (☐ does ☐ does not) have a linear relationship, because: (☐ changes in x are not constant ☐ rates of change between data points are constant ☐ rates of change between data points are not constant)

The slope-intercept form of the equation for this line is ☐.

Explanation. To assess whether the relationship is linear, we examine rates of change between data points.

- The first rate of change is $\frac{-6}{3} = -2$.
- The second rate of change is $\frac{-4}{2} = -2$.
- The third rate of change is $\frac{-6}{3} = -2$.

Since the rates of change are all the same, -2, the relationship is linear and the slope m is -2.

So we know that the slope-intercept equation is $y = -2x + b$, but what number is b? The table does not directly tell us what the initial y-value is.

One approach is to use any point that we know the line passes through, and use algebra to solve for b. We know the line passes through $(3, -2)$, so

$$\begin{aligned} y &= -2x + b \\ -2 &= -2(3) + b \\ -2 &= -6 + b \\ 4 &= b \end{aligned}$$

So the equation is $y = -2x + 4$.

4.5.2 Graphing Slope-Intercept Equations

Example 4.5.10 The conversion formula for a Celsius temperature into Fahrenheit is $F = \frac{9}{5}C + 32$. This appears to be in slope-intercept form, except that x and y are replaced with C and F. Suppose you are asked to graph this equation. How will you proceed? You *could* make a table of values as we do in Section 4.2 but that takes time and effort. Since the equation here is in slope-intercept form, there is a nicer way.

Since this equation is for starting with a Celsius temperature and obtaining a Fahrenheit temperature, it makes sense to let C be the horizontal axis variable and F be the vertical axis variable. Note the slope is $\frac{9}{5}$ and the vertical intercept (here, the F-intercept) is $(0, 32)$.

1. Set up the axes using an appropriate window and labels. Considering the freezing and boiling temperatures of water, it's reasonable to let C run through at least 0 to 100. Similarly it's reasonable to let F run through at least 32 to 212.

2. Plot the F-intercept, which is at $(0, 32)$.
3. Starting at the F-intercept, use slope triangles to reach the next point. Since our slope is $\frac{9}{5}$, that suggests a "run" of 5 and a "rise" of 9 might work. But as Figure 4.5.11 indicates, such slope triangles are too tiny. Since $\frac{9}{5} = \frac{90}{50}$, we can try a "run" of 50 and a rise of 90.
4. Connect your points with a straight line, use arrowheads, and label the equation.

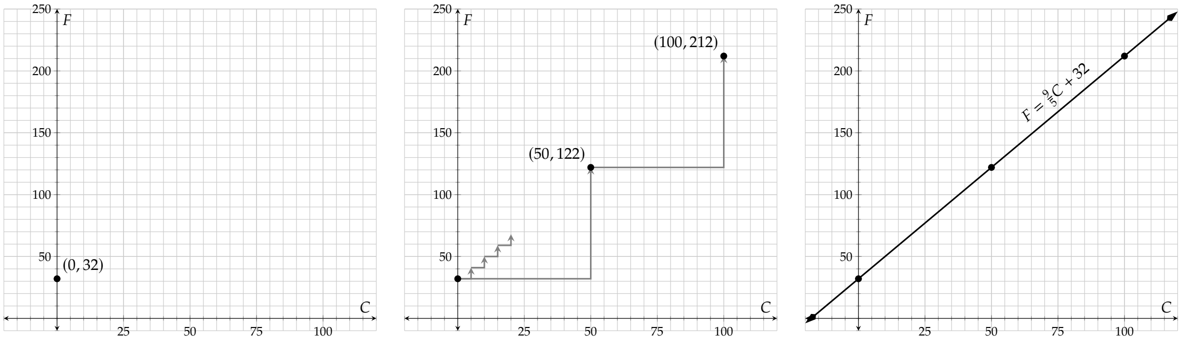
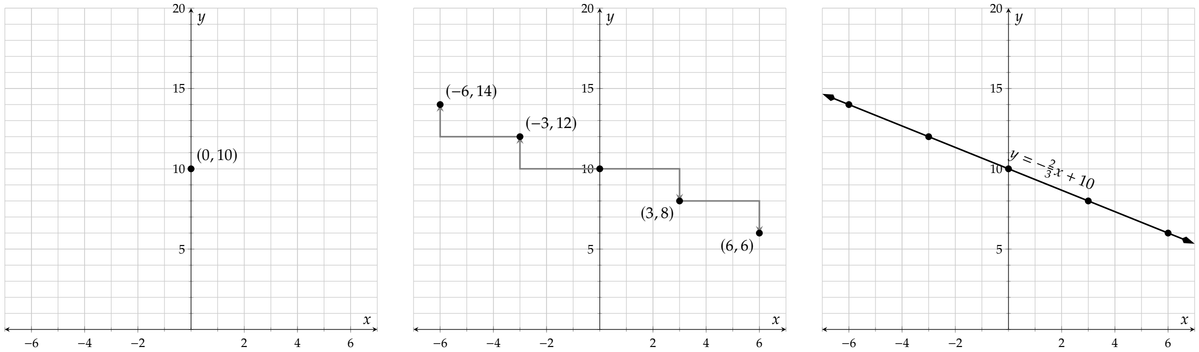


Figure 4.5.11: Graphing $F = \frac{9}{5}C + 32$

Example 4.5.12 Graph $y = -\frac{2}{3}x + 10$.



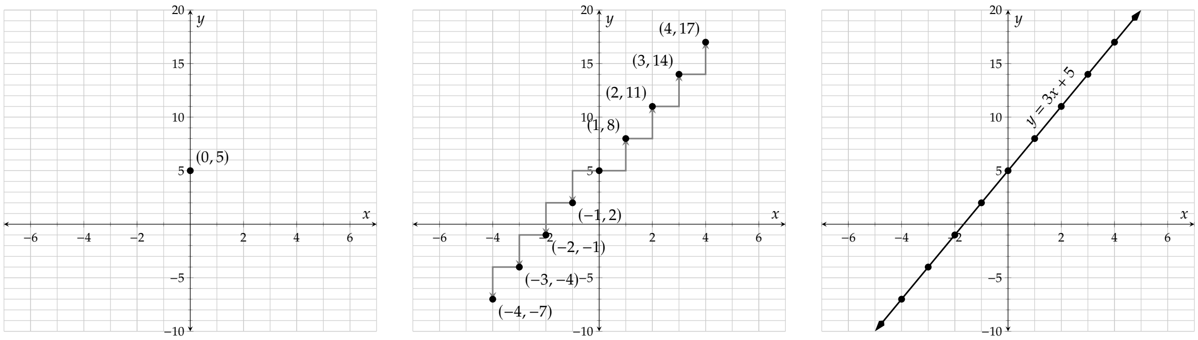
(a) Setting up the axes in an appropriate window and making sure that the y-intercept will be visible, and that any "run" and "rise" amounts we wish to use will not make triangles that are too big or too small.

(b) The slope is $-\frac{2}{3} = \frac{-2}{3} = \frac{2}{-3}$. So we can try using a "run" of 3 and a "rise" of −2 or a "run" of −3 and a "rise" of 2.

(c) Connecting the points with a straight line and adding labels.

Figure 4.5.12: Graphing $y = -\frac{2}{3}x + 10$

Example 4.5.13 Graph $y = 3x + 5$.



(a) Setting up the axes to make sure that the y-intercept will be visible, and that any "run" and "rise" amounts we wish to use will not make triangles that are too big or too small.

(b) The slope is a whole number 3. Every 1 unit forward causes a change of positive 3 in the y-values.

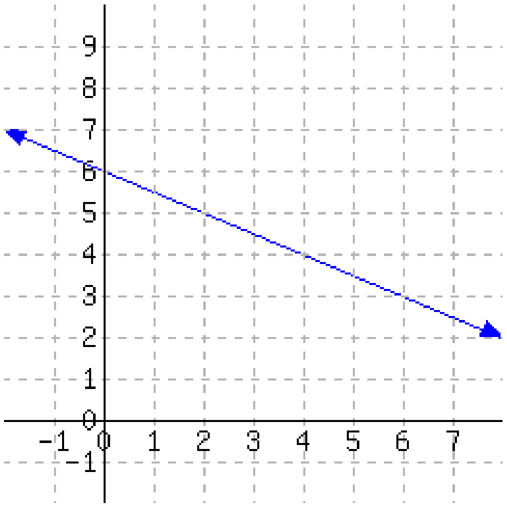
(c) Connecting the points with a straight line and adding labels.

Figure 4.5.13: Graphing $y = 3x + 5$

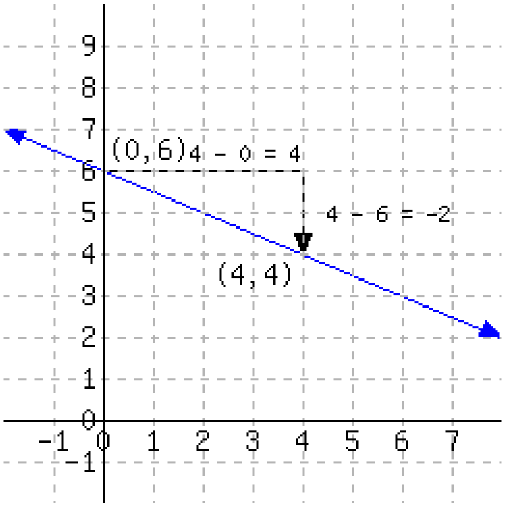
4.5.3 Writing a Slope-Intercept Equation Given a Graph

We can write a linear equation in slope-intercept form based on its graph. We need to be able to calculate the line's slope and see its y-intercept.

Checkpoint 4.5.14. Use the graph to write an equation of the line in slope-intercept form.



Explanation. On the line, pick two points with easy-to-read integer coordinates so that we can calculate slope. It doesn't matter which two points we use; the slope will be the same.



Using the slope triangle, we can calculate the line's slope:

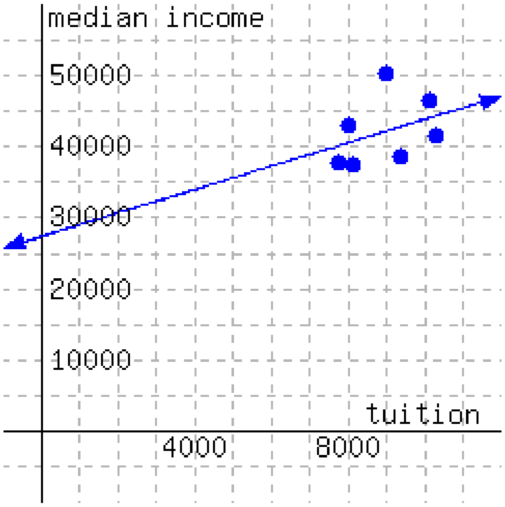
$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{-2}{4} = -\frac{1}{2}.$$

From the graph, we can see the y-intercept is $(0, 6)$.

With the slope and y-intercept found, we can write the line's equation:

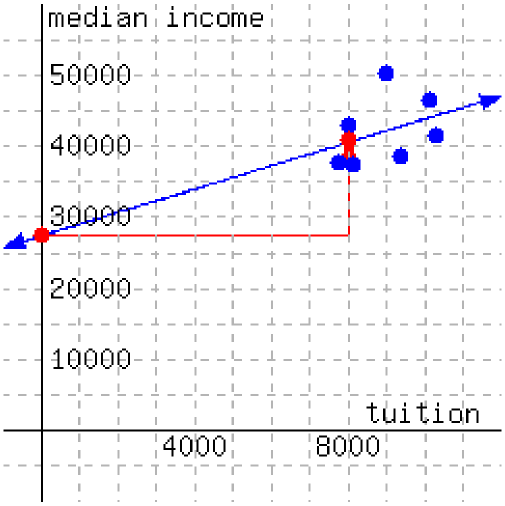
$$y = -\frac{1}{2}x + 6.$$

Checkpoint 4.5.15. There are seven public four-year colleges in Oregon. The graph plots the annual in-state tuition for each school on the x-axis, and the median income of former students ten years after first enrolling on the y-axis.



Write an equation for this line in slope-intercept form.

Explanation. Do your best to identify two points on the line. We go with $(0, 27500)$ and $(8000, 41000)$.



$$\frac{\Delta y}{\Delta x} = \frac{41000 - 27500}{8000 - 0} = \frac{13500}{8000} = 1.6875$$

So the slope is about 1.6875 dollars of median income per dollar of tuition. This is only an estimate since we are not all certain the two points we chose are actually on the line.

Estimating the y-intercept to be at $(0, 27500)$, we have $y = 1.6875x + 27500$.

4.5.4 Writing a Slope-Intercept Equation Given Two Points

The idea that any two points uniquely determine a line has been understood for thousands of years in many cultures around the world. Once you have two specific points, there is a straightforward process to find the slope-intercept form of the equation of the line that connects them.

Example 4.5.16 Find the slope-intercept form of the equation of the line that passes through the points $(0, 5)$ and $(8, -5)$.

Explanation. We are trying to write down $y = mx + b$, but with specific numbers for m and b. So the first step is to find the slope, m. To do this, recall the slope formula (4.4.3) from Section 4.4. It says that if a line passes through the points (x_1, y_1) and (x_2, y_2), then the slope is found by the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Applying this to our two points $(\overset{x_1}{0}, \overset{y_1}{5})$ and $(\overset{x_2}{8}, \overset{y_2}{-5})$, we see that the slope is:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-5 - 5}{8 - 0} \\ &= \frac{-10}{8} \\ &= -\frac{5}{4} \end{aligned}$$

We are trying to write $y = mx + b$. Since we already found the slope, we know that we want to write $y = -\frac{5}{4}x + b$ but we need a specific number for b. We *happen* to know that one point on this line is $(0, 5)$, which is on the y-axis because its x-value is 0. So $(0, 5)$ is this line's y-intercept, and therefore $b = 5$. (We're only able to make this conclusion because this point has 0 for its x-coordinate.) So, our equation is

$$y = -\frac{5}{4}x + 5.$$

Example 4.5.17 Find the slope-intercept form of the equation of the line that passes through the points $(3, -8)$ and $(-6, 1)$.

Explanation. The first step is always to find the slope between our two points: $(\overset{x_1}{3}, \overset{y_1}{-8})$ and $(\overset{x_2}{-6}, \overset{y_2}{1})$. Using the slope formula (4.4.3) again, we have:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1-(-8)}{-6-3} \\ &= \frac{9}{-9} \\ &= -1 \end{aligned}$$

Now that we have the slope, we can write $y = -1x + b$, which simplifies to $y = -x + b$. Unlike in Example 4.5.16, we are not given the value of b because neither of our two given points have an x-value of 0. The trick to finding b is to remember that we have two points that we know make the equation true! This means all we have to do is substitute *either* point into the equation for x and y and solve for b. Let's arbitrarily choose $(3, -8)$ to plug in.

$$\begin{aligned} y &= -x + b \\ -8 &= -(3) + b && \text{(Now solve for } b\text{.)} \\ -8 &= -3 + b \\ -8 + 3 &= -3 + b + 3 \\ -5 &= b \end{aligned}$$

In conclusion, the equation for which we were searching is $y = -x - 5$.

Don't be tempted to plug in values for x and y at this point. The general equation of a line in any form should have (at least one, and in this case two) variables in the final answer.

Checkpoint 4.5.18. Find the slope-intercept form of the equation of the line that passes through the points $(-3, 150)$ and $(0, 30)$.

Explanation. The first step is always to find the slope between our points: $(\overset{x_1}{-3}, \overset{y_1}{150})$ and $(\overset{x_2}{0}, \overset{y_2}{30})$. Using the slope formula, we have:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{30 - 150}{0-(-3)} \\ &= \frac{-120}{3} \\ &= -40 \end{aligned}$$

Now we can write $y = -40x + b$ and to find b we need look no further than one of the given points: $(0, 30)$. Since the x-value is 0, the value of b must be 30. So, the slope-intercept form of the line is

$$y = -40x + 30$$

Checkpoint 4.5.19. Find the slope-intercept form of the equation of the line that passes through the points $\left(-3, \frac{3}{4}\right)$ and $\left(-6, -\frac{17}{4}\right)$.

Explanation. First find the slope through our points: $\left(-3, \frac{3}{4}\right)$ and $\left(-6, -\frac{17}{4}\right)$. For this problem, we choose to do all of our algebra with improper fractions as it often simplifies the process.

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1}\\
&= \frac{-\frac{17}{4} - \frac{3}{4}}{-6 - (-3)}\\
&= \frac{\frac{-20}{4}}{-3}\\
&= \frac{-5}{-3}\\
&= \frac{5}{3}
\end{aligned}$$

So far we have $y = \frac{5}{3}x + b$. Now we need to solve for b since neither of the points given were the vertical intercept. Recall that to do this, we will choose one of the two points and plug it into our equation. We choose $\left(-3, \frac{3}{4}\right)$.

$$\begin{aligned}
y &= \frac{5}{3}x + b\\
\frac{3}{4} &= \frac{5}{3}(-3) + b\\
\frac{3}{4} &= -5 + b\\
\frac{3}{4} + 5 &= -5 + b + 5\\
\frac{3}{4} + \frac{20}{4} &= b\\
\frac{23}{4} &= b
\end{aligned}$$

Lastly, we write our equation.

$$y = \frac{5}{3}x + \frac{23}{4}$$

4.5.5 Modeling with Slope-Intercept Form

We can model many relatively simple relationships using slope-intercept form, and then solve related questions using algebra. Here are a few examples.

> **Example 4.5.20** Uber is a ride-sharing company. Its pricing in Portland factors in how much time and how many miles a trip takes. But if you assume that rides average out at a speed of 30 mph, then their pricing scheme boils down to a base of \$7.35 for the trip, plus \$3.85 per mile. Use a slope-intercept equation and algebra to answer these questions.
>
> a. How much is the fare if a trip is 5.3 miles long?

b. With \$100 available to you, how long of a trip can you afford?

Explanation. The rate of change (slope) is \$3.85 per mile, and the starting value is \$7.35. So the slope-intercept equation is

$$y = 3.85x + 7.35.$$

In this equation, x stands for the number of miles in a trip, and y stands for the amount of money to be charged.

If a trip is 5 miles long, we substitute $x = 5$ into the equation and we have:

$$\begin{aligned} y &= 3.85x + 7.35 \\ &= 3.85(5) + 7.35 \\ &= 19.25 + 7.35 \\ &= 26.60 \end{aligned}$$

And the 5-mile ride will cost you about \$26.60. (We say "about," because this was all assuming you average 30 mph.)

Next, to find how long of a trip would cost \$100, we substitute $y = 100$ into the equation and solve for x:

$$\begin{aligned} y &= 3.85x + 7.35 \\ 100 &= 3.85x + 7.35 \\ 100 - 7.35 &= 3.85x \\ 92.65 &= 3.85x \\ \frac{92.65}{3.85} &= x \\ 24.06 &\approx x \end{aligned}$$

So with \$100 you could afford a little more than a 24-mile trip.

Checkpoint 4.5.21. In a certain wildlife reservation in Africa, there are approximately 2400 elephants. Unfortunately, the population has been decreasing by 30 elephants per year. Use a slope-intercept equation and algebra to answer these questions.

a. If the trend continues, what would the elephant population be 15 years from now?

☐ elephants

b. If the trend continues, how many years will it be until the elephant population dwindles to 1200?

☐ years

Explanation. The rate of change (slope) is -30 elephants per year. Notice that since we are losing elephants, the slope is a negative number. The starting value is 2400 elephants. So the slope-intercept equation is

$$y = -30x + 2400.$$

In this equation, x stands for a number of years into the future, and y stands for the elephant population.

To estimate the elephant population 15 years later, we substitute x in the equation with 15, and we have:

$$\begin{aligned} y &= -30x + 2400 \\ &= -30(15) + 2400 \\ &= -450 + 2400 \\ &= 1950 \end{aligned}$$

So if the trend continues, there would be 1950 elephants on this reservation 15 years later.

Next, to find when the elephant population would decrease to 1200, we substitute y in the equation with 1200, and solve for x:

$$\begin{aligned} y &= -30x + 2400 \\ 1200 &= -30x + 2400 \\ 1200 - 2400 &= -30x \\ -1200 &= -30x \\ \frac{-1200}{-30} &= x \\ 40 &= x \end{aligned}$$

So if the trend continues, 40 years later, the elephant population would dwindle to 1,200.

Exercises

Review and Warmup

1. Evaluate $10B + 2c$ for $B = 7$ and $c = -4$.

2. Evaluate $-9C - a$ for $C = -5$ and $a = -10$.

3. Evaluate

$$\frac{y_2 - y_1}{x_2 - x_1}$$

for $x_1 = 19$, $x_2 = 9$, $y_1 = 8$, and $y_2 = 11$:

4. Evaluate

$$\frac{y_2 - y_1}{x_2 - x_1}$$

for $x_1 = -18$, $x_2 = -5$, $y_1 = -16$, and $y_2 = -1$:

Identifying Slope and y-Intercept Find the line's slope and y-intercept.

5. A line has equation $y = 3x + 1$.

This line's slope is $\boxed{}$.

This line's y-intercept is $\boxed{}$.

6. A line has equation $y = 4x + 7$.

This line's slope is $\boxed{}$.

This line's y-intercept is $\boxed{}$.

7. A line has equation $y = -7x - 7$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

8. A line has equation $y = -6x - 1$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

9. A line has equation $y = x + 3$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

10. A line has equation $y = x + 5$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

11. A line has equation $y = -x + 7$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

12. A line has equation $y = -x + 9$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

13. A line has equation $y = -\frac{2}{3}x + 8$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

14. A line has equation $y = -\frac{2}{9}x - 5$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

15. A line has equation $y = \frac{1}{2}x + 8$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

16. A line has equation $y = \frac{1}{4}x - 7$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

17. A line has equation $y = 7 + 6x$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

18. A line has equation $y = 9 + 7x$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

19. A line has equation $y = 8 - x$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

20. A line has equation $y = 9 - x$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

Graphs and Slope-Intercept Form

21. Graph the equation $y = 4x$.

22. Graph the equation $y = 5x$.

23. Graph the equation $y = -3x$.

24. Graph the equation $y = -2x$.

25. Graph the equation $y = \frac{5}{2}x$.

26. Graph the equation $y = \frac{1}{4}x$.

27. Graph the equation $y = -\frac{1}{3}x$.

28. Graph the equation $y = -\frac{5}{4}x$.

29. Graph the equation $y = 5x + 2$.

30. Graph the equation $y = 3x + 6$.

31. Graph the equation $y = -4x + 3$.

32. Graph the equation $y = -2x + 5$.

33. Graph the equation $y = x - 4$.

34. Graph the equation $y = x + 2$.

35. Graph the equation $y = -x + 3$.

36. Graph the equation $y = -x - 5$.

37. Graph the equation $y = \frac{2}{3}x + 4$.

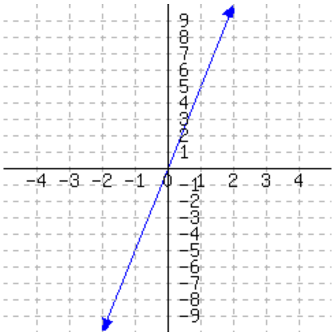
38. Graph the equation $y = \frac{3}{2}x - 5$.

39. Graph the equation $y = -\frac{3}{5}x - 1$.

40. Graph the equation $y = -\frac{1}{5}x + 1$.

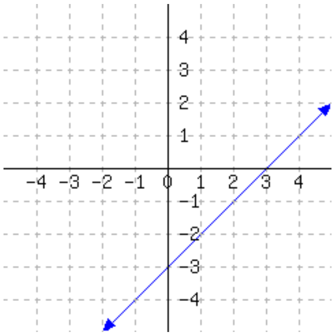
A line's graph is given.

41.



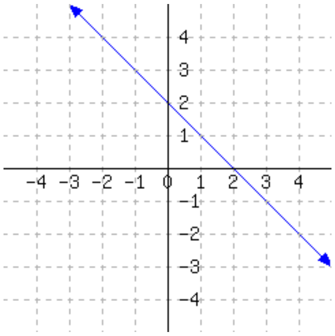
This line's slope-intercept equation is ☐

42.



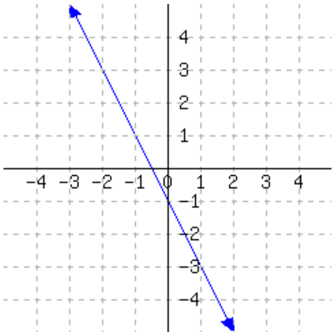
This line's slope-intercept equation is ☐

43.



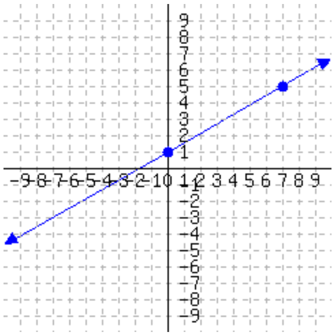
This line's slope-intercept equation is ☐

44.



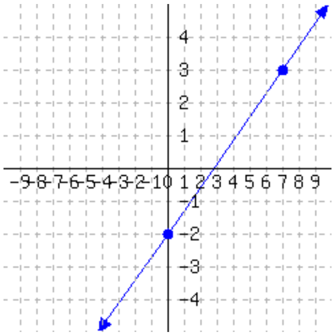
This line’s slope-intercept equation is

45.



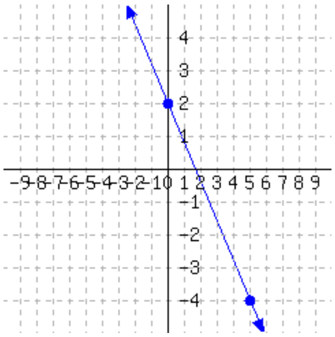
This line’s slope-intercept equation is

46.



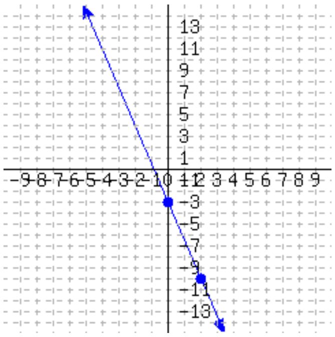
This line’s slope-intercept equation is

47.



This line’s slope-intercept equation is

48.



This line’s slope-intercept equation is

Writing a Slope-Intercept Equation Given Two Points

49. A line passes through the points $(3, 22)$ and $(4, 27)$. Find this line’s equation in slope-intercept form.

This line’s slope-intercept equation is .

50. A line passes through the points $(5, 29)$ and $(2, 14)$. Find this line’s equation in slope-intercept form.

This line’s slope-intercept equation is .

51. A line passes through the points $(-2, 20)$ and $(-5, 35)$. Find this line’s equation in slope-intercept form.

This line’s slope-intercept equation is .

52. A line passes through the points $(3, -12)$ and $(2, -7)$. Find this line’s equation in slope-intercept form.

This line’s slope-intercept equation is .

53. A line passes through the points $(-2, -3)$ and $(-3, -2)$. Find this line’s equation in slope-intercept form.

This line’s slope-intercept equation is .

54. A line passes through the points $(5, -7)$ and $(1, -3)$. Find this line’s equation in slope-intercept form.

This line’s slope-intercept equation is .

55. A line passes through the points $(18, 16)$ and $(0, 1)$. Find this line's equation in slope-intercept form.

This line's slope-intercept equation is .

56. A line passes through the points $(0, 7)$ and $(-15, -11)$. Find this line's equation in slope-intercept form.

This line's slope-intercept equation is .

57. A line passes through the points $(-9, 16)$ and $(0, 9)$. Find this line's equation in slope-intercept form.

This line's slope-intercept equation is .

58. A line passes through the points $(-5, 9)$ and $(-15, 25)$. Find this line's equation in slope-intercept form.

This line's slope-intercept equation is .

Applications

59. A gym charges members \$40 for a registration fee, and then \$24 per month. You became a member some time ago, and now you have paid a total of \$448 to the gym. How many months have passed since you joined the gym?

months have passed since you joined the gym.

60. Your cell phone company charges a \$11 monthly fee, plus \$0.18 per minute of talk time. One month your cell phone bill was \$68.60. How many minutes did you spend talking on the phone that month?

You spent talking on the phone that month.

61. A school purchased a batch of T-shirts from a company. The company charged \$4 per T-shirt, and gave the school a \$75 rebate. If the school had a net expense of \$1,565 from the purchase, how many T-shirts did the school buy?

The school purchased T-shirts.

62. Izabelle hired a face-painter for a birthday party. The painter charged a flat fee of \$65, and then charged \$2.50 per person. In the end, Izabelle paid a total of \$137.50. How many people used the face-painter's service?

people used the face-painter's service.

63. A certain country has 406.56 million acres of forest. Every year, the country loses 4.84 million acres of forest mainly due to deforestation for farming purposes. If this situation continues at this pace, how many years later will the country have only 227.48 million acres of forest left? (Use an equation to solve this problem.)

After years, this country would have 227.48 million acres of forest left.

64. Anthony has \$80 in his piggy bank. He plans to purchase some Pokemon cards, which costs \$1.75 each. He plans to save \$60.75 to purchase another toy. At most how many Pokemon cards can he purchase?

Write an equation to solve this problem.

Anthony can purchase at most Pokemon cards.

65. By your cell phone contract, you pay a monthly fee plus \$0.06 for each minute you spend on the phone. In one month, you spent 220 minutes over the phone, and had a bill totaling \$25.20.

Let x be the number of minutes you spend on the phone in a month, and let y be your total cell phone bill for that month, in dollars. Use a linear equation to model your monthly bill based on the number of minutes you spend on the phone.

a. This line's slope-intercept equation is ☐.

b. If you spend 140 minutes on the phone in a month, you would be billed ☐.

c. If your bill was \$39.60 one month, you must have spent ☐ minutes on the phone in that month.

66. A company set aside a certain amount of money in the year 2000. The company spent exactly \$42,000 from that fund each year on perks for its employees. In 2003, there was still \$782,000 left in the fund.

Let x be the number of years since 2000, and let y be the amount of money, in dollars, left in the fund that year. Use a linear equation to model the amount of money left in the fund after so many years.

a. The linear model's slope-intercept equation is ☐.

b. In the year 2009, there was ☐ left in the fund.

c. In the year ☐, the fund will be empty.

67. A biologist has been observing a tree's height. This type of tree typically grows by 0.27 feet each month. Ten months into the observation, the tree was 17.1 feet tall.

Let x be the number of months passed since the observations started, and let y be the tree's height at that time, in feet. Use a linear equation to model the tree's height as the number of months pass.

a. This line's slope-intercept equation is ☐.

b. 26 months after the observations started, the tree would be ☐ feet in height.

c. ☐ months after the observation started, the tree would be 29.52 feet tall.

68. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way. Each minute, they lose 1.7 grams. Seven minutes since the experiment started, the remaining gas had a mass of 73.1 grams.

Let x be the number of minutes that have passed since the experiment started, and let y be the mass of the gas in grams at that moment. Use a linear equation to model the weight of the gas over time.

a. This line's slope-intercept equation is ☐.

b. 33 minutes after the experiment started, there would be ☐ grams of gas left.

c. If a linear model continues to be accurate, ☐ minutes since the experiment started, all gas in the container will be gone.

69. A company set aside a certain amount of money in the year 2000. The company spent exactly the same amount from that fund each year on perks for its employees. In 2004, there was still \$807,000 left in the fund. In 2005, there was \$786,000 left.

Let x be the number of years since 2000, and let y be the amount of money, in dollars, left in the fund that year. Use a linear equation to model the amount of money left in the fund after so many years.

a. The linear model's slope-intercept equation is ☐.

b. In the year 2009, there was ☐ left in the fund.

c. In the year ☐, the fund will be empty.

70. By your cell phone contract, you pay a monthly fee plus some money for each minute you use the phone during the month. In one month, you spent 230 minutes on the phone, and paid \$17.45. In another month, you spent 380 minutes on the phone, and paid \$19.70.

Let x be the number of minutes you talk over the phone in a month, and let y be your cell phone bill, in dollars, for that month. Use a linear equation to model your monthly bill based on the number of minutes you talk over the phone.

a. This linear model's slope-intercept equation is ☐.

b. If you spent 130 minutes over the phone in a month, you would pay ☐.

c. If in a month, you paid \$20.15 of cell phone bill, you must have spent ☐ minutes on the phone in that month.

71. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way.

Nine minutes since the experiment started, the gas had a mass of 42.9 grams.

Fifteen minutes since the experiment started, the gas had a mass of 35.1 grams.

Let x be the number of minutes that have passed since the experiment started, and let y be the mass of the gas in grams at that moment. Use a linear equation to model the weight of the gas over time.

a. This line's slope-intercept equation is ☐.

b. 32 minutes after the experiment started, there would be ☐ grams of gas left.

c. If a linear model continues to be accurate, ☐ minutes since the experiment started, all gas in the container will be gone.

72. A biologist has been observing a tree's height. 10 months into the observation, the tree was 18.2 feet tall. 16 months into the observation, the tree was 19.22 feet tall.

Let x be the number of months passed since the observations started, and let y be the tree's height at that time, in feet. Use a linear equation to model the tree's height as the number of months pass.

a. This line's slope-intercept equation is ☐.

b. 26 months after the observations started, the tree would be ☐ feet in height.

c. ☐ months after the observation started, the tree would be 25 feet tall.

Challenge

73. Line S has the equation $y = ax+b$ and Line T has the equation $y = cx+d$. Suppose $a > b > c > d > 0$.

a. What can you say about Line S and Line T, given that $a > c$? Give as much information about Line S and Line T as possible.

b. What can you say about Line S and Line T, given that $b > d$? Give as much information about Line S and Line T as possible.

4.6 Point-Slope Form

In Section 4.5, we learned that a linear equation can be written in slope-intercept form, $y = mx + b$. This section covers an alternative that can often be more useful depending on the application: **point-slope form**.

4.6.1 Point-Slope Motivation and Definition

Starting in 1990, the population of the United States has been growing by about 2.865 million per year. Also, back in 1990, the population was 253 million. Since the rate of growth has been roughly constant, a linear model is appropriate. Let's try to write an equation to model this.

We consider using slope-intercept form (4.5.1), but we would need to know the y-intercept, and nothing in the background tells us that. We'd need to know the population of the United States in the year 0, before there even was a United States.

We could do some side work to calculate the y-intercept, but let's try something else. Here are some things we know:

1. The slope equation is $m = \frac{y_2 - y_1}{x_2 - x_1}$.
2. The slope is $m = 2.865$ (million per year).
3. One point on the line is $(1990, 253)$, because in 1990, the population was 253 million.

If we use the generic (x, y) to represent a point *somewhere* on this line, then the rate of change between $(1990, 253)$ and (x, y) has to be 2.865. So

$$\frac{y - 253}{x - 1990} = 2.865.$$

There is good reason[1] to want to isolate y in this equation:

$$\begin{aligned} \frac{y - 253}{x - 1990} &= 2.865 \\ y - 253 &= 2.865 \cdot (x - 1990) && \text{(could distribute, but not going to)} \\ y &= 2.865(x - 1990) + 253 \end{aligned}$$

This is a good place to stop. We have isolated y, and three *meaningful* numbers appear in the population: the rate of growth, a certain year, and the population in that year. This is a specific example of **point-slope form**. Before we look deeper at point-slope form, let's continue reducing the line equation into slope-intercept form.

$$\begin{aligned} y &= 2.865(x - 1990) + 253 \\ y &= 2.865x - 5701.35 + 253 \\ y &= 2.865x - 5448.35 \end{aligned}$$

One concern with slope-intercept form (4.5.1) is that it uses the y-intercept, which might be somewhat meaningless in the context of an application. For example, here we have found that the y-intercept is at $(0, -5448.35)$, but what practical use is that? It's nonsense to say that in the year 0, the population of the United States was -5448.35 million. It doesn't make sense to have a negative population. It doesn't make sense to talk about the United States population before there even was a United States. And it doesn't make sense to use this model for years earlier than 1990 because the background information says clearly that the rate of change we have applies to years 1990 and later.

---

[1] It will help us to see that y (population) *depends* on x (whatever year it is).

For all these reasons, we prefer the equation when it was in the form

$$y = 2.865(x - 1990) + 253$$

Definition 4.6.2 Point-Slope Form. When x and y have a linear relationship where m is the slope and (x_0, y_0) is some specific point that the line passes through, one equation for this relationship is

$$y = m\,(x - x_0) + y_0 \tag{4.6.1}$$

and this equation is called the **point-slope form** of the line. It is called this because the slope and one point on the line are immediately discernible from the numbers in the equation.

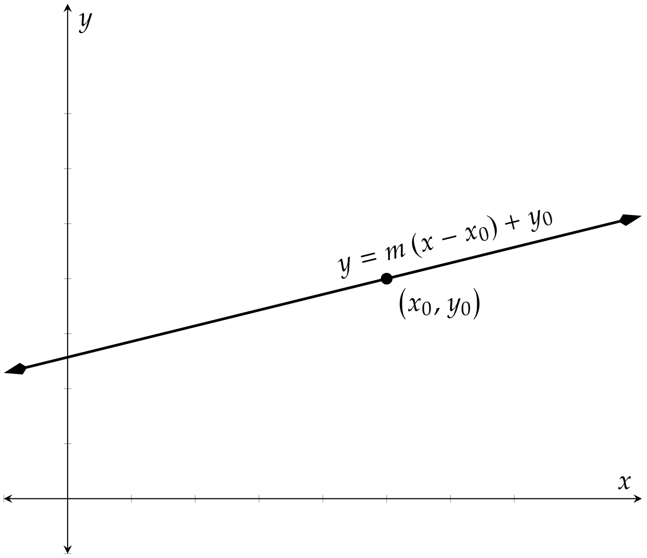


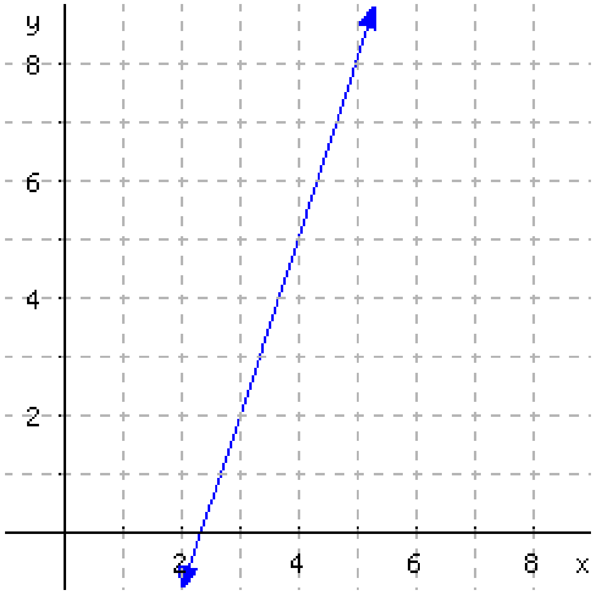
Figure 4.6.3

Remark 4.6.4 Alternative Point-Slope Form. It is also common to define point-slope form as

$$y - y_0 = m\,(x - x_0) \tag{4.6.2}$$

by subtracting y_0 from each side. Some exercises may appear using this form.

Checkpoint 4.6.5. Consider the line in this graph:



a. Identify a point visible on this line that has integer coordinates.

b. What is the slope of the line?

c. Use point-slope form to write an equation for this line, making use of a point with integer coordinates.

Explanation.

a. The visible points with integer coordinates are $(2,-1)$, $(3,2)$, $(4,5)$, and $(5,8)$.

b. Several slope triangles are visible where the "run" is 1 and the "rise" is 3. So the slope is $\frac{3}{1} = 3$.

c. Using $(3,2)$, the point-slope equation is $y = 3(x-3)+2$. (You could use other points, like $(2,-1)$, and get a different-looking equation like $y = 3(x-2)+(-1)$ which simplifies to $y = 3(x-2)-1$.)

In Checkpoint 4.6.5, the solution explains that each of the following are acceptable equations for the same line:

$$y = 3(x-3)+2 \qquad\qquad y = 3(x-2)-1$$

The first uses $(3,2)$ as a point on the line, and the second uses $(2,-1)$. Are those two equations really equivalent? Let's distribute and simplify each of them to get slope-intercept form (4.5.1).

$$\begin{aligned} y &= 3(x-3)+2 & \qquad y &= 3(x-2)-1 \\ y &= 3x-9+2 & y &= 3x-6-1 \\ y &= 3x-7 & y &= 3x-7 \end{aligned}$$

So, yes. It didn't matter which point we used to write a point-slope equation. We get different-looking equations that still represent the same line.

Point-slope form is preferable when we know a line's slope and a point on it, but we don't know the y-intercept.

Example 4.6.6 A spa chain has been losing customers at a roughly constant rate since the year 2010. In 2013, it had 2,975 customers; in 2016, it had 2,585 customers. Management estimated that the company will go out of business once its customer base decreases to 1,800. If this trend continues, when will the company close?

The given information tells us two points on the line: $(2013, 2975)$ and $(2016, 2585)$. The slope formula (4.4.3) will give us the slope. After labeling those two points as $(\overset{x_1}{2013}, \overset{y_1}{2975})$ and $(\overset{x_2}{2016}, \overset{y_2}{2585})$, we have:

$$\begin{aligned}\text{slope} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{2585 - 2975}{2016 - 2013}\\ &= \frac{-390}{3}\\ &= -130\end{aligned}$$

And considering units, this means they are losing 130 customers per year.

Let's note that we could try to make an equation for this line in slope-intercept form, but then we would need to calculate the y-intercept, which in context would correspond to the number of customers in year 0. We could do it, but we'd be working with numbers that have no real-world meaning in this context.

For point-slope form, since we calculated the slope, we know at least this much:

$$y = -130(x - x_0) + y_0.$$

Now we can pick one of those two given points, say $(2013, 2975)$, and get the equation

$$y = -130(x - 2013) + 2975.$$

Note that all three numbers in this equation have meaning in the context of the spa chain.

We're ready to answer the question about when the chain might go out of business. Substitute y in the equation with 1800 and solve for x, and we will get the answer we seek.

$$\begin{aligned}y &= -130(x - 2013) + 2975\\ 1800 &= -130(x - 2013) + 2975\\ 1800 - 2975 &= -130(x - 2013)\\ -1175 &= -130(x - 2013)\\ \frac{-1175}{-130} &= \frac{-130(x - 2013)}{-130}\\ 9.038 &\approx x - 2013\\ 9.038 + 2013 &\approx x\\ 2022 &\approx x\end{aligned}$$

And so we find that at this rate, the company is headed toward a collapse in 2022.

Shown is a graph that represents the scenario. Note that to make a graph of $y = -130(x - 2013) + 2975$, we must first find the point $(2013, 2975)$ and from there use the slope of -130 to draw the line.

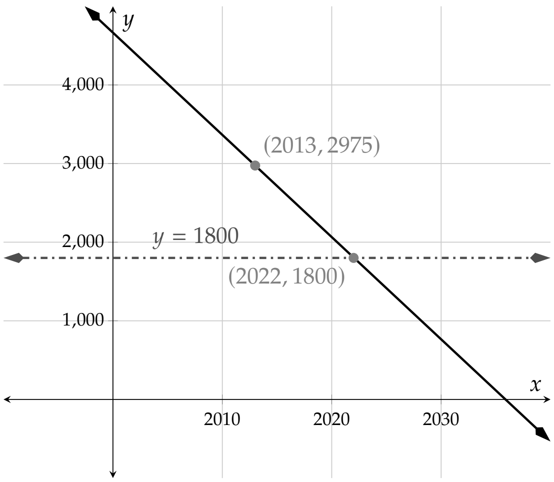
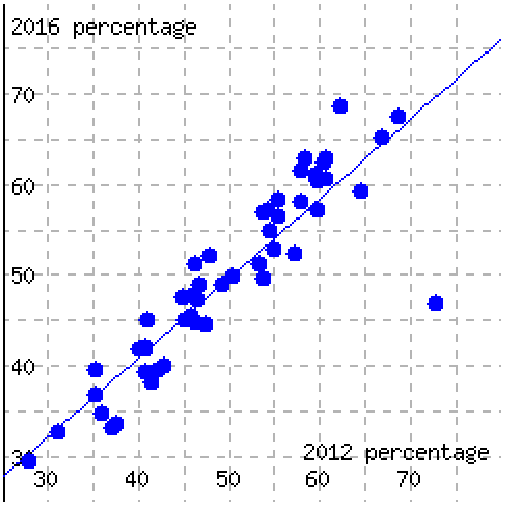


Figure 4.6.7: A Graph of $y = -130(x - 2013) + 2975$

Checkpoint 4.6.8. If we go state by state and compare the Republican candidate's 2012 vote share (x) to the Republican candidate's 2016 vote share (y), we get the following graph where a trendline has been superimposed.



Find a point-slope equation for this line. (Note that a slope-intercept equation would use the y-intercept cooridnate b, and that would not be meaningful in context, since no state had anywhere near zero percent Republican vote.)

Explanation. We need to calculate slope first. And for that, we need to identify two points on the line. conveniently, the line appears to pass right through $(50, 50)$. We have to take a second point somewhere, and $(75, 72)$ seems like a reasonable roughly accurate choice. The slope equation gives us that

$$m = \frac{72 - 50}{75 - 50} = \frac{22}{25} = 0.88.$$

Using $(50, 50)$ as the point, the point-slope equation would then be

$$y = 0.88(x - 50) + 50.$$

4.6.2 Using Two Points to Build a Linear Equation

Since two points can determine a line's location, we can calculate a line's equation using just the coordinates from any two points it passes through.

Example 4.6.9 A line passes through $(-6, 0)$ and $(9, -10)$. Find this line's equation in both point-slope and slope-intercept form.

Explanation. We will use the slope formula (4.4.3) to find the slope first. After labeling those two points as $(\overset{x_1}{-6}, \overset{y_1}{0})$ and $(\overset{x_2}{9}, \overset{y_2}{-10})$, we have:

$$\begin{aligned} \text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-10 - 0}{9 - (-6)} \\ &= \frac{-10}{15} \\ &= -\frac{2}{3} \end{aligned}$$

The point-slope equation is $y = -\frac{2}{3}(x - x_0) + y_0$. Next, we will use $(9, -10)$ and substitute x_0 with 9 and y_0 with -10, and we have:

$$\begin{aligned} y &= -\frac{2}{3}(x - x_0) + y_0 \\ y &= -\frac{2}{3}(x - 9) + (-10) \\ y &= -\frac{2}{3}(x - 9) - 10 \end{aligned}$$

Next, we will change the point-slope equation into slope-intercept form:

$$\begin{aligned} y &= -\frac{2}{3}(x - 9) - 10 \\ y &= -\frac{2}{3}x + 6 - 10 \\ y &= -\frac{2}{3}x - 4 \end{aligned}$$

Remark 4.6.10. Note that many other resources use the alternate point-slope form (4.6.2) to write their equations. Those equations will always be equivalent to those created using our point-slope form. In Example 4.6.9, we found the point-slope form $y = -\frac{2}{3}(x - 9) - 10$. The alternate point-slope form equation[2]would have given us $y + 10 = -\frac{2}{3}(x - 9)$. If you solve this equation for y and simplify, you should still get $y = -\frac{2}{3}x - 4$, as we did earlier.

Checkpoint 4.6.11. A line passes through $(13, -108)$ and $(-42, 23)$. Find equations for this line using both point-slope and slope-intercept form.

A point-slope equation:

A slope-intercept equation:

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[2] khanacademy.org/math/algebra/two-var-linear-equations/point-slope/a/point-slope-form-review

Explanation. First, use the slope formula to find the slope of this line:

$$\begin{aligned} m = \frac{y_2 - y_1}{x_2 - x_1} &= \frac{23 - (-108)}{-42 - 13} \\ &= \frac{131}{-55} \\ &= -\frac{131}{55}. \end{aligned}$$

The generic point-slope equation is $y = m(x - x_0) + y_0$. We have found the slope, m, and we may use $(13, -108)$ for (x_0, y_0). So an equation in point-slope form is $y = \frac{-131}{55}(x - 13) - 108$.

To find a slope-intercept form equation, we can take the generic $y = mx + b$ and substitute in the value of m we found. Also, we know that $(x, y) = (13, -108)$ should make the equation true. So we have

$$\begin{aligned} y &= mx + b \\ -108 &= -\frac{131}{55}(13) + b && \text{Now we may solve for } b. \\ -108 \cdot 55 &= \left(-\frac{131}{55}(13) + b\right) \cdot 55 \\ -5940 &= -131(13) + 55b \\ -5940 &= -1703 + 55b \\ -5940 + 1703 &= -1703 + 55b + 1703 \\ -4237 &= 55b \\ \frac{-4237}{55} &= \frac{55b}{55} \\ b &= -\frac{4237}{55}. \end{aligned}$$

So the slope-intercept equation is $y = \frac{-131}{55}x - \frac{4237}{55}$.

4.6.3 More on Point-Slope Form

We can tell a lot about a linear equation now that we have learned both slope-intercept form (4.5.1) and point-slope form (4.6.1). For example, we can know that $y = 4x + 2$ is in slope-intercept form because it looks like $y = mx + b$. It will graph as a line with slope 4 and vertical intercept $(0, 2)$. Likewise, we know that the equation $y = -5(x - 3) + 2$ is in point-slope form because it looks like $y = m(x - x_0) + y_0$. It will graph as a line that has slope -5 and will pass through the point $(3, 2)$.

Example 4.6.12 For the equations below, state whether they are in slope-intercept form or point-slope form. Then identify the slope of the line and at least one point that the line will pass through.

a. $y = -3x + 2$

b. $y = 9(x + 1) - 6$

c. $y = 5 - x$

d. $y = -\frac{12}{5}(x - 9) + 1$

Explanation.

a. The equation $y = -3x + 2$ is in slope-intercept form. The slope is -3 and the vertical intercept is $(0, 2)$.

b. The equation $y = 9(x + 1) - 6$ is in point-slope form. The slope is 9 and the line passes through the point $(-1, -6)$.

c. The equation $y = 5 - x$ is almost in slope-intercept form. If we rearrange the right hand side to be $y = -x + 5$, we can see that the slope is -1 and the vertical intercept is $(0, 5)$.

d. The equation $y = -\frac{12}{5}(x - 9) + 1$ is in point-slope form. The slope is $-\frac{12}{5}$ and the line passes through the point $(9, 1)$.

Remark 4.6.13. Again, we should note that the alternate point-slope form (4.6.2) can be used to identify equations. For example, the equation $y + 10 = -\frac{2}{3}(x - 9)$ matches the alternate point-slope form equation[3] with slope $-\frac{2}{3}$ and the line passes through the point $(9, -10)$. Note that both coordinates are the opposite of what they appear to be in the equation with this form.

Consider the graph in Figure 4.6.15.

Example 4.6.14 a. Find three equations that describe the line shown written in point-slope form. Three integer-valued points are shown for convenience.

b. Determine the slope-intercept form of the equation of this line.

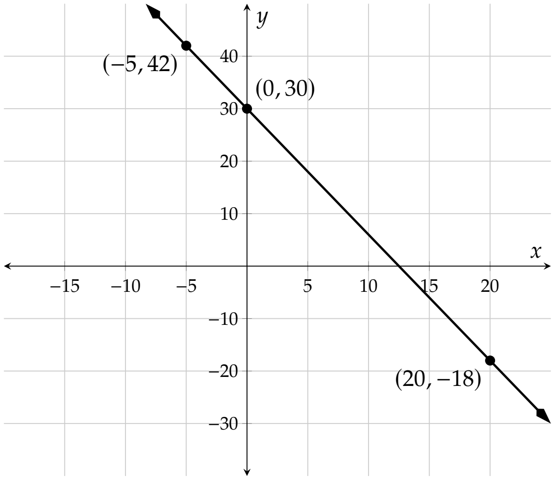

Figure 4.6.15

Explanation.

a. To write *any* of the equations representing this line in point-slope form, we must first find the slope of the line and we can use the slope formula (4.4.3) to do so. We will arbitrarily choose $(0, 30)$ and $(-5, 42)$ as the two points. Inputting these points into the slope formula yields:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{42 - 30}{-5 - 0} \\ &= \frac{12}{-5} \end{aligned}$$

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[3] en.wikipedia.org/wiki/Linear_equation#Point–slope_form

$$= -\frac{12}{5}$$

Thus the slope of the line is $-\frac{12}{5}$.

Next, we need to write an equation in point-slope form based on each point shown. Using the point $(0, 30)$, we have:

$$y = -\frac{12}{5}(x - 0) + 30$$

(This simplifies to $y = -\frac{12}{5}x + 30$.)

The next point is $(20, -18)$. Using this point, we can write an equation for this line as:

$$y = -\frac{12}{5}(x - 20) - 18$$

Finally, we can also use the point $(-5, 42)$ to write an equation for this line:

$$y = -\frac{12}{5}(x - (-5)) + 42$$

which can also be written as:

$$y = -\frac{12}{5}(x + 5) + 42$$

b. As $(0, 30)$ is the vertical intercept, we can write the equation of this line in slope-intercept form as $y = -\frac{12}{5}x + 30$. It's important to note that each of the equations that were written in point-slope form simplify to this, making all four equations equivalent.

Exercises

Review and Warmup

1. Evaluate $-5C - 7b$ for $C = 6$ and $b = -7$.

2. Evaluate $-a + 3A$ for $a = 2$ and $A = -7$.

3. Evaluate

$$\frac{y_2 - y_1}{x_2 - x_1}$$

for $x_1 = -14$, $x_2 = -10$, $y_1 = -19$, and $y_2 = -10$:

4. Evaluate

$$\frac{y_2 - y_1}{x_2 - x_1}$$

for $x_1 = -10$, $x_2 = 17$, $y_1 = -2$, and $y_2 = 19$:

Point-Slope Form

5. A line's equation is given in point-slope form:

$y = 5(x - 5) + 28$

This line's slope is $\boxed{}$.

A point on this line that is apparent from the given equation is $\boxed{}$.

6. A line's equation is given in point-slope form:

$y = 2(x - 1) + 5$

This line's slope is $\boxed{}$.

A point on this line that is apparent from the given equation is $\boxed{}$.

7. A line's equation is given in point-slope form:

$y = -2(x + 2) + 5$

This line's slope is $\boxed{}$.

A point on this line that is apparent from the given equation is $\boxed{}$.

8. A line's equation is given in point-slope form:

$y = -3(x + 4) + 7$

This line's slope is $\boxed{}$.

A point on this line that is apparent from the given equation is $\boxed{}$.

9. A line's equation is given in point-slope form:

$y = \frac{8}{3}(x + 9) - 23$

This line's slope is $\boxed{}$.

A point on this line that is apparent from the given equation is $\boxed{}$.

10. A line's equation is given in point-slope form:

$y = \frac{9}{8}(x + 24) - 29$

This line's slope is $\boxed{}$.

A point on this line that is apparent from the given equation is $\boxed{}$.

11. A line passes through the points $(2, 9)$ and $(1, 7)$. Find this line's equation in point-slope form.

Using the point $(2, 9)$, this line's point-slope form equation is $\boxed{}$.

Using the point $(1, 7)$, this line's point-slope form equation is $\boxed{}$.

12. A line passes through the points $(4, 10)$ and $(1, 4)$. Find this line's equation in point-slope form.

Using the point $(4, 10)$, this line's point-slope form equation is $\boxed{}$.

Using the point $(1, 4)$, this line's point-slope form equation is $\boxed{}$.

13. A line passes through the points $(-3, 17)$ and $(0, 8)$. Find this line's equation in point-slope form.

Using the point $(-3, 17)$, this line's point-slope form equation is $\boxed{}$.

Using the point $(0, 8)$, this line's point-slope form equation is $\boxed{}$.

14. A line passes through the points $(1, 2)$ and $(-3, 14)$. Find this line's equation in point-slope form.

Using the point $(1, 2)$, this line's point-slope form equation is $\boxed{}$.

Using the point $(-3, 14)$, this line's point-slope form equation is $\boxed{}$.

15. A line passes through the points $(6,5)$ and $(-6,-25)$. Find this line's equation in point-slope form.

Using the point $(6,5)$, this line's point-slope form equation is ☐.

Using the point $(-6,-25)$, this line's point-slope form equation is ☐.

16. A line passes through the points $(7,5)$ and $(21,17)$. Find this line's equation in point-slope form.

Using the point $(7,5)$, this line's point-slope form equation is ☐.

Using the point $(21,17)$, this line's point-slope form equation is ☐.

17. A line's slope is 4. The line passes through the point $(5,22)$. Find an equation for this line in both point-slope and slope-intercept form.

An equation for this line in point-slope form is: ☐.

An equation for this line in slope-intercept form is: ☐.

18. A line's slope is 5. The line passes through the point $(2,11)$. Find an equation for this line in both point-slope and slope-intercept form.

An equation for this line in point-slope form is: ☐.

An equation for this line in slope-intercept form is: ☐.

19. A line's slope is -2. The line passes through the point $(2,-2)$. Find an equation for this line in both point-slope and slope-intercept form.

An equation for this line in point-slope form is: ☐.

An equation for this line in slope-intercept form is: ☐.

20. A line's slope is -5. The line passes through the point $(-4,18)$. Find an equation for this line in both point-slope and slope-intercept form.

An equation for this line in point-slope form is: ☐.

An equation for this line in slope-intercept form is: ☐.

21. A line's slope is 1. The line passes through the point $(5,1)$. Find an equation for this line in both point-slope and slope-intercept form.

An equation for this line in point-slope form is: ☐.

An equation for this line in slope-intercept form is: ☐.

22. A line's slope is 1. The line passes through the point $(2,-1)$. Find an equation for this line in both point-slope and slope-intercept form.

An equation for this line in point-slope form is: ☐.

An equation for this line in slope-intercept form is: ☐.

23. A line's slope is -1. The line passes through the point $(-2, 1)$. Find an equation for this line in both point-slope and slope-intercept form.

An equation for this line in point-slope form is: .

An equation for this line in slope-intercept form is: .

24. A line's slope is -1. The line passes through the point $(3, 2)$. Find an equation for this line in both point-slope and slope-intercept form.

An equation for this line in point-slope form is: .

An equation for this line in slope-intercept form is: .

25. A line's slope is $\frac{6}{5}$. The line passes through the point $(10, 8)$. Find an equation for this line in both point-slope and slope-intercept form.

An equation for this line in point-slope form is: .

An equation for this line in slope-intercept form is: .

26. A line's slope is $\frac{7}{4}$. The line passes through the point $(12, 22)$. Find an equation for this line in both point-slope and slope-intercept form.

An equation for this line in point-slope form is: .

An equation for this line in slope-intercept form is: .

27. A line's slope is $-\frac{8}{9}$. The line passes through the point $(9, -13)$. Find an equation for this line in both point-slope and slope-intercept form.

An equation for this line in point-slope form is: .

An equation for this line in slope-intercept form is: .

28. A line's slope is $-\frac{9}{5}$. The line passes through the point $(10, -20)$. Find an equation for this line in both point-slope and slope-intercept form.

An equation for this line in point-slope form is: .

An equation for this line in slope-intercept form is: .

Point-Slope and Slope-Intercept Change this equation from point-slope form to slope-intercept form.

29. $y = 2(x - 4) + 5$

In slope-intercept form:

30. $y = 2(x + 2)$

In slope-intercept form:

31. $y = -4(x - 3) - 10$

In slope-intercept form:

32. $y = -4(x + 4) + 12$

In slope-intercept form:

33. $y = \dfrac{5}{8}(x - 16) + 15$

In slope-intercept form:

34. $y = \dfrac{6}{5}(x - 5) + 11$

In slope-intercept form:

35. $y = -\frac{7}{3}(x + 3) + 6$

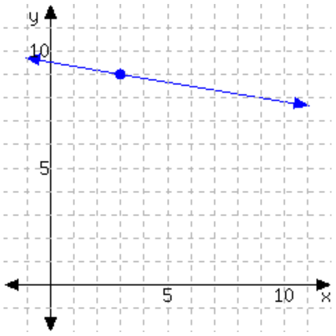
In slope-intercept form:

36. $y = -\frac{8}{7}(x + 21) + 29$

In slope-intercept form:

Point-Slope Form and Graphs Determine the point-slope form of the linear equation from its graph.

37.



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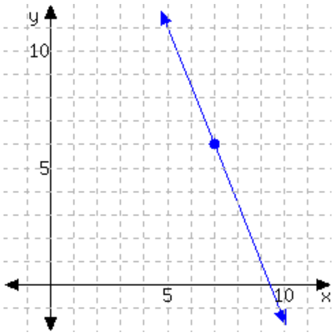

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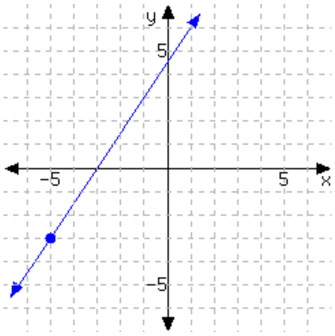

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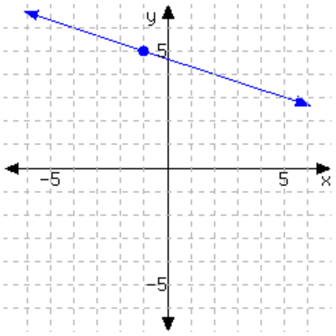

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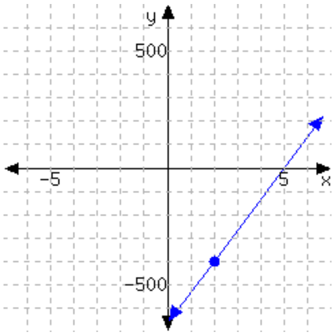

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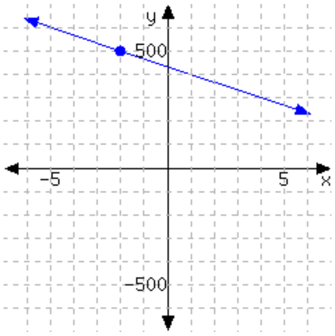

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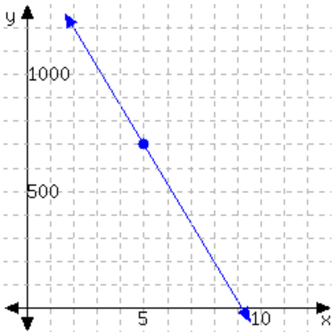

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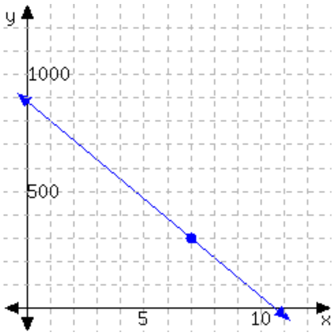

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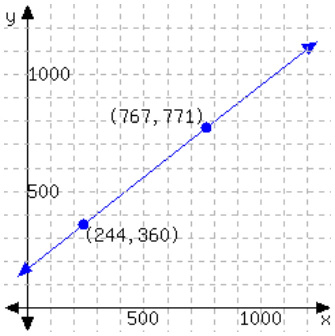

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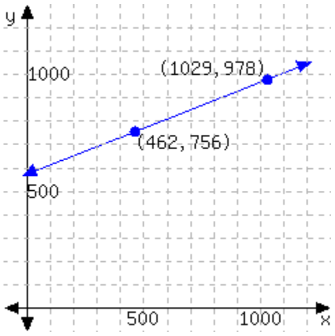

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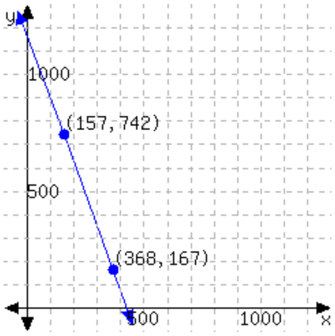

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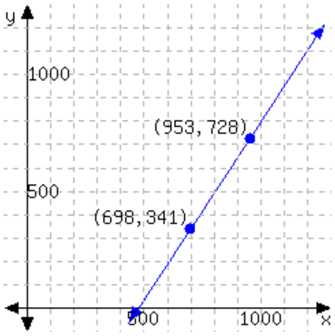


49. Graph the linear equation $y = -\frac{8}{3}(x - 4) - 5$ by identifying the slope and one point on this line.

50. Graph the linear equation $y = \frac{5}{7}(x + 3) + 2$ by identifying the slope and one point on this line.

51. Graph the linear equation $y = \frac{3}{4}(x + 2) + 1$ by identifying the slope and one point on this line.

52. Graph the linear equation $y = -\frac{5}{2}(x - 1) - 5$ by identifying the slope and one point on this line.

53. Graph the linear equation $y = -3(x - 9) + 4$ by identifying the slope and one point on this line.

54. Graph the linear equation $y = 7(x + 3) - 10$ by identifying the slope and one point on this line.

55. Graph the linear equation $y = 8(x + 12) - 20$ by identifying the slope and one point on this line.

56. Graph the linear equation $y = -5(x - 20) - 70$ by identifying the slope and one point on this line.

Applications

57. By your cell phone contract, you pay a monthly fee plus \$0.04 for each minute you spend on the phone. In one month, you spent 230 minutes over the phone, and had a bill totaling \$22.20.

Let x be the number of minutes you spend on the phone in a month, and let y be your total cell phone bill for that month, in dollars. Use a linear equation to model your monthly bill based on the number of minutes you spend on the phone.

a. A point-slope equation to model this is ☐.

b. If you spend 160 minutes on the phone in a month, you would be billed ☐.

c. If your bill was \$30.60 one month, you must have spent ☐ minutes on the phone in that month.

58. A company set aside a certain amount of money in the year 2000. The company spent exactly \$31,000 from that fund each year on perks for its employees. In 2003, there was still \$872,000 left in the fund.

Let x be the number of years since 2000, and let y be the amount of money, in dollars, left in the fund that year. Use a linear equation to model the amount of money left in the fund after so many years.

a. A point-slope equation to model this is .

b. In the year 2010, there was left in the fund.

c. In the year , the fund will be empty.

59. A biologist has been observing a tree's height. This type of tree typically grows by 0.19 feet each month. Ten months into the observation, the tree was 17.4 feet tall.

Let x be the number of months passed since the observations started, and let y be the tree's height at that time, in feet. Use a linear equation to model the tree's height as the number of months pass.

a. A point-slope equation to model this is .

b. 28 months after the observations started, the tree would be feet in height.

c. months after the observation started, the tree would be 25.76 feet tall.

60. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way. Each minute, they lose 3 grams. Seven minutes since the experiment started, the remaining gas had a mass of 117 grams.

Let x be the number of minutes that have passed since the experiment started, and let y be the mass of the gas in grams at that moment. Use a linear equation to model the weight of the gas over time.

a. A point-slope equation to model this is .

b. 35 minutes after the experiment started, there would be grams of gas left.

c. If a linear model continues to be accurate, minutes since the experiment started, all gas in the container will be gone.

61. A company set aside a certain amount of money in the year 2000. The company spent exactly the same amount from that fund each year on perks for its employees. In 2004, there was still \$783,000 left in the fund. In 2006, there was \$701,000 left.

Let x be the number of years since 2000, and let y be the amount of money, in dollars, left in the fund that year. Use a linear equation to model the amount of money left in the fund after so many years.

a. A point-slope equation to model this is ☐.

b. In the year 2010, there was ☐ left in the fund.

c. In the year ☐, the fund will be empty.

62. By your cell phone contract, you pay a monthly fee plus some money for each minute you use the phone during the month. In one month, you spent 230 minutes on the phone, and paid \$27.65. In another month, you spent 320 minutes on the phone, and paid \$32.60.

Let x be the number of minutes you talk over the phone in a month, and let y be your cell phone bill, in dollars, for that month. Use a linear equation to model your monthly bill based on the number of minutes you talk over the phone.

a. A point-slope equation to model this is ☐.

b. If you spent 150 minutes over the phone in a month, you would pay ☐.

c. If in a month, you paid \$41.95 of cell phone bill, you must have spent ☐ minutes on the phone in that month.

63. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way.

Nine minutes since the experiment started, the gas had a mass of 106.6 grams.

Eighteen minutes since the experiment started, the gas had a mass of 83.2 grams.

Let x be the number of minutes that have passed since the experiment started, and let y be the mass of the gas in grams at that moment. Use a linear equation to model the weight of the gas over time.

a. A point-slope equation to model this is ☐.

b. 35 minutes after the experiment started, there would be ☐ grams of gas left.

c. If a linear model continues to be accurate, ☐ minutes since the experiment started, all gas in the container will be gone.

64. A biologist has been observing a tree's height. 10 months into the observation, the tree was 18.6 feet tall. 18 months into the observation, the tree was 19.4 feet tall.

Let x be the number of months passed since the observations started, and let y be the tree's height at that time, in feet. Use a linear equation to model the tree's height as the number of months pass.

a. A point-slope equation to model this is .

b. 27 months after the observations started, the tree would be feet in height.

c. months after the observation started, the tree would be 23.5 feet tall.

4.7 Standard Form

We've seen that a linear relationship can be expressed with an equation in slope-intercept form (4.5.1) or with an equation in point-slope-form (4.6.1). There is a third standard form that you can use to write line equations. It's so "standard" that it's actually known as **standard form**.

4.7.1 Standard Form Definition

Imagine trying to gather donations to pay for a \$10,000 medical procedure you cannot afford. Oversimplifying the mathematics a bit, suppose that there were only two types of donors in the world: those who will donate \$20 and those who will donate \$100. How many of each, or what combination, do you need to reach the funding goal? As in, if x people donate \$20 and y people donate \$100, what numbers could x and y be? The donors of the first type have collectively donated $20x$ dollars, and the donors of the second type have collectively donated $100y$. So altogether you'd need

$$20x + 100y = 10000.$$

This is an example of a line equation in **standard form**.

Definition 4.7.2 Standard Form. It is always possible to write an equation for a line in the form

$$Ax + By = C \tag{4.7.1}$$

where A, B, and C are three numbers (each of which might be 0, although at least one of A and B must be nonzero). This form of a line equation is called **standard form**. In the context of an application, the meaning of A, B, and C depends on that context. This equation is called **standard form** perhaps because *any* line can be written this way, even vertical lines which cannot be written using the two previous forms we've studied.

Checkpoint 4.7.3. For each of the following equations, identify what form they are in.

$2.7x + 3.4y = -82$ (☐ slope-intercept ☐ point-slope ☐ standard ☐ other linear ☐ not linear)
$y = \frac{2}{7}(x-3) + \frac{1}{10}$ (☐ slope-intercept ☐ point-slope ☐ standard ☐ other linear ☐ not linear)
$12x - 3 = y + 2$ (☐ slope-intercept ☐ point-slope ☐ standard ☐ other linear ☐ not linear)
$y = x^2 + 5$ (☐ slope-intercept ☐ point-slope ☐ standard ☐ other linear ☐ not linear)
$x - y = 10$ (☐ slope-intercept ☐ point-slope ☐ standard ☐ other linear ☐ not linear)
$y = 4x + 1$ (☐ slope-intercept ☐ point-slope ☐ standard ☐ other linear ☐ not linear)

Explanation. $2.7x + 3.4y = -82$ is in standard form, with $A = 2.7$, $B = 3.4$, and $C = -82$.

$y = \frac{2}{7}(x-3) + \frac{1}{10}$ is in point-slope form, with slope $\frac{2}{7}$, and passing through $\left(3, \frac{1}{10}\right)$.

$12x - 3 = y + 2$ is linear, but not in any of the forms we have studied. Using algebra, you can rearrange it to read $y = 12x - 5$.

$y = x^2 + 5$ is not linear. The exponent on x is a dead giveaway.

$x - y = 10$ is in standard form, with $A = 1$, $B = -1$, and $C = 10$.

$y = 4x + 1$ is in slope-intercept form, with slope 4 and y-intercept at $(0, 1)$.

Returning to the example with donations for the medical procedure, let's examine the equation

$$20x + 100y = 10000.$$

What units are attached to all of the parts of this equation? Both x and y are numbers of people. The 10000 is in dollars. Both the 20 and the 100 are in dollars per person. Note how both sides of the equation are in dollars. On the right, that fact is clear. On the left, $20x$ is in dollars since 20 is in dollars per person, and x is in people. The same is true for $100y$, and the two dollar amounts $20x$ and $100y$ add to a dollar amount.

What is the slope of the linear relationship? It's not immediately visible since m is not part of the standard form equation. But we can use algebra to isolate y:

$$\begin{aligned}20x+100y&=10000\\100y&=-20x+10000\\y&=\frac{-20x+10000}{100}\\y&=\frac{-20x}{100}+\frac{10000}{100}\\y&=-\frac{1}{5}x+100.\end{aligned}$$

And we see that the slope is $-\frac{1}{5}$. OK, what units are on that slope? As always, the units on slope are $\frac{y\text{-unit}}{x\text{-unit}}$. In this case that's $\frac{\text{person}}{\text{person}}$, which sounds a little weird and seems like it should be simplified away to unitless. But this slope of $-\frac{1}{5}\frac{\text{person}}{\text{person}}$ is saying that for every one extra person who donates \$20, you only need $\frac{1}{5}$ fewer people donating \$100 to still reach your goal.

What is the y-intercept? Since we've already converted the equation into slope-intercept form, we can see that it is at $(0,100)$. This tells us that if 0 people donate \$20, then you will need 100 people to each donate \$100.

What does a graph for this line look like? We've already converted into slope-intercept form, and we could use that to make the graph. But when given a line in standard form, there is another approach that is often used. Returning to

$$20x+100y=10000,$$

let's calculate the y-intercept and the x-intercept. Recall that these are *points* where the line crosses the y-axis and x-axis. To be on the y-axis means that $x=0$, and to be on the x-axis means that $y=0$. All these zeros make the resulting algebra easy to solve:

$$\begin{aligned}20x+100y&=10000 & 20x+100y&=10000\\20(0)+100y&=10000 & 20x+100(0)&=10000\\100y&=10000 & 20x&=10000\\y&=\frac{10000}{100} & x&=\frac{10000}{20}\\y&=100 & x&=500\end{aligned}$$

So we have a y-intercept at $(0, 100)$ and an x-intercept at $(500, 0)$. If we plot these, we get to mark especially relevant points given the context, and then drawing a straight line between them gives us Figure 4.7.4.

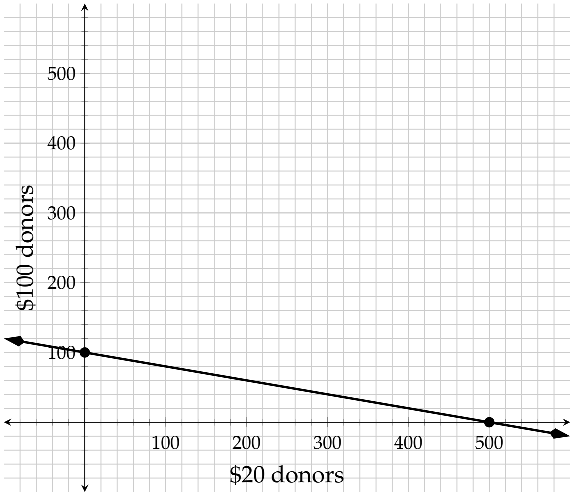


Figure 4.7.4

4.7.2 The x- and y-Intercepts

With a linear relationship (and other types of equations), we are often interested in the x-intercept and y-intercept because they are important in the context. For example, in Figure 4.7.4, the x-intercept implies that if *no one* donates \$100, you need 500 people to donate \$20 to get us to \$10,000. And the y-intercept implies if *no one* donates \$20, you need 100 people to donate \$100. Let's look at another example.

Example 4.7.5 James owns a restaurant that uses about 32 lb of flour every day. He just purchased 1200 lb of flour. Model the amount of flour that remains x days later with a linear equation, and interpret the meaning of its x-intercept and y-intercept.

Since the rate of change is constant (−32 lb every day), and we know the initial value, we can model the amount of flour at the restaurant with a slope-intercept equation (4.5.1):

$$y = -32x + 1200$$

where x represents the number of days passed since the initial purchase, and y represents the amount of flour left (in lb.)

A line's x-intercept is in the form of $(x, 0)$, since to be on the x-axis, the y-coordinate must be 0. To find this line's x-intercept, we substitute y in the equation with 0, and solve for x:

$$\begin{aligned} y &= -32x + 1200 \\ 0 &= -32x + 1200 \\ 0 - 1200 &= -32x \\ -1200 &= -32x \\ \frac{-1200}{-32} &= x \\ 37.5 &= x \end{aligned}$$

So the line's x-intercept is at $(37.5, 0)$. In context this means the flour would last for 37.5 days.

A line's y-intercept is in the form of $(0, y)$. This line equation is already in slope-intercept form, so we can just see that its y-intercept is at $(0, 1200)$. In general though, we would substitute x in the equation with 0, and we have:

$$\begin{aligned} y &= -32x + 1200 \\ y &= -32(0) + 1200 \\ y &= 1200 \end{aligned}$$

So yes, the line's y-intercept is at $(0, 1200)$. This means that when the flour was purchased, there was 1200 lb of it. In other words, the y-intercept tells us one of the original pieces of information: in the beginning, James purchased 1200 lb of flour.

If a line is in standard form, it's often easiest to graph it using its two intercepts.

Example 4.7.6 Graph $2x - 3y = -6$ using its intercepts. And then use the intercepts to calculate the line's slope.

Explanation. To graph a line by its x-intercept and y-intercept, it might help to first set up a table like Table 4.7.7:

	x-value	y-value	Intercepts
x-intercept		0	
y-intercept	0		

Table 4.7.7: Intercepts of $2x - 3y = -6$

A table like this might help you stay focused on the fact that we are searching for *two* points. As we've noted earlier, an x-intercept is on the x-axis, and so its y-coordinate must be 0. This is worth taking special note of: to find an x-intercept, y must be 0. This is why we put 0 in the y-value cell of the x-intercept. Similarly, a line's y-intercept has $x = 0$, and we put 0 into the x-value cell of the y-intercept.

Next, we calculate the line's x-intercept by substituting $y = 0$ into the equation

$$\begin{aligned} 2x - 3y &= -6 \\ 2x - 3(0) &= -6 \\ 2x &= -6 \\ x &= -3 \end{aligned}$$

So the line's x-intercept is $(-3, 0)$.

Similarly, we substitute $x = 0$ into the equation to calculate the y-intercept:

$$\begin{aligned} 2x - 3y &= -6 \\ 2(0) - 3y &= -6 \\ -3y &= -6 \\ y &= 2 \end{aligned}$$

So the line's y-intercept is $(0, 2)$.

Now we can complete the table:

	x-value	y-value	Intercepts
x-intercept	-3	0	$(-3, 0)$
y-intercept	0	2	$(0, 2)$

Table 4.7.8: Intercepts of $2x - 3y = -6$

With both intercepts' coordinates, we can graph the line:

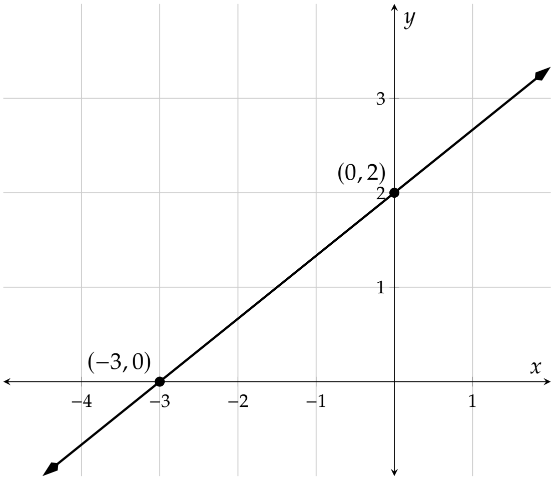


Figure 4.7.9: Graph of $2x - 3y = -6$

There is a slope triangle from the x-intercept to the origin up to the y-intercept. It tells us that the slope is

$$m = \frac{\Delta y}{\Delta x} = \frac{2}{3}.$$

This last example generalizes to a fact worth noting.

Fact 4.7.10. *If a line's x-intercept is at $(r, 0)$ and its y-intercept is at $(0, b)$, then the slope of the line is $-\frac{b}{r}$. (Unless the line passes through the origin. Then both r and b equal 0, and then this fraction is undefined. And the slope of the line could be anything.)*

Checkpoint 4.7.11. Consider the line with equation $2x + 4.3y = \frac{1000}{99}$.

a. What is its x-intercept?

b. What is its y-intercept?

c. What is its slope?

Explanation.

a. To find the x-intercept:

$$\begin{aligned} 2x + 4.3y &= \frac{1000}{99} \\ 2x + 4.3(0) &= \frac{1000}{99} \\ 2x &= \frac{1000}{99} \\ x &= \frac{500}{99} \end{aligned}$$

So the x-intercept is at $\left(\frac{500}{99}, 0\right)$.

b. To find the y-intercept:

$$\begin{aligned} 2x + 4.3y &= \frac{1000}{99} \\ 2(0) + 4.3y &= \frac{1000}{99} \\ 4.3y &= \frac{1000}{99} \\ y &= \frac{1}{4.3} \cdot \frac{1000}{99} \\ y &\approx 2.349\ldots \end{aligned}$$

So the y-intercept is at about $(0, 2.349)$.

c. Since we have the x- and y-intercepts, we can calulate the slope:

$$m \approx -\frac{2.349}{\frac{500}{99}} = -\frac{2.349 \cdot 99}{500} \approx -0.4561.$$

4.7.3 Transforming between Standard Form and Slope-Intercept Form

Sometimes a linear equation arises in standard form (4.7.1), but it would be useful to see that equation in slope-intercept form (4.5.1). Or perhaps, vice versa.

A linear equation in slope-intercept form (4.5.1) tells us important information about the line: its slope m and y-intercept $(0, b)$. However, a line's standard form does not show those two important values. As a result, we often need to change a line's equation from standard form to slope-intercept form. Let's look at some examples.

Example 4.7.12 Change $2x - 3y = -6$ to slope-intercept form, and then graph it.

Explanation. Since a line in slope-intercept form looks like $y = \ldots$, we will solve for y in $2x - 3y = -6$:

$$\begin{aligned} 2x - 3y &= -6 \\ -3y &= -6 - 2x \\ -3y &= -2x - 6 \\ y &= \frac{-2x - 6}{-3} \\ y &= \frac{-2x}{-3} - \frac{6}{-3} \\ y &= \frac{2}{3}x + 2 \end{aligned}$$

In the third line, we wrote $-2x - 6$ on the right side, instead of $-6 - 2x$. The only reason we did this is because we are headed to slope-intercept form, where the x-term is traditionally written first.

Now we can see that the slope is $\frac{2}{3}$ and the y-intercept is at $(0, 2)$. With these things found, we can graph the line using slope triangles.

Compare this graphing method with the Graphing by Intercepts method in Example 4.7.6. We have more points in this graph, thus we can graph the line more accurately.

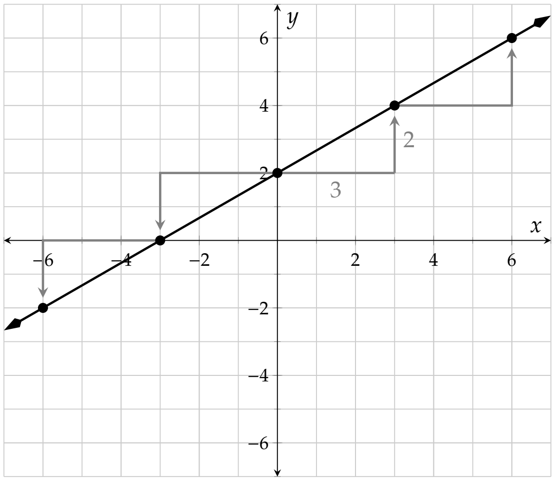


Figure 4.7.13: Graphing $2x - 3y = -6$ with Slope Triangles

Example 4.7.14 Graph $2x - 3y = 0$.

Explanation. First, we will try (and fail) to graph this line using its x- and y-intercepts.

Trying to find the x-intercept:

$$\begin{aligned}2x - 3y &= 0\\ 2x - 3(0) &= 0\\ 2x &= 0\\ x &= 0\end{aligned}$$

So the line's x-intercept is at $(0, 0)$, at the origin.

Huh, that is *also* on the y-axis…

Trying to find the y-intercept:

$$\begin{aligned}2x - 3y &= 0\\ 2(0) - 3y &= 0\\ -3y &= 0\\ y &= 0\end{aligned}$$

So the line's y-intercept is also at $(0, 0)$.

Since both intercepts are the same point, there is no way to use the intercepts alone to graph this line. So what can be done?

Several approaches are out there, but one is to convert the line equation into slope-intercept form:

$$\begin{aligned}2x-3y&=0\\-3y&=0-2x\\-3y&=-2x\\y&=\frac{-2x}{-3}\\y&=\frac{2}{3}x\end{aligned}$$

So the line's slope is $\frac{2}{3}$, and we can graph the line using slope triangles and the intercept at $(0,0)$, as in Figure 4.7.15.

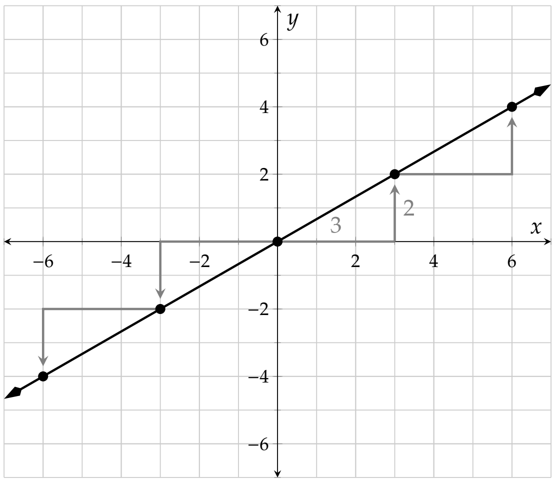


Figure 4.7.15: Graphing $2x-3y=0$ with Slope Triangles

In summary, if $C=0$ in a standard form equation (4.7.1), it's convenient to graph it by first converting the equation to slope-intercept form (4.5.1).

Example 4.7.16 Write the equation $y=\frac{2}{3}x+2$ in standard form.

Explanation. Once we subtract $\frac{2}{3}x$ on both sides of the equation, we have

$$-\frac{2}{3}x+y=2$$

Technically, this equation is already in standard form $Ax+By=C$. However, you might like to end up with an equation that has no fractions, and so you can take some extra steps.

$$\begin{aligned}y&=\frac{2}{3}x+2\\y-\frac{2}{3}x&=2\\-\frac{2}{3}x+y&=2\end{aligned}$$

This is in standard form, but we keep going to clear away the fraction.

$$\begin{aligned}3\cdot\left(-\frac{2}{3}x+y\right)&=3\cdot2\\-2x+3y&=6\end{aligned}$$

Exercises

Review and Warmup Solve the linear equation for y.

1. $3y - 6x = 27$

2. $-18x - 3y = 6$

3. $-x - y = 16$

4. $-4x - y = -9$

5. $2x - 8y = -5$

6. $-5x - 9y = -5$

Slope and y-intercept Find the line's slope and y-intercept.

7. A line has equation $-2x + y = 4$.

This line's slope is ☐.

This line's y-intercept is ☐.

8. A line has equation $-x - y = -8$.

This line's slope is ☐.

This line's y-intercept is ☐.

9. A line has equation $2x + 2y = 4$.

This line's slope is ☐.

This line's y-intercept is ☐.

10. A line has equation $12x - 3y = -9$.

This line's slope is ☐.

This line's y-intercept is ☐.

11. A line has equation $x + 3y = -6$.

This line's slope is ☐.

This line's y-intercept is ☐.

12. A line has equation $7x + 6y = -24$.

This line's slope is ☐.

This line's y-intercept is ☐.

13. A line has equation $7x - 6y = 24$.

This line's slope is ☐.

This line's y-intercept is ☐.

14. A line has equation $20x + 12y = -36$.

This line's slope is ☐.

This line's y-intercept is ☐.

15. A line has equation $12x - 10y = 0$.

This line's slope is ☐.

This line's y-intercept is ☐.

16. A line has equation $3x - 12y = 0$.

This line's slope is ☐.

This line's y-intercept is ☐.

17. A line has equation $2x + 6y = 5$.

This line's slope is ☐.

This line's y-intercept is ☐.

18. A line has equation $8x + 12y = 5$.

This line's slope is ☐.

This line's y-intercept is ☐.

Converting to Standard Form

19. Rewrite $y = 4x + 7$ in standard form.

20. Rewrite $y = 5x - 6$ in standard form.

21. Rewrite $y = \frac{6}{7}x - 6$ in standard form.

22. Rewrite $y = -\frac{7}{5}x - 7$ in standard form.

Graphs and Standard Form

23. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter `none` into all the answer blanks for that row.

$$7x + 2y = 28$$

	x-value	y-value	Location
y-intercept			
x-intercept			

24. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter `none` into all the answer blanks for that row.

$$8x + 7y = -168$$

	x-value	y-value	Location
y-intercept			
x-intercept			

25. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter `none` into all the answer blanks for that row.

$$2x - 5y = -20$$

	x-value	y-value	Location
y-intercept	____________	____________	________________________
x-intercept	____________	____________	________________________

26. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter `none` into all the answer blanks for that row.

$$x - 3y = -3$$

	x-value	y-value	Location
y-intercept	____________	____________	________________________
x-intercept	____________	____________	________________________

27. Find the x- and y-intercepts of the line with equation $4x + 6y = 24$. Then find one other point on the line. Use your results to graph the line.

28. Find the x- and y-intercepts of the line with equation $4x + 5y = -40$. Then find one other point on the line. Use your results to graph the line.

29. Find the x- and y-intercepts of the line with equation $5x - 2y = 10$. Then find one other point on the line. Use your results to graph the line.

30. Find the x- and y-intercepts of the line with equation $5x - 6y = -90$. Then find one other point on the line. Use your results to graph the line.

31. Find the x- and y-intercepts of the line with equation $x + 5y = -15$. Then find one other point on the line. Use your results to graph the line.

32. Find the x- and y-intercepts of the line with equation $6x + y = -18$. Then find one other point on the line. Use your results to graph the line.

33. Make a graph of the line $x + y = 2$.

34. Make a graph of the line $-5x - y = -3$.

35. Make a graph of the line $x + 5y = 5$.

36. Make a graph of the line $x - 2y = 2$.

37. Make a graph of the line $20x - 4y = 8$.

38. Make a graph of the line $3x + 5y = 10$.

39. Make a graph of the line $-3x + 2y = 6$.

40. Make a graph of the line $-4x - 5y = 10$.

41. Make a graph of the line $4x - 5y = 0$.

42. Make a graph of the line $5x + 7y = 0$.

Interpreting Intercepts in Context

43. Scot is buying some tea bags and some sugar bags. Each tea bag costs 6 cents, and each sugar bag costs 2 cents. He can spend a total of $1.80.

Assume Scot will purchase x tea bags and y sugar bags. Use a linear equation to model the number of tea bags and sugar bags he can purchase.

Find this line's x-intercept, and interpret its meaning in this context.

- ⊙ *A*. The x-intercept is (0, 90). It implies Scot can purchase 90 sugar bags with no tea bags.
- ⊙ *B*. The x-intercept is (30,0). It implies Scot can purchase 30 tea bags with no sugar bags.
- ⊙ *C*. The x-intercept is (90,0). It implies Scot can purchase 90 tea bags with no sugar bags.
- ⊙ *D*. The x-intercept is (0,30). It implies Scot can purchase 30 sugar bags with no tea bags.

44. Douglas is buying some tea bags and some sugar bags. Each tea bag costs 2 cents, and each sugar bag costs 9 cents. He can spend a total of $0.90.

Assume Douglas will purchase x tea bags and y sugar bags. Use a linear equation to model the number of tea bags and sugar bags he can purchase.

Find this line's y-intercept, and interpret its meaning in this context.

- ⊙ *A*. The y-intercept is (0,45). It implies Douglas can purchase 45 sugar bags with no tea bags.
- ⊙ *B*. The y-intercept is (45,0). It implies Douglas can purchase 45 tea bags with no sugar bags.
- ⊙ *C*. The y-intercept is (10,0). It implies Douglas can purchase 10 tea bags with no sugar bags.
- ⊙ *D*. The y-intercept is (0, 10). It implies Douglas can purchase 10 sugar bags with no tea bags.

45. An engine's tank can hold 70 gallons of gasoline. It was refilled with a full tank, and has been running without breaks, consuming 3.5 gallons of gas per hour.

Assume the engine has been running for x hours since its tank was refilled, and assume there are y gallons of gas left in the tank. Use a linear equation to model the amount of gas in the tank as time passes.

Find this line's x-intercept, and interpret its meaning in this context.

- ⊙ *A*. The x-intercept is (20,0). It implies the engine will run out of gas 20 hours after its tank was refilled.
- ⊙ *B*. The x-intercept is (0,70). It implies the engine started with 70 gallons of gas in its tank.
- ⊙ *C*. The x-intercept is (0,20). It implies the engine started with 20 gallons of gas in its tank.
- ⊙ *D*. The x-intercept is (70,0). It implies the engine will run out of gas 70 hours after its tank was refilled.

46. An engine's tank can hold 120 gallons of gasoline. It was refilled with a full tank, and has been running without breaks, consuming 3 gallons of gas per hour.

Assume the engine has been running for x hours since its tank was refilled, and assume there are y gallons of gas left in the tank. Use a linear equation to model the amount of gas in the tank as time passes.

Find this line's y-intercept, and interpret its meaning in this context.

- ⊙ *A*. The y-intercept is (40,0). It implies the engine will run out of gas 40 hours after its tank was refilled.
- ⊙ *B*. The y-intercept is (120,0). It implies the engine will run out of gas 120 hours after its tank was refilled.
- ⊙ *C*. The y-intercept is (0,120). It implies the engine started with 120 gallons of gas in its tank.
- ⊙ *D*. The y-intercept is (0,40). It implies the engine started with 40 gallons of gas in its tank.

47. A new car of a certain model costs \$43,200.00. According to Blue Book, its value decreases by \$2,400.00 every year.

Assume x years since its purchase, the car's value is y dollars. Use a linear equation to model the car's value.

Find this line's x-intercept, and interpret its meaning in this context.

- ⊙ *A*. The x-intercept is (0,43200). It implies the car's initial value was 43200.
- ⊙ *B*. The x-intercept is (0,18). It implies the car would have no more value 18 years since its purchase.
- ⊙ *C*. The x-intercept is (18,0). It implies the car would have no more value 18 years since its purchase.
- ⊙ *D*. The x-intercept is (43200,0). It implies the car's initial value was 43200.

48. A new car of a certain model costs \$39,000.00. According to Blue Book, its value decreases by \$2,600.00 every year.

Assume x years since its purchase, the car's value is y dollars. Use a linear equation to model the car's value.

Find this line's y-intercept, and interpret its meaning in this context.

- ⊙ *A*. The y-intercept is (15,0). It implies the car would have no more value 15 years since its purchase.
- ⊙ *B*. The y-intercept is (0,39000). It implies the car's initial value was 39000.
- ⊙ *C*. The y-intercept is (0,15). It implies the car would have no more value 15 years since its purchase.
- ⊙ *D*. The y-intercept is (39000,0). It implies the car's initial value was 39000.

Challenge

49. Fill in the variables A, B, and C in $Ax + By = C$ with the numbers $10, 11$ and 14. You may only use each number once.

a. To make a line with the steepest slope possible, A must equal ☐, B must equal ☐, and C must equal ☐.

b. To make a line with the shallowest slope possible, A must equal ☐, B must equal ☐, and C must equal ☐.

4.8 Horizontal, Vertical, Parallel, and Perpendicular Lines

Horizontal and vertical lines have some special features worth our attention. Also if a pair of lines are parallel or perpendicular to each other, we have some interesting things to say about them. This section looks at these geometric features that lines may have.

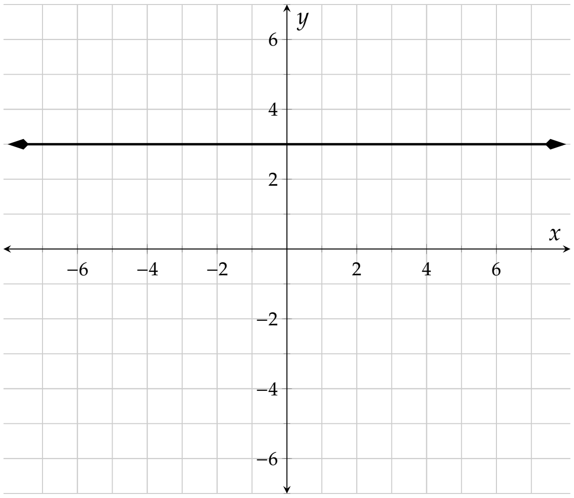


Figure 4.8.2: Horizontal Line

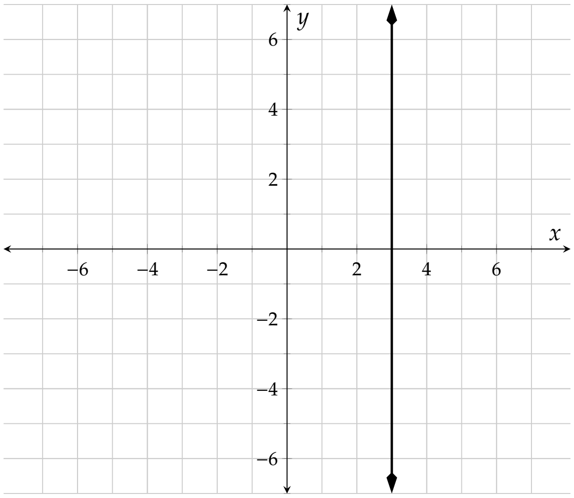


Figure 4.8.3: Vertical Line

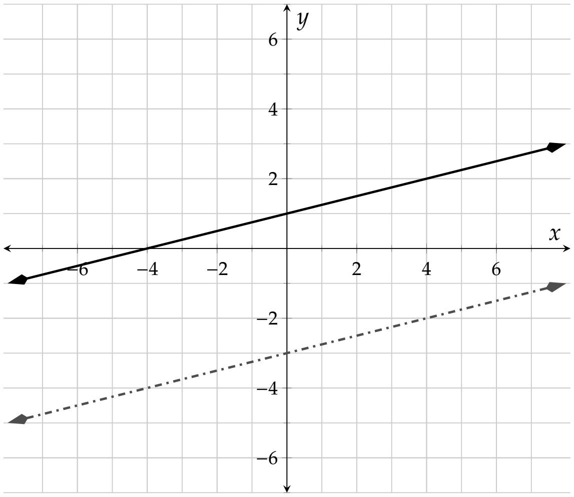


Figure 4.8.4: Parallel Lines

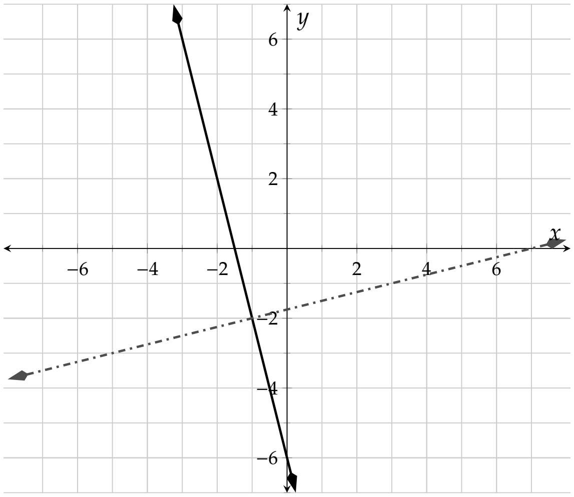


Figure 4.8.5: Perpendicular Lines

4.8.1 Horizontal Lines and Vertical Lines

We learned in Section 4.7 that all lines can be written in standard form (4.7.1). When either A or B equal 0, we end up with a horizontal or vertical line, as we will soon see. Let's take the standard form (4.7.1) line equation, let $A = 0$ and $B = 0$ one at a time and simplify each equation.

$$\begin{aligned} Ax + By &= C \\ 0x + By &= C \\ By &= C \\ y &= \frac{C}{B} \\ y &= k \end{aligned} \qquad\qquad \begin{aligned} Ax + By &= C \\ Ax + 0y &= C \\ Ax &= C \\ x &= \frac{C}{A} \\ x &= h \end{aligned}$$

At the end we just renamed the constant numbers $\frac{C}{B}$ and $\frac{C}{A}$ to k and h because of tradition. What is important, is that you view h and k (as well as A, B, and C) as constants: numbers that have some specific value and don't change in the context of one problem.

Think about just one of these last equations: $y = k$. It says that the y-value is the same no matter where you are on the line. If you wanted to plot points on this line, you are free to move far to the left or far to the right on the x-axis, but then you always move up (or down) to make the y-value equal k. What does such a line look like?

Example 4.8.6 Let's plot the line with equation $y = 3$. (Note that this is the same as $0x + 1y = 3$.)

To plot some points, it doesn't matter what x-values we use. All that matters is that y is *always* 3.

A line like this is **horizontal**, parallel to the horizontal axis. All lines with an equation in the form

$$y = k$$

(or, in standard form, $0x + By = C$) are **horizontal**.

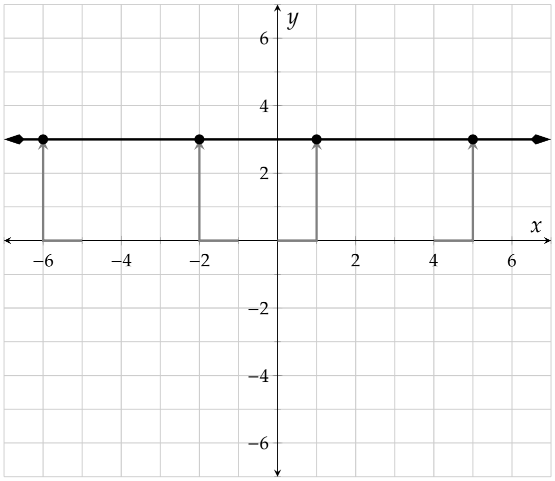


Figure 4.8.7: $y = 3$

Example 4.8.8 Let's plot the line with equation $x = 5$. (Note that this is the same as $1x + 0y = 5$.)

The line has $x = 5$, so to plot points, we are *required* to move over to $x = 5$. From there, we have complete freedom to move however far we like up or down.

A line like this is **vertical**, parallel to the vertical axis. All lines with an equation in the form

$$x = h$$

(or, in standard form, $Ax + 0y = C$) are vertical.

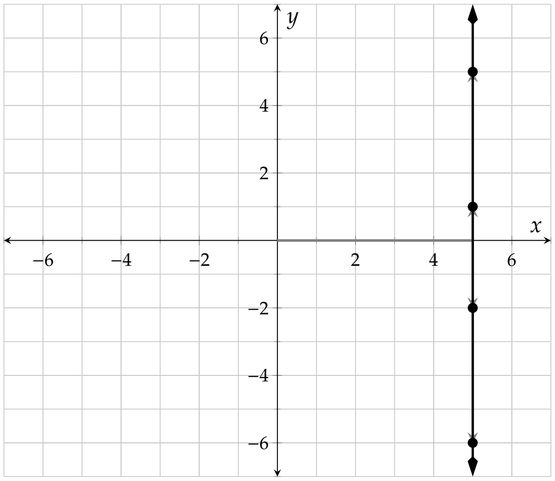


Figure 4.8.9: $x = 5$

Example 4.8.10 Zero Slope. In Checkpoint 4.4.17, we learned that a horizontal line's slope is 0, because the distance doesn't change as time moves on. So the numerator in the slope formula (4.4.3) is 0. Now, if we know a line's slope and its y-intercept, we can use slope-intercept form (4.5.1) to write its equation:

$$\begin{aligned}y &= mx + b\\ y &= 0x + b\\ y &= b\end{aligned}$$

This provides us with an alternative way to think about equations of horizontal lines. They have a certain y-intercept b, and they have slope 0.

We use horizontal lines to model scenarios where there is no change in y-values, like when Kato stopped for 12 hours (he deserved a rest)!

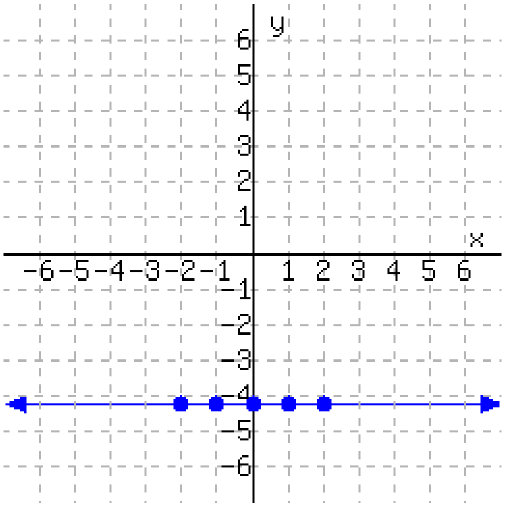
Checkpoint 4.8.11 Plotting Points. Suppose you need to plot the equation $y = -4.25$. Since the equation is in "$y =$" form, you decide to make a table of points. Fill out some points for this table.

x	y

Explanation. We can use whatever values for x that we like, as long as they are all different. The equation tells us the y-value has to be -4.25 each time.

x	y
-2	-4.25
-1	-4.25
0	-4.25
1	-4.25
2	-4.25

The reason we made a table was to help with plotting the line.



Example 4.8.12 Undefined Slope. What is the slope of a vertical line? Figure 4.8.13 shows three lines passing through the origin, each steeper than the last. In each graph, you can see a slope triangle that uses a “rise” of 4 each time.

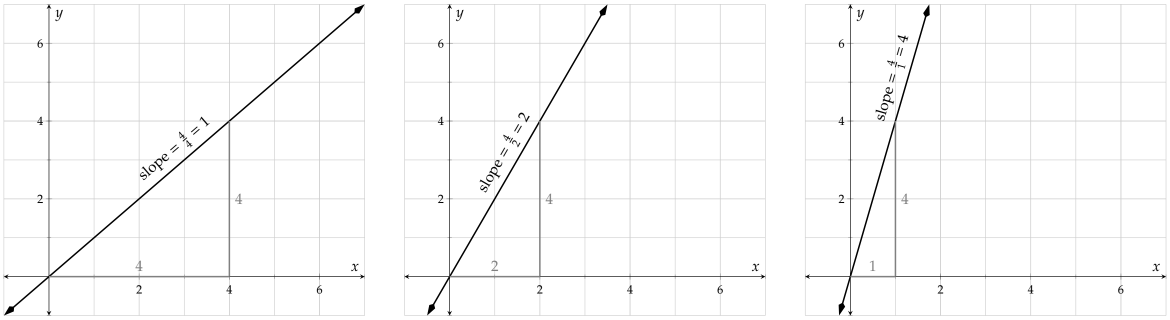


Figure 4.8.13

If we continued making the line steeper and steeper until it was vertical, the slope triangle would still have a “rise” of 4, but the “run” would become smaller and smaller, closer to 0. And then the slope would be $m = \frac{4}{\text{very small}} =$ very large. So the slope of a vertical line can be thought of as “infinitely

> large.”
>
> If we actually try to compute the slope using the slope triangle when the run is 0, we would have $\frac{4}{0}$, which is undefined. So we also say that the slope of a vertical line is *undefined*. Some people say that a vertical line *has no slope.*

Remark 4.8.14. Be careful not to mix up “no slope” (which means “its slope is undefined”) with “has slope 0.” If a line has slope 0, it *does* have a slope.

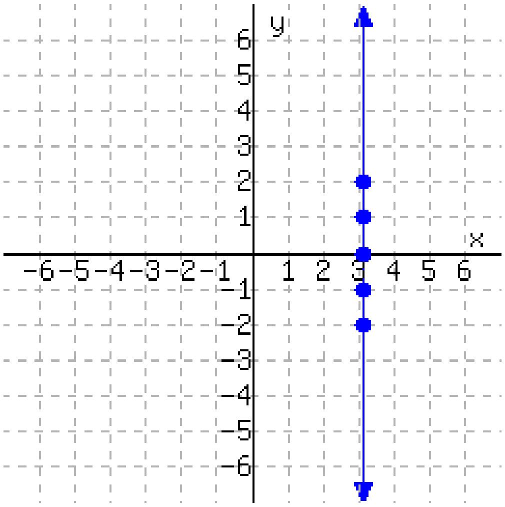
Checkpoint 4.8.15 Plotting Points. Suppose you need to plot the equation $x = 3.14$. You decide to try making a table of points. Fill out some points for this table.

x	y

Explanation. Since the equation says x is always the number 3.14, we have to use this for the x value in all the points. This is different from how we would plot a “$y =$” equation, where we would use several different x-values. We can use whatever values for y that we like, as long as they are all different.

x	y
3.14	−2
3.14	−1
3.14	0
3.14	1
3.14	2

The reason we made a table was to help with plotting the line.



Example 4.8.16 Let x represent the price of a new 60-inch television at Target on Black Friday (which was \$650), and let y be the number of hours you will watch something on this TV over its lifetime. What is the relationship between x and y?

Well, there is no getting around the fact that $x = 650$. As for y, without any extra information about your viewing habits, it could theoretically be as low as 0 or it could be anything larger than that. If we graph this scenario, we have to graph the equation $x = 650$ which we now know to give a vertical line, and we get Figure 4.8.17.

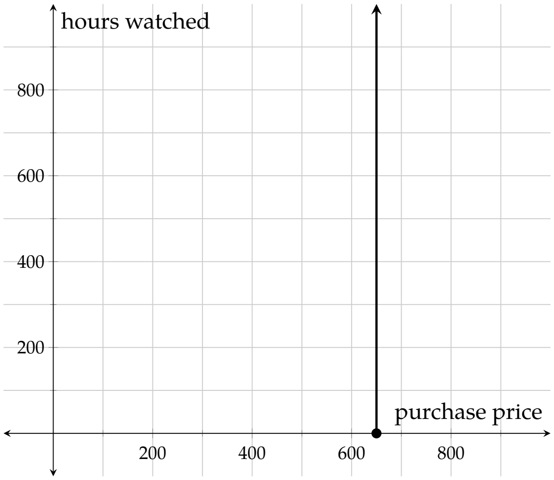


Figure 4.8.17: New TV: hours watched versus purchase price; negative y-values omitted since they make no sense in context

Summary of Horizontal and Vertical Line Equations

Horizontal Lines	Vertical Lines
A line is **horizontal** if and only if its equation can be written $$y = k$$ for some constant k.	A line is **vertical** if and only if its equation can be written $$x = h$$ for some constant h.
In standard form (4.7.1), any line with equation $$0x + By = C$$ is horizontal.	In standard form (4.7.1), any line with equation $$Ax + 0y = C$$ is vertical.
If the line with equation $y = k$ is horizontal, it has a y-intercept at $(0, k)$ and has slope 0.	If the line with equation $x = h$ is vertical, it has an x-intercept at $(h, 0)$ and its slope is *undefined*. Some say it has *no* slope, and some say the slope is *infinitely large*.
In slope-intercept form (4.5.1), any line with equation $$y = 0x + b$$ is horizontal.	It's impossible to write the equation of a vertical line in slope-intercept form (4.5.1), because vertical lines do not have a defined slope.

4.8.2 Parallel Lines

Two trees were planted in the same year, and their growth over time is modeled by the two lines in Figure 4.8.19. Use linear equations to model each tree's growth, and interpret their meanings in this context.

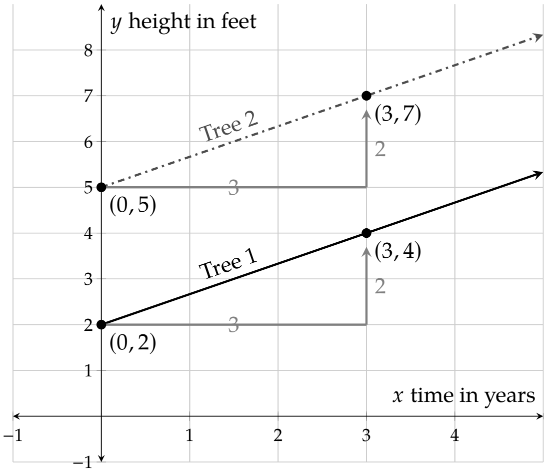


Figure 4.8.19: Two Trees' Growth Chart

Example 4.8.18 We can see Tree 1's equation is $y = \frac{2}{3}x + 2$, and Tree 2's equation is $y = \frac{2}{3}x + 5$. Tree 1 was 2 feet tall when it was planted, and Tree 2 was 5 feet tall when it was planted. Both trees have been growing at the same rate, $\frac{2}{3}$ feet per year, or 2 feet every 3 years.

An important observation right now is that those two lines are parallel. Why? For lines with positive slopes, the bigger a line's slope, the steeper the line is slanted. As a result, if two lines have the same slope, they are slanted at the same angle, thus they are parallel.

Fact 4.8.20. *Any two vertical lines are parallel to each other. For two non-vertical lines, they are parallel if and only if they have the same slope.*

Checkpoint 4.8.21. A line ℓ is parallel to the line with equation $y = 17.2x - 340.9$, but ℓ has y-intercept at $(0, 128.2)$. What is an equation for ℓ?

Explanation. Parallel lines have the same slope, and the slope of $y = 17.2x - 340.9$ is 17.2. So ℓ has slope 17.2. And we have been given that ℓ's y-intercept is at $(0, 128.2)$. So we can use slope-intercept form to write its equation as

$$y = 17.2x + 128.2.$$

Checkpoint 4.8.22. A line κ is parallel to the line with equation $y = -3.5x + 17$, but κ passes through the point $(-12, 23)$. What is an equation for κ?

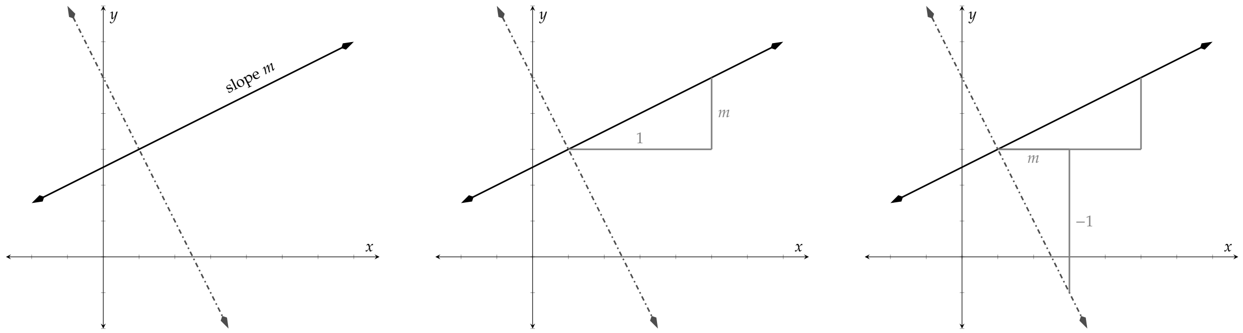
Explanation. Parallel lines have the same slope, and the slope of $y = -3.5x + 17$ is -3.5. So κ has slope -3.5. And we know a point that κ passes through, so we can use point-slope form to write its equation as

$$y = -3.5(x + 12) + 23.$$

4.8.3 Perpendicular Lines

The slopes of two perpendicular lines have a special relationship too.

Figure 4.8.22 walks you through an explanation of this realationship.



(a) Two generic perpendicular lines, where one has slope m.

(b) Since the one slope is m, we can draw a slope triangle with "run" 1 and "rise" m.

(c) A *congruent* slope triangle can be drawn for the perpendicular line. It’s legs have the same lengths, but in different positions, and one is negative.

Figure 4.8.22: The relationship between slopes of perpendicular lines

The second line in Figure 4.8.22 has slope

$$\frac{\Delta y}{\Delta x} = \frac{-1}{m} = -\frac{1}{m}.$$

Fact 4.8.23. *A vertical line and a horizontal line are perpendicular. For lines that are neither vertical nor horizontal, they are perpendicular if and only if the slope of one is the negative reciprocal of the slope of the other. That is, if one has slope m, the other has slope $-\frac{1}{m}$.*

Another way to say this is that the product of the slopes of two perpendicular lines is -1 (assuming both of the lines have a slope in the first place).

Not convinced? Here are three pairs of perpendicular lines where we can see if the pattern holds.

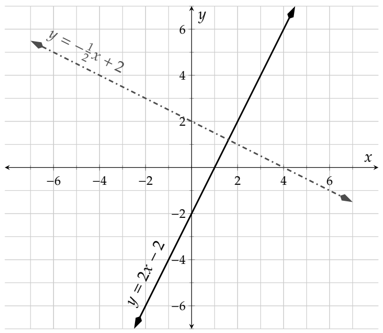


Figure 4.8.24: Graphing $y = 2x - 2$ and $y = -\frac{1}{2}x + 2$. Note the relationship between their slopes: $2 = -\frac{1}{-1/2}$

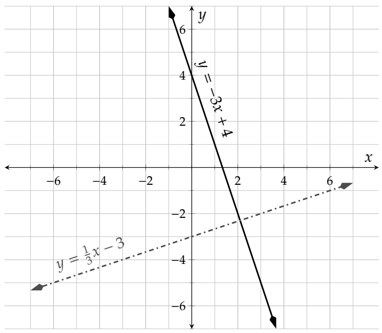


Figure 4.8.25: Graphing $y = -3x + 4$ and $y = \frac{1}{3}x - 3$. Note the relationship between their slopes: $-3 = -\frac{1}{1/3}$

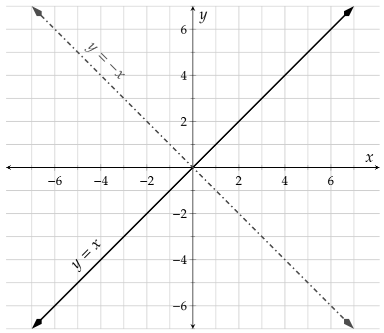


Figure 4.8.26: Graphing $y = x$ and $y = -x$. Note the relationship between their slopes: $1 = -\frac{1}{-1}$

Example 4.8.27 Line A passes through $(-2, 10)$ and $(3, -10)$. Line B passes through $(-4, -4)$ and $(8, -1)$. Determine whether these two lines are parallel, perpendicular or neither.

Explanation. We will use the slope formula to find both lines' slopes:

$$\begin{aligned}\text{Line } A\text{'s slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-10-10}{3-(-2)} \\ &= \frac{-20}{5} \\ &= -4\end{aligned} \qquad \begin{aligned}\text{Line } B\text{'s slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1-(-4)}{8-(-4)} \\ &= \frac{3}{12} \\ &= \frac{1}{4}\end{aligned}$$

Their slopes are not the same, so those two lines are not parallel.

The product of their slopes is $(-4) \cdot \frac{1}{4} = -1$, which means the two lines are perpendicular.

Checkpoint 4.8.28. Line A and Line B are perpendicular. Line A's equation is $2x + 3y = 12$. Line B passes through the point $(4, -3)$. Find an equation for Line B.

Explanation. First, we will find Line A's slope by rewriting its equation from standard form to slope-intercept form:

$$\begin{aligned}2x + 3y &= 12 \\ 3y &= 12 - 2x \\ 3y &= -2x + 12 \\ y &= \frac{-2x + 12}{3} \\ y &= -\frac{2}{3}x + 4\end{aligned}$$

So Line A's slope is $-\frac{2}{3}$. Since Line B is perpendicular to Line A, its slope is $-\frac{1}{-\frac{2}{3}} = \frac{3}{2}$. It's also given that

Line B passes through $(4, -3)$, so we can write Line B's point-slope form equation:

$$y = m(x - x_0) + y_0$$
$$y = \frac{3}{2}(x - 4) - 3$$

Exercises

Review and Warmup

1. Evaluate the following expressions. If the answer is undefined, you may answer with `DNE` (meaning "does not exist").

a. $\frac{6}{0} =$

b. $\frac{0}{6} =$

2. Evaluate the following expressions. If the answer is undefined, you may answer with `DNE` (meaning "does not exist").

a. $\frac{0}{7} =$

b. $\frac{7}{0} =$

3. A line passes through the points $(5, 6)$ and $(-5, 6)$. Find this line's slope.

4. A line passes through the points $(3, 8)$ and $(-3, 8)$. Find this line's slope.

5. A line passes through the points $(-10, -5)$ and $(-10, 5)$. Find this line's slope.

6. A line passes through the points $(-8, -1)$ and $(-8, 2)$. Find this line's slope.

7. Consider the equation:

$$y = 1$$

Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

☐ $(0, 9)$ ☐ $(-6, 1)$ ☐ $(1, 4)$
☐ $(4, 1)$

8. Consider the equation:

$$y = 1$$

Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

☐ $(-8, 1)$ ☐ $(0, 7)$ ☐ $(5, 1)$
☐ $(1, 2)$

9. Consider the equation:

$$x + 1 = 0$$

Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

☐ $(-1, 3)$ ☐ $(-1, 0)$
☐ $(1, -1)$ ☐ $(0, -6)$

10. Consider the equation:

$$x + 1 = 0$$

Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

☐ $(-1, 3)$ ☐ $(0, -8)$
☐ $(1, -1)$ ☐ $(-1, 0)$

Tables for Horizontal and Vertical Lines

11. Fill out this table for the equation $y = 7$. The first row is an example.

x	y	Points
-3	7	$(-3, 7)$
-2	______	______________
-1	______	______________
0	______	______________
1	______	______________
2	______	______________

12. Fill out this table for the equation $y = 8$. The first row is an example.

x	y	Points
-3	8	$(-3, 8)$
-2	______	______________
-1	______	______________
0	______	______________
1	______	______________
2	______	______________

13. Fill out this table for the equation $x = -2$. The first row is an example.

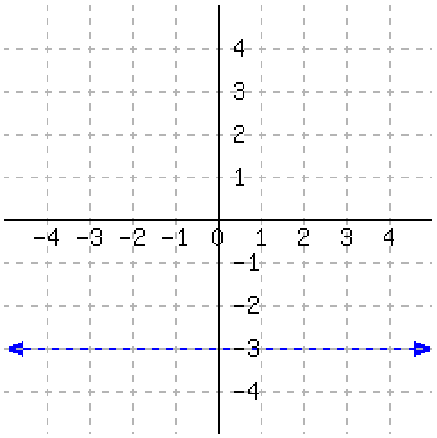
x	y	Points
-2	-3	$(-2, -3)$
______	-2	______________
______	-1	______________
______	0	______________
______	1	______________
______	2	______________

14. Fill out this table for the equation $x = -10$. The first row is an example.

x	y	Points
-10	-3	$(-10, -3)$
______	-2	______________
______	-1	______________
______	0	______________
______	1	______________
______	2	______________

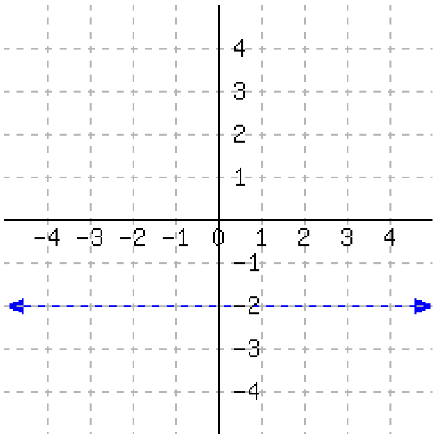
Line Equations

15. A line's graph is given.



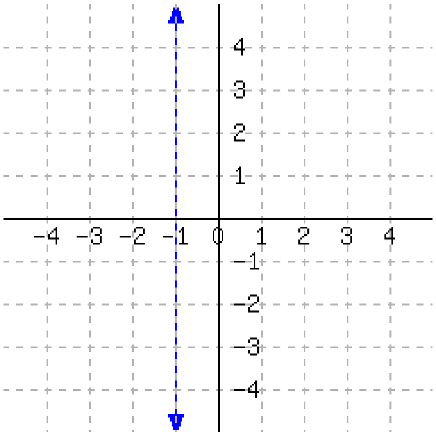
This line's equation is

16. A line's graph is given.



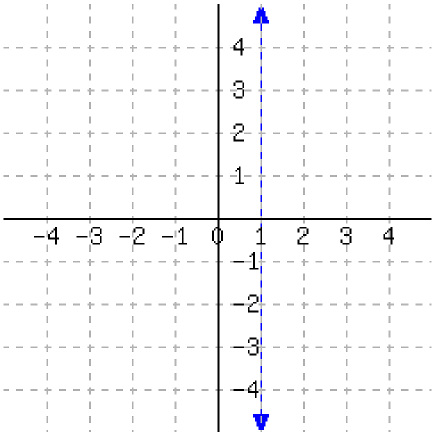
This line's equation is

17. A line's graph is given.



This line's equation is ☐

18. A line's graph is given.



This line's equation is ☐

19. A line passes through the points $(-2, 1)$ and $(3, 1)$. Find an equation for this line.

An equation for this line is ☐.

20. A line passes through the points $(5, 4)$ and $(-4, 4)$. Find an equation for this line.

An equation for this line is ☐.

21. A line passes through the points $(6, 1)$ and $(6, 2)$. Find an equation for this line.

An equation for this line is ☐.

22. A line passes through the points $(8, -3)$ and $(8, 5)$. Find an equation for this line.

An equation for this line is ☐.

Intercepts

23. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter `none` into all the answer blanks for that row.

$$x = 10$$

	x-value	y-value	Location
y-intercept	______	______	____________
x-intercept	______	______	____________

24. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter `none` into all the answer blanks for that row.

$$x = -8$$

	x-value	y-value	Location
y-intercept			
x-intercept			

25. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter `none` into all the answer blanks for that row.

$$y = -6$$

	x-value	y-value	Location
y-intercept			
x-intercept			

26. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter `none` into all the answer blanks for that row.

$$y = -4$$

	x-value	y-value	Location
y-intercept			
x-intercept			

Graphs of Horizontal and Vertical Lines

27. Graph the line $y = 1$.

28. Graph the line $y + 5 = 0$.

29. Graph the line $x = 2$.

30. Graph the line $x - 3 = 0$.

Parallel or Perpendicular?

31. Line m passes points $(-5,-14)$ and $(5,16)$.

Line n passes points $(-4,-10)$ and $(4,14)$.

Determine how the two lines are related.

These two lines are

- ⊙ parallel
- ⊙ perpendicular
- ⊙ neither parallel nor perpendicular

32. Line m passes points $(5,-1)$ and $(15,-13)$.

Line n passes points $(20,-20)$ and $(-5,10)$.

Determine how the two lines are related.

These two lines are

- ⊙ parallel
- ⊙ perpendicular
- ⊙ neither parallel nor perpendicular

33. Line m passes points $(12,4)$ and $(-8,9)$.

Line n passes points $(-4,-26)$ and $(-2,-18)$.

Determine how the two lines are related.

These two lines are

- ⊙ parallel
- ⊙ perpendicular
- ⊙ neither parallel nor perpendicular

34. Line m passes points $(-10,12)$ and $(5,-12)$.

Line n passes points $(16,2)$ and $(8,-3)$.

Determine how the two lines are related.

These two lines are

- ⊙ parallel
- ⊙ perpendicular
- ⊙ neither parallel nor perpendicular

35. Line m passes points $(3,-11)$ and $(4,-12)$.

Line n passes points $(-3,-16)$ and $(5,8)$.

Determine how the two lines are related.

These two lines are

- ⊙ parallel
- ⊙ perpendicular
- ⊙ neither parallel nor perpendicular

36. Line m passes points $(6,10)$ and $(-7,10)$.

Line n passes points $(1,2)$ and $(3,2)$.

Determine how the two lines are related.

These two lines are

- ⊙ parallel
- ⊙ perpendicular
- ⊙ neither parallel nor perpendicular

37. Line m passes points $(-8,-1)$ and $(-8,1)$.

Line n passes points $(-4,0)$ and $(-4,-7)$.

Determine how the two lines are related.

These two lines are

- ⊙ parallel
- ⊙ perpendicular
- ⊙ neither parallel nor perpendicular

38. Line m passes points $(-6,-8)$ and $(-6,10)$.

Line n passes points $(-9,0)$ and $(-9,3)$.

Determine how the two lines are related.

These two lines are

- ⊙ parallel
- ⊙ perpendicular
- ⊙ neither parallel nor perpendicular

Parallel and Perpendicular Line Equations

39. A line passes through the point $(-2, 5)$, and it's parallel to the line $y = -4$. Find an equation for this line.

An equation for this line is $\boxed{}$.

41. A line passes through the point $(-9, -6)$, and it's parallel to the line $x = 1$. Find an equation for this line.

An equation for this line is $\boxed{}$.

43. Line k has the equation $y = 3x + 4$.

Line ℓ is parallel to line k, but passes through the point $(-5, -17)$.

Find an equation for line ℓ in both slope-intercept form and point-slope form.

An equation for ℓ in slope-intercept form is: $\boxed{}$.

An equation for ℓ in point-slope form is: $\boxed{}$.

45. Line k has the equation $y = -\frac{9}{7}x - 2$.

Line ℓ is parallel to line k, but passes through the point $(-21, 31)$.

Find an equation for line ℓ in both slope-intercept form and point-slope form.

An equation for ℓ in slope-intercept form is: $\boxed{}$.

An equation for ℓ in point-slope form is: $\boxed{}$.

40. A line passes through the point $(7, -2)$, and it's parallel to the line $y = -1$. Find an equation for this line.

An equation for this line is $\boxed{}$.

42. A line passes through the point $(4, 3)$, and it's parallel to the line $x = 3$. Find an equation for this line.

An equation for this line is $\boxed{}$.

44. Line k has the equation $y = 4x - 2$.

Line ℓ is parallel to line k, but passes through the point $(-1, -9)$.

Find an equation for line ℓ in both slope-intercept form and point-slope form.

An equation for ℓ in slope-intercept form is: $\boxed{}$.

An equation for ℓ in point-slope form is: $\boxed{}$.

46. Line k has the equation $y = -\frac{1}{4}x + 3$.

Line ℓ is parallel to line k, but passes through the point $(8, -4)$.

Find an equation for line ℓ in both slope-intercept form and point-slope form.

An equation for ℓ in slope-intercept form is: $\boxed{}$.

An equation for ℓ in point-slope form is: $\boxed{}$.

47. Line k has the equation $y = -x + 10$.

Line ℓ is perpendicular to line k, and passes through the point $(1, 4)$.

Find an equation for line ℓ in both slope-intercept form and point-slope form.

An equation for ℓ in slope-intercept form is: $\boxed{}$.

An equation for ℓ in point-slope form is: $\boxed{}$.

48. Line k has the equation $y = 3x - 4$.

Line ℓ is perpendicular to line k and passes through the point $(-3, -1)$.

Find an equation for ℓ in both slope-intercept form and point-slope forms.

An equation for ℓ in slope-intercept form is: $\boxed{}$.

An equation for ℓ in point-slope form is: $\boxed{}$.

49. Line k's equation is $y = \frac{5}{4}x - 5$.

Line ℓ is perpendicular to line k and passes through the point $(-15, 10)$.

Find an equation for line ℓ in both slope-intercept form and point-slope forms.

An equation for ℓ in slope-intercept form is: $\boxed{}$.

An equation for ℓ in point-slope form is: $\boxed{}$.

50. Line k has the equation $x - 6y = -30$.

Line ℓ is perpendicular to line k and passes through the point $(-2, 15)$.

Find line ℓ's equation in both slope-intercept form and point-slope form.

An equation for ℓ in slope-intercept form is: $\boxed{}$.

An equation for ℓ in point-slope form is: $\boxed{}$.

Challenge

51. Prove that a triangle with vertices at the points $(1, 1)$, $(-4, 4)$, and $(-3, 0)$ is a right triangle.

4.9 Summary of Graphing Lines

The previous several sections have demonstrated several methods for plotting a graph of a linear equation. In this section, we review these methods.

We have learned three forms to write a linear equation:

- slope-intercept form

  $y = mx + b$
- standard form

  $Ax + By = C$
- point-slope form

  $y = m(x - x_0) + y_0$

We have studied two special types of line:

- horizontal line: $y = k$
- vertical line: $x = h$

We have practiced three ways to graph a line:

- building a table of x- and y-values
- plotting *one* point (often the y-intercept) and drawing slope triangles
- plotting its x-intercept and y-intercept

4.9.1 Graphing Lines in Slope-Intercept Form

In the following examples we will graph $y = -2x + 1$, which is in slope-intercept form (4.5.1), with different methods and compare them.

Example 4.9.2 Building a Table of x- and y-values. First, we will graph $y = -2x + 1$ by building a table of values. In theory this method can be used for any type of equation, linear or not. Every student must feel comfortable with building a table of values based on an equation.

x-value	y-value	Point
-2	$y = -2(-2) + 1 = 5$	$(-2, 5)$
-1	$y = -2(-1) + 1 = 3$	$(-1, 3)$
0	$y = -2(0) + 1 = 1$	$(0, 1)$
1	$y = -2(1) + 1 = -1$	$(1, -1)$
2	$y = -2(2) + 1 = -3$	$(2, -3)$

Table 4.9.3: Table for $y = -2x + 1$

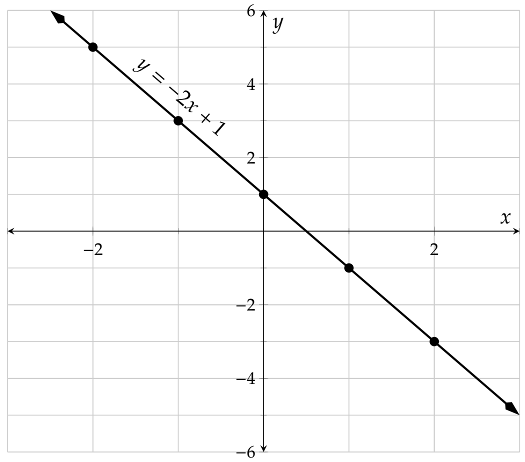


Figure 4.9.4: Graphing $y = -2x + 1$ by Building a Table of Values

Example 4.9.5 Using Slope Triangles. Although making a table is straightforward, the slope triangle method is both faster and reinforces the true meaning of slope. In the slope triangle method, we first identify some point on the line. With a line in slope-intercept form (4.5.1), we know the y-intercept, which is $(0, 1)$. Then, we can draw slope triangles in both directions to find more points.

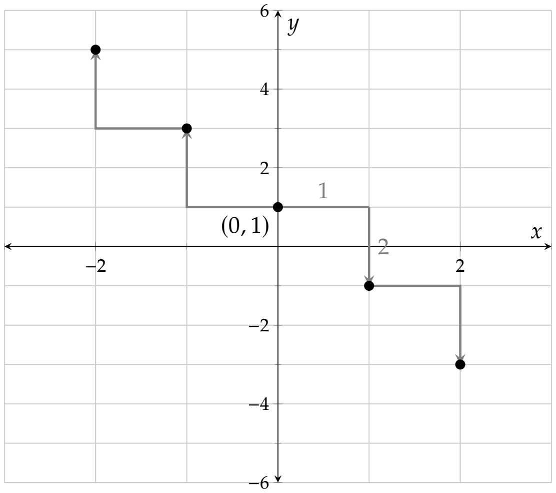


Figure 4.9.6: Marking a point and some slope triangles

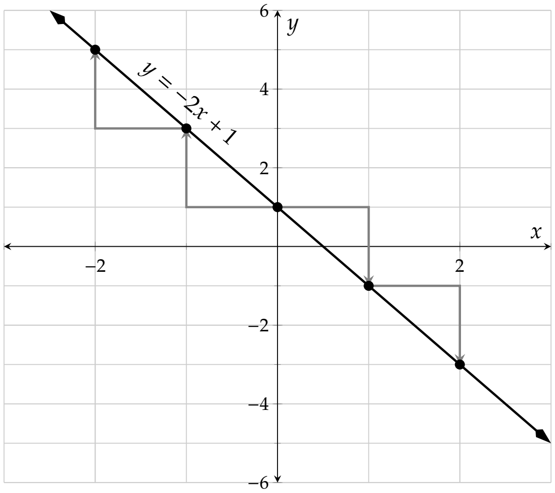


Figure 4.9.7: Graphing $y = -2x + 1$ by slope triangles

Compared to the table method, the slope triangle method:

- is less straightforward
- doesn't take the time and space to make a table
- doesn't involve lots of calculations where you might make a human error
- shows slope triangles, which reinforces the meaning of slope

Example 4.9.8 Using intercepts. If we use the x- and y-intercepts to plot $y = -2x + 1$, we have some calculation to do. While it is apparent that the y-intercept is at $(0, 1)$, where is the x-intercept? Here are two methods to find it.

Set $y = 0$.

$$\begin{aligned} y &= -2x + 1 \\ 0 &= -2x + 1 \\ 0 - 1 &= -2x \\ -1 &= -2x \\ \frac{-1}{-2} &= x \\ \frac{1}{2} &= x \end{aligned}$$

Factor out the coefficient of x.

$$\begin{aligned} y &= -2x + 1 \\ y &= -2x + (-2)\left(-\frac{1}{2}\right)1 \\ y &= -2\left(x + \left(-\frac{1}{2}\right)1\right) \\ y &= -2\left(x - \frac{1}{2}\right) \end{aligned}$$

And now it is easy to see that substituting $x = \frac{1}{2}$ would make $y = 0$.

So the x-intercept is at $\left(\frac{1}{2}, 0\right)$. Plotting both intercepts:

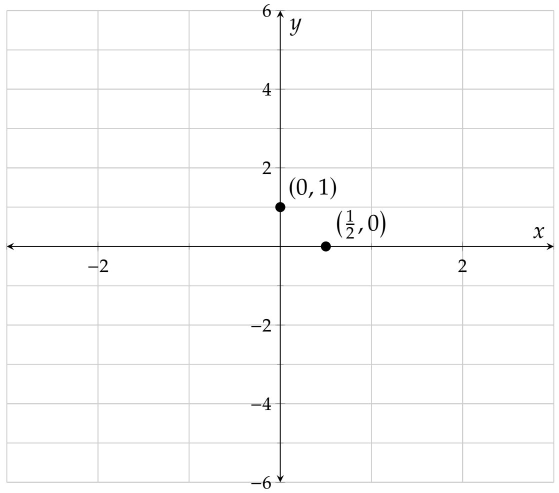


Figure 4.9.9: Marking intercepts

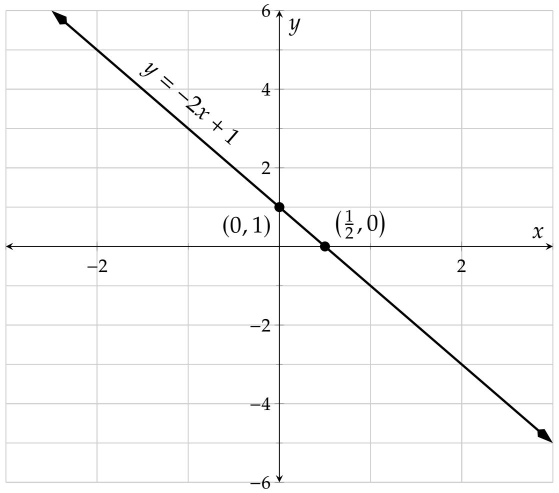


Figure 4.9.10: Using slope triangles

This worked, but here are some observations about why this method is not the greatest.

- We had to plot a point with fractional coordinates.
- We only plotted two points and they turned out very close to each other, so even the slightest inaccuracy in our drawing skills could result in a line that is way off.

When a line is presented in slope-intercept form (4.5.1), our opinion is that the slope triangle method is the best choice for making its graph.

4.9.2 Graphing Lines in Standard Form

In the following examples we will graph $3x + 4y = 12$, which is in standard form (4.7.1), with different methods and compare them.

Example 4.9.11 Building a Table of x- and y-values. To make a table, we could substitute x for various numbers and use algebra to find the corresponding y-values. Let's start with $x = -2$, planning to move on to $x = -1, 0, 1, 2$.

$$\begin{aligned} 3x + 4y &= 12 \\ 3(-2) + 4y &= 12 \\ -6 + 4y &= 12 \\ 4y &= 18 \\ y &= \frac{18}{4} = \frac{9}{2} \end{aligned}$$

The first point we found is $\left(-2, \frac{9}{2}\right)$. This has been a lot of calculation, and we ended up with a fraction we will have to plot. *And* we have to repeat this process a few more times to get more points for the table. The table method is generally not a preferred way to graph a line in standard form (4.7.1). Let's look at other options.

Example 4.9.12 Using intercepts. Next, we will try graphing $3x + 4y = 12$ using intercepts. We set up a small table to record the two intercepts:

	x-value	y-value	Intercept
x-intercept		0	
y-intercept	0		

We have to calculate the line's x-intercept by substituting $y = 0$ into the equation:

$$\begin{aligned} 3x + 4y &= 12 \\ 3x + 4(0) &= 12 \\ 3x &= 12 \\ x &= \frac{12}{3} \\ x &= 4 \end{aligned}$$

And similarly for the y-intercept:

$$\begin{aligned} 3x + 4y &= 12 \\ 3(0) + 4y &= 12 \\ 4y &= 12 \\ y &= \frac{12}{4} \\ y &= 3 \end{aligned}$$

So the line's x-intercept is at $(4, 0)$ and its y-intercept is at $(0, 3)$. Now we can complete the table and then graph the line:

	x-value	y-value	Intercepts
x-intercept	4	0	$(4, 0)$
y-intercept	0	3	$(0, 3)$

Table 4.9.13: Intercepts of $3x + 4y = 12$

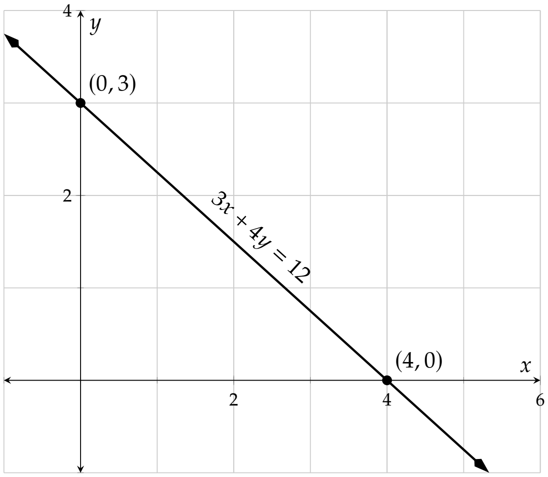


Figure 4.9.14: Graph of $3x + 4y = 12$

We can always rearrange $3x + 4y = 12$ into slope-intercept form (4.5.1), and then graph it with the slope triangle method:

$$\begin{aligned} 3x + 4y &= 12 \\ 4y &= 12 - 3x \\ 4y &= -3x + 12 \\ y &= \frac{-3x + 12}{4} \\ y &= -\frac{3}{4}x + 3 \end{aligned}$$

Example 4.9.15 With Slope Triangles. With the y-intercept at $(0, 3)$ and slope $-\frac{3}{4}$, we can graph the line using slope triangles:

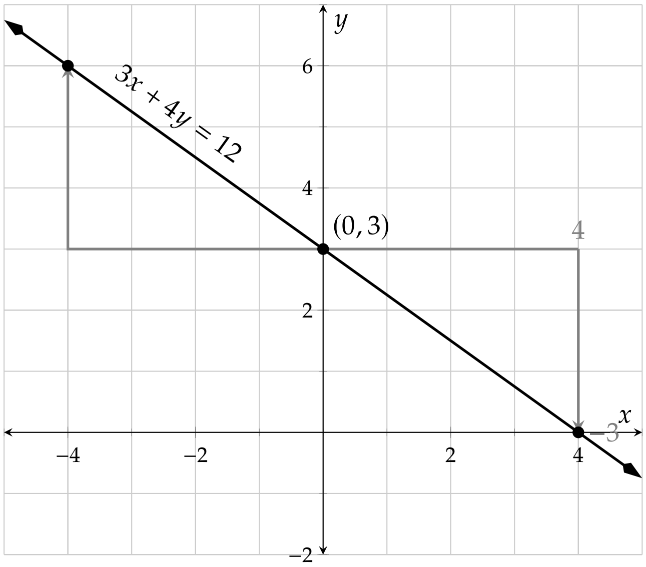


Figure 4.9.16: Graphing $3x + 4y = 12$ using slope triangles

Compared with the intercepts method, the slope triangle method takes more time, but shows more points with slope triangles, and thus a more accurate graph. Also sometimes (as with Example 4.7.14) when we graph a standard form equation like $2x - 3y = 0$, the intercepts method doesn't work because both intercepts are actually at the same point, and we have to resort to something else like slope triangles anyway.

Here are some observations about graphing a line equation that is in standard form (4.7.1):

- The intercepts method might be the quickest approach.
- The intercepts method only tells us two intercepts of the line. When we need to know more information, like the line's slope, and get a more accurate graph, we should spend more time and use the slope triangle method.
- When $C = 0$ in a standard form equation (4.7.1) we have to use something else like slope triangles anyway.

4.9.3 Graphing Lines in Point-Slope Form

When we graph a line in point-slope form (4.6.1) like $y = \frac{2}{3}(x + 1) + 3$, the slope triangle method is the obvious choice. We can see a point on the line, $(-1, 3)$, and the slope is apparent: $\frac{2}{3}$. Here is the graph:

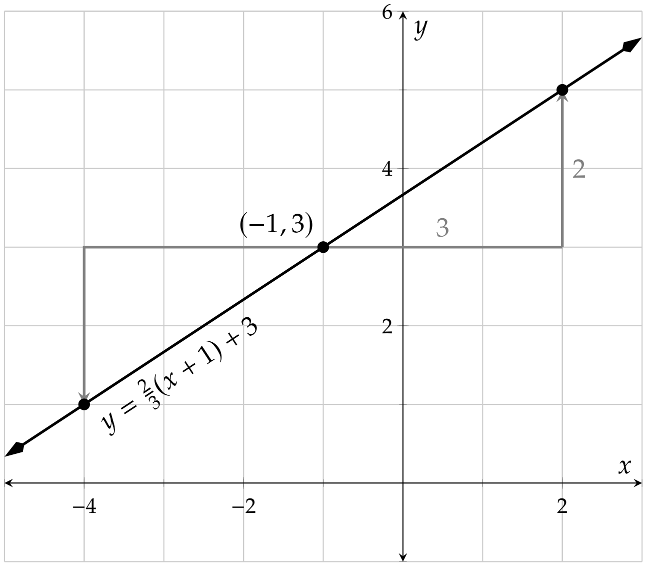


Figure 4.9.17: Graphing $y - 3 = \frac{2}{3}(x + 1)$ using slope triangles

Other graphing methods would take more work and miss the purpose of point-slope form (4.6.1). To graph a line in point-slope form (4.6.1), we recommend always using slope triangles.

4.9.4 Graphing Horizontal and Vertical Lines

We learned in Section 4.8 that equations in the form $x = h$ and $y = k$ make vertical and horizontal lines. But perhaps you will one day find yourself not remembering which is which. Making a table and plotting points can quickly remind you which type of equation makes which type of line. Let's build a table for $y = 2$ and another one for $x = -3$:

x-value	y-value	Point
0	2	$(0, 2)$
1	2	$(1, 2)$

Table 4.9.18: Table of Data for $y = 2$

x-value	y-value	Point
−3	0	$(-3, 0)$
−3	1	$(-3, 1)$

Table 4.9.19: Table of Data for $x = -3$

With two points on each line, we can graph them:

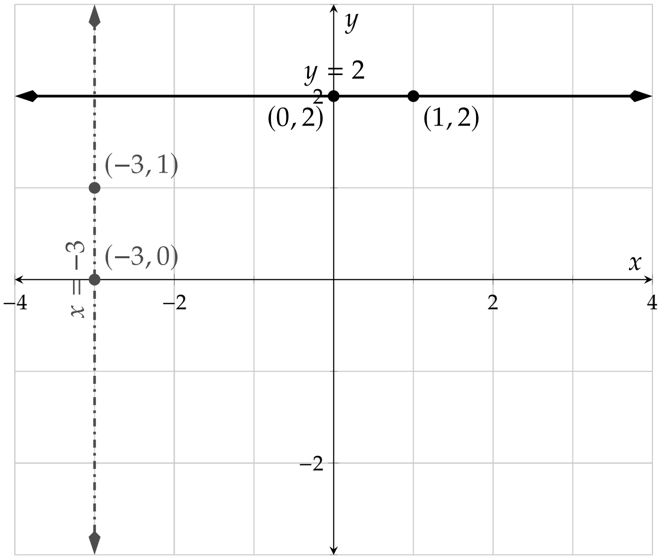


Figure 4.9.20: Graphing $y = 2$ and $x = -3$

Exercises

Graphing by Table

1. Use a table to make a plot of $y = 4x + 3$.

2. Use a table to make a plot of $y = -5x - 1$.

3. Use a table to make a plot of $y = -\frac{3}{4}x - 1$.

4. Use a table to make a plot of $y = \frac{5}{3}x + 3$.

Graphing Standard Form Equations

5. First find the x- and y-intercepts of the line with equation $6x + 5y = -90$. Then find one other point on the line. Use your results to graph the line.

6. First find the x- and y-intercepts of the line with equation $2x - 3y = -6$. Then find one other point on the line. Use your results to graph the line.

7. First find the x- and y-intercepts of the line with equation $3x+y = -9$. Then find one other point on the line. Use your results to graph the line.

8. First find the x- and y-intercepts of the line with equation $-15x + 3y = -3$. Then find one other point on the line. Use your results to graph the line.

9. First find the x- and y-intercepts of the line with equation $4x + 3y = -3$. Then find one other point on the line. Use your results to graph the line.

10. First find the x- and y-intercepts of the line with equation $-4x - 5y = 5$. Then find one other point on the line. Use your results to graph the line.

11. First find the x- and y-intercepts of the line with equation $5x-3y=0$. Then find one other point on the line. Use your results to graph the line.

12. First find the x- and y-intercepts of the line with equation $2x+9y=0$. Then find one other point on the line. Use your results to graph the line.

Graphing Slope-Intercept Equations

13. Use the slope and y-intercept from the line $y=-5x$ to plot the line. Use slope triangles.

14. Use the slope and y-intercept from the line $y=3x-6$ to plot the line. Use slope triangles.

15. Use the slope and y-intercept from the line $y=-\frac{2}{5}x+2$ to plot the line. Use slope triangles.

16. Use the slope and y-intercept from the line $y=\frac{10}{3}x-3$ to plot the line. Use slope triangles.

Graphing Horizontal and Vertical Lines

17. Plot the line $y=1$.

18. Plot the line $y=-4$.

19. Plot the line $x=-8$.

20. Plot the line $x=5$.

Choosing the Best Method to Graph Lines

21. Use whatever method you think best to plot $y=2x+2$.

22. Use whatever method you think best to plot $y=-3x+6$.

23. Use whatever method you think best to plot $y=-\frac{3}{4}x-1$.

24. Use whatever method you think best to plot $y=\frac{5}{3}x-3$.

25. Use whatever method you think best to plot $y=-\frac{3}{4}(x-5)+2$.

26. Use whatever method you think best to plot $y=\frac{2}{5}(x+1)-3$.

27. Use whatever method you think best to plot $3x+2y=6$.

28. Use whatever method you think best to plot $5x-4y=8$.

29. Use whatever method you think best to plot $3x-4y=0$.

30. Use whatever method you think best to plot $9x+6y=0$.

31. Use whatever method you think best to plot $x = -3$.

32. Use whatever method you think best to plot $x = 2$.

33. Use whatever method you think best to plot $y = -7$.

34. Use whatever method you think best to plot $y = 5$.

4.10 Linear Inequalities in Two Variables

We have learned how to graph lines like $y = 2x + 1$. In this section, we will learn how to graph linear inequalities like $y > 2x + 1$.

Example 4.10.2 Office Supplies. Isabel has a budget of \$133.00 to purchase some staplers and markers for the office supply closet. Each stapler costs \$19.00, and each marker costs \$1.75. We will define the variables so that she will purchase x staplers and y markers. Write and plot a linear inequality to model the relationship between the number of staplers and markers Isabel can purchase. Keep in mind that she might not spend all of the \$133.00.

The cost of buying x staplers would be $19x$ dollars. Similarly, the cost of buying y markers would be $1.75y$ dollars. Since whatever Isabel spends needs to be no more than 133 dollars, we have the inequality

$$19x + 1.75y \leq 133.$$

This is a standard-form inequality, similar to Equation (4.7.1). Next, let's graph it.

The first method to graph the inequality is to graph the corresponding equation, $19x + 1.75y = 133$. Its x- and y-intercepts can be found this way:

$$\begin{aligned} 19x + 1.75y &= 133 \\ 19x + 1.75(0) &= 133 \\ 19x &= 133 \\ \frac{19x}{19} &= \frac{133}{19} \\ x &= 7 \end{aligned} \qquad \begin{aligned} 19x + 1.75y &= 133 \\ 19(0) + 1.75y &= 133 \\ 1.75y &= 133 \\ \frac{1.75y}{1.75} &= \frac{133}{1.75} \\ y &= 76 \end{aligned}$$

So the intercepts are $(7, 0)$ and $(0, 76)$, and we can plot the line in Figure 4.10.3.

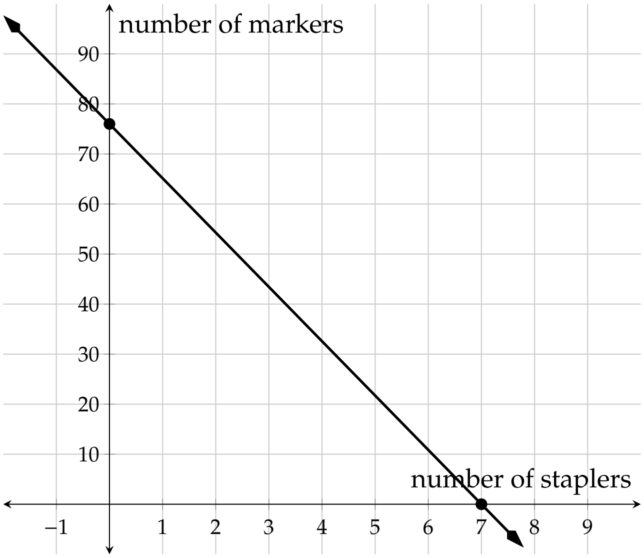


Figure 4.10.3: $19x + 1.75y = 133$

The points *on* this line represent ways in which Isabel can spend exactly all of the \$133. But what does a point like $(2, 40)$ in Figure 4.10.4, which is not on the line, mean in this context? That would mean Isabel bought 2 staplers and 40 markers, spending $19 \cdot 2 + 1.75 \cdot 40 = 108$ dollars. That is within her budget.

In fact, any point on the lower left side of this line represents a total purchase within Isabel's budget. The shading in Figure 4.10.5 captures *all* solutions to $19x + 1.75y \leq 133$. Some of those solutions have negative x- and y-values, which make no sense in context. So in Figure 4.10.6, we restrict the shading to solutions which make physical sense.

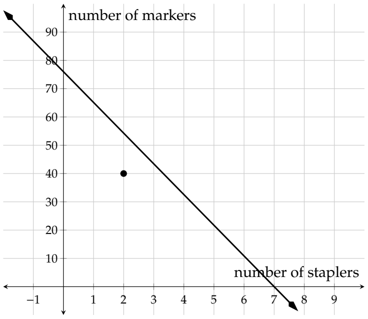


Figure 4.10.4: The line $19x + 1.75y = 133$ with a point identified that is within Isabel's budget.

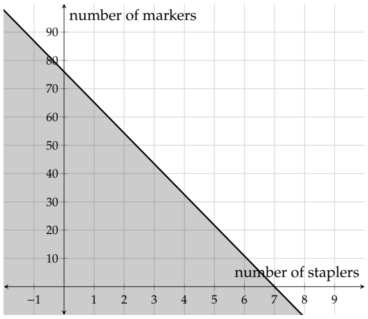


Figure 4.10.5: Shading all points that solve the inequality.

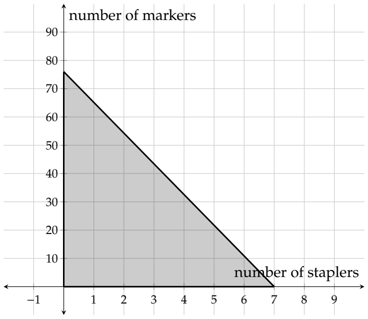


Figure 4.10.6: Shading restricted to points that make physical sense in context.

Let's look at some more examples of graphing linear inequalities in two variables.

Example 4.10.7 Is the point $(1, 2)$ a solution of $y > 2x + 1$?

In the inequality $y > 2x + 1$, substitute x with 1 and y with 2, and we will see whether the inequality is true:

$$
\begin{aligned}
y &> 2x + 1 \\
2 &\overset{?}{>} 2(2) + 1 \\
2 &\overset{\text{no}}{>} 5
\end{aligned}
$$

Since $2 > 5$ is not true, $(1, 2)$ is not a solution of $y > 2x + 1$.

Example 4.10.8 Graph $y > 2x + 1$.

There are two steps to graphing this linear inequality in two variables.

1. Graph the line $y = 2x + 1$. Because the inequality symbol is $>$ (instead of $\geq$), the line should be dashed (instead of solid).
2. Next, we need to decide whether to shade the region above $y = 2x + 1$ or below it. We will choose a point to test whether $y > 2x + 1$ is true. As long as the line doesn't cross $(0, 0)$, we will use $(0, 0)$ to test because the number 0 is the easiest number for calculation.

$$
\begin{aligned}
y &> 2x + 1 \\
0 &\overset{?}{>} 2(0) + 1
\end{aligned}
$$

$$0 \overset{\text{no}}{>} 1$$

Because $0 > 1$ is not true, the point $(0, 0)$ is not a solution and should not be shaded. As a result, we shade the region *without* $(0, 0)$.

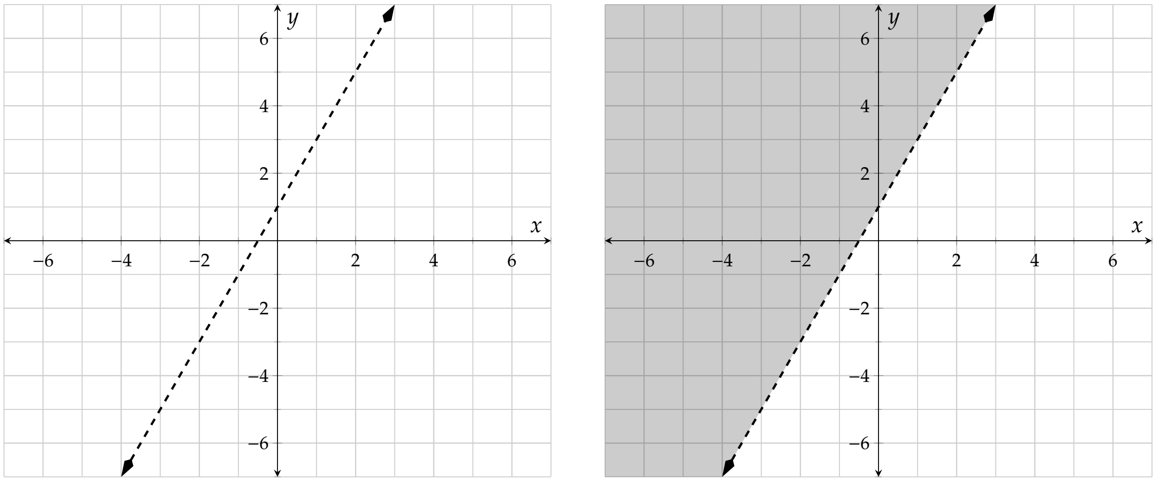


Figure 4.10.9: Step 1 of graphing $y > 2x + 1$ **Figure 4.10.10:** Complete graph of $y > 2x + 1$

Example 4.10.11 Graph $y \le -\frac{5}{3}x + 2$.

There are two steps to graphing this linear inequality in two variables.

1. Graph the line $y = -\frac{5}{3}x + 2$. Because the inequality symbol is $\le$ (instead of $<$), the line should be solid.
2. Next, we need to decide whether to shade the region above $y = -\frac{5}{3}x + 2$ or below it. We will choose a point to test whether $y \le -\frac{5}{3}x + 2$ is true there. Using $(0, 0)$ as a test point:

$$\begin{aligned} y &\le -\frac{5}{3}x + 2 \\ 0 &\overset{?}{\le} -\frac{5}{3}(0) + 2 \\ 0 &\overset{\checkmark}{\le} 2 \end{aligned}$$

Because $0 \le 2$ is true, the point $(0, 0)$ is a solution. As a result, we shade the region *with* $(0, 0)$.

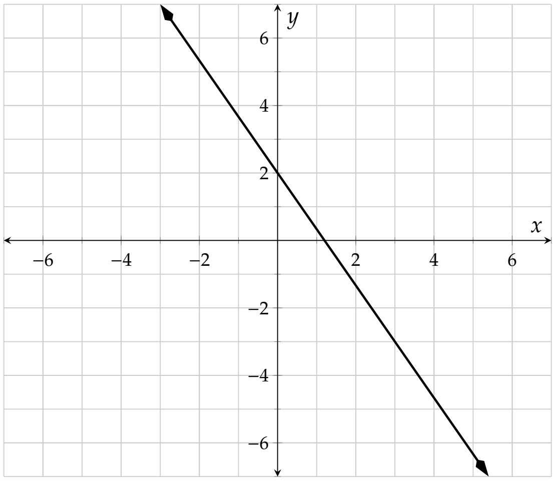


Figure 4.10.12: Step 1 of graphing $y \le -\frac{5}{3}x + 2$

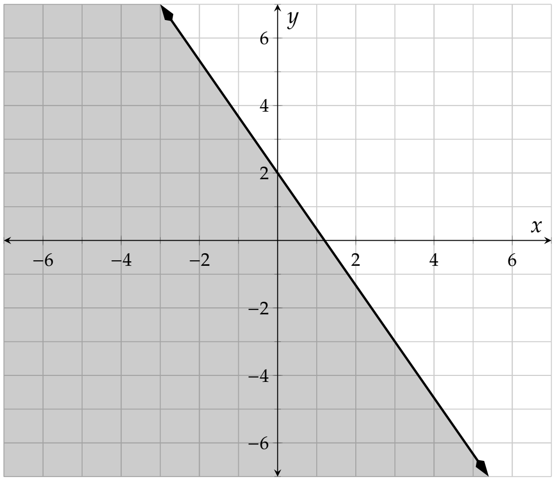


Figure 4.10.13: Complete graph of $y \le -\frac{5}{3}x + 2$

Exercises

Review and Warmup Find the line's slope and y-intercept.

1. A line has equation $y = 8x + 3$.

This line's slope is ☐.

This line's y-intercept is ☐.

2. A line has equation $y = 9x + 9$.

This line's slope is ☐.

This line's y-intercept is ☐.

3. A line has equation $y = -2x - 5$.

This line's slope is ☐.

This line's y-intercept is ☐.

4. A line has equation $y = -10x - 9$.

This line's slope is ☐.

This line's y-intercept is ☐.

5. A line has equation $y = -x - 6$.

This line's slope is ☐.

This line's y-intercept is ☐.

6. A line has equation $y = -x - 4$.

This line's slope is ☐.

This line's y-intercept is ☐.

7. A line has equation $y = -\frac{4x}{3} - 4$.

This line's slope is ☐.

This line's y-intercept is ☐.

8. A line has equation $y = -\frac{6x}{7} - 6$.

This line's slope is ☐.

This line's y-intercept is ☐.

Graphing Two-Variable Inequalities

9. Graph the linear inequality $y \geq -4x$.

10. Graph the linear inequality $y \leq -\frac{1}{2}x - 3$.

11. Graph the linear inequality $y < 3x + 5$.

12. Graph the linear inequality $y > \frac{4}{3}x + 1$.

13. Graph the linear inequality $2x + y \geq 3$.

14. Graph the linear inequality $3x + 2y < -6$.

15. Graph the linear inequality $y \geq 3$.

16. Graph the linear inequality $x < -1$.

Applications

17. You fed your grandpa's cat while he was on vacation. When he was back, he took out a huge bank of coins, including quarters and dimes. He said you can take as many coins as you want, but the total value must be less than \$30.00.

(a) Write an inequality to model this situation, with q representing the number of quarters you will take, and d representing the number of dimes.

(b) Graph this linear inequality.

18. A couple is planning their wedding. They want the cost of the reception and the ceremony to be no more than \$8,000.

(a) Write an inequality to model this situation, with r as the cost of the reception (in dollars) and c as the cost of the ceremony (in dollars).

(b) Graph this linear inequality.

4.11 Graphing Lines Chapter Review

4.11.1 Cartesian Coordinates

In Section 4.1 we covered the definition of the Cartesian Coordinate System and how to plot points using the x and y-axes.

Example 4.11.1 On paper, sketch a Cartesian coordinate system with units, and then plot the following points: $(3,2), (-5,-1), (0,-3), (4,0)$.

Explanation.

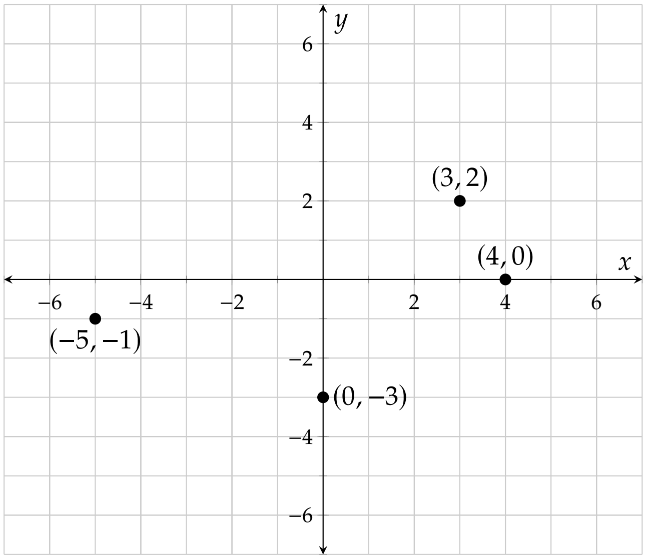


Figure 4.11.2: A Cartesian grid with the four points plotted.

4.11.2 Graphing Equations

In Section 4.2 we covered how to plot solutions to equations to produce a graph of the equation.

Example 4.11.3 Graph the equation $y = -2x + 5$.

Explanation.

x	$y = -2x + 5$	Point
-2		
-1		
0		
1		
2		

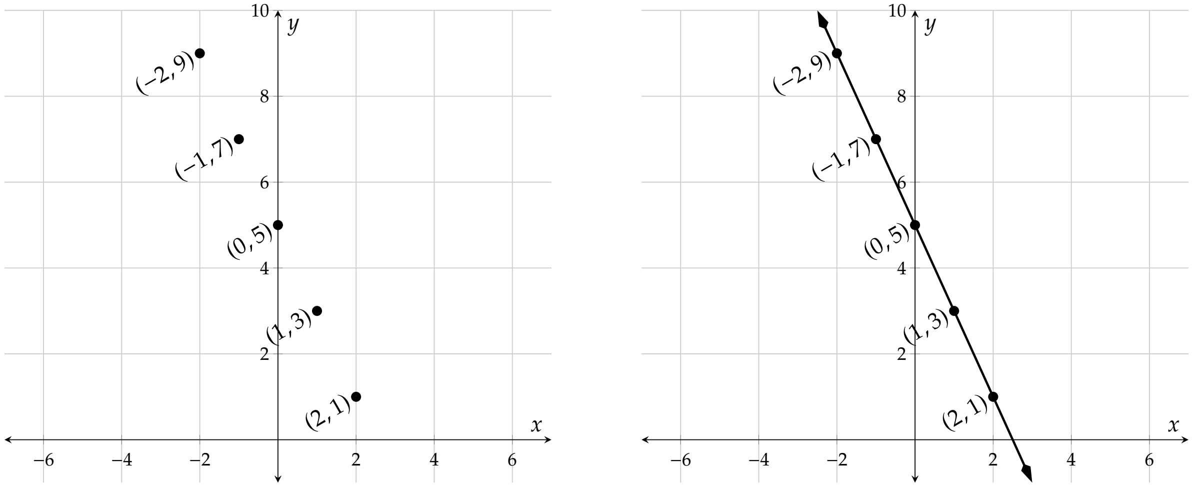
(a) Set up the table

x	$y = -2x + 5$	Point
-2	$-2(-2) + 5 = 9$	$(-2, 9)$
-1	$-2(-1) + 5 = 7$	$(-1, 7)$
0	$-2(0) + 5 = 5$	$(0, 5)$
1	$-2(1) + 5 = 3$	$(1, 3)$
2	$-2(2) + 5 = 1$	$(2, 1)$

(b) Complete the table

Figure 4.11.3: Making a table for $y = -2x + 5$

We use points from the table to graph the equation. First, plot each point carefully. Then, connect the points with a smooth curve. Here, the curve is a straight line. Lastly, we can communicate that the graph extends further by sketching arrows on both ends of the line.



(a) Use points from the table

(b) Connect the points in whatever pattern is apparent

Figure 4.11.3: Graphing the Equation $y = -2x + 5$

4.11.3 Exploring Two-Variable Data and Rate of Change

In Section 4.3 we covered how to find patterns in tables of data and how to calculate the rate of change between points in data.

Exploring Two-Variable Data and Rate of Change For a linear relationship, by its data in a table, we can see the rate of change (slope) and the line's y-intercept, thus writing the equation.

Write an equation in the form $y = \ldots$ suggested by the pattern in the table.

x	y
0	−4
1	−6
2	−8
3	−10

Table 4.11.5: A table of linear data.

Example 4.11.4 Explanation.

We consider how the values change from one row to the next. From row to row, the x-value increases by 1. Also, the y-value decreases by 2 from row to row.

	x	y	
	0	−4	
$+1 \rightarrow$	1	−6	$\leftarrow -2$
$+1 \rightarrow$	2	−8	$\leftarrow -2$
$+1 \rightarrow$	3	−10	$\leftarrow -2$

Since row-to-row change is always 1 for x and is always -2 for y, the rate of change from one row to another row is always the same: -2 units of y for every 1 unit of x.

We know that the output for $x = 0$ is $y = -4$. And our observation about the constant rate of change tells us that if we increase the input by x units from 0, the ouput should decrease by $\overbrace{(-2) + (-2) + \cdots + (-2)}^{x \text{ times}}$, which is $-2x$. So the output would be $-4 - 2x$.

So the equation is $y = -2x - 4$.

4.11.4 Slope

In Section 4.4 we covered the definition of slope 4.4.3 and how to use slope-triangles to calculate slope. There is also the slope formula (4.4.3) which helps find the slope through any two points.

Example 4.11.6 Find the slope of the line in the following graph.

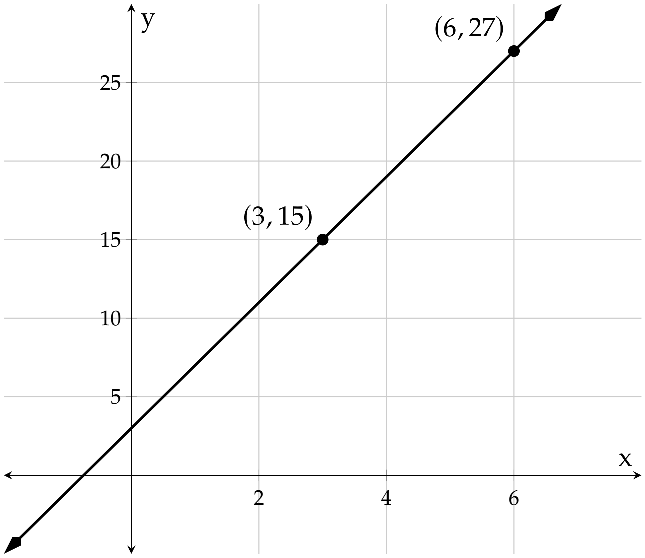


Figure 4.11.7: The line with two points indicated.

Explanation.

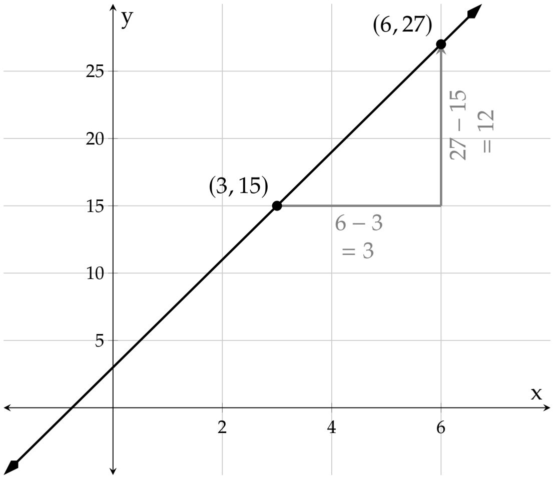


Figure 4.11.8: The line with a slope triangle drawn.

We picked two points on the line, and then drew a slope triangle. Next, we will do:

$$\text{slope} = \frac{12}{3} = 4$$

The line's slope is 4.

Example 4.11.9 Finding a Line's Slope by the Slope Formula. Use the slope formula (4.4.3) to find the slope of the line that passes through the points $(-5, 25)$ and $(4, -2)$.

Explanation.

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$
\begin{aligned}
&= \frac{-2-(25)}{4-(-5)}\\
&= \frac{-27}{9}\\
&= -3
\end{aligned}
$$

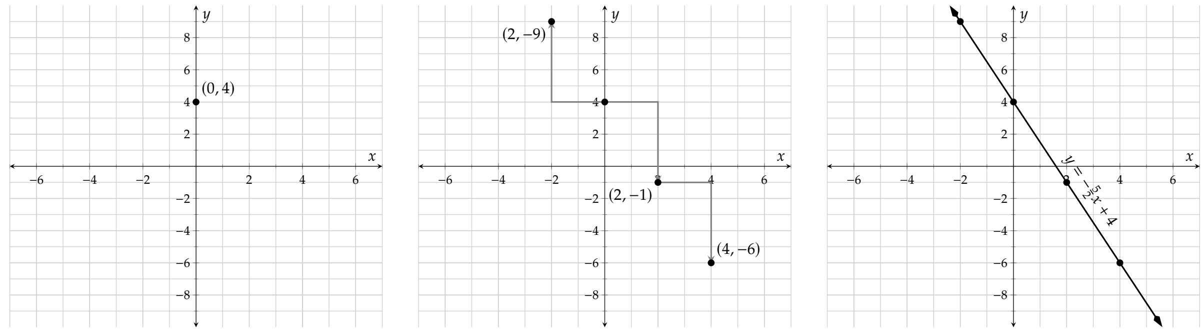
The line's slope is −3.

4.11.5 Slope-Intercept Form

In Section 4.5 we covered the definition of slope intercept-form and both wrote equations in slope-intercept form and graphed lines given in slope-intercept form.

Example 4.11.10 Graph the line $y = -\frac{5}{2}x + 4$.

Explanation.



(a) First, plot the line's y-intercept, $(0, 4)$.

(b) The slope is $-\frac{5}{2} = \frac{-5}{2} = \frac{5}{-2}$. So we can try using a "run" of 2 and a "rise" of −5 or a "run" of −2 and a "rise" of 5.

(c) Arrowheads and labels are encouraged.

Figure 4.11.10: Graphing $y = -\frac{5}{2}x + 4$

Writing a Line's Equation in Slope-Intercept Form Based on Graph Given a line's graph, we can identify its y-intercept, and then find its slope by a slope triangle. With a line's slope and y-intercept, we can write its equation in the form of $y = mx + b$.

Find the equation of the line in the graph.

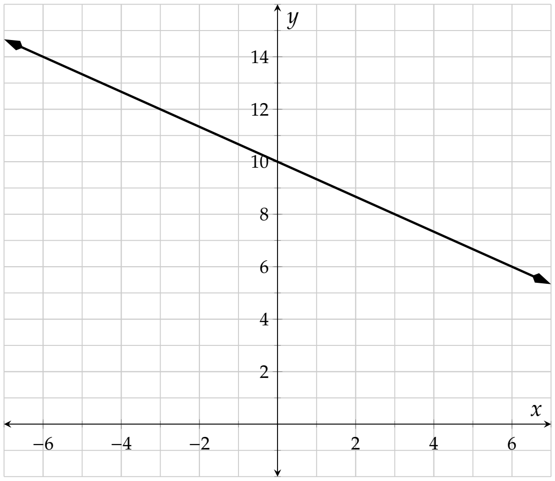


Figure 4.11.12: Graph of a line

Example 4.11.11 Explanation.

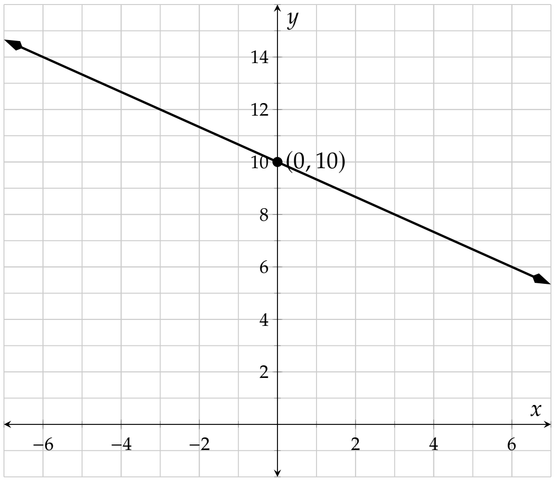


Figure 4.11.13: Identify the line's y-intercept, 10.

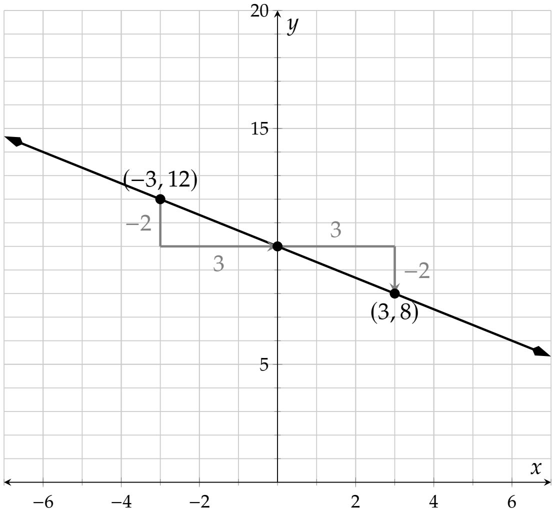


Figure 4.11.14: Identify the line's slope by a slope triangle. Note that we can pick any two points on the line to create a slope triangle. We would get the same slope: $-\frac{2}{3}$

With the line's slope $-\frac{2}{3}$ and y-intercept 10, we can write the line's equation in slope-intercept form: $y = -\frac{2}{3}x + 10$.

4.11.6 Point-Slope Form

In Section 4.6 we covered the definition of point-slope form and both wrote equations in point-slope form and graphed lines given in point-slope form.

Example 4.11.15 A line passes through $(-6, 0)$ and $(9, -10)$. Find this line's equation in point-slope.

Explanation. We will use the slope formula (4.4.3) to find the slope first. After labeling those two points as $(\overset{x_1}{-6}, \overset{y_1}{0})$ and $(\overset{x_2}{9}, \overset{y_2}{-10})$, we have:

$$\begin{aligned} \text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-10 - 0}{9 - (-6)} \\ &= \frac{-10}{15} \\ &= -\frac{2}{3} \end{aligned}$$

Now the point-slope equation looks like $y = -\frac{2}{3}(x - x_0) + y_0$. Next, we will use $(9, -10)$ and substitute x_0 with 9 and y_0 with -10, and we have:

$$\begin{aligned} y &= -\frac{2}{3}(x - x_0) + y_0 \\ y &= -\frac{2}{3}(x - 9) + (-10) \\ y &= -\frac{2}{3}(x - 9) - 10 \end{aligned}$$

4.11.7 Standard Form

In Section 4.7 we covered the definition of standard form of a linear equation. We converted equations from standard form to slope-intercept form and vice versa. We also graphed lines from standard form by finding the intercepts of the line.

Example 4.11.16

a. Convert $2x + 3y = 6$ into slope-intercept form.

b. Convert $y = -\frac{4}{7}x - 3$ into standard form.

Explanation.

a.

$$\begin{aligned} 2x + 3y &= 6 \\ 2x + 3y - 2x &= 6 - 2x \\ 3y &= -2x + 6 \\ \frac{3y}{3} &= \frac{-2x + 6}{3} \end{aligned}$$

$$\begin{aligned} y &= \frac{-2x}{3} + \frac{6}{3} \\ y &= -\frac{2}{3}x + 2 \end{aligned}$$

The line's equation in slope-intercept form is $y = -\frac{2}{3}x + 2$.

b.

$$\begin{aligned} y &= -\frac{4}{7}x - 3 \\ 7 \cdot y &= 7 \cdot (-\frac{4}{7}x - 3) \\ 7y &= 7 \cdot (-\frac{4}{7}x) - 7 \cdot 3 \\ 7y &= -4x - 21 \\ 7y + 4x &= -4x - 21 + 4x \\ 4x + 7y &= -21 \end{aligned}$$

The line's equation in standard form is $4x + 7y = -21$.

To graph a line in standard form, we could first change it to slope-intercept form, and then graph the line by its y-intercept and slope triangles. A second method is to graph the line by its x-intercept and y-intercept.

Example 4.11.17 Graph $2x - 3y = -6$ using its intercepts. And then use the intercepts to calculate the line's slope.

Explanation. We calculate the line's x-intercept by substituting $y = 0$ into the equation

$$\begin{aligned} 2x - 3y &= -6 \\ 2x - 3(0) &= -6 \\ 2x &= -6 \\ x &= -3 \end{aligned}$$

So the line's x-intercept is $(-3, 0)$.

Similarly, we substitute $x = 0$ into the equation to calculate the y-intercept:

$$\begin{aligned} 2x - 3y &= -6 \\ 2(0) - 3y &= -6 \\ -3y &= -6 \\ y &= 2 \end{aligned}$$

So the line's y-intercept is $(0, 2)$.

With both intercepts' coordinates, we can graph the line:

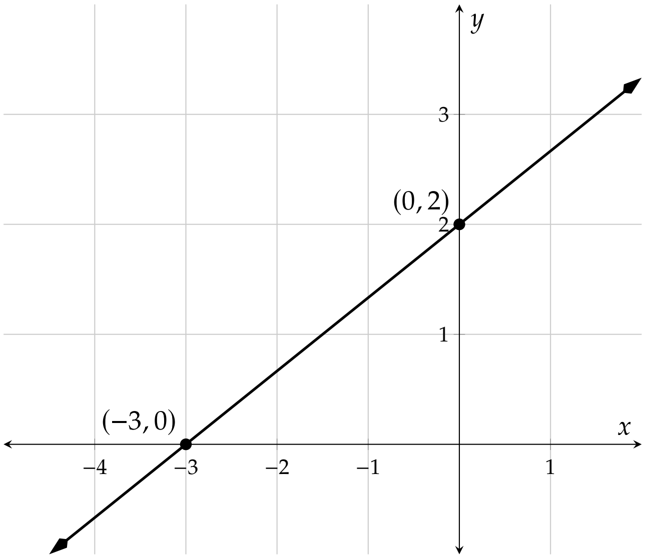


Figure 4.11.18: Graph of $2x - 3y = -6$

Now that we have graphed the line we can read the slope. The rise is 2 units and the run is 3 units so the slope is $\frac{2}{3}$.

4.11.8 Horizontal, Vertical, Parallel, and Perpendicular Lines

In Section 4.8 we covered what equations of horizontal and vertical lines. We also covered the relationships between the slopes of parallel and perpendicular lines.

Example 4.11.19 Line m's equation is $y = -2x + 20$. Line n is parallel to m, and line n also passes the point $(4, -3)$. Find an equation for line n in point-slope form.

Explanation. Since parallel lines have the same slope, line n's slope is also -2. Since line n also passes the point $(4, -3)$, we can write line n's equation in point-slope form:

$$\begin{aligned} y &= m(x - x_1) + y_1 \\ y &= -2(x - 4) + (-3) \\ y &= -2(x - 4) - 3 \end{aligned}$$

Two lines are perpendicular if and only if the product of their slopes is -1.

Example 4.11.20 Line m's equation is $y = -2x + 20$. Line n is perpendicular to m, and line q also passes the point $(4, -3)$. Find an equation for line q in slope-intercept form.

Explanation. Since line m and q are perpendicular, the product of their slopes is -1. Because line m's slope is given as -2, we can find line q's slope is $\frac{1}{2}$.

Since line q also passes the point $(4, -3)$, we can write line q's equation in point-slope form:

$$y = m(x - x_1) + y_1$$

$$\begin{aligned}y &= \frac{1}{2}(x-4)+(-3)\\ y &= \frac{1}{2}(x-4)-3\end{aligned}$$

We can now convert this equation to slope-intercept form:

$$\begin{aligned}y &= \frac{1}{2}(x-4)-3\\ y &= \frac{1}{2}x-2-3\\ y &= \frac{1}{2}x-5\end{aligned}$$

4.11.9 Linear Inequalities in Two Variables

In Section 4.10 we covered how to graph the solution set for an inequality with two variables as a region in the plane.

Example 4.11.21 Graph $y > 2x+1$.

Explanation. There are two steps to graph an inequality.

1. Graph the line $y = 2x+1$. Because the inequality symbol is $>$, (instead of $\geq$) the line should be dashed (instead of solid).
2. Next, we need to decide whether to shade the region above $y = 2x+1$ or below it. We will choose a point to test whether $y > 2x+1$ is true. As long as the line doesn't cross $(0,0)$, we will use $(0,0)$ to test, because the number 0 is the easiest number for calculation.

$$\begin{aligned}y &> 2x+1\\ 0 &\overset{?}{>} 2(0)+1\\ 0 &\overset{\text{no}}{>} 1\end{aligned}$$

   Because $0 > 1$ is not true, the point $(0,0)$ is not a solution and should not be shaded. As a result, we shade the region without $(0,0)$.

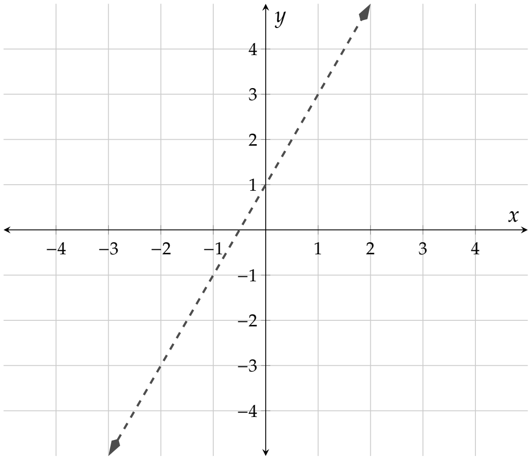


Figure 4.11.22: Step 1 of graphing $y > 2x + 1$

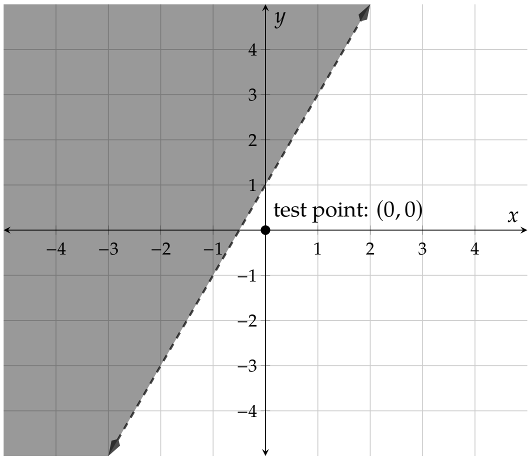
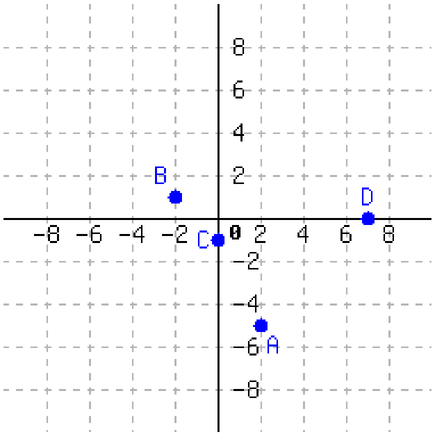


Figure 4.11.23: Step 2 of graphing $y > 2x + 1$

Exercises

1. Sketch the points $(8, 2)$, $(5, 5)$, $(-3, 0)$, $\left(0, -\frac{14}{3}\right)$, $(3, -2.5)$, and $(-5, 7)$ on a Cartesian plane.

2. Locate each point in the graph:



Write each point's position as an ordered pair, like $(1, 2)$.

$A =$ ______ $B =$ ______

$C =$ ______ $D =$ ______

3. Consider the equation

$y = -\frac{3}{8}x - 3$

Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

☐ $(-24, 9)$ ☐ $(32, -13)$ ☐ $(0, -3)$
☐ $(-40, 12)$

4. Consider the equation

$y = -\frac{7}{8}x - 5$

Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

☐ $(24, -25)$ ☐ $(0, -5)$ ☐ $(-16, 14)$
☐ $(-24, 16)$

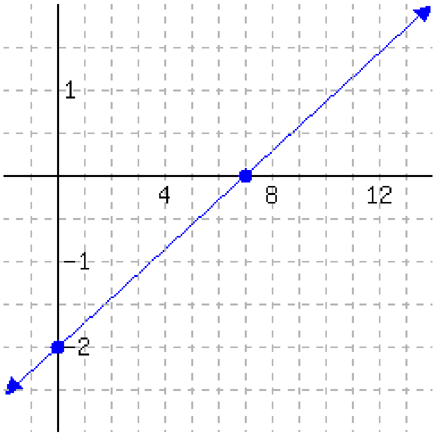
5. Write an equation in the form $y = \ldots$ suggested by the pattern in the table.

x	y
0	-4
1	-2
2	0
3	2

6. Write an equation in the form $y = \ldots$ suggested by the pattern in the table.

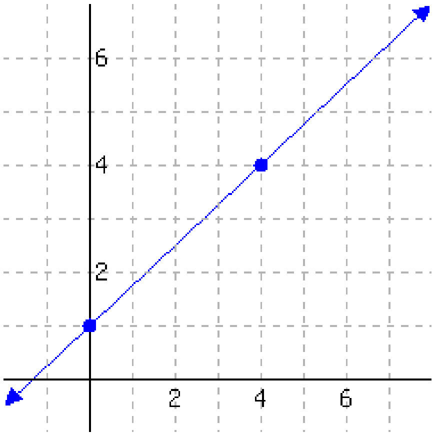
x	y
0	3
1	-2
2	-7
3	-12

7. Below is a line's graph.



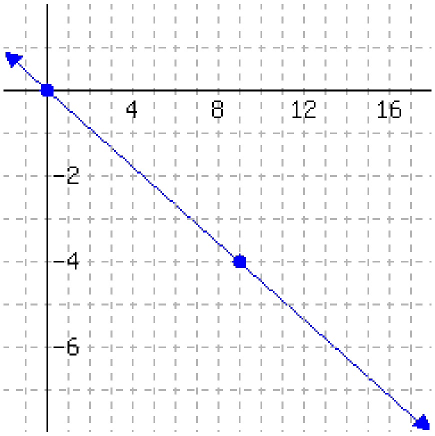
The slope of this line is .

8. Below is a line's graph.



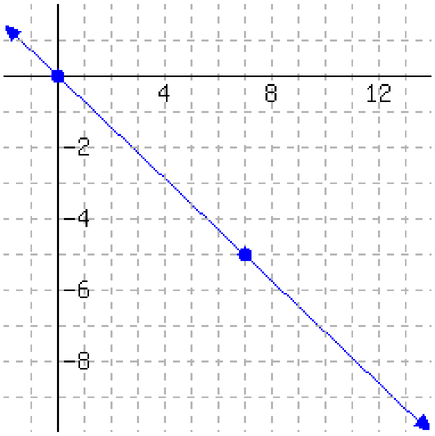
The slope of this line is .

9. Below is a line's graph.



The slope of this line is .

10. Below is a line's graph.



The slope of this line is .

11. A line passes through the points $(-8, 23)$ and $(4, 2)$. Find this line's slope.

12. A line passes through the points $(-24, 23)$ and $(8, -13)$. Find this line's slope.

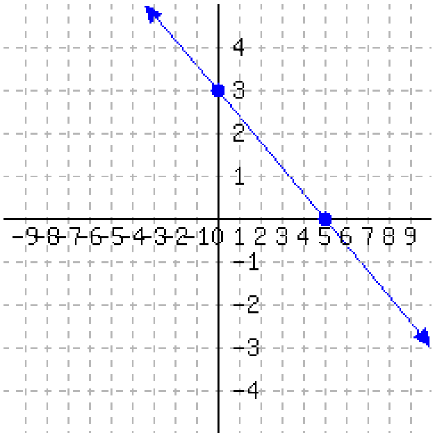
13. A line passes through the points $(4, 8)$ and $(-2, 8)$. Find this line's slope.

14. A line passes through the points $(2, 10)$ and $(-5, 10)$. Find this line's slope.

15. A line passes through the points $(-9, -1)$ and $(-9, 3)$. Find this line's slope.

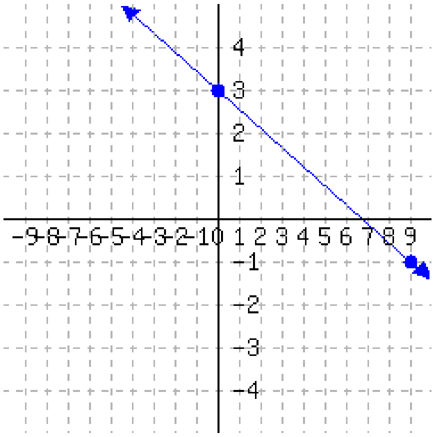
16. A line passes through the points $(-6, -3)$ and $(-6, 5)$. Find this line's slope.

17. A line's graph is given.



This line's slope-intercept equation is

18. A line's graph is given.



This line's slope-intercept equation is

19. Find the line's slope and y-intercept.

A line has equation $3x - 5y = -15$.

This line's slope is ☐.

This line's y-intercept is ☐.

20. Find the line's slope and y-intercept.

A line has equation $5x - 6y = -30$.

This line's slope is ☐.

This line's y-intercept is ☐.

21. A line passes through the points $(8, -2)$ and $(24, 12)$. Find this line's equation in point-slope form.

Using the point $(8, -2)$, this line's point-slope form equation is ☐.

Using the point $(24, 12)$, this line's point-slope form equation is ☐.

22. A line passes through the points $(15, 22)$ and $(0, -2)$. Find this line's equation in point-slope form.

Using the point $(15, 22)$, this line's point-slope form equation is ☐.

Using the point $(0, -2)$, this line's point-slope form equation is ☐.

23. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way. Each minute, they lose 2.4 grams. Ten minutes since the experiment started, the remaining gas had a mass of 96 grams.

Let x be the number of minutes that have passed since the experiment started, and let y be the mass of the gas in grams at that moment. Use a linear equation to model the weight of the gas over time.

a. This line's slope-intercept equation is ☐.

b. 40 minutes after the experiment started, there would be ☐ grams of gas left.

c. If a linear model continues to be accurate, ☐ minutes since the experiment started, all gas in the container will be gone.

24. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way. Each minute, they lose 8.2 grams. Six minutes since the experiment started, the remaining gas had a mass of 278.8 grams.

Let x be the number of minutes that have passed since the experiment started, and let y be the mass of the gas in grams at that moment. Use a linear equation to model the weight of the gas over time.

a. This line's slope-intercept equation is ☐.

b. 40 minutes after the experiment started, there would be ☐ grams of gas left.

c. If a linear model continues to be accurate, ☐ minutes since the experiment started, all gas in the container will be gone.

25. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter `none` into all the answer blanks for that row.

$$2x + 5y = -30$$

	x-value	y-value	Location
y-intercept	________	________	________________
x-intercept	________	________	________________

26. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter `none` into all the answer blanks for that row.

$$4x + 3y = -12$$

	x-value	y-value	Location
y-intercept	________	________	________________
x-intercept	________	________	________________

27. Find the line's slope and y-intercept.

A line has equation $-5x + y = 5$.

This line's slope is ☐.

This line's y-intercept is ☐.

28. Find the line's slope and y-intercept.

A line has equation $-x - y = 1$.

This line's slope is ☐.

This line's y-intercept is ☐.

29. Find the line's slope and y-intercept.

A line has equation $8x + 10y = 1$.

This line's slope is ☐.

This line's y-intercept is ☐.

30. Find the line's slope and y-intercept.

A line has equation $10x + 12y = 5$.

This line's slope is ☐.

This line's y-intercept is ☐.

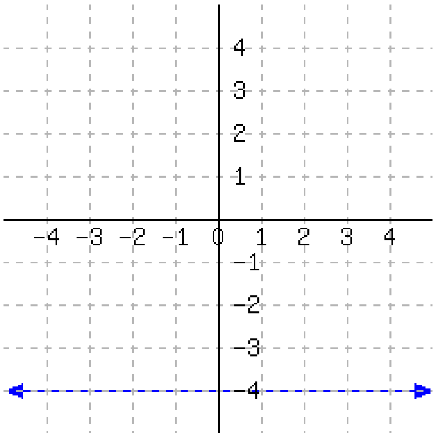
31. Fill out this table for the equation $x = -2$. The first row is an example.

x	y	Points
-2	-3	$(-2, -3)$
______	-2	____________
______	-1	____________
______	0	____________
______	1	____________
______	2	____________

32. Fill out this table for the equation $x = -1$. The first row is an example.

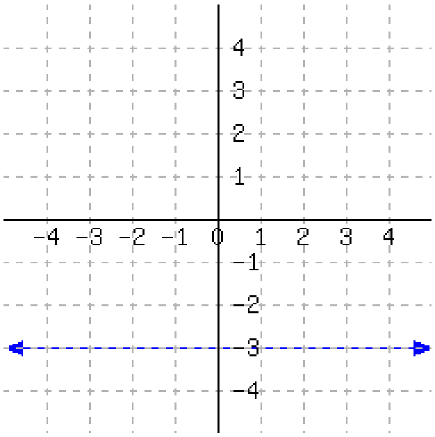
x	y	Points
-1	-3	$(-1, -3)$
______	-2	____________
______	-1	____________
______	0	____________
______	1	____________
______	2	____________

33. A line's graph is given.



This line's equation is

34. A line's graph is given.



This line's equation is

35. Line m passes points $(-4, 10)$ and $(-4, 9)$.

Line n passes points $(7, 6)$ and $(7, -10)$.

Determine how the two lines are related.

These two lines are

- ⊙ parallel
- ⊙ perpendicular
- ⊙ neither parallel nor perpendicular

36. Line m passes points $(-2, 3)$ and $(-2, -4)$.

Line n passes points $(1, 6)$ and $(1, 0)$.

Determine how the two lines are related.

These two lines are

- ⊙ parallel
- ⊙ perpendicular
- ⊙ neither parallel nor perpendicular

37. Line k's equation is $y = -\frac{6}{7}x + 3$.

Line ℓ is perpendicular to line k and passes through the point $(-6, -2)$.

Find an equation for line ℓ in both slope-intercept form and point-slope forms.

An equation for ℓ in slope-intercept form is: $\boxed{\qquad\qquad\qquad\qquad}$.

An equation for ℓ in point-slope form is: $\boxed{\quad}$.

38. Line k's equation is $y = -\frac{7}{9}x + 5$.

Line ℓ is perpendicular to line k and passes through the point $(7, 12)$.

Find an equation for line ℓ in both slope-intercept form and point-slope forms.

An equation for ℓ in slope-intercept form is: $\boxed{\qquad\qquad\qquad\qquad}$.

An equation for ℓ in point-slope form is: $\boxed{\quad}$.

39. Graph the linear inequality $y > \frac{4}{3}x + 1$.

40. Graph the linear inequality $y \leq -\frac{1}{2}x - 3$.

41. Graph the linear inequality $y \geq 3$.

42. Graph the linear inequality $3x + 2y < -6$.

CHAPTER 5

Systems of Linear Equations

5.1 Solving Systems of Linear Equations by Graphing

We have learned how to graph a line given its equation. In this section, we will learn what a *system* of *two* linear equations is, and how to use graphing to solve such a system.

5.1.1 Solving Systems of Equations by Graphing

Example 5.1.2

Fabiana and David are running at constant speeds in parallel lanes on a track. David starts out ahead of Fabiana, but Fabiana is running faster. We want to determine when Fabiana will catch up with David. Let's start by looking at the graph of each runner's distance over time, in Figure 5.1.3.

Each of the two lines has an equation, as discussed in Chapter 4. The line representing David appears to have y-intercept $(0,4)$ and slope $\frac{4}{3}$, so its equation is $y = \frac{4}{3}t + 4$. The line representing Fabiana appears to have y-intercept $(0,0)$ and slope 2, so its equation is $y = 2t$.

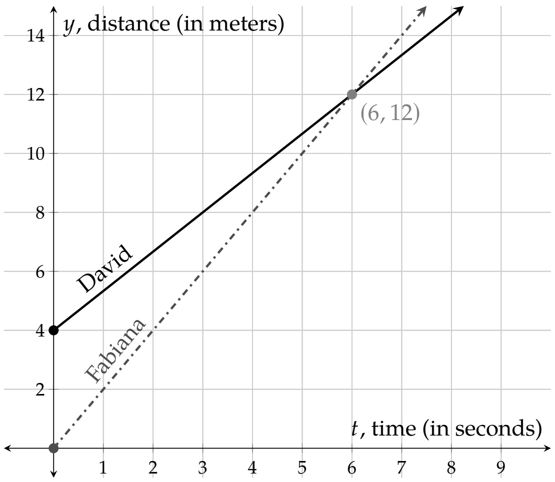


Figure 5.1.3: David and Fabiana's distances.

When these two equations are together as a package, we have what is called a **system of linear equations**:

$$\begin{cases} y = \frac{4}{3}t + 4 \\ y = 2t \end{cases}$$

The large left brace indicates that this is a collection of two distinct equations, not one equation that was somehow algebraically manipulated into an equivalent equation.

As we can see in Figure 5.1.3, the graphs of the two equations cross at the point $(6,12)$. We refer to the point $(6,12)$ as the **solution** to this system of linear equations. To denote the **solution set**, we write

$\{(6, 12)\}$. But it's much more valuable to interpret these numbers in context whenever possible: it took 6 seconds for the two runners to meet up, and when they met they were 12 meters up the track.

Remark 5.1.4. In Example 5.1.2, we stated that the solution was the point $(6, 12)$. It makes sense to write this as an ordered pair when we're given a graph. In some cases when we have no graph, particularly when our variables are not x and y, it might not be clear which variable "comes first" and we won't be able to write an ordered pair. Nevertheless, given the context we can write meaningful summary statements.

Example 5.1.5 Determine the solution to the system of equations graphed in Figure 5.1.6.

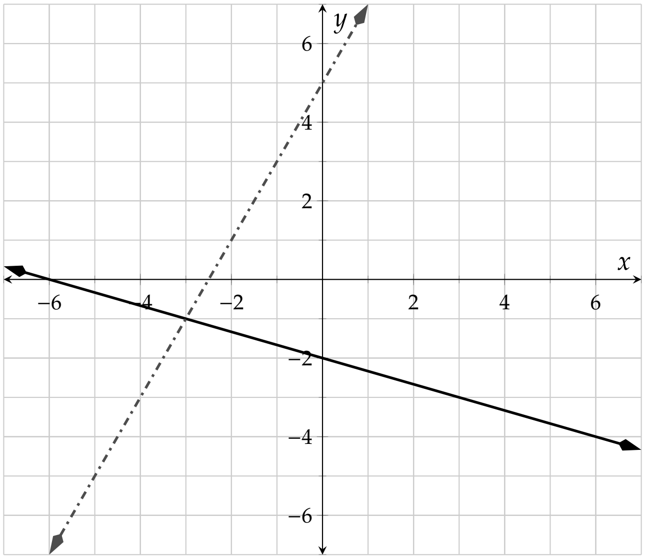
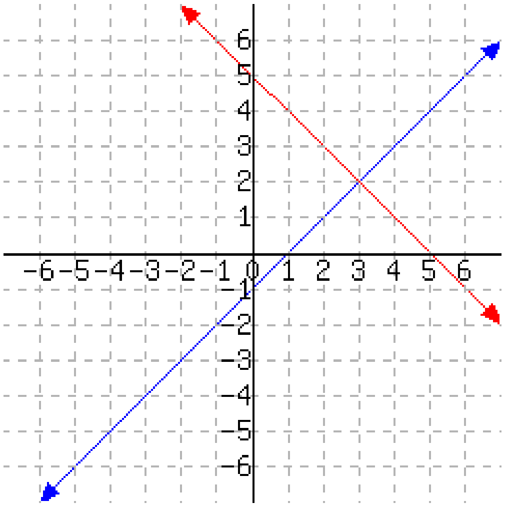


Figure 5.1.6: Graph of a System of Equations

Explanation. The two lines intersect where $x = -3$ and $y = -1$, so the solution is the point $(-3, -1)$. We write the solution set as $\{(-3, -1)\}$.

Checkpoint 5.1.7. Determine the solution to the system of equations graphed below.



The solution is the point ☐.

Explanation. The two lines intersect where $x = 3$ and $y = 2$, so the solution is the point $(3, 2)$. We write the solution set as $\{(3, 2)\}$.

Now let's look at an example where *we* need to make a graph to find the solution.

Example 5.1.8 Solve the following system of equations by graphing:

$$\begin{cases} y = \frac{1}{2}x + 4 \\ y = -x - 5 \end{cases}$$

Notice that each of these equations is written in slope-intercept form. The first equation, $y = \frac{1}{2}x + 4$, is a linear equation with a slope of $\frac{1}{2}$ and a y-intercept of $(0, 4)$. The second equation, $y = -x - 5$, is a linear equation with a slope of -1 and a y-intercept of $(0, -5)$. We'll use this information to graph both lines.

The two lines intersect where $x = -6$ and $y = 1$, so the solution of the system of equations is the point $(-6, 1)$. We write the solution set as $\{(-6, 1)\}$.

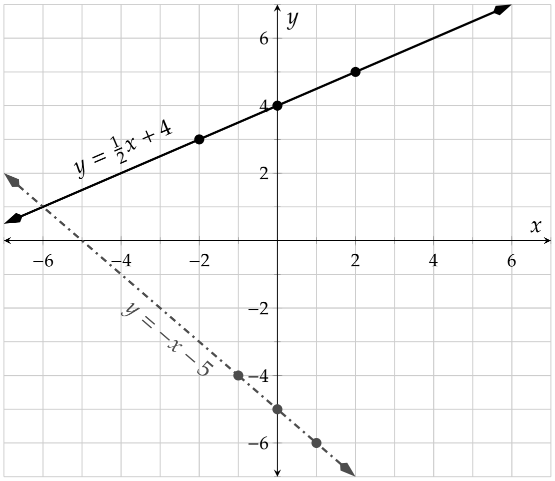


Figure 5.1.9: $y = \frac{1}{2}x + 4$ and $y = -x - 5$.

Example 5.1.10 Solve the following system of equations by graphing:

$$\begin{cases} x - 3y = -12 \\ 2x + 3y = 3 \end{cases}$$

Explanation. Since both line equations are given in standard form, we'll graph each one by finding the intercepts. Recall that to find the x-intercept of each equation, replace y with 0 and solve for x. Similarly, to find the y-intercept of each equation, replace x with 0 and solve for y.

For our first linear equation, we have:

$$\begin{aligned} x - 3(0) &= -12 \\ x &= -12 \end{aligned} \qquad \begin{aligned} 0 - 3y &= -12 \\ -3y &= -12 \\ y &= 4. \end{aligned}$$

So the intercepts are $(-12, 0)$ and $(0, 4)$.

For our second linear equation, we have:

$$\begin{aligned} 2x + 3(0) &= 3 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned} \qquad \begin{aligned} 2(0) + 3y &= 3 \\ 3y &= 3 \\ y &= 1. \end{aligned}$$

So the intercepts are $\left(\frac{3}{2}, 0\right)$ and $(0, 1)$.

Now we can graph each line by plotting the intercepts and connecting these points:

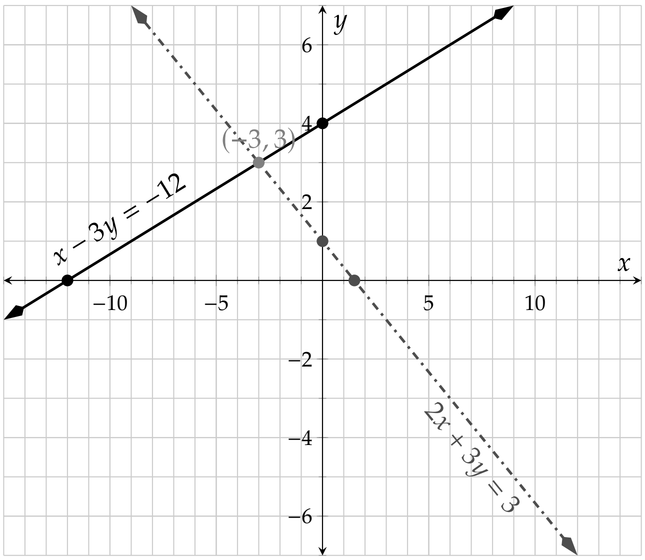


Figure 5.1.11: Graphs of $x-3y=-12$ and $2x+3y=3$

It appears that the solution of the system of equations is the point of intersection of those two lines, which is $(-3,3)$. It's important to check this is correct, because when making a hand-drawn graph, it would be easy to be off by a little bit. To check, we can substitute the values of x and y from the point $(-3,3)$ into each equation:

$$\begin{aligned} x-3y&=-12 & 2x+3y&=3\\ -3-3(3)&\overset{?}{=}-12 & 2(-3)+3(3)&\overset{?}{=}3\\ -3-9&\overset{\checkmark}{=}-12 & -6+9&\overset{\checkmark}{=}3 \end{aligned}$$

So we have checked that $(-3,3)$ is indeed the solution for the system. We write the solution set as $\{(-3,3)\}$.

Example 5.1.12 A college has a north campus and a south campus. The north campus has 18,000 students, and the south campus has 4,000 students. In the past five years, the north campus lost 4,000 students, and the south campus gained 3,000 students. If these trends continue, in how many years would the two campuses have the same number of students? Write and solve a system of equations modeling this problem.

Explanation. Since all the given student counts are in the thousands, we make the decision to measure student population in thousands. So for instance, the north campus starts with a student population of 18 (thousand students).

The north campus lost 4 thousand students in 5 years. So it is losing students at a rate of $\frac{4\text{ thousand}}{5\text{ year}}$, or $\frac{4}{5}\,\frac{\text{thousand}}{\text{year}}$. This rate of change should be interpreted as a negative number, because the north campus is losing students over time. So we have a linear model with starting value 18 thousand students, and a slope of $-\frac{4}{5}$ thousand students per year. In other words,

$$y = -\frac{4}{5}t + 18,$$

where y stands for the number of students in thousands, and t stands for the number of years into the future.

Similarly, the number of students at the south campus can be modeled by $y = \frac{3}{5}t + 4$. Now we have a system of equations:

$$\begin{cases} y = -\dfrac{4}{5}t + 18 \\ y = \dfrac{3}{5}t + 4 \end{cases}$$

We will graph both lines using their slopes and y-intercepts.

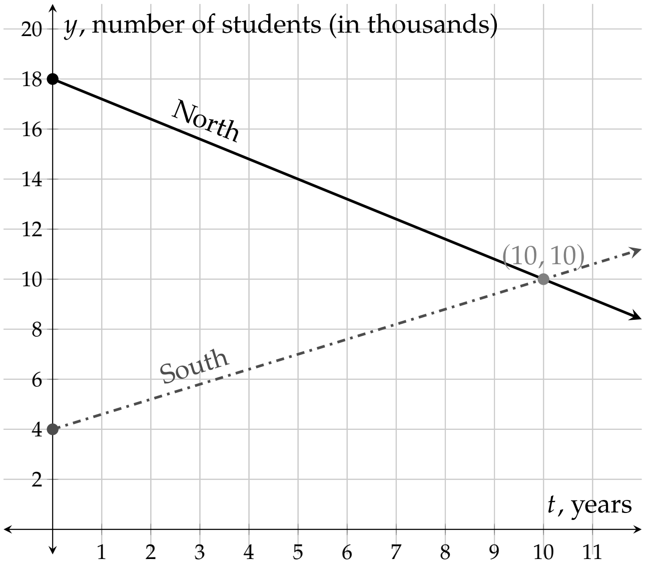


Figure 5.1.13: Number of Students at the South Campus and North Campus

According to the graph, the lines intersect at (10, 10). So if the trends continue, both campuses will have 10,000 students 10 years from now.

Example 5.1.14 Solve the following system of equations by graphing:

$$\begin{cases} y = 3(x-2)+1 \\ y = -\frac{1}{2}(x+1)-1 \end{cases}$$

Explanation. Since both line equations are given in point-slope form, we can start by graphing the point indicated in each equation and use the slope to determine the rest of the line.

For our first equation, $y = 3(x - 2) + 1$, the point indicated in the equation is $(2, 1)$ and the slope is 3.

For our second equation, $y = -\frac{1}{2}(x + 1) - 1$, the point indicated in the equation is $(-1, -1)$ and the slope is $-\frac{1}{2}$.

Now we can graph each line by plotting the points and using their slopes.

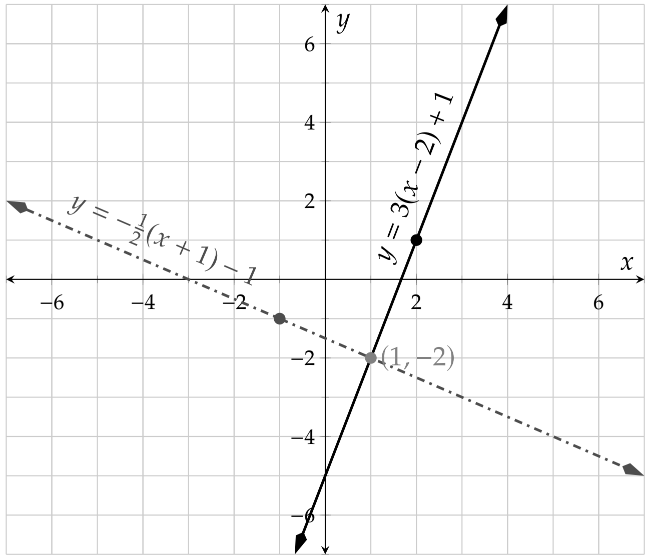


Figure 5.1.15: Graphs of $y = 3(x - 2) + 1$ and $y = -\frac{1}{2}(x + 1) - 1$

It appears that the solution of the system of equations is the point of intersection of those two lines, which is $(1, -2)$. It's important to check this is correct, because when making a hand-drawn graph, it would be easy to be off by a little bit. To check, we can substitute the values of x and y from the point $(1, -2)$ into each equation:

$$\begin{aligned} y &= 3(x - 2) + 1 \\ -2 &\overset{?}{=} 3(1 - 2) + 1 \\ -2 &\overset{?}{=} 3(-1) + 1 \\ -2 &\overset{\checkmark}{=} -3 + 1 \end{aligned}$$

$$\begin{aligned} y &= -\frac{1}{2}(x + 1) - 1 \\ -2 &\overset{?}{=} -\frac{1}{2}(1 + 1) - 1 \\ -2 &\overset{?}{=} -\frac{1}{2}(2) - 1 \\ -2 &\overset{\checkmark}{=} -1 - 1 \end{aligned}$$

So we have checked that $(2, -1)$ is indeed the solution for the system. We write the solution set as $\{(2, -1)\}$.

5.1.2 Special Systems of Equations

Recall that when we solved linear equations in one variable, we had two special cases. In one special case there was no solution and in the other case, there were infinitely many solutions. When solving systems of equations in two variables, we have two similar special cases.

Example 5.1.16 Parallel Lines. Let's look at the graphs of two lines with the same slope, $y = 2x - 4$ and $y = 2x + 1$:

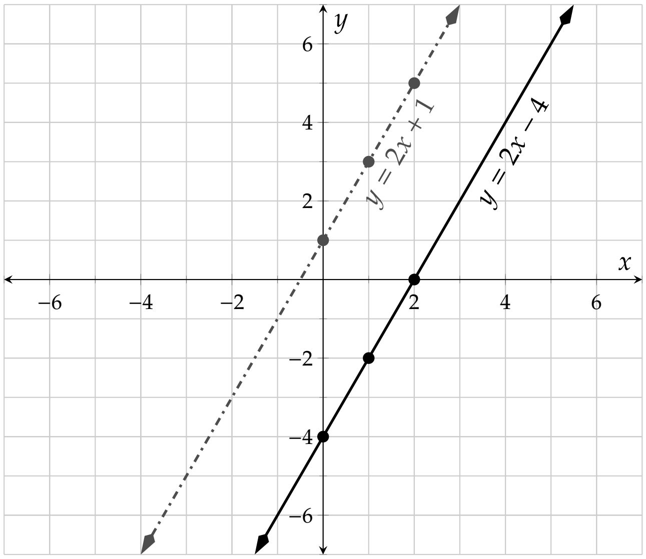


Figure 5.1.17: Graphs of $y = 2x - 4$ and $y = 2x + 1$

For this system of equations, what is the solution? Since the two lines have the same slope they are **parallel lines** and will never intersect. This means that there is *no solution* to this system of equations. We write the solution set as $\emptyset$.

The symbol $\emptyset$ is a special symbol that represents the **empty set**, a *set* that has no numbers in it. This symbol is *not* the same thing as the number zero. The *number* of eggs in an empty egg carton is zero whereas the empty carton itself could represent the empty set. The symbols for the empty set and the number zero may look similar depending on how you write the number zero. Try to keep the concepts separate.

Example 5.1.18 Coinciding Lines. Next we'll look at the other special case. Let's start with this system of equations:

$$\begin{cases} y = 2x - 4 \\ 6x - 3y = 12 \end{cases}$$

To solve this system of equations, we want to graph each line. The first equation is in slope-intercept form and can be graphed easily using its slope of 2 and its y-intercept of $(0, -4)$.

The second equation, $6x - 3y = 12$, can either be graphed by solving for y and using the slope-intercept form or by finding the intercepts. If we use the intercept method, we'll find that this line has an x-intercept of $(2, 0)$ and a y-intercept of $(0, -4)$. When we graph both lines we get Figure 5.1.19.

Now we can see these are actually the *same* line, or **coinciding lines**. To determine the solution to this system, we'll note that they overlap everywhere. This means that we have an infinite number of solutions: *all* points that fall on the line. It may be enough to report that there are infinitely many solutions. In order to be more specific, all we can do is say that any ordered pair (x, y) satisfying the line equation is a solution. In set-builder notation, we would write $\{(x, y) \mid y = 2x - 4\}$.

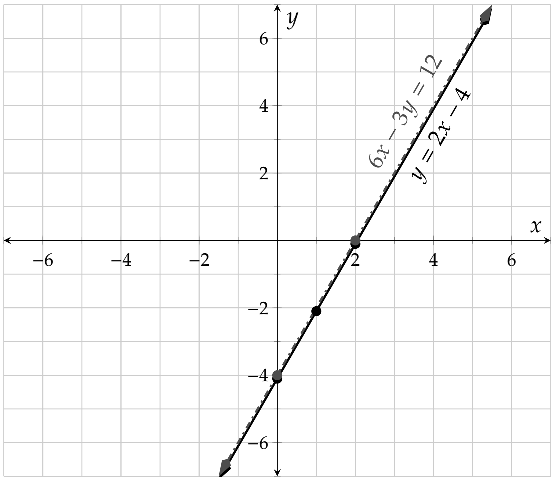


Figure 5.1.19: Graphs of $y = 2x - 4$ and $6x - 3y = 12$

Remark 5.1.20. In Example 5.1.18, what would have happened if we had decided to convert the second line equation into slope-intercept form?

$$\begin{aligned} 6x - 3y &= 12 \\ 6x - 3y - 6x &= 12 - 6x \\ -3y &= -6x + 12 \\ -\frac{1}{3} \cdot (-3y) &= -\frac{1}{3} \cdot (-6x + 12) \\ y &= 2x - 4 \end{aligned}$$

This is the literally the same as the first equation in our system. This is a different way to show that these two equations are equivalent and represent the same line. Any time we try to solve a system where the equations are equivalent, we'll have an infinite number of solutions.

Warning 5.1.21. Notice that for a system of equations with infinite solutions like Example 5.1.18, we didn't say that *every* point was a solution. Rather, every point that falls on that line is a solution. It would be incorrect to state this solution set as "all real numbers" or as "all ordered pairs."

Intersecting Lines: If two linear equations have different slopes, the system has one solution.

Parallel Lines: If the linear equations have the same slope with different y-intercepts, the system has no solution.

Coinciding Lines: If two linear equations have the same slope and the same y-intercept (in other words, they are equivalent equations), the system has infinitely many solutions. This solution set consists of all ordered pairs on that line.

List 5.1.22: A summary of the three types of systems of equations and their solution sets:

Exercises

Warmup and Review Find the line’s slope and y-intercept.

1. A line has equation $y = 9x + 3$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

2. A line has equation $y = 10x + 9$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

3. A line has equation $y = -x - 9$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

4. A line has equation $y = -x - 6$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

5. A line has equation $y = -\dfrac{4x}{9} + 3$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

6. A line has equation $y = -\dfrac{4x}{5} - 9$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

7. A line has equation $y = \dfrac{x}{6} - 7$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

8. A line has equation $y = \dfrac{x}{8} + 6$.

This line’s slope is ☐.

This line’s y-intercept is ☐.

9. Graph the equation $y = -3x$.

10. Graph the equation $y = \frac{1}{4}x$.

11. Graph the equation $y = \frac{2}{3}x + 4$.

12. Graph the equation $y = -2x + 5$.

Solve the linear equation for y.

13.

$$12x - 4y = 44$$

14.

$$-25x - 5y = -85$$

15.

$$3x + 9y = 27$$

16.

$$-6x - 2y = -6$$

Checking Solutions for System of Equations

17. Decide whether $(-3, -5)$ is a solution to the system of equations:

$$\begin{cases} 5x - 4y = 5 \\ x + 4y = -25 \end{cases}$$

The point $(-3, -5)$ (☐ is ☐ is not) a solution.

18. Decide whether $(-2, 2)$ is a solution to the system of equations:

$$\begin{cases} -x + 4y = 10 \\ x + 5y = 8 \end{cases}$$

The point $(-2, 2)$ (☐ is ☐ is not) a solution.

19. Decide whether $(-1, -1)$ is a solution to the system of equations:

$$\begin{cases} 4x + y = -8 \\ y = x \end{cases}$$

The point $(-1, -1)$ (☐ is ☐ is not) a solution.

20. Decide whether $(0, -5)$ is a solution to the system of equations:

$$\begin{cases} -3x - 3y = 15 \\ y = -5x - 6 \end{cases}$$

The point $(0, -5)$ (☐ is ☐ is not) a solution.

21. Decide whether $\left(\frac{7}{4}, \frac{9}{4}\right)$ is a solution to the system of equations:

$$\begin{cases} -8x - 12y = -41 \\ 8x - 12y = -13 \end{cases}$$

The point $\left(\frac{7}{4}, \frac{9}{4}\right)$ (☐ is ☐ is not) a solution.

22. Decide whether $\left(\frac{7}{4}, \frac{5}{4}\right)$ is a solution to the system of equations:

$$\begin{cases} 12x - 4y = 16 \\ -4x - 12y = -19 \end{cases}$$

The point $\left(\frac{7}{4}, \frac{5}{4}\right)$ (☐ is ☐ is not) a solution.

Using a Graph to Solve a System Use a graph to solve the system of equations.

23.

$$\begin{cases} y = -\dfrac{7}{2}x - 8 \\ y = 5x + 9 \end{cases}$$

24.

$$\begin{cases} y = \dfrac{2}{3}x + 5 \\ y = -2x - 11 \end{cases}$$

25.

$$\begin{cases} y = 12x + 7 \\ 3x + y = -8 \end{cases}$$

26.

$$\begin{cases} y = -3x + 5 \\ 4x + y = 8 \end{cases}$$

27.

$$\begin{cases} x + y = 0 \\ 3x - y = 8 \end{cases}$$

28.

$$\begin{cases} 4x - 2y = 4 \\ x + 2y = 6 \end{cases}$$

29.
$$\begin{cases} y = 4x - 5 \\ y = -1 \end{cases}$$

30.
$$\begin{cases} 3x - 4y = 12 \\ y = 3 \end{cases}$$

31.
$$\begin{cases} x + y = -1 \\ x = 2 \end{cases}$$

32.
$$\begin{cases} x - 2y = -4 \\ x = -4 \end{cases}$$

33.
$$\begin{cases} y = 2(x + 3) - 5 \\ y = -\frac{4}{3}(x - 4) - 1 \end{cases}$$

34.
$$\begin{cases} y = -\frac{2}{3}(x - 6) - 2 \\ y = -\frac{1}{2}(x - 1) + 2 \end{cases}$$

35.
$$\begin{cases} y = -\frac{1}{2}(x - 6) + 4 \\ y = 4(x + 1) - 6 \end{cases}$$

36.
$$\begin{cases} y = \frac{5}{6}(x - 6) + 4 \\ y = 2(x + 1) + 4 \end{cases}$$

37.
$$\begin{cases} y = -\frac{4}{5}x + 8 \\ 4x + 5y = -35 \end{cases}$$

38.
$$\begin{cases} 2x - 7y = 28 \\ y = \frac{2}{7}x - 3 \end{cases}$$

39.
$$\begin{cases} -10x + 15y = 60 \\ 6x - 9y = 36 \end{cases}$$

40.
$$\begin{cases} 6x - 8y = 32 \\ 9x - 12y = 12 \end{cases}$$

41.
$$\begin{cases} y = -\frac{3}{5}x + 7 \\ 9x + 15y = 105 \end{cases}$$

42.
$$\begin{cases} 9y - 12x = 18 \\ y = \frac{4}{3}x + 2 \end{cases}$$

Determining the Number of Solutions in a System of Equations

43. Simply by looking at this system of equations, decide the number of solutions it has.

$$\begin{cases} y = -6x \\ y = -6x \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

44. Simply by looking at this system of equations, decide the number of solutions it has.

$$\begin{cases} y = \frac{2}{5}x - 4 \\ y = \frac{2}{5}x - 4 \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

45. Without graphing this system of equations, decide the number of solutions it has.

$$\begin{cases} y = \frac{3}{2}x + 4 \\ 9x - 6y = 0 \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

46. Without graphing this system of equations, decide the number of solutions it has.

$$\begin{cases} y = 2x + 1 \\ 8x - 4y = 4 \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

47. Without graphing this system of equations, decide the number of solutions it has.

$$\begin{cases} 3x + 12y = -24 \\ 4x + 16y = -16 \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

48. Without graphing this system of equations, decide the number of solutions it has.

$$\begin{cases} 12x + 4y = -20 \\ 8x + 2y = 10 \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

49. Simply by looking at this system of equations, decide the number of solutions it has.

$$\begin{cases} x = 4 \\ y = -5 \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

50. Simply by looking at this system of equations, decide the number of solutions it has.

$$\begin{cases} x = 1 \\ y = -1 \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

5.2 Substitution

In Section 5.1, we focused on solving systems of equations by graphing. In addition to being time consuming, graphing can be an awkward method to determine the exact solution when the solution has large numbers or fractions. There are two symbolic methods for solving systems of linear equations, and in this section we will use one of them: **substitution**.

5.2.1 Solving Systems of Equations Using Substitution

Example 5.2.2 The Interview. In 2014, the New York Times[a] posted the following about the movie, "The Interview":

> "The Interview" generated roughly \$15 million in online sales and rentals during its first four days of availability, Sony Pictures said on Sunday.
>
> Sony did not say how much of that total represented \$6 digital rentals versus \$15 sales. The studio said there were about two million transactions overall.

A few days later, Joey Devilla cleverly pointed out in his blog[b], that there is enough information given to find the amount of sales versus rentals. Using algebra, we can write a system of equations and solve it to find the two quantities.[c]

First, we will define variables. We need two variables, because there are two unknown quantities: how many sales there were and how many rentals there were. Let r be the number of rental transactions and let s be the number of sales transactions.

If you are unsure how to write an equation from the background information, use the units to help you. The units of each term in an equation must match because we can only add like quantities. Both r and s are in transactions. The article says that the total number of transactions is 2 million. So our first equation will add the total number of rental and sales transactions and set that equal to 2 million. Our equation is:

$$(r \text{ transactions}) + (s \text{ transactions}) = 2{,}000{,}000 \text{ transactions}$$

Without the units:

$$r + s = 2{,}000{,}000$$

The price of each rental was \$6. That means the problem has given us a *rate* of $6\,\frac{\text{dollars}}{\text{transaction}}$ to work with. The rate unit suggests this should be multiplied by something measured in transactions. It makes sense to multiply by r, and then the number of dollars generated from rentals was $6r$. Similarly, the price of each sale was \$15, so the revenue from sales was $15s$. The total revenue was \$15 million, which we can represent with this equation:

$$\left(6\,\frac{\text{dollars}}{\text{transaction}}\right)(r \text{ transactions}) + \left(15\,\frac{\text{dollars}}{\text{transaction}}\right)(s \text{ transactions}) = \$15{,}000{,}000$$

Without the units:

$$6r + 15s = 15{,}000{,}000$$

Here is our system of equations:

$$\begin{cases} r + s = 2{,}000{,}000 \\ 6r + 15s = 15{,}000{,}000 \end{cases}$$

To solve the system, we will use the **substitution method**. The idea is to use *one* equation to find an expression that is equal to r but, cleverly, does not use the variable "r." Then, substitute this for r into the *other* equation. This leaves you with *one* equation that only has *one* variable.

The first equation from the system is an easy one to solve for r:

$$\begin{aligned} r + s &= 2{,}000{,}000 \\ r &= 2{,}000{,}000 - s \end{aligned}$$

This tells us that the expression $2{,}000{,}000 - s$ is equal to r, so we can *substitute* it for r in the second equation:

$$\begin{aligned} 6r + 15s &= 15{,}000{,}000 \\ 6(2{,}000{,}000 - s) + 15s &= 15{,}000{,}000 \end{aligned}$$

Now we have an equation with only one variable, s, which we will solve for:

$$\begin{aligned} 6(2{,}000{,}000 - s) + 15s &= 15{,}000{,}000 \\ 12{,}000{,}000 - 6s + 15s &= 15{,}000{,}000 \\ 12{,}000{,}000 + 9s &= 15{,}000{,}000 \\ 9s &= 3{,}000{,}000 \\ \frac{9s}{9} &= \frac{3{,}000{,}000}{9} \\ s &= 333{,}333.\overline{3} \end{aligned}$$

At this point, we know that $s = 333{,}333.\overline{3}$. This tells us that out of the 2 million transactions, roughly 333,333 were from online sales. Recall that we solved the first equation for r, and found $r = 2{,}000{,}000 - s$.

$$\begin{aligned} r &= 2{,}000{,}000 - s \\ r &= 2{,}000{,}000 - 333{,}333.\overline{3} \\ r &= 1{,}666{,}666.\overline{6} \end{aligned}$$

To check our answer, we will see if $s = 333{,}333.\overline{3}$ and $r = 1{,}666{,}666.\overline{6}$ make the original equations true:

$$\begin{aligned} r + s &= 2{,}000{,}000 \\ 1{,}666{,}666.\overline{6} + 333{,}333.\overline{3} &\stackrel{?}{=} 2{,}000{,}000 \\ 2{,}000{,}000 &\stackrel{\checkmark}{=} 2{,}000{,}000 \end{aligned}$$

$$\begin{aligned} 6r + 15s &= 15{,}000{,}000 \\ 6\left(1{,}666{,}666.\overline{6}\right) + 15\left(333{,}333.\overline{3}\right) &\stackrel{?}{=} 15{,}000{,}000 \\ 10{,}000{,}000 + 5{,}000{,}000 &\stackrel{\checkmark}{=} 15{,}000{,}000 \end{aligned}$$

In summary, there were roughly 333,333 copies sold and roughly 1,666,667 copies rented.

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[a] (nyti.ms/2pupebT)

[b] http://www.joeydevilla.com/2014/12/31/

[c] Although since the given information uses approximate values, the solutions we will find will only be approximations too.

Remark 5.2.3. In Example 5.2.2, we *chose* to solve the equation $r + s = 2{,}000{,}000$ for r. We could just as easily have instead solved for s and substituted that result into the second equation instead. The summary conclusion would have been the same.

Remark 5.2.4. In Example 5.2.2, we rounded the solution values because only whole numbers make sense in the context of the problem. It was OK to round, because the original information we had to work with were rounded. In fact, it would be OK to round even more to $s = 330{,}000$ and $r = 1{,}700{,}000$, as long as we communicate clearly that we rounded and our values are rough.

In other exercises where there is no context and nothing suggests the given numbers are approximations, it is not OK to round and all answers should be communicated with their exact values.

Example 5.2.5 Solve the system of equations using substitution:

$$\begin{cases} x + 2y = 8 \\ 3x - 2y = 8 \end{cases}$$

Explanation. To use substitution, we need to solve for *one* of the variables in *one* of our equations. Looking at both equations, it will be easiest to solve for x in the first equation:

$$\begin{aligned} x + 2y &= 8 \\ x &= 8 - 2y \end{aligned}$$

Next, we replace x in the second equation with $8 - 2y$, giving us a linear equation in only one variable, y, that we may solve:

$$\begin{aligned} 3x - 2y &= 8 \\ 3(8 - 2y) - 2y &= 8 \\ 24 - 6y - 2y &= 8 \\ 24 - 8y &= 8 \\ -8y &= -16 \\ y &= 2 \end{aligned}$$

Now that we have the value for y, we need to find the value for x. We have already solved the first equation for x, so that is the easiest equation to use.

$$\begin{aligned} x &= 8 - 2y \\ x &= 8 - 2(2) \\ x &= 8 - 4 \\ x &= 4 \end{aligned}$$

To check this solution, we replace x with 4 and y with 2 in each equation:

$$\begin{aligned} x + 2y &= 8 \\ 4 + 2(2) &\overset{?}{=} 8 \end{aligned} \qquad \begin{aligned} 3x - 2y &= 8 \\ 3(4) - 2(2) &\overset{?}{=} 8 \end{aligned}$$

$$4+4 \overset{\checkmark}{=} 8 \qquad\qquad 12-4 \overset{\checkmark}{=} 8$$

We conclude then that this system of equations is true when $x = 4$ and $y = 2$. Our solution is the point $(4, 2)$ and we write the solution set as $\{(4, 2)\}$.

Checkpoint 5.2.6. Try a similar exercise.

Solve the following system of equations.

$$\begin{cases} 5x + y = -3 \\ \quad\;\; 0 = -1 + 4x + y \end{cases}$$

Explanation. These equations have no fractions; let's try to keep it that way.

$$\begin{cases} 5x + y = -3 \\ \quad\;\; 0 = -1 + 4x + y \end{cases}$$

Since one of the coefficients of y is 1, it is wise to solve for y in terms of the other variable and then use substitution to complete the problem.

$$y = -5x - 3 \quad \text{(from the first equation)}$$

which we can substitute in for y into the second equation:

$$\begin{aligned} &0 = 4x - 1 + (-5x) - 3 \quad \text{(from the second equation)} \\ &0 = -x - 4 \\ &x = -4 \\ &x = -4 \end{aligned}$$

We can substitute this back for x into the first equation to find y.

$$\begin{aligned} &y = -5(-4) - 3 \quad \text{(from the first equation, after we had solved for } y \text{ in terms of } x\text{)} \\ &y = 20 + (-3) \\ &y = 17 \end{aligned}$$

So the solution is $x = -4$, $y = 17$.

Example 5.2.7 Solve this system of equations using substitution:

$$\begin{cases} \;\;\, 3x - 7y = \;\;5 \\ -5x + 2y = 11 \end{cases}$$

Explanation. We need to solve for *one* of the variables in *one* of our equations. Looking at both equations, it will be easiest to solve for y in the second equation. The coefficient of y in that equation is smallest.

$$-5x + 2y = 11$$

$$\begin{aligned} 2y &= 11 + 5x \\ \frac{2y}{2} &= \frac{11 + 5x}{2} \\ y &= \frac{11}{2} + \frac{5}{2}x \end{aligned}$$

Note that in this example, there are fractions once we solve for y. We should take care with the steps that follow that the fraction arithmetic is correct.

Replace y in the first equation with $\frac{11}{2} + \frac{5}{2}x$, giving us a linear equation in only one variable, x, that we may solve:

$$\begin{aligned} 3x - 7y &= 5 \\ 3x - 7\left(\frac{11}{2} + \frac{5}{2}x\right) &= 5 \\ 3x - 7 \cdot \frac{11}{2} - 7 \cdot \frac{5}{2}x &= 5 \\ 3x - \frac{77}{2} - \frac{35}{2}x &= 5 \\ \frac{6}{2}x - \frac{77}{2} - \frac{35}{2}x &= 5 \\ -\frac{29}{2}x - \frac{77}{2} &= 5 \\ -\frac{29}{2}x &= \frac{10}{2} + \frac{77}{2} \\ -\frac{29}{2}x &= \frac{87}{2} \\ -\frac{2}{29} \cdot \left(-\frac{29}{2}x\right) &= -\frac{2}{29} \cdot \left(\frac{87}{2}\right) \\ x &= -3 \end{aligned}$$

Now that we have the value for x, we need to find the value for y. We have already solved the second equation for y, so that is the easiest equation to use.

$$\begin{aligned} y &= \frac{11}{2} + \frac{5}{2}x \\ y &= \frac{11}{2} + \frac{5}{2}(-3) \\ y &= \frac{11}{2} - \frac{15}{2} \\ y &= -\frac{4}{2} \\ y &= -2 \end{aligned}$$

To check this solution, we replace x with -3 and y with -2 in each equation:

$$\begin{aligned} 3x - 7y &= 5 & \qquad -5x + 2y &= 11 \\ 3(-3) - 7(-2) &\stackrel{?}{=} 5 & \qquad -5(-3) + 2(-2) &\stackrel{?}{=} 11 \end{aligned}$$

$$-9 + 14 \overset{\checkmark}{=} 5 \qquad 15 - 4 \overset{\checkmark}{=} 11$$

We conclude then that this system of equations is true when $x = -3$ and $y = -2$. Our solution is the point $(-3, -2)$ and we write the solution set as $\{(-3, -2)\}$.

Example 5.2.8 Clearing Fraction Denominators Before Solving. Solve the system of equations using the substitution method:

$$\begin{cases} \frac{x}{3} - \frac{1}{2}y = \frac{5}{6} \\ \frac{1}{4}x = \frac{y}{2} + 1 \end{cases}$$

Explanation. When a system of equations has fraction coefficients, it can be helpful to take steps that replace the fractions with whole numbers. With each equation, we may multiply each side by the least common multiple of all the denominators.

In the first equation, the least common multiple of the denominators is 6, so:

$$\begin{aligned} \frac{x}{3} - \frac{1}{2}y &= \frac{5}{6} \\ 6 \cdot \left(\frac{x}{3} - \frac{1}{2}y\right) &= 6 \cdot \frac{5}{6} \\ 6 \cdot \frac{x}{3} - 6 \cdot \frac{1}{2}y &= 6 \cdot \frac{5}{6} \\ 2x - 3y &= 5 \end{aligned}$$

In the second equation, the least common multiple of the denominators is 4, so:

$$\begin{aligned} \frac{1}{4}x &= \frac{y}{2} + 1 \\ 4 \cdot \frac{1}{4}x &= 4 \cdot \frac{y}{2} + 4 \cdot 1 \\ 4 \cdot \frac{1}{4}x &= 4 \cdot \frac{y}{2} + 4 \cdot 1 \\ x &= 2y + 4 \end{aligned}$$

Now we have this system that is equivalent to the original system of equations, but there are no fraction coefficients:

$$\begin{cases} 2x - 3y = 5 \\ x = 2y + 4 \end{cases}$$

The second equation is already solved for x, so we will substitute x in the first equation with $2y + 4$, and we have:

$$\begin{aligned} 2x - 3y &= 5 \\ 2(2y + 4) - 3y &= 5 \\ 4y + 8 - 3y &= 5 \\ y + 8 &= 5 \\ y &= -3 \end{aligned}$$

And we have solved for y. To find x, we know $x = 2y + 4$, so we have:

$$\begin{aligned} x &= 2y + 4 \\ x &= 2(-3) + 4 \\ x &= -6 + 4 \\ x &= -2 \end{aligned}$$

The solution is $(-2, -3)$. Checking this solution is left as an exercise.

Checkpoint 5.2.9. Try a similar exercise.

Solve the following system of equations.

$$\begin{cases} -3 = -m + \frac{1}{2}r \\ -m + \frac{4}{3} = -r \end{cases}$$

Explanation. If an equation involves fractions, it is helpful to clear denominators by multiplying both sides of the equation by a common multiple of the denominators.

$$\begin{cases} 2\left(-3\right) = 2\left(-m + \frac{1}{2}r\right) \\ 3\left(-m + \frac{4}{3}\right) = 3\left(-r\right) \end{cases}$$

$$\begin{cases} -6 = -2m + r \\ -3m + 4 = -3r \end{cases}$$

Since one of the coefficients of r is 1, it is wise to solve for r in terms of the other variable and then use substitution to complete the problem.

$$2m - 6 = r \quad \text{(from the first equation)}$$

which we can substitute in for r into the second equation:

$$\begin{aligned} 4 - 3m &= -3(2m - 6) \quad \text{(from the second equation)} \\ 4 - 3m &= 18 - 6m \\ 3m &= 14 \\ m &= \frac{14}{3} \end{aligned}$$

We can substitute this back for m into the first equation to find r.

$$\begin{aligned} 2\left(\frac{14}{3}\right) - 6 &= r \quad \text{(from the first equation, after we had solved for } r \text{ in terms of } m\text{)} \\ \frac{28}{3} + (-6) &= r \\ \frac{10}{3} &= r \end{aligned}$$

So the solution is $m = \frac{14}{3}, r = \frac{10}{3}$.

5.2.2 Applications of Systems of Equations

In Example 5.2.2, we set up and solved a system of linear equations for a real-world application. The quantities in that problem included rate units (dollars per transaction). Here are some more scenarios that we can model with systems of linear equations.

Example 5.2.10 Two Different Interest Rates. Notah made some large purchases with his two credit cards one month and took on a total of \$8,400 in debt from the two cards. He didn't make any payments the first month, so the two credit card debts each started to accrue interest. That month, his Visa card charged 2% interest and his Mastercard charged 2.5% interest. Because of this, Notah's total debt grew by \$178. How much money did Notah charge to each card?

Explanation. To start, we will define two variables based on our two unknowns. Let v be the amount charged to the Visa card (in dollars) and let m be the amount charged to the Mastercard (in dollars).

To determine our equations, notice that we are given two different totals. We will use these to form our two equations. The total amount charged is \$8,400 so we have:

$$(v \text{ dollars}) + (m \text{ dollars}) = \$8400$$

Or without units:

$$v + m = 8400$$

The other total we were given is the total amount of interest, \$178, which is also in dollars. The Visa had v dollars charged to it and accrues 2% interest. So $0.02v$ is the dollar amount of interest that comes from using this card. Similarly, $0.025m$ is the dollar amount of interest from using the Mastercard. Together:

$$0.02(v \text{ dollars}) + 0.025(m \text{ dollars}) = \$178$$

Or without units:

$$0.02v + 0.025m = 178$$

As a system, we write:

$$\begin{cases} v + m = 8400 \\ 0.02v + 0.025m = 178 \end{cases}$$

To solve this system by substitution, notice that it will be easier to solve for one of the variables in the first equation. We'll solve that equation for v:

$$\begin{aligned} v + m &= 8400 \\ v &= 8400 - m \end{aligned}$$

Now we will substitute $8400 - m$ for v in the second equation:

$$\begin{aligned} 0.02v + 0.025m &= 178 \\ 0.02(8400 - m) + 0.025m &= 178 \\ 168 - 0.02m + 0.025m &= 178 \\ 168 + 0.005m &= 178 \\ 0.005m &= 10 \\ \frac{0.005m}{0.005} &= \frac{10}{0.005} \\ m &= 2000 \end{aligned}$$

Lastly, we can determine the value of v by using the earlier equation where we isolated v:

$$\begin{aligned} v &= 8400 - m \\ v &= 8400 - 2000 \end{aligned}$$

$$v = 6400$$

In summary, Notah charged \$6400 to the Visa and \$2000 to the Mastercard. We should check that these numbers work as solutions to our original system *and* that they make sense in context. (For instance, if one of these numbers were negative, or was something small like \$0.50, they wouldn't make sense as credit card debt.)

The next two examples are called **mixture problems**, because they involve mixing two quantities together to form a combination and we want to find out how much of each quantity to mix.

Example 5.2.11 Mixing Solutions with Two Different Concentrations. LaVonda is a meticulous bartender and she needs to serve 600 milliliters of Rob Roy, an alcoholic cocktail that is 34% alcohol by volume. The main ingredients are scotch that is 42% alcohol and vermouth that is 18% alcohol. How many milliliters of each ingredient should she mix together to make the concentration she needs?

Explanation. The two unknowns are the quantities of each ingredient. Let s be the amount of scotch (in mL) and let v be the amount of vermouth (in mL).

One quantity given to us in the problem is 600 mL. Since this is the total volume of the mixed drink, we must have:

$$(s\,\text{mL}) + (v\,\text{mL}) = 600\,\text{mL}$$

Or without units:

$$s + v = 600$$

To build the second equation, we have to think about the alcohol concentrations for the scotch, vermouth, and Rob Roy. It can be tricky to think about percentages like these correctly. One strategy is to focus on the *amount* (in mL) of *alcohol* being mixed. If we have s milliliters of scotch that is 42% alcohol, then $0.42s$ is the actual *amount* (in mL) of alcohol in that scotch. Similarly, $0.18v$ is the amount of alcohol in the vermouth. And the final cocktail is 600 mL of liquid that is 34% alcohol, so it has $0.34(600) = 204$ milliliters of alcohol. All this means:

$$0.42(s\,\text{mL}) + 0.18(v\,\text{mL}) = 204\,\text{mL}$$

Or without units:

$$0.42s + 0.18v = 204$$

So our system is:

$$\begin{cases} s + v = 600 \\ 0.42s + 0.18v = 204 \end{cases}$$

To solve this system, we'll solve for s in the first equation:

$$\begin{aligned} s + v &= 600 \\ s &= 600 - v \end{aligned}$$

And then substitute s in the second equation with $600 - v$:

$$\begin{aligned} 0.42s + 0.18v &= 204 \\ 0.42(600 - v) + 0.18v &= 204 \\ 252 - 0.42v + 0.18v &= 204 \end{aligned}$$

$$\begin{aligned} 252 - 0.24v &= 204 \\ -0.24v &= -48 \\ \frac{-0.24v}{-0.24} &= \frac{-48}{-0.24} \\ v &= 200 \end{aligned}$$

As a last step, we will determine s using the equation where we had isolated s:

$$\begin{aligned} s &= 600 - v \\ s &= 600 - 200 \\ s &= 400 \end{aligned}$$

In summary, LaVonda needs to combine 400 mL of scotch with 200 mL of vermouth to create 600 mL of Rob Roy that is 34% alcohol by volume.

As a check for Example 5.2.11, we will use **estimation** to see that our solution is reasonable. Since LaVonda is making a 34% solution, she would need to use more of the 42% concentration than the 18% concentration, because 34% is closer to 42% than to 18%. This agrees with our answer because we found that she needed 400 mL of the 42% solution and 200 mL of the 18% solution. This is an added check that we have found reasonable answers.

Example 5.2.12 Mixing a Coffee Blend. Desi owns a coffee shop and they want to mix two different types of coffee beans to make a blend that sells for \$12.50 per pound. They have some coffee beans from Columbia that sell for \$9.00 per pound and some coffee beans from Honduras that sell for \$14.00 per pound. How many pounds of each should they mix to make 30 pounds of the blend?

Explanation. Before we begin, it may be helpful to try to estimate the solution. Let's compare the three prices. Since \$12.50 is between the prices of \$9.00 and \$14.00, this mixture is possible. Now we need to estimate the amount of each type needed. The price of the blend (\$12.50 per pound) is closer to the higher priced beans (\$14.00 per pound) than the lower priced beans (\$9.00 per pound). So we will need to use more of that type. Keeping in mind that we need a total of 30 pounds, we roughly estimate 20 pounds of the \$14.00 Honduran beans and 10 pounds of the \$9.00 Columbian beans. How good is our estimate? Next we will solve this exercise exactly.

To set up our system of equations we define variables, letting C be the amount of Columbian coffee beans (in pounds) and H be the amount of Honduran coffee beans (in pounds).

The equations in our system will come from the total amount of beans and the total cost. The equation for the total amount of beans can be written as:

$$(C\,\text{lb}) + (H\,\text{lb}) = 30\,\text{lb}$$

Or without units:

$$C + H = 30$$

To build the second equation, we have to think about the cost of all these beans. If we have C pounds of Columbian beans that cost \$9.00 per pound, then $9C$ is the cost of those beans in dollars. Similarly, $14H$ is the cost of the Honduran beans. And the total cost is for 30 pounds of beans priced at \$12.50 per pound, totaling $12.5(30) = 37.5$ dollars. All this means:

$$\left(9\,\tfrac{\text{dollars}}{\text{lb}}\right)(C\,\text{lb}) + \left(14\,\tfrac{\text{dollars}}{\text{lb}}\right)(H\,\text{lb}) = \left(12.50\,\tfrac{\text{dollars}}{\text{lb}}\right)(30\,\text{lb})$$

Or without units and carrying out the multiplication on the right:

$$9C + 14H = 37.5$$

Now our system is:

$$\begin{cases} C + H = 30 \\ 9C + 14H = 12.50(30) \end{cases}$$

To solve the system, we'll solve the first equation for C:

$$\begin{aligned} C + H &= 30 \\ C &= 30 - H \end{aligned}$$

Next, we'll substitute C in the second equation with $30 - H$:

$$\begin{aligned} 9C + 14H &= 375 \\ 9(30 - H) + 14H &= 375 \\ 270 - 9H + 14H &= 375 \\ 270 + 5H &= 375 \\ 5H &= 105 \\ H &= 21 \end{aligned}$$

Since $H = 21$, we can conclude that $C = 9$.

In summary, Desi needs to mix 21 pounds of the Honduran coffee beans with 9 pounds of the Columbian coffee beans to create this blend. Our estimate at the beginning was pretty close, so we feel this answer is reasonable.

5.2.3 Solving Special Systems of Equations with Substitution

Remember the two special cases we encountered when solving by graphing in Section 5.1? If the two lines represented by a system of equations have the same slope, then they might be separate lines that never meet, meaning the system has no solutions. Or they might coincide as the same line, in which case there are infinitely many solutions represented by all the points on that line. Let's see what happens when we use the substitution method on each of the special cases.

Example 5.2.13 A System with No Solution. Solve the system of equations using the substitution method:

$$\begin{cases} y = 2x - 1 \\ 4x - 2y = 3 \end{cases}$$

Explanation. Since the first equation is already solved for y, we will substitute $2x-1$ for y in the second equation, and we have:

$$\begin{aligned} 4x - 2y &= 3 \\ 4x - 2(2x - 1) &= 3 \\ 4x - 4x + 2 &= 3 \\ 2 &= 3 \end{aligned}$$

Even though we were only intending to substitute away y, we ended up with an equation where there are no variables at all. This will happen whenever the lines have the same slope. This tells us the system represents either parallel or coinciding lines. Since $2 = 3$ is false no matter what values x and y might be, there can be no solution to the system. So the lines are parallel and *distinct*. We write the solution set using the empty set symbol: the solution set is $\emptyset$.

To verify this, re-write the second equation, $4x - 2y = 3$, in slope-intercept form:

$$\begin{aligned} 4x - 2y &= 3 \\ -2y &= -4x + 3 \\ \frac{-2y}{-2} &= \frac{-4x + 3}{-2} \\ y &= \frac{-4x}{-2} + \frac{3}{-2} \\ y &= 2x - \frac{3}{2} \end{aligned}$$

So the system is equivalent to:

$$\begin{cases} y = 2x - 1 \\ y = 2x - \dfrac{3}{2} \end{cases}$$

Now it is easier to see that the two lines have the same slope but different y-intercepts. They are parallel and distinct lines, so the system has no solution.

Example 5.2.14 A System with Infinitely Many Solutions. Solve the system of equations using the substitution method:

$$\begin{cases} y = 2x - 1 \\ 4x - 2y = 2 \end{cases}$$

Explanation. Since $y = 2x - 1$, we will substitute $2x - 1$ for y in the second equation and we have:

$$\begin{aligned} 4x - 2y &= 2 \\ 4x - 2(2x - 1) &= 2 \\ 4x - 4x + 2 &= 2 \\ 2 &= 2 \end{aligned}$$

Even though we were only intending to substitute away y, we ended up with an equation where there are no variables at all. This will happen whenever the lines have the same slope. This tells us the system represents either parallel or coinciding lines. Since $2 = 2$ is true no matter what values x and y might be, the system equations are true no matter what x is, as long as $y = 2x - 1$. So the lines *coincide*. We write the solution set as $\{(x, y) \mid y = 2x - 1\}$.

To verify this, re-write the second equation, $4x - 2y = 2$, in slope-intercept form:

$$\begin{aligned} 4x - 2y &= 2 \\ -2y &= -4x + 2 \end{aligned}$$

$$\frac{-2y}{-2} = \frac{-4x}{-2} + \frac{2}{-2}$$
$$y = 2x - 1$$

The system looks like:

$$\begin{cases} y = 2x - 1 \\ y = 2x - 1 \end{cases}$$

Now it is easier to see that the two equations represent the same line. Every point on the line is a solution to the system, so the system has infinitely many solutions. The solution set is $\{(x, y) \mid y = 2x - 1\}$.

Exercises

Review and Warmup Solve the equation.

1. $\frac{7}{6} - 8q = 5$

2. $\frac{3}{2} - 5y = 5$

3. $\frac{7}{10} - \frac{1}{10}r = 2$

4. $\frac{5}{6} - \frac{1}{6}a = 10$

Solve the linear equation for y.

5. $$9x - 3y = -6$$

6. $$20x + 4y = -32$$

7. $$24x - 4y = 76$$

8. $$-8x - 4y = -36$$

9. $$-7x - y = -9$$

10. $$-9x - y = 5$$

Solving System of Equations Using Substitution Solve the following system of equations.

11. $$\begin{cases} x = -5 \\ 5 + 5x = -5n \end{cases}$$

12. $$\begin{cases} 0 = -C \\ 0 = 1 + t + 2C \end{cases}$$

13. $$\begin{cases} -2a = -8 \\ 4C - 8 = -3a \end{cases}$$

14. $$\begin{cases} 0 = 5q + 10 \\ -4q - 2y = -20 \end{cases}$$

15. $$\begin{cases} y = -16 - 4x \\ y = 4x \end{cases}$$

16. $$\begin{cases} y = -2x + 6 \\ y = x + 6 \end{cases}$$

17.

$$\begin{cases} y = -5x - 8 \\ 2y + 2x = -16 \end{cases}$$

18.

$$\begin{cases} y = -x - 3 \\ 4y + 5x = -1 \end{cases}$$

19.

$$\begin{cases} y = -36 - 2x \\ 2x - 5y = -24 \end{cases}$$

20.

$$\begin{cases} B = -38 - 3p \\ -5B + 4p = 19 \end{cases}$$

21.

$$\begin{cases} a = 7 + 2y \\ 3y - 5a = 7 \end{cases}$$

22.

$$\begin{cases} r = -4b + 18 \\ -r + 5b = 9 \end{cases}$$

23.

$$\begin{cases} 2t - 6 = a \\ 6 + 2t = 2a \end{cases}$$

24.

$$\begin{cases} -10 + 4y = 3x \\ x + 5 = 3y \end{cases}$$

25.

$$\begin{cases} y = x - \dfrac{1}{4} \\ y = x + \dfrac{2}{5} \end{cases}$$

26.

$$\begin{cases} y = 1 - 3x \\ y = -3x - 1 \end{cases}$$

27.

$$\begin{cases} -3x - 9 = 2y \\ 3x - y = -9 \end{cases}$$

28.

$$\begin{cases} 5x + 54 = -y \\ -3x = 18 - 3y \end{cases}$$

29.

$$\begin{cases} 2 = 4a + 2m \\ 0 = -1 - 3m \end{cases}$$

30.

$$\begin{cases} 0 = -3 + 4m - 4b \\ -1 = -4m \end{cases}$$

31.

$$\begin{cases} -5a - 2 = 2q \\ -3q + a = -2 \end{cases}$$

32.

$$\begin{cases} -3m = -3B + 5 \\ B + 2m = 4 \end{cases}$$

33.

$$\begin{cases} -x + 5y = 2 \\ 2y - 4 = 4x \end{cases}$$

34.

$$\begin{cases} -3x + 4y = -4 \\ -3 = -x + 5y \end{cases}$$

35.

$$\begin{cases} -\dfrac{4}{5} = -\dfrac{2}{5}x + \dfrac{3}{5}y \\ x + \dfrac{5}{4} = 0 \end{cases}$$

36.

$$\begin{cases} x = -\dfrac{3}{5} \\ -\dfrac{4}{5}y = -x - 1 \end{cases}$$

37.

$$\begin{cases} 1 = -\dfrac{5}{4}x \\ \dfrac{3}{4} - \dfrac{1}{2}y = \dfrac{2}{3}x \end{cases}$$

38.
$$\begin{cases} y - \frac{1}{5}B = -\frac{2}{5} \\ 0 = -\frac{5}{3}B + \frac{1}{2} \end{cases}$$

39.
$$\begin{cases} \frac{1}{4}a - 1 = -A \\ \frac{1}{4}A = -\frac{2}{3} + a \end{cases}$$

40.
$$\begin{cases} 0 = \frac{5}{4} + \frac{5}{2}q + p \\ -\frac{4}{5} = -\frac{5}{3}q + 3p \end{cases}$$

41.
$$\begin{cases} 5 = -B - 3a \\ -a = -\frac{3}{2}B + 1 \end{cases}$$

42.
$$\begin{cases} \frac{2}{3} - x = -2y \\ 0 = 1 - \frac{3}{5}y - x \end{cases}$$

43.
$$\begin{cases} -5y + 4x = 1 \\ -2x + 5y = -2 \end{cases}$$

44.
$$\begin{cases} x + 3y = -4 \\ -2y + 3x = -2 \end{cases}$$

45.
$$\begin{cases} -4x + 2y = -7 \\ -4x + 3y = -\frac{13}{2} \end{cases}$$

46.
$$\begin{cases} 4x - 2y = -\frac{37}{11} \\ x + y = \frac{97}{22} \end{cases}$$

47.
$$\begin{cases} -\frac{1}{5}x + \frac{1}{3}y = \frac{13}{42} \\ -\frac{1}{3}x + \frac{1}{4}y = \frac{151}{840} \end{cases}$$

48.
$$\begin{cases} \frac{1}{5}x + \frac{1}{3}y = \frac{41}{120} \\ \frac{1}{2}x + \frac{1}{4}y = \frac{47}{96} \end{cases}$$

49.
$$\begin{cases} 3x + 2y = -1 \\ 5x + y = -18 \end{cases}$$

50.
$$\begin{cases} 4x + 4y = -12 \\ 2x + 4y = -8 \end{cases}$$

51.
$$\begin{cases} 4x - 4y = 32 \\ 5x + 2y = -16 \end{cases}$$

52.
$$\begin{cases} 6x + 5y = 42 \\ 6x - 5y = -18 \end{cases}$$

53.
$$\begin{cases} -4x - 4y = -12 \\ -2x - 4y = -2 \end{cases}$$

54.
$$\begin{cases} -2x - y = -5 \\ -2x - 2y = 4 \end{cases}$$

55.
$$\begin{cases} -4x + 2y = -26 \\ 2x = 18 \end{cases}$$

56.
$$\begin{cases} x - y = -7 \\ 2x = -18 \end{cases}$$

57.
$$\begin{cases} x + y = 3 \\ -2x - 2y = 3 \end{cases}$$

58.
$$\begin{cases} 2x + 4y = 2 \\ 6x + 12y = 2 \end{cases}$$

59.

$$\begin{cases} 2x + 2y = 2 \\ 8x + 8y = 8 \end{cases}$$

60.

$$\begin{cases} 3x + 5y = 2 \\ -6x - 10y = -4 \end{cases}$$

Applications

61. A rectangle's length is 4 feet longer than four times its width. The rectangle's perimeter is 358 feet. Find the rectangle's length and width.

The rectangle's length is feet, and its width is feet.

62. A school fund raising event sold a total of 208 tickets and generated a total revenue of \$808.80. There are two types of tickets: adult tickets and child tickets. Each adult ticket costs \$4.75, and each child ticket costs \$3.35. Write and solve a system of equations to answer the following questions.

adult tickets and child tickets were sold.

63. Phone Company A charges a monthly fee of \$38.70, and \$0.05 for each minute of talk time. Phone Company B charges a monthly fee of \$30.00, and \$0.08 for each minute of talk time. Write and solve a system equation to answer the following questions.

These two companies would charge the same amount on a monthly bill when the talk time was minutes.

64. Company A's revenue this fiscal year is \$840,000, but its revenue is decreasing by \$14,000 each year. Company B's revenue this fiscal year is \$180,000, and its revenue is increasing by \$19,000 each year. Write and solve a system of equations to answer the following question.

After years, Company B will catch up with Company A in revenue.

65. A test has 23 problems, which are worth a total of 96 points. There are two types of problems in the test. Each multiple-choice problem is worth 2 points, and each short-answer problem is worth 7 points. Write and solve a system equation to answer the following questions.

This test has multiple-choice problems and short-answer problems.

66. Penelope invested a total of \$5,500 in two accounts. One account pays 7% interest annually; the other pays 6% interest annually. At the end of the year, Penelope earned a total of \$365 in interest. Write and solve a system of equations to find how much money Penelope invested in each account.

Penelope invested in the 7% account and in the 6% account.

67. Michael invested a total of $11,000 in two accounts. After a year, one account lost 7.2%, while the other account gained 4.5%. In total, Michael lost $441. Write and solve a system of equations to find how much money Michael invested in each account.

Michael invested ☐ in the account with 7.2% loss and ☐ in the account with 4.5% gain.

68. Town A and Town B were located close to each other, and recently merged into one city. Town A had a population with 6% native Americans. Town B had a population with 12% native Americans. After the merge, the new city has a total of 4800 residents, with 10.5% native Americans. Write and solve a system of equations to find how many residents Town A and Town B used to have.

Town A used to have ☐ residents, and Town B used to have ☐ residents.

69. You poured some 8% alcohol solution and some 6% alcohol solution into a mixing container. Now you have 600 grams of 6.6% alcohol solution. How many grams of 8% solution and how many grams of 6% solution did you pour into the mixing container?

Write and solve a system equation to answer the following questions.

You mixed ☐ grams of 8% solution with ☐ grams of 6% solution.

70. Neil invested a total of $8,000 in two accounts. One account pays 5% interest annually; the other pays 4% interest annually. At the end of the year, Neil earned a total of $355 in interest. How much money did Neil invest in each account?

Neil invested ☐ in the 5% account and ☐ in the 4% account.

71. Hannah invested a total of $72,000 in two accounts. One account pays 3.8% interest annually; the other pays 2.4% interest annually. At the end of the year, Hannah earned a total interest of $2,092. How much money did Hannah invest in each account?

Hannah invested ☐ in the 3.8% account and ☐ in the 2.4% account.

72. Charity invested a total of $5,500 in two accounts. One account pays 5% interest annually; the other pays 6% interest annually. At the end of the year, Charity earned the same amount of interest from both accounts. How much money did Charity invest in each account?

Charity invested ☐ in the 5% account and ☐ in the 6% account.

73. Page invested a total of $86,000 in two accounts. One account pays 5.3% interest annually; the other pays 4.7% interest annually. At the end of the year, Page earned the same amount of interest from both accounts. How much money did Page invest in each account?

Page invested ☐ in the 5.3% account ☐ in the 4.7% account.

74. Laurie invested a total of \$10,000 in two accounts. After a year, one account had *earned* 10.1%, while the other account had *lost* 5.4%. In total, Laurie had a net gain of \$855. How much money did Laurie invest in each account?

Laurie invested ☐ in the account that grew by 10.1% and ☐ in the account that fell by 5.4%.

75. You've poured some 6% (by mass) alcohol solution and some 12% alcohol solution into a large glass mixing container. Now you have 680 grams of 9% alcohol solution. How many grams of 6% solution and how many grams of 12% solution did you pour into the mixing container?

You poured ☐ grams of 6% solution and ☐ grams of 12% solution into the mixing container.

76. A store has some beans selling for \$2.30 per pound, and some vegetables selling for \$5.00 per pound. The store plans to use them to produce 13 pounds of mixture and sell for \$3.92 per pound. How many pounds of beans and how many pounds of vegetables should be used?

To produce 13 pounds of mixture, the store should use ☐ pounds of beans and ☐ pounds of vegetables.

77. Town A and Town B were located close to each other, and recently merged into one city. Town A had a population with 6% native Americans. Town B had a population with 10% native Americans. After the merge, the new city has a total of 4000 residents, with 8.8% native Americans. How many residents did Town A and Town B used to have?

Town A used to have ☐ residents, and Town B used to have ☐ residents.

5.3 Elimination

We learned how to solve a system of linear equations using substitution in Section 5.2. In this section, we will learn a second symbolic method for solving systems of linear equations.

5.3.1 Solving Systems of Equations by Elimination

Example 5.3.2 Alicia has \$1000 to give to her two grandchildren for New Year's. She would like to give the older grandchild \$120 more than the younger grandchild, because that is the cost of the older grandchild's college textbooks this term. How much money should she give to each grandchild?

To answer this question, we will demonstrate a new technique. You may have a very good way for finding how much money Alicia should give to each grandchild, but right now we will try to see this new method.

Let A be the dollar amount she gives to her older grandchild, and B be the dollar amount she gives to her younger grandchild. (As always, we start solving a word problem like this by defining the variables, including their units.) Since the total she has to give is \$1000, we can say that $A + B = 1000$. And since she wants to give \$120 more to the older grandchild, we can say that $A - B = 120$. So we have the system of equations:

$$\begin{cases} A + B = 1000 \\ A - B = 120 \end{cases}$$

We could solve this system by substitution as we learned in Section 5.2, but there is an easier method. If we add together the *left* sides from the two equations, it should equal the sum of the *right* sides:

$$\begin{array}{rcr} A + B & & 1000 \\ & = & \\ + A - B & & +120 \end{array}$$

So we have:

$$2A = 1120$$

Note that the variable B is eliminated. This happened because the " $+B$" and the " $-B$" were perfectly in shape to cancel each other out. With only one variable left, it doesn't take much to finish:

$$\begin{aligned} 2A &= 1120 \\ A &= 560 \end{aligned}$$

To finish solving this system of equations, we need the value of B. For now, an easy way to find B is to substitute in our value of A into one of the original equations:

$$\begin{aligned} A + B &= 1000 \\ 560 + B &= 1000 \\ B &= 440 \end{aligned}$$

To check our work, substitute $A = 560$ and $B = 440$ into the original equations:

$$A + B = 1000 \qquad\qquad A - B = 120$$

$$\begin{aligned} 560+440 &\overset{?}{=} 1000 \\ 1000 &\overset{\checkmark}{=} 1000 \end{aligned} \qquad \begin{aligned} 560-440 &\overset{?}{=} 120 \\ 120 &\overset{\checkmark}{=} 120 \end{aligned}$$

This confirms that our solution is correct. In summary, Alicia should give \$560 to her older grandchild, and \$440 to her younger grandchild.

This method for solving the system of equations in Example 5.3.2 worked because B and $-B$ add to zero. Once the B-terms were eliminated we were able to solve for A. This method is called the **elimination method**. Some textbooks call it the **addition method**, because we added the corresponding sides from the two equations to eliminate a variable.

If neither variable can be immediately eliminated, we can still use this method but it will require that we first adjust one or both of the equations. Let's look at an example where we need to adjust one of the equations.

Example 5.3.3 Scaling One Equation. Solve the system of equations using the elimination method.

$$\begin{cases} 3x-4y=2 \\ 5x+8y=18 \end{cases}$$

Explanation. To start, we want to see whether it will be easier to eliminate x or y. We see that the coefficients of x in each equation are 3 and 5, and the coefficients of y are -4 and 8. Because 8 is a multiple of 4 and the coefficients already have opposite signs, the y variable will be easier to eliminate.

To eliminate the y terms, we will multiply each side of the first equation by 2 so that we will have $-8y$. We can call this process **scaling** the first equation by 2.

$$\begin{cases} 2\cdot(3x-4y)=2\cdot(2) \\ \quad 5x+\ 8y=\quad 18 \end{cases}$$

$$\begin{cases} 6x-8y=4 \\ 5x+8y=18 \end{cases}$$

We now have an equivalent system of equations where the y-terms can be eliminated:

$$\begin{array}{rcr} 6x-8y & = & 4 \\ +\,5x+8y & & +\,18 \end{array}$$

So we have:

$$\begin{aligned} 11x &= 22 \\ x &= 2 \end{aligned}$$

To solve for y, we will substitute 2 for x into either of the original equations or the new one. We will use the original first equation, $3x-4y=2$:

$$\begin{aligned} 3x-4y &= 2 \\ 3(2)-4y &= 2 \\ 6-4y &= 2 \end{aligned}$$

$$\begin{aligned}-4y &= -4\\ y &= 1\end{aligned}$$

Our solution is $x = 2$ and $y = 1$. We will check this in both of the original equations:

$$\begin{aligned}5x + 8y &= 18 & 3x - 4y &= 2\\ 5(2) + 8(1) &\overset{?}{=} 18 & 3(2) - 4(1) &\overset{?}{=} 2\\ 10 + 8 &\overset{\checkmark}{=} 18 & 6 - 4 &\overset{\checkmark}{=} 2\end{aligned}$$

The solution to this system is $(2, 1)$ and the solution set is $\{(2, 1)\}$.

Checkpoint 5.3.4. Try a similar exercise.

Solve the following system of equations.

$$\begin{cases}5x + 4y = -7\\ 5x + 2y = -1\end{cases}$$

Explanation.

1. We subtract the two equations, which will cancel the terms in involving x and give $4y - 2y = -7-(-1)$.
2. This gives $y = -3$.
3. Now that we have y, we find x using either equation. Let's use the first: $5x - 12 = -7$, so $x = 1$.
4. The solution to the system is $(1, -3)$. It is left as an exercise to check. Please also note that you may have solved this problem a different way.

Here's an example where we have to scale both equations.

Example 5.3.5 Scaling Both Equations. Solve the system of equations using the elimination method.

$$\begin{cases}2x + 3y = 10\\ -3x + 5y = -15\end{cases}$$

Explanation. Considering the coefficients of x (2 and -3) and the coefficients of y (3 and 5) we see that we cannot eliminate the x or the y variable by scaling a single equation. We will need to scale *both*.

The x-terms already have opposite signs, so we choose to eliminate x. The least common multiple of 2 and 3 is 6. We can scale the first equation by 3 and the second equation by 2 so that the equations have terms $6x$ and $-6x$, which will cancel when added.

$$\begin{cases}3 \cdot (2x + 3y) = 3 \cdot (10)\\ 2 \cdot (-3x + 5y) = 2 \cdot (-15)\end{cases}$$

$$\begin{cases}6x + 9y = 30\\ -6x + 10y = -30\end{cases}$$

At this point we can add the corresponding sides from the two equations and solve for y:

$$\begin{array}{r} 6x + 9y \\ -6x + 10y \end{array} = \begin{array}{r} 30 \\ -30 \end{array}$$

So we have:

$$\begin{aligned} 19y &= 0 \\ y &= 0 \end{aligned}$$

To solve for x, we'll replace y with 0 in $2x + 3y = 10$:

$$\begin{aligned} 2x + 3y &= 10 \\ 2x + 3(0) &= 10 \\ 2x &= 10 \\ x &= 5 \end{aligned}$$

We'll check the system using $x = 5$ and $y = 0$ in each of the original equations:

$$\begin{aligned} 2x + 3y &= 10 & \qquad -3x + 5y &= -15 \\ 2(5) + 3(0) &\overset{?}{=} 10 & \qquad -3(5) + 5(0) &\overset{?}{=} -15 \\ 10 + 0 &\overset{\checkmark}{=} 10 & \qquad -15 + 0 &\overset{\checkmark}{=} -15 \end{aligned}$$

So the system's solution is $(5, 0)$ and the solution set is $\{(5, 0)\}$.

Checkpoint 5.3.6. Try a similar exercise.

Solve the following system of equations.

$$\begin{cases} 3x + 4y = -26 \\ 5x + 5y = -40 \end{cases}$$

Explanation.

1. Let's multiply the *first* equation by 5 and the *second* equation by 3

$$\begin{aligned} 15x + 20y &= -130 \\ 15x + 15y &= -120 \end{aligned}$$

2. Subtracting these two equations gives $20y - 15y = -10$, so $y = -2$.
3. Now that we have y, we can use either equation to find x; let's use the first one:

$$3x + (4) \cdot (-2) = -26$$

so $x = -6$.

4. The solution to the system is $(-6, -2)$. It is left as an exercise to check. Please also note that you may have solved this problem a different way.

Example 5.3.7 Meal Planning. Javed is on a meal plan and needs to consume 600 calories and 20 grams of fat for breakfast. A small avocado contains 300 calories and 30 grams of fat. He has bagels that contain 400 calories and 8 grams of fat. Write and solve a system of equations to determine how much bagel and avocado would combine to make his target calories and fat.

Explanation. To write this system of equations, we first need to define our variables. Let A be the number of avocados consumed and let B be the number of bagels consumed. Both A and B might be fractions. For our first equation, we will count calories from the avocados and bagels:

$$\left(300\,\frac{\text{calories}}{\text{avocado}}\right)(A\text{ avocados})+\left(400\,\frac{\text{calories}}{\text{bagel}}\right)(B\text{ bagel})=600\,\text{calories}$$

Or, without the units:

$$300A+400B=600$$

Similarly, for our second equation, we will count the grams of fat:

$$\left(30\,\frac{\text{g fat}}{\text{avocado}}\right)(A\text{ avocados})+\left(8\,\frac{\text{g fat}}{\text{bagel}}\right)(B\text{ bagel})=20\,\text{g fat}$$

Or, without the units:

$$30A+8B=20$$

So the system of equations is:

$$\begin{cases}300A+400B=600\\ 30A+8B=20\end{cases}$$

Since none of the coefficients are equal to 1, it will be easier to use the elimination method to solve this system. Looking at the terms $300A$ and $30A$, we can eliminate the A variable if we multiply the second equation by -10 to get $-300A$:

$$\begin{cases}300A+400B=600\\ -10\cdot(30A+8B)=-10\cdot(20)\end{cases}$$

$$\begin{cases}300A+400B=600\\ -300A+(-80B)=-200\end{cases}$$

When we add the corresponding sides from the two equations together we have:

$$\begin{array}{rcr}300A+400B & = & 600\\ -300A-80B & & -200\end{array}$$

So we have:

$$\begin{aligned}320B&=400\\ \frac{320B}{320}&=\frac{400}{320}\\ B&=\frac{5}{4}\end{aligned}$$

We now know that Javed should eat $\frac{5}{4}$ bagels (or one and one-quarter bagels). To determine the number of avocados, we will substitute B with $\frac{5}{4}$ in either of our original equations.

$$\begin{aligned}
300A + 400B &= 600\\
300A + 400\left(\frac{5}{4}\right) &= 600\\
300A + 500 &= 600\\
300A &= 100\\
\frac{300A}{300} &= \frac{100}{300}\\
A &= \frac{1}{3}
\end{aligned}$$

To check this result, try using $B = \frac{5}{4}$ and $A = \frac{1}{3}$ in each of the original equations:

$$\begin{aligned}
300A + 400B &= 600 & \qquad 30A + 8B &= 20\\
300\left(\frac{1}{3}\right) + 400\left(\frac{5}{4}\right) &\stackrel{?}{=} 600 & \qquad 30\left(\frac{1}{3}\right) + 8\left(\frac{5}{4}\right) &\stackrel{?}{=} 20\\
100 + 500 &\stackrel{\checkmark}{=} 600 & \qquad 10 + 10 &\stackrel{\checkmark}{=} 20
\end{aligned}$$

In summary, Javed can eat $\frac{5}{4}$ of a bagel (so one and one-quarter bagel) and $\frac{1}{3}$ of an avocado in order to consume exactly 600 calories and 20 grams of fat.

5.3.2 Solving Special Systems of Equations with Elimination

Remember the two special cases we encountered when solving by graphing and substitution? Sometimes a system of equations has no solutions at all, and sometimes the solution set is infinite with all of the points on one line satisfying the equations. Let's see what happens when we use the elimination method on each of the special cases.

Example 5.3.8 A System with Infinitely Many Solutions. Solve the system of equations using the elimination method.

$$\begin{cases} 3x + 4y = 5 \\ 6x + 8y = 10 \end{cases}$$

Explanation. To eliminate the x-terms, we multiply each term in the first equation by -2, and we have:

$$\begin{cases} -2 \cdot (3x + 4y) = -2 \cdot 5 \\ \qquad 6x + \ 8y = \qquad 10 \end{cases}$$

$$\begin{cases} -6x + -8y = -10 \\ \ \ 6x + \ \ 8y = \ \ 10 \end{cases}$$

We might notice that the equations look very similar. Adding the respective sides of the equation, we have:

$$0 = 0$$

Both of the variables have been eliminated. Since the statement $0 = 0$ is true no matter what x and y are, the solution set is infinite. Specifically, you just need any (x, y) satisfying *one* of the two equations, since the two equations represent the same line. We can write the solution set as $\{(x, y) \mid 3x + 4y = 5\}$.

Example 5.3.9 A System with No Solution. Solve the system of equations using the elimination method.

$$\begin{cases} 10x + 6y = 9 \\ 25x + 15y = 4 \end{cases}$$

Explanation. To eliminate the x-terms, we will scale the first equation by -5 and the second by 2:

$$\begin{cases} -5 \cdot (10x + 6y) = -5 \cdot (9) \\ 2 \cdot (25x + 15y) = 2 \cdot (4) \end{cases}$$

$$\begin{cases} -50x + (-30y) = -45 \\ 50x + 30y = 8 \end{cases}$$

Adding the respective sides of the equation, we have:

$$0 = -37$$

Both of the variables have been eliminated. In this case, the statement $0 = -37$ is just false, no matter what x and y are. So the system has no solution.

5.3.3 Deciding to Use Substitution versus Elimination

In every example so far from this section, both equations were in standard form, $Ax + By = C$. And all of the coefficients were integers. If none of the coefficients are equal to 1 then it is usually easier to use the elimination method, because otherwise you will probably have some fraction arithmetic to do in the middle of the substitution method. If there *is* a coefficient of 1, then it is a matter of preference.

Example 5.3.10 A college used to have a north campus with 6000 students and a south campus with 15,000 students. The percentage of students at the north campus who self-identify as LGBTQ was three times the percentage at the south campus. After the merge, 5.5% of students identify as LGBTQ. What percentage of students on each campus identified as LGBTQ before the merge?

Explanation. We will define N as the percentage (as a decimal) of students at the north campus and S as the percentage (as a decimal) of students at the south campus that identified as LGBTQ. Since the percentage of students at the north campus was three times the percentage at the south campus, we have:

$$N = 3S$$

For our second equation, we will count LGBTQ students at the various campuses. At the north campus, multiply the population, 6000, by the percentage N to get $6000N$. This must be the actual number of LGBTQ students. Similarly, the south campus has $15000S$ LGBTQ students, and the combined school has $21000(0.055) = 1155$. When we combine the two campuses, we have:

$$6000N + 15000S = 1155$$

We write the system as:

$$\begin{cases} N = 3S \\ 6000N + 15000S = 1155 \end{cases}$$

Because the first equation is already solved for N, this is a good time to *not* use the elimination method. Instead we can substitute N in our second equation with $3S$ and solve for S:

$$\begin{aligned} 6000N + 15000S &= 1155 \\ 6000(3S) + 15000S &= 1155 \\ 18000S + 15000S &= 1155 \\ 33000S &= 1155 \\ \frac{33000S}{33000} &= \frac{1155}{33000} \\ S &= 0.035 \end{aligned}$$

We can determine N using the first equation:

$$\begin{aligned} N &= 3S \\ N &= 3(0.035) \\ N &= 0.105 \end{aligned}$$

Before the merge, 10.5% of the north campus students self-identified as LGBTQ, and 3.5% of the south campus students self-identified as LGBTQ.

If you need to solve a system, and one of the equations is not in standard form, substitution may be easier. But you also may find it easier to convert the equations into standard form. Additionally, if the system's coefficients are fractions or decimals, you may take an additional step to scale the equations so that they only have integer coefficients.

Example 5.3.11 Solve the system of equations using the method of your choice.

$$\begin{cases} -\frac{1}{3}y = \frac{1}{15}x + \frac{1}{5} \\ \frac{5}{2}x - y = 6 \end{cases}$$

Explanation. First, we can cancel the fractions by using the least common multiple of the denominators in each equation, similarly to the topic of Section 3.3. We have:

$$\begin{cases} 15 \cdot -\frac{1}{3}y = 15 \cdot \left(\frac{1}{15}x + \frac{1}{5}\right) \\ 2 \cdot \left(\frac{5}{2}x - y\right) = 2 \cdot (6) \end{cases}$$

$$\begin{cases} -5y = x + 3 \\ 5x - 2y = 12 \end{cases}$$

We could put convert the first equation into standard form by subtracting x from both sides, and then use elimination. However, the x-variable in the first equation has a coefficient of 1, so the substitution method may be faster. Solving for x in the first equation we have:

$$\begin{aligned}-5y &= x+3\\ -5y-3 &= x+3-3\\ -5y-3 &= x\end{aligned}$$

Substituting $-5y-3$ for x in the second equation we have:

$$\begin{aligned}5(-5y-3)-2y &= 12\\ -25y-15-2y &= 12\\ -27y-15 &= 12\\ -27y &= 27\\ y &= -1\end{aligned}$$

Using the equation where we isolated x and substituting -1 for y, we have:

$$\begin{aligned}-5(-1)-3 &= x\\ 5-3 &= x\\ 2 &= x\end{aligned}$$

The solution is $(2,-1)$. Checking the solution is left as an exercise.

Example 5.3.12 A penny is made by combining copper and zinc. A chemistry reference source says copper has a density of $9\,\frac{\text{g}}{\text{cm}^3}$ and zinc has a density of $7.1\,\frac{\text{g}}{\text{cm}^3}$. A penny's mass is 2.5 g and its volume is $0.35\,\text{cm}^3$. How many cm^3 each of copper and zinc go into one penny?

Explanation. Let c be the volume of copper and z be the volume of zinc in one penny, both measured in cm^3. Since the total volume is $0.35\,\text{cm}^3$, one equation is:

$$\left(c\,\text{cm}^3\right)+\left(z\,\text{cm}^3\right)=0.35\,\text{cm}^3$$

Or without units:

$$c+z=0.35.$$

For the second equation, we will examine the masses of copper and zinc. Since copper has a density of $9\,\frac{\text{g}}{\text{cm}^3}$ and we are using c to represent the volume of copper, the mass of copper is $9c$. Similarly, the mass of zinc is 7.1. Since the total mass is 2.5 g, we have the equation:

$$\left(9\,\frac{\text{g}}{\text{cm}^3}\right)\left(c\,\text{cm}^3\right)+\left(7.1\,\frac{\text{g}}{\text{cm}^3}\right)\left(z\,\text{cm}^3\right)=2.5\,\text{g}$$

Or without units:

$$9c+7.1z=2.5.$$

So we have a system of equations:

$$\begin{cases}c+z=0.35\\ 9c+7.1z=2.5\end{cases}$$

Since the coefficient of c (or z) in the first equation is 1, we could solve for one of these variables and use substitution to complete the problem. Some decimal arithmetic would be required. Alternatively, we can scale the equations by the right power of 10 to make all the coefficients integers:

$$\begin{cases} 100\cdot(c + z) = 100\cdot(0.35) \\ 10\cdot(9c+7.1z) = 10\cdot(2.5) \end{cases}$$
$$\begin{cases} 100c+100z = 35 \\ 90c+71z = 25 \end{cases}$$

Now to set up elimination, scale each equation again to eliminate c:

$$\begin{cases} 9\cdot(100c+100z) = 9\cdot(35) \\ -10\cdot(90c+71z) = -10\cdot(25) \end{cases}$$
$$\begin{cases} 900c+900z = 315 \\ -900c+(-710z) = -250 \end{cases}$$

Adding the corresponding sides from the two equations gives

$$190z = 65,$$

from which we find $z = \frac{65}{190} \approx 0.342$. So there is about $0.342\,\text{cm}^3$ of zinc in a penny.

To solve for c, we can use one of the original equations:

$$\begin{aligned} c + z &= 0.35 \\ c + 0.342 &\approx 0.35 \\ c &\approx 0.008 \end{aligned}$$

Therefore there is about $0.342\,\text{cm}^3$ of zinc and $0.008\,\text{cm}^3$ of copper in a penny.

To summarize, if a variable is already isolated or has a coefficient of 1, consider using the substitution method. If both equations are in standard form or none of the coefficients are equal to 1, we suggest using the elimination method. Either way, if you have fraction or decimal coefficients, it may help to scale your equations so that only integer coefficients remain.

Exercises

Review and Warmup Solve the equation.

1. $\frac{7}{2} - 8C = 4$

2. $\frac{3}{8} - 6n = 4$

3. $\frac{5}{4} - \frac{1}{4}q = 3$

4. $\frac{9}{10} - \frac{1}{10}x = 10$

5. $\frac{4r}{7} - 8 = -\frac{76}{7}$

6. $\frac{6t}{5} - 6 = -\frac{72}{5}$

Solving System of Equations by Elimination Solve the following system of equations.

7.
$$\begin{cases} x + y = 6 \\ 2x + 4y = 30 \end{cases}$$

8.
$$\begin{cases} 3x + 5y = 10 \\ 2x + y = 2 \end{cases}$$

9.
$$\begin{cases} 6x + 3y = -3 \\ -x + 5y = -27 \end{cases}$$

10.
$$\begin{cases} -2x + 3y = 19 \\ 2x + 2y = 26 \end{cases}$$

11.
$$\begin{cases} -2x - 5y = -19 \\ -5x - 2y = -37 \end{cases}$$

12.
$$\begin{cases} -5x - 2y = -33 \\ -5x - 5y = -15 \end{cases}$$

13.
$$\begin{cases} x - y = -18 \\ -4x = 40 \end{cases}$$

14.
$$\begin{cases} -5x - 4y = 35 \\ -4x = 28 \end{cases}$$

15.
$$\begin{cases} 2x + y = -8 \\ 6x + 3y = -8 \end{cases}$$

16.
$$\begin{cases} 2x + 5y = -8 \\ 8x + 20y = -8 \end{cases}$$

17.
$$\begin{cases} 3x + 3y = -8 \\ -6x - 6y = 16 \end{cases}$$

18.
$$\begin{cases} 3x + y = -8 \\ -12x - 4y = 32 \end{cases}$$

19.
$$\begin{cases} -2y = 2x + 8 \\ -5x = -8 + y \end{cases}$$

20.
$$\begin{cases} 2x = -18 + y \\ -5y + 4x + 18 = 0 \end{cases}$$

21.
$$\begin{cases} -5x - 3y = 3 \\ -4y + x = 1 \end{cases}$$

22.
$$\begin{cases} -4y + 5x = -2 \\ 2x + 3y = 3 \end{cases}$$

23.
$$\begin{cases} 4m = 2b + 30 \\ 30 - 3b = -m \end{cases}$$

24.
$$\begin{cases} -4x + 18 - r = 0 \\ 2r + 2x - 6 = 0 \end{cases}$$

25.
$$\begin{cases} 3B + C = 3 \\ -2C = 5B + 4 \end{cases}$$

26.
$$\begin{cases} -5 - 3n = q \\ 4 = -2n - 3q \end{cases}$$

27.
$$\begin{cases} -y = -5 + 4x \\ 0 = 2x + 3y + 1 \end{cases}$$

28.

$$\begin{cases} -1 = -5x + 4y \\ -y = -2x \end{cases}$$

29.

$$\begin{cases} -5 - \frac{2}{5}x + 4y = 0 \\ 0 = \frac{3}{5}x + \frac{1}{5} + y \end{cases}$$

30.

$$\begin{cases} 0 = \frac{5}{3} - 3x + \frac{1}{2}y \\ y = -\frac{4}{5} + \frac{3}{4}x \end{cases}$$

31.

$$\begin{cases} -B = y - 1 \\ -\frac{4}{5}y - \frac{1}{5} = -\frac{5}{4}B \end{cases}$$

32.

$$\begin{cases} -2t + \frac{5}{3} = 2A \\ -t = \frac{3}{2}A + 1 \end{cases}$$

33.

$$\begin{cases} 0 = 4q - 2p + 12 \\ -2p + 12 = -4q \end{cases}$$

34.

$$\begin{cases} 5 = 4b + 4A \\ -A - b - 2 = 0 \end{cases}$$

35.

$$\begin{cases} 5x + 2y = \frac{21}{2} \\ 4x - 5y = -\frac{129}{20} \end{cases}$$

36.

$$\begin{cases} -x - 4y = -\frac{72}{11} \\ -3x - 3y = -\frac{135}{22} \end{cases}$$

37.

$$\begin{cases} \frac{1}{5}x + \frac{1}{4}y = \frac{127}{280} \\ \frac{1}{3}x - \frac{1}{2}y = -\frac{29}{420} \end{cases}$$

38.

$$\begin{cases} -\frac{1}{5}x - \frac{1}{4}y = -\frac{19}{24} \\ \frac{1}{4}x - \frac{1}{5}y = \frac{11}{15} \end{cases}$$

Applications

39. A test has 20 problems, which are worth a total of 130 points. There are two types of problems in the test. Each multiple-choice problem is worth 5 points, and each short-answer problem is worth 10 points. Write and solve a system of equations to answer the following questions.

This test has ☐ multiple-choice problems and ☐ short-answer problems.

40. Barbara invested a total of $5,000 in two accounts. One account pays 5% interest annually; the other pays 6% interest annually. At the end of the year, Barbara earned a total of $255 in interest. Write and solve a system of equations to find how much money Barbara invested in each account.

Barbara invested ☐ in the 5% account and ☐ in the 6% account.

41. Diane invested a total of $10,000 in two accounts. After a year, one account lost 6.3%, while the other account gained 6.8%. In total, Diane lost $499. Write and solve a system of equations to find how much money Diane invested in each.

Diane invested ☐ in the account with 6.3% loss and ☐ in the account with 6.8% gain.

42. Town A and Town B were located close to each other, and recently merged into one city. Town A had a population with 10% Hispanics. Town B had a population with 6% Hispanics. After the merge, the new city has a total of 5000 residents, with 7.12% Hispanics. Write and solve a system of equations to find how many residents Town A and Town B used to have.

Town A used to have ☐ residents, and Town B used to have ☐ residents.

43. You poured some 8% alcohol solution and some 12% alcohol solution into a mixing container. Now you have 680 grams of 10% alcohol solution. Write and solve a system of equations to find how many grams of 8% solution and how many grams of 12% solution you poured into the mixing container.

You mixed ☐ grams of 8% solution with ☐ grams of 12% solution.

44. You will purchase some CDs and DVDs. If you purchase 13 CDs and 5 DVDs, it will cost you $85.20; if you purchase 5 CDs and 13 DVDs, it will cost you $130.80. Write and solve a system of equations to answer the following questions.

Each CD costs ☐ and each DVD costs ☐.

45. A school fund raising event sold a total of 211 tickets and generated a total revenue of $861.30. There are two types of tickets: adult tickets and child tickets. Each adult ticket costs $6.60, and each child ticket costs $2.75. Write and solve a system of equations to answer the following questions.

☐ adult tickets and ☐ child tickets were sold.

46. Phone Company A charges a monthly fee of $35.80, and $0.03 for each minute of talk time. Phone Company B charges a monthly fee of $25.00, and $0.07 for each minute of talk time. Write and solve a system equation to answer the following questions.

These two companies would charge the same amount on a monthly bill when the talk time was ☐ minutes.

47. Company A's revenue this fiscal year is $805,000, but its revenue is decreasing by $5,000 each year. Company B's revenue this fiscal year is $409,000, and its revenue is increasing by $17,000 each year. Write and solve a system of equations to answer the following question.

After ☐ years, Company B will catch up with Company A in revenue.

48. If a boat travels from Town A to Town B, it has to travel 990 mi along a river. A boat traveled from Town A to Town B along the river's current with its engine running at full speed. This trip took 27.5 hr. Then the boat traveled back from Town B to Town A, again with the engine at full speed, but this time against the river's current. This trip took 45 hr. Write and solve a system of equations to answer the following questions.

The boat's speed in still water with the engine running at full speed is .

The river current's speed was .

49. A small fair charges different admission for adults and children. It charges \$3.75 for adults, and \$1 for children. On a certain day, the total revenue is \$384.25 and the fair admits 200 people. How many adults and children were admitted?

There were adults and children at the fair.

Challenge

50. Find the value of b so that the system of equations has an infinite number of solutions.

$$\begin{cases} -10x + 35y = 25 \\ \quad 2x - by = -5 \end{cases}$$

5.4 Systems of Linear Equations Chapter Review

5.4.1 Solving Systems of Linear Equations by Graphing

In Section 5.1 we covered the definition of system of linear equations and how a solution to a system of linear equation is a point where the graphs of the two equations cross. We also considered special systems of equations that overlap or never touch.

Example 5.4.1 Solving Systems of Linear Equations by Graphing. Solve the following system of equations by graphing:

$$\begin{cases} y = -\dfrac{2}{3}x - 4 \\ y = -4x - 14 \end{cases}$$

Explanation.

The first equation, $y = -\frac{2}{3}x - 4$, is a linear equation in slope-intercept form with a slope of $-\frac{2}{3}$ and a y-intercept of $(0, -4)$. The second equation, $y = -4x - 14$, is a linear equation in slope-intercept form with a slope of -4 and a y-intercept of $(0, -14)$. We'll use this information to graph both lines in Figure 5.4.2.

The two lines intersect where $x = -3$ and $y = -2$, so the solution of the system of equations is the point $(-3, -2)$. We write the solution set as $\{(-3, -2)\}$.

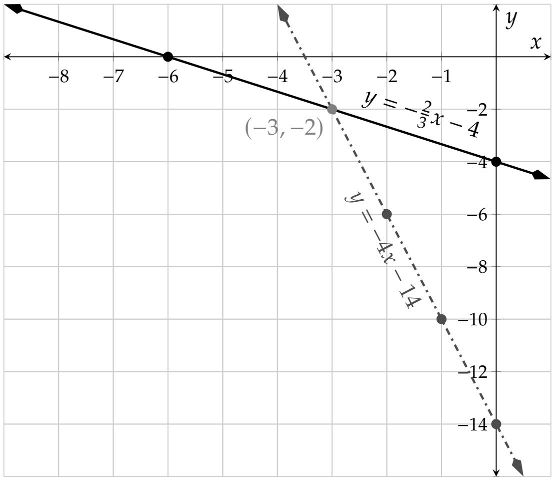

Figure 5.4.2: Graphs of $y = -\frac{2}{3}x - 4$ and $y = -4x - 14$.

Example 5.4.3 Special Systems of Equations. Solve the following system of equations by graphing:

$$\begin{cases} y = \dfrac{3}{2}(x - 1) + 4 \\ 3x - 2y = 4 \end{cases}$$

Explanation. The first equation, $y = \frac{3}{2}(x-1)+4$, is a linear equation in point-slope form with a slope of $\frac{3}{2}$ that passes through the point $(1, 4)$. The second equation, $3x - 2y = 4$, is a linear equation in standard form To graph this line, we either need to find the intercepts or put the equation into slope-intercept form. Just for practice, we will put the line in slope-intercept form.

$$\begin{aligned} 3x - 2y &= 4 \\ -2y &= -3x + 4 \end{aligned}$$

$$\begin{aligned}\frac{-2y}{-2} &= \frac{-3x}{-2} + \frac{4}{-2}\\ y &= \frac{3}{2}x - 2\end{aligned}$$

We'll use this information to graph both lines:

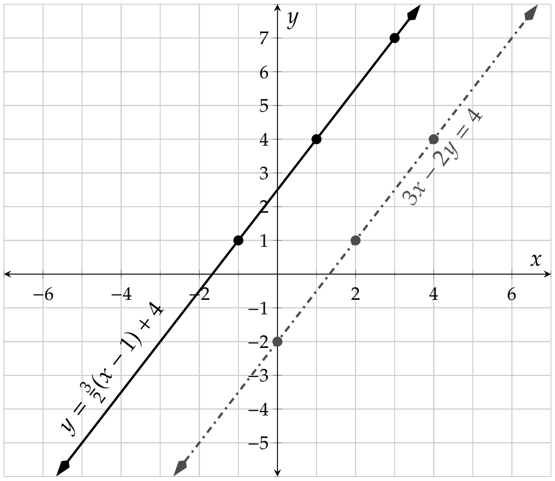

Figure 5.4.4: Graphs of $y = \frac{3}{2}(x-1)+4$ and $3x-2y=4$.

The two lines never intersect: they are parallel. So there are no solutions to the system of equations. We write the solution set as $\emptyset$.

5.4.2 Substitution

In Section 5.2, we covered the substitution method of solving systems of equations. We isolated one variable in one equation and then substituted into the other equation to solve for one variable.

Example 5.4.5 Solving Systems of Equations Using Substitution. Solve this system of equations using substitution:

$$\begin{cases}-5x+6y = -10\\ \quad 4x-3y = \quad -1\end{cases}$$

Explanation. We need to solve for *one* of the variables in *one* of our equations. Looking at both equations, it will be best to solve for y in the second equation. The coefficient of y in that equation is smallest.

$$\begin{aligned}4x-3y &= -1\\ -3y &= -1-4x\\ \frac{-3y}{-3} &= \frac{-1}{-3} - \frac{4x}{-3}\\ y &= \frac{1}{3} + \frac{4}{3}x\end{aligned}$$

Replace y in the first equation with $\frac{1}{3}+\frac{4}{3}x$, giving us a linear equation in only one variable, x, that we may solve:

$$\begin{aligned}-5x+6y&=-10\\ -5x+6\left(\frac{1}{3}+\frac{4}{3}x\right)&=-10\\ -5x+2+8x&=-10\\ 3x+2&=-10\\ 3x&=-12\\ x&=-4\end{aligned}$$

Now that we have the value for x, we need to find the value for y. We have already solved the second equation for y, so that is the easiest equation to use.

$$\begin{aligned}y&=\frac{1}{3}+\frac{4}{3}x\\ y&=\frac{1}{3}+\frac{4}{3}(-4)\\ y&=\frac{1}{3}-\frac{16}{3}\\ y&=-\frac{15}{3}\\ y&=-5\end{aligned}$$

To check this solution, we replace x with -4 and y with -5 in each equation:

$$\begin{aligned}-5x+6y&=-10\\ -5(-4)+6(-5)&\overset{?}{=}-10\\ 20-30&\overset{\checkmark}{=}-10\end{aligned}$$

$$\begin{aligned}4x-3y&=-1\\ 4(-4)-3(-4)&\overset{?}{=}-1\\ -16+15&\overset{\checkmark}{=}-1\end{aligned}$$

We conclude then that this system of equations is true when $x=-4$ and $y=-5$. Our solution is the point $(-4,-5)$ and we write the solution set as $\{(-4,-5)\}$.

Example 5.4.6 Applications of Systems of Equations. The Rusk Ranch Nature Center[a] in south-western Oregon is a volunteer run nonprofit that exists to promote the wellbeing of the local communities and conserve local nature with an emphasis on native butterflies. They sell admission tickets: \$6 for adults and \$4 for children. Amanda, who was working at the front desk, counted that one day she sold a total of 79 tickets and had \$384 in the register from those ticket sales. She didn't bother to count how many were adult tickets and how many were child tickets because she knew she could use math to figure it out at the end of the day. So, how many of the 79 tickets were adult and how many were child?

Explanation. Let's let a represent the number of adult tickets sold and c represent the number of child tickets sold. We need to build two equations to solve a system for both variables.

The first equation we will build relates to the fact that there were 79 total tickets sold. If we combine both the number of adult tickets and child tickets, the total is 79. This fact becomes:

$$a+c=79$$

For the second equation we need to use the per-ticket dollar amounts to generate the total cost of \$384. The amount of money that was made from adult tickets is found my multiplying the number of adult tickets sold, a, by the price per ticket, \$6. Similarly, the amount of money from child tickets is found my multiplying the number of child tickets sold, c, by the price per ticket, \$4. These two amounts will add to be \$384. This fact becomes:

$$6a+4c=384$$

And so, our system is

$$\begin{cases} a + c = 79 \\ 6a + 4c = 384 \end{cases}$$

To solve, we will use the substitution method and solve the first equation for the variable a.

$$\begin{aligned} a + c &= 79 \\ a &= 79 - c \end{aligned}$$

Now we will substitute $79 - c$ for a in the second equation.

$$\begin{aligned} 6a + 4c &= 384 \\ 6\left(79 - c\right) + 4c &= 384 \\ 474 - 6c + 4c &= 384 \\ 474 - 2c &= 384 \\ -2c &= -90 \\ c &= 45 \end{aligned}$$

Last, we will solve for a by substituting 45 in for c in the equation $a = 79 - c$.

$$\begin{aligned} a &= 79 - c \\ a &= 79 - 45 \\ a &= 34 \end{aligned}$$

Our conclusion is that Amanda sold 34 adult tickets and 45 child tickets.

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[a] ruskranchnaturecenter.org

Example 5.4.7 Solving Special Systems of Equations with Substitution. Solve the systems of linear equations using substitution.

a. $\begin{cases} 3x - 5y = 9 \\ x = \dfrac{5}{3}y + 3 \end{cases}$

b. $\begin{cases} y + 7 = 4x \\ 2y - 8x = 7 \end{cases}$

Explanation. To solve the systems using substitution, we first need to solve for one variable in one equation, then substitute into the *other* equation.

a. In this case, x is already solved for in the second equation so we can substitute $\frac{5}{3}y + 3$ everywhere we see x in the first equation. Then simplify and solve for y.

$$\begin{aligned} 3x - 5y &= 9 \\ 3\left(\frac{5}{3}y + 3\right) - 5y &= 9 \\ 5y + 9 - 5y &= 9 \\ 9 &= 9 \end{aligned}$$

We will stop here since we have eliminated all of the variables in the equation and ended with a *true* statement. Since 9 always equals 9, no matter what, then any value of y must make the original equation, $3\left(\frac{5}{3}y+3\right)-5y=9$ true. If you recall from the section on substitution, this means that both lines $3x-5y=9$ and $x=\frac{5}{3}y+3$ are in fact the same line. Since a solution to a system of linear equations is any point where the lines touch, *all* points along both lines are solutions. We can say this in English as, "The solutions are all points on the line $3x-5y=9$," or in math as, "The solution set is $\{(x,y)|3x-5y=9\}$."

b. We will first solve the top equation for y.

$$\begin{aligned} y+7&=4x\\ y&=4x-7 \end{aligned}$$

Now we can substitute $4x-7$ wherever we see y in the second equation.

$$\begin{aligned} 2y-8x&=7\\ 2(4x-7)-8x&=7\\ 8x-14-8x&=7\\ -14&=7 \end{aligned}$$

We will stop here since we have eliminated all of the variables in the equation and ended with a *false* statement. Since -14 never equals 7, then no values of x and y can make the original system true. If you recall from the section on substitution, this means that the lines $y+7=4x$ and $2y-8x=7$ are parallel. Since a solution to a system of linear equations is any point where the lines touch, and parallel lines never touch, *no* points are solutions. We can say this in English as, "There are no solutions," or in math as, "The solution set is $\emptyset$."

5.4.3 Elimination

In Section 5.3, we explored a third way of solving systems of linear equations called elimination where we add two equations together to cancel a variable.

Example 5.4.8 Solving Systems of Equations by Elimination. Solve the system using elimination.

$$\begin{cases} 4x-6y= 13\\ 5x+4y=-1 \end{cases}$$

Explanation. To solve the system using elimination, we first need to scale one or both of the equations so that one variable has equal but opposite coefficients in the system. In this case, we will choose to make y have opposite coefficients because the signs are already opposite for that variable in the system.

We need to multiply the first equation by 2 and the second equation by 3.

$$\begin{cases} 4x-6y= 13\\ 5x+4y=-1 \end{cases}$$

$$\begin{cases} 2\cdot(4x-6y)= 2\cdot(13)\\ 3\cdot(5x+4y)=3\cdot(-1) \end{cases}$$

$$\begin{cases} 8x - 12y = 26 \\ 15x + 12y = -3 \end{cases}$$

We now have an equivalent system of equations where the y-terms can be eliminated:

$$\begin{array}{r} 8x - 12y \\ +\ 15x + 12y \end{array} = \begin{array}{r} 26 \\ +\ (-3) \end{array}$$

So we have:

$$\begin{aligned} 23x &= 23 \\ x &= 1 \end{aligned}$$

To solve for y, we will substitute 1 for x into either of the original equations. We will use the first equation, $4x - 6y = 13$:

$$\begin{aligned} 4x - 6y &= 13 \\ 4(1) - 6y &= 13 \\ 4 - 6y &= 13 \\ -6y &= 9 \\ \frac{-6y}{-6} &= \frac{9}{-6} \\ y &= -\frac{3}{2} \end{aligned}$$

To verify this, we substitute the x and y values into both of the original equations.

$$\begin{aligned} 4x - 6y &= 13 \\ 4(1) - 6\left(-\frac{3}{2}\right) &\stackrel{?}{=} 13 \\ 4 + 9 &\stackrel{\checkmark}{=} 13 \end{aligned} \qquad \begin{aligned} 5x + 4y &= -1 \\ 5(1) + 4\left(-\frac{3}{2}\right) &\stackrel{?}{=} -1 \\ 5 - 6 &\stackrel{\checkmark}{=} -1 \end{aligned}$$

So the solution is the point $\left(-\frac{3}{2}, 1\right)$ and the solution set is $\left\{\left(-\frac{3}{2}, 1\right)\right\}$.

Example 5.4.9 Solving Special Systems of Equations with Elimination. Solve the system of equations using the elimination method.

$$\begin{cases} 24x + 6y = 9 \\ 8x + 2y = 2 \end{cases}$$

Explanation. To eliminate the x-terms, we will scale the second equation by -3.

$$\begin{cases} 24x + \ 6y = 9 \\ -3 \cdot (8x + 2y) = -3 \cdot (2) \end{cases}$$

$$\begin{cases} 24x + 6y = 9 \\ -24x - 6y = -6 \end{cases}$$

Adding the respective sides of the equation, we have:

$$0 = 3$$

Both of the variables have been eliminated. In this case, the statement $0 = 3$ is just false, no matter what x and y are. So the system has no solution. The solution set is $\emptyset$.

Example 5.4.10 Deciding to Use Substitution versus Elimination. Decide which method would be easiest to solve the systems of linear equations: substitution or elimination.

a. $\begin{cases} 2x + 3y = -11 \\ 5x - 6y = 13 \end{cases}$ b. $\begin{cases} x - 7y = 10 \\ 9x - 16y = -4 \end{cases}$ c. $\begin{cases} 6x + 30y = 15 \\ 4x + 20y = 10 \end{cases}$ d. $\begin{cases} y = 3x - 2 \\ y = 7x + 6 \end{cases}$

Explanation.

a. Elimination is probably easiest here. Multiply the first equation by 2 and eliminate the y variables. The solution to this one is $(-1, -3)$ if you want to solve it for practice.

b. Substitution is probably easiest here. Solve the first equation for x and substitute it into the second equation. We *could* use elimination if we multiplied the first equation by -9 and eliminate the x variable, but it's probably a little more work than substitution. The solution to this one is $(-4, -2)$ if you want to solve it for practice.

c. Elimination is probably easiest here. Multiply the first equation by 2 and the second equation by -3. Doing this will eliminate both variables and leave you with $0 = 0$. This should mean that all points on the line are solutions. So the solution set is $\{(x, y) | 6x + 30y = 15\}$.

d. Substitution is definitely easiest here. Substituting y from one equation into y in the other equation gives you $3x - 2 = 7x + 6$. Solve this and find then find y and you should get the solution to the system to be $(-2, -8)$ if you want to solve it for practice.

Exercises

Solving Systems of Linear Equations by Graphing Use a graph to solve the system of equations.

1. $\begin{cases} x + y = 0 \\ 3x - y = 8 \end{cases}$

2. $\begin{cases} 4x - 2y = 4 \\ x + 2y = 6 \end{cases}$

3. $\begin{cases} x + y = -1 \\ x = 2 \end{cases}$

4. $\begin{cases} x - 2y = -4 \\ x = -4 \end{cases}$

5. $\begin{cases} -10x + 15y = 60 \\ 6x - 9y = 36 \end{cases}$

6. $\begin{cases} 6x - 8y = 32 \\ 9x - 12y = 12 \end{cases}$

7. $\begin{cases} y = -\frac{3}{5}x + 7 \\ 9x + 15y = 105 \end{cases}$

8. $\begin{cases} 9y - 12x = 18 \\ y = \frac{4}{3}x + 2 \end{cases}$

Substitution Solve the following system of equations.

9.
$$\begin{cases} c = -46 - 5C \\ 2C + 5c = 0 \end{cases}$$

10.
$$\begin{cases} p = -2a \\ 4a + 2p = 0 \end{cases}$$

11.
$$\begin{cases} 2x + 2y = 18 \\ 4x + 3y = 26 \end{cases}$$

12.
$$\begin{cases} x + 3y = 8 \\ 3x + 2y = 10 \end{cases}$$

13.
$$\begin{cases} 2x + 5y = -17 \\ 2x - 4y = 28 \end{cases}$$

14.
$$\begin{cases} 4x - 4y = -12 \\ 3x + 3y = 45 \end{cases}$$

15.
$$\begin{cases} 5x + 3y = 2 \\ -20x - 12y = -8 \end{cases}$$

16.
$$\begin{cases} x + 2y = 2 \\ 3x + 6y = 6 \end{cases}$$

17. A rectangle's length is 6 feet longer than five times its width. The rectangle's perimeter is 192 feet. Find the rectangle's length and width.

The rectangle's length is feet, and its width is feet.

18. A school fund raising event sold a total of 178 tickets and generated a total revenue of \$420.90. There are two types of tickets: adult tickets and child tickets. Each adult ticket costs \$4.45, and each child ticket costs \$1.55. Write and solve a system of equations to answer the following questions.

adult tickets and child tickets were sold.

19. A test has 21 problems, which are worth a total of 78 points. There are two types of problems in the test. Each multiple-choice problem is worth 3 points, and each short-answer problem is worth 6 points. Write and solve a system equation to answer the following questions.

This test has multiple-choice problems and short-answer problems.

20. A test has 20 problems, which are worth a total of 130 points. There are two types of problems in the test. Each multiple-choice problem is worth 3 points, and each short-answer problem is worth 10 points. Write and solve a system equation to answer the following questions.

This test has multiple-choice problems and short-answer problems.

21. Kristen invested a total of \$8,000 in two accounts. One account pays 5% interest annually; the other pays 4% interest annually. At the end of the year, Kristen earned a total of \$355 in interest. Write and solve a system of equations to find how much money Kristen invested in each account.

Kristen invested ☐ in the 5% account and ☐ in the 4% account.

22. Lily invested a total of \$13,000 in two accounts. After a year, one account lost 6.8%, while the other account gained 2.5%. In total, Lily lost \$419. Write and solve a system of equations to find how much money Lily invested in each account.

Lily invested ☐ in the account with 6.8% loss and ☐ in the account with 2.5% gain.

23. Town A and Town B were located close to each other, and recently merged into one city. Town A had a population with 12% African Americans. Town B had a population with 6% African Americans. After the merge, the new city has a total of 4000 residents, with 8.1% African Americans. Write and solve a system of equations to find how many residents Town A and Town B used to have.

Town A used to have ☐ residents, and Town B used to have ☐ residents.

24. You poured some 12% alcohol solution and some 6% alcohol solution into a mixing container. Now you have 600 grams of 8% alcohol solution. How many grams of 12% solution and how many grams of 6% solution did you pour into the mixing container?

Write and solve a system equation to answer the following questions.

You mixed ☐ grams of 12% solution with ☐ grams of 6% solution.

Elimination Solve the following system of equations.

25.
$$\begin{cases} x + 3y = -37 \\ 3x + 3y = -57 \end{cases}$$

26.
$$\begin{cases} 3x + 2y = -14 \\ 3x + 5y = 1 \end{cases}$$

27.
$$\begin{cases} 6x - 3y = -24 \\ 3x + 5y = -25 \end{cases}$$

28.
$$\begin{cases} 3x + 4y = -45 \\ -3x + 3y = -18 \end{cases}$$

29.
$$\begin{cases} 3x + 4y = -2 \\ -12x - 16y = -2 \end{cases}$$

30.
$$\begin{cases} 3x + 2y = -2 \\ -6x - 4y = -2 \end{cases}$$

31.
$$\begin{cases} 4x + y = -2 \\ 12x + 3y = -6 \end{cases}$$

32.
$$\begin{cases} 4x + 4y = -3 \\ 8x + 8y = -6 \end{cases}$$

33. A test has 20 problems, which are worth a total of 112 points. There are two types of problems in the test. Each multiple-choice problem is worth 5 points, and each short-answer problem is worth 7 points. Write and solve a system of equations to answer the following questions.

This test has ☐ multiple-choice problems and ☐ short-answer problems.

34. Wenwu invested a total of $5,000 in two accounts. One account pays 7% interest annually; the other pays 6% interest annually. At the end of the year, Wenwu earned a total of $345 in interest. Write and solve a system of equations to find how much money Wenwu invested in each account.

Wenwu invested ☐ in the 7% account and ☐ in the 6% account.

35. Town A and Town B were located close to each other, and recently merged into one city. Town A had a population with 6% Hispanics. Town B had a population with 8% Hispanics. After the merge, the new city has a total of 5000 residents, with 7.52% Hispanics. Write and solve a system of equations to find how many residents Town A and Town B used to have.

Town A used to have ☐ residents, and Town B used to have ☐ residents.

36. You poured some 6% alcohol solution and some 10% alcohol solution into a mixing container. Now you have 800 grams of 8.4% alcohol solution. Write and solve a system of equations to find how many grams of 6% solution and how many grams of 10% solution you poured into the mixing container.

You mixed ☐ grams of 6% solution with ☐ grams of 10% solution.

37. If a boat travels from Town A to Town B, it has to travel 797.5 mi along a river. A boat traveled from Town A to Town B along the river's current with its engine running at full speed. This trip took 27.5 hr. Then the boat traveled back from Town B to Town A, again with the engine at full speed, but this time against the river's current. This trip took 72.5 hr. Write and solve a system of equations to answer the following questions.

The boat's speed in still water with the engine running at full speed is ☐.

The river current's speed was ☐.

38. A small fair charges different admission for adults and children. It charges $3.50 for adults, and $2 for children. On a certain day, the total revenue is $5,957.50 and the fair admits 2300 people. How many adults and children were admitted?

There were ☐ adults and ☐ children at the fair.

CHAPTER 6

Exponents and Polynomials

6.1 Exponent Rules

6.1.1 Review of Exponent Rules for Products and Exponents

In Section 2.9, we introduced three basic rules involving products and exponents. We'll begin with a brief recap and explanation of these three exponent rules.

Product Rule When multiplying two expressions that have the same base, simplify the product by adding the exponents.

$$x^m \cdot x^n = x^{m+n}$$

Power to a Power Rule When a base is raised to an exponent and that expression is raised to another exponent, multiply the exponents.

$$\left(x^m\right)^n = x^{m\cdot n}$$

Product to a Power Rule When a product is raised to an exponent, apply the exponent to each factor in the product.

$$\left(x \cdot y\right)^n = x^n \cdot y^n$$

List 6.1.2: Summary of Exponent Rules

Checkpoint 6.1.3.

a. Simplify $r^{16} \cdot r^5$.

b. Simplify $\left(x^{11}\right)^{10}$.

c. Simplify $(3r)^4$.

d. Simplify $\left(3y^2\right)^2 \left(y^3\right)^5$.

Explanation.

a. We *add* the exponents because this is a product of powers with the same base:

$$\begin{aligned} r^{16} \cdot r^5 &= r^{16+5} \\ &= r^{21} \end{aligned}$$

b. We *multiply* the exponents because this is a power being raised to a power:

$$\begin{aligned}\left(x^{11}\right)^{10} &= x^{11\cdot 10}\\ &= x^{110}\end{aligned}$$

c. We apply the power to each factor in the product:

$$\begin{aligned}(3r)^4 &= 3^4r^4\\ &= 81r^4\end{aligned}$$

d. All three exponent rules must be used, one at a time:

$$\begin{aligned}\left(3y^2\right)^2\left(y^3\right)^5 &= 3^2\left(y^2\right)^2\left(y^3\right)^5\\ &= 9\left(y^2\right)^2\left(y^3\right)^5\\ &= 9y^{2\cdot 2}y^{3\cdot 5}\\ &= 9y^4y^{15}\\ &= 9y^{4+15}\\ &= 9y^{19}\end{aligned}$$

6.1.2 Quotients and Exponents

Since division is a form of multiplication, it should seem natural that there are some exponent rules for division as well. Not only are there division rules, these rules for division and exponents are direct counterparts for some of the product rules for exponents.

Quotient of Powers When we multiply the same base raised to powers, we end up adding the exponents, as in $2^2 \cdot 2^3 = 2^5$ since $4 \cdot 8 = 32$. What happens when we divide the same base raised to powers?

> **Example 6.1.4** Simplify $\frac{x^5}{x^2}$ by first writing out what each power means.
>
> **Explanation**. Without knowing a rule for simplifying this quotient of powers, we can write the expressions without exponents and simplify.
>
> $$\begin{aligned}\frac{x^5}{x^2} &= \frac{x\cdot x\cdot x\cdot x\cdot x}{x\cdot x}\\ &= \frac{\cancel{x}\cdot\cancel{x}\cdot x\cdot x\cdot x}{\cancel{x}\cdot\cancel{x}\cdot 1}\\ &= \frac{x\cdot x\cdot x}{1}\\ &= x^3\end{aligned}$$
>
> Notice that the difference of the exponents of the numerator and the denominator (5 and 2, respectively) is 3, which is the exponent of the simplified expression.

When we divide as we've just done, we end up canceling factors from the numerator and denominator one-for-one. These common factors cancel to give us factors of 1. The general rule for this is:

Fact 6.1.5 Quotient of Powers Rule. *For any non-zero real number a and integers m and n,*

$$\frac{a^n}{a^m} = a^{n-m}$$

This rule says that when you're dividing two expressions that have the same base, you can simplify the quotient by subtracting the exponents. In Example 6.1.4, this means that we can directly compute $\frac{x^5}{x^2}$:

$$\begin{aligned}\frac{x^5}{x^2} &= x^{5-2}\\ &= x^3\end{aligned}$$

Quotient to a Power Another rule we have learned is the product to a power rule, which applies the outer exponent to each factor in the product inside the parentheses. We can use the rules of fractions to extend this property to a *quotient* raised to a power.

> **Example 6.1.6** Let y be a real number, where $y \neq 0$. Find another way to write $\left(\frac{7}{y}\right)^4$.
>
> **Explanation**. Writing the expression without an exponent and then simplifying, we have:
>
> $$\begin{aligned}\left(\frac{7}{y}\right)^4 &= \left(\frac{7}{y}\right)\left(\frac{7}{y}\right)\left(\frac{7}{y}\right)\left(\frac{7}{y}\right)\\ &= \frac{7\cdot 7\cdot 7\cdot 7}{y\cdot y\cdot y\cdot y}\\ &= \frac{7^4}{y^4}\\ &= \frac{2401}{y^4}\end{aligned}$$
>
> Similar to the product to a power rule, we essentially applied the outer exponent to the "factors" inside the parentheses—to factors of the numerator *and* factors of the denominator. The general rule is:

Fact 6.1.7 Quotient to a Power Rule. *For real numbers a and b (with $b \neq 0$) and integer n,*

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

This rule says that when you raise a fraction to a power, you may separately raise the numerator and denominator to that power. In Example 6.1.6, this means that we can directly calculate $\left(\frac{7}{y}\right)^4$:

$$\begin{aligned}\left(\frac{7}{y}\right)^4 &= \frac{7^4}{y^4}\\ &= \frac{2401}{y^4}\end{aligned}$$

Practice Try these exercises that use the quotient rules for exponents.

Checkpoint 6.1.8.

a. Simplify $\dfrac{3^7x^9}{3^2x^4}$.

b. Simplify $\left(\dfrac{p}{2}\right)^6$.

c. Simplify $\left(\dfrac{5^6w^7}{5^2w^4}\right)^9$. If you end up with a large power of a specific number, leave it written that way.

d. Simplify $\dfrac{\left(2r^5\right)^7}{\left(2^2r^8\right)^3}$. If you end up with a large power of a specific number, leave it written that way.

Explanation.

a. We can use the quotient of powers rule separately on the 3s and on the xs:

$$\begin{aligned}\frac{3^7x^9}{3^2x^4} &= 3^{7-2}x^{9-4}\\ &= 3^5x^5\\ &= 243x^5\end{aligned}$$

b. We can use the quotient to a power rule:

$$\begin{aligned}\left(\frac{p}{2}\right)^6 &= \frac{p^6}{2^6}\\ &= \frac{p^6}{64}\end{aligned}$$

c. If we stick closely to the order of operations, we should first simplify inside the parentheses and then work with the outer exponent. Going this route, we will first use the quotient rule:

$$\begin{aligned}\left(\frac{5^6w^7}{5^2w^4}\right)^9 &= \left(5^{6-2}w^{7-4}\right)^9\\ &= \left(5^4w^3\right)^9\end{aligned}$$

Now we can apply the outer exponent to each factor inside the parentheses using the product to a power rule.

$$= \left(5^4\right)^9 \cdot \left(w^3\right)^9$$

To finish, we need to use the power to a power rule.

$$\begin{aligned}&= 5^{4\cdot 9} \cdot w^{3\cdot 9}\\ &= 5^{36} \cdot w^{27}\end{aligned}$$

d. According to the order of operations, we should simplify inside parentheses first, then apply exponents, then divide. Since we cannot simplify inside the parentheses, we must apply the outer exponents to each factor inside the respective set of parentheses first:

$$\frac{\left(2r^5\right)^7}{\left(2^2r^8\right)^3} = \frac{2^7\left(r^5\right)^7}{\left(2^2\right)^3\left(r^8\right)^3}$$

At this point, we need to use the power-to-a-power rule:

$$\begin{aligned} &= \frac{2^7 r^{5\cdot 7}}{2^{2\cdot 3} r^{8\cdot 3}} \\ &= \frac{2^7 r^{35}}{2^6 r^{24}} \end{aligned}$$

To finish simplifying, we'll conclude with the quotient rule:

$$\begin{aligned} &= 2^{7-6} r^{35-24} \\ &= 2^1 r^{11} \\ &= 2r^{11} \end{aligned}$$

6.1.3 The Zero Exponent

So far, we have been working with exponents that are natural numbers $(1, 2, 3, \ldots)$. By the end of this chapter, we will expand our understanding to include exponents that are any integer, including 0 and negative numbers. As a first step, we will focus on understanding how 0 should behave as an exponent by considering the pattern of decreasing powers of 2 below.

power		product		value	
2^4	=	$2\cdot2\cdot2\cdot2$	=	16	(divide by 2)
2^3	=	$2\cdot2\cdot2$	=	8	(divide by 2)
2^2	=	$2\cdot2$	=	4	(divide by 2)
2^1	=	2	=	2	(divide by 2)
2^0	=	?	=	?	

Table 6.1.9: Descending Powers of 2

As we move down from one row to the row below it, we reduce the power by 1 and we remove a factor of 2. The question then becomes, "What happens when you remove the only remaining factor of 2, when you have no factors of 2?" We can see that "removing a factor of 2" really means that we're dividing the value by 2. Following that pattern, we can see that moving from 2^1 to 2^0 means that we need to divide the value 2 by 2. Since $2 \div 2 = 1$, we have:

$$2^0 = 1$$

Fact 6.1.10 The Zero Exponent Rule. *For any non-zero real number a,*

$$a^0 = 1$$

We exclude the case where $a = 0$ from this rule, because our reasoning for this rule with the table had us dividing by the base. And we cannot divide by 0.

Checkpoint 6.1.11. Simplify the following expressions. Assume all variables represent non-zero real numbers.

a. $\left(173x^4y^{251}\right)^0$ b. $(-8)^0$ c. -8^0 d. $3x^0$

Explanation. To simplify any of these expressions, it is critical that we remember an exponent only applies to what it is touching or immediately next to.

a. In the expression $\left(173x^4y^{251}\right)^0$, the exponent 0 applies to everything inside the parentheses.

$$\left(173x^4y^{251}\right)^0 = 1$$

b. In the expression $(-8)^0$ the exponent applies to everything inside the parentheses, -8.

$$(-8)^0 = 1$$

c. In contrast to the previous example, the exponent only applies to the 8. The exponent has a higher priority than negation in the order of operations. We should consider that $-8^0 = -\left(8^0\right)$, and so:

$$\begin{aligned}-8^0 &= -\left(8^0\right)\\ &= -1\end{aligned}$$

d. In the expression $3x^0$, the exponent 0 only applies to the x:

$$\begin{aligned}3x^0 &= 3\cdot x^0\\ &= 3\cdot 1\\ &= 3\end{aligned}$$

6.1.4 Negative Exponents

In Section 2.9, we developed rules for simplifying expressions with whole number exponents, like 0, 1, 2, 3, etc. It turns out that these same rules apply even if the exponent is a negative integer, like -1, -2, -3, etc.

To consider the effects of negative integer exponents, let's extend the pattern we examined in Table 6.1.9. In that table, each time we move down a row, we reduce the power by 1 and we divide the value by 2. We can continue this pattern in the power and value columns, paying particular attention to the values for negative exponents:

Power	Value	
2^3	8	(divide by 2)
2^2	4	(divide by 2)
2^1	2	(divide by 2)
2^0	1	(divide by 2)
2^{-1}	$1/2 = 1/2^1$	(divide by 2)
2^{-2}	$1/4 = 1/2^2$	(divide by 2)
2^{-3}	$1/8 = 1/2^3$	

Note that the choice of base 2 was arbitrary, and this pattern works for all bases except 0, since we cannot divide by 0 in moving from one row to the next.

Fact 6.1.12 The Negative Exponent Rule. *For any non-zero real number* a *and any integer* n,

$$a^{-n} = \frac{1}{a^n}$$

Note that if we take reciprocals of both sides, we have another helpful fact:

$$\frac{1}{a^{-n}} = a^n.$$

Taken together, these facts tell us that a negative exponent power in the numerator belongs in the denominator (with a positive exponent) and a negative exponent power in the denominator belongs in the numerator (with a positive exponent). In other words, you can see a negative exponent as telling you to move things in and out of the numerator and denominator of an expression.

Remark 6.1.13. You may be expected to simplify expressions so that they do not have any negative exponents. This can always be accomplished using the negative exponent rule.

Try these exercises that involve negative exponents.

Checkpoint 6.1.14.

a. Write $4y^{-6}$ without using negative exponents.

b. Write $\dfrac{3x^{-4}}{yz^{-2}}$ without using negative exponents.

c. Simplify $\left(-5x^{-5}\right)\left(-8x^{4}\right)$ and write it without using negative exponents.

Explanation.

a. Always remember that an exponent only applies to what it is touching. In the expression $4y^{-6}$, only the y has an exponent of -6.

$$\begin{aligned} 4y^{-6} &= 4 \cdot \frac{1}{y^6} \\ &= \frac{4}{y^6} \end{aligned}$$

b. Negative exponents tell us to move some variables between the numerator and denominator to make the exponents positive.

$$\frac{3x^{-4}}{yz^{-2}} = \frac{3z^2}{yx^4}$$

Notice that the factors of 3 and y did not move, as both of those factors had positive exponents.

c. The product of powers rule still applies, and we can add exponents even when one or both are negative:

$$\begin{aligned} \left(-5x^{-5}\right)\left(-8x^{4}\right) &= (-5)(-8)x^{-5}x^{4} \\ &= 40x^{-5+4} \\ &= 40x^{-1} \\ &= \frac{40}{x^1} \\ &= \frac{40}{x} \end{aligned}$$

6.1.5 Summary of Exponent Rules

Now that we have some new exponent rules beyond those from Section 2.9, let's summarize.

If a and b are real numbers, and n and m are integers, then we have the following rules:

Product Rule $a^n \cdot a^m = a^{n+m}$

Power to a Power Rule $(a^n)^m = a^{n \cdot m}$

Product to a Power Rule $(ab)^n = a^n \cdot b^n$

Quotient Rule $\dfrac{a^n}{a^m} = a^{n-m}$, as long as $a \neq 0$

Quotient to a Power Rule $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$, as long as $b \neq 0$

Zero Exponent Rule $a^0 = 1$ for $a \neq 0$

Negative Exponent Rule $a^{-n} = \frac{1}{a^n}$

Negative Exponent Reciprocal Rule $\frac{1}{a^{-n}} = a^n$

List 6.1.15: Summary of the Rules of Exponents for Multiplication and Division

Remark 6.1.16 Why we have "$a \neq 0$" and "$b \neq 0$" for some rules. Whenever we're working with division, we have to be careful to make sure the rules we state don't ever imply that we might be dividing by zero. Dividing by zero leads us to expressions that have no meaning. For example, both $\frac{9}{0}$ and $\frac{0}{0}$ are *undefined*, meaning no one has defined what it means to divide a number by 0. Also, we established that $a^0 = 1$ using repeated division by a in table rows, so that reasoning doesn't work if $a = 0$.

Warning 6.1.17 A Common Mistake. It may be tempting to apply the rules of exponents to expressions containing addition or subtraction. However, none of the rules of exponents 6.1.15 involve addition or subtraction in the initial expression. Because whole number exponents mean repeated multiplication, not repeated addition or subtraction, trying to apply exponent rules in situations that do not use multiplication simply doesn't work.

Can we say something like $a^n + a^m = a^{n+m}$? How would that work out when $a = 2$?

$$\begin{aligned} 2^3 + 2^4 &\overset{?}{=} 2^{3+4} \\ 8 + 16 &\overset{?}{=} 2^7 \\ 24 &\neq 128 \end{aligned}$$

As we can see, that's not even close. This attempt at a "sum rule" falls apart. In fact, without knowing values for a, n, and m, there's no way to simplify the expression $a^n + a^m$.

Checkpoint 6.1.18. Decide whether each statements is true or false.

a. $(7+8)^3 = 7^3 + 8^3$

   (☐ true ☐ false)

b. $(xy)^3 = x^3 y^3$

   (☐ true ☐ false)

c. $2x^3 \cdot 4x^2 \cdot 5x^6 = (2 \cdot 4 \cdot 5)x^{3+2+6}$

   (☐ true ☐ false)

d. $\left(x^3 y^5\right)^4 = x^{3+4} y^{5+4}$

   (☐ true ☐ false)

e. $2\left(x^2 y^5\right)^3 = 8x^6 y^{15}$

   (☐ true ☐ false)

f. $x^2 + x^3 = x^5$

   (☐ true ☐ false)

g. $x^3 + x^3 = 2x^3$

   (☐ true ☐ false)

h. $x^3 \cdot x^3 = 2x^6$

   (☐ true ☐ false)

i. $3^2 \cdot 2^3 = 6^5$

   (☐ true ☐ false)

j. $3^{-2} = -\frac{1}{9}$

   (☐ true ☐ false)

Explanation.

a. False, $(7+8)^3 \neq 7^3 + 8^3$. Following the order of operations, on the left $(7+8)^3$ would simplify as 15^3, which is 3375. However, on the right side, we have

$$\begin{aligned} 7^3 + 8^3 &= 343 + 512 \\ &= 855 \end{aligned}$$

   Since $3375 \neq 855$, the equation is false.

b. True. As the cube applies to the product of x and y, $(xy)^3 = x^3 y^3$.

c. True. The coefficients do get multiplied together and the exponents added when the expressions are multiplied, so $2x^3 \cdot 4x^2 \cdot 5x^6 = (2 \cdot 4 \cdot 5)x^{3+2+6}$.

d. False, $\left(x^3 y^5\right)^4 \neq x^{3+4} y^{5+4}$. When we have a power to a power, we multiply the exponents rather than adding them. So

$$\left(x^3 y^5\right)^4 = x^{3\cdot 4} y^{5\cdot 4}$$

e. False, $2\left(x^2 y^5\right)^3 \neq 8x^6 y^{15}$. The exponent of 3 applies to x^2 and y^5, but does not apply to the 2. So

$$\begin{aligned} 2\left(x^2 y^5\right)^3 &= 2x^{2\cdot 3}6y^{5\cdot 3} \\ &= 2x^6 y^{15} \end{aligned}$$

f. False, $x^2 + x^3 \neq x^5$. The two terms on the left hand side are not like terms and there is no way to combine them.

g. True. The terms x^3 and x^3 are like terms, so $x^3 + x^3 = 2x^3$.

h. False, $x^3 \cdot x^3 \neq 2x^6$. When x^3 and x^3 are multiplied, their coefficients are each 1. So the coefficient of their product is still 1, and we have $x^3 \cdot x^3 = x^6$.

i. False, $3^2 \cdot 2^3 \neq 6^5$. Note that neither the bases nor the exponents are the same. Following the order of operations, on the left $3^2 \cdot 2^3$ would simplify as $9 \cdot 8$, which is 72. However, on the right side, we have $6^5 = 7776$. Since $72 \neq 7776$, the equation is false.

j. False, $3^{-2} \neq -\frac{1}{9}$. The exponent of -2 on the number 3 does not result in a negative number. Instead, $3^{-2} = \frac{1}{3^2}$, which is $\frac{1}{9}$.

As we mentioned before, many situations we'll come across will require us to use more than one exponent rule. In these situations, we'll have to decide which rule to use first. There are often different, correct approaches we could take. But if we rely on order of operations, we will have a straightforward approach to simplify the expression correctly. To bring it all together, try these exercises.

Checkpoint 6.1.19.

a. Simplify $\dfrac{6x^3}{2x^7}$ and write it without using negative exponents.

b. Simplify $4\left(\frac{1}{5}tv^{-4}\right)^2$ and write it without using negative exponents.

c. Simplify $\left(\dfrac{3^0y^4 \cdot y^5}{6y^2}\right)^3$ and write it without using negative exponents.

d. Simplify $\left(7^4x^{-6}t^2\right)^{-5}\left(7x^{-2}t^{-7}\right)^4$ and write it without using negative exponents. Leave larger numbers (such as 7^{10}) in exponent form.

Explanation.

a. In the expression $\frac{6x^3}{2x^7}$, the coefficients reduce using the properties of fractions. One way to simplify the variable powers is:

$$\begin{aligned}
\frac{6x^3}{2x^7} &= \frac{6}{2} \cdot \frac{x^3}{x^7}\\
&= 3 \cdot x^{3-7}\\
&= 3 \cdot x^{-4}\\
&= 3 \cdot \frac{1}{x^4}\\
&= \frac{3}{x^4}
\end{aligned}$$

b. In the expression $4\left(\frac{1}{5}tv^{-4}\right)^2$, the exponent 2 applies to each factor inside the parentheses.

$$\begin{aligned}
4\left(\frac{1}{5}tv^{-4}\right)^2 &= 4\left(\frac{1}{5}\right)^2(t)^2\left(v^{-4}\right)^2\\
&= 4\left(\frac{1}{25}\right)\left(t^2\right)\left(v^{-4\cdot 2}\right)\\
&= 4\left(\frac{1}{25}\right)\left(t^2\right)\left(v^{-8}\right)\\
&= 4\left(\frac{1}{25}\right)\left(t^2\right)\left(\frac{1}{v^8}\right)\\
&= \frac{4t^2}{25v^8}
\end{aligned}$$

c. To follow the order of operations in the expression $\left(\frac{3^0y^4 \cdot y^5}{6y^2}\right)^3$, the numerator inside the parentheses should be dealt with first. After that, we'll simplify the quotient inside the parentheses. As a final step, we'll apply the exponent to that simplified expression:

$$\begin{aligned}
\left(\frac{3^0 y^4 \cdot y^5}{6y^2}\right)^3 &= \left(\frac{1 \cdot y^{4+5}}{6y^2}\right)^3 \\
&= \left(\frac{y^9}{6y^2}\right)^3 \\
&= \left(\frac{y^{9-2}}{6}\right)^3 \\
&= \left(\frac{y^7}{6}\right)^3 \\
&= \frac{\left(y^7\right)^3}{6^3} \\
&= \frac{y^{7\cdot 3}}{216} \\
&= \frac{y^{21}}{216}
\end{aligned}$$

d. We'll again rely on the order of operations, and look to simplify anything inside parentheses first and then apply exponents. In this example, we will begin by applying the product to a power rule, followed by the power to a power rule.

$$\begin{aligned}
\left(7^4 x^{-6} t^2\right)^{-5} \left(7x^{-2} t^{-7}\right)^4 &= \left(7^4\right)^{-5} \left(x^{-6}\right)^{-5} \left(t^2\right)^{-5} \cdot (7)^4 \left(x^{-2}\right)^4 \left(t^{-7}\right)^4 \\
&= 7^{-20} x^{30} t^{-10} \cdot 7^4 x^{-8} t^{-28} \\
&= 7^{-20+4} x^{30-8} t^{-10-28} \\
&= 7^{-16} x^{22} t^{-38} \\
&= \frac{x^{22}}{7^{16} t^{38}}
\end{aligned}$$

Exercises

Review and Warmup Evaluate the following.

1. a. $2^2 =$ ☐

b. $3^3 =$ ☐

c. $(-2)^2 =$ ☐

d. $(-5)^3 =$ ☐

2. a. $3^2 =$ ☐

b. $5^3 =$ ☐

c. $(-2)^2 =$ ☐

d. $(-3)^3 =$ ☐

Use the properties of exponents to simplify the expression.

3. $5 \cdot 5^5$

4. $6 \cdot 6^2$

5. $7^8 \cdot 7^3$

6. $8^5 \cdot 8^6$

Simplifying Products and Quotients Involving Exponents Use the properties of exponents to simplify the expression.

7. $t^{19} \cdot t^3$

8. $y^2 \cdot y^{15}$

9. $\left(t^3\right)^5$

10. $\left(x^4\right)^2$

11. $\left(4x^5\right)^3$

12. $\left(2t^6\right)^2$

13. $(6x^{12}) \cdot (-2x^{20})$

14. $(-9x^{14}) \cdot (3x^{13})$

15. $\left(-\frac{x^{16}}{4}\right) \cdot \left(-\frac{x^7}{2}\right)$

16. $\left(\frac{y^{18}}{8}\right) \cdot \left(\frac{y^{19}}{3}\right)$

17. $-3\left(-9t^2\right)^3$

18. $-5\left(-6t^3\right)^2$

19. $(-40)^0 = \boxed{\quad}$

20. $(-34)^0 = \boxed{\quad}$

21. $-23^0 = \boxed{\quad}$

22. $-29^0 = \boxed{\quad}$

23. $34^0 + (-34)^0 = \boxed{\quad}$

24. $40^0 + (-40)^0 = \boxed{\quad}$

25. $45n^0 = \boxed{\quad}$

26. $50c^0 = \boxed{\quad}$

27. $(-792y)^0 = \boxed{\quad}$

28. $(-571m)^0 = \boxed{\quad}$

29. $\left(\frac{x^5}{4}\right)^2 = \boxed{\quad}$

30. $\left(\frac{x^2}{5}\right)^3 = \boxed{\quad}$

31. $\left(\frac{-7}{8x^{10}}\right)^3 = \boxed{\quad}$

32. $\left(\frac{-7}{4x^5}\right)^2 = \boxed{\quad}$

33. $\left(\frac{9x^8}{2}\right)^2 = \boxed{\quad}$

34. $\left(\frac{3x^9}{10}\right)^3 = \boxed{\quad}$

35. $\frac{45y^{12}}{15y^9} = \boxed{\quad}$

36. $\frac{-18t^7}{3t^5} = \boxed{\quad}$

37. $\frac{5r^{11}}{25r^{10}} = \boxed{\quad}$

38. $\frac{6r^{18}}{18r^{12}} = \boxed{\quad}$

39. $\frac{y^{10}}{y^9} = \boxed{\quad}$

40. $\frac{r^{12}}{r^7} = \boxed{\quad}$

41. $\frac{16^{12}}{16^3} = \boxed{\quad}$

42. $\frac{18^{19}}{18^{10}} = \boxed{\quad}$

43. $\left(\frac{x^6}{2y^{10}z^4}\right)^3 = \boxed{\quad}$

44. $\left(\frac{x^3}{2y^5z^2}\right)^3 = \boxed{\quad}$

45. $\left(\frac{-7x^2}{4y^9}\right)^3 = \boxed{\quad}$

46. $\left(\frac{-7x^3}{10y^6}\right)^2 = \boxed{\quad}$

47. $\frac{12^7 \cdot 11^{14}}{12^4 \cdot 11^5} = \boxed{\quad}$

48. $\frac{14^{17} \cdot 20^{14}}{14^2 \cdot 20^{10}} = \boxed{\quad}$

49. $\frac{-48x^{16}y^9z^{11}}{12x^{11}y^4z^3} = \boxed{\quad}$

50. $\frac{28x^{12}y^{17}z^{18}}{14x^8y^7z^7} = \boxed{\quad}$

51. $\frac{-80x^4y^8}{16x^3y^2} = \boxed{\quad}$

52. $\frac{72x^{17}y^{12}}{18x^{10}y^7} = \boxed{\quad}$

Rewrite the expression simplified and using only positive exponents.

53. $\left(\frac{1}{10}\right)^{-2} = \boxed{}$

54. $\left(\frac{1}{2}\right)^{-3} = \boxed{}$

55. $\frac{3^{-2}}{4^{-3}} = \boxed{}$

56. $\frac{4^{-3}}{2^{-2}} = \boxed{}$

57. $5^{-1} - 8^{-1} = \boxed{}$

58. $6^{-1} - 4^{-1} = \boxed{}$

59. $20x^{-9} = \boxed{}$

60. $15x^{-10} = \boxed{}$

61. $\frac{9}{x^{-11}} = \boxed{}$

62. $\frac{19}{x^{-12}} = \boxed{}$

63. $\frac{14x^{-3}}{x} = \boxed{}$

64. $\frac{8x^{-5}}{x} = \boxed{}$

65. $\frac{7x^{-8}}{x^{-26}} = \boxed{}$

66. $\frac{13x^{-10}}{x^{-17}} = \boxed{}$

67. $\frac{20x^{-18}}{21x^{-9}} = \boxed{}$

68. $\frac{11x^{-21}}{12x^{-35}} = \boxed{}$

69. $\frac{t^{-17}}{y^{-3}} = \boxed{}$

70. $\frac{t^{-6}}{x^{-17}} = \boxed{}$

71. $\frac{t^{-12}}{r^{7}} = \boxed{}$

72. $\frac{x^{-3}}{y^{7}} = \boxed{}$

73. $\frac{1}{7x^{-10}} = \boxed{}$

74. $\frac{1}{39y^{-18}} = \boxed{}$

75. $\frac{y^{8}}{y^{22}} = \boxed{}$

76. $\frac{r^{4}}{r^{6}} = \boxed{}$

77. $\frac{-27r^{6}}{9r^{37}} = \boxed{}$

78. $\frac{21t^{12}}{7t^{20}} = \boxed{}$

79. $\frac{-7t^{3}}{5t^{4}} = \boxed{}$

80. $\frac{-3t^{14}}{13t^{35}} = \boxed{}$

81. $\frac{x^{5}}{\left(x^{6}\right)^{9}} = \boxed{}$

82. $\frac{x^{4}}{\left(x^{12}\right)^{6}} = \boxed{}$

83. $\frac{y^{-6}}{\left(y^{9}\right)^{4}} = \boxed{}$

84. $\frac{y^{-2}}{\left(y^{5}\right)^{10}} = \boxed{}$

85. $r^{-19} \cdot r^{10} = \boxed{}$

86. $r^{-13} \cdot r^{11} = \boxed{}$

87. $(-3t^{-7}) \cdot (3t^{3}) = \boxed{}$

88. $(-9t^{-19}) \cdot (8t^{14}) = \boxed{}$

89. $\left(\frac{7}{2}\right)^{-2} = \boxed{}$

90. $\left(\frac{10}{7}\right)^{-2} = \boxed{}$

91. $(-3)^{-3} = \boxed{}$

92. $(-4)^{-3} = \boxed{}$

93. $\frac{1}{(-5)^{-2}} = \boxed{}$

94. $\frac{1}{(-6)^{-3}} = \boxed{}$

95. $\frac{6}{(-3)^{-3}} = \boxed{}$

96. $\frac{-3}{(-4)^{-3}} = \boxed{}$

97. $9^{-3} = \boxed{}$

98. $10^{-2} = \boxed{}$

99. $2^{-1} + 4^{-1} = \boxed{}$

100. $3^{-1} + 8^{-1} = \boxed{}$

101. $\dfrac{1}{4^{-2}} = \boxed{}$

102. $\dfrac{1}{5^{-2}} = \boxed{}$

103. $-6^{-3} = \boxed{}$

104. $-7^{-2} = \boxed{}$

105. $\dfrac{\left(6r^{12}\right)^{3}}{r^{37}} = \boxed{}$

106. $\dfrac{\left(6t^{9}\right)^{2}}{t^{37}} = \boxed{}$

107. $\dfrac{\left(6t^{5}\right)^{3}}{t^{-10}} = \boxed{}$

108. $\dfrac{\left(6x^{12}\right)^{2}}{x^{-6}} = \boxed{}$

109. $\left(\dfrac{x^{13}}{x^{7}}\right)^{-2} = \boxed{}$

110. $\left(\dfrac{y^{7}}{y^{6}}\right)^{-4} = \boxed{}$

111. $\left(\dfrac{10y^{19}}{5y^{7}}\right)^{-3} = \boxed{}$

112. $\left(\dfrac{25r^{12}}{5r^{9}}\right)^{-2} = \boxed{}$

113. $\left(-2r^{-6}\right)^{-2} \boxed{}$

114. $\left(-3r^{-18}\right)^{-3} \boxed{}$

115. $\left(3t^{-12}\right)^{-3} \boxed{}$

116. $\left(2t^{-5}\right)^{-2} \boxed{}$

117. $\dfrac{9x^{9} \cdot 5x^{9}}{2x^{14}} = \boxed{}$

118. $\dfrac{8x^{6} \cdot 7x^{4}}{5x^{6}} = \boxed{}$

119. $\left(y^{4}\right)^{4} \cdot y^{-5} = \boxed{}$

120. $\left(y^{12}\right)^{2} \cdot y^{-23} = \boxed{}$

121. $\left(3r^{8}\right)^{3} \cdot r^{-17} = \boxed{}$

122. $\left(3r^{3}\right)^{4} \cdot r^{-6} = \boxed{}$

123. $\dfrac{\left(r^{8}\right)^{2}}{\left(r^{4}\right)^{5}} = \boxed{}$

124. $\dfrac{\left(t^{6}\right)^{4}}{\left(t^{14}\right)^{4}} = \boxed{}$

125. $\left(t^{3}\right)^{-2} = \boxed{}$

126. $\left(x^{15}\right)^{-4} = \boxed{}$

127. $\left(x^{15}y^{3}\right)^{-4} = \boxed{}$

128. $\left(y^{8}t^{12}\right)^{-4} = \boxed{}$

129. $\left(y^{-13}r^{9}\right)^{-4} = \boxed{}$

130. $\left(r^{-6}y^{5}\right)^{-4} = \boxed{}$

131. $\left(\dfrac{r^{15}}{4}\right)^{-3} = \boxed{}$

132. $\left(\dfrac{r^{10}}{3}\right)^{-2} = \boxed{}$

133. $\left(\dfrac{t^{9}}{x^{7}}\right)^{-3} = \boxed{}$

134. $\left(\dfrac{t^{9}}{x^{10}}\right)^{-5} = \boxed{}$

135. $\dfrac{\left(x^{4}r^{-7}\right)^{-3}}{\left(x^{-3}r^{8}\right)^{-3}} = \boxed{}$

136. $\dfrac{\left(x^{6}r^{-5}\right)^{-3}}{\left(x^{-8}r^{7}\right)^{-3}} = \boxed{}$

Rewrite the expression simplified and using only positive exponents.

137. $7x^{-4}y^{5}z^{-2}\left(2x^{2}\right)^{-4} = \boxed{}$

138. $5x^{-5}y^{5}z^{-3}\left(3x^{5}\right)^{-4} = \boxed{}$

139. $\left(\dfrac{x^{7}y^{4}z^{2}}{x^{-3}y^{-8}z^{-4}}\right)^{-4} = \boxed{}$

140. $\left(\dfrac{x^{5}y^{4}z^{5}}{x^{-4}y^{-4}z^{-6}}\right)^{-2} = \boxed{}$

Challenge

141. Consider the exponential expression $\frac{x^a \cdot x^b}{x^c}$ where $a > 0, b < 0$, and $c > 0$.

a. Are there values for a, b, and c so that the expression equals x^6? If so, fill in the blanks below with possible values for a, b, and c. If not, fill in the blanks below with the word none.

$a =$ ☐, $b =$ ☐, and $c =$ ☐

b. Are there values for a, b, and c so that the exponential expression equals $\frac{1}{x^{13}}$? If so, fill in the blanks below with possible values for a, b, and c. If not, fill in the blanks below with the word none.

$a =$ ☐, $b =$ ☐, and $c =$ ☐

142. Consider the exponential expression $\frac{x^a \cdot x^b}{x^c}$ where $a < 0, b < 0$, and $c > 0$.

a. Are there values for a, b, and c so that the expression equals x^7? If so, fill in the blanks below with possible values for a, b, and c. If not, fill in the blanks below with the word none.

$a =$ ☐, $b =$ ☐, and $c =$ ☐

b. Are there values for a, b, and c so that the expression equals $\frac{1}{x^9}$? If so, fill in the blanks below with possible values for a, b, and c. If not, fill in the blanks below with the word none.

$a =$ ☐, $b =$ ☐, and $c =$ ☐

143. Consider the exponential expression $\frac{x^a \cdot x^b}{x^c}$ where $a > 0, b > 0$, and $c < 0$.

a. Are there values for a, b, and c so that the expression equals x^6? If so, fill in the blanks below with possible values for a, b, and c. If not, fill in the blanks below with the word none.

$a =$ ☐, $b =$ ☐, and $c =$ ☐

b. Are there values for a, b, and c so that the expression equals $\frac{1}{x^{14}}$? If so, fill in the blanks below with possible values for a, b, and c. If not, fill in the blanks below with the word none.

$a =$ ☐, $b =$ ☐, and $c =$ ☐

6.2 Scientific Notation

Having just learned more about exponents, including negative exponents, we can discuss a format used for very large and very small numbers called **scientific notation**.

6.2.1 The Basics of Scientific Notation

An October 3, 2016 CBS News headline[1] read:

> Federal Debt in FY 2016 Jumped \$1,422,827,047,452.46—that's \$12,036 Per Household.

The article also later states:

> By the close of business on Sept. 30, 2016, the last day of fiscal 2016, it had climbed to \$19,573,444,713,936.79.

When presented in this format, trying to comprehend the value of these numbers can be overwhelming. More commonly, such numbers would be presented in a descriptive manner:

- The federal debt climbed by 1.42 trillion dollars in 2016.
- The federal debt was 19.6 trillion dollars at the close of business on Sept. 30, 2016.

Unless we're presented with such news items, most of us deal with numbers no larger than the thousands in our daily life. In science, government, business, and many other disciplines, it's not uncommon to deal with much larger numbers. When numbers get this large, it can be hard to distinguish between a number that has nine or twelve digits. On the other hand, we have descriptive language that allows us grasp the value and not be lost in the sheer size of the number.

We have descriptive language for all numbers, based on the place value of the different digits: ones, tens, thousands, ten thousands, etc. We tend to rely upon this language more when we start dealing with larger numbers. Here's a chart for some of the most common numbers we see and use in the world around us:

Number	US English Name	Power of 10
1	one	10^0
10	ten	10^1
100	hundred	10^2
1,000	one thousand	10^3
10,000	ten thousand	10^4
100,000	one hundred thousand	10^5
1,000,000	one million	10^6
1,000,000,000	one billion	10^9

Table 6.2.2: Whole Number Powers of 10

Each number above has a corresponding power of ten and this power of ten will be important as we start to work with the content in this section.

This descriptive language also covers even larger numbers: trillion, quadrillion, quintillion, sextillion, septillion, and so on. There's also corresponding language to describe very small numbers, such as thousandth, millionth, billionth, trillionth, etc.

Through centuries of scientific progress, humanity became increasingly aware of very large numbers and very small measurements. As one example, the star that is nearest to our sun is Proxima Centauri[2]. Proxima

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[1] `http://www.cnsnews.com/news/article/terence-p-jeffrey/federal-debt-fy-2016-jumped-142282704745246`

[2] imagine.gsfc.nasa.gov/features/cosmic/nearest_star_info.html

Centauri is about 25,000,000,000,000 miles from our sun. Again, many will find the descriptive language easier to digest: Proxima Centauri is about 25 trillion miles from our sun.

To make computations involving such numbers more manageable, a standardized notation called **scientific notation** was established. The foundation of scientific notation is the fact that multiplying or dividing by a power of 10 will move the decimal point of a number so many places to the right or left, respectively.

Checkpoint 6.2.3. Perform the following operations:

a. Multiply 5.7 by 10.

b. Multiply 3.1 by 10,000.

Explanation.

a. $5.7 \times 10 = 57$

   $10 = 10^1$and multiplying by 10^1 moved the decimal point one place to the right.

b. $3.1 \times 10000 = 31{,}000$

   $10{,}000 = 10^4$ and multiplying by 10^4 moved the decimal point four places to the right.

Multiplying a number by 10^n where n is a positive integer had the effect of moving the decimal point n places to the right.

Every number can be written as a product of a number between 1 and 10 and a power of 10. For example, $650 = 6.5 \times 100$. Since $100 = 10^2$, we can also write

$$650 = 6.5 \times 10^2$$

and this is our first example of writing a number in **scientific notation**.

Definition 6.2.4. A positive number is written in **scientific notation** when it has the form $a \times 10^n$ where n is an integer and $1 \leq a < 10$. In other words, a has precisely one digit to the left of the decimal place. The exponent n used here is called the number's **order of magnitude**. The number a is sometimes called the **significand** or the **mantissa**.

Some conventions do not require a to be between 1 and 10, excluding both values, but that is the convention used in this book.

6.2.2 Scientific Notation for Large Numbers

To write a numbers larger than 10 in scientific notation, we write a decimal point after the first non-zero digit of the number and then count the number of places between where the decimal point originally was and where it now is. Scientific notation communicates the size of a number and the order of magnitude just as quickly, but with no need to write long strings of zeros or to try to decipher the language of quintillions, sextillions, etc.

> **Example 6.2.5** To get a sense of how scientific notation works, let's consider familiar lengths of time converted to seconds.

Length of Time	Length in Seconds	Scientific Notation
one second	1 second	1×10^0 second
one minute	60 seconds	6×10^1 seconds
one hour	3600 seconds	3.6×10^3 seconds
one month	2,628,000 seconds	2.628×10^6 seconds
ten years	315,400,000 seconds	3.154×10^8 seconds
79 years (about a lifetime)	2,491,000,000 seconds	2.491×10^9 seconds

Checkpoint 6.2.6. Write each of the following in scientific notation.

a. The federal debt at the close of business on Sept. 30, 2016: about 19,600,000,000,000 dollars.

b. The world's population in 2016: about 7,418,000,000 people.

Explanation.

a. To convert the federal debt to scientific notation, we will count the number of digits after the first non-zero digit (which happens to be a 1 here). Since there are 13 places after the first non-zero digit, we write:

$$1\,\overbrace{9{,}600{,}000{,}000{,}000}^{\text{13 places}} \text{ dollars} = 1.96 \times 10^{13} \text{ dollars}$$

b. Since there are nine places after the first non-zero digit of 7, the world's population in 2016 was about

$$7,\overbrace{418{,}000{,}000}^{\text{9 places}} \text{ people} = 7.418 \times 10^{9} \text{ people}$$

Checkpoint 6.2.7. Convert each of the following from scientific notation to decimal notation (without any exponents).

a. The earth's diameter: about 1.27×10^7 meters.

b. As of 2013, known digits of π: 1.21×10^{13}.

Explanation.

a. To convert this number to decimal notation we will move the decimal point after the digit 1 seven places to the right, including zeros where necessary. The earth's diameter is:

$$1.27 \times 10^7 \text{ meters} = 1\,\overbrace{2{,}700{,}000}^{\text{7 places}} \text{ meters.}$$

b. As of 2013 there are

$$1.21 \times 10^{13} = 1\,\overbrace{2{,}100{,}000{,}000{,}000}^{\text{13 places}}$$

known digits of π.

6.2.3 Scientific Notation for Small Numbers

Scientific notation can also be useful when working with numbers smaller than 1. As we saw in Table 6.2.2, we can denote thousands, millions, billions, trillions, etc., with positive integer exponents on 10. We can similarly denote numbers smaller than 1 (which are written as tenths, hundredths, thousandths, millionths, billionths, trillionths, etc.), with *negative* integer exponents on 10. This relationship is outlined in Table 6.2.8.

Number	English Name	Power of 10
1	one	10^0
0.1	one tenth	$\frac{1}{10} = 10^{-1}$
0.01	one hundredth	$\frac{1}{100} = 10^{-2}$
0.001	one thousandth	$\frac{1}{1{,}000} = 10^{-3}$
0.0001	one ten thousandth	$\frac{1}{10{,}000} = 10^{-4}$
0.00001	one hundred thousandth	$\frac{1}{100{,}000} = 10^{-5}$
0.000001	one millionth	$\frac{1}{1{,}000{,}000} = 10^{-6}$
0.000000001	one billionth	$\frac{1}{1{,}000{,}000{,}000} = 10^{-9}$

Table 6.2.8: Negative Integer Powers of 10

To see how this works with a digit other than 1, let's look at 0.05. When we state 0.05 as a number, we say "5 hundredths." Thus $0.05 = 5 \times \frac{1}{100}$. The fraction $\frac{1}{100}$ can be written as $\frac{1}{10^2}$, which we know is equivalent to 10^{-2}. Using negative exponents, we can then rewrite 0.05 as 5×10^{-2}. This is the scientific notation for 0.05.

In practice, we won't generally do that much computation. To write a small number in scientific notation we start as we did before and place the decimal point behind the first non-zero digit. We then count the number of decimal places between where the decimal had originally been and where it now is. Keep in mind that negative powers of ten are used to help represent very small numbers (smaller than 1) and positive powers of ten are used to represent very large numbers (larger than 1). So to convert 0.05 to scientific notation, we have:

$$0\overbrace{.05}^{2\text{ places}} = 5 \times 10^{-2}$$

Example 6.2.9 In quantum mechanics, there is an important value called the Planck Constant[a]. Written as a decimal, the value of the Planck constant (rounded to 4 significant digits) is

$$0.0000000000000000000000000000000006626.$$

In scientific notation, this number will be $6.626 \times 10^{?}$. To determine the exponent, we need to count the number of places from where the decimal is when the number is written as

$$0.0000000000000000000000000000000006626$$

to where it will be when written in scientific notation:

$$0\overbrace{.0000000000000000000000000000000006}^{34\text{ places}}626$$

As a result, in scientific notation, the Planck Constant value is 6.626×10^{-34}. It will be much easier to

> use 6.626×10^{-34} in a calculation, and an added benefit is that scientific notation quickly communicates both the value and the order of magnitude of the Planck constant.
>
> ---
>
> [a] en.wikipedia.org/wiki/Planck_constant

Checkpoint 6.2.10. Write each of the following in scientific notation.

a. The weight of a single grain of long grain rice: about 0.029 grams.

b. The gate pitch of a microprocessor: 0.000000014 meters

Explanation.

a. To convert this weight to scientific notation, we must first move the decimal behind the first non-zero digit to obtain 2.9, which requires that we move the decimal point 2 places. Thus we have:

$$0\overbrace{.02}^{\text{2 places}}9 \text{ grams} = 2.9 \times 10^{-2} \text{ grams}$$

b. The gate pitch of a microprocessor is:

$$0\overbrace{.00000001}^{\text{8 places}}4 \text{ meters} = 1.4 \times 10^{-8} \text{ meters}$$

Checkpoint 6.2.11. Convert each of the following from scientific notation to decimal notation (without any exponents).

a. A download speed of 7.53×10^{-3} Gigabyte per second

b. The weight of a poppy seed: about 3×10^{-7} kilograms

Explanation.

a. To convert a download speed of 7.53×10^{-3} Gigabyte per second to decimal notation, we will move the decimal point 3 places to the left and include the appropriate number of zeros:

$$7.53 \times 10^{-3} \text{ Gigabyte per second} = 0\overbrace{.007}^{\text{3 places}}53 \text{ Gigabyte per second}$$

b. The weight of a poppy seed is:

$$3 \times 10^{-7} \text{ kilograms} = 0\overbrace{.0000003}^{\text{7 places}} \text{ kilograms}$$

Checkpoint 6.2.12. Decide if the numbers are written in scientific notation or not. Use Definition 6.2.4.

a. The number $7 \times 10^{1.9}$ is (☐ in scientific notation ☐ not in scientific notation) .

b. The number 2.6×10^{-31} is (☐ in scientific notation ☐ not in scientific notation) .

c. The number 10×7^{4} is (☐ in scientific notation ☐ not in scientific notation) .

d. The number 0.93×10^{3} is (☐ in scientific notation ☐ not in scientific notation) .

e. The number 4.2×10^{0} is (☐ in scientific notation ☐ not in scientific notation) .

f. The number 12.5×10^{-6} is (☐ in scientific notation ☐ not in scientific notation) .

Explanation.

a. The number $7 \times 10^{1.9}$ *is not* in scientific notation. The exponent on the 10 is required to be an integer and 1.9 is not.

b. The number 2.6×10^{-31} *is* in scientific notation.

c. The number 10×7^4 *is not* in scientific notation. The base must be 10, not 7.

d. The number 0.93×10^3 *is not* in scientific notation. The coefficient of the 10 must be between 1 (inclusive) and 10.

e. The number 4.2×10^0 *is* in scientific notation.

f. The number 12.5×10^{-6} *is not* in scientific notation. The coefficient of the 10 must be between 1 (inclusive) and 10.

6.2.4 Multiplying and Dividing Using Scientific Notation

One main reason for having scientific notation is to make calculations involving immensely large or small numbers easier to perform. By having the order of magnitude separated out in scientific notation, we can separate any calculation into two components.

Example 6.2.13 On Sept. 30th, 2016, the US federal debt was about \$19,600,000,000,000 and the US population was about 323,000,000. What was the average debt per person that day?

a. Calculate the answer using the numbers provided, which are not in scientific notation.

b. First, confirm that the given values in scientific notation are 1.96×10^{13} and 3.23×10^8. Then calculate the answer using scientific notation.

Explanation. We've been asked to answer the same question, but to perform the calculation using two different approaches. In both cases, we'll need to divide the debt by the population.

a. We may need to be working a calculator to handle such large numbers and we have to be careful that we type the correct number of `0`s.

$$\frac{19600000000000}{323000000} \approx 60681.11$$

b. To perform this calculation using scientific notation, our work would begin by setting up the quotient $\frac{1.96 \times 10^{13}}{3.23 \times 10^8}$. Dividing this quotient follows the same process we did with variable expressions of the same format, such as $\frac{1.96w^{13}}{3.23w^8}$. In both situations, we'll divide the coefficients and then use exponent rules to simplify the powers.

$$\begin{aligned}\frac{1.96 \times 10^{13}}{3.23 \times 10^8} &= \frac{1.96}{3.23} \times \frac{10^{13}}{10^8} \\ &\approx 0.6068111 \times 10^5 \\ &\approx 60681.11\end{aligned}$$

The federal debt per capita in the US on September 30th, 2016 was about \$60,681.11 per person. Both calculations give us the same answer, but the calculation relying upon scientific notation has less room for error and allows us to perform the calculation as two smaller steps.

Whenever we multiply or divide numbers that are written in scientific notation, we must separate the calculation for the coefficients from the calculation for the powers of ten, just as we simplified earlier expressions using variables and the exponent rules.

Example 6.2.14

a. Multiply $\left(2 \times 10^5\right)\left(3 \times 10^4\right)$.

b. Divide $\frac{8 \times 10^{17}}{4 \times 10^2}$.

Explanation. We will simplify the significand/mantissa parts as one step and then simplify the powers of 10 as a separate step.

a.

$$\begin{aligned}\left(2 \times 10^5\right)\left(3 \times 10^4\right) &= (2 \times 3) \times \left(10^5 \times 10^4\right)\\ &= 6 \times 10^9\end{aligned}$$

b.

$$\begin{aligned}\frac{8 \times 10^{17}}{4 \times 10^2} &= \frac{8}{4} \times \frac{10^{17}}{10^2}\\ &= 2 \times 10^{15}\end{aligned}$$

Often when we multiply or divide numbers in scientific notation, the resulting value will not be in scientific notation. Suppose we were multiplying $\left(9.3 \times 10^{17}\right)\left(8.2 \times 10^{-6}\right)$ and need to state our answer using scientific notation. We would start as we have previously:

$$\begin{aligned}\left(9.3 \times 10^{17}\right)\left(8.2 \times 10^{-6}\right) &= (9.3 \times 8.2) \times \left(10^{17} \times 10^{-6}\right)\\ &= 76.26 \times 10^{11}\end{aligned}$$

While this is a correct value, it is not written using scientific notation. One way to covert this answer into scientific notation is to turn just the coefficient into scientific notation and momentarily ignore the power of ten:

$$\begin{aligned}&= 76.26 \times 10^{11}\\ &= 7.626 \times 10^1 \times 10^{11}\end{aligned}$$

Now that the coefficient fits into the proper format, we can combine the powers of ten and have our answer written using scientific notation.

$$\begin{aligned}&= 7.626 \times 10^1 \times 10^{11}\\ &= 7.626 \times 10^{12}\end{aligned}$$

Example 6.2.15 Multiply or divide as indicated. Write your answer using scientific notation.

a. $\left(8 \times 10^{21}\right)\left(2 \times 10^{-7}\right)$

b. $\frac{2 \times 10^{-6}}{8 \times 10^{-19}}$

Explanation. Again, we'll separate out the work for the significand/mantissa from the work for the powers of ten. If the resulting coefficient is not between 1 and 10, we'll need to adjust that coefficient to put it into scientific notation.

a.

$$\begin{aligned}\left(8\times10^{21}\right)\left(2\times10^{-7}\right) &= (8\times2)\times\left(10^{21}\times10^{-7}\right)\\ &= 16\times10^{14}\\ &= 1.6\times10^{1}\times10^{14}\\ &= 1.6\times10^{15}\end{aligned}$$

We need to remember to apply the product rule for exponents to the powers of ten.

b.

$$\begin{aligned}\frac{2\times10^{-6}}{8\times10^{-19}} &= \frac{2}{8}\times\frac{10^{-6}}{10^{-19}}\\ &= 0.25\times10^{13}\\ &= 2.5\times10^{-1}\times10^{13}\\ &= 2.5\times10^{12}\end{aligned}$$

There are times where we will have to raise numbers written in scientific notation to a power. For example, suppose we have to find the area of a square whose radius is 3×10^7 feet. To perform this calculation, we first remember the formula for the area of a square, $A = s^2$ and then substitute 3×10^7 for s: $A = \left(3\times10^7\right)^2$. To perform this calculation, we'll need to remember to use the product to a power rule and the power to a power rule:

$$\begin{aligned}A &= \left(3\times10^7\right)^2\\ &= (3)^2\times\left(10^7\right)^2\\ &= 9\times10^{14}\end{aligned}$$

Exercises

Converting To and From Scientific Notation Write the following number in scientific notation.

1. $7500 =$ ☐

2. $850000 =$ ☐

3. $95000 =$ ☐

4. $1500 =$ ☐

5. $0.026 =$ ☐

6. $0.0036 =$ ☐

7. $0.00046 =$ ☐

8. $0.055 =$ ☐

Write the following number in decimal notation without using exponents.

9. $6.5\times10^4 =$ ☐

10. $7.5\times10^2 =$ ☐

11. $8.53\times10^5 =$ ☐

12. $9.53\times10^{4} =$ ☐ **13.** $1.52\times10^{0} =$ ☐ **14.** $2.51\times10^{0} =$ ☐

15. $3.5\times10^{-3} =$ ☐ **16.** $4.5\times10^{-4} =$ ☐ **17.** $5.51\times10^{-2} =$ ☐

18. $6.5\times10^{-3} =$ ☐

Arithmetic with Scientific Notation Multiply the following numbers, writing your answer in scientific notation.

19. $(7\times10^{2})(2\times10^{4}) =$ ☐ **20.** $(8\times10^{4})(5\times10^{3}) =$ ☐ **21.** $(9\times10^{3})(2\times10^{5}) =$ ☐

22. $(2\times10^{2})(4\times10^{5}) =$ ☐ **23.** $(3\times10^{4})(7\times10^{4}) =$ ☐ **24.** $(4\times10^{3})(3\times10^{3}) =$ ☐

Divide the following numbers, writing your answer in scientific notation.

25. $\dfrac{4.5\times10^{5}}{5\times10^{2}} =$ ☐ **26.** $\dfrac{3.5\times10^{6}}{5\times10^{4}} =$ ☐ **27.** $\dfrac{2.4\times10^{4}}{6\times10^{3}} =$ ☐

28. $\dfrac{6.3\times10^{6}}{7\times10^{2}} =$ ☐ **29.** $\dfrac{4.8\times10^{2}}{8\times10^{-4}} =$ ☐ **30.** $\dfrac{3.6\times10^{4}}{9\times10^{-3}} =$ ☐

31. $\dfrac{1.8\times10^{2}}{2\times10^{-4}} =$ ☐ **32.** $\dfrac{1.8\times10^{3}}{3\times10^{-5}} =$ ☐ **33.** $\dfrac{1.2\times10^{-2}}{4\times10^{5}} =$ ☐

34. $\dfrac{4\times10^{-4}}{5\times10^{4}} =$ ☐ **35.** $\dfrac{3\times10^{-3}}{5\times10^{2}} =$ ☐ **36.** $\dfrac{1.8\times10^{-5}}{6\times10^{5}} =$ ☐

Simplify the following expression, writing your answer in scientific notation.

37. $(4\times10^{9})^{3} =$ ☐ **38.** $(5\times10^{6})^{4} =$ ☐ **39.** $(5\times10^{3})^{3} =$ ☐

40. $(2\times10^{9})^{4} =$ ☐ **41.** $(2\times10^{5})^{2} =$ ☐ **42.** $(3\times10^{2})^{4} =$ ☐

6.3 Adding and Subtracting Polynomials

A polynomial is a particular type of algebraic expression used for things all around us.

- A company's sales, s (in millions of dollars), can be modeled by $2.2t + 5.8$, where t stands for the number of years since 2010.
- The height of an object from the ground, h (in feet), launched upward from the top of a building can be modeled by $-16t^2 + 32t + 300$, where t represents the amount of time (in seconds) since the launch.
- The volume of an open-top box with a square base, V (in cubic inches), can be calculated by $30s^2 - \frac{1}{2}s^2$, where s stands for the length of the square base, and the box sides have to be cut from a certain square piece of metal.

All of the expressions above are **polynomials**. In this section, we will learn some basic vocabulary relating to polynomials and we'll then learn how to add and subtract polynomials.

6.3.1 Polynomial Vocabulary

Definition 6.3.2. A **polynomial** is an expression that consists of terms summed together. Each term must be the product of a number and one or more variables raised to whole number powers. Since 0 is a whole number, a term can just be a number. A polynomial may have just one term. The expression 0 is also considered a polynomial, with zero terms.

Some examples of polynomials in one variable are:

$$x^2 - 5x + 2 \qquad t^3 - 1 \qquad 7y.$$

The expression $3x^4y^3 + 7xy^2 - 12xy$ is an example of a polynomial in several variables.

Definition 6.3.3. A **term** of a polynomial is the product of a numerical coefficient and one or more variables raised to whole number powers. Since 0 is a whole number, a term can just be a number.

For example:

- the polynomial $x^2 - 5x + 3$ has three terms: x^2, $-5x$, and 3;
- the polynomial $3x^4 + 7xy^2 - 12xy$ also has three terms;
- the polynomial $t^3 - 1$ has two terms.

Definition 6.3.4. The **coefficient** (or numerical coefficient) of a term is the numerical factor in the term.

For example:

- the coefficient of the term $\frac{4}{3}x^6$ is $\frac{4}{3}$;
- the coefficient of the second term of the polynomial $x^2 - 5x + 3$ is -5;
- the coefficient of the term $\frac{y^7}{4}$ is $\frac{1}{4}$.

Remark 6.3.5. Because variables in polynomials must have whole number exponents, a polynomial will never have a variable in the denominator of a fraction or under a square root (or any other radical).

Checkpoint 6.3.6. Identify which of the following are polynomials and which are not.

a. The expression $-2x^9 - \frac{7}{13}x^3 - 1$ (☐ is ☐ is not) a polynomial.

b. The expression $5x^{-2} - 5x^2 + 3$ (☐ is ☐ is not) a polynomial.

c. The expression $\sqrt{2}x - \frac{3}{5}$ (☐ is ☐ is not) a polynomial.

d. The expression $5x^3 - 5^{-5}x - x^4$ (☐ is ☐ is not) a polynomial.

e. The expression $\frac{25}{x^2} + 23 - x$ (☐ is ☐ is not) a polynomial.

f. The expression $37x^6 - x + 8^{\frac{4}{3}}$ (☐ is ☐ is not) a polynomial.

g. The expression $\sqrt{7x} - 4x^3$ (☐ is ☐ is not) a polynomial.

h. The expression $6x^{\frac{3}{2}} + 1$ (☐ is ☐ is not) a polynomial.

i. The expression $6^x - 3x^6$ (☐ is ☐ is not) a polynomial.

Explanation.

a. The expression $-2x^9 - \frac{7}{13}x^3 - 1$ is a polynomial.

b. The expression $5x^{-2} - 5x^2 + 3$ is not a polynomial because it has negative exponents on a variable.

c. The expression $\sqrt{2}x - \frac{3}{5}$ is a polynomial. Note that *coefficients* can have radicals even though variables cannot, and the square root here is *only* applied to the 2.

d. The expression $5x^3 - 5^{-5}x - x^4$ is a polynomial. Note that *coefficients* can have negative exponents even though variables cannot.

e. The expression $\frac{25}{x^2} + 23 - x$ is not a polynomial because it has a variable in a denominator.

f. The expression $37x^6 - x + 8^{\frac{4}{3}}$ is a polynomial. Note that *coefficients* can have fractional exponents even though variables cannot.

g. The expression $\sqrt{7x} - 4x^3$ is not a polynomial because it has a variable inside a radical.

h. The expression $6x^{\frac{3}{2}} + 1$ is not a polynomial because a variable has a fractional exponent.

i. The expression $6^x - 3x^6$ is not a polynomial because it has a variable in an exponent.

Definition 6.3.7. When a term only has one variable, its **degree** is the exponent on that variable. When a term has more than on variable, its **degree** is the sum of the exponents on the variables in that term. When a term has no variables, its **degree** is 0.

For example:

- the degree of $5x^2$ is 2;
- the degree of $-\frac{4}{7}y^5$ is 5.
- the degree of $-4x^2y^3$ is 5.

Polynomial terms are often classified by their degree. In doing so, we would refer to $5x^2$ as a second-degree term.

Definition 6.3.8. The **degree of a polynomial** is the greatest degree that appears amongst its terms. If the polynomial is just 0, it has no terms, and we say its degree is -1.

Definition 6.3.9. The **leading term** of a polynomial is the term with the greatest degree (assuming there is one, and there is no tie).

For example, the degree of the polynomial $x^2 - 5x + 3$ is 2 because the terms have degrees 2, 1, and 0, and 2 is the largest. Its leading term is x^2. Polynomials are often classified by their degree, and we would say that

$x^2 - 5x + 3$ is a second-degree polynomial.

The coefficient of a polynomial's leading term is called the polynomial's **leading coefficient**. For example, the leading coefficient of $x^2 - 5x + 3$ is 1 (because $x^2 = 1 \cdot x^2$).

Definition 6.3.10. A term with no variable factor is called a **constant term**.

For example, the constant term of the polynomial $x^2 - 5x + 3$ is 3.

There are some special names for polynomials with certain degrees:

- A zero-degree polynomial is called a **constant polynomial** or simply a **constant**.

  An example is the polynomial 7, which has degree zero because it can be viewed as $7x^0$.
- A first-degree polynomial is called a **linear polynomial**.

  An example is $-2x + 7$.
- A second-degree polynomial is called a **quadratic polynomial**.

  An example is $4x^2 - 2x + 7$.
- A third-degree polynomial is called a **cubic polynomial**.

  An example is $x^3 + 4x^2 - 2x + 7$.

Fourth-degree and fifth-degree polynomials are called quartic and quintic polynomials, respectively. If the degree of the polynomial, n, is greater than five, we'll simply call it an nth-degree polynomial. For example, the polynomial $5x^8 - 4x^5 + 1$ is an 8th-degree polynomial.

Remark 6.3.11. To help us recognize a polynomial's degree, it is the standard convention to write a polynomial's terms in order from greatest-degree term to lowest-degree term. When a polynomial is written in this order, it is written in **standard form**. For example, it is standard practice to write $7 - 4x - x^2$ as $-x^2 - 4x + 7$ since $-x^2$ is the leading term. By writing the polynomial in standard form, we can look at the first term to determine both the polynomial's degree and leading term.

There are special names for polynomials with a small number of terms:

Definition 6.3.12.

- A polynomial with one term, such as $3x^5$ or 9, is called a **monomial**.
- A polynomial with two terms, such as $3x^5 + 2x$ or $-2x + 1$, is called a **binomial**.
- A polynomial with three terms, such as $x^2 - 5x + 3$, is called a **trinomial**.

6.3.2 Adding and Subtracting Polynomials

> **Example 6.3.13 Production Costs.** Bayani started a company that is devoted to one product: ketchup. The company's production costs only involve two components: supplies and labor. The cost of supplies, S (in thousands of dollars), can be modeled by $S = 0.05x^2 + 2x + 30$, where x is number of thousands of jars of ketchup produced. The labor cost for his employees, L (in thousands of dollars), can be modeled by $0.1x^2 + 4x$, where x again represents the number of jars they produce (in thousands of jars). Find a model for the company's total production costs.
>
> Since Bayani's company only has these two costs, we can find a model for the total production costs, C (in thousands of dollars), by adding the supply costs and the labor costs:
>
> $$C = \left(0.05x^2 + 2x + 30\right) + \left(0.1x^2 + 4x\right)$$

To finish simplifying our total production cost model, we'll combine the like terms:

$$\begin{aligned}C &= 0.05x^2 + 0.1x^2 + 2x + 4x + 30\\ &= 0.15x^2 + 6x + 30\end{aligned}$$

This simplified model can now calculate Bayani's total production costs C (in thousands of dollars) when the company produces x thousand jars of ketchup.

In short, the process of adding two or more polynomials involves recognizing and then combining the like terms.

Checkpoint 6.3.14. Add the polynomials.

$\left(-6x^2 - 2x\right) + \left(2x^2 + 3x\right)$

Explanation. We combine like terms as follows

$$\begin{aligned}\left(-6x^2 - 2x\right) + \left(2x^2 + 3x\right) &= \left(2x^2 - 6x^2\right) + (3x - 2x)\\ &= -4x^2 + x\end{aligned}$$

Example 6.3.15 Simplify the expression $\left(\frac{1}{2}x^2 - \frac{2}{3}x - \frac{3}{2}\right) + \left(\frac{3}{2}x^2 + \frac{7}{2}x - \frac{1}{4}\right)$.

Explanation.

$$\begin{aligned}&\left(\frac{1}{2}x^2 - \frac{2}{3}x - \frac{3}{2}\right) + \left(\frac{3}{2}x^2 + \frac{7}{2}x - \frac{1}{4}\right)\\ &= \left(\frac{1}{2}x^2 + \frac{3}{2}x^2\right) + \left(\left(-\frac{2}{3}x\right) + \frac{7}{2}x\right) + \left(\left(-\frac{3}{2}\right) + \left(-\frac{1}{4}\right)\right)\\ &= \left(\frac{4}{2}x^2\right) + \left(\left(-\frac{4}{6}x\right) + \frac{21}{6}x\right) + \left(\left(-\frac{6}{4}\right) + \left(-\frac{1}{4}\right)\right)\\ &= \left(2x^2\right) + \left(\frac{17}{6}x\right) + \left(-\frac{7}{4}\right)\\ &= 2x^2 + \frac{17}{6}x - \frac{7}{4}\end{aligned}$$

Example 6.3.16 Profit, Revenue, and Costs. From Example 6.3.13, we know Bayani's ketchup company's production costs, C (in thousands of dollars), for producing x thousand jars of ketchup is modeled by $C = 0.15x^2 + 6x + 30$. The revenue, R (in thousands of dollars), from selling the ketchup can be modeled by $R = 13x$, where x stands for the number of thousands of jars of ketchup sold. The company's net profit can be calculated using the concept:

$$\text{net profit} = \text{revenue} - \text{costs}$$

Assuming all products produced will be sold, a polynomial to model the company's net profit, P (in thousands of dollars) is:

$$\begin{aligned}P &= R - C\\ &= (13x) - \left(0.15x^2 + 6x + 30\right)\\ &= 13x - 0.15x^2 - 6x - 30\end{aligned}$$

$$\begin{aligned}
&= -0.15x^2 + (13x + (-6x)) - 30\\
&= -0.15x^2 + 7x - 30
\end{aligned}$$

The key distinction between the addition and subtraction of polynomials is that when we subtract a polynomial, we must subtract each term in that polynomial.

Remark 6.3.17. Notice that our first step in simplifying the expression in Example 6.3.16 was to subtract *every* term in the second expression. We can also think of this as distributing a factor of -1 across the second polynomial, $0.15x^2 + 6x + 30$, and then adding these terms as follows:

$$\begin{aligned}
P &= R - C\\
&= (13x) - \bigl(0.15x^2 + 6x + 30\bigr)\\
&= 13x + (-1)(0.15x^2) + (-1)(6x) + (-1)(30)\\
&= 13x - 0.15x^2 - 6x - 30\\
&= -0.15x^2 + (13x + (-6x)) - 30\\
&= -0.15x^2 + 7x - 30
\end{aligned}$$

Example 6.3.18 Subtract $\bigl(5x^3 + 4x^2 - 6x\bigr) - \bigl(-3x^2 + 9x - 2\bigr)$.

Explanation. We must first subtract every term in $\bigl(-3x^2 + 9x - 2\bigr)$ from $\bigl(5x^3 + 4x^2 - 6x\bigr)$. Then we can combine like terms.

$$\begin{aligned}
&\bigl(5x^3 + 4x^2 - 6x\bigr) - \bigl(-3x^2 + 9x - 2\bigr)\\
&= 5x^3 + 4x^2 - 6x + 3x^2 - 9x + 2\\
&= 5x^3 + \bigl(4x^2 + 3x^2\bigr) + (-6x + (-9x)) + 2\\
&= 5x^3 + 7x^2 - 15x + 2
\end{aligned}$$

Checkpoint 6.3.19. Subtract the polynomials.

$\bigl(-3x^6 + 8x^4\bigr) - (-2x + 3)$

Explanation. We combine like terms as follows

$$\bigl(-3x^6 + 8x^4\bigr) - (-2x + 3) = -3x^6 + 8x^4 + 2x - 3$$

Let's look at one more example involving multiple variables. Remember that like terms must have the same variable(s) with the same exponent.

Example 6.3.20 Subtract $\bigl(3x^2y + 8xy^2 - 17y^3\bigr) - \bigl(2x^2y + 11xy^2 + 4y^2\bigr)$.

Explanation. Again, we'll begin by subtracting each term in $\bigl(2x^2y + 11xy^2 + 4y^2\bigr)$. Once we've done this, we'll need to identify and combine like terms.

$$\begin{aligned}
&\bigl(3x^2y + 8xy^2 - 17y^3\bigr) - \bigl(2x^2y + 11xy^2 + 4y^2\bigr)\\
&= 3x^2y + 8xy^2 - 17y^3 - 2x^2y - 11xy^2 - 4y^2\\
&= \bigl(3x^2y + \bigl(-2x^2y\bigr)\bigr) + \bigl(8xy^2 + \bigl(-11xy^2\bigr)\bigr) + \bigl(-17y^3\bigr) + \bigl(-4y^2\bigr)
\end{aligned}$$

$$= x^2y - 3xy^2 - 17y^3 - 4y^2$$

6.3.3 Evaluating Polynomial Expressions

Evaluating expressions was introduced in Section 2.1, and involves replacing the variable(s) in an expression with specific numbers and calculating the result. Here, we will look at evaluating polynomial expressions.

Example 6.3.21 Evaluate the expression $-12y^3 + 4y^2 - 9y + 2$ for $y = -5$.

Explanation. We will replace y with -5 and simplify the result:

$$\begin{aligned} -12y^3 + 4y^2 - 9y + 2 &= -12(-5)^3 + 4(-5)^2 - 9(-5) + 2 \\ &= -12(-125) + 4(25) + 45 + 2 \\ &= 1647 \end{aligned}$$

Remark 6.3.22. Recall that $(-5)^2$ and -5^2 are not the same expressions. The first expression, $(-5)^2$, represents the number -5 squared, and is $(-5)(-5) = 25$. The second expression, -5^2, is the *opposite* of the number 5 squared, and is $-5^2 = -(5 \cdot 5) = -25$.

Example 6.3.23 Evaluate the expression $C = 0.15x^2 + 6x + 30$ from Example 6.3.13 for $x = 10$ and explain what this means in context.

Explanation. We will replace x with 10:

$$\begin{aligned} C &= 0.15x^2 + 6x + 30 \\ &= 0.15(10)^2 + 6(10) + 30 \\ &= 105 \end{aligned}$$

In context, we can interpret this as it costing \$105,000 to produce 10,000 jars of ketchup.

Checkpoint 6.3.24. Evaluate the following expressions.

a. Evaluate $(-y)^2$ when $y = -2$.

$(-y)^2 = \boxed{}$

b. Evaluate $(-y)^3$ when $y = -2$.

$(-y)^3 = \boxed{}$

Explanation.

a. $$\begin{aligned} (-y)^2 &= \left(-1(-2)\right)^2 \\ &= (2)^2 \\ &= 4 \end{aligned}$$

b. $$\begin{aligned} (-y)^3 &= \left(-1(-2)\right)^3 \\ &= (2)^3 \\ &= 8 \end{aligned}$$

Exercises

Review and Warmup

1. List the terms in each expression.

a. $-3.4t + 0.1x + 8.2t + 3.6s$

b. $-2.8s^2 - 0.8z$

c. $7.5s + 7.6 - 5.4t^2$

d. $2.9y + 4t^2 + 5.2y + 2.5s$

2. List the terms in each expression.

a. $-1.7t^2 + 8.3z^2 + 6.2y + 2.4x$

b. $3.2z^2 - 2.4 + 8.9t - 8z^2$

c. $-6.6x$

d. $-6.6z^2 + 5.6y^2 + 0.4x^2$

3. List the terms in each expression.

a. $-0.1t$

b. $-1.6s + 4.2s + 1.2x^2$

c. $7.2x + 4.8 + 3x$

d. $-7.3s^2 - 7.4$

4. List the terms in each expression.

a. $1.5t^2 + 6.6y$

b. $2.2z + 1.4x$

c. $-7.9t^2$

d. $-3.7t - 1.4x + 8.1s^2$

Simplify each expression, if possible, by combining like terms.

5. a. $3t - 3t^2$

b. $-6x - 7 - 7x + 8x$

c. $-4z^2 + 9 + s^2$

d. $-4s - 4s^2 + 8x + 7y$

6. a. $5z^2 + 5z^2 - 2z^2$

b. $-3y - 5 + 4z$

c. $-4z^2 + 7y + 8y - 9$

d. $-4t^2 - 2y^2$

7. a. $7z + \frac{1}{2}x + \frac{5}{4}x$

b. $\frac{9}{4}y^2 + 1$

c. $-x - \frac{7}{5}x$

d. $\frac{2}{7}s + \frac{1}{2}s - \frac{9}{2}s$

8. a. $\frac{8}{9}z^2 + 4t - \frac{6}{5}t$

b. $-2s^2 + \frac{1}{9}s + 3z - \frac{4}{3}x$

c. $-3s + 8s^2 - \frac{6}{5}s^2$

d. $\frac{1}{8}y^2 + \frac{2}{9}s^2 - 2s^2 - \frac{9}{7}y$

Vocabulary Questions Is the following expression a monomial, binomial, or trinomial?

9. $3r^{14} - 17r^6$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree ☐

10. $-12t^{11} - 3t^2$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree ☐

11. 37 is a (☐ monomial ☐ binomial ☐ trinomial) of degree ☐

12. 2 is a (☐ monomial ☐ binomial ☐ trinomial) of degree ☐

13. $-18y^{11} - 9y^{7} - 20y^{6}$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree $\boxed{\quad\quad}$

14. $-20r^{10} - 2r^{9} - 10r^{2}$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree $\boxed{\quad\quad}$

15. $-y^{4} + 4y^{7} - 19y^{2}$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree $\boxed{\quad\quad}$

16. $-15y^{6} - 20y^{7} + 10y^{4}$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree $\boxed{\quad\quad}$

17. $12r^{9}$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree $\boxed{\quad\quad}$

18. $-3r^{17}$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree $\boxed{\quad\quad}$

Find the degree of the following polynomial.

19. $13x^{7}y^{6} - 17x^{4}y^{2} - 10x^{2} + 17$

The degree of this polynomial is $\boxed{\quad\quad}$.

20. $17x^{7}y^{9} + 10xy^{4} + 7x^{2} + 6$

The degree of this polynomial is $\boxed{\quad\quad}$.

Simplifying Polynomials Add the polynomials.

21. $(-10x - 2) + (-9x - 3)$

22. $(-7x - 10) + (-9x - 2)$

23. $\left(-5x^{2} + 4x\right) + \left(9x^{2} + 7x\right)$

24. $\left(-3x^{2} + x\right) + \left(-3x^{2} + 7x\right)$

25. $\left(-6x^{2} - 8x - 3\right) + \left(-4x^{2} + 9x - 3\right)$

26. $\left(7x^{2} + 3x - 3\right) + \left(9x^{2} - 4x + 7\right)$

27. $\left(4r^{3} - 8r^{2} - 4\right) + \left(7r^{3} - 6r^{2} - 10\right)$

28. $\left(-10t^{3} - 5t^{2} + 7\right) + \left(-8t^{3} + 9t^{2} + 4\right)$

29. $\left(-7t^{6} - 3t^{4} - 4t^{2}\right) + \left(9t^{6} - 3t^{4} - 4t^{2}\right)$

30. $\left(4x^{6} - 9x^{4} + 7x^{2}\right) + \left(10x^{6} + 6x^{4} + 9x^{2}\right)$

31. $\left(0.8x^{5} - 0.5x^{4} + 0.2x^{2} - 0.4\right) + \left(-0.4x^{5} + 0.9x^{3} - 0.3\right)$

32. $\left(0.1y^{5} + 0.3y^{4} - 0.3y^{2} - 0.5\right) + \left(0.6y^{5} - 0.8y^{3} + 0.4\right)$

33. $\left(-5x^{3} - 2x^{2} - 4x + \frac{1}{2}\right) + \left(-5x^{3} + 9x^{2} - 5x + \frac{3}{2}\right)$

34. $\left(6x^{3} + 4x^{2} - 8x + \frac{1}{2}\right) + \left(4x^{3} - 6x^{2} + 6x + \frac{3}{2}\right)$

Subtract the polynomials.

35. $(2x+1)-(9x-6)$

36. $(4x-7)-(-3x-6)$

37. $\left(6x^2+7x\right)-\left(5x^2+9x\right)$

38. $\left(9x^2-7x\right)-\left(3x^2-7x\right)$

39. $\left(-2x^5-7x^4\right)-\left(-5x^2+4\right)$

40. $\left(-8x^5-7x^4\right)-(6x+10)$

41. $\left(-4x^3+5x^2-8x+(-5)\right)-\left(-9x^2+9x+8\right)$

42. $\left(-5x^3+9x^2-8x+6\right)-\left(10x^2-2x+3\right)$

43. $\left(6x^2-4x-8\right)-\left(-4x^2-3x+1\right)$

44. $\left(-7x^2+8x-8\right)-\left(-8x^2+7x+10\right)$

45. $\left(-2r^6-8r^4-5r^2\right)-\left(-4r^6-2r^4+r^2\right)$

46. $\left(8t^6-5t^4+6t^2\right)-\left(-4t^6+4t^4-10t^2\right)$

Add or subtract the given polynomials as indicated.

47. $\left[4t^{13}-2t^7+2t^6-\left(-10t^{13}+3t^7-7t^6\right)\right]-\left(-3t^{13}-5t^7-7t^6\right)$

48. $\left[x^6-7x^4+9x^3-\left(-10x^6+8x^4-6x^3\right)\right]-\left(-8x^6-2x^4-3x^3\right)$

49. $\left[7x^5-4x^4+7x^3-\left(-10x^5+3x^4-5x^3\right)\right]-\left[-5x^5-8x^4+3x^3+\left(-2x^5-6x^4-5x^3\right)\right]$

50. $\left[4x^6+4x^5-\left(-10x^6-4x^5\right)\right]-\left[-10x^6+8x^5+\left(-6x^6-6x^5\right)\right]$

51. $\left(5x^8y^4-7xy\right)+\left(-2x^8y^4-8xy\right)$

52. $\left(6x^3y^7+2xy\right)+\left(10x^3y^7+9xy\right)$

53. $\left(-5x^4y^8+4xy-9\right)+\left(-8x^4y^8-5xy-6\right)$

54. $\left(8x^3y^6-9xy+8\right)+\left(10x^3y^6+2xy-2\right)$

55. $\left(9x^9y^8+4x^2y^2+2xy\right)+\left(-10x^9y^8-3x^2y^2-8xy\right)$

56. $\left(-10x^8y^9+8x^5y^3-4xy\right)+\left(-3x^8y^9-5x^5y^3+9xy\right)$

57. $\left(2x^8-2xy+7y^6\right)-\left(-10x^8+4xy-10y^6\right)$

58. $\left(3x^7-6xy-10y^9\right)-\left(10x^7-5xy-7y^9\right)$

59. $\left(-4x^8y^9+10x^4y^3+4xy\right)-\left(-10x^8y^9+6x^4y^3-3xy\right)$

60. $\left(5x^7y^9+5x^2y^4-7xy\right)-\left(10x^7y^9-7x^2y^4-9xy\right)$

61. $\left(-6x^4-8y^8\right)-\left(-9x^4+7x^7y^8+9x^4y^8-6y^8\right)$

62. $\left(-7x^3-3y^8\right)-\left(9x^3+8x^4y^8-3x^3y^8-2y^8\right)$

63. Subtract $-8r^{15}-6r^{14}-5r^{12}$ from the sum of $8r^{15}-5r^{14}+8r^{12}$ and $-3r^{15}+3r^{14}-3r^{12}$.

64. Subtract $-4t^8-10t^7$ from the sum of $5t^8+5t^7$ and $-3t^8-2t^7$.

65. Subtract $-10x^3y^5-5xy$ from $-9x^3y^5-2xy$

66. Subtract $2x^5y^8+9xy$ from $9x^5y^8+2xy$

Evaluating Polynomials

67. Evaluate the expression x^2:

a. When $x=5$, $x^2=$ ☐

b. When $x=-4$, $x^2=$ ☐

68. Evaluate the expression x^2:

a. When $x=2$, $x^2=$ ☐

b. When $x=-7$, $x^2=$ ☐

69. Evaluate the expression $-y^2$:

a. When $y=4$, $-y^2=$ ☐

b. When $y=-2$, $-y^2=$ ☐

70. Evaluate the expression $-y^2$:

a. When $y=3$, $-y^2=$ ☐

b. When $y=-4$, $-y^2=$ ☐

71. Evaluate the expression r^3:

a. When $r=2$, $r^3=$ ☐

b. When $r=-5$, $r^3=$ ☐

72. Evaluate the expression r^3:

a. When $r=4$, $r^3=$ ☐

b. When $r=-3$, $r^3=$ ☐

73. Evaluate the following expressions.

a. Evaluate $(-t)^2$ when $t=-4$.

$(-t)^2=$ ☐

b. Evaluate $(-t)^3$ when $t=-4$.

$(-t)^3=$ ☐

74. Evaluate the following expressions.

a. Evaluate $(-t)^2$ when $t=-4$.

$(-t)^2=$ ☐

b. Evaluate $(-t)^3$ when $t=-4$.

$(-t)^3=$ ☐

75. Evaluate the expression $\frac{1}{5}(x+1)^2 - 6$ when $x = -6$.

76. Evaluate the expression $\frac{1}{8}(x+1)^2 - 4$ when $x = -9$.

77. Evaluate the expression $\frac{1}{6}(x+1)^2 - 9$ when $x = -7$.

78. Evaluate the expression $\frac{1}{3}(x+2)^2 - 6$ when $x = -5$.

79. Evaluate the expression $-16t^2+64t+128$ when $t = -2$.

80. Evaluate the expression $-16t^2+64t+128$ when $t = 5$.

81. Evaluate the expression $-16t^2+64t+128$ when $t = 4$.

82. Evaluate the expression $-16t^2+64t+128$ when $t = 2$.

Applications of Simplifying Polynomials The formula

$$y = \frac{1}{2}a\,t^2 + v_0\,t + y_0$$

gives the vertical position of an object, at time t, thrown with an initial velocity v_0, from an initial position y_0 in a place where the acceleration of gravity is a. The acceleration of gravity on earth is $-9.8\ \frac{\text{m}}{\text{s}^2}$. It is negative, because we consider the upward direction as positive in this situation, and gravity pulls down.

83. What is the height of a baseball thrown with an initial velocity of $v_0 = 95\ \frac{\text{m}}{\text{s}}$, from an initial position of $y_0 = 92$ m, and at time $t = 19$ s?

Nineteen seconds after the baseball was thrown, it was ☐ high in the air.

84. What is the height of a baseball thrown with an initial velocity of $v_0 = 50\ \frac{\text{m}}{\text{s}}$, from an initial position of $y_0 = 74$ m, and at time $t = 4$ s?

Four seconds after the baseball was thrown, it was ☐ high in the air.

85. An auto company's sales volume can be modeled by $4.1x^2 + 7.1x + 3.3$, and its cost can be modeled by $2.3x^2 + 2.7x + 3.3$, where x represents the number of cars produced, and y stands for money in thousand dollars. We can calculate the company's net profit by subtracting cost from sales. Find the polynomial which models the company's sales in thousands of dollars.

The company's profit can be modeled by ☐ dollars.

86. An auto company's sales volume can be modeled by $7.9x^2 + 1.8x + 3.2$, and its cost can be modeled by $2.7x^2 - 3.2x + 3.2$, where x represents the number of cars produced, and y stands for money in thousand dollars. We can calculate the company's net profit by subtracting cost from sales. Find the polynomial which models the company's sales in thousands of dollars.

The company's profit can be modeled by ☐ dollars.

87. A handyman is building two pig pens sharing the same side. Assume the length of the shared side is x meters. The cost of building one pen would be $33.5x^2 - 2x + 4$ dollars, and the cost of building the other pen would be $30.5x^2 + 2x - 42$ dollars. What's the total cost of building those two pens?

A polynomial representing the total cost of building those two pens is dollars.

88. A handyman is building two pig pens sharing the same side. Assume the length of the shared side is x meters. The cost of building one pen would be $23x^2 + 10x + 2.5$ dollars, and the cost of building the other pen would be $33.5x^2 - 10x + 30.5$ dollars. What's the total cost of building those two pens?

A polynomial representing the total cost of building those two pens is dollars.

89. A farmer is building fence around a triangular area. The cost of building the shortest side is $40x$ dollars, where x stands for the length of the side in feet. The cost of building the other two sides can be modeled by $8x^2 - 0.5x + 40$ dollars and $4x^3 - 3.5x + 20$ dollars, respectively. What's the total cost of building fence for all three sides?

The cost of building fence for all three sides would be dollars.

90. A farmer is building fence around a triangular area. The cost of building the shortest side is $45x$ dollars, where x stands for the length of the side in feet. The cost of building the other two sides can be modeled by $5x^2 + 3.5x + 35$ dollars and $4x^3 + 1.5x + 25$ dollars, respectively. What's the total cost of building fence for all three sides?

The cost of building fence for all three sides would be dollars.

91. An architect is designing a house on an empty plot. The area of the plot can be modeled by the polynomial $5x^4 + 5x^2 - 4.5x$, and the area of the house's base can be modeled by $6x^3 - 4.5x - 40$. The rest of the plot is the yard. What's the yard's area?

The area of the yard can be modeled by the polynomial .

92. An architect is designing a house on an empty plot. The area of the plot can be modeled by the polynomial $6x^4 + 16x^2 + 4x$, and the area of the house's base can be modeled by $5x^3 + 4x - 5$. The rest of the plot is the yard. What's the yard's area?

The area of the yard can be modeled by the polynomial .

6.4 Multiplying Polynomials

Previously, we have learned to multiply monomials in Section 6.1 (such as $(4xy)\left(3x^2\right)$) and to add and subtract polynomials in Section 6.3 (such as $(4x^2 - 3x) + (5x^2 + x - 2)$). In this section, we will learn how to multiply polynomials.

> **Example 6.4.2 Revenue.** Avery owns a local organic jam company that currently sells about 1500 jars a month at a price of \$13 per jar. Avery has found that every time they raise the price by 25 cents a jar, they will sell 50 fewer jars of jam each month.
>
> In general, this company's revenue can be calculated by multiplying the cost per jar by the total number of jars of jam sold.
>
> If we let x represent the number of 25-cent increases in the price, then the price per jar will be the current price of thirteen dollars/jar plus x times 0.25 dollars/jar, or $13 + 0.25x$.
>
> Continuing with x representing the number of 25-cent increases in the price, we know the company will sell 50 fewer jars each time the price increases by 25 cents. The number of jars the company will sell will be the 1500 they currently sell each month, minus 50 jars times x, the number of price increases. This gives us the expression $1500 - 50x$ to represent how many jars the company will sell after x 25-cent price increases.
>
> Combining this, we can now write a formula for our revenue model:
>
> $$\begin{aligned} \text{revenue} &= \left(\text{price per item}\right)(\text{number of items sold}) \\ R &= (13 + 0.25x)(1500 - 50x) \end{aligned}$$
>
> To simplify the expression $(13 + 0.25x)(1500 - 50x)$, we'll need to multiply $13 + 0.25x$ by $1500 - 50x$. In this section, we'll learn how to multiply these two expressions that each have multiple terms.

6.4.1 Review of the Distributive Property

The first step in almost every polynomial multiplication exercise will be a step of distribution. Let's quickly review the distributive property from Section 2.10, which states that $a(b + c) = ab + ac$ where a, b, and c are real numbers or variable expressions.

When we multiply a monomial with a binomial, we apply this property by distributing the monomial to each term in the binomial. For example,

$$\begin{aligned} -4x(3x^2 + 5) &= (-4x) \cdot \left(3x^2\right) + (-4x) \cdot (5) \\ &= -12x^3 - 20x \end{aligned}$$

A visual approach to the distributive property is to treat the product as finding a rectangle's area. Such rectangles are referred to as **generic rectangles** and they can be used to model polynomial multiplication.

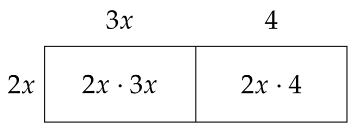


Figure 6.4.3: A Generic Rectangle Modeling $2x(3x + 4)$

The big rectangle consists of two smaller rectangles. The big rectangle's area is $2x(3x + 4)$, and the sum of those two smaller rectangles is $2x \cdot 3x + 2x \cdot 4$. Since the sum of the areas of those two smaller rectangles is the same as the bigger rectangle's area, we have:

$$\begin{aligned} 2x(3x + 4) &= 2x \cdot 3x + 2x \cdot 4 \\ &= 6x^2 + 8x \end{aligned}$$

Generic rectangles are frequently used to visualize the distributive property.

Multiplying a monomial with a polynomial involves two steps: distribution and monomial multiplication. We also need to rely on the rules of exponents 6.1.15 when simplifying.

Checkpoint 6.4.4. Simplify the expression so that the result does not have parentheses.

a. $2(xy) =$

b. $2(x + y) =$

Explanation.

a. $2\left(xy\right) = 2xy$

b. $2\left(x + y\right) = 2x + 2y$

Checkpoint 6.4.5. Multiply the polynomials.

$-3x\left(-6x + 6\right) =$

Explanation. We multiply the monomial by each term in the binomial, using the properties of exponents to help us.

$$-3x(-6x + 6) = 18x^2 - 18x$$

Checkpoint 6.4.6. Multiply the polynomials.

$(4a^4)(10a^3 - 5a^{10}b^6 - 8b^7) =$

Explanation. We multiply the polynomials using the rule $a^m \cdot a^n = a^{m+n}$ to guide us.

$$\begin{aligned} \left(4a^4\right)\left(10a^3 - 5a^{10}b^6 - 8b^7\right) &= 40a^{4+3} - 20a^{4+10}b^6 - 32a^4b^7 \\ &= 40a^7 - 20a^{14}b^6 - 32a^4b^7 \end{aligned}$$

Note that we are using the *distributive* property of multiplication in this problem: $x(y + z) = xy + xz$.

Remark 6.4.7. We can use the distributive property when multiplying on either the left or the right. This means that we can state $a(b + c) = ab + ac$, or that $(b + c)a = ba + ca$, which is equivalent to $ab + ac$. As an example,

$$\begin{aligned} (3x^2 + 5)(-4x) &= (3x^2) \cdot (-4x) + (5) \cdot (-4x) \\ &= -12x^3 - 20x \end{aligned}$$

6.4.2 Approaches to Multiplying Binomials

Multiplying Binomials Using Distribution Whether we're multiplying a monomial with a polynomial or two larger polynomials together, the first step to carrying out the multiplication is a step of distribution.

We'll start with multiplying binomials and then move to working with larger polynomials.

We know we can distribute the 3 in $(x+2)3$ to obtain $(x+2)\cdot 3 = x\cdot 3 + 2\cdot 3$. We can actually distribute anything across $(x+2)$. For example:

$$(x+2)\text{😺} = x\cdot\text{😺} + 2\cdot\text{😺}$$

With this in mind, we can begin multiplying $(x+2)(x+3)$ by distributing the $(x+3)$ across $(x+2)$:

$$(x+2)(x+3) = x(x+3) + 2(x+3)$$

To finish multiplying, we'll continue by distributing again, but this time across $(x+3)$:

$$\begin{aligned}(x+2)(x+3) &= x(x+3) + 2(x+3)\\ &= x\cdot x + x\cdot 3 + 2\cdot x + 2\cdot 3\\ &= x^2 + 3x + 2x + 6\\ &= x^2 + 5x + 6\end{aligned}$$

To multiply a binomial by another binomial, we simply had to repeat the step of distribution and simplify the resulting terms. In fact, multiplying any two polynomials will rely upon these same steps.

Multiplying Binomials Using FOIL While multiplying two binomials requires two applications of the distributive property, people often remember this distribution process using the mnemonic FOIL. FOIL refers to the pairs of terms from each binomial that end up distributed to each other.

If we take another look at the example we just completed, $(x+2)(x+3)$, we can highlight how the FOIL process works. FOIL is the acronym for "First, Outer, Inner, Last".

$$\begin{aligned}(x+2)(x+3) &= (\overbrace{x\cdot x}^{\text{F}}) + (\overbrace{3\cdot x}^{\text{O}}) + (\overbrace{2\cdot x}^{\text{I}}) + (\overbrace{2\cdot 3}^{\text{L}})\\ &= x^2 + 3x + 2x + 6\\ &= x^2 + 5x + 6\end{aligned}$$

F: x^2 The x^2 term was the result of the product of *first* terms from each binomial.

O: $3x$ The $3x$ was the result of the product of the *outer* terms from each binomial. This was from the x in the front of the first binomial and the 3 in the back of the second binomial.

I: $2x$ The $2x$ was the result of the product of the *inner* terms from each binomial. This was from the 2 in the back of the first binomial and the x in the front of the second binomial.

L: 6 The constant term 6 was the result of the product of the *last* terms of each binomial.

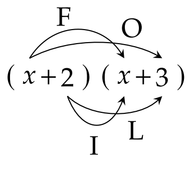


Figure 6.4.8: Using FOIL Method to multiply $(x+2)(x+3)$

Multiplying Binomials Using Generic Rectangles We can also approach this same example using the generic rectangle method. To use generic rectangles, we treat $x + 2$ as the base of a rectangle, and $x + 3$ as the height. Their product, $(x+2)(x+3)$, represents the rectangle's area. The next diagram shows how to set up generic rectangles to multiply $(x + 2)(x + 3)$.

	x	2
x		
3		

Figure 6.4.9: Setting up Generic Rectangles to Multiply $(x + 2)(x + 3)$

The big rectangle consists of four smaller rectangles. We will find each small rectangle's area in the next diagram by the formula area = base · height.

	x	2
x	x^2	$2x$
3	$3x$	6

Figure 6.4.10: Using Generic Rectangles to Multiply $(x + 2)(x + 3)$

To finish finding this product, we need to add the areas of the four smaller rectangles:

$$\begin{aligned}(x+2)(x+3) &= x^2 + 3x + 2x + 6\\ &= x^2 + 5x + 6\end{aligned}$$

Notice that the areas of the four smaller rectangles are exactly the same as the four terms we obtained using distribution, which are also the same four terms that came from the FOIL method. Both the FOIL method and generic rectangles approach are different ways to represent the distribution that is occurring.

Example 6.4.11 Multiply $(2x - 3y)(4x - 5y)$ using distribution.

Explanation. To use the distributive property to multiply those two binomials, we'll first distribute the second binomial across $(2x - 3y)$. Then we'll distribute again, and simplify the terms that result.

$$\begin{aligned}(2x-3y)(4x-5y) &= 2x(4x-5y) - 3y(4x-5y)\\ &= 8x^2 - 10xy - 12xy + 15y^2\\ &= 8x^2 - 22xy + 15y^2\end{aligned}$$

Example 6.4.12 Multiply $(2x - 3y)(4x - 5y)$ using FOIL.

Explanation. First, Outer, Inner, Last: Either with arrows on paper or mentally in our heads, we'll pair up the four pairs of monomials and multiply those pairs together.

$$\begin{aligned}(2x-3y)(4x-5y) &= \overbrace{(2x\cdot 4x)}^{\text{F}} + \overbrace{(2x\cdot(-5y))}^{\text{O}} + \overbrace{(-3y\cdot 4x)}^{\text{I}} + \overbrace{(-3y\cdot(-5y))}^{\text{L}}\\ &= 8x^2-10xy-12xy+15y^2\\ &= 8x^2-22xy+15y^2\end{aligned}$$

Example 6.4.13 Multiply $(2x-3y)(4x-5y)$ using generic rectangles.

Explanation. We begin by drawing four rectangles and marking their bases and heights with terms in the given binomials:

	$2x$	$-3y$
$4x$		
$-5y$		

Figure 6.4.14: Setting up Generic Rectangles to Multiply $(2x-3y)(4x-5y)$

Next, we calculate each rectangle's area by multiplying its base with its height:

	$2x$	$-3y$
$4x$	$8x^2$	$-12xy$
$-5y$	$-10xy$	$15y^2$

Figure 6.4.15: Using Generic Rectangles to Multiply $(2x-3y)(4x-5y)$

Finally, we add up all rectangles' area to find the product:

$$\begin{aligned}(2x-3y)(4x-5y) &= 8x^2-10xy-12xy+15y^2\\ &= 8x^2-22xy+15y^2\end{aligned}$$

6.4.3 More Examples of Multiplying Binomials

When multiplying binomials, all of the approaches shown in Subsection 6.4.2 will have the same result. The FOIL method is the most direct and will be used in the examples that follow.

Checkpoint 6.4.16. Multiply the polynomials.

$(5x-4)(x+9) =$

Explanation. We use the FOIL technique: *F*irst *O*utside *I*nside *L*ast.

$$\begin{aligned}(5x-4)(x+9) &= 5x^2+45x-4x-36\\ &= 5x^2+41x-36\end{aligned}$$

Checkpoint 6.4.17. Multiply the polynomials.

$\left(4x^2-5\right)\left(6x^2-2\right) =$

Explanation. We use the FOIL technique: *F*irst *O*utside *I*nside *L*ast.

$$\begin{aligned}\left(4x^2-5\right)\left(6x^2-2\right) &= 24x^4-8x^2-30x^2+10\\ &= 24x^4-38x^2+10\end{aligned}$$

Example 6.4.18 Multiply and simplify the formula for Avery's jam company's revenue, R (in dollars), from Example 6.4.2 where $R = (13+0.25x)(1500-50x)$ and x represents the number of 25-cent price increases to the selling price of a jar of jam.

Explanation. To multiply this, we'll use FOIL:

$$\begin{aligned}R &= (13+0.25x)(1500-50x)\\ &= (13\cdot 1500)+(13\cdot(-50x))+(0.25x\cdot 1500)+(0.25x\cdot(-50x))\\ &= 19500-650x+375x-12.5x^2\\ &= -12.5x^2-275x+19500\end{aligned}$$

Example 6.4.19 Tyrone is an artist and he sells each of his paintings for \$200. Currently, he can sell 100 paintings per year. Thus, his annual income from paintings is $200\cdot 100 = 20000$ dollars. He plans to raise the price. However, for each \$20 price increase per painting, his customers would buy 5 fewer paintings annually.

Assume Tyrone would raise the price of his paintings x times, each time by \$20. Use an expanded polynomial to represent his new income per year.

Explanation. Currently, each painting costs \$200. After raising the price x times, each time by \$20, each painting's new price would be $200+20x$ dollars.

Currently, Tyrone sells 100 paintings per year. After raising the price x times, each time selling 5 fewer paintings, he would sell $100-5x$ paintings per year.

His annual income can be calculated by multiplying each painting's price by the number of paintings he would sell:

$$\begin{aligned}\text{annual income} &= (200+20x)(100-5x)\\ &= 200(100)+200(-5x)+20x(100)+20x(-5x)\\ &= 20000-1000x+2000x-100x^2\\ &= -100x^2+1000x+20000\end{aligned}$$

After raising the price x times, each time by \$20, Tyrone's annual income from paintings would be $-100x^2+1000x+20000$ dollars.

6.4.4 Multiplying Polynomials Larger Than Binomials

The foundation for multiplying any pair of polynomials is distribution and monomial multiplication. Whether we are working with binomials, trinomials, or larger polynomials, the process is fundamentally the same.

Example 6.4.20 Multiply $(x+5)\left(x^2-4x+6\right)$.

We can approach this product using either distribution generic rectangles. We cannot directly use the FOIL method, although it can be helpful to draw arrows to the six pairs of products that will occur.

Using the distributive property, we begin by distributing across $\left(x^2-4x+6\right)$, perform a second step of distribution, and then combine like terms.

$$\begin{aligned}(x+5)\left(x^2-4x+6\right) &= x\left(x^2-4x+6\right)+5\left(x^2-4x+6\right)\\ &= x\cdot x^2 - x\cdot 4x + x\cdot 6 + 5\cdot x^2 - 5\cdot 4x + 5\cdot 6\\ &= x^3-4x^2+6x+5x^2-20x+30\\ &= x^3+x^2-14x+30\end{aligned}$$

With the foundation of monomial multiplication and understanding how distribution applies in this context, we are able to find the product of any two polynomials.

Checkpoint 6.4.21. Multiply the polynomials.

$(a-3b)(a^2+7ab+9b^2) =$

Explanation. We multiply the polynomials by using the terms from $a-3b$ successively.

$$\begin{aligned}(a-3b)\left(a^2+7ab+9b^2\right) &= aa^2 + a\cdot 7ab + a\cdot 9b^2 - 3ba^2 - 3b\cdot 7ab - 3b\cdot 9b^2\\ &= a^3+4a^2b-12ab^2-27b^3\end{aligned}$$

Exercises

Review and Warmup Use the properties of exponents to simplify the expression.

1. $r^{14}\cdot r^7$

2. $x^{16}\cdot x^{19}$

3. $(-5x^{19})\cdot(-4x^{13})$

4. $(9y^2)\cdot(5y^6)$

5. $\left(-8r^3\right)^3$

6. $\left(-4y^4\right)^3$

7. Count the number of terms in each expression.

a. $-2y-5y$

b. $7y-x$

c. $7s$

d. $4z-s+3y^2$

8. Count the number of terms in each expression.

a. $3y-8t-3s$

b. $s^2+6s-5x+5z$

c. $6s-s^2$

d. $-4y^2-5x+8+7x$

9. List the terms in each expression.

a. $-5.5y^2-0.5s+8.2x$

b. $1.1y+3.1t$

c. $2.2s^2-0.5z-7.3y+8.5s$

d. $8.3x^2-5.2y-5y^2$

10. List the terms in each expression.

a. $6y + 8.6 + 8.6x^2$

b. $-2.5z^2$

c. $0.9y - 0.1t$

d. $-5.5x^2 + 2.1s + 4.5y$

11. List the terms in each expression.

a. $-8.9y^2 + 4.2z - 8.2 + 3.4z$

b. $5.9y^2 - 7y^2 - 7.8y$

c. $-7.5z + 1.4y - 3t$

d. $1.8z - 2s + 8.7z + 6.6x^2$

12. List the terms in each expression.

a. $-8.9y + 6.9z^2 + 7.3 - 3.8x$

b. $7.5y + 6.5y$

c. $-4.6s^2$

d. $-6.2y^2 - 5.3y - 6s^2$

13. Simplify each expression, if possible, by combining like terms.

a. $-9y^2 - 8y^2$

b. $-5t^2 + 4t^2$

c. $-9s^2 + 7t^2 - 4t$

d. $-7x - z + 9z$

14. Simplify each expression, if possible, by combining like terms.

a. $-6y + 5$

b. $-6t^2 - 2x^2 - 2x^2 - 7x^2$

c. $-t^2 + 2$

d. $-y + 4y^2$

15. Simplify each expression, if possible, by combining like terms.

a. $y - 9y$

b. $-y^2 + \frac{9}{5}z^2 - \frac{7}{9}t^2$

c. $-\frac{4}{9}t^2 + \frac{2}{3}t + t$

d. $\frac{1}{4}x^2 - 9z^2 + 4x^2 + \frac{2}{3}z$

16. Simplify each expression, if possible, by combining like terms.

a. $\frac{1}{3}x - \frac{7}{9}s$

b. $-\frac{2}{9}z + \frac{9}{4}y$

c. $-\frac{3}{5}z + \frac{5}{2}y - \frac{9}{8}s - \frac{1}{4}z$

d. $\frac{1}{3}z^2 + \frac{1}{8}z^2 + \frac{6}{5}z^2$

Multiplying Monomials with Binomials Multiply the polynomials.

17. $-x(x-7) =$

18. $x(x+5) =$

19. $-7x(3x-3) =$

20. $8x(-6x+3) =$

21. $8x^2(x+5) =$

22. $10x^2(x-9) =$

23. $7x^2\left(-7x^2+2x\right) =$

24. $-4x^2\left(-5x^2-6x\right) =$

25. $10y^2\left(-2y^2+8y-9\right) =$

26. $-7y^2\left(-9y^2-6y+2\right) =$

27. $(-6x^{8}y^{15})(-2x^{17}+5y^{16}) =$

28. $(7x^{10}y^{19})(-9x^{13}+2y^{5}) =$

29. $(8a^{12}b^{8})(4a^{19}b^{13}-2a^{4}b^{15}) =$

30. $(-9a^{13}b^{16})(8a^{6}b^{3}+2a^{15}b^{19}) =$

31. $(10a^{7})(-3a^{6}-10a^{5}b^{8}-8b^{9}) =$

32. $(-2a^{9})(-6a^{10}+10a^{10}b^{8}+9b^{8}) =$

Applications of Multiplying Monomials with Binomials

33. A rectangle's length is 3 feet shorter than twice its width. If we use w to represent the rectangle's width, use a polynomial to represent the rectangle's area in expanded form.

area = square feet

34. A rectangle's length is 4 feet shorter than 5 times its width. If we use w to represent the rectangle's width, use a polynomial to represent the rectangle's area in expanded form.

area = square feet

35. A triangle's height is 6 feet longer than 4 times its base. If we use b to represent the triangle's base, use a polynomial to represent the triangle's area in expanded form. A triangle's area can be calculated by $A = \frac{1}{2}bh$, where b stands for base, and h stands for height.

area = square feet

36. A triangle's height is 6 feet longer than twice its base. If we use b to represent the triangle's base, use a polynomial to represent the triangle's area in expanded form. A triangle's area can be calculated by $A = \frac{1}{2}bh$, where b stands for base, and h stands for height.

area = square feet

37. A trapezoid's top base is 2 feet longer than its height, and its bottom base is 6 feet longer than its height. If we use h to represent the trapezoid's height, use a polynomial to represent the trapezoid's area in expanded form. A trapezoid's area can be calculated by $A = \frac{1}{2}(a+b)h$, where a stands for the top base, b stands for the bottom base, and h stands for height.

area = square feet

38. A trapezoid's top base is 10 feet longer than its height, and its bottom base is 6 feet longer than its height. If we use h to represent the trapezoid's height, use a polynomial to represent the trapezoid's area in expanded form. A trapezoid's area can be calculated by $A = \frac{1}{2}(a+b)h$, where a stands for the top base, b stands for the bottom base, and h stands for height.

area = square feet

Multiplying Binomials

Multiply the polynomials.

39. $(t+1)(t+5) =$

40. $(t+8)(t+9) =$

41. $(5x+3)(x+7) =$

42. $(2x+8)(x+4) =$

43. $(y+7)(y-9) =$

44. $(y+4)(y-5) =$

45. $(r-1)(r-5) =$

46. $(r-4)(r-7) =$

47. $(3t+8)(t+8) =$

48. $(6t+3)(5t+8) =$

49. $(4t-6)(5t-1) =$

50. $(3x-10)(2x-3) =$

51. $(10x-6)(x-1) =$

52. $(6y-2)(y-4) =$

53. $(3y-8)(y+4) =$

54. $(9r-3)(r+1) =$

55. $(4r-9)(5r^2-4) =$

56. $(2t-5)(4t^2-6) =$

57. $(9t^3+10)(t^2+3) =$

58. $(6t^3+4)(t^2+10) =$

59. $(2x^2-3)(5x^2-4) =$

60. $(5x^2-8)(3x^2-4) =$

61. $(a-4b)(a-7b) =$

62. $(a-5b)(a+2b) =$

63. $(a-8b)(6a+7b) =$

64. $(a+5b)(7a+4b) =$

65. $(8a-4b)(2a+6b) =$

66. $(9a+8b)(8a-6b) =$

67. $(10ab-3)(4ab+6) =$

68. $(2ab-6)(10ab-6) =$

69. $4(x+10)(x+9) =$

70. $-2(y-3)(y+3) =$

71. $y(y+9)(y+6) =$

72. $-3r(r-9)(r-1) =$

73. $-3(4r+3)(r-1) =$

74. $4(2t-1)(t-1) =$

Applications of Multiplying Binomials

75. An artist sells his paintings at \$19.00 per piece. Currently, he can sell 130 paintings per year. Thus, his annual income from paintings is $19 \cdot 130 = 2470$ dollars. He plans to raise the price. However, for each \$3.00 of price increase per painting, his customers would buy 6 fewer paintings annually.

Assume the artist would raise the price of his painting x times, each time by \$3.00. Use an expanded polynomial to represent his new income per year.

new annual income =

76. An artist sells his paintings at \$20.00 per piece. Currently, he can sell 110 paintings per year. Thus, his annual income from paintings is $20 \cdot 110 = 2200$ dollars. He plans to raise the price. However, for each \$5.00 of price increase per painting, his customers would buy 10 fewer paintings annually.

Assume the artist would raise the price of his painting x times, each time by \$5.00. Use an expanded polynomial to represent his new income per year.

new annual income =

77. A rectangle's base can be modeled by $x + 2$ meters, and its height can be modeled by $x + 4$ meters. Use a polynomial to represent the rectangle's area in expanded form.

area = ☐
square meters

78. A rectangle's base can be modeled by $x + 3$ meters, and its height can be modeled by $x - 8$ meters. Use a polynomial to represent the rectangle's area in expanded form.

area = ☐
square meters

Multiplying Larger Polynomials Multiply the polynomials.

79. $(-3x - 2)\left(x^2 + 3x + 5\right) =$ ☐

80. $(3x + 4)\left(x^2 - 3x - 2\right) =$ ☐

81. $(-4x + 5)\left(3x^3 - 2x^2 - 4x + 5\right) =$ ☐

82. $(-4x - 3)\left(2x^3 + 5x^2 + 3x - 2\right) =$ ☐

83. $\left(x^2 + 5x - 5\right)\left(x^2 + 3x - 3\right) =$ ☐

84. $\left(x^2 - 5x + 2\right)\left(x^2 - 3x + 3\right) =$ ☐

85. $(a - 10b)(a^2 - 7ab + 4b^2) =$ ☐

86. $(a + 2b)(a^2 - 2ab - 4b^2) =$ ☐

87. $(a + b - 3)(a + b + 3) =$ ☐

88. $(a + b - 4)(a + b + 4) =$ ☐

Challenge

89. Fill in the blanks with algebraic expressions that make the equation true. You may not use 0 or 1 in any of the blank spaces. An example is ? + ? = $8x$, where one possible answer is $3x + 5x = 8x$. There are infinitely many correct answers to this problem. *Be creative.* After finding a correct answer, see if you can come up with a different answer that is also correct.

a. ☐ + ☐ $= -13xy$

b. ☐ + ☐ $= -19x^{15}y^9$

c. ☐ · ☐ · ☐ · ☐ · ☐ $= 17x^{50}y^{75}$

6.5 Special Cases of Multiplying Polynomials

Since we are now able to multiply polynomials together, we will look at a few special cases of polynomial multiplication.

6.5.1 Squaring a Binomial

Example 6.5.2 To "square a binomial" is to take a binomial and multiply it by itself. We know that exponent notation means that $4^2 = 4 \cdot 4$. Applying this to a binomial, we'll see that $(x+4)^2 = (x+4)(x+4)$. To expand this expression, we'll simply distribute $(x+4)$ across $(x+4)$:

$$\begin{aligned}(x+4)^2 &= (x+4)(x+4)\\ &= x^2 + 4x + 4x + 16\\ &= x^2 + 8x + 16\end{aligned}$$

Similarly, to expand $(y-7)^2$, we'll have:

$$\begin{aligned}(y-7)^2 &= (y-7)(y-7)\\ &= y^2 - 7y - 7y + 49\\ &= y^2 - 14y + 49\end{aligned}$$

These two examples might look like any other example of multiplying binomials, but looking closely we can see that something very specific (or *special*) happened. Focusing on the original expression and the simplified one, we can see that a specific pattern occurred in each:

$$\begin{aligned}(x+4)^2 &= x^2 + 4x + 4x + 4 \cdot 4\\ (x+4)^2 &= x^2 + 2(4x) + 4^2\end{aligned}$$

And:

$$\begin{aligned}(y-7)^2 &= y^2 - 7y - 7y + 7 \cdot 7\\ (y-7)^2 &= y^2 - 2(7y) + 7^2\end{aligned}$$

Notice that the two middle terms are not only the same, they are also exactly the product of the two terms in the binomial. Furthermore, the last term is the square of the second term in each original binomial.

What we're seeing is a pattern that relates to two important phrases: The process is called **squaring a binomial**, and the result is called a **perfect square trinomial**. The first phrase is a description of what we're doing, we are literally squaring a binomial. The second phrase is a description of what you end up with. This second name will become important in a future chapter.

Example 6.5.3 The general way this pattern is presented is by squaring the two most general binomials possible, $(a+b)$ and $(a-b)$. We will establish the pattern for $(a+b)^2$ and $(a-b)^2$. Once we have done so, we will be able to substitute anything in place of a and b and rely upon the general pattern to simplify squared binomials.

We first must expand $(a+b)^2$ as $(a+b)(a+b)$ and then we can multiply those binomials:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2+ab+ba+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

Notice the final simplification step was to add $ab+ba$. Since these are like terms, we can combine them into $2ab$.

Similarly, we can find a general formula for $(a-b)^2$:

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a^2-ab-ba+b^2\\ &= a^2-2ab+b^2\end{aligned}$$

Fact 6.5.4 Squaring a Binomial Formulas. *If a and b are real numbers or variable expressions, then we have the following formulas:*

$$(a+b)^2 = a^2+2ab+b^2$$

$$(a-b)^2 = a^2-2ab+b^2$$

These formulas will allow us to multiply this type of special product more quickly.

Remark 6.5.5. Notice that when both $(a+b)^2$ and $(a-b)^2$ are expanded in Example 6.5.3, the last term was a *positive* b^2 in both. This is because any number or expression, regardless of its sign, is positive after it is squared.

6.5.2 Further Examples of Squaring Binomials

Example 6.5.6 Expand $(2x-3)^2$ using the squaring a binomial formula.

For this example we need to recognize that to apply the formula $(a-b)^2 = a^2-2ab+b^2$ in this situation, $a=2x$ and $b=3$. Expanding this, we have:

$$\begin{aligned}(a-b)^2 &= a^2 \quad -2ab \quad +b^2\\ (2x-3)^2 &= (2x)^2 - 2(2x)(3) + (3)^2\\ &= 4x^2 \quad -12x \quad +9\end{aligned}$$

Remark 6.5.7. While we rely on the formula for squaring a binomial in Example 6.5.6, we will often omit the step of formally writing the formula and jump to the simplification, in this way:

$$(2x-3)^2 = 4x^2-12x+9$$

Example 6.5.8 Multiply the following using the squaring a binomial formula:

a. $(5xy+1)^2$

b. $4(3x-7)^2$

Explanation.

a. $$\begin{aligned}(5xy+1)^2 &= (5xy)^2 + 2(5xy)(1) + 1^2\\ &= 25x^2y^2 + 10xy + 1\end{aligned}$$

b. With this expression, we will first note that the factor of 4 is *outside* the portion of the expression that is squared. Using the order of operations, we will first expand $(3x-7)^2$ and then multiply that expression by 4:

$$\begin{aligned}4(3x-7)^2 &= 4\left((3x)^2 - 2(3x)(7) + 7^2\right)\\ &= 4\left(9x^2 - 42x + 49\right)\\ &= 36x^2 - 168x + 196\end{aligned}$$

Example 6.5.9 A circle's area can be calculated by the formula

$$A = \pi r^2$$

where A stands for area, and r stands for radius. If a certain circle's radius can be modeled by $x-5$ feet, use an expanded polynomial to model the circle's area.

Explanation. The circle's area would be:

$$\begin{aligned}A &= \pi r^2\\ &= \pi(x-5)^2\\ &= \pi\left[(x)^2 - 2(x)(5) + (5)^2\right]\\ &= \pi\left[x^2 - 10x + 25\right]\\ &= \pi x^2 - 10\pi x + 25\pi\end{aligned}$$

The circle's area can be modeled by $\pi x^2 - 10\pi x + 25\pi$ square feet.

Checkpoint 6.5.10. Expand the square of a *bi*nomial.

$\left(y^3-12\right)^2 =$ ☐

Explanation. We use the FOIL technique: *F*irst *O*utside *I*nside *L*ast

$$\begin{aligned}\left(y^3-12\right)^2 &= \left(y^3-12\right)\left(y^3-12\right)\\ &= y^6 - 12y^3 - 12y^3 + 144\\ &= y^6 - 24y^3 + 144\end{aligned}$$

Alternatively, we might observe that this is the square of *the difference of two terms*, in which case we may use the formula

$$(a-b)^2 = a^2 - 2ab + b^2$$

and write

$$\begin{aligned}\left(y^3-12\right)\left(y^3-12\right) &= \left(y^3\right)^2 - 2\cdot y^3\cdot 12 + 12^2\\ &= y^6 - 24y^3 + 144\end{aligned}$$

which is the same result we obtained using the FOIL method.

6.5.3 The Product of the Sum and Difference of Two Terms

To identify the next “special case” for multiplying polynomials, we’ll look at a couple of examples.

Example 6.5.11 Multiply the following binomials:

a. $(x+5)(x-5)$

b. $(y-8)(y+8)$

Explanation. We can approach these as using distribution, FOIL, or generic rectangles, and obtain the following:

a. $$\begin{aligned}(x+5)(x-5)&=x^2-5x+5x-25\\&=x^2-25\end{aligned}$$

b. $$\begin{aligned}(y+8)(y-8)&=y^2-8y+8y-4\\&=y^2-64\end{aligned}$$

Notice that for each of these products, we multiplied the sum of two terms by the difference of the *same* two terms. Notice also in these three examples that once these expressions were multiplied, the two middle terms were opposites and thus canceled to zero.

These pairs, generally written as $(a+b)$ and $(a-b)$, are known as **conjugates**. If we multiply $(a+b)(a-b)$, we can see this general pattern more clearly:

$$\begin{aligned}(a+b)(a-b)&=a^2-ab+ab-b^2\\&=a^2-b^2\end{aligned}$$

As with the previous special case, this one also has two names. This can be called the **product of the sum and difference of two terms**, because this pattern is built on multiplying two binomials that have the same two terms, except one binomial is a sum and the other binomial is a difference. The second name is a **difference of squares**, because the end result of the multiplication is a binomial that is the difference of two perfect squares. As before, the second name will become useful in a future chapter when using exactly the technique described in this section will be pertinent.

Fact 6.5.12 The Product of the Sum and Difference of Two Terms Formula. *If a and b are real numbers or variable expressions, then we have the following formula:*

$$(a+b)(a-b)=a^2-b^2$$

Checkpoint 6.5.13. Multiply the polynomials.

$(4x+2)(4x-2)=$

Explanation. We use the FOIL technique: *F*irst *O*utside *I*nside *L*ast

$$\begin{aligned}(4x+2)(4x-2)&=16x^2-8x+8x-4\\&=16x^2-4\end{aligned}$$

Alternatively, we might observe that this is the product of *the sum and difference of two terms*, in which case we may use the formula

$$(a-b)(a+b)=a^2-b^2$$

and write

$$\begin{aligned}(4x+2)(4x-2)&=(4x)^2-2^2\\&=16x^2-4\end{aligned}$$

which is the same result we obtained using the FOIL method.

Example 6.5.14 Multiply the following using Fact 6.5.12.

a. $(4x-7y)(4x+7y)$

b. $-2(3x+1)(3x-1)$

Explanation. The first step to using this method is to identify the values of a and b.

a. In this instance, $a=4x$ and $b=7y$. Using the formula,

$$\begin{aligned}(4x-7y)(4x+7y)&=(4x)^2-(7y)^2\\&=16x^2-49y^2\end{aligned}$$

b. In this instance, we have a constant factor as well as a product in the form $(a+b)(a-b)$. We will first expand $(3x+1)(3x-1)$ by identifying $a=3x$ and $b=1$ and using the formula. Then we will multiply the factor of -2 through this expression. So,

$$\begin{aligned}-2(3x+1)(3x-1)&=-2\left((3x)^2-1^2\right)\\&=-2\left(9x^2-1\right)\\&=-18x^2+2\end{aligned}$$

Checkpoint 6.5.15. Multiply the polynomials.

$\left(x^7-2\right)\left(x^7+2\right)=$

Explanation. We use the FOIL technique: *F*irst *O*utside *I*nside *L*ast

$$\begin{aligned}\left(x^7-2\right)\left(x^7+2\right)&=x^{14}+2x^7-2x^7-4\\&=x^{14}-4\end{aligned}$$

Alternatively, we might observe that this is the product of *the sum and difference of two terms*, in which case we may use the formula

$$(a-b)(a+b)=a^2-b^2$$

and write

$$\begin{aligned}\left(x^7-2\right)\left(x^7+2\right)&=\left(x^7\right)^2-2^2\\&=x^{14}-4\end{aligned}$$

which is the same result we obtained using the FOIL method.

If a and b are real numbers or variable expressions, then we have the following formulas:

Squaring a Binomial (Sum) $(a+b)^2=a^2+2ab+b^2$

Squaring a Binomial (Difference) $(a-b)^2=a^2-2ab+b^2$

Product of the Sum and Difference of Two Terms $(a+b)(a-b)=a^2-b^2$

List 6.5.16: Special Cases of Multiplication Formulas

Warning 6.5.17 Common Mistakes. We've found that

$$(a+b)(a-b) = a^2 - b^2$$

However,

$$(a-b)^2 \neq a^2 - b^2 \text{ because } (a-b)^2 = a^2 - 2ab + b^2$$

Similarly,

$$(a+b)^2 \neq a^2 + b^2 \text{ because } (a+b)^2 = a^2 + 2ab + b^2$$

6.5.4 Binomials Raised to Other Powers

Example 6.5.18 Simplify the expression $(x+5)^3$ into an expanded polynomial.

Before we start expanding this expression, it is important to recognize that $(x+5)^3 \neq x^3 + 5^3$. We can see that this doesn't work by inputting 1 for x and applying the order of operations:

$$\begin{aligned}(1+5)^3 &= 6^3 \\ &= 216\end{aligned} \qquad\qquad \begin{aligned}1^3 + 5^3 &= 1 + 125 \\ &= 126\end{aligned}$$

With this in mind, we will need to rely on distribution to expand this expression. The first step in expanding $(x+5)^3$ is to remember that the exponent of 3 indicates that

$$(x+5)^3 = \overbrace{(x+5)(x+5)(x+5)}^{\text{3 times}}$$

Once we rewrite this in an expanded form, we next multiply the two binomials on the left and then finish by multiplying that result by the remaining binomial:

$$\begin{aligned}(x+5)^3 &= [(x+5)(x+5)](x+5) \\ &= \left[x^2 + 10x + 25\right](x+5) \\ &= x^3 + 5x^2 + 10x^2 + 50x + 25x + 125 \\ &= x^3 + 15x^2 + 75x + 125\end{aligned}$$

Checkpoint 6.5.19. Simplify the given expression into an expanded polynomial.

$(2y-6)^3 =$

Explanation. The main thing to notice on this problem is that we can write $(2y-6)^3$ as

$$(2y-6)^3 = (2y-6)\,(2y-6)^2$$

This means that we can use the FOIL technique on the second binomial multiplication, and then multiply the first factor $2y-6$ by the result.

$$\begin{aligned}(2y-6)^3 &= (2y-6)\left[(2y-6)\,(2y-6)\right] \\ &= (2y-6)\left[4y^2 - 12y - 12y + 36\right] \\ &= (2y-6)\left[4y^2 - 24y + 36\right] \\ &= 8y^3 - 48y^2 + 72y - 24y^2 + 144y - 216 \\ &= 8y^3 - 72y^2 + 216y - 216\end{aligned}$$

You might like to know that the formula for the cube of the *difference* of two terms is

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

If you have the time, you can verify that this formula works in this problem as an exercise.

If we wanted to expand a binomial raised to any power, we always start by rewriting the expression without an exponent.

To multiply $(x-3)^4$, we'd start by rewriting $(x-3)^4$ in expanded form as:

$$(x-3)^4 = \overbrace{(x-3)(x-3)(x-3)(x-3)}^{\text{4 times}}$$

We will then multiply pairs of polynomials from the left to the right.

$$\begin{aligned}(x-3)^4 &= [(x-3)(x-3)](x-3)(x-3)\\ &= \left[(x^2-6x+9)(x-3)\right](x-3)\\ &= \left[x^3-9x^2+27x-27\right](x-3)\\ &= x^2-9x^3+27x^2-27x-3x^3+27x^2-81x+81\\ &= x^4-12x^3+54x^2-108x+81\end{aligned}$$

Exercises

Review and Warmup Use the properties of exponents to simplify the expression.

1. $\left(3t^{12}\right)^4$

2. $\left(5r^2\right)^3$

3. $(2x)^2$

4. $(5r)^2$

5. $\left(-10t^6\right)^3$

6. $\left(-6r^7\right)^2$

7. $-4\left(-3x^8\right)^3$

8. $-5\left(-8x^{10}\right)^3$

Simplify each expression, if possible, by combining like terms.

9.
- a. $-9x+3s$
- b. $-4x^2+8x^2$
- c. $5x^2+8x^2$
- d. $2y^2-5y^2$

10.
- a. $-8x^2-7$
- b. $-3t^2-3s^2$
- c. $-9t-6t^2$
- d. $-3z^2+6y^2$

11.
- a. $-6x+t$
- b. $-2y^2-9y^2+7z^2$
- c. $4x^2-2-7t$
- d. $3z^2+9y^2$

12.
- a. $-4x^2-9x^2$
- b. $-s+2s^2$
- c. $3x+6t-3x+5s$
- d. $-7x-8x+5t+9s$

13. Determine if the following statements are true or false.

a. $(a-b)^2 = a^2 - b^2$

(☐ True ☐ False)

b. $(a+b)^2 = a^2 + b^2$

(☐ True ☐ False)

c. $(a+b)(a-b) = a^2 - b^2$

(☐ True ☐ False)

14. Determine if the following statements are true or false.

a. $(2(a-b))^2 = 4(a-b)^2$

(☐ True ☐ False)

b. $2(a+b)^2 = 2a^2 + 2b^2$

(☐ True ☐ False)

c. $2(a+b)(a-b) = 2a^2 - 2b^2$

(☐ True ☐ False)

Perfect Square Trinomial Formula Expand the square of a *bi*nomial.

15. $(r+1)^2 =$ ☐

16. $(r+8)^2 =$ ☐

17. $(5r+1)^2 =$ ☐

18. $(2t+5)^2 =$ ☐

19. $(t-4)^2 =$ ☐

20. $(x-7)^2 =$ ☐

21. $(10x-3)^2 =$ ☐

22. $(7y-9)^2 =$ ☐

23. $(4y^2-5)^2 =$ ☐

24. $(10r^2-1)^2 =$ ☐

25. $(r^6+3)^2 =$ ☐

26. $(r^9-7)^2 =$ ☐

27. $(9a-10b)^2 =$ ☐

28. $(10a+6b)^2 =$ ☐

29. $(2ab-3)^2 =$ ☐

30. $(3ab-9)^2 =$ ☐

31. $(x^2+4y^2)^2 =$ ☐

32. $(x^2-5y^2)^2 =$ ☐

Difference of Squares Formula Multiply the polynomials.

33. $(r+9)(r-9) =$ ☐

34. $(r-1)(r+1) =$ ☐

35. $(2r+1)(2r-1) =$ ☐

36. $(5t+10)(5t-10) =$ ☐

37. $(5-5t)(5+5t) =$ ☐

38. $(1+9x)(1-9x) =$ ☐

39. $(x^4+6)(x^4-6) =$ ☐

40. $(y^8-3)(y^8+3) =$ ☐

41. $(2y^{10}-12)(2y^{10}+12) =$ ☐

42. $(5r^7-1)(5r^7+1) =$ ☐

43. $(1+7r^5)(1-7r^5) =$ ☐

44. $(1-13r^3)(1+13r^3) =$ ☐

45. $(9x+7y)(9x-7y) =$

46. $(10x-4y)(10x+4y) =$

47. $(ab+2)(ab-2) =$

48. $(ab+3)(ab-3) =$

49. $3\left(y-7\right)\left(y+7\right) =$

50. $5\left(y+11\right)\left(y-11\right) =$

51. $2\,(4r-6)\,(4r+6) =$

52. $6\,(3r+3)\,(3r-3) =$

53. $2\,(r+9)^2 =$

54. $4\,(t+6)^2 =$

55. $2\,(3t+9)^2 =$

56. $6\,(9x+4)^2 =$

57. $(x^2-3y^2)(x^2+3y^2) =$

58. $(x^2-4y^2)(x^2+4y^2) =$

59. $(5x^8+9y^4)(5x^8-9y^4) =$

60. $(6x^6+5y^4)(6x^6-5y^4) =$

61. $(7x^4y^3-2y^4)(7x^4y^3+2y^4) =$

62. $(8x^2y^6+8y^4)(8x^2y^6-8y^4) =$

Binomials Raised to Other Powers Simplify the given expression into an expanded polynomial.

63. $(t+3)^3 =$

64. $(t+2)^3 =$

65. $(x-5)^3 =$

66. $(x-3)^3 =$

67. $\left(6y+5\right)^3 =$

68. $\left(5y+3\right)^3 =$

69. $(3r-6)^3 =$

70. $(6r-2)^3 =$

6.6 Dividing by a Monomial

Now that we know how to add, subtract, and multiply polynomials, we will learn how to divide a polynomial by a monomial.

6.6.1 Dividing a Polynomial by a Monomial

One example of dividing a polynomial is something we already studied in Section 4.7, when we rewrote an equation in standard form in slope-intercept form. We'll briefly review this process.

> **Example 6.6.2** Rewrite $4x - 2y = 10$ in slope-intercept form.
>
> In being asked to rewrite this equation in slope-intercept form, we're really being asked to solve the equation $4x - 2y = 10$ for y.
>
> $$\begin{aligned} 7x - 2y &= 10 \\ 7x - 2y - 7x &= 10 - 7x \\ -2y &= -7x + 10 \\ \frac{-2y}{-2} &= \frac{-7x + 10}{-2} \\ y &= -\frac{7}{2}x - 5 \end{aligned}$$
>
> In the final step of work, we divided each term on the right side of the equation by -2.
>
> This is an example of polynomial division that we have already done. We'll extend it to more complicated examples, many of which involve dividing polynomials by variables (instead of just numbers).

Like polynomial multiplication, polynomial division will rely upon distribution.

It's important to remember that dividing by a number c is the same as multiplying by the reciprocal $\frac{1}{c}$:

$$\frac{8}{2} = \frac{1}{2} \cdot 8 \quad \text{and} \quad \frac{9}{3} = \frac{1}{3} \cdot 9$$

If we apply this idea to a situation involving polynomials, say $\frac{a+b}{c}$, we can show that distribution works for division as well:

$$\begin{aligned} \frac{a+b}{c} &= \frac{1}{c} \cdot (a + b) \\ &= \frac{1}{c} \cdot a + \frac{1}{2} \cdot b \\ &= \frac{a}{c} + \frac{b}{c} \end{aligned}$$

Once we recognize that the division distributes just as multiplication distributed, we are left with individual monomial pairs that we will divide.

> **Example 6.6.3** Simplify $\dfrac{2x^3 + 4x^2 - 10x}{2}$.
>
> The first step will be to recognize that the 2 we're dividing by will be divided into every term of the

numerator. Once we recognize that, we will simply perform that division.

$$\begin{aligned}\frac{2x^3+4x^2-10x}{2} &= \frac{2x^3}{2}+\frac{4x^2}{2}+\frac{-10x}{2}\\ &= x^3+2x^2-5x\end{aligned}$$

Example 6.6.4 Simplify $\dfrac{15x^4-9x^3+12x^2}{3x^2}$

Explanation. The key to simplifying $\frac{15x^4-9x^3+12x^2}{3x^2}$ is to recognize that each term in the numerator will be divided by $3x^2$. In doing this, each coefficient and exponent will change. Performing this division by distributing, we get:

$$\begin{aligned}\frac{15x^4-9x^3+12x^2}{3x^2} &= \frac{15x^4}{3x^2}+\frac{-9x^3}{3x^2}+\frac{12x^2}{3x^2}\\ &= 5x^2-3x+4\end{aligned}$$

Remark 6.6.5. Once you become comfortable with this process, you will often leave out the step where we wrote out the distribution. You will do the distribution in your head and this will often become a one-step problem. Here's how Example 6.6.4 would be visualized:

$$\frac{15x^4-9x^3+12x^2}{3x^2} = \square x^{\square} - \square x^{\square} + \square x^{\square}$$

And when calculated, we'd get:

$$\frac{15x^4-9x^3+12x^2}{3x^2} = 5x^2-3x+4$$

(Note that $\frac{x^2}{x^2}$ is technically x^0, which is equivalent to 1.)

Example 6.6.6 Simplify $\dfrac{20x^3y^4+30x^2y^3-5x^2y^2}{-5xy^2}$

Explanation.

$$\begin{aligned}\frac{20x^3y^4+30x^2y^3-5x^2y^2}{-5xy^2} &= \frac{20x^3y^4}{-5xy^2}+\frac{30x^2y^3}{-5xy^2}+\frac{-5x^2y^2}{-5xy^2}\\ &= -4x^2y^2-6xy+x\end{aligned}$$

Checkpoint 6.6.7. Simplify the following expression

$$\frac{18r^{20}+18r^{16}-54r^{14}}{-6r^2} = \boxed{}$$

Explanation. We divide each term by $-6r^2$ as follows.

$$\begin{aligned}\frac{18r^{20}+18r^{16}-54r^{14}}{-6r^2} &= \frac{18r^{20}}{-6r^2}+\frac{18r^{16}}{-6r^2}+\frac{-54r^{14}}{-6r^2}\\ &= -\frac{18}{6}r^{18}-\frac{18}{6}r^{14}+\frac{54}{6}r^{12}\\ &= -3r^{18}-3r^{14}+9r^{12}\end{aligned}$$

Example 6.6.8 A rectangular prism's volume can be calculated by the formula

$$V = Bh$$

where V stands for volume, B stands for base area, and h stands for height. A certain rectangular prism's volume can be modeled by $4x^3-6x^2+8x$ cubic units. If its height is $2x$ units, find the prism's base area.

Explanation. Since $V = Bh$, we can use $B = \frac{V}{h}$ to calculate the base area. After substitution, we have:

$$\begin{aligned} B &= \frac{V}{h} \\ &= \frac{4x^3-6x^2+8x}{2x} \\ &= \frac{4x^3}{2x} - \frac{6x^2}{2x} + \frac{8x}{2x} \\ &= 2x^2-3x+4 \end{aligned}$$

The prism's base area can be modeled by $2x^2-3x+4$ square units.

Exercises

Review and Warmup Use the properties of exponents to simplify the expression.

1. $\frac{t^{17}}{t^5} =$ ☐

2. $\frac{y^{19}}{y^{17}} =$ ☐

3. $\frac{8r^{16}}{2r^4} =$ ☐

4. $\frac{-12x^{11}}{4x^3} =$ ☐

5. $\frac{5x^{14}}{30x^3} =$ ☐

6. $\frac{7x^4}{28x^3} =$ ☐

Rewrite the expression simplified and using only positive exponents.

7. $\frac{y^6}{y^{11}} =$ ☐

8. $\frac{r^{37}}{r^{42}} =$ ☐

9. $\frac{-3r^9}{11r^{25}} =$ ☐

10. $\frac{-13t^7}{3t^9} =$ ☐

Dividing Polynomials by Monomials Simplify the following expression

11. $\frac{56t^5+7t^3}{7} =$ ☐

12. $\frac{-6x^{13}-18x^4}{3} =$ ☐

13. $\frac{-66x^{10}+78x^8+78x^7}{6x^3} =$ ☐

14. $\frac{-54y^{10}+27y^7-54y^6}{-9y^3} =$ ☐

15. $\dfrac{-21y^{18} - 14y^4}{7y} = \boxed{}$

16. $\dfrac{-24y^8 - 18y^4}{2y} = \boxed{}$

17. $\dfrac{25r^{20} - 60r^{18} - 60r^9 - 60r^8}{5r^4} = \boxed{}$

18. $\dfrac{-40r^{11} - 100r^{10} - 10r^9 - 40r^8}{10r^4} = \boxed{}$

19. $\dfrac{18x^2y^2 - 26xy - 10xy^2}{2xy} = \boxed{}$

20. $\dfrac{24x^2y^2 + 8xy + 22xy^2}{2xy} = \boxed{}$

21. $\dfrac{60x^{25}y^{18} + 25x^{19}y^9 - 35x^{15}y^7}{-5x^5y^2} = \boxed{}$

22. $\dfrac{72x^{18}y^{14} - 96x^{17}y^6 - 48x^{10}y^9}{-8x^5y^2} = \boxed{}$

23. $\dfrac{18y^8 + 12y^5 + 12y^4}{6y^2} = \boxed{}$

24. $\dfrac{60y^{15} + 110y^{14} - 40y^{13}}{-10y^2} = \boxed{}$

Application Problems

25. A rectangular prism's volume can be calculated by the formula $V = Bh$, where V stands for volume, B stands for base area, and h stands for height. A certain rectangular prism's volume can be modeled by $16x^5 - 24x^3 - 40x$ cubic units. If its height is $4x$ units, find the prism's base area.

$B = \boxed{}$ square units

26. A rectangular prism's volume can be calculated by the formula $V = Bh$, where V stands for volume, B stands for base area, and h stands for height. A certain rectangular prism's volume can be modeled by $28x^6 - 20x^4 - 40x^2$ cubic units. If its height is $4x$ units, find the prism's base area.

$B = \boxed{}$ square units

27. A cylinder's volume can be calculated by the formula $V = Bh$, where V stands for volume, B stands for base area, and h stands for height. A certain cynlinder's volume can be modeled by $12\pi x^5 - 36\pi x^4 + 40\pi x^3$ cubic units. If its base area is $4\pi x^2$ square units, find the prism's height.

$h = \boxed{}$ units

28. A cylinder's volume can be calculated by the formula $V = Bh$, where V stands for volume, B stands for base area, and h stands for height. A certain cynlinder's volume can be modeled by $45\pi x^6 + 20\pi x^4 - 50\pi x^2$ cubic units. If its base area is $5\pi x^2$ square units, find the prism's height.

$h = \boxed{}$ units

6.7 Exponents and Polynomials Chapter Review

6.7.1 Exponent Rules

In Section 6.1 we covered the exponent rules and how to use them.

Example 6.7.1 Quotients and Exponents. Let t and q be real numbers, where $q \neq 0$ and $t \neq 0$. Find another way to write $\left(\frac{q^9}{t \cdot q^3}\right)^2$.

Explanation. We first use the Quotient Rule, then the Quotient to a Power Rule, then the Power to a Power Rule.

$$\begin{aligned}\left(\frac{q^9}{t \cdot q^3}\right)^2 &= \left(\frac{q^{9-3}}{t}\right)^2\\ &= \left(\frac{q^6}{t}\right)^2\\ &= \frac{q^{6\cdot 2}}{t^2}\\ &= \frac{q^{12}}{t^2}\end{aligned}$$

Example 6.7.2 The Zero Exponent. Recall that the Zero Exponent Rule says that any real number raised to the 0-power is 1. Using this, and the other exponent rules, find another way to write -9^0.

Explanation. Remember that in expressions like -9^0, the exponent only applies to what it is directly next to! In this case, the 0 only applies to the 9 and not the negative sign. So,

$$-9^0 = -1$$

Example 6.7.3 Negative Exponents. Write $5x^{-3}$ without any negative exponents.

Explanation. Recall that the Negative Exponent Rule says that a factor in the numerator with a negative exponent can be flipped into the denominator. So

$$5x^{-3} = \frac{5}{x^3}$$

Note that the 5 does not move to the denominator because the -3 exponent *only applies* to the x to which it is directly attached.

Example 6.7.4 Summary of Exponent Rules. Use the exponent rules in List 6.1.15 to write the expressions in a different way. Reduce and simplify when possible. Always find a way to write your final simplification without any negative exponents.

a. $\dfrac{24p^3}{20p^{12}}$

b. $\left(\dfrac{2v^5}{4g^{-2}}\right)^4$

c. $12n^7\left(m^0 \cdot n^2\right)^2$

d. $\dfrac{k^5}{k^{-4}}$

Explanation.

a. $$\begin{aligned}\frac{24p^3}{20p^{12}} &= \frac{24}{20}\cdot\frac{p^3}{p^{12}}\\ &= \frac{6}{5}\cdot p^{3-12}\\ &= \frac{6}{5}\cdot p^{-9}\\ &= \frac{6}{5}\cdot\frac{1}{p^9}\\ &= \frac{6}{5p^9}\end{aligned}$$

b. $$\begin{aligned}\left(\frac{2v^5}{4g^{-2}}\right)^4 &= \left(\frac{v^5}{2g^{-2}}\right)^4\\ &= \left(\frac{v^5g^2}{2}\right)^4\\ &= \frac{v^{5\cdot4}g^{2\cdot4}}{2^4}\\ &= \frac{v^{20}g^8}{16}\end{aligned}$$

c. $$\begin{aligned}12n^7\left(m^0\cdot n^2\right)^2 &= 12n^7\left(1\cdot n^2\right)^2\\ &= 12n^7\left(n^2\right)^2\\ &= 12n^7n^{2\cdot2}\\ &= 12n^7n^4\\ &= 12n^{7+4}\\ &= 12n^{11}\end{aligned}$$

d. $$\begin{aligned}\frac{k^5}{k^{-4}} &= k^5\cdot k^4\\ &= k^{5+4}\\ &= k^9\end{aligned}$$

6.7.2 Scientific Notation

In Section 6.2 we covered the definition of scientific notation, how to convert to and from scientific notation, and how to do some calculations in scientific notation.

Example 6.7.5 Scientific Notation for Large Numbers.

a. The distance to the star Betelgeuse is about 3,780,000,000,000,000 miles. Write this number in scientific notation.

b. The gross domestic product (GDP) of California in the year 2017 was about \$$2.746\times10^{13}$. Write this number in standard notation.

Explanation.

a. $3{,}780{,}000{,}000{,}000{,}000 = 3.78\times10^{15}$.

b. \$$2.746\times10^{13}$ = \$2,746,000,000,000.

Example 6.7.6 Scientific Notation for Small Numbers.

a. Human DNA forms a double helix with diameter 2×10^{-9} meters. Write this number in standard notation.

b. A single grain of Forget-me-not (Myosotis) pollen is about 0.00024 inches in diameter. Write this number in scientific notation.

Explanation.

a. $2\times10^{-9} = 0.000000002$.

b. $0.00024 = 2.4\times10^{-4}$.

Example 6.7.7 Multiplying and Dividing Using Scientific Notation. The fastest spacecraft so far have traveled about 5×10^6 miles per day.

a. If that spacecraft traveled at that same speed for 2×10^4 days (which is about 55 years), how far would it have gone? Write your answer in scientific notation.

b. The nearest star to Earth, besides the Sun, is Proxima Centauri, about 2.5×10^{13} miles from Earth. How many days would you have to fly in that spacecraft at top speed to reach Proxima Centauri

Explanation.

a. Remember that you can find the distance traveled by multiplying the rate of travel times the time traveled: $d = r \cdot t$. So this problem turns into

$$\begin{aligned} d &= r \cdot t \\ d &= \left(5 \times 10^6\right) \cdot \left(2 \times 10^4\right) \end{aligned}$$

Multiply coefficient with coefficient and power of 10 with power of 10.

$$\begin{aligned} &= (5 \cdot 2)\left(10^6 \times 10^4\right) \\ &= 10 \times 10^{10} \end{aligned}$$

Remember that this still isn't in scientific notation. So we convert like this:

$$\begin{aligned} &= 1.0 \times 10^1 \times 10^{10} \\ &= 1.0 \times 10^{11} \end{aligned}$$

So, after traveling for 2×10^4 days (55 years), we will have traveled about 1.0×10^{11} miles. That's one-hundred million miles. I hope someone remembered the snacks.

b. Since we are looking for time, let's solve the equation $d = r \cdot t$ for t by dividing by r on both sides: $t = \frac{d}{r}$. So we have:

$$\begin{aligned} t &= \frac{d}{r} \\ t &= \frac{2.5 \times 10^{13}}{5 \times 10^6} \end{aligned}$$

Now we can divide coefficient by coefficient and power of 10 with power of 10.

$$\begin{aligned} t &= \frac{2.5}{5} \times \frac{10^{13}}{10^6} \\ t &= 0.5 \times 10^7 \\ t &= 5 \times 10^{-1} \times 10^7 \\ t &= 5 \times 10^6 \end{aligned}$$

This means that to get to Proxima Centauri, even in our fastest spacecraft, would take 5×10^6 years. Converting to standard form, this is 5,000,000 years. I think we're going to need a faster ship.

6.7.3 Adding and Subtracting Polynomials

In Section 6.3 we covered the definitions of a polynomial, a term of a polynomial, a coefficient of a term, the degree of a term, the degree of a polynomial, theleading term of a polynomial, a constant term, monomials, binomials, and trinomials, and how to write a polynomial in standard form.

Example 6.7.8 Polynomial Vocabulary. Decide if the following statements are true or false.

a. The expression $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is a polynomial.

b. The expression $4x^6 - 3x^{-2} - x + 1$ is a polynomial.

c. The degree of the polynomial $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is 10.

d. The degree of the term $5x^2y^4$ is 6.

e. The leading coefficient of $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is $\frac{3}{5}$.

f. There are 4 terms in the polynomial $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$.

g. The polynomial $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is in standard form.

Explanation.

a. True. The expression $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is a polynomial.

b. False. The expression $4x^6 - 3x^{-2} - x + 1$ is *not* a polynomial. Variables are only allowed to have whole number exponents in polynomials and the second term has a -2 exponent.

c. False. The degree of the polynomial $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is *not* 10. It is 7, which is the highest power of any variable in the expression.

d. True. The degree of the term $5x^2y^4$ is 6.

e. False. The leading coefficient of $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is *not* $\frac{3}{5}$.The leading coefficient comes from the degree 7 term which is $-\frac{1}{5}$.

f. True. There are 4 terms in the polynomial $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$.

g. False. The polynomial $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is *not* in standard form. The exponents have to be written from highest to lowest, i.e. $-\frac{1}{5}x^7 + \frac{3}{5}x^2 + \frac{x}{2} - 4$.

Example 6.7.9 Adding and Subtracting Polynomials. Simplify the expression $\left(\frac{2}{9}x - 4x^2 - 5\right) + \left(6x^2 - \frac{1}{6}x - 3\right)$.

Explanation. First identify like terms and group them either physically or mentally. Then we will look for common denominators for these like terms and combine appropriately.

$$\begin{aligned}
&\left(\frac{2}{9}x - 4x^2 - 5\right) + \left(6x^2 - \frac{1}{6}x - 3\right)\\
&= \frac{2}{9}x - 4x^2 - 5 + 6x^2 - \frac{1}{6}x - 3\\
&= \left(-4x^2 + 6x^2\right) + \left(\frac{2}{9}x - \frac{1}{6}x\right) + (-3 - 5)\\
&= 2x^2 + \left(\frac{4}{18}x - \frac{3}{18}x\right) - 8\\
&= 2x^2 + \frac{1}{18}x - 8
\end{aligned}$$

6.7.4 Multiplying Polynomials

In Section 6.4 we covered how to multiply two polynomials together using distribution, FOIL, and generic rectangles.

Example 6.7.10 Multiplying Binomials. Expand the expression $(5x - 6)(3 + 2x)$ using the binomial multiplication method of your choice: distribution, FOIL, or generic rectangles.

Explanation. We will show work using the FOIL method.

$$\begin{aligned}(5x-6)(3-2x)&=(5x\cdot 3)+(5x\cdot(-2x))+(-6\cdot 3)+(-6\cdot(-2x))\\&=15x-10x^2-18+12x\\&=-10x^2+27x-18\end{aligned}$$

Example 6.7.11 Multiplying Polynomials Larger than Binomials. Expand the expression $(3x-2)\left(4x^2-2x+5\right)$ by multiplying every term in the first factor with every term in the second factor.

Explanation. $(3x-2)\left(4x^2-2x+5\right)$

$$\begin{aligned}&=3x\cdot 4x^2+3x\cdot(-2x)+3x\cdot 5+(-2)\cdot 4x^2+(-2)\cdot(-2x)+(-2)\cdot 5\\&=12x^3-6x^2+15x-8x^2+4x-10\\&=12x^3-14x^2+19x-10\end{aligned}$$

6.7.5 Special Cases of Multiplying Polynomials

In Section 6.5 we covered how to square a binomial and how to find the product of the sum or difference of two terms.

Example 6.7.12 Squaring a Binomial. Recall that Fact 6.5.4 gives formulas that help square a binomial.

Simplify the expression $(2x+3)^2$.

Explanation. Remember that you *can* use FOIL to do these problems, but in the interest of understanding concepts at a higher level for use in later chapters, we will use the relevant formula from Fact 6.5.4. In this case, since we have a sum of two terms being squared, we will use $(a+b)^2=a^2+2ab+b^2$.

First identify a and b. In this case, $a=2x$ and $b=3$. So, we have:

$$\begin{aligned}(a+b)^2&=(a)^2\quad+2(a)(b)\quad+(b)^2\\(2x+3)^2&=(2x)^2+2(2x)(3)+(3)^2\\&=4x^2\quad+12x\qquad+9\end{aligned}$$

Example 6.7.13 The Product of the Sum and Difference of Two Terms. Recall that Fact 6.5.12 gives a formula to help multiply things that look like $(a+b)(a-b)$.

Simplify the expression $(7x+4)(7x-4)$.

Explanation. Remember that you *can* use FOIL to do these problems, but in the interest of understanding concepts at a higher level for use in later chapters, we will use the formula from Fact 6.5.12. In this case, that means we will use $(a+b)(a-b)=a^2-b^2$.

First identify a and b. In this case, $a=7x$ and $b=4$. So, we have:

$$\begin{aligned}(a+b)(a-b)&=(a)^2\quad-(b)^2\\(7x+4)(7x-4)&=(7x)^2-(4)^2\\&=49x^2\;-16\end{aligned}$$

Example 6.7.14 Binomials Raised to Other Powers. To raise binomials to powers higher than 2, we start by expanding the expression and multiplying all factors together from left to right.

Expand the expression $(2x-5)^3$.

Explanation.

$$\begin{aligned}
&(2x-5)^3\\
&=(2x-5)(2x-5)(2x-5)\\
&=\left[(2x)^2-2(2x)(5)+5^2\right](2x-5)\\
&=\left[4x^2-20x+25\right](2x-5)\\
&=\left[4x^2\right](2x)+\left[4x^2\right](-5)+[-20x](2x)+[-20x](-5)+[25](2x)+[25](-5)\\
&=8x^3-20x^2-40x^2+100x+50x-125\\
&=8x^3-60x^2+150x-125
\end{aligned}$$

6.7.6 Dividing by a Monomial

In Section 6.6 we covered how you can split a fraction up into multiple terms if there is a sum or difference in the numerator. Mathematically, this happens using the rule $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$. This formula can be used for any number of terms in the numerator, and for both sums and differences.

Example 6.7.15 Expand the expression $\frac{12x^5+2x^3-4x^2}{4x^2}$.

Explanation.

$$\begin{aligned}
\frac{12x^5+2x^3-4x^2}{4x^2}&=\frac{12x^5}{4x^2}+\frac{2x^3}{4x^2}-\frac{4x^2}{4x^2}\\
&=3x^3+\frac{x}{2}-1
\end{aligned}$$

Exercises

Exponent Rules Use the properties of exponents to simplify the expression.

1. $\left(4r^{12}\right)^3$

2. $\left(2y^2\right)^2$

3. $\left(6y^4\right)\cdot\left(9y^{17}\right)$

4. $\left(10y^7\right)\cdot\left(-8y^{11}\right)$

5. $\left(-\frac{y^9}{4}\right)\cdot\left(\frac{y^4}{9}\right)$

6. $\left(-\frac{r^{11}}{7}\right)\cdot\left(-\frac{r^{16}}{8}\right)$

7. $(-21)^0=$

8. $(-16)^0=$

9. $-42^0=$

10. $-47^0=$

11. $\left(\frac{-3}{2x^9}\right)^3=$

12. $\left(\frac{-3}{8x^4}\right)^3=$

13. $\dfrac{5r^{14}}{15r^3} =$

14. $\dfrac{7r^4}{14r^3} =$

15. $\left(\dfrac{x^7}{2y^6z^8}\right)^3 =$

16. $\left(\dfrac{x^4}{2y^{10}z^5}\right)^3 =$

Rewrite the expression simplified and using only positive exponents.

17. $\left(\dfrac{1}{8}\right)^{-3} =$

18. $\left(\dfrac{1}{9}\right)^{-3} =$

19. $9x^{-12} =$

20. $19x^{-3} =$

21. $\dfrac{14}{x^{-4}} =$

22. $\dfrac{8}{x^{-5}} =$

23. $\dfrac{18x^{-9}}{x^{-26}} =$

24. $\dfrac{8x^{-11}}{x^{-17}} =$

25. $\dfrac{r^{-3}}{(r^4)^{10}} =$

26. $\dfrac{r^{-2}}{(r^{11})^7} =$

27. $t^{-11} \cdot t^6 =$

28. $t^{-5} \cdot t^4 =$

29. $(9x^{-17}) \cdot (6x^6) =$

30. $(6x^{-10}) \cdot (10x^7) =$

31. $\left(-5y^{-4}\right)^{-2}$

32. $\left(-2y^{-16}\right)^{-3}$

33. $\left(3y^8\right)^4 \cdot y^{-22} =$

34. $\left(3r^3\right)^3 \cdot r^{-4} =$

35. $\left(r^4t^8\right)^{-3} =$

36. $\left(t^6y^{14}\right)^{-3} =$

37. $\left(t^{-11}x^{10}\right)^{-3} =$

38. $\left(x^{-4}r^6\right)^{-3} =$

39. $\left(\dfrac{x^6}{2}\right)^{-3} =$

40. $\left(\dfrac{y^{15}}{4}\right)^{-4} =$

Scientific Notation Write the following number in scientific notation.

41. 42000 =

42. 5200 =

43. 0.062 =

44. 0.0072 =

Write the following number in decimal notation without using exponents.

45. $8.24 \times 10^2 =$

46. $9.24 \times 10^5 =$

47. $1.23 \times 10^0 =$

48. $2.23 \times 10^{0} = \boxed{}$ **49.** $3.23 \times 10^{-2} = \boxed{}$ **50.** $4.23 \times 10^{-3} = \boxed{}$

Multiply the following numbers, writing your answer in scientific notation.

51. $(5 \times 10^{2})(7 \times 10^{2}) = \boxed{}$ **52.** $(6 \times 10^{5})(9 \times 10^{3}) = \boxed{}$

Divide the following numbers, writing your answer in scientific notation.

53. $\dfrac{3.5 \times 10^{3}}{7 \times 10^{-5}} = \boxed{}$ **54.** $\dfrac{1.6 \times 10^{4}}{8 \times 10^{-2}} = \boxed{}$

Adding and Subtracting Polynomials Is the following expression a monomial, binomial, or trinomial?

55. $10t^{12} - 14t^{10}$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree $\boxed{}$

56. $-4x^{7} + 3x^{6}$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree $\boxed{}$

57. $-18x^{6} - 19x^{5} + 20x^{3}$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree $\boxed{}$

58. $8x^{6} + 11x^{5} - 3x$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree $\boxed{}$

Find the degree of the following polynomial.

59. $-6x^{5}y^{7} - 6x^{2}y^{3} + 14x^{2} - 1$

The degree of this polynomial is $\boxed{}$.

60. $-x^{5}y^{9} - 20x^{2}y^{4} - 10x^{2} - 12$

The degree of this polynomial is $\boxed{}$.

Add the polynomials.

61. $\left(7x^{2} - 2x - 1\right) + \left(4x^{2} - 2x + 4\right)$

62. $\left(-8x^{2} + 2x - 9\right) + \left(10x^{2} - 8x + 9\right)$

63. $\left(-10t^{6} - 5t^{4} - 2t^{2}\right) + \left(-7t^{6} + 6t^{4} + t^{2}\right)$

64. $\left(7t^{6} - 2t^{4} + 8t^{2}\right) + \left(8t^{6} - 9t^{4} - 7t^{2}\right)$

65. $\left(-2x^{3} + 5x^{2} - 2x + \dfrac{7}{10}\right) + \left(3x^{3} + 9x^{2} + 6x + \dfrac{1}{8}\right)$

66. $\left(3x^{3} + 8x^{2} + 2x + \dfrac{9}{4}\right) + \left(-5x^{3} + 6x^{2} + 8x + \dfrac{1}{6}\right)$

Subtract the polynomials.

67. $\left(-5x^{2} + 2x\right) - \left(-7x^{2} - 5x\right)$

68. $\left(-3x^{2} - 6x\right) - \left(2x^{2} + 10x\right)$

69. $\left(6x^2+2x+10\right)-\left(-5x^2-7x+4\right)$

70. $\left(7x^2-5x+10\right)-\left(-9x^2+10x-8\right)$

71. $\left(3r^6-5r^4-2r^2\right)-\left(10r^6-7r^4+7r^2\right)$

72. $\left(-9t^6+9t^4+8t^2\right)-\left(5t^6-4t^4-6t^2\right)$

73. Add or subtract the given polynomials as indicated.

$$\left(-10x^4+8xy-9y^2\right)-\left(-6x^4-6xy+10y^2\right)$$

74. Add or subtract the given polynomials as indicated.

$$\left(-2x^8-2xy+9y^9\right)-\left(10x^8+7xy-6y^9\right)$$

75. A handyman is building two pig pens sharing the same side. Assume the length of the shared side is x meters. The cost of building one pen would be $44x^2+9.5x-13.5$ dollars, and the cost of building the other pen would be $23.5x^2-9.5x+41.5$ dollars. What's the total cost of building those two pens?

A polynomial representing the total cost of building those two pens is $\boxed{}$ dollars.

76. A handyman is building two pig pens sharing the same side. Assume the length of the shared side is x meters. The cost of building one pen would be $33.5x^2-8.5x+40.5$ dollars, and the cost of building the other pen would be $27x^2+8.5x+27$ dollars. What's the total cost of building those two pens?

A polynomial representing the total cost of building those two pens is $\boxed{}$ dollars.

Multiplying Polynomials Multiply the polynomials.

77. $-3x\,(x-2)=\boxed{}$

78. $-x\,(x+9)=\boxed{}$

79. $-5r^2\left(10r^2-5r-7\right)=\boxed{}$

80. $2r^2\left(7r^2-9r+2\right)=\boxed{}$

81. $(8t+1)\,(t+4)=\boxed{}$

82. $(5t+2)\,(t+9)=\boxed{}$

83. $(x+1)\,(x-2)=\boxed{}$

84. $(x+7)\,(x-8)=\boxed{}$

85. $(3x-4)\,(2x-3)=\boxed{}$

86. $\left(6y-10\right)\left(5y-2\right)=\boxed{}$

87. $3(x+2)(x+3)=\boxed{}$

88. $-3(x+2)(x+3)=\boxed{}$

89. $x(x-2)(x+2)=\boxed{}$

90. $-x(x+2)(x+3)=\boxed{}$

91. $(a-10b)(a^2+3ab-8b^2)=\boxed{}$

92. $(a+2b)(a^2-7ab+8b^2)=\boxed{}$

93. A rectangle's length is 2 feet shorter than 4 times its width. If we use w to represent the rectangle's width, use a polynomial to represent the rectangle's area in expanded form.

area = square feet

94. A rectangle's length is 3 feet shorter than twice its width. If we use w to represent the rectangle's width, use a polynomial to represent the rectangle's area in expanded form.

area = square feet

Special Cases of Multiplying Polynomials Expand the square of a *bi*nomial.

95. $(9y+6)^2 =$

96. $(6y+3)^2 =$

97. $(r-9)^2 =$

98. $(r-2)^2 =$

99. $(9a+6b)^2 =$

100. $(10a+2b)^2 =$

Multiply the polynomials.

101. $(x+7)(x-7) =$

102. $(x-2)(x+2) =$

103. $(1-3x)(1+3x) =$

104. $(8-7y)(8+7y) =$

105. $(3y^8-13)(3y^8+13) =$

106. $(2r^6-2)(2r^6+2) =$

Simplify the given expression into an expanded polynomial.

107. $(r+5)^3 =$

108. $(t+3)^3 =$

Dividing by a Monomial Simplify the following expression

109. $\dfrac{52t^{15}+12t^8}{4} =$

110. $\dfrac{24x^4-32x^3}{8} =$

111. $\dfrac{-12x^{12}-24x^{11}-2x^9}{2x^3} =$

112. $\dfrac{-24x^{15}-24x^{13}-16x^{12}}{-2x^3} =$

113. $\dfrac{4y^{10}-4y^5}{2y} =$

114. $\dfrac{-56y^{17}-56y^9}{8y} =$

CHAPTER 7

Factoring

7.1 Factoring out the Common Factor

In Chapter 6, we learned how to multiply polynomials, such as when you start with $(x+2)(x+3)$ and obtain x^2+5x+6. This chapter, starting with this section, is about the *opposite* process—factoring. For example, starting with x^2+5x+6 and obtaining $(x+2)(x+3)$. We will start with the simplest kind of factoring: for example starting with x^2+2x and obtaining $x(x+2)$.

7.1.1 Motivation for Factoring

When you write x^2+2x, you have an algebraic expression built with two terms—two parts that are *added* together. When you write $x(x+2)$, you have an algebraic expression built with two factors—two parts that are *multiplied* together. Factoring is useful, because sometimes (but not always) having your expression written as parts that are *multiplied* together makes it easy to simplify the expression.

You've seen this with fractions. To simplify $\frac{15}{35}$, breaking down the numerator and denominator into factors is useful: $\frac{3\cdot5}{7\cdot5}$. Now you can see that the factors of 5 cancel.

There are a few other reasons to appreciate the value of factoring that will float to the surface in this chapter and beyond.

7.1.2 Identifying the Greatest Common Factor

The most basic technique for factoring involves recognizing the **greatest common factor** between two expressions, which is the largest factor that goes in evenly to both expressions. For example, the greatest common factor between 6 and 8 is 2 since 2 goes in nicely into both 6 and 8 and no larger number would divide both 6 and 8 nicely.

Similarly, the greatest common factor between $4x$ and $3x^2$ is x. If you write $4x$ as a product of its factors, you have $2\cdot2\cdot x$. And if you fully factor $3x^2$, you have $3\cdot x\cdot x$. The only factor they have in common is x, so that is the greatest common factor. No larger expression goes in nicely to both expressions.

> **Example 7.1.2 Finding the Greatest Common Factor.** What is the common factor between $6x^2$ and $70x$? Break down each of these into its factors:
>
> $$6x^2=2\cdot3\cdot x\cdot x \qquad 70x=2\cdot5\cdot7\cdot x$$

> And identify the common factors:
>
> $$6x^2 = \overset{\downarrow}{2} \cdot 3 \cdot \overset{\downarrow}{x} \cdot x \qquad\qquad 70x = \overset{\downarrow}{2} \cdot 5 \cdot 7 \cdot \overset{\downarrow}{x}$$
>
> With 2 and x in common, the greatest common factor is $2x$.

Let's try a few more examples.

Checkpoint 7.1.3.

a. The greatest common factor between $6x$ and $8x$ is $\boxed{}$.

b. The greatest common factor between $14x^2$ and $10x$ is $\boxed{}$.

c. The greatest common factor between $6y^2$ and $7y^2$ is $\boxed{}$.

d. The greatest common factor between $12xy^2$ and $9xy$ is $\boxed{}$.

e. The greatest common factor between $6x^3$, $2x^2$, and $8x$ is $\boxed{}$.

Explanation.

a. Since $6x$ completely factors as $\overset{\downarrow}{2} \cdot 3 \cdot \overset{\downarrow}{x}$ ...

   ... and $8x$ completely factors as $\overset{\downarrow}{2} \cdot 2 \cdot 2 \cdot \overset{\downarrow}{x}$, ...

   ... the greatest common factor is $2x$.

b. Since $14x^2$ completely factors as $\overset{\downarrow}{2} \cdot 7 \cdot \overset{\downarrow}{x} \cdot x$ ...

   ... and $10x$ completely factors as $\overset{\downarrow}{2} \cdot 5 \cdot \overset{\downarrow}{x}$, ...

   ... the greatest common factor is $2x$.

c. Since $6y^2$ completely factors as $2 \cdot 3 \cdot \overset{\downarrow}{y} \cdot \overset{\downarrow}{y}$ ...

   ... and $7y^2$ completely factors as $7 \cdot \overset{\downarrow}{y} \cdot \overset{\downarrow}{y}$, ...

   ... the greatest common factor is y^2.

d. Since $12xy^2$ completely factors as $2 \cdot 2 \cdot \overset{\downarrow}{3} \cdot \overset{\downarrow}{x} \cdot \overset{\downarrow}{y} \cdot y$ ...

   ... and $9xy$ completely factors as $\overset{\downarrow}{3} \cdot 3 \cdot \overset{\downarrow}{x} \cdot \overset{\downarrow}{y}$, ...

   ... the greatest common factor is $3xy$.

e. Since $6x^3$ completely factors as $\overset{\downarrow}{2} \cdot 3 \cdot \overset{\downarrow}{x} \cdot x \cdot x$ ...

   ..., $2x^2$ completely factors as $\overset{\downarrow}{2} \cdot \overset{\downarrow}{x} \cdot x$, ...

   ... and $8x$ completely factors as $\overset{\downarrow}{2} \cdot 2 \cdot 2 \cdot \overset{\downarrow}{x}$, ...

   ... the greatest common factor is $2x$.

7.1.3 Factoring Out the Greatest Common Factor

We have learned the distributive property: $a(b+c) = ab + ac$. Perhaps you have thought of this as a way to "distribute" the number a to each of b and c. In this section, we will use the distributive property in the opposite way. If you have an expression $ab + ac$, it is equal to $a(b+c)$. In that example, we factored out a, which is the common factor between ab and ac.

The following steps use the distributive property to factor out the greatest common factor between two or more terms.

Factoring Out the Greatest Common Factor by Filling in the Blank

Process 7.1.4.

1. *Identify the common factor in all terms.*
2. *Write the common factor outside a pair of parentheses with the appropriate addition or subtraction signs inside.*
3. *For each term from the original expression, what would you multiply the greatest common factor by to result in that term? Write your answer in the parentheses.*

Example 7.1.5 To factor $12x^2 + 15x$:

1. The common factor between $12x^2$ and $15x$ is $3x$.
2. $3x(\quad + \quad)$
3. $3x(4x+5)$

Let's look at a few examples.

Example 7.1.6 Factor the polynomial $3x^3 + 3x^2 - 9$.

a. We identify the common factor as 3, because 3 is the only common factor between $3x^3$, $3x^2$ and 9.

b. We write:

$$3x^3 + 3x^2 - 9 = 3(\quad + \quad - \quad).$$

c. We ask the question "3 times what gives $3x^3$?" The answer is x^3. Now we have:

$$3x^3 + 3x^2 - 9 = 3(x^3 + \quad - \quad).$$

We ask the question "3 times what gives $3x^2$?" The answer is x^2. Now we have:

$$3x^3 + 3x^2 - 9 = 3(x^3 + x^2 - \quad).$$

We ask the question "3 times what gives 9?" The answer is 3. Now we have:

$$3x^3 + 3x^2 - 9 = 3(x^3 + x^2 - 3).$$

To check that this is correct, multiplying through $3(x^3 + x^2 - 3)$ should give the original expression $3x^3 + 3x^2 - 9$. We check this, and it does.

Checkpoint 7.1.7. Factor the polynomial $4x^3 + 12x^2 - 12x$.

Explanation. In this exercise, $4x$ is the greatest common factor. We find

$$\begin{aligned}4x^3+12x^2-12x &= 4x(\quad + \quad - \quad)\\ &= 4x(x^2 + \quad - \quad)\\ &= 4x(x^2+3x- \quad)\\ &= 4x(x^2+3x-3)\end{aligned}$$

Note that you might fail to recognize that $4x$ is the greatest common factor. At first you might only find that, say, 4 is a common factor. This is OK—you can factor out the 4 and continue from there:

$$\begin{aligned}4x^3+12x^2-12x &= 4(\quad + \quad - \quad)\\ &= 4(x^3 + \quad - \quad)\\ &= 4(x^3+3x^2- \quad)\\ &= 4(x^3+3x^2-3x)\end{aligned}$$

Now examine the second factor here and there is *still* a common factor, x. Factoring this out too.

$$= 4x(x^2+3x-3)$$

So there is more than one way to find the answer here.

7.1.4 Visualizing With Rectangles

In Section 6.4, we learned one way to multiply polynomials using rectangle diagrams. Similarly, we can factor a polynomial with a rectangle diagram.

Factoring Out the Greatest Common Factor Using Rectangles

Process 7.1.8 Factoring Out the Greatest Common Factor Using Rectangles.

1. *Put the terms into adjacent rectangles. Think of these as labeling the areas of each rectangle.*
2. *Identify the common factor, and mark the height of the overall rectangle with it.*
3. *Mark the base of each rectangle based on each rectangle's area and height.*
4. *Since the overall rectangle's area equals its base times its height, the height is one factor, and the sum of the widths is another factor.*

> **Example 7.1.9** We will factor $12x^2+15x$, the same polynomial from the example in Algorithm 7.1.4, so that you may compare the two styles.
>
> $$\begin{array}{|c|c|}\hline 12x^2 & 15x \\ \hline\end{array} \qquad\qquad \begin{array}{c|c|c|}\cline{2-3} 3x & 12x^2 & 15x \\ \cline{2-3}\end{array} \qquad\qquad \begin{array}{c|c|c|} & 4x & 5 \\ \cline{2-3} 3x & 12x^2 & 15x \\ \cline{2-3}\end{array}$$
>
> So $12x^2+15x$ factors as $3x(4x+5)$.

7.1.5 More Examples of Factoring out the Common Factor

Previous examples did not cover every nuance with factoring out the greatest common factor. Here are a few more factoring examples that attempt to do so.

Example 7.1.10 Factor $-35m^5 + 5m^4 - 10m^3$.

First, we identify the common factor. The number 5 is the greatest common factor of the three coefficients (which were -35, 5, and -10) and also m^3 is the largest expression that divides m^5, m^4, and m^3. Therefore the greatest common factor is $5m^3$.

In this example, the leading term is a negative number. When this happens, we will make it common practice to take that negative as part of the greatest common factor. So we will proceed by factoring out $-5m^3$. Note the sign changes.

$$\begin{aligned}
-35m^5 + 5m^4 - 10m^3 &= -5m^3(- +)\\
&= -5m^3(7m^2 - +)\\
&= -5m^3(7m^2 - m +)\\
&= -5m^3(7m^2 - m + 2)
\end{aligned}$$

Example 7.1.11 Factor $14 - 7n^2 + 28n^4 - 21n$.

Notice that the terms are not in a standard order, with powers of n decreasing as you read left to right. It is usually a best practice to rearrange the terms into the standard order first.

$$14 - 7n^2 + 28n^4 - 21n = 28n^4 - 7n^2 - 21n + 14.$$

The number 7 divides all of the numerical coefficients. Separately, no power of n is part of the greatest common factor because the 14 term has no n factors. So the greatest common factor is just 7. We proceed by factoring that out:

$$\begin{aligned}
14 - 7n^2 + 28n^4 - 21n &= 28n^4 - 7n^2 - 21n + 14\\
&= 7\left(4n^4 - n^2 - 3n + 2\right)
\end{aligned}$$

Example 7.1.12 Factor $24ab^2 + 16a^2b^3 - 12a^3b^2$.

There are two variables in this polynomial, but that does not change the factoring strategy. The greatest numerical factor between the three terms is 4. The variable a divides all three terms, and b^2 divides all three terms. So we have:

$$24ab^2 + 16a^2b^3 - 12a^3b^2 = 4ab^2\left(6 + 4ab - 3a^2\right)$$

Example 7.1.13 Factor $4m^2n - 3xy$.

There are no common factors in those two terms (unless you want to count 1 or -1, but we do not count these for the purposes of identifying a greatest common factor). In this situation we can say the polynomial is **prime** or **irreducible**, and leave it as it is.

> **Example 7.1.14** Factor $-x^3 + 2x + 18$.
>
> There are no common factors in those three terms, and it would be correct to state that this polynomial is prime or irreducible. However, since its leading coefficient is negative, it may be wise to factor out a negative sign. So, it could be factored as $-\left(x^3 - 2x - 18\right)$. Note that *every* term is negated as the leading negative sign is extracted.

Exercises

Review and Warmup Multiply the polynomials.

1. $6x(x-3) =$

2. $8x(x-8) =$

3. $10x(9x-5) =$

4. $-2x(4x-8) =$

5. $-6x^2(x+9) =$

6. $-4x^2(x-5) =$

7. $2y^2\left(-8y^2+4y\right) =$

8. $8r^2\left(-6r^2-8r\right) =$

Identifying Common Factors Find the greatest common factor of the following terms.

9. 5 and $20r$

10. 2 and $18t$

11. $8t$ and $32t^2$

12. $5t$ and $35t^2$

13. $2x^3$ and $-20x^4$

14. $8x^3$ and $-56x^4$

15. $4y^{19}$ and $-24y^{10}$

16. $10y^{12}$ and $-80y^9$

17. $7r^{17}, -14r^{10}, 21r^8$

18. $4r^{11}, -16r^9, 32r^8$

19. $5x^{11}y^9, -45x^9y^{17}, 35x^8y^{18}$

20. $3x^{11}y^{13}, -12x^9y^{14}, 27x^6y^{19}$

Factoring out the Common Factor Factor the given polynomial.

21. $4t+4 =$

22. $10x+10 =$

23. $6x-6 =$

24. $3y-3 =$

25. $-9y-9=$

26. $-6r-6=$

27. $3r-27=$

28. $9t+36=$

29. $18t^2+48=$

30. $21t^2-6=$

31. $14x^2-20x+8=$

32. $50x^2+90x+70=$

33. $16y^4+24y^3+72y^2=$

34. $12y^4+6y^3+16y^2=$

35. $50r^5-70r^4+100r^3=$

36. $40r^5+12r^4+4r^3=$

37. $24t-64t^2+80t^3=$

38. $8t+6t^2+18t^3=$

39. $27t^2+5=$

40. $20x^2+9=$

41. $3xy+3y=$

42. $4xy+4y=$

43. $5x^{13}y^9+35y^9=$

44. $6x^7y^9+24y^9=$

45. $20x^5y^6-32x^4y^6+40x^3y^6=$

46. $80x^5y^9+10x^4y^9+60x^3y^9=$

47. $24x^5y^7z^6+20x^4y^7z^5+32x^3y^7z^4=$

48. $50x^5y^8z^8+35x^4y^8z^7+10x^3y^8z^6=$

7.2 Factoring by Grouping

This section covers a technique for factoring polynomials like x^3+3x^2+2x+6, which factors as $\left(x^2+2\right)(x+3)$. If there are four terms, the technique in this section *might* help you to factor the polynomial. Additionally, this technique is a stepping stone to a factoring technique in Section 7.3 and Section 7.4.

7.2.1 Factoring out Common Polynomials

Recall that to factor $3x+6$, we factor out the common factor 3:

$$\begin{aligned} 3x+6 &= \overset{\downarrow}{3}x+\overset{\downarrow}{3}\cdot 2 \\ &= 3(x+2) \end{aligned}$$

The "3" here could have been something more abstract, and it still would be valid to factor it out:

$$\begin{aligned} xA+2A &= x\overset{\downarrow}{A}+2\overset{\downarrow}{A} \\ &= A(x+2) \end{aligned} \qquad \begin{aligned} x\text{🍎}+2\text{🍎} &= x\overset{\downarrow}{\text{🍎}}+2\overset{\downarrow}{\text{🍎}} \\ &= \text{🍎}(x+2) \end{aligned}$$

In fact, even "larger" things can be factored out, as in this example:

$$\begin{aligned} x(a+b)+2(a+b) &= x\overbrace{(a+b)}^{\downarrow}+2\overbrace{(a+b)}^{\downarrow} \\ &= (a+b)(x+2) \end{aligned}$$

In this last example, we factored out the binomial factor $(a+b)$. Factoring out binomials is the essence of this section, so let's see that a few more times:

$$\begin{aligned} x(x+2)+3(x+2) &= x\overbrace{(x+2)}^{\downarrow}+3\overbrace{(x+2)}^{\downarrow} \\ &= (x+2)(x+3) \end{aligned}$$

$$\begin{aligned} z^2(2y+5)+3(2y+5) &= z^2\overbrace{(2y+5)}^{\downarrow}+3\overbrace{(2y+5)}^{\downarrow} \\ &= (2y+5)(z^2+3) \end{aligned}$$

And even with an expression like $Q^2(Q-3)+Q-3$, if we re-write it in the right way using a 1 and some parentheses, then it too can be factored:

$$\begin{aligned} Q^2(Q-3)+Q-3 &= Q^2(Q-3)+1(Q-3) \\ &= Q^2\overbrace{(Q-3)}^{\downarrow}+1\overbrace{(Q-3)}^{\downarrow} \\ &= (Q-3)(Q^2+1) \end{aligned}$$

The truth is you are unlikely to come upon an expression like $x(x+2)+3(x+2)$, as in these examples. Why wouldn't someone have multiplied that out already? Or factored it all the way? So far in this section, we have only been looking at a stepping stone to a real factoring technique called **factoring by grouping**.

7.2.2 Factoring by Grouping

Factoring by grouping is a factoring technique that *sometimes* works on polynomials with four terms. Here is an example.

Example 7.2.2 Suppose we must factor $x^3 - 3x^2 + 5x - 15$. Note that there are four terms, and they are written in descending order of the powers of x. "Grouping" means to group the first two terms and the last two terms together:

$$x^3 - 3x^2 + 5x - 15 = \left(x^3 - 3x^2\right) + (5x - 15)$$

Now, each of these two groups has its own greatest common factor we can factor out:

$$= x^2(x-3) + 5(x-3)$$

In a sense, we are "lucky" because we now see matching binomials that can themselves be factored out:

$$\begin{aligned} &= x^2\overset{\downarrow}{\overbrace{(x-3)}} + 5\overset{\downarrow}{\overbrace{(x-3)}} \\ &= (x-3)\left(x^2+5\right) \end{aligned}$$

And so we have factored $x^3 - 3x^2 + 5x - 15$ as $(x-3)\left(x^2+5\right)$. But to be sure, if we multiply this back out, it should recover the original $x^3 - 3x^2 + 5x - 15$. To confirm your answers are correct, you should always make checks like this.

Checkpoint 7.2.3. Factor $x^3 + 4x^2 + 2x + 8$.

Explanation. We will break the polynomial into two groups: $x^3 + 4x^2$ and $2x + 8$.

$$x^3 + 4x^2 + 2x + 8 = \left(x^3 + 4x^2\right) + (2x + 8)$$

and now each group has its own greatest common factor to factor out:

$$= x^2(x+4) + 2(x+4)$$

and now the binomial $(x + 4)$ appears twice and can be factored out:

$$= (x+4)\left(x^2+2\right)$$

Example 7.2.4 Factor $t^3 - 5t^2 - 3t + 15$. This example has a complication with negative signs. If we try to break up this polynomial into two groups as $\left(t^3 - 5t^2\right) - (3t + 15)$, then we've made an error! In that last expression, we are *subtracting* a group with the term 15, so overall it subtracts 15. The original polynomial *added* 15, so we are off course.

One way to handle this is to treat subtraction as addition of a negative:

$$\begin{aligned} t^3 - 5t^2 - 3t + 15 &= t^3 - 5t^2 + (-3t) + 15 \\ &= \left(t^3 - 5t^2\right) + (-3t + 15) \end{aligned}$$

Now we can proceed to factor out common factors from each group. Since the second group leads with a negative coefficient, we'll factor out -3. This will result in the " + 15" becoming " − 5."

$$= t^2(t-5) + (-3)(t-5)$$

$$\begin{aligned}&= t^2\overset{\downarrow}{\overbrace{(t-5)}} - 3\overset{\downarrow}{\overbrace{(t-5)}}\\ &= (t-5)\left(t^2-3\right)\end{aligned}$$

And remember that we can confirm this is correct by multiplying it out. If we made no mistakes, it should result in the original $t^3 - 5t^2 - 3t + 15$.

Checkpoint 7.2.5. Factor $6q^3 - 9q^2 - 4q + 6$.

Explanation. We will break the polynomial into two groups: $6q^3 - 9q^2$ and $-4q + 6$.

$$6q^3 - 9q^2 - 4q + 6 = \left(6q^3 - 9q^2\right) + (-4q + 6)$$

and now each group has its own greatest common factor to factor out:

$$= 3q^2(2q-3) - 2(2q-3)$$

and now the binomial $(2q - 3)$ appears twice and can be factored out:

$$= (2q-3)\left(3q^2-2\right)$$

Example 7.2.6 Factor x^3-3x^2+x-3. To succeed with this example, we will need to "factor out" a trivial number 1 that isn't apparent until we make it so.

$$\begin{aligned}x^3 - 3x^2 + x - 3 &= \left(x^3 - 3x^2\right) + (x-3)\\ &= x^2(x-3) + 1(x-3)\\ &= x^2\overset{\downarrow}{\overbrace{(x-3)}} + 1\overset{\downarrow}{\overbrace{(x-3)}}\\ &= (x-3)\left(x^2+1\right)\end{aligned}$$

Notice how we changed $x - 3$ to $+1(x - 3)$, so we wouldn't forget the $+1$ in the final factored form. As always, we should check this is correct by multiplying it out.

Checkpoint 7.2.7. Factor $6t^6 + 9t^4 + 2t^2 + 3$.

Explanation. We will break the polynomial into two groups: $6t^6 + 9t^4$ and $2t^2 + 3$.

$$6t^6 + 9t^4 + 2t^2 + 3 = \left(6t^6 + 9t^4\right) + \left(2t^2 + 3\right)$$

the first group has its own greatest common factor to factor out, and with the second group we can write a 1:

$$= 3t^4\left(2t^2+3\right) + 1\left(2t^2+3\right)$$

and now the binomial $\left(2t^2 + 3\right)$ appears twice and can be factored out:

$$= \left(2t^2+3\right)\left(3t^4+1\right)$$

Example 7.2.8 Factor $xy^2 - 10y^2 - 2x + 20$. The technique can work when there are multiple variables too.

$$xy^2 - 10y^2 - 2x + 20 = \left(xy^2 - 10y^2\right) + (-2x + 20)$$

$$\begin{aligned}
&= y^2(x-10)+(-2)(x-10)\\
&= y^2\overbrace{(x-10)}^{\downarrow}-2\overbrace{(x-10)}^{\downarrow}\\
&= (x-10)\left(y^2-2\right).
\end{aligned}$$

Unfortunately, this technique is not guaranteed to work on every polynomial with four terms. In fact, *most* randomly selected four-term polynomials will not factor using this method and those selected here should be considered "nice." Here is an example that will not factor with grouping:

$$\begin{aligned}
x^3+6x^2+11x+6 &= \left(x^3+6x^2\right)+(11x+6)\\
&= x^2\underbrace{(x+6)}_{?}+1\underbrace{(11x+6)}_{?}
\end{aligned}$$

In this example, at the step where we hope to see the same binomial appearing twice, we see two different binomials. It doesn't mean that this kind of polynomial can't be factored, but it does mean that "factoring by grouping" is not going to help. This polynomial actually factors as $(x+1)(x+2)(x+3)$. So the fact that grouping fails to factor the polynomial doesn't tell us whether or not it is prime.

Exercises

Review and Warmup Factor the given polynomial.

1. $-6y-6=$

2. $-3r-3=$

3. $8r+16=$

4. $5t-30=$

5. $14t^2-20=$

6. $56t^2+40=$

Factoring out Common Polynomials Factor the given polynomial.

7. $x(x+5)+7(x+5)=$

8. $x(x-2)+5(x-2)=$

9. $x(y+4)-7(y+4)=$

10. $x(y-5)+2(y-5)=$

11. $6x(x+y)+7(x+y)=$

12. $7x(x+y)+4(x+y)=$

13. $4t^7(4t-7)+4t-7=$

14. $10t^{10}(8t+5)+8t+5=$

15. $42t^4(t+18)+21t^3(t+18)+56t^2(t+18)=$

16. $14x^4(x+2) + 21x^3(x+2) + 14x^2(x+2) = \boxed{}$

Factoring by Grouping Factor the given polynomial.

17. $x^2 + 10x - 9x - 90 = \boxed{}$

18. $y^2 - 6y - 7y + 42 = \boxed{}$

19. $y^2 + 3y - 4y - 12 = \boxed{}$

20. $r^2 - 9r - 2r + 18 = \boxed{}$

21. $r^3 + 6r^2 - 8r - 48 = \boxed{}$

22. $r^3 - 3r^2 - 6r + 18 = \boxed{}$

23. $t^3 - 8t^2 - 3t + 24 = \boxed{}$

24. $t^3 + 6t^2 - 10t - 60 = \boxed{}$

25. $xy - 2x + 4y - 8 = \boxed{}$

26. $xy + 3x - 8y - 24 = \boxed{}$

27. $xy - 4x - 3y + 12 = \boxed{}$

28. $xy - 5x + 6y - 30 = \boxed{}$

29. $6x^2 + 30xy + 5xy + 25y^2 = \boxed{}$

30. $7x^2 + 14xy + 4xy + 8y^2 = \boxed{}$

31. $8x^2 - 80xy + 7xy - 70y^2 = \boxed{}$

32. $9x^2 - 36xy + 8xy - 32y^2 = \boxed{}$

33. $x^3 - 10 - 7x^3y + 70y = \boxed{}$

34. $x^3 + 2 - 4x^3y - 8y = \boxed{}$

35. $x^3 + 3 + 6x^3y + 18y = \boxed{}$

36. $x^3 - 4 + 9x^3y - 36y = \boxed{}$

37. $18y^5 + 36y^4 - 9y^4 - 18y^3 + 18y^3 + 36y^2 = \boxed{}$

38. $8r^5 + 24r^4 - 8r^4 - 24r^3 + 12r^3 + 36r^2 = \boxed{}$

7.3 Factoring Trinomials with Leading Coefficient One

In Chapter 6, we learned how to multiply binomials like $(x+2)(x+3)$ and obtain the trinomial x^2+5x+6. In this section, we will learn how to undo that. So we'll be starting with a trinomial like x^2+5x+6 and obtaining its factored form $(x+2)(x+3)$. The trinomials that we'll factor in this section all have leading coefficient 1, but Section 7.4 will cover some more general trinomials.

7.3.1 Factoring Trinomials by Listing Factor Pairs

Consider the example $x^2+5x+6=(x+2)(x+3)$. There are at least three things that are important to notice:

- The leading coefficient of x^2+5x+6 is 1.
- The two factors on the right use the numbers 2 and 3, and when you *multiply* these you get the 6.
- The two factors on the right use the numbers 2 and 3, and when you *add* these you get the 5.

So the idea is that if you need to factor x^2+5x+6 and you somehow discover that 2 and 3 are special numbers (because $2\cdot3=6$ and $2+3=5$), then you can conclude that $(x+2)(x+3)$ is the factored form of the given polynomial.

Example 7.3.2 Factor $x^2+13x+40$. Since the leading coefficient is 1, we are looking to write this polynomial as $(x+?)(x+?)$ where the question marks are two possibly different, possibly negative, numbers. We need these two numbers to multiply to 40 and add to 13. How can you track these two numbers down? Since the numbers need to multiply to 40, one method is to list all **factor pairs** of 40 in a table just to see what your options are. We'll write every *pair of factors* that multiply to 40.

$1\cdot40$	$-1\cdot(-40)$
$2\cdot20$	$-2\cdot(-20)$
$4\cdot10$	$-4\cdot(-10)$
$5\cdot8$	$-5\cdot(-8)$

We wanted to find *all* factor pairs. To avoid missing any, we started using 1 as a factor, and then slowly increased that first factor. The table skips over using 3 as a factor, because 3 is not a factor of 40. Similarly the table skips using 6 and 7 as a factor. And there would be no need to continue with 8 and beyond, because we already found "large" factors like 8 as the partners of "small" factors like 5.

There is an entire second column where the signs are reversed, since these are also ways to multiply two numbers to get 40. In the end, there are eight factor pairs.

We need a pair of numbers that *also* adds to 13. So we check what each of our factor pairs add up to:

Factor Pair	Sum of the Pair
$1\cdot40$	41
$2\cdot20$	22
$4\cdot10$	14
$5\cdot8$	13 (what we wanted)

Factor Pair	Sum of the Pair
$-1\cdot(-40)$	(no need to go this far)
$-2\cdot(-20)$	(no need to go this far)
$-4\cdot(-10)$	(no need to go this far)
$-5\cdot(-8)$	(no need to go this far)

The winning pair of numbers is 5 and 8. Again, what matters is that $5\cdot8=40$, and $5+8=13$. So we can conclude that $x^2+13x+40=(x+5)(x+8)$.

To ensure that we made no mistakes, here are some possible checks.

Multiply it Out Multiplying out our answer $(x+5)(x+8)$ should give us $x^2+13x+40$.

$$\begin{aligned}(x+5)(x+8) &= (x+5)\cdot x + (x+5)\cdot 8\\ &= x^2+5x+8x+40\\ &\overset{\checkmark}{=} x^2+13x+40\end{aligned}$$

We could also use a rectangular area diagram to verify the factorization is correct:

	x	5
x	x^2	$5x$
8	$8x$	40

Evaluating If the answer really is $(x+5)(x+8)$, then notice how evaluating at -5 would result in 0. So the original expression should also result in 0 if we evaluate at -5. And similarly, if we evaluate it at -8, $x^2+13x+40$ should be 0.

$$\begin{aligned}(-5)^2+13(-5)+40 &\overset{?}{=} 0\\ 25-65+40 &\overset{?}{=} 0\\ 0 &\overset{\checkmark}{=} 0\end{aligned} \qquad \begin{aligned}(-8)^2+13(-8)+40 &\overset{?}{=} 0\\ 64-104+40 &\overset{?}{=} 0\\ 0 &\overset{\checkmark}{=} 0.\end{aligned}$$

This also gives us evidence that the factoring was correct.

Example 7.3.3 Factor $y^2-11y+24$. The negative coefficient is a small complication from Example 7.3.2, but the process is actually still the same.

Explanation. We need a pair of numbers that multiply to 24 and add to -11. Note that we *do* care to keep track that they sum to a negative total.

Factor Pair	Sum of the Pair
$1\cdot 24$	25
$2\cdot 12$	14
$3\cdot 8$	11 (close; wrong sign)
$4\cdot 6$	10

Factor Pair	Sum of the Pair
$-1\cdot(-24)$	-25
$-2\cdot(-12)$	-14
$-3\cdot(-8)$	-11 (what we wanted)
$-4\cdot(-6)$	(no need to go this far)

So $y^2-11y+24=(y-3)(y-8)$. To confirm that this is correct, we should check. Either by multiplying out the factored form:

$$\begin{aligned}(y-3)(y-8) &= (y-3)\cdot y-(y-3)\cdot 8\\ &= y^2-3y-8y+24\\ &\overset{\checkmark}{=} y^2-11y+24\end{aligned}$$

	y	-3
y	y^2	$-3y$
-8	$-8y$	24

Or by evaluating the original expression at 3 and 8:

$$3^2-11(3)+24 \overset{?}{=} 0 \qquad 8^2-11(8)+24 \overset{?}{=} 0$$

$$9 - 33 + 24 \overset{?}{=} 0 \qquad\qquad 64 - 88 + 24 \overset{?}{=} 0$$

$$0 \overset{\checkmark}{=} 0 \qquad\qquad 0 \overset{\checkmark}{=} 0.$$

Our factorization passes the tests.

Example 7.3.4 Factor $z^2 + 5z - 6$. The negative coefficient is again a small complication from Example 7.3.2, but the process is actually still the same.

Explanation. We need a pair of numbers that multiply to -6 and add to 5. Note that we *do* care to keep track that they multiply to a negative product.

Factor Pair	Sum of the Pair
$1 \cdot (-6)$	-5 (close; wrong sign)
$2 \cdot (-3)$	14

Factor Pair	Sum of the Pair
$-1 \cdot 6$	5 (what we wanted)
$-2 \cdot 3$	(no need to go this far)

So $z^2 + 5z - 6 = (z - 1)(z + 6)$. To confirm that this is correct, we should check. Either by multiplying out the factored form:

$$\begin{aligned}(z - 1)(z + 6) &= (z - 1) \cdot z + (z - 1) \cdot 6 \\ &= z^2 - z + 6z - 6 \\ &\overset{\checkmark}{=} z^2 + 5z - 6\end{aligned}$$

	z	-1
z	z^2	$-z$
6	$6z$	-6

Or by evaluating the original expression at 1 and -6:

$$\begin{aligned}1^2 + 5(1) - 6 &\overset{?}{=} 0 \\ 1 + 5 - 6 &\overset{?}{=} 0 \\ 0 &\overset{\checkmark}{=} 0\end{aligned} \qquad\qquad \begin{aligned}(-6)^2 + 5(-6) - 6 &\overset{?}{=} 0 \\ 36 - 30 - 6 &\overset{?}{=} 0 \\ 0 &\overset{\checkmark}{=} 0.\end{aligned}$$

Our factorization passes the tests.

Checkpoint 7.3.5. Factor $m^2 - 6m - 40$.

Explanation. We need a pair of numbers that multiply to -40 and add to -6. Note that we *do* care to keep track that they multiply to a negative product and sum to a negative total.

Factor Pair	Sum of the Pair
$1 \cdot (-40)$	-39
$2 \cdot (-20)$	-18
$4 \cdot (-10)$	-6 (what we wanted)
(no need to continue)	...

So $m^2 - 6m - 40 = (m + 4)(m - 10)$.

7.3.2 Connection to Grouping

The factoring method we just learned takes a bit of a shortcut. To prepare yourself for a more complicated factoring technique in Section 7.4, you may want to try taking the "scenic route" instead of that shortcut.

> **Example 7.3.6** Let's factor $x^2 + 13x + 40$ again (the polynomial from Example 7.3.2). As before, it is important to discover that 5 and 8 are important numbers, because they multiply to 40 and add to 13. As before, listing out all of the factor pairs is one way to discover the 5 and the 8.
>
> Instead of jumping to the factored answer, we can show how $x^2 + 13x + 40$ factors in a more step-by-step fashion using 5 and 8. Since they add up to 13, we can write:
>
> $$x^2 + \overset{\downarrow}{13}x + 40 = x^2 + \overbrace{5x + 8x}^{\downarrow} + 40$$
>
> We have intentionally split up the trinomial into an unsimplified polynomial with four terms. In Section 7.2, we handled such four-term polynomials by grouping:
>
> $$= \left(x^2 + 5x\right) + (8x + 40)$$
>
> Now we can factor out each group's greatest common factor:
>
> $$\begin{aligned} &= x(x + 5) + 8(x + 5) \\ &= x\overbrace{(x + 5)}^{\downarrow} + 8\overbrace{(x + 5)}^{\downarrow} \\ &= (x + 5)(x + 8) \end{aligned}$$
>
> And we have found that $x^2 + 13x + 40$ factors as $(x + 5)(x + 8)$ without memorizing the shortcut.

This approach takes more time, and ultimately you may not use it much. However, if you try a few examples this way, it may make you more comfortable with the more complicated technique in Section 7.4.

7.3.3 Trinomials with Higher Powers

So far we have only factored examples of **quadratic** trinomials: trinomials whose highest power of the variable is 2. However, this technique can also be used to factor trinomials where there is a larger highest power of the variable. It only requires that the highest power is even, that the next highest power is half of the highest power, and that the third term is a constant term.

In the four examples below, check:

1. if the highest power is even
2. if the next highest power is half of the highest power
3. if the last term is constant

Factor pairs *will* help with…

- $y^6 - 23y^3 - 50$
- $h^{16} + 22h^8 + 105$

Factor pairs *won't* help with…

- $y^5 - 23y^3 - 50$
- $h^{16} + 22h^8 + 105h^2$

Example 7.3.7 Factor $h^{16}+22h^{8}+105$. This polynomial is one of the examples above where using factor pairs will help. We find that $7\cdot 15=105$, and $7+15=22$, so the numbers 7 and 15 can be used:

$$\begin{aligned}h^{16}+22h^{8}+105&=h^{16}+\overbrace{7h^{8}+15h^{8}}+105\\&=\left(h^{16}+7h^{8}\right)+\left(15h^{8}+105\right)\\&=h^{8}\left(h^{8}+7\right)+15\left(h^{8}+7\right)\\&=\left(h^{8}+7\right)\left(h^{8}+15\right)\end{aligned}$$

Actually, once we settled on using 7 and 15, we could have concluded that $h^{16}+22h^{8}+105$ factors as $\left(h^{8}+7\right)\left(h^{8}+15\right)$, if we know which power of h to use. We'll always use half the highest power in these factorizations.

In any case, to confirm that this is correct, we should check by multiplying out the factored form:

$$\begin{aligned}(h^{8}+7)(h^{8}+15)&=(h^{8}+7)\cdot h^{8}+(h^{8}+7)\cdot 15\\&=h^{16}+7h^{8}+15h^{8}+105\\&\overset{\checkmark}{=}h^{16}+22h^{8}+15\end{aligned}$$

	h^{8}	7
h^{8}	h^{16}	$7h^{8}$
15	$15h^{8}$	105

Our factorization passes the tests.

Checkpoint 7.3.8. Factor $y^{6}-23y^{3}-50$.

Explanation. We need a pair of numbers that multiply to -50 and add to -23. Note that we *do* care to keep track that they multiply to a negative product and sum to a negative total.

Factor Pair	Sum of the Pair
$1\cdot(-50)$	-49
$2\cdot(-25)$	-23 (what we wanted)
(no need to continue)	...

So $y^{6}-23y^{3}-50=\left(y^{3}-25\right)\left(y^{3}+2\right)$.

7.3.4 Factoring in Stages

Sometimes factoring a polynomial will take two or more "stages." Always begin factoring a polynomial by factoring out its greatest common factor, and *then* apply a second stage where you use a technique from this section. The process of factoring a polynomial is not complete until each of the factors cannot be factored further.

Example 7.3.9 Factor $2z^{2}-6z-80$.

Explanation. We will first factor out the common factor, 2:

$$2z^{2}-6z-80=2\left(z^{2}-3z-40\right)$$

Now we are left with a factored expression that might factor more. Looking inside the parentheses, we ask ourselves, "what two numbers multiply to be −40 and add to be −3?" Since 5 and −8 do the job the full factorization is:

$$\begin{aligned}2z^2-6z-80&=2\left(z^2-3z-40\right)\\&=2(z+5)(z-8)\end{aligned}$$

Example 7.3.10 Factor $-r^2+2r+24$.

Explanation. The three terms don't exactly have a common factor, but as discussed in Section 7.1, when the leading term has a negative sign, it is often helpful to factor out that negative sign:

$$-r^2+2r+24=-\left(r^2-2r-24\right).$$

Looking inside the parentheses, we ask ourselves, "what two numbers multiply to be −24 and add to be −2?" Since −6 and 4 work here and the full factorization is shown:

$$\begin{aligned}-r^2+2r+24&=-\left(r^2-2r-24\right)\\&=-(r-6)(r+4)\end{aligned}$$

Example 7.3.11 Factor $p^2q^3+4p^2q^2-60p^2q$.

Explanation. First, always look for the greatest common factor: in this trinomial it is p^2q. After factoring this out, we have

$$p^2q^3+4p^2q^2-60p^2q=p^2q\left(q^2+4q-60\right).$$

Looking inside the parentheses, we ask ourselves, "what two numbers multiply to be −60 and add to be 4?" Since 10 and −6 fit the bill, the full factorization can be shown below:

$$\begin{aligned}p^2q^3+4p^2q^2-60p^2q&=p^2q\left(q^2+4q-60\right)\\&=p^2q(q+10)(q-6)\end{aligned}$$

7.3.5 More Trinomials with Two Variables

You might encounter a trinomial with two variables that can be factored using the methods we've discussed in this section. It can be tricky though: $x^2+5xy+6y^2$ has two variables and it *can* factor using the methods from this section, but $x^2+5x+6y^2$ also has two variables and it *cannot* be factored. So in examples of this nature, it is even more important to check that factorizations you find actually work.

Example 7.3.12 Factor $x^2+5xy+6y^2$. This is a trinomial, and the coefficient of x is 1, so maybe we can factor it. We want to write $(x+?)(x+?)$ where the question marks will be *something* that makes it all multiply out to $x^2+5xy+6y^2$.

Since the last term in the polynomial has a factor of y^2, it is natural to wonder if there is a factor of y in each of the two question marks. If there were, these two factors of y would multiply to y^2. So it is natural to wonder if we are looking for $(x+?y)(x+?y)$ where now the question marks are just numbers.

At this point we can think like we have throughout this section. Are there some numbers that multiply to 6 and add to 5? Yes, specifically 2 and 3. So we suspect that $(x+2y)(x+3y)$ might be the factorization. To confirm that this is correct, we should check by multiplying out the factored form:

$$\begin{aligned}(x+2y)(x+3y) &= (x+2y)\cdot x + (x+2y)\cdot 3y\\ &= x^2+2xy+3xy+6y^2\\ &\overset{\checkmark}{=} x^2+5xy+6y^2\end{aligned}$$

	x	$2y$
x	x^2	$2xy$
$3y$	$3xy$	$6y^2$

Our factorization passes the tests.

In Section 7.4, there is a more definitive method for factoring polynomials of this form.

Exercises

Review and Warmup Multiply the polynomials.

1. $(t+6)(t+10) =$ ☐
2. $(t+3)(t+4) =$ ☐
3. $(x+9)(x-3) =$ ☐
4. $(x+6)(x-8) =$ ☐
5. $(y-9)(y-4) =$ ☐
6. $(y-2)(y-10) =$ ☐
7. $3(x+2)(x+3) =$ ☐
8. $-4(y-1)(y-9) =$ ☐
9. $2(y-10)(y-3) =$ ☐
10. $-2(r+7)(r+6) =$ ☐

Factoring Trinomials with Leading Coefficient One Factor the given polynomial.

11. $t^2+3t+2 =$ ☐
12. $x^2+14x+48 =$ ☐
13. $x^2+14x+40 =$ ☐
14. $y^2+5y+4 =$ ☐
15. $y^2+3y-28 =$ ☐
16. $r^2-3r-4 =$ ☐
17. $r^2+2r-80 =$ ☐
18. $r^2-2r-35 =$ ☐
19. $t^2-9t+20 =$ ☐
20. $t^2-19t+90 =$ ☐
21. $x^2-10x+21 =$ ☐
22. $x^2-11x+24 =$ ☐
23. $y^2+12y+20 =$ ☐
24. $y^2+15y+54 =$ ☐
25. $r^2+13r+30 =$ ☐

26. $r^2 + 13r + 36 = \boxed{}$

27. $r^2 - 4r - 60 = \boxed{}$

28. $t^2 + 6t - 16 = \boxed{}$

29. $t^2 + 4t - 45 = \boxed{}$

30. $x^2 - 3x - 10 = \boxed{}$

31. $x^2 - 7x + 10 = \boxed{}$

32. $y^2 - 17y + 72 = \boxed{}$

33. $y^2 - 8y + 15 = \boxed{}$

34. $r^2 - 9r + 8 = \boxed{}$

35. $r^2 + 2r + 7 = \boxed{}$

36. $r^2 + 5 = \boxed{}$

37. $t^2 - 2t + 5 = \boxed{}$

38. $t^2 + 9 = \boxed{}$

39. $x^2 + 10x + 25 = \boxed{}$

40. $x^2 + 24x + 144 = \boxed{}$

41. $y^2 + 16y + 64 = \boxed{}$

42. $y^2 + 8y + 16 = \boxed{}$

43. $r^2 - 24r + 144 = \boxed{}$

44. $r^2 - 16r + 64 = \boxed{}$

45. $r^2 - 8r + 16 = \boxed{}$

46. $t^2 - 22t + 121 = \boxed{}$

47. $9t^2 - 9t - 18 = \boxed{}$

48. $7x^2 + 7x - 42 = \boxed{}$

49. $6x^2 - 6x - 12 = \boxed{}$

50. $4y^2 - 16 = \boxed{}$

51. $2y^2 - 8y + 6 = \boxed{}$

52. $2y^2 - 14y + 20 = \boxed{}$

53. $4r^2 - 20r + 16 = \boxed{}$

54. $2r^2 - 12r + 10 = \boxed{}$

55. $9t^4 + 27t^3 + 18t^2 = \boxed{}$

56. $2t^4 + 22t^3 + 20t^2 = \boxed{}$

57. $5x^6 + 20x^5 + 15x^4 = \boxed{}$

58. $2x^6 + 18x^5 + 16x^4 = \boxed{}$

59. $5y^6 - 10y^5 - 40y^4 = \boxed{}$

60. $6y^4 - 18y^3 - 24y^2 = \boxed{}$

61. $4y^{10} - 12y^9 - 16y^8 = \boxed{}$

62. $2r^8 + 12r^7 - 32r^6 = \boxed{}$

63. $4r^6 - 12r^5 + 8r^4 = \boxed{}$

64. $7t^9 - 28t^8 + 21t^7 = \boxed{}$

65. $4t^{10} - 20t^9 + 16t^8 = \boxed{}$

66. $9x^6 - 27x^5 + 18x^4 = \boxed{}$

67. $-x^2 + 16 = \boxed{}$

68. $-y^2 + 4y + 21 = \boxed{}$

69. $-y^2 + 6y + 40 = \boxed{}$

70. $-y^2 - 5y + 6 = \boxed{}$

71. $r^2 + 10rx + 16x^2 =$

72. $r^2 + 9rt + 18t^2 =$

73. $t^2 - 7tr - 30r^2 =$

74. $t^2 + tx - 12x^2 =$

75. $x^2 - 6xt + 8t^2 =$

76. $x^2 - 11xy + 24y^2 =$

77. $y^2 + 16yx + 64x^2 =$

78. $y^2 + 2yt + t^2 =$

79. $y^2 - 4yr + 4r^2 =$

80. $r^2 - 22rx + 121x^2 =$

81. $4r^2 + 20r + 16 =$

82. $2t^2 + 14t + 12 =$

83. $2x^2y + 10xy + 8y =$

84. $8x^2y + 24xy + 16y =$

85. $3a^2b - 21ab - 24b =$

86. $7a^2b - 7b =$

87. $2x^2y - 6xy + 4y =$

88. $2x^2y - 16xy + 14y =$

89. $2x^3y + 12x^2y + 16xy =$

90. $2x^3y + 14x^2y + 20xy =$

91. $x^2y^2 - 9x^2z^2 =$

92. $x^2y^2 - 5x^2yz - 14x^2z^2 =$

93. $x^2 + x + 0.24 =$

94. $x^2 + x + 0.21 =$

95. $y^2t^2 + 12yt + 20 =$

96. $y^2r^2 + 11yr + 28 =$

97. $y^2x^2 + 7yx - 30 =$

98. $r^2t^2 + 8rt - 20 =$

99. $r^2y^2 - 12ry + 20 =$

100. $t^2x^2 - 11tx + 24 =$

101. $2t^2r^2 + 22tr + 20 =$

102. $2x^2y^2 + 20xy + 18 =$

103. $10x^2y^2 - 10 =$

104. $5y^2r^2 - 10yr - 15 =$

105. $2x^2y^3 - 16xy^2 + 30y =$

106. $2x^2y^3 - 6xy^2 + 4y =$

107. $(a + b)\,r^2 + 10(a + b)\,r + 16(a + b) =$

108. $(a + b)\,r^2 + 10(a + b)\,r + 24(a + b) =$

Challenge

109. What integers can go in the place of b so that the quadratic expression $x^2 + bx - 16$ is factorable?

7.4 Factoring Trinomials with a Nontrivial Leading Coefficient

In Section 7.3, we learned how to factor ax^2+bx+c when $a = 1$. In this section, we will examine the situation when $a \neq 1$. The techniques are similar to those in the last section, but there are a few important differences that will make-or-break your success in factoring these.

7.4.1 The AC Method

The AC Method is a technique for factoring trinomials like $4x^2+5x-6$, where there is no greatest common factor, and the leading coefficient is not 1.

Please note at this point that if we try the method in the previous section and ask ourselves the question "what two numbers multiply to be −6 and add to be 5?," we might come to the *erroneous* conclusion that $4x^2+5x-6$ factors as $(x+6)(x-1)$. If we expand $(x+6)(x-1)$, we get

$$(x+6)(x-1) = x^2+5x-6$$

This expression is *almost* correct, except for the missing leading coefficient, 4. Dealing with this missing coefficient requires starting over with the AC method. If you are only interested in the steps for using the technique, skip ahead to Algorithm 7.4.3.

The example below explains *why* the AC Method works. Understanding all of the details might take a few rereads, and coming back to this example after mastering the algorithm may be the best course of action.

Example 7.4.2 Expand the expression $(px+q)(rx+s)$ and analyze the result to gain an insight into *why* the AC method works. Then use this information to factor $4x^2+5x-6$.

Explanation. Factoring is the opposite process from multiplying polynomials together. We can gain some insight into how to factor complicated polynomials by taking a closer look at what happens when two generic polynomials are multiplied together:

$$\begin{aligned}(px+q)(rx+s) &= (px+q)(rx)+(px+q)s\\ &= (px)(rx)+q(rx)+(px)s+qs\\ &= (pr)x^2+qrx+psx+qs\\ &= (pr)x^2+(qr+ps)x+qs \qquad (7.4.1)\end{aligned}$$

When you encounter a trinomial like $4x^2+5x-6$ and you wish to factor it, the leading coefficient, 4, is the (pr) from Equation (7.4.1). Similarly, the −6 is the qs, and the 5 is the $(qr+ps)$.

Now, if you multiply the leading coefficient and constant term from Equation (7.4.1), you have $(pr)(qs)$, which equals $pqrs$. Notice that if we factor this number in just the right way, $(qr)(ps)$, then we have two factors that add to the middle coefficient from Equation (7.4.1), $(qr+ps)$.

Can we do all this with the example $4x^2+5x-6$? Multiplying 4 and −6 makes −24. Is there some way to factor −24 into two factors which add to 5? We make a table of factor pairs for −24 to see:

Factor Pair	Sum of the Pair
$-1 \cdot 24$	23
$-2 \cdot 12$	10
$-3 \cdot 8$	5 (what we wanted)
$-4 \cdot 6$	(no need to go this far)

Factor Pair	Sum of the Pair
$1 \cdot (-24)$	(no need to go this far)
$2 \cdot (-12)$	(no need to go this far)
$3 \cdot (-8)$	(no need to go this far)
$4 \cdot (-6)$	(no need to go this far)

So that 5 in $4x^2 + 5x - 6$, which is equal to the abstract $(qr + ps)$ from Equation (7.4.1), breaks down as $-3 + 8$. We can take -3 to be the qr and 8 to be the ps. Once we intentionally break up the 5 this way, factoring by grouping (see Section 7.2) can take over and is guaranteed to give us a factorization.

$$4x^2 \overbrace{+5x} -6 = 4x^2 \overbrace{-3x + 8x} -6$$

Now that there are four terms, group them and factor out each group's greatest common factor.

$$\begin{aligned}
&= \left(4x^2 - 3x\right) + (8x - 6)\\
&= x(4x - 3) + 2(4x - 3)\\
&= (4x - 3)(x + 2)
\end{aligned}$$

And this is the factorization of $4x^2 + 5x - 6$. This whole process is known as the "AC method," since it begins by multiplying a and c from the generic $ax^2 + bx + c$.

The AC Method Here is a summary of the algorithm:

Process 7.4.3 The AC Method. *To factor $ax^2 + bx + c$:*

1. *Multiply $a \cdot c$.*
2. *Make a table of factor pairs for ac. Look for a pair that adds to b. If you cannot find one, the polynomial is irreducible.*
3. *If you did find a factor pair summing to b, replace b with an explicit sum, and distribute x. With the four terms you have at this point, use factoring by grouping to continue. You are guaranteed to find a factorization.*

Example 7.4.4 Factor $10x^2 + 23x + 6$.

1. $10 \cdot 6 = 60$
2. Use a list of factor pairs for 60 to find that 3 and 20 are a pair that sums to 23.
3. Intentionally break up the 23 as $3 + 20$:

$$\begin{aligned}
&10x^2 \overbrace{+23x} +6\\
&= 10x^2 \overbrace{+3x + 20x} +6\\
&= \left(10x^2 + 3x\right) + (20x + 6)\\
&= x(10x + 3) + 2(10x + 3)\\
&= (10x + 3)(x + 2)
\end{aligned}$$

Example 7.4.5 Factor $2x^2 - 5x - 3$.

Explanation. Always start the factoring process by examining if there is a greatest common factor. Here there is not one. Next, note that this is a trinomial with a leading coefficient that is not 1. So the AC Method may be of help.

1. Multiply $2 \cdot (-3) = -6$.
2. Examine factor pairs that multiply to -6, looking for a pair that sums to -5:

Factor Pair	Sum of the Pair
$1 \cdot -6$	-5 (what we wanted)
$2 \cdot -3$	(no need to go this far)

Factor Pair	Sum of the Pair
$-1 \cdot 6$	(no need to go this far)
$-2 \cdot 3$	(no need to go this far)

3. Intentionally break up the -5 as $1 + (-6)$:

$$\begin{aligned} 2x^2 \overbrace{-5x}^{} -3 &= 2x^2 \overbrace{+x-6x}^{} -3 \\ &= \left(2x^2 + x\right) + (-6x - 3) \\ &= x(2x+1) - 3(2x+1) \\ &= (2x+1)(x-3) \end{aligned}$$

So we believe that $2x^2 - 5x - 3$ factors as $(2x+1)(x-3)$, and we should check by multiplying out the factored form:

$$\begin{aligned} (2x+1)(x-3) &= (2x+1) \cdot x + (2x+1) \cdot (-3) \\ &= 2x^2 + x - 6x - 3 \\ &\overset{\checkmark}{=} 2x^2 - 5x - 3 \end{aligned}$$

	$2x$	1
x	$2x^2$	x
-3	$-6x$	-3

Our factorization passes the tests.

Example 7.4.6 Factor $6p^2 + 5pq - 6q^2$. Note that this example has two variables, but that does not really change our approach.

Explanation. There is no greatest common factor. Since this is a trinomial, we try the AC Method.

1. Multiply $6 \cdot (-6) = -36$.

2. Examine factor pairs that multiply to −36, looking for a pair that sums to 5:

Factor Pair	Sum of the Pair
$1 \cdot -36$	-35
$2 \cdot -18$	-16
$3 \cdot -12$	-9
$4 \cdot -9$	-5 (close; wrong sign)
$6 \cdot -6$	0

Factor Pair	Sum of the Pair
$-1 \cdot 36$	35
$-2 \cdot 18$	16
$-3 \cdot 12$	9
$-4 \cdot 9$	5 (what we wanted)

3. Intentionally break up the 5 as $-4 + 9$:

$$\begin{aligned}
6p^2 \overbrace{+5pq} -6q^2 &= 6p^2 \overbrace{-4pq + 9pq} -6q^2 \\
&= \left(6p^2 - 4pq\right) + \left(9pq - 6q^2\right) \\
&= 2p(3p - 2q) + 3q(3p - 2q) \\
&= (3p - 2q)(2p + 3q)
\end{aligned}$$

So we believe that $6p^2 + 5pq - 6q^2$ factors as $(3p - 2q)(2p + 3q)$, and we should check by multiplying out the factored form:

$$\begin{aligned}
(3p - 2q)(2p + 3q) &= (3p - 2q) \cdot 2p + (3p - 2q) \cdot 3q \\
&= 6p^2 - 4pq + 9pq - 6q^2 \\
&\overset{\checkmark}{=} 6p^2 + 5pq - 6q^2
\end{aligned}$$

	$3p$	$-2q$
$2p$	$6p^2$	$-4pq$
$3q$	$9pq$	$-6q^2$

Our factorization passes the tests.

7.4.2 Factoring in Stages

Sometimes factoring a polynomial will take two or more "stages." For instance you may need to begin factoring a polynomial by factoring out its greatest common factor, and *then* apply a second stage where you use a technique from this section. The process of factoring a polynomial is not complete until each of the factors cannot be factored further.

Example 7.4.7 Factor $18n^2 - 21n - 60$.

Explanation. Notice that 3 is a common factor in this trinomial. We should factor it out first:

$$18n^2 - 21n - 60 = 3\left(6n^2 - 7n - 20\right)$$

Now we are left with two factors, one of which is $6n^2 - 7n - 20$, which might factor further. Using the AC Method:

1. $6 \cdot -20 = -120$

2. Examine factor pairs that multiply to −120, looking for a pair that sums to −7:

Factor Pair	Sum of the Pair
$1 \cdot -120$	-119
$2 \cdot -60$	-58
$3 \cdot -40$	-37
$4 \cdot -30$	-26
$5 \cdot -24$	-19
$6 \cdot -20$	-14
$8 \cdot -15$	-7 (what we wanted)
$10 \cdot -12$	(no need to go this far)

Factor Pair	Sum of the Pair
$-1 \cdot 120$	(no need to go this far)
$-2 \cdot 60$	(no need to go this far)
$-3 \cdot 40$	(no need to go this far)
$-4 \cdot 30$	(no need to go this far)
$-5 \cdot 24$	(no need to go this far)
$-6 \cdot 20$	(no need to go this far)
$-8 \cdot 15$	(no need to go this far)
$-10 \cdot 12$	(no need to go this far)

3. Intentionally break up the −7 as 8 + (−15):

$$\begin{aligned}
18n^2 - 21n - 60 &= 3\left(6n^2 \overbrace{-7n} -20\right)\\
&= 3\left(6n^2 \overbrace{+8n - 15n} -20\right)\\
&= 3\left((6n^2 + 8n) + (-15n - 20)\right)\\
&= 3\left(2n(3n+4) - 5(3n+4)\right)\\
&= 3(3n+4)(2n-5)
\end{aligned}$$

So we believe that $18n^2 - 21n - 60$ factors as $3(3n+4)(2n-5)$, and you should check by multiplying out the factored form.

Example 7.4.8 Factor $-16x^3y - 12x^2y + 18xy$.

Explanation. Notice that $2xy$ is a common factor in this trinomial. Also the leading coefficient is negative, and as discussed in Section 7.1, it is wise to factor that out as well. So we find:

$$-16x^3y - 12x^2y + 18xy = -2xy\left(8x^2 + 6x - 9\right)$$

Now we are left with one factor being $8x^2 + 6x - 9$, which might factor further. Using the AC Method:

1. $8 \cdot -9 = -72$
2. Examine factor pairs that multiply to −72, looking for a pair that sums to 6:

Factor Pair	Sum of the Pair
$1 \cdot -72$	-71
$2 \cdot -36$	-34
$3 \cdot -24$	-21
$4 \cdot -18$	-14
$6 \cdot -12$	-6 (close; wrong sign)
$8 \cdot -9$	-1

Factor Pair	Sum of the Pair
$-1 \cdot 72$	71
$-2 \cdot 36$	34
$-3 \cdot 24$	21
$-4 \cdot 18$	14
$-6 \cdot 12$	6 (what we wanted)
$-8 \cdot 9$	(no need to go this far)

3. Intentionally break up the 6 as −6 + 12:

$$\begin{aligned}
-16x^3y - 12x^2y + 18xy &= -2xy\left(8x^2 \overbrace{+6x} -9\right)\\
&= -2xy\left(8x^2 \overbrace{-6x + 12x} -9\right)\\
&= -2xy\left((8x^2 - 6x) + (12x - 9)\right)\\
&= -2xy\left(2x(4x - 3) + 3(4x - 3)\right)\\
&= -2xy(4x - 3)(2x + 3)
\end{aligned}$$

So we believe that $-16x^3y - 12x^2y + 18xy$ factors as $-2xy(4x - 3)(2x + 3)$, and you should check by multiplying out the factored form.

Exercises

Review and Warmup Multiply the polynomials.

1. $(5t + 4)(6t + 8) = \boxed{\quad}$

2. $(3x + 9)(5x + 8) = \boxed{\quad}$

3. $(6x - 8)(3x - 4) = \boxed{\quad}$

4. $(5y - 4)(y - 3) = \boxed{\quad}$

5. $(4y - 10)(y + 9) = \boxed{\quad}$

6. $(10y - 6)(y + 7) = \boxed{\quad}$

7. $(7r^3 + 9)(r^2 + 4) = \boxed{\quad}$

8. $(4r^3 + 1)(r^2 + 7) = \boxed{\quad}$

Factoring Trinomials with a Nontrivial Leading Coefficient Factor the given polynomial.

9. $3t^2 + 22t + 7 = \boxed{\quad}$

10. $3t^2 + 17t + 10 = \boxed{\quad}$

11. $2x^2 + 5x - 7 = \boxed{\quad}$

12. $5x^2 + 3x - 14 = \boxed{\quad}$

13. $2y^2 - 15y + 7 = \boxed{\quad}$

14. $3y^2 - 7y + 2 = \boxed{\quad}$

15. $2y^2 + y + 1 = \boxed{\quad}$

16. $3r^2 + 3r + 7 = \boxed{\quad}$

17. $4r^2 + 25r + 6 = \boxed{\quad}$

18. $6t^2 + 7t + 1 = \boxed{\quad}$

19. $6t^2 + t - 7 = \boxed{\quad}$

20. $4x^2 + 5x - 9 = \boxed{\quad}$

21. $8x^2 - 21x + 10 = \boxed{\quad}$

22. $6x^2 - 11x + 5 = \boxed{\quad}$

23. $6y^2 + 11y + 3 = \boxed{\quad}$

24. $15y^2 + 29y + 8 =$

25. $9r^2 - 3r - 20 =$

26. $6r^2 + 7r - 20 =$

27. $12t^2 - 25t + 12 =$

28. $10t^2 - 17t + 3 =$

29. $6x^2 + 8x + 2 =$

30. $10x^2 + 25x + 15 =$

31. $15x^2 - 12x - 27 =$

32. $4y^2 - 14y - 18 =$

33. $6y^2 - 27y + 12 =$

34. $6r^2 - 20r + 16 =$

35. $4r^9 + 14r^8 + 6r^7 =$

36. $6t^9 + 15t^8 + 6t^7 =$

37. $4t^6 + 30t^5 - 16t^4 =$

38. $4x^7 - 14x^6 - 30x^5 =$

39. $6x^{10} - 27x^9 + 21x^8 =$

40. $6x^5 - 26x^4 + 24x^3 =$

41. $2y^2t^2 + 15yt + 18 =$

42. $2y^2r^2 + 19yr + 9 =$

43. $5r^2x^2 + rx - 4 =$

44. $2r^2y^2 - 13ry - 7 =$

45. $3t^2y^2 - 4ty + 1 =$

46. $2t^2x^2 - 17tx + 8 =$

47. $3x^2 + 20xr + 12r^2 =$

48. $5x^2 + 7xy + 2y^2 =$

49. $5x^2 - 8xt - 21t^2 =$

50. $5y^2 - 14yr - 3r^2 =$

51. $3y^2 - 10yx + 8x^2 =$

52. $2r^2 - 13rt + 18t^2 =$

53. $8r^2 + 17rx + 9x^2 =$

54. $6t^2 + 7tx + x^2 =$

55. $4t^2 - 11tx - 3x^2 =$

56. $4x^2 - 3xr - r^2 =$

57. $4x^2 - 17xr + 4r^2 =$

58. $4x^2 - 9xt + 2t^2 =$

59. $8y^2 + 14yr + 3r^2 =$

60. $12y^2 + 17yx + 6x^2 =$

61. $15r^2 + 2rt - 24t^2 =$

62. $25r^2 - y^2 =$

63. $8t^2 - 22tx + 15x^2 =$

64. $8t^2 - 14tr + 5r^2 =$

65. $10x^2y^2 + 18xy + 8 =$

66. $10x^2y^2 + 34xy + 12 =$

67. $6x^2r^2 + 3xr - 18 =$

68. $6y^2t^2 - 9yt - 6 =$

69. $21y^{10}x^2 - 35y^9x + 14y^8 =$ ☐

70. $25r^6x^2 - 35r^5x + 10r^4 =$ ☐

71. $10x^2 + 25xy + 15y^2 =$ ☐

72. $9x^2 + 24xy + 12y^2 =$ ☐

73. $4a^2 + 6ab - 4b^2 =$ ☐

74. $6a^2 - 9ab - 15b^2 =$ ☐

75. $10x^2 - 15xy + 5y^2 =$ ☐

76. $4x^2 - 18xy + 20y^2 =$ ☐

77. $6x^2y + 26xy^2 + 20y^3 =$ ☐

78. $6x^2y + 38xy^2 + 12y^3 =$ ☐

79. $12x^2(y-3) + 20x(y-3) + 8(y-3) =$ ☐

80. $10x^2(y+3) + 34x(y+3) + 12(y+3) =$ ☐

81. $6x^2(y-6) + 15x(y-6) + 6(y-6) =$ ☐

82. $15x^2(y+7) + 18x(y+7) + 3(y+7) =$ ☐

83.a. Factor the given polynomial.

$2x^2 + 17x + 21 =$ ☐

b. Use your previous answer to factor

$2(y-1)^2 + 17(y-1) + 21 =$ ☐

84.a. Factor the given polynomial.

$5x^2 + 11x + 6 =$ ☐

b. Use your previous answer to factor

$5(y+7)^2 + 11(y+7) + 6 =$ ☐

Challenge

85. What integers can go in the place of b so that the quadratic expression $3x^2 + bx + 4$ is factorable?

7.5 Factoring Special Polynomials

Certain polynomials have patterns that you can train yourself to recognize. And when they have these patterns, there are formulas you can use to factor them, much more quickly than using the techniques from Section 7.3 and Section 7.4.

7.5.1 Difference of Squares

If b is some positive integer, then when you multiply $(x-b)(x+b)$:

$$\begin{aligned}(x-b)(x+b)&=x^2-bx+bx-b^2\\&=x^2-b^2.\end{aligned}$$

The $-bx$ and the $+bx$ cancel each other out. So this is telling us that

$$x^2-b^2=(x-b)(x+b).$$

And so if we ever encounter a polynomial of the form x^2-b^2 (a "difference of squares") then we have a quick formula for factoring it. Just identify what "b" is, and use that in $(x-b)(x+b)$.

To use this formula, it's important to recognize which numbers are perfect squares, as in Table 1.3.7.

> **Example 7.5.2** Factor x^2-16.
>
> **Explanation**. The "16" being subtracted here is a perfect square. It is the same as 4^2. So we can take $b=4$ and write:
>
> $$\begin{aligned}x^2-16&=(x-b)(x+b)\\&=(x-4)(x+4)\end{aligned}$$

Checkpoint 7.5.3. Try to factor one yourself:

Factor x^2-49.

Explanation. The "49" being subtracted here is a perfect square. It is the same as 7^2. So we can take $b=7$ and write:

$$\begin{aligned}x^2-49&=(x-b)(x+b)\\&=(x-7)(x+7)\end{aligned}$$

We can do a little better. There is nothing special about starting with "x^2" in these examples. In full generality:

Fact 7.5.4 The Difference of Squares Formula. *If A and B are any algebraic expressions, then:*

$$A^2-B^2=(A-B)(A+B).$$

> **Example 7.5.5** Factor $1-p^2$.
>
> **Explanation**. The "1" at the beginning of this expression is a perfect square; it's the same as 1^2. The "p^2" being subtracted here is also perfect square. We can take $A=1$ and $B=p$, and use The Difference

of Squares Formula:

$$\begin{aligned}1 - p^2 &= (A - B)(A + B)\\ &= (1 - p)(1 + p)\end{aligned}$$

Example 7.5.6 Factor $m^2n^2 - 4$.

Explanation. Is the "m^2n^2" at the beginning of this expression a perfect square? By the rules for exponents, it is the same as $(mn)^2$, so yes, it is a perfect square and we may take $A = mn$. The "4" being subtracted here is also perfect square. We can take $B = 2$. The Difference of Squares Formula tells us:

$$\begin{aligned}m^2n^2 - 4 &= (A - B)(A + B)\\ &= (mn - 2)(mn + 2)\end{aligned}$$

Checkpoint 7.5.7. Try to factor one yourself:

Factor $4z^2 - 9$.

Explanation. The "$4z^2$" at the beginning here is a perfect square. It is the same as $(2z)^2$. So we can take $A = 2z$. The "9" being subtracted is also a perfect square, so we can take $B = 3$:

$$\begin{aligned}4z^2 - 9 &= (A - B)(A + B)\\ &= (2z - 3)(2z + 3)\end{aligned}$$

Example 7.5.8 Factor $x^6 - 9$.

Explanation. Is the "x^6" at the beginning of this expression is a perfect square? It may appear to be a *sixth* power, but it is *also* a perfect square because we can write $x^6 = \left(x^3\right)^2$. So we may take $A = x^3$. The "9" being subtracted here is also perfect square. We can take $B = 3$. The Difference of Squares Formula tells us:

$$\begin{aligned}x^6 - 9 &= (A - B)(A + B)\\ &= (x^3 - 3)(x^3 + 3)\end{aligned}$$

Warning 7.5.9. It's a common mistake to write something like $x^2 + 16 = (x + 4)(x - 4)$. This is not what The Difference of Squares Formula allows you to do, and this is in fact incorrect. The issue is that $x^2 + 16$ is a *sum* of squares, not a *difference*. And it happens that $x^2 + 16$ is actually prime. In fact, any sum of squares without a common factor will always be prime.

7.5.2 Perfect Square Trinomials

If we expand $(A + B)^2$:

$$\begin{aligned}(A + B)^2 &= (A + B)(A + B)\\ &= A^2 + BA + AB + B^2\\ &= A^2 + 2AB + B^2.\end{aligned}$$

The BA and the AB equal each other and double up when added together. So this is telling us that

$$A^2 + 2AB + B^2 = (A + B)^2.$$

And so if we ever encounter a polynomial of the form $A^2 + 2AB + B^2$ (a "perfect square trinomial") then we have a quick formula for factoring it.

The tricky part is recognizing when a trinomial you have encountered is in this special form. Ask yourself:

1. Is the first term a perfect square? If so, jot down what A would be.
2. Is the second term a perfect square? If so, jot down what B would be.
3. When you multiply 2 with what you wrote down for A and B, i.e. $2AB$, do you have the middle term? If you have this middle term exactly, then your polynomial factors as $(A + B)^2$. If the middle term is the negative of $2AB$, then the sign on your B can be reversed, and your polynomial factors as $(A - B)^2$.

Fact 7.5.10 The Perfect Square Trinomial Formula. *If A and B are any algebraic expressions, then:*

$$A^2 + 2AB + B^2 = (A + B)^2$$

and

$$A^2 - 2AB + B^2 = (A - B)^2$$

> **Example 7.5.11** Factor $x^2 + 6x + 9$.
>
> **Explanation**. The first term, x^2, is clearly a perfect square. So we could take $A = x$. The last term, 9, is also a perfect square since it is equal to 3^2. So we could take $B = 3$. Now we multiply $2AB = 2 \cdot x \cdot 3$, and the result is $6x$. This is the middle term, which is what we hope to see.
>
> So we can use The Perfect Square Trinomial Formula:
>
> $$\begin{aligned} x^2 + 6x + 9 &= (A + B)^2 \\ &= (x + 3)^2 \end{aligned}$$

> **Example 7.5.12** Factor $4x^2 - 20xy + 25y^2$.
>
> **Explanation**. The first term, $4x^2$, is a perfect square because it equals $(2x)^2$. So we could take $A = 2x$. The last term, $25y^2$, is also a perfect square since it is equal to $(5y)^2$. So we could take $B = 5y$. Now we multiply $2AB = 2 \cdot (2x) \cdot (5y)$, and the result is $20xy$. This is the *negative* of the middle term, which we can work with. The factored form will be $(A - B)^2$ instead of $(A + B)^2$.
>
> So we can use The Perfect Square Trinomial Formula:
>
> $$\begin{aligned} 4x^2 - 20xy + 25y^2 &= (A - B)^2 \\ &= (2x - 5y)^2 \end{aligned}$$

Checkpoint 7.5.13. Try to factor one yourself:

Factor $16q^2 + 56q + 49$.

Explanation. The first term, $16q^2$, is a perfect square because it equals $(4q)^2$. So we could take $A = 4q$. The last term, 49, is also a perfect square since it is equal to 7^2. So we could take $B = 7$. Now we multiply $2AB = 2 \cdot (4q) \cdot 7$, and the result is $56q$. This is the middle term, which is what we hope to see.

So we can use The Perfect Square Trinomial Formula:

$$\begin{aligned} 16q^2 + 56q + 49 &= (A + B)^2 \\ &= (4q + 7)^2 \end{aligned}$$

Warning 7.5.14. It is not enough to just see that the first and last terms are perfect squares. For example, $9x^2 + 10x + 25$ has its first term equal to $(3x)^2$ and its last term equal to 5^2. But when you examine $2 \cdot (3x) \cdot 5$ the result is $30x$, *not* equal to the middle term. So The Perfect Square Trinomial Formula doesn't apply here. In fact, this polynomial doesn't factor at all.

Remark 7.5.15. To factor these perfect square trinomials, we *could* use methods from Section 7.3 and Section 7.4. As an exercise for yourself, try to factor each of the three previous examples using those methods. The advantage to using The Perfect Square Trinomial Formula is that it is much faster. With some practice, all of the work for using it can be done mentally.

7.5.3 Difference/Sum of Cubes Formulas

The following calculation may seem to come from nowhere at first, but see it through. If we multiply $(A - B)\left(A^2 + AB + B^2\right)$:

$$\begin{aligned}
(A - B)\left(A^2 + AB + B^2\right) &= A^3 - BA^2 + A^2B - BAB + AB^2 - B^3 \\
&= A^3 - A^2B + A^2B - AB^2 + AB^2 - B^3 \\
&= A^3 \overbrace{- A^2B + A^2B} \overbrace{- AB^2 + AB^2} - B^3 \\
&= A^3 - B^3.
\end{aligned}$$

This is telling us that

$$A^3 - B^3 = (A - B)\left(A^2 + AB + B^2\right).$$

And so if we ever encounter a polynomial of the form $A^3 - B^3$ (a "difference of cubes") then we have a quick formula for factoring it.

A similar formula exists for factoring a *sum* of cubes, $A^3 + B^3$. Here are both formulas, followed by some tips on how to memorize them.

Fact 7.5.16 The Difference/Sum of Cubes Formula. *If A and B are any algebraic expressions, then:*

$$A^3 - B^3 = (A - B)\left(A^2 + AB + B^2\right)$$

and

$$A^3 + B^3 = (A + B)\left(A^2 - AB + B^2\right)$$

To memorize this, focus on:

- The factorization is a binomial times a trinomial.
- The sign you start with appears again in the binomial.
- The three terms in the trinomial are all quadratic: A^2, AB, and B^2.
- In the trinomial, it's *always* adding A^2 and B^2. But the sign on AB is always the *opposite* of the sign in the sum/difference of cubes.

Differences of cubes *are* sums of cubes. Technically a difference of cubes, $A^3 - B^3$, is equal to $A^3 + (-B)^3$. So you can treat any difference of cubes as a sum of cubes, $A^3 + B^3$, where B is negative and you only need to memorize the *sum* of cubes formula.

To use these formulas effectively, we need to recognize when numbers are perfect cubes. Perfect cubes become large fast before you can list too many of them. Try to memorize as many of these as you can:

$1^3 = 1$ $\quad$ $2^3 = 8$ $\quad$ $3^3 = 27$ $\quad$ $4^3 = 64$

$5^3 = 125$ $6^3 = 216$ $7^3 = 343$ $8^3 = 512$

$9^3 = 729$ $10^3 = 1000$ $11^3 = 1331$ $12^3 = 1728$

Let's look at a few examples.

Example 7.5.17 Factor $x^3 - 27$.

Explanation. We recognize that x^3 is a perfect cube and $27 = 3^3$, so we can use The Difference/Sum of Cubes Formula to factor the binomial. Note that since we have a *difference* of cubes, the binomial factor from the formula will *subtract* two terms, and the middle term from the trinomial will be $+AB$.

$$\begin{aligned} x^3 - 27 &= (A - B)\left(A^2 + AB + B^2\right) \\ &= (x - 3)\left(x^2 + (x)(3) + 3^2\right) \\ &= (x - 3)\left(x^2 + 3x + 9\right) \end{aligned}$$

Example 7.5.18 Factor $27m^3 + 64n^3$.

Explanation. We recognize that $27m^3 = (3m)^3$ and $64n^3 = (4n)^3$ are both perfect cubes, so we can use The Difference/Sum of Cubes Formula to factor the binomial. Note that since we have a *sum* of cubes, the binomial factor from the formula will *add* two terms, and the middle term from the trinomial will be $-AB$.

$$\begin{aligned} 27m^3 + 64n^3 &= (A + B)\ \left(A^2 - AB + B^2\right) \\ &= (3m + 4n)\left((3m)^2 - (3m)(4n) + (4n)^2\right) \\ &= (3m + 4n)\left(9m^2 - 12mn + 16n^2\right) \end{aligned}$$

Checkpoint 7.5.19. Try to factor one yourself:

Factor $y^3 + 1000$.

Explanation. We recognize that y^3 is a perfect cube and $1000 = 10^3$, so we can use The Difference/Sum of Cubes Formula to factor the binomial. Note that since we have a *sum* of cubes, the binomial factor from the formula will *add* two terms, and the middle term from the trinomial will be $-AB$.

$$\begin{aligned} y^3 + 1000 &= (A + B)\left(A^2 - AB + B^2\right) \\ &= (y + 10)\left(y^2 - (y)(10) + 10^2\right) \\ &= (y + 10)\left(y^2 - 10y + 100\right) \end{aligned}$$

7.5.4 Factoring in Stages

Sometimes factoring a polynomial will take two or more "stages." For instance you might use one of the special formulas from this section to factor something into two factors, and *then* those factors might be factor even more. When the task is to *factor* a polynomial, the intention is that you *fully* factor it, breaking down the pieces into even smaller pieces when that is possible.

Example 7.5.20 Factor out any greatest common factor. Factor $12z^3 - 27z$.

Explanation. The two terms of this polynomial have greatest common factor $3z$, so the first step in factoring should be to factor this out:

$$3z\left(4z^2-9\right).$$

Now we have two factors. There is nothing for us to do with $3z$, but we should ask if $\left(4z^2-9\right)$ can factor further. And in fact, that is a difference of squares. So we can apply The Difference of Squares Formula. The full process would be:

$$\begin{aligned}12z^3-27z&=3z\left(4z^2-9\right)\\&=3z(2z-3)(2z+3)\end{aligned}$$

Example 7.5.21 Recognize a *second* special pattern. Factor p^4-1.

Explanation. Since p^4 is the same as $\left(p^2\right)^2$, we have a difference of squares here. We can apply The Difference of Squares Formula:

$$p^4-1=\left(p^2-1\right)\left(p^2+1\right)$$

It doesn't end here. Of the two factors we found, $\left(p^2+1\right)$ cannot be factored further. But the other one, $\left(p^2-1\right)$ is *also* a difference of squares. So we should apply The Difference of Squares Formula again:

$$=(p-1)(p+1)\left(p^2+1\right)$$

Checkpoint 7.5.22. Try to factor one yourself:

Factor $3x^3-3$.

Explanation. The two terms of this polynomial have greatest common factor 3, so the first step in factoring should be to factor this out:

$$3\left(x^3-1\right).$$

Now we have two factors. There is nothing for us to do with 3, but we should ask if $\left(x^3-1\right)$ can factor further. And in fact, that is a difference of cubes. So we can apply The Difference/Sum of Cubes Formula. The full process would be:

$$\begin{aligned}3x^3-3&=3\left(x^3-1\right)\\&=3(x-1)(x^2+x+1)\end{aligned}$$

The trinomial from The Difference/Sum of Cubes Formula will *never* factor further. However, in some cases, the binomial from that formula will factor further, and you should look for this.

Example 7.5.23 Factor $64p^6-729$.

Explanation. We recognize that $64p^6$ is a perfect square because it equals $\left(8p^3\right)^2$. And 729 is also a perfect square because $729=27^2$. So we can use The Difference of Squares Formula to factor the binomial.

$$\begin{aligned}64p^6-729&=(A-B)(A+B)\\&=\left(8p^3-27\right)\left(8p^3+27\right)\end{aligned}$$

Next, note that in each of the factors we have an $8p^3$ and a 27, both of which can be viewed as perfect cubes: $8p^3 = (2p)^3$ and a $27 = 3^3$. So we will use *both* the difference of cubes and the sum of cubes formulas.

$$\begin{aligned}
&= (A - B)\left(A^2 + AB + B^2\right)(A + B)\left(A^2 - AB + B^2\right) \\
&= (2p - 3)\left((2p)^2 + (2p) \cdot 3 + 3^2\right)(2p + 3)\left((2p)^2 - (2p) \cdot 3 + 3^2\right) \\
&= (2p - 3)\left(4p^2 + 6p + 9\right)(2p + 3)\left(4p^2 - 6p + 9\right)
\end{aligned}$$

Example 7.5.24 Factor $32x^6y^2 - 48x^5y + 18x^4$.

Explanation. The first step of factoring any polynomial is to factor out the common factor if possible. For this trinomial, the common factor is $2x^4$, so we write

$$32x^6y^2 - 48x^5y + 18x^4 = 2x^4(16x^2y^2 - 24xy + 9).$$

The square numbers 16 and 9 in $16x^2y^2 - 24xy + 9$ hint that maybe we could use The Perfect Square Trinomial Formula. Taking $A = 4xy$ and $B = 3$, we multiply $2AB = 2 \cdot (4xy) \cdot 3$. The result is $24xy$, which is the negative of our middle term. So the whole process is:

$$\begin{aligned}
32x^6y^2 - 48x^5y + 18x^4 &= 2x^4(16x^2y^2 - 24xy + 9) \\
&= 2x^4(a - b)^2 \\
&= 2x^4(4xy - 3)^2
\end{aligned}$$

Exercises

Review and Warmup Expand the square of a *bi*nomial.

1. $(5t + 10)^2 =$

2. $(2t + 4)^2 =$

3. $(x - 4)^2 =$

4. $(x - 7)^2 =$

5. $\left(y^7 + 13\right)^2 =$

6. $\left(y^{10} + 4\right)^2 =$

Multiply the polynomials.

7. $(r - 6)(r + 6) =$

8. $(r + 12)(r - 12) =$

9. $(4t - 5)(4t + 5) =$

10. $(3t + 3)(3t - 3) =$

11. $\left(6t^5 - 11\right)\left(6t^5 + 11\right) =$

12. $\left(4x^6 - 13\right)\left(4x^6 + 13\right) =$

Factoring Factor the given polynomial.

13. $x^2 - 9 =$

14. $y^2 - 121 =$

15. $49y^2 - 81 =$

16. $4r^2 - 121 =$

17. $r^2t^2 - 49 =$

18. $t^2y^2 - 144 =$

19. $36t^2x^2 - 1 =$

20. $121t^2y^2 - 64 =$

21. $36 - x^2 =$

22. $1 - x^2 =$

23. $4 - 81y^2 =$

24. $49 - 25y^2 =$

25. $r^4 - 4 =$

26. $r^4 - 25 =$

27. $25t^4 - 121 =$

28. $144t^4 - 49 =$

29. $t^{10} - 64 =$

30. $x^6 - 25 =$

31. $16x^4 - 9y^4 =$

32. $25x^4 - 64y^4 =$

33. $x^6 - 36y^{10} =$

34. $x^{14} - 49y^{10} =$

35. $r^2 + 14r + 49 =$

36. $t^2 + 6t + 9 =$

37. $t^2 - 22t + 121 =$

38. $t^2 - 14t + 49 =$

39. $9x^2 + 6x + 1 =$

40. $100x^2 + 20x + 1 =$

41. $49y^2 - 14y + 1 =$

42. $9y^2 - 6y + 1 =$

43. $36r^2t^2 - 12rt + 1 =$

44. $121r^2y^2 - 22ry + 1 =$

45. $t^2 + 8tx + 16x^2 =$

46. $t^2 + 18tr + 81r^2 =$

47. $t^2 - 4ty + 4y^2 =$

48. $x^2 - 2xr + r^2 =$

49. $49x^2 + 84xr + 36r^2 =$

50. $4y^2 + 20yt + 25t^2 =$

51. $16y^2 - 24yt + 9t^2 =$

52. $4r^2 - 12rx + 9x^2 =$

53. $r^3 + 125 =$

54. $t^3 + 1728 =$
Hint: $1728 = 12^3$.

55. $t^3 - 512 =$
Hint: $-512 = -8^3$.

56. $t^3 - 64 =$

57. $1000x^3 + 1 =$
Hint: $1000 = 10^3$.

58. $343x^3 + 1 =$
Hint: $343 = 7^3$.

59. $27y^3 - 1 =$

60. $729y^3 - 1 =$
Hint: $729 = 9^3$.

61. $216r^3 + 125 =$
Hint: $216 = 6^3$.

62. $27r^3 + 125 =$

63. $729t^3 - 512 =$
Hint: $729 = 9^3$, $-512 = -8^3$.

64. $216t^3 - 343 =$
Hint: $216 = 6^3$, $-343 = -7^3$.

65. $x^3y^3 + 1728 =$
Hint: $1728 = 12^3$.

66. $x^3y^3 + 8 =$

67. $x^3 - 64y^3 =$

68. $x^3 - 125y^3 =$

69. $81y^4 - 16 =$

70. $16r^4 - 81 =$

71. $3r^2 - 3 =$

72. $4r^2 - 64 =$

73. $9t^3 - 36t =$

74. $3t^3 - 147t =$

75. $9x^4r^3 - 81x^2r =$

76. $7x^4y^4 - 63x^2y^2 =$

77. $108 - 12y^2 =$

78. $128 - 2y^2 =$

79. $80r^2 + 40r + 5 =$

80. $63r^2 + 42r + 7 =$

81. $64r^2y^2 + 32ry + 4 =$

82. $98t^2x^2 + 28tx + 2 =$

83. $96t^2 - 48t + 6 =$

84. $24x^2 - 24x + 6 =$

85. $9x^{10} + 6x^9 + x^8 =$

86. $121y^6 + 22y^5 + y^4 =$

87. $49y^9 - 14y^8 + y^7 =$

88. $9r^5 - 6r^4 + r^3 =$

89. $18r^8 + 12r^7 + 2r^6 =$

90. $72r^6 + 24r^5 + 2r^4 =$

91. $20t^4 - 20t^3 + 5t^2 =$

92. $36t^9 - 36t^8 + 9t^7 =$

93. $125x^4 + 64x =$

94. $8x^4 + 125x =$

95. $64x^3 + 125y^3 =$

96. $125x^3 + 216y^3 =$
Hint: $216 = 6^3$.

97. $3r^4 - 243 = \boxed{}$

98. $3r^4 - 48 = \boxed{}$

99. $r^{10} + 125r^7 = \boxed{}$

100. $t^6 + 1728t^3 = \boxed{}$

Hint: $1728 = 12^3$.

101. $t^2 + 64 = \boxed{}$

102. $x^2 + 16 = \boxed{}$

103. $5x^3 + 20x = \boxed{}$

104. $4y^3 + 100y = \boxed{}$

105. $0.09y - y^3 = \boxed{}$

106. $0.81r - r^3 = \boxed{}$

107. $(r - 8)^2 - 81 = \boxed{}$

108. $(r - 7)^2 - 49 = \boxed{}$

109. $x^2 + 14x + 49 - 121y^2 = \boxed{}$

110. $x^2 + 20x + 100 - 49y^2 = \boxed{}$

7.6 Factoring Strategies

7.6.1 Factoring Strategies

Deciding which method to use when factoring a random polynomial can seem like a daunting task. Understanding all of the techniques that we have learned and how they fit together can be done using a decision tree.

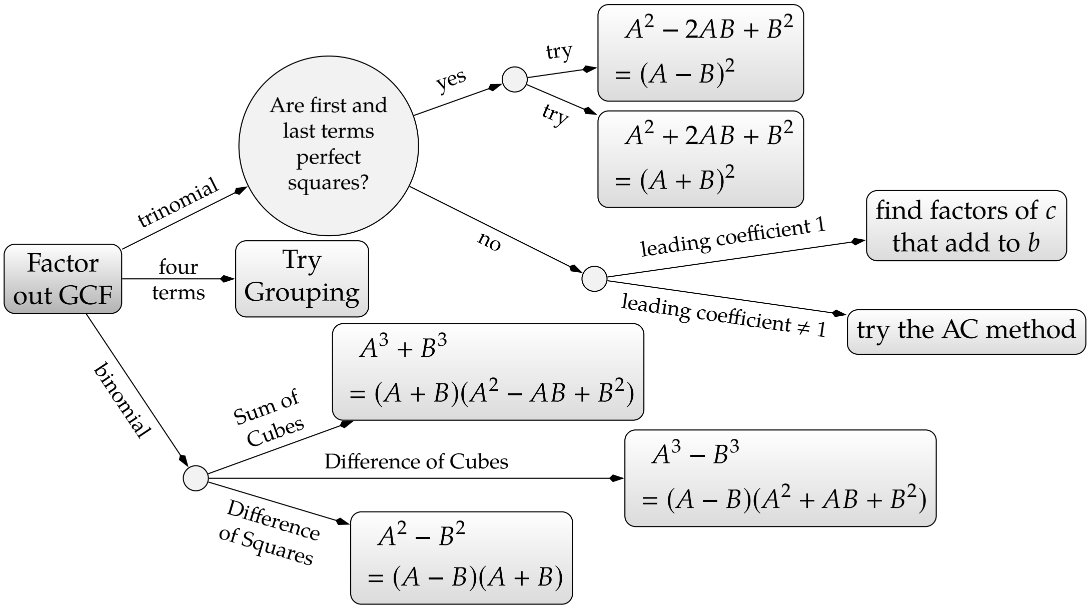

Factor out GCF
trinomial
four terms
binomial
Try Grouping
Are first and last terms perfect squares?
yes
no
try
try
$A^2 - 2AB + B^2 = (A - B)^2$
$A^2 + 2AB + B^2 = (A + B)^2$
leading coefficient 1
leading coefficient ≠ 1
find factors of c that add to b
try the AC method
Sum of Cubes
Difference of Cubes
Difference of Squares
$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$
$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$
$A^2 - B^2 = (A - B)(A + B)$

Figure 7.6.2: Factoring Decision Tree

Using the decision tree can guide us when we are given an expression to factor.

> **Example 7.6.3** Factor the expression $4k^2 + 12k - 40$ completely.
>
> **Explanation**. Start by noting that the GCF is 4. Factoring this out, we get
>
> $$4k^2 + 12k - 40 = 4\left(k^2 + 3k - 10\right).$$
>
> Following the decision tree, we now have a trinomial where the leading coefficient is 1 and we need to look for factors of −10 that add to 3. We find that −2 and 5 work. So, the full factorization is:
>
> $$\begin{aligned} 4k^2 + 12k - 40 &= 4\left(k^2 + 3k - 10\right) \\ &= 4(k - 2)(k + 5) \end{aligned}$$

> **Example 7.6.4** Factor the expression $64d^2 + 144d + 81$ completely.
>
> **Explanation**. Start by noting that the GCF is 1, and there is no GCF to factor out. We continue along the decision tree for a trinomial. Notice that both 64 and 81 are perfect squares and that this expression might factor using the pattern $A^2 + 2AB + B^2 = (A + B)^2$. To find A and B, take the square roots of the first and last terms, so $A = 8d$ and $B = 9$. We have to check that the middle term is correct: since

$2AB = 2(8d)(9) = 144d$ matches our middle term, the expression must factor as

$$64d^2 + 144d + 81 = (8d + 9)^2.$$

Example 7.6.5 Factor the expression $10x^2y - 12xy^2$ completely.

Explanation. Start by noting that the GCF is $2xy$. Factoring this out, we get

$$10x^2y - 12xy^2 = 2xy(5x - 6y).$$

Since we have a binomial inside the parentheses, the only options on the decision tree for a binomial involve squares or cubes. Since there are none, we conclude that $2xy(5x - 6y)$ is the complete factorization.

Example 7.6.6 Factor the expression $9b^2 - 25y^2$ completely.

Explanation. Start by noting that the GCF is 1, and there is no GCF to factor out. We continue along the decision tree for a binomial and notice that we now have a difference of squares, $A^2 - B^2 = (A - B)(A + B)$. To find the values for A and B that fit the patterns, just take the square roots. So $A = 3b$ since $(3b)^2 = 9b^2$ and $B = 5y$ since $(5y)^2 = 25y^2$. So, the expression must factor as

$$9b^2 - 25y^2 = (3b - 5y)(3b + 5y).$$

Example 7.6.7 Factor the expression $24w^3 + 6w^2 - 9w$ completely.

Explanation. Start by noting that the GCF is $3w$. Factoring this out, we get

$$24w^3 + 6w^2 - 9w = 3w\left(8w^2 + 2w - 3\right).$$

Following the decision tree, we now have a trinomial inside the parentheses where $a \neq 1$. We should try the AC method because neither 8 nor -3 are perfect squares. In this case, $ac = -24$ and we must find two factors of -24 that add to be 2. The numbers 6 and -4 work in this case. The rest of the factoring process is:

$$\begin{aligned}
24w^3 + 6w^2 - 9w &= 3w\left(8w^2 \overbrace{+\,2w} -3\right)\\
&= 3w\left(8w^2 \overbrace{+\,6w - 4w} -3\right)\\
&= 3w\left(\left(8w^2 + 6w\right) + (-4w - 3)\right)\\
&= 3w\left(2w(4w + 3) - 1(4w + 3)\right)\\
&= 3w(4w + 3)(2w - 1)
\end{aligned}$$

Example 7.6.8 Factor the expression $q^5 + q^2$ completely.

Explanation. Start by noting that the GCF is q^2. Factoring this out, we find

$$q^5 + q^2 = q^2\left(q^3 + 1\right).$$

Following the decision tree, we now have a binomial with a sum of cubes. (Notice that $1^3 = 1$.) So using the sum of cubes formula, we have the complete factorization:

$$\begin{aligned} q^5 + q^2 &= q^2\left(q^3 + 1\right) \\ &= q^2(q+1)\left(q^2 - q + 1\right). \end{aligned}$$

Example 7.6.9 Factor the expression $-6xy + 9y + 2x - 3$ completely.

Explanation. Start by noting that the GCF is 1, and there is no GCF to factor out. We continue along the decision tree. Since we have a four-term polynomial, we should try to factor by grouping. The full process is:

$$\begin{aligned} -6xy + 9y + 2x - 3 &= (-6xy + 9y) + (2x - 3) \\ &= -3y(2x - 3) + 1(2x - 3) \\ &= (2x - 3)(-3y + 1) \end{aligned}$$

Note that the negative sign in front of the $3y$ can be factored out if you wish. That would look like:

$$= -(2x - 3)(3y - 1)$$

Example 7.6.10 Factor the expression $4w^3 - 20w^2 + 24w$ completely.

Explanation. Start by noting that the GCF is $4w$. Factoring this out, we get

$$4w^3 - 20w^2 + 24w = 4w\left(w^2 - 5w + 6\right).$$

Following the decision tree, we now have a trinomial with $a = 1$ inside the parentheses. So, we can look for factors of 6 that add up to -5. Since -3 and -2 fit the requirements, the full factorization is:

$$\begin{aligned} 4w^3 - 20w^2 + 24w &= 4w\left(w^2 - 5w + 6\right) \\ &= 4w(w - 3)(w - 2) \end{aligned}$$

Example 7.6.11 Factor the expression $9 - 24y + 16y^2$ completely.

Explanation. Start by noting that the GCF is 1, and there is no GCF to factor out. Continue along the decision tree. We now have a trinomial where both the first term, 9, and last term, $16y^2$, look like perfect squares. To use the perfect squares difference pattern, $A^2 - 2AB + B^2 = (A - B)^2$, recall that we need to mentally take the square roots of these two terms to find A and B. So, $A = 3$ since $3^2 = 9$, and $B = 4y$ since $(4y)^2 = 16y^2$. Now we have to check that $2AB$ matches $24y$:

$$2AB = 2(3)(4y) = 24y.$$

So the full factorization is:

$$9 - 24y + 16y^2 = (3 - 4y)^2.$$

Example 7.6.12 Factor the expression $9 - 25y + 16y^2$ completely.

Explanation. Start by noting that the GCF is 1, and there is no GCF to factor out. Since we now have a trinomial where both the first term and last term are perfect squares in exactly the same way as in

Example 11. However, we cannot apply the perfect squares method to this problem because it worked when $2AB = 24y$. Since our middle term is $25y$, we can be certain that it won't be a perfect square.

Continuing on with the decision tree, our next option is to use the AC method. You might be tempted to rearrange the order of the terms, but that is unnecessary. In this case, $ac = 144$ and we need to come up with two factors of 144 that add to be -25. After a brief search, we conclude that those values are -16 and -9. The remainder of the factorization is:

$$\begin{aligned} 9 \overbrace{-25y} + 16y^2 &= 9 \overbrace{-16y - 9y} + 16y^2 \\ &= \big(9 - 16y\big) + \big(-9y + 16y^2\big) \\ &= 1\big(9 - 16y\big) - y\big(9 + 16y\big) \\ &= \big(9 - 16y\big)(1 - y) \end{aligned}$$

Example 7.6.13 Factor the expression $20x^4 + 13x^3 - 21x^2$ completely.

Explanation. Start by noting that the GCF is x^2. Factoring this out, we get

$$20x^4 + 13x^3 - 21x^2 = x^2\left(20x^2 + 13x - 21\right).$$

Following the decision tree, we now have a trinomial inside the parentheses where $a \neq 1$ and we should try the AC method. In this case, $ac = -420$ and we need factors of -420 that add to 13.

Factor Pair	Sum
$1 \cdot -420$	-419
$2 \cdot -210$	-208
$3 \cdot -140$	-137
$4 \cdot -105$	-101

Factor Pair	Sum
$5 \cdot -84$	-79
$6 \cdot -70$	-64
$7 \cdot -60$	-53
$10 \cdot -42$	-32

Factor Pair	Sum
$12 \cdot -35$	-23
$14 \cdot -30$	-16
$15 \cdot -28$	-13
$20 \cdot -21$	-1

In the table of the factor pairs of -420 we find $15 + (-28) = -13$, the opposite of what we want, so we want the opposite numbers: -15 and 28. The rest of the factoring process is shown:

$$\begin{aligned} 20x^4 + 13x^3 - 21x^2 &= x^2\left(20x^2 \overbrace{+13x} - 21\right) \\ &= x^2\left(20x^2 \overbrace{-15x + 28x} - 21\right) \\ &= x^2\left(\big(20x^2 - 15x\big) + (28x - 21)\right) \\ &= x^2\left(5x(4x - 3) + 7(4x - 3)\right) \\ &= x^2(4x - 3)(5x + 7) \end{aligned}$$

Exercises

Strategies

1. In factoring $7x^3 - 2401y^3$, which factoring techniques/tools will be useful? Check all that apply.

   ☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to c and sum to b ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

2. In factoring $-7r - 4$, which factoring techniques/tools will be useful? Check all that apply.

   ☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to c and sum to b ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

3. In factoring $8a^2 - 8$, which factoring techniques/tools will be useful? Check all that apply.

   ☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to c and sum to b ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

4. In factoring $16b^2 + 8b + 1$, which factoring techniques/tools will be useful? Check all that apply.

   ☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to c and sum to b ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

5. In factoring $A^2 - 16A + 64$, which factoring techniques/tools will be useful? Check all that apply.

   ☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to c and sum to b ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

6. In factoring $2560B^3 + 3645$, which factoring techniques/tools will be useful? Check all that apply.

   ☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to c and sum to b ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

7. In factoring $mx - m - 5x + 5$, which factoring techniques/tools will be useful? Check all that apply.

   ☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to c and sum to b ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

8. In factoring $200n^2 - 32C^2$, which factoring techniques/tools will be useful? Check all that apply.

☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to c and sum to b ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

9. In factoring $q^2 - 4qa + 4a^2$, which factoring techniques/tools will be useful? Check all that apply.

☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to c and sum to b ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

10. In factoring $-36n - 6$, which factoring techniques/tools will be useful? Check all that apply.

☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to c and sum to b ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

11. In factoring $9r^3 + 9C^3$, which factoring techniques/tools will be useful? Check all that apply.

☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to c and sum to b ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

12. In factoring $56ta + 21t + 8a + 3$, which factoring techniques/tools will be useful? Check all that apply.

☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to c and sum to b ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

13. In factoring $4b^4 - 4b$, which factoring techniques/tools will be useful? Check all that apply.

☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to c and sum to b ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

14. In factoring $16A^2 + 72A + 81$, which factoring techniques/tools will be useful? Check all that apply.

☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to c and sum to b ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

Factoring Factor the given polynomial.

15. $4r+4=$

16. $-10r-10=$

17. $14t^2-35=$

18. $30t^4+80t^3+80t^2=$

19. $20x-50x^2+90x^3=$

20. $3xy+3y=$

21. $64x^5y^8-24x^4y^8+72x^3y^8=$

22. $y^2+9y+8y+72=$

23. $xy-6x-10y+60=$

24. $x^3-7-5x^3y+35y=$

25. $r^2-2r-80=$

26. $5t^2-23t-10=$

27. $3t^2x^2+tx-4=$

28. $3x^2+x+8=$

29. $6x^2-17x-3=$

30. $4y^2+16y+7=$

31. $9y^2-15y+4=$

32. $3y^2+5yr+2r^2=$

33. $2r^2-9rx+9x^2=$

34. $8r^2-17ry-21y^2=$

35. $20t^2+13ty+2y^2=$

36. $12t^2-7ty+y^2=$

37. $4x^2+6x-28=$

38. $15x^2y^2+27xy-6=$

39. $8y^{10}+28y^9+20y^8=$

40. $10y^5-22y^4+4y^3=$

41. $9x^2+24xy+15y^2=$

42. $9x^2-24xy+12y^2=$

43. $r^2+9r+18=$

44. $t^2-9t+14=$

45. $t^2+10tx+24x^2=$

46. $x^2y^2-xy-30=$

47. $x^2-7xt+10t^2=$

48. $2y^2t^2+10yt+8=$

49. $3y^2-12y-36=$

50. $10y^7+30y^6+20y^5=$

51. $10r^6-30r^5+20r^4=$

52. $2x^2y+10xy+12y=$

53. $2x^2y-16xy+14y=$

54. $3x^2y^3-15xy^2+12y=$

55. $x^2y^2+2x^2yz-3x^2z^2=$

56. $x^2+0.9x+0.14=$

57. $y^2 - 64 =$ ☐

58. $y^2r^2 - 100 =$ ☐

59. $36 - y^2 =$ ☐

60. $r^4 - 64 =$ ☐

61. $r^{14} - 100 =$ ☐

62. $x^{14} - 121y^{10} =$ ☐

63. $16t^4 - 81 =$ ☐

64. $2x^3 - 18x =$ ☐

65. $x^2 + 121 =$ ☐

66. $40 - 10y^2 =$ ☐

67. $y^2 + 6y + 9 =$ ☐

68. $y^2 - 18yt + 81t^2 =$ ☐

69. $r^2 - 12r + 36 =$ ☐

70. $9r^2 - 6r + 1 =$ ☐

71. $t^2 + 6tx + 9x^2 =$ ☐

72. $16t^2 + 40ty + 25y^2 =$ ☐

73. $48x^2t^2 + 24xt + 3 =$ ☐

74. $100x^5 + 20x^4 + x^3 =$ ☐

75. $12y^{10} + 12y^9 + 3y^8 =$ ☐

76. $y^3 + 8 =$ ☐

77. $512y^3 + 1 =$ ☐
Hint: $512 = 8^3$.

78. $125r^3 + 729 =$ ☐
Hint: $729 = 9^3$.

79. $x^3y^3 + 729 =$ ☐
Hint: $729 = 9^3$.

80. $343t^4 + 512t =$ ☐
Hint: $343 = 7^3, 512 = 8^3$.

81. $0.16t - t^3 =$ ☐

82. $3x^4 - 243 =$ ☐

83. $x^2 - 14x + 49 - 64y^2 =$ ☐

84. $30x^2(y+2) + 36x(y+2) + 6(y+2) =$ ☐

Challenge

85. Select the expression which is equivalent to the following expression: $333^2 - 888^2$ (☐ 1221(1221) ☐ 1221(-555) ☐ -555(-555) ☐ none of the above)

7.7 Factoring Chapter Review

7.7.1 Review of Factoring out the GCF

In Section 7.1 we covered how to factor out the greatest common factor. Recall that the **greatest common factor** between two expressions is the largest factor that goes in evenly to both expressions.

Example 7.7.1 Finding the Greatest Common Factor. What is the greatest common factor between $12x^3y$ and $42x^2y^2$?

Explanation. Break down each of these into its factors:

$$12x^3y = (2 \cdot 2) \cdot 3 \cdot (x \cdot x \cdot x) \cdot y \qquad 42x^2y^2 = 2 \cdot 3 \cdot 7 \cdot (x \cdot x) \cdot (y \cdot y)$$

Identify the common factors:

$$12x^3y = \overset{\downarrow}{2} \cdot 2 \cdot \overset{\downarrow}{3} \cdot \overset{\downarrow}{x} \cdot \overset{\downarrow}{x} \cdot x \cdot \overset{\downarrow}{y} \qquad 42x^2y^2 = \overset{\downarrow}{2} \cdot \overset{\downarrow}{3} \cdot 7 \cdot \overset{\downarrow}{x} \cdot \overset{\downarrow}{x} \cdot \overset{\downarrow}{y} \cdot y$$

With 2, 3, two x's and a y in common, the greatest common factor is $6x^2y$.

Example 7.7.2 What is the greatest common factor between $18c^3y^2$ and $27y^3c$?

Explanation. Break down each into factors. You can definitely do this mentally with practice.

$$18c^3y^2 = 2 \cdot 3 \cdot 3 \cdot c \cdot c \cdot c \cdot y \cdot y \qquad 27y^3c = 3 \cdot 3 \cdot 3 \cdot y \cdot y \cdot y \cdot c$$

And take note of the common factors.

$$18c^3y^2 = 2 \cdot \overset{\downarrow}{3} \cdot \overset{\downarrow}{3} \cdot \overset{\downarrow}{c} \cdot c \cdot c \cdot \overset{\downarrow}{y} \cdot \overset{\downarrow}{y} \qquad 27y^3c = \overset{\downarrow}{3} \cdot \overset{\downarrow}{3} \cdot 3 \cdot \overset{\downarrow}{y} \cdot \overset{\downarrow}{y} \cdot y \cdot c$$

And so the GCF is $9y^2c$

Example 7.7.3 Factoring out the Greatest Common Factor. Factor out the GCF from the expression $32mn^2 - 24m^2n - 12mn$.

Explanation. To factor out the GCF from the expression $32mn^2 - 24m^2n - 12mn$, first note that the GCF to all three terms is $4mn$. Begin by writing that in front of a blank pair of parentheses and fill in the missing pieces.

$$\begin{aligned} 32mn^2 - 24m^2n - 12mn &= 4mn(\quad - \quad - \quad) \\ &= 4mn(8n - 6m - 3) \end{aligned}$$

Example 7.7.4 Factor out the GCF from the expression $14x^3 - 35x^2$.

Explanation. First note that the GCF of the terms in $14x^3 - 35x^2$ is $7x^2$. Factoring this out, we have:

$$\begin{aligned} 14x^3 - 35x^2 &= 7x^2\,(\quad - \quad) \\ &= 7x^2\,(2x - 5) \end{aligned}$$

Example 7.7.5 Factor out the GCF from the expression $36m^3n^2 - 18m^2n^5 + 24mn^3$.

Explanation. First note that the GCF of the terms in $36m^3n^2 - 18m^2n^5 + 24mn^3$ is $6mn^2$. Factoring this out, we have:

$$\begin{aligned}36m^3n^2 - 18m^2n^5 + 24mn^3 &= 6mn^2\left(\quad - \quad + \quad\right)\\&= 6mn^2\left(6m^2 - 3mn^3 + 4n\right)\end{aligned}$$

Example 7.7.6 Factor out the GCF from the expression $42f^3w^2 - 8w^2 + 9f^3$.

Explanation. First note that the GCF of the terms in $42f^3w^2 - 8w^2 + 9f^3$ is 1. The only way to factor the GCF out of this expression is:

$$42f^3w^2 - 8w^2 + 9f^3 = 1\left(42f^3w^2 - 8w^2 + 9f^3\right)$$

7.7.2 Review of Factoring by Grouping

In Section 7.2 we covered how to factor by grouping. Recall that factoring using grouping is used on four-term polynomials, and also later in the AC method in Section 7.4. Begin by grouping two pairs of terms and factoring out their respective GCF; if all is well, we should be left with two matching pieces in parentheses that can be factored out in their own right.

Example 7.7.7 Factor the expression $2x^3 + 5x^2 + 6x + 15$ using grouping.

Explanation.

$$\begin{aligned}2x^3 + 5x^2 + 6x + 15 &= \left(2x^3 + 5x^2\right) + (6x + 15)\\&= x^2(2x + 5) + 3(2x + 5)\\&= \left(x^2 + 3\right)(2x + 5)\end{aligned}$$

Example 7.7.8 Factor the expression $2xy - 3x - 8y + 12$ using grouping.

Explanation.

$$\begin{aligned}2xy - 3x + 8y - 12 &= \left(2xy - 3x\right) + \left(-8y + 12\right)\\&= x\left(2y - 3\right) - 4\left(2y - 3\right)\\&= (x - 4)(2y - 3)\end{aligned}$$

Example 7.7.9 Factor the expression $xy - 2 - 2x + y$ using grouping.

Explanation. This is a special example because if we try to simply follow the algorithm without considering the bigger context, we will fail:

$$xy - 2 - 2x + y = \left(xy - 2\right) + \left(-2x + y\right)$$

Note that there is no common factor in either grouping, besides 1, but the groupings themselves don't match. We should now recognize that whatever we are doing isn't working and try something else. It turns out that this polynomial *isn't* prime; all we need to do is rearrange the polynomial into standard form where the degrees decrease from left to right before grouping.

$$\begin{aligned} xy - 2 - 2x + y &= xy - 2x + y - 2 \\ &= \left(xy - 2x\right) + \left(y - 2\right) \\ &= x\left(y - 2\right) + 1\left(y - 2\right) \\ &= (x + 1)(y - 2) \end{aligned}$$

Example 7.7.10 Factor the expression $15m^2 - 3m - 10mn + 2n$ using grouping.

Explanation.

$$\begin{aligned} 15m^2 - 3m - 10mn + 2n &= \left(15m^2 - 3m\right) + (-10mn + 2n) \\ &= 3m\,(5m - 1) - 2n\,(5m - 1) \\ &= (3m - 2n)(5m - 1) \end{aligned}$$

7.7.3 Review of Factoring Trinomials with Leading Coefficient 1

In Section 7.3 we covered factoring expressions that look like $x^2 + bx + c$. The trick was to look for two numbers whose product was c and whose sum was b. Always remember to look for a greatest common factor first, before looking for factor pairs.

Example 7.7.11 Answer the questions to practice for the factor pairs method.

a. What two numbers multiply to be 6 and add to be 5?
b. What two numbers multiply to be −6 and add to be 5?
c. What two numbers multiply to be −6 and add to be −1?
d. What two numbers multiply to be 24 and add to be −10?
e. What two numbers multiply to be −24 and add to be 2?
f. What two numbers multiply to be −24 and add to be −5?
g. What two numbers multiply to be 420 and add to be 44?
h. What two numbers multiply to be −420 and add to be −23?
i. What two numbers multiply to be 420 and add to be −41?

Explanation.

a. What two numbers multiply to be 6 and add to be 5? The numbers are 2 and 3.
b. What two numbers multiply to be −6 and add to be 5? The numbers are 6 and −1.
c. What two numbers multiply to be −6 and add to be −1? The numbers are −3 and 2.
d. What two numbers multiply to be 24 and add to be −10? The numbers are −6 and −4.
e. What two numbers multiply to be −24 and add to be 2? The numbers are 6 and −4.
f. What two numbers multiply to be −24 and add to be −5? The numbers are −8 and 3.

g. What two numbers multiply to be 420 and add to be 44? The numbers are 30 and 14.

h. What two numbers multiply to be −420 and add to be −23? The numbers are −35 and 12.

i. What two numbers multiply to be 420 and add to be −41? The numbers are −20 and −21.

Note that for parts g–i, the factors of 420 are important. Below is a table of factors of 420 which will make it much clearer how the answers were found. To generate a table like this, we start with 1, and we work our way up the factors of 420.

Factor Pair
$1 \cdot 420$
$2 \cdot 210$
$3 \cdot 140$
$4 \cdot 105$

Factor Pair
$5 \cdot 84$
$6 \cdot 70$
$7 \cdot 60$
$10 \cdot 42$

Factor Pair
$12 \cdot 35$
$14 \cdot 30$
$15 \cdot 28$
$20 \cdot 21$

It is now much easier to see how to find the numbers in question. For example, to find two numbers that multiply to be −420 and add to be −23, simply look in the table for two factors that are 23 apart and assign a negative sign appropriately. As we found earlier, the numbers that are 23 apart are 12 and 35, and making the larger one negative, we have our answer: 12 and −35.

Example 7.7.12 Factor the expression $x^2 - 3x - 28$

Explanation. To factor the expression $x^2 - 3x - 28$, think of two numbers that multiply to be −28 and add to be −3. In the Section 7.3, we created a table of all possibilities of factors, like the one shown, to be sure that we never missed the right numbers; however, we encourage you to try this mentally for most problems.

Factor Pair	Sum of the Pair
$-1 \cdot 28$	27
$-2 \cdot 14$	12
$-4 \cdot 7$	3 (close; wrong sign)

Factor Pair	Sum of the Pair
$1 \cdot (-28)$	−27
$2 \cdot (-14)$	−12
$4 \cdot (-7)$	−3 (what we wanted)

Since the two numbers in question are 4 and −7 that means simply that

$$x^2 - 3x - 28 = (x + 4)(x - 7)$$

Remember that you can always multiply out your factored expression to verify that you have the correct answer. We will use the FOIL expansion.

$$\begin{aligned}(x + 4)(x - 7) &= x^2 - 7x + 4x - 28\\ &\overset{\checkmark}{=} x^2 - 3x - 28\end{aligned}$$

Example 7.7.13 Factoring in Stages. Completely factor the expression $4x^3 - 4x^2 - 120x$.

Explanation. Remember that some expressions require more than one step to completely factor. To factor $4x^3 - 4x^2 - 120x$, first, always look for any GCF; after that is done, consider other options. Since the GCF is $4x$, we have that

$$4x^3 - 4x^2 - 120x = 4x\left(x^2 - x - 30\right).$$

Now the factor inside parentheses might factor further. The key here is to consider what two numbers multiply to be -30 and add to be -1. In this case, the answer is -6 and 5. So, to completely write the factorization, we have:

$$\begin{aligned}4x^3 - 4x^2 - 120x &= 4x\left(x^2 - x - 30\right)\\ &= 4x(x-6)(x+5)\end{aligned}$$

Example 7.7.14 Factoring Expressions with Higher Powers. Completely factor the expression $p^{10} - 6p^5 - 72$.

Explanation. If we have a trinomial with an even exponent on the leading term, and the middle term has an exponent that is half the leading term exponent, we can still use the factor pairs method. To factor $p^{10} - 6p^5 - 72$, we note that the middle term exponent 5 is half of the leading term exponent 10, and that two numbers that multiply to be -72 and add to be -6 are -12 and 6. So the factorization of the expression is

$$p^{10} - 6p^5 - 72 = \left(p^5 - 12\right)\left(p^5 + 6\right)$$

Example 7.7.15 Factoring Expressions with Two Variables. Completely factor the expression $x^2 - 3xy - 70y^2$.

Explanation. If an expression has two variables, like $x^2 - 3xy - 70y^2$, we pretend for a moment that the expression is $x^2 - 3x - 70$. To factor this expression we ask ourselves "what two numbers multiply to be -70 and add to be -3?" The two numbers in question are 7 and -10. So $x^2 - 3x - 70$ factors as $(x+7)(x-10)$.

To go back to the original problem now, simply make the two numbers $7y$ and $-10y$. So, the full factorization is

$$x^2 - 3xy - 70y^2 = (x+7y)(x-10y)$$

With problems like this, it is important to verify the your answer to be sure that all of the variables ended up where they were supposed to. So, to verify, simply FOIL your answer.

$$\begin{aligned}(x+7y)(x-10y) &= x^2 - 10xy + 7yx - 70y^2\\ &= x^2 - 10xy + 7xy - 70y^2\\ &\overset{\checkmark}{=} x^2 - 3xy - 70y^2\end{aligned}$$

Example 7.7.16 Completely factor the expressions.

a. $x^2 - 11x + 30$

b. $-s^2 + 3s + 28$

c. $g^2 - 3g - 24$

d. $w^2 - wr - 30r^2$

e. $z^8 + 2z^4 - 63$

Explanation.

a. $x^2 - 11x + 30 = (x-6)(x-5)$

b. $$\begin{aligned}-s^2+3s+28 &= -\left(s^2-3s-28\right)\\ &= -(s-7)(s+4)\end{aligned}$$

c. $g^2-3g-24$ is prime. No two integers multiply to be -24 and add to be -3.

d. $w^2-wr-30r^2=(w-6r)(w+5r)$

e. $z^8+2z^4-63=\left(z^4-7\right)\left(z^4+9\right)$

7.7.4 Review of Factoring Trinomials with Non-Trivial Leading Coefficient

In Section 7.4 we covered factoring trinomials of the form ax^2+bx+c when $a\neq 1$ using the AC method.

Example 7.7.17 Using the AC Method. Completely factor the expression $9x^2-6x-8$.

Explanation. To factor the expression $9x^2-6x-8$, we first find ac:

1. $9\cdot(-8)=-72$.
2. Examine factor pairs that multiply to -72, looking for a pair that sums to -6:

Factor Pair	Sum of the Pair
$1\cdot -72$	-71
$2\cdot -36$	-34
$3\cdot -24$	-21
$4\cdot -18$	-14
$6\cdot -12$	-6
$8\cdot -9$	(no need to go this far)

Factor Pair	Sum of the Pair
$-1\cdot 72$	(no need to go this far)
$-2\cdot 36$	(no need to go this far)
$-3\cdot 24$	(no need to go this far)
$-4\cdot 18$	(no need to go this far)
$-6\cdot 12$	(no need to go this far)
$-8\cdot 9$	(no need to go this far)

3. Intentionally break up the -6 as $6+(-12)$ and then factor using grouping:

$$\begin{aligned}9x^2\overbrace{-6x}-8 &= 9x^2\overbrace{+6x-12x}-8\\ &= \left(9x^2+6x\right)+(-12x-8)\\ &= 3x(3x+2)-4(3x+2)\\ &= (3x+2)(3x-4)\end{aligned}$$

Example 7.7.18 Completely factor the expression $3x^2+5x-6$.

Explanation. First note that there is no GCF besides 1 and that $ac=-18$. To look for two factors of -18 that add up to 5, we will make a factor pair table.

Factor Pair	Sum of the Pair
$1\cdot -18$	-17
$2\cdot -9$	-7
$3\cdot -6$	-3

Factor Pair	Sum of the Pair
$-1\cdot 18$	17
$-2\cdot 9$	7
$-3\cdot 6$	3

Since none of the factor pairs of -18 sum to 5, we must conclude that this trinomial is prime. The only way to factor it is $3x^2 + 5x - 6 = 1\left(3x^2 + 5x - 6\right)$.

Example 7.7.19 Completely factor the expression $3y^2 + 20y - 63$.

Explanation. First note that $ac = -189$. Looking for two factors of -189 that add up to 20, we find 27 and -7. Breaking up the $+20$ into $+27 - 7$, we can factor using grouping.

$$\begin{aligned} 3y^2 \overbrace{+20y} -63 &= 3y^2 \overbrace{+27y - 7y} -63 \\ &= \left(3y^2 + 27y\right) + \left(-7y - 63\right) \\ &= 3y\left(y + 9\right) - 7\left(y + 9\right) \\ &= \left(y + 9\right)(3y - 7) \end{aligned}$$

Example 7.7.20 Factoring in Stages with the AC Method. Completely factor the expression $8y^3 + 54y^2 + 36y$.

Explanation. Recall that some trinomials need to be factored in stages: the first stage is always to factor out the GCF. To factor $8y^3 + 54y^2 + 36y$, first note that the GCF of the three terms in the expression is $2y$. Then apply the AC method:

$$8y^3 + 54y^2 + 36y = 2y\left(4y^2 + 27y + 18\right)$$

Now we find $ac = 4 \cdot 18 = 72$. What two factors of 72 add up to 27? After checking a few numbers, we find that 3 and 24 fit the requirements. So:

$$\begin{aligned} &= 2y\left(4y^2 \overbrace{+27y} +18\right) \\ &= 2y\left(4y^2 \overbrace{+3y + 24y} +18\right) \\ &= 2y\left(\left(4y^2 + 3y\right) + \left(24y + 18\right)\right) \\ &= 2y\left(y\left(4y + 3\right) + 6\left(4y + 3\right)\right) \\ &= 2y\left(4y + 3\right)\left(y + 6\right) \end{aligned}$$

Example 7.7.21 Completely factor the expression $18x^3 + 26x^2 + 4x$.

Explanation. First note that there is a GCF of $2x$ which should be factored out first. Doing this leaves us with $18x^3 + 26x^2 + 8x = 2x\left(9x^2 + 13x + 4\right)$. Now we apply the AC method on the factor in the parentheses. So, $ac = 36$, and we must find two factors of 36 that sum to be 13. These two factors are 9 and 4. Now we can use grouping.

$$\begin{aligned} 18x^3 + 26x^2 + 8x &= 2x\left(9x^2 \overbrace{+13x} +4\right) \\ &= 2x\left(9x^2 \overbrace{+9x + 4x} +4\right) \end{aligned}$$

$$\begin{aligned}&= 2x\left(\left(9x^2+9x\right)+(4x+4)\right)\\&= 2x\left(9x(x+1)+4(x+1)\right)\\&= 2x(x+1)(9x+4)\end{aligned}$$

7.7.5 Review of Factoring Special Forms

In Section 7.5 we covered how to factor binomials and trinomials using formulas. Using these formulas, when appropriate, often drastically increased the speed of factoring. Below is a summary of the formulas covered. For each, consider that A and B could be any algebraic expressions.

Difference of Squares $A^2 - B^2 = (A+B)(A-B)$

Perfect Square Sum $A^2 + 2AB + B^2 = (A+B)^2$

Perfect Square Difference $A^2 - 2AB + B^2 = (A+B)^2$

Difference of Cubes $A^3 - B^3 = (A-B)\left(A^2 + AB + B^2\right)$

Sum of Cubes $A^3 + B^3 = (A+B)\left(A^2 - AB + B^2\right)$

Example 7.7.22 Factoring the Form $A^2 - 2AB + B^2$. Completely factor the expression $16y^2 - 24y + 9$.

Explanation. To factor $16y^2-24y+9$ we notice that the expression might be of the form $A^2-2AB+B^2$. To find A and B, we mentally take the square root of both the first and last terms of the original expression. The square root of $16y^2$ is $4y$ since $(4y)^2 = 4^2y^2 = 16y^2$. The square root of 9 is 3. So, we conclude that $A = 4y$ and $B = 3$. Recall that we now need to check that the $24y$ matches our $2AB$. Using our values for A and B, we indeed see that $2AB = -2(4y)(3) = 24y$. So, we conclude that

$$16y^2 - 24y + 9 = (4y-3)^2.$$

Example 7.7.23 Mixed Special Forms Factoring.

a. Completely factor the expression $v^3 - 27$.
b. Completely factor the expression $9w^2 + 12w + 4$.
c. Completely factor the expression $4q^2 - 81$.
d. Completely factor the expression $9p^2 + 25$.
e. Completely factor the expression $121b^2 - 36$.
f. Completely factor the expression $25u^2 - 70u + 49$.
g. Completely factor the expression $64q^3 - 27y^3$.

Explanation. The first step for each problem is to try to fit the expression to one of the special factoring forms.

a. To factor $v^3 - 27$ we notice that the expression is of the form $A^3 - B^3$. To find values for A and B, take mentally take the cube root of both terms. So, $A = v$ and $B = 3$. So, using the form $A^3 - B^3 = (A-B)(A^2 + AB + B^2)$, we have that

$$\begin{aligned}v^3 - 27 &= (v-3)\left(v^2 + (v)(3) + 3^2\right)\\&= (v-3)\left(v^2 + 3v + 9\right)\end{aligned}$$

b. To factor $9w^2 + 12w + 4$ we notice that the expression might be of the form $A^2 + 2AB + B^2$ where $A = 3w$ and $B = 2$. With this formula we need to check the value of $2AB$ which in this case is $2AB = 2(3w)(2) = 12w$. Since the value of $2AB$ is correct, the expression must factor as

$$9w^2 + 12w + 4 = (3w + 2)^2$$

c. To factor $4q^2 - 81$ we notice that the expression is of the form $A^2 - B^2$ where $A = 2q$ and $B = 9$. Thus, the expression must factor as

$$4q^2 - 81 = (2q - 9)(2q + 9)$$

d. To factor $9p^2 + 25$ we notice that the expression is of the form $A^2 + B^2$. This is called a sum of squares. If you recall from the section, the sum of squares is always prime. So $9p^2 + 25$ is prime.

e. To completely factor the expression $121b^2 - 36$ first note that the expression is of the form $A^2 - B^2$ where $A = 11b$ and $B = 6$. So, the expression factors as

$$121b^2 - 36 = (11b + 6)(11b - 6).$$

f. To completely factor the expression $25u^2 - 70u + 49$ first note that the expression might be of the form $A^2 - 2AB + B^2$ where $A = 5u$ and $B = 7$. Now, we check that $2AB$ matches the middle term: $2AB = 2(5u)(7) = 70u$. So, the expression factors as

$$25u^2 - 70u + 49 = (5u - 7)^2.$$

g. To completely factor the expression $64q^3 - 27y^3$ first note that the expression is of the form $A^3 - B^3$ where $A = 4q$ and $B = 3y$. So, the expression factors as

$$\begin{aligned} 64q^3 - 27y^3 &= (4q - 3y)\left((4q)^2 + (4q)(3y) + (3y)^2\right) \\ &= (4q - 3y)\left(16q^2 + 12qy + 9y^2\right) \end{aligned}$$

7.7.6 Review of Factoring Strategies

In Section 7.6 we covered a factoring decision tree to help us decide what methods to try when factoring a given expression. Remember to always factor out the GCF first.

Example 7.7.24 Factor the expressions using an effective method.

a. $24xy - 20x - 18y + 15$.

b. $12t^2 + 36t + 27$.

c. $8u^2 + 14u - 9$.

d. $18c^2 - 98p^2$.

Explanation.

a. To factor the expression $24xy - 20x - 18y + 15$, we first look for a GCF. Since the GCF is 1, we can move further on the flowchart. Since this is a four-term polynomial, we will try grouping.

$$\begin{aligned} 24xy - 20x - 18y + 15 &= 24xy + (-20x) + (-18y) + 15 \\ &= (24xy - 20x) + (-18y + 15) \end{aligned}$$

$$\begin{aligned} &= 4x(6y-5)+(-3)(6y-5) \\ &= 4x\overbrace{(6x-5)}-3\overbrace{(6x-5)} \\ &= (6x-5)(4x-3) \end{aligned}$$

b. To factor the expression $12t^2+36t+27$, we first look for a GCF. Since the GCF is 3, first we will factor that out.

$$12t^2+36t+27=3\left(4t^2+12t+9\right)$$

Next, we can note that the first and last terms are perfect squares where $A^2=4t^2$ and $B=9$; so $A=2t$ and $B=3$. To check the middle term, $2AB=12t$. So the expression factors as a perfect square.

$$\begin{aligned} 12t^2+36t+27 &= 3\left(4t^2+12t+9\right) \\ &= 3(2t+3)^2 \end{aligned}$$

c. To factor the expression $8u^2+14u-9$, we first look for a GCF. Since the GCF is 1, we can move further on the flowchart. Since the expression is a trinomial with leading coefficient other than 1, we should try the AC method. Note that $AC=-72$ and factor pairs of -72 that add up to 14 are 18 and -4.

$$\begin{aligned} 8u^2+14u-9 &= 8u^2+18u-4u-9 \\ &= \left(8u^2+18\right)+(-4u-9) \\ &= 2u(4u+9)-1(4u+9) \\ &= (2u-1)(4u+9) \end{aligned}$$

d. To factor the expression $18c^2-98p^2$, we first look for a GCF. Since the GCF is 2, first we will factor that out.

$$18c^2-98p^2=2\left(9c^2-49p^2\right)$$

Now we notice that we have a binomial where both the first and second terms can be written as squares: $9c^2=(3c)^2$ and $49p^2=(7p)^2$.

$$\begin{aligned} 18c^2-98p^2 &= 2\left(9c^2-49p^2\right) \\ &= 2(3c-7p)(3c+7p) \end{aligned}$$

Exercises

Find the greatest common factor of the following terms.

1. $7y$ and $28y^2$

2. $4r$ and $28r^2$

3. $9r^{10}$ and $-54r^9$

4. $6t^{15}$ and $-54t^9$

5. $10x^{20}y^9, -70x^{18}y^{11}, 80x^7y^{20}$

6. $7x^{20}y^4, -21x^{19}y^9, 21x^{13}y^{10}$

Greatest Common Factor Factor the given polynomial.

7. $6x-6=$

8. $3y-3=$

9. $9y+54=$

10. $6y+60=$

11. $28r^2-16r+16=$

12. $9r^2-6r+12=$

13. $13t^2+8=$

14. $3t^2-20=$

15. $x(x-8)+3(x-8)=$

16. $x(x-5)+9(x-5)=$

17. $4x\left(x+y\right)+7\left(x+y\right)=$

18. $5x\left(x+y\right)-8\left(x+y\right)=$

Factor by Grouping Factor the given polynomial.

19. $y^2+4y+2y+8=$

20. $r^2-10r+8r-80=$

21. $r^3+7r^2+6r+42=$

22. $t^3+3t^2-8t-24=$

23. $xy+10x+5y+50=$

24. $xy+2x+9y+18=$

25. $3x^2+18xy+5xy+30y^2=$

26. $4x^2+32xy+9xy+72y^2=$

One-variable Polynomials Factor the given polynomial.

27. $y^2+7y+6=$

28. $y^2+7y+10=$

29. $r^2+6r-27=$

30. $r^2+5r-50=$

31. $t^2-10t+16=$

32. $t^2-11t+18=$

33. $x^2-x+1=$

34. $x^2+5x+7=$

35. $x^2-20x+100=$

36. $y^2-10y+25=$

37. $7y^2-7=$

38. $6r^2+18r-24=$

39. $3r^2-21r+18=$

40. $2t^2-16t+30=$

41. $2t^8-6t^7-36t^6=$

42. $6x^4-12x^3-18x^2=$

43. $-x^2-6x+40=$

44. $-x^2+5x+14=$

Multivariable Polynomials Factor the given polynomial.

45. $y^2 - 6yr - 27r^2 =$

46. $y^2 + 4yt - 32t^2 =$

47. $r^2 - 12ry + 27y^2 =$

48. $r^2 - 3rx + 2x^2 =$

49. $2a^2b + 2ab - 24b =$

50. $2a^2b + 6ab - 8b =$

51. $2x^3y + 6x^2y + 4xy =$

52. $4x^3y + 16x^2y + 12xy =$

Non-trivial Leading Coefficient Factor the given polynomial.

53. $3x^2 + 2x - 8 =$

54. $2y^2 - 3y - 9 =$

55. $2y^2 + y + 2 =$

56. $3r^2 - 8r + 6 =$

57. $6r^2 + 19r + 10 =$

58. $4t^2 + 12t + 5 =$

59. $6t^2 - 19t + 10 =$

60. $9x^2 - 21x + 10 =$

61. $8x^2 + 20x + 8 =$

62. $15x^2 + 25x + 10 =$

63. $4y^2 - 22y + 28 =$

64. $6y^2 - 10y + 4 =$

65. $4r^4 - 14r^3 + 12r^2 =$

66. $6r^6 - 14r^5 + 8r^4 =$

67. $5t^2r^2 + 18tr + 16 =$

68. $2t^2y^2 + 9ty + 10 =$

69. $3x^2 - 14xt + 15t^2 =$

70. $2x^2 - 5xr + 2r^2 =$

71. $8x^2 - 25xy + 3y^2 =$

72. $8y^2 - 25yt + 3t^2 =$

73. $6y^2r^2 + 15yr + 6 =$

74. $4r^2x^2 + 18rx + 14 =$

75. $12r^2t^2 + 18rt - 12 =$

76. $20t^2r^2 - 30tr - 20 =$

77. $8x^2 + 28xy + 12y^2 =$

78. $8x^2 + 28xy + 20y^2 =$

Factor Special Forms Factor the given polynomial.

79. $x^2 - 36 =$

80. $x^2 - 4 =$

81. $100y^2 - 49 =$

82. $36y^2 - 49 = \boxed{\quad}$

83. $25 - r^2 = \boxed{\quad}$

84. $100 - 81r^2 = \boxed{\quad}$

85. $t^6 - 36 = \boxed{\quad}$

86. $t^{12} - 4 = \boxed{\quad}$

87. $x^{12} - 4y^{10} = \boxed{\quad}$

88. $x^{10} - 9y^{12} = \boxed{\quad}$

89. $144x^2 - 24x + 1 = \boxed{\quad}$

90. $81y^2 - 18y + 1 = \boxed{\quad}$

91. $4y^2 - 44yr + 121r^2 = \boxed{\quad}$

92. $9r^2 - 60rt + 100t^2 = \boxed{\quad}$

93. $x^3 - 729y^3 = \boxed{\quad}$
Hint: $-729 = -9^3$.

94. $x^3 - 1000y^3 = \boxed{\quad}$
Hint: $-1000 = -10^3$.

95. $81t^4 - 16 = \boxed{\quad}$

96. $16t^4 - 81 = \boxed{\quad}$

97. $4x^3 - 36x = \boxed{\quad}$

98. $10x^3 - 40x = \boxed{\quad}$

99. $4y^4r^4 - 4y^2r^2 = \boxed{\quad}$

100. $3y^4x^4 - 48y^2x^2 = \boxed{\quad}$

101. $24 - 6r^2 = \boxed{\quad}$

102. $9 - 9r^2 = \boxed{\quad}$

103. $72t^9 + 48t^8 + 8t^7 = \boxed{\quad}$

104. $18t^7 + 12t^6 + 2t^5 = \boxed{\quad}$

105. $t^8 + 216t^5 = \boxed{\quad}$
Hint: $216 = 6^3$.

106. $x^{11} + 8x^8 = \boxed{\quad}$

107. $x^2 + 81 = \boxed{\quad}$

108. $y^2 + 25 = \boxed{\quad}$

CHAPTER 8

Solving Quadratic Equations

8.1 Solving Quadratic Equations by Factoring

We have learned how to factor trinomials like $x^2 + 5x + 6$ into $(x + 2)(x + 3)$. This skill is needed to solve an equation like $x^2+5x+6 = 0$, which is a quadratic equation. A **quadratic equation** is is an equation in the form $ax^2+bx+c = 0$ with $a \neq 0$. We also consider equations such as $x^2 = x+3$ and $5x^2+3 = (x+1)^2+(x+1)(x-3)$ to be quadratic equations, because we can expand any multiplication, add or subtract terms from both sides, and combine like terms to get the form $ax^2 + bx + c = 0$. The form $ax^2 + bx + c = 0$ is called the **standard form** of a quadratic equation.

Before we begin exploring the method of solving quadratic equations by factoring, we'll identify what types of equations are quadratic and which are not.

Checkpoint 8.1.2. Identify which of the items are quadratic equations.

a. The equation $2x^2 + 5x = 7$ (☐ is ☐ is not) a quadratic equation.
b. The equation $5 - 2x = 3$ (☐ is ☐ is not) a quadratic equation.
c. The equation $15 - x^3 = 3x^2 + 9x$ (☐ is ☐ is not) a quadratic equation.
d. The equation $(x + 3)(x - 4) = 0$ (☐ is ☐ is not) a quadratic equation.
e. The equation $x(x + 1)(x - 1) = 0$ (☐ is ☐ is not) a quadratic equation.
f. The expression $x^2 - 5x + 6$ (☐ is ☐ is not) a quadratic equation.
g. The equation $(2x - 3)(x + 5) = 12$ (☐ is ☐ is not) a quadratic equation.

Explanation.

a. The equation $2x^2+5x = 7$ *is* a quadratic equation. To write it in standard form, simply subtract 7 from both sides.
b. The equation $5 - 2x = 3$ *is not* quadratic. It is a linear equation.
c. The equation $15 + x^3 = 3x^2 + 9x$ *is not* a quadratic equation because of the x^3 term.
d. The equation $(x+3)(x-4) = 0$ *is* a quadratic equation. If we expand the left-hand side of the equation, we would get something in standard form.
e. The equation $x(x + 1)(x - 1) = 0$ *is not* a quadratic equation. If we expanded the left-hand side of the equation, we would have an expression with an x^3 term, which automatically makes it not quadratic.
f. The expression $x^2 - 5x + 6$ *is not* a quadratic equation; it's not an *equation* at all. Instead, this is a quadratic *expression*.
g. The equation $(2x - 3)(x + 5) = 12$ *is* a quadratic equation. Multiplying out the left-hand side, and subtracting 12 form both sides, we would have a quadratic equation in standard form.

Now we'll look at an application that demonstrates the need and method for solving a quadratic equation by factoring.

Nita is in a physics class that launches a tennis ball from a rooftop 80 feet above the ground. They fire it directly upward at a speed of 64 feet per second and measure the time it takes for the ball to hit the ground below. We can model the height of the tennis ball, h, in feet, with the quadratic equation

$$h = -16t^2 + 64t + 80,$$

where t represents the time in seconds after the launch. Using the model we can predict when the ball will hit the ground.

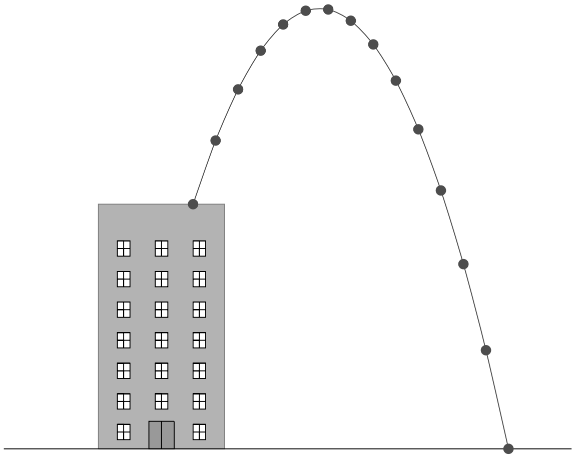


Figure 8.1.3: A Diagram of the Ball Thrown from the Roof

The ground has a height of 0, or $h = 0$. We will substitute 0 for h in the equation and we have

$$-16t^2 + 64t + 80 = 0$$

We need to solve this quadratic equation for t to find when the ball will hit the ground.

The key strategy for solving a *linear* equation is to separate the variable terms from the constant terms on either side of the equal sign. It turns out that this same method *will not work* for quadratic equations. Fortunately, we already have spent a good amount of time discussing a method that *will* work: factoring. If we can factor the polynomial on the left-hand side, we will be on the home stretch to solving the whole equation.

We will look for a common factor first, and see that we can factor out -16. Then we can finish factoring the trinomial:

$$\begin{aligned}-16t^2 + 64t + 80 &= 0\\ -16(t^2 - 4t - 5) &= 0\\ -16(t + 1)(t - 5) &= 0\end{aligned}$$

In order to finish solving the equation, we need to understand the following property. This property explains why it was *incredibly important* to *not* move the 80 in our example over to the other side of the equation before trying to factor.

Fact 8.1.4 Zero Product Property. *If the product of two or more numbers is equal to zero, then at least one of the numbers must be zero.*

One way to understand this property is to think about the equation $a \cdot b = 0$. Maybe $b = 0$, because that would certainly cause the equation to be true. But suppose that $b \neq 0$. Then it is safe to divide both sides by b, and the resulting equation says that $a = 0$. So no matter what, either $a = 0$ or $b = 0$.

To understand this property more, let's look at a few products:

$$\begin{aligned} &4 \cdot 7 = 28 && 4 \cdot 0 = 0 && 4 \cdot 7 \cdot 3 = 84 \\ &0 \cdot 7 = 0 && -4 \cdot 0 = 0 && 4 \cdot 0 \cdot 3 = 0 \end{aligned}$$

When none of the factors are 0, the result is never 0. The only way to get a product of 0 is when one of the factors is 0. This property is unique to the number 0 and can be used no matter how many numbers are multiplied together.

Now we can see the value of factoring. We have three factors in our equation

$$-16(t+1)(t-5) = 0.$$

The first factor is the number -16. The second and third factors, $t+1$ and $t-5$, are expressions that represent numbers. Since the product of the three factors is equal to 0, one of the factors must be zero.

Since -16 is not 0, either $t+1$ or $t-5$ must be 0. This gives us two equations to solve:

$$\begin{aligned} t+1 &= 0 &&\text{or} & t-5 &= 0 \\ t+1-1 &= 0-1 &&\text{or} & t-5+5 &= 0+5 \\ t &= -1 &&\text{or} & t &= 5 \end{aligned}$$

We have found two solutions, -1 and 5. A quadratic expression will have at most two linear factors, not including any constants, so it can have up to two solutions.

Let's check each of our two solutions -1 and 5:

$$\begin{aligned} -16t^2 + 64t + 80 &= 0 & -16t^2 + 64t + 80 &= 0 \\ -16(-1)^2 + 64(-1) + 80 &\stackrel{?}{=} 0 & -16(5)^2 + 64(5) + 80 &\stackrel{?}{=} 0 \\ -16(1) - 64 + 80 &\stackrel{?}{=} 0 & -16(25) + 320 + 80 &\stackrel{?}{=} 0 \\ -16 - 64 + 80 &\stackrel{?}{=} 0 & -400 + 320 + 80 &\stackrel{?}{=} 0 \\ 0 &\stackrel{\checkmark}{=} 0 & 0 &\stackrel{\checkmark}{=} 0 \end{aligned}$$

We have verified our solutions. While there are two solutions to the equation, the solution -1 is not relevant to this physics model because it is a negative time which would tell us something about the ball's height *before* it was launched. The solution 5 does make sense. According to the model, the tennis ball will hit the ground 5 seconds after it is launched.

8.1.1 Further Examples

We'll now look at further examples of solving quadratic equations by factoring. The general process is outlined here:

Process 8.1.5 Solving Quadratic Equations by Factoring.

Simplify *Simplify the equation using distribution and by combining like terms.*

Isolate *Move all terms onto one side of the equation so that the other side has* 0.

Factor *Factor the quadratic expression.*

Apply the Zero Product Property *Apply the Zero Product Property.*

Solve *Solve the equation(s) that result after the zero product property was applied.*

Example 8.1.6 Solve $x^2 - 5x - 14 = 0$ by factoring.

Explanation.

$$\begin{aligned} x^2 - 5x - 14 &= 0 \\ (x-7)(x+2) &= 0 \end{aligned}$$

$$\begin{aligned} x - 7 &= 0 & &\text{or} & x + 2 &= 0 \\ x - 7 + 7 &= 0 + 7 & &\text{or} & x + 2 - 2 &= 0 - 2 \\ x &= 7 & &\text{or} & x &= -2 \end{aligned}$$

The solutions are -2 and 7, so the solution set is written as $\{-2, 7\}$.

If the two factors of a polynomial happen to be the same, the equation will only have one solution. Let's look at an example of that.

Example 8.1.7 A Quadratic Equation with Only One Solution. Solve $x^2 - 10x + 25 = 0$ by factoring.

Explanation.

$$\begin{aligned} x^2 - 10x + 25 &= 0 \\ (x-5)(x-5) &= 0 \\ (x-5)^2 &= 0 \\ x - 5 &= 0 \\ x - 5 + 5 &= 0 + 5 \\ x &= 5 \end{aligned}$$

The solution is 5, so the solution set is written as $\{5\}$.

Example 8.1.8 Factor Out a Common Factor. Solve $5x^2 + 55x + 120 = 0$ by factoring.

Explanation. Note that the terms are all divisible by 5, so we can factor that out to start.

$$\begin{aligned} 5x^2 + 55x + 120 &= 0 \\ 5(x^2 + 11x + 24) &= 0 \\ 5(x+8)(x+3) &= 0 \end{aligned}$$

$$\begin{aligned} x + 8 &= 0 & &\text{or} & x + 3 &= 0 \\ x &= -8 & &\text{or} & x &= -3 \end{aligned}$$

The solution set is $\{-8, -3\}$.

Example 8.1.9 Factoring Using the AC Method. Solve $3x^2 - 7x + 2 = 0$ by factoring.

Explanation. Recall that we multiply $3 \cdot 2 = 6$ and find a factor pair that multiplies to 6 and adds to

−7. The factors are −6 and −1. We use the two factors to replace the middle term with $-6x$ and $-x$.

$$\begin{aligned}3x^2-7x+2&=0\\3x^2-6x-x+2&=0\\(3x^2-6x)+(-x+2)&=0\\3x(x-2)-1(x-2)&=0\\(3x-1)(x-2)&=0\end{aligned}$$

$$\begin{aligned}3x-1&=0 &&\text{or}&& x-2=0\\3x&=1 &&\text{or}&& x=2\\x&=\frac{1}{3} &&\text{or}&& x=2\end{aligned}$$

The solution set is $\left\{\frac{1}{3},2\right\}$.

So far the equations have been written in standard form, which is

$$ax^2+bx+c=0$$

If an equation is not given in standard form then we must rearrange it in order to use the Zero Product Property.

Example 8.1.10 Writing in Standard Form. Solve $x^2-10x=24$ by factoring.

Explanation. There is nothing like the Zero Product Property for the number 24. We must have a 0 on one side of the equation to solve quadratic equations using factoring.

$$\begin{aligned}x^2-10x&=24\\x^2-10x-24&=24-24\\x^2-10x-24&=0\\(x-12)(x+2)&=0\end{aligned}$$

$$\begin{aligned}x-12&=0 &&\text{or}&& x+2=0\\x&=12 &&\text{or}&& x=-2\end{aligned}$$

The solution set is $\{-2,12\}$.

Example 8.1.11 Writing in Standard Form. Solve $(x+4)(x-3)=18$ by factoring.

Explanation. Again, there is nothing like the Zero Product Property for a number like 18. We must expand the left side and subtract 18 from both sides.

$$\begin{aligned}(x+4)(x-3)&=18\\x^2+x-12&=18\\x^2+x-12-18&=18-18\\x^2+x-30&=0\end{aligned}$$

$$(x+6)(x-5)=0$$

$$\begin{aligned} x+6&=0 &\quad\text{or}\quad&& x-5&=0\\ x&=-6 &\quad\text{or}\quad&& x&=5 \end{aligned}$$

The solution set is $\{-6,5\}$.

Example 8.1.12 A Quadratic Equation with No Constant Term. Solve $2x^2=5x$ by factoring.

Explanation. It may be tempting to divide both sides of the equation by x. But x is a variable, and for all we know, maybe $x=0$. So it is not safe to divide by x. As a general rule, never divide an equation by a variable in the solving process. Instead, we will put the equation in standard form.

$$\begin{aligned} 2x^2&=5x\\ 2x^2-5x&=5x-5x\\ 2x^2-5x&=0 \end{aligned}$$

We can factor out x.

$$x(2x-5)=0$$

$$\begin{aligned} x&=0 &\quad\text{or}\quad&& 2x-5&=0\\ x&=0 &\quad\text{or}\quad&& 2x&=5\\ x&=0 &\quad\text{or}\quad&& x&=\frac{5}{2} \end{aligned}$$

The solution set is $\left\{0,\frac{5}{2}\right\}$. In general, if a quadratic equation does not have a constant term, then 0 will be one of the solutions.

Example 8.1.13 Factoring a Special Polynomial. Solve $x^2=9$ by factoring.

Explanation. We can put the equation in standard form and use factoring. In this case, we find a difference of squares.

$$\begin{aligned} x^2&=9\\ x^2-9&=0\\ (x+3)(x-3)&=0 \end{aligned}$$

$$\begin{aligned} x+3&=0 &\quad\text{or}\quad&& x-3&=0\\ x&=-3 &\quad\text{or}\quad&& x&=3 \end{aligned}$$

The solution set is $\{-3,3\}$.

Example 8.1.14 Solving an Equation with a Higher Degree. Solve $2x^3-10x^2-28x=0$ by factoring.

Explanation. Although this equation is not quadratic, it does factor so we can solve it by factoring.

$$\begin{aligned} 2x^3 - 10x^2 - 28x &= 0 \\ 2x(x^2 - 5x - 14) &= 0 \\ 2x(x - 7)(x + 2) &= 0 \end{aligned}$$

$$\begin{aligned} 2x &= 0 &\quad \text{or} \quad&& x - 7 &= 0 &\quad \text{or} \quad&& x + 2 &= 0 \\ x &= 0 &\quad \text{or} \quad&& x &= 7 &\quad \text{or} \quad&& x &= -2 \end{aligned}$$

The solution set is $\{-2, 0, 7\}$.

8.1.2 Applications

Example 8.1.15 Kicking it on Mars. Some time in the recent past, Filip traveled to Mars for a vacation with his kids, Henrik and Karina, who wanted to kick a soccer ball around in the comparatively reduced gravity. Karina stood at point K and kicked the ball over her dad standing at point F to Henrik standing at point H. The height of the ball off the ground, h in feet, can be modeled by the equation $h = -0.01\left(x^2 - 70x - 1800\right)$, where x is how far to the right the ball is from Filip. Note that distances to the left of Filip will be negative.

a. Find out how high the ball was above the ground when it passed over Filip's head.

b. Find the distance from Karina to Henrik.

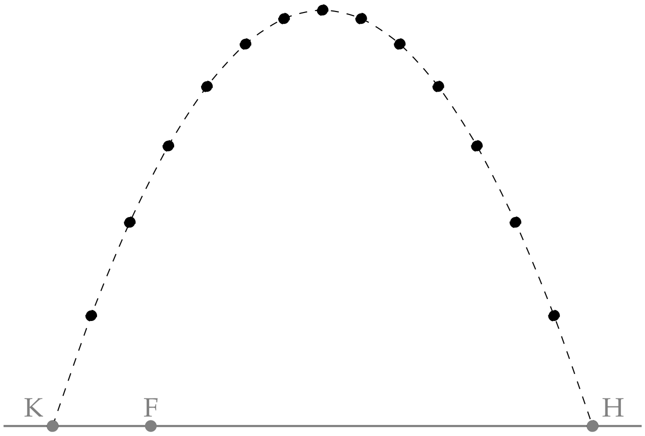


Figure 8.1.16: A Soccer Kick on Mars

Explanation.

a. The ball was neither left nor right of Filip when it went over him, so $x = 0$. Plugging that value into our equation for x,

$$\begin{aligned} h &= -0.01\left(0^2 - 70(0) - 1800\right) \\ &= -0.01(-1800) \\ &= 18 \end{aligned}$$

It seems that the soccer ball was 18 feet above the ground when it flew over Filip.

b. The distance from Karina to Henrik is the same as the distance from point K to point H. These are the horizontal intercepts of the graph of the given formula: $h = -0.01\left(x^2 - 70x - 1800\right)$. To find the horizontal intercepts, set $h = 0$ and solve for x.

$$0 = -0.01\left(x^2 - 70x - 1800\right)$$

Note that we can divide by -0.01 on both sides of the equation to simplify.

$$\begin{aligned} 0 &= x^2 - 70x - 1800 \\ 0 &= (x - 90)(x + 20) \end{aligned}$$

So, either:

$$\begin{aligned} x - 90 &= 0 & \text{or} && x + 20 &= 0 \\ x &= 90 & \text{or} && x &= -20 \end{aligned}$$

Since the x-values are how far right or left the points are from Filip, Karina is standing 20 feet left of Filip and Henrik is standing 90 feet right of Filip. Thus, the two kids are 110 feet apart.

It is worth noting that if this same kick, with same initial force at the same angle, took place on Earth, the ball would have traveled less than 30 feet from Karina before landing!

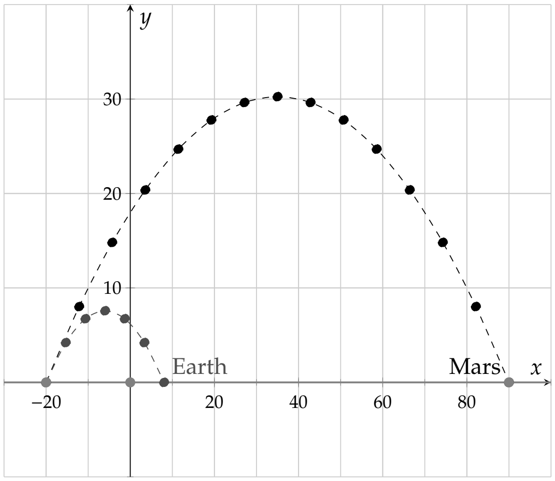


Figure 8.1.17: A Soccer Kick on Mars and the Same Kick on Earth

Example 8.1.18 An Area Application. Rajesh has a hot tub and he wants to build a deck around it. The hot tub is 7 ft by 5 ft and it is covered by a roof that is 99 ft^2. How wide can he make the deck so that it will be covered by the roof?

Explanation. We will define x to represent the width of the deck (in feet). Here is a diagram to help us understand the scenario.

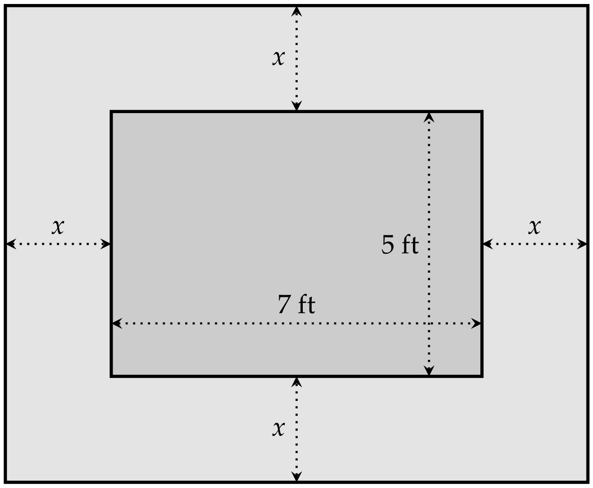


Figure 8.1.19: Diagram for the Deck

The overall length is $7+2x$ feet, because Rajesh is adding x feet on each side. Similarly, the overall width is $5+2x$ feet.

The formula for the area of a rectangle is area = length · width. Since the total area of the roof is $99\,\text{ft}^2$, we can write and solve the equation:

$$\begin{aligned}(7+2x)(5+2x)&=99\\4x^2+24x+35&=99\\4x^2+24x+35-99&=99-99\\4x^2+24x-64&=0\\4(x^2+6x-16)&=0\\4(x+8)(x-2)&=0\end{aligned}$$

$$\begin{aligned}x+8&=0 &\quad\text{or}\quad&& x-2&=0\\x&=-8 &\quad\text{or}\quad&& x&=2\end{aligned}$$

Since a length cannot be negative, we take $x=2$ as the only applicable solution. Rajesh should make the deck 2 ft wide on each side to fit under the roof.

Exercises

Warmup and Review Factor the given polynomial.

1. $8r+56=$ ☐

2. $5r+10=$ ☐

3. $t^2-7t-30=$ ☐

4. $t^2-3t-70=$ ☐

5. $2t^2-13t+21=$ ☐

6. $3x^2-11x+8=$ ☐

7. $54x^2 + 9x + 54 = \boxed{\qquad\qquad}$

8. $24y^2 + 6y + 48 = \boxed{\qquad\qquad}$

9. $121y^4 - 144 = \boxed{\qquad\qquad}$

10. $64r^4 - 25 = \boxed{\qquad\qquad}$

Solve Quadratic Equations by Factoring Solve the equation.

11. $(x + 4)(x - 7) = 0$

12. $(x + 7)(x + 9) = 0$

13. $-74(x + 8)(11x - 7) = 0$

14. $-60(x + 10)(5x - 1) = 0$

15. $x^2 + 10x + 9 = 0$

16. $x^2 + 8x + 15 = 0$

17. $x^2 - 2x - 8 = 0$

18. $x^2 + 7x - 8 = 0$

19. $x^2 - 11x + 30 = 0$

20. $x^2 - 14x + 40 = 0$

21. $x^2 + 18x = -80$

22. $x^2 + 13x = -36$

23. $x^2 + 9x = 10$

24. $x^2 - 2x = 63$

25. $x^2 - 15x = -56$

26. $x^2 - 8x = -7$

27. $x^2 = 2x$

28. $x^2 = -x$

29. $7x^2 = -63x$

30. $8x^2 = -16x$

31. $9x^2 = 5x$

32. $10x^2 = -9x$

33. $x^2 - 4x + 4 = 0$

34. $x^2 - 6x + 9 = 0$

35. $x^2 = 8x - 16$

36. $x^2 = 12x - 36$

37. $49x^2 = -42x - 9$

38. $4x^2 = -28x - 49$

39. $5x^2 = -36x - 36$

40. $5x^2 = -52x - 20$

41. $x^2 - 100 = 0$

42. $x^2 - 81 = 0$

43. $9x^2 - 1 = 0$

44. $16x^2 - 81 = 0$

45. $9x^2 = 16$

46. $36x^2 = 1$

47. $x(x - 5) = 36$

48. $x(x + 1) = 30$

49. $x(5x + 62) = -120$

50. $x(5x + 54) = -40$

51. $(x - 7)(x - 1) = -8$

52. $(x + 2)(x + 4) = 3$

53. $(x+2)(4x-7)=6+3x^2$

54. $(x-2)(2x-3)=x^2-6$

55. $x(x+4)=2x-1$

56. $x(x+12)=3(2x-3)$

57. $64x^2+48x+9=0$

58. $81x^2+198x+121=0$

59. $(x+7)\left(x^2+16x+60\right)=0$

60. $(x-5)\left(x^2+3x+2\right)=0$

61. $x\left(x^2-4\right)=0$

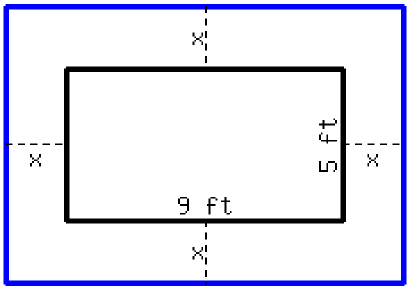
62. $x\left(x^2-16\right)=0$

63. $x^3-10x^2+16x=0$

64. $x^3+15x^2+54x=0$

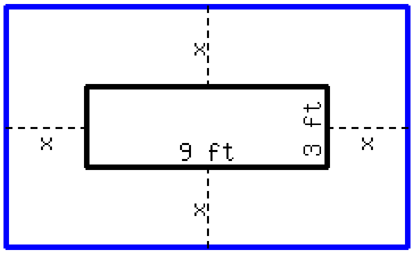
Quadratic Equation Application Problems

65. There is a rectangular lot in the garden, with 9 ft in length and 5 ft in width. You plan to expand the lot by an equal length around its four sides, and make the area of the expanded rectangle 117 ft^2. How long should you expand the original lot in four directions?



You should expand the original lot by
in four directions.

66. There is a rectangular lot in the garden, with 9 ft in length and 3 ft in width. You plan to expand the lot by an equal length around its four sides, and make the area of the expanded rectangle 135 ft^2. How long should you expand the original lot in four directions?



You should expand the original lot by
in four directions.

67. Two numbers' sum is 16, and their product is 63. Find these two numbers.

These two numbers are .

68. Two numbers' sum is 4, and their product is −60. Find these two numbers.

These two numbers are .

69. A rectangle's base is 5 cm longer than its height. The rectangle's area is 50 cm^2. Find this rectangle's dimensions.

The rectangle's height is .

The rectangle's base is .

70. A rectangle's base is 8 cm longer than its height. The rectangle's area is 65 cm^2. Find this rectangle's dimensions.

The rectangle's height is .

The rectangle's base is .

71. A rectangle's base is 8 in shorter than three times its height. The rectangle's area is 3 in^2. Find this rectangle's dimensions.

The rectangle's height is ☐.

The rectangle's base is ☐.

72. A rectangle's base is 3 in shorter than twice its height. The rectangle's area is 9 in^2. Find this rectangle's dimensions.

The rectangle's height is ☐.

The rectangle's base is ☐.

Challenge

73. Give an example of a cubic equation that has three solutions: one solution is $x = 6$, the second solution is $x = -4$, and the third solution is $x = \frac{2}{3}$.

74. Solve for x in the equation $27x^{14} - 3x^{12} = 0$.

8.2 Square Root Properties

In this chapter, we will learn how to both simplify square roots and to do operations with square roots.

Definition 8.2.2 The Definition of the Square Root of a Number. If $y^2 = x$ for a positive number y, then y is called the **square root** of x, and we write $y = \sqrt{x}$, where the $\sqrt{}$ symbol is called the **radical** or the **root**. We call expressions with a root symbol **radical expressions**. The number inside the radical is called the **radicand**.

For example, since $4^2 = 16$, then $\sqrt{16} = 4$. Both $\sqrt{2}$ and $3\sqrt{2}$ are radical expressions. In both expressions, the number 2 is the radicand. You can review the square root basics in Section 1.3.

The word "radical" means something like "on the fringes" when used in politics, sports, and other places. It actually has that same meaning in math, when you consider a square with area A as in Figure 8.2.3.

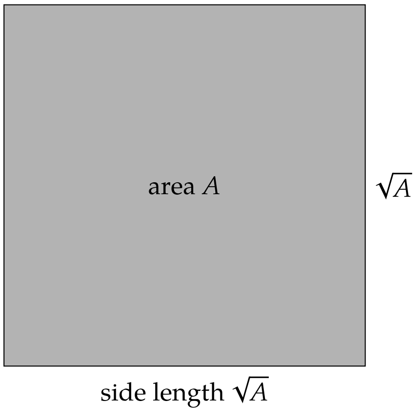


Figure 8.2.3: "Radical" means "off to the side."

8.2.1 Estimating Square Roots

When the radicand is a perfect square, its square root is a rational number. If the radicand is not a perfect square, the square root is irrational. We want to be able to estimate square roots without using a calculator.

To estimate $\sqrt{10}$, we can find the nearest perfect squares that are whole numbers on either side of 10. Recall that the perfect squares are $1, 4, 9, 16, 25, 36, 49, 64, \ldots$ The perfect square that is just below 10 is 9 and the perfect square just above 10 is 16. This tells us that $\sqrt{10}$ is between $\sqrt{9}$ and $\sqrt{16}$, or between 3 and 4. We can also say that $\sqrt{10}$ is much closer to 3 than 4 because 10 is closer to 9, so we think 3.1 or 3.2 would be a good estimate.

To check our estimate, let's find $\sqrt{10}$ with a calculator:

$$\sqrt{10} \approx 3.162$$

The actual value is just above 3 as we estimated, and between 3.1 and 3.2. Let's look at some more examples.

Checkpoint 8.2.4. Estimate $\sqrt{19}$ without a calculator.

Explanation. The radicand, 19, is between 16 and 25, so $\sqrt{19}$ is between $\sqrt{16}$ and $\sqrt{25}$, or between 4 and 5.

To be more precise, we notice that 19 is in the middle between 16 and 25 but closer to 16. We estimate $\sqrt{19}$

to be about 4.4.

We will check our estimate with a calculator:

$$\sqrt{19} \approx 4.358$$

Example 8.2.5 Estimate $\sqrt{3.2}$ without a calculator.

Explanation. The radicand 3.2 is between 1 and 4, so $\sqrt{3.2}$ is between $\sqrt{1}$ and $\sqrt{4}$, or between 1 and 2. To be more precise, we notice that 3.2 is much closer to 4 than 1. We estimate $\sqrt{3.2}$ to be about 1.8.

We will check our estimate with a calculator:

$$\sqrt{3.2} \approx 1.788$$

8.2.2 Multiplication and Division Properties of Square Roots

Here is an example using perfect squares and the rules of exponents to show a relationship between the product of two square roots:

$$\sqrt{9 \cdot 16} = \sqrt{3^2 \cdot 4^2} = \sqrt{(3 \cdot 4)^2} = 3 \cdot 4 = 12$$

and

$$\sqrt{9} \cdot \sqrt{16} = \sqrt{3^2} \cdot \sqrt{4^2} = 3 \cdot 4 = 12$$

Whether we multiply the radicands first or take the square roots first, we get the same result. This tells us that in multiplication with radicals, we can combine factors into a single radical or separate them as needed.

Now let's look at division. When we learned how to find the square root of a fraction in Section 1.3, we saw that the numerators and denominators could be simplified separately. We multiply the numerators and denominators independently. Here is an example of two different ways to simplify a fraction in a square root:

$$\sqrt{\frac{25}{9}} = \sqrt{\left(\frac{5}{3}\right)^2} = \frac{5}{3}$$

and

$$\frac{\sqrt{25}}{\sqrt{9}} = \frac{\sqrt{5^2}}{\sqrt{3^2}} = \frac{5}{3}$$

Just like with multiplication, we can separate the numerators and denominators in a radical expression or combine them as needed. Note that we worked with expressions that were perfect squares, but these properties will work regardless of the number inside the radical. Let's summarize these properties.

Fact 8.2.6 Multiplication and Division Properties of Square Roots. *For any positive real numbers x and y we have the following properties:*

Multiplication Property of Square Roots $\sqrt{x \cdot y} = \sqrt{x} \cdot \sqrt{y}$

Division Property of Square Roots $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

8.2.3 Simplifying Square Roots

We can use Multiplication and Division Properties of Square Roots to simplify a radicand that is not a perfect square. Simplifying radicals is similar to simplifying fractions because we want the radicand to be as small as possible.

To understand why we can simplify radicals, let's use a calculator to compare $\sqrt{12}$ and $2\sqrt{3}$.

$$\sqrt{12} = 3.4641\ldots \qquad \text{and} \qquad 2\sqrt{3} = 3.4641\ldots$$

These are equivalent expressions so let's see how we can simplify $\sqrt{12}$ to $2\sqrt{3}$.

First, we will make a table of factor pairs for the number 12, as we did in Section 7.3.

$1 \cdot 12$
$2 \cdot 6$
$3 \cdot 4$

The factor pair with the largest perfect square is $3 \cdot 4$. We will use the property of multiplying radicals to separate the perfect square from the other factor. We write the perfect square first because it will end up in front of the radical.

$$\begin{aligned}\sqrt{12} &= \sqrt{4} \cdot \sqrt{3}\\ &= 2 \cdot \sqrt{3}\\ &= 2\sqrt{3}\end{aligned}$$

This process can be used to simplify any square root, or to determine that it is fully simplified. Let's look at a few more examples.

Example 8.2.7 Simplify $\sqrt{72}$.

Explanation.

Here is a table of factor pairs for the number 72.

$1 \cdot 72$
$2 \cdot 36$
$3 \cdot 24$
$4 \cdot 18$
$6 \cdot 12$
$8 \cdot 9$

The largest perfect square is 36 so we will rewrite 72 as $36 \cdot 2$.

$$\begin{aligned}\sqrt{72} &= \sqrt{36 \cdot 2}\\ &= \sqrt{36} \cdot \sqrt{2}\\ &= 6\sqrt{2}\end{aligned}$$

Notice that if we had chosen $4 \cdot 18$ we could simplify the radical partially but we would need to continue and find the perfect square of 9 in 18.

Checkpoint 8.2.8. Simplify $\sqrt{125}$.

Explanation. Here is a table of factor pairs for the number 125.

$1 \cdot 125$
$5 \cdot 25$

The largest perfect square is 25 so we will rewrite 125 as $25 \cdot 5$.

$$\begin{aligned}\sqrt{125} &= \sqrt{25 \cdot 5}\\ &= \sqrt{25} \cdot \sqrt{5}\\ &= 5\sqrt{5}\end{aligned}$$

Example 8.2.9 Simplify $\sqrt{30}$.

Explanation.

Here is a table of factor pairs for the number 30.

$1 \cdot 30$	$3 \cdot 10$
$2 \cdot 15$	$5 \cdot 6$

The number 30 does not have any factors that are perfect squares so it cannot be simplified further.

We can also use Division Property of Square Roots to simplify expressions.

Example 8.2.10

a. Simplify $\sqrt{\frac{9}{16}}$.

b. Simplify $\frac{\sqrt{50}}{\sqrt{2}}$.

Explanation.

a. For the first expression, we will use the Division Property of Square Roots:

$$\begin{aligned}\sqrt{\frac{9}{16}} &= \frac{\sqrt{9}}{\sqrt{16}}\\ &= \frac{3}{4}\end{aligned}$$

b. For the second expression, we use the same property in reverse: $\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$:

$$\begin{aligned}\frac{\sqrt{50}}{\sqrt{2}} &= \sqrt{\frac{50}{2}}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

8.2.4 Multiplying Square Root Expressions

If we use the Multiplication Property of Square Roots and the Division Property of Square Roots in the reverse order as

$$\sqrt{x} \cdot \sqrt{y} = \sqrt{xy} \qquad \text{and} \qquad \frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}},$$

we can use these properties to multiply and divide square root expressions. We want to simplify each radical first to keep the radicands as small as possible. Let's look at a few examples.

Example 8.2.11 Multiply $\sqrt{8} \cdot \sqrt{54}$.

Explanation. We will simplify each radical first, and then multiply them together. We do not want to multiply $8 \cdot 54$ because we will end up with a larger number that is harder to factor.

$$\begin{aligned}\sqrt{8} \cdot \sqrt{54} &= \sqrt{4 \cdot 2} \cdot \sqrt{9 \cdot 6}\\ &= 2\sqrt{2} \cdot 3\sqrt{6}\\ &= 2 \cdot 3\sqrt{2 \cdot 6}\\ &= 2 \cdot 3\sqrt{2 \cdot 2 \cdot 3}\\ &= 6 \cdot 2\sqrt{3}\\ &= 12\sqrt{3}\end{aligned}$$

We could have multiplied $2 \cdot 6$ inside the radical to get 12 and then factored 12 into $4 \cdot 3$. Whenever you find a pair of identical factors, this is a perfect square.

Checkpoint 8.2.12. Multiply $2\sqrt{7} \cdot 3\sqrt{21}$.

Explanation. First multiply the non-radical factors together and the radical factors together. Then look for further simplifications.

$$\begin{aligned}2\sqrt{7} \cdot 3\sqrt{21} &= 2 \cdot 3 \cdot \sqrt{7} \cdot \sqrt{21}\\ &= 6 \cdot \sqrt{7} \cdot \sqrt{7 \cdot 3}\\ &= 6\sqrt{7 \cdot 7 \cdot 3}\\ &= 6 \cdot 7 \cdot \sqrt{3}\\ &= 42\sqrt{3}\end{aligned}$$

Example 8.2.13 Multiply $\sqrt{\frac{6}{5}} \cdot \sqrt{\frac{3}{5}}$.

Explanation.

$$\begin{aligned}\sqrt{\frac{6}{5}} \cdot \sqrt{\frac{3}{5}} &= \sqrt{\frac{6}{5} \cdot \frac{3}{5}}\\ &= \sqrt{\frac{18}{25}}\\ &= \frac{\sqrt{18}}{\sqrt{25}}\\ &= \frac{\sqrt{9 \cdot 2}}{5}\\ &= \frac{3\sqrt{2}}{5}\end{aligned}$$

8.2.5 Adding and Subtracting Square Root Expressions

We learned the Multiplication Property of Square Roots previously and applied this to multiplication of square roots, but we cannot apply this property to the operations of addition or subtraction. Here are two examples to demonstrate this:

$$\begin{aligned}\sqrt{9+16} &\overset{?}{=} \sqrt{9}+\sqrt{16} \\ \sqrt{25} &\overset{?}{=} 3+4 \\ 5 &\overset{\text{no}}{=} 7\end{aligned} \qquad\qquad \begin{aligned}\sqrt{169-25} &\overset{?}{=} \sqrt{169}-\sqrt{25} \\ \sqrt{144} &\overset{?}{=} 13-5 \\ 12 &\overset{\text{no}}{=} 8\end{aligned}$$

We do not get the same result if we separate the radicals, so we must complete any additions and subtractions inside the radical first.

To add and subtract radical expressions, we will need to recognize that we can only add and subtract like terms. In this case, we will call them **like radicals**. In fact, adding like radicals will work just like adding like terms

$$x+x=2x$$

and

$$\sqrt{5}+\sqrt{5}=2\sqrt{5}$$

We can verify that the second equation is true by replacing x with $\sqrt{5}$ in the second equation. Let's look at a few more examples.

> **Example 8.2.14** Simplify $\sqrt{2}+\sqrt{8}$.
>
> **Explanation**.
>
> $$\begin{aligned}\sqrt{2}+\sqrt{8} &= \sqrt{2}+\sqrt{4\cdot 2} \\ &= \sqrt{2}+2\sqrt{2} \\ &= 3\sqrt{2}\end{aligned}$$
>
> To help understand $\sqrt{2}+2\sqrt{2}=3\sqrt{2}$, think of $x+2x=3x$ or "a taco plus two tacos is three tacos."

Checkpoint 8.2.15. Simplify $2\sqrt{3}-3\sqrt{48}$.

Explanation. First we will simplify the radical term where 48 is the radicand, and we may see that we then have like radicals.

$$\begin{aligned}2\sqrt{3}-3\sqrt{48} &= 2\sqrt{3}-3\sqrt{16\cdot 3} \\ &= 2\sqrt{3}-3\cdot 4\sqrt{3} \\ &= 2\sqrt{3}-12\sqrt{3} \\ &= -10\sqrt{3}\end{aligned}$$

> **Example 8.2.16** Simplify $\sqrt{2}+\sqrt{27}$.
>
> **Explanation**.
>
> $$\begin{aligned}\sqrt{2}+\sqrt{27} &= \sqrt{2}+\sqrt{9\cdot 3} \\ &= \sqrt{2}+3\sqrt{3}\end{aligned}$$

We cannot simplify the expression further because $\sqrt{2}$ and $\sqrt{3}$ are not like radicals.

Example 8.2.17 Simplify $\sqrt{6}-\sqrt{18}\cdot\sqrt{12}$.

Explanation. In this example, we should multiply the latter two square roots first and then see if we have like radicals.

$$\begin{aligned}\sqrt{6}-\sqrt{18}\cdot\sqrt{12}&=\sqrt{6}-\sqrt{9\cdot 2}\cdot\sqrt{4\cdot 3}\\&=\sqrt{6}-3\sqrt{2}\cdot 2\sqrt{3}\\&=\sqrt{6}-3\cdot 2\cdot\sqrt{2}\cdot\sqrt{3}\\&=\sqrt{6}-6\sqrt{6}\\&=-5\sqrt{6}\end{aligned}$$

8.2.6 Rationalizing the Denominator

When simplifying square root expressions, we have seen that we need to make the radicand as small as possible. Another rule is that we do not leave any irrational numbers, such as $\sqrt{3}$ or $2\sqrt{5}$, in the denominator of a fraction. In other words, we want the denominator to be rational. The process of dealing with such numbers in the denominator is called **rationalizing the denominator**.

Let's see how we can remove the square root symbol from the denominator in $\frac{1}{\sqrt{5}}$. If we multiply a radical by itself, the result is the radicand, by Definition 8.2.2. As an example:

$$\sqrt{5}\cdot\sqrt{5}=5$$

To write $\frac{1}{\sqrt{5}}$ as an equivalent fraction, we must multiply both the numerator and denominator by the same number. If we multiply the numerator and denominator by $\sqrt{5}$, we have:

$$\begin{aligned}\frac{1}{\sqrt{5}}&=\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}\\&=\frac{1\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}}\\&=\frac{\sqrt{5}}{5}\end{aligned}$$

We can use a calculator to verify that $\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}$. They both are $0.4472\ldots$. Let's look at a few more examples.

Example 8.2.18 Rationalize the denominator in $\frac{6}{\sqrt{3}}$.

Explanation. We will rationalize this denominator by multiplying the numerator and denominator by $\sqrt{3}$:

$$\frac{6}{\sqrt{3}}=\frac{6}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$

$$\begin{aligned} &= \frac{6 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} \\ &= \frac{6\sqrt{3}}{3} \\ &= 2\sqrt{3} \end{aligned}$$

Note that we reduced any fractions that are outside the radical.

Checkpoint 8.2.19. Rationalize the denominator in $\frac{2}{\sqrt{10}}$.

Explanation. We will rationalize the denominator by multiplying the numerator and denominator by $\sqrt{10}$:

$$\begin{aligned} \frac{2}{\sqrt{10}} &= \frac{2}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} \\ &= \frac{2 \cdot \sqrt{10}}{\sqrt{10} \cdot \sqrt{10}} \\ &= \frac{2\sqrt{10}}{10} \\ &= \frac{\sqrt{10}}{5} \end{aligned}$$

Again note that the fraction was simplified in the last step.

Example 8.2.20 Rationalize the denominator in $\sqrt{\frac{2}{7}}$.

Explanation.

$$\begin{aligned} \sqrt{\frac{2}{7}} &= \frac{\sqrt{2}}{\sqrt{7}} \\ &= \frac{\sqrt{2}}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} \\ &= \frac{\sqrt{2} \cdot \sqrt{7}}{\sqrt{7} \cdot \sqrt{7}} \\ &= \frac{\sqrt{14}}{7} \end{aligned}$$

8.2.7 More Complicated Square Root Operations

In Section 6.4, we learned how to multiply polynomials like $2(x + 3)$ and $(x + 2)(x + 3)$. All the methods we learned apply when we multiply square root expressions. We will look at a few examples done with different methods.

Example 8.2.21 Multiply $\sqrt{5}\left(\sqrt{3} - \sqrt{2}\right)$.

Explanation. We will use the distributive property to do this problem:

$$\begin{aligned}\sqrt{5}\left(\sqrt{3}-\sqrt{2}\right) &= \sqrt{5}\sqrt{3}-\sqrt{5}\sqrt{2}\\ &= \sqrt{15}-\sqrt{10}\end{aligned}$$

Example 8.2.22 Multiply $\left(\sqrt{6}+\sqrt{12}\right)\left(\sqrt{3}-\sqrt{2}\right)$.

Explanation. We will use the FOIL Method to do this problem:

$$\begin{aligned}\left(\sqrt{6}+\sqrt{12}\right)\left(\sqrt{3}-\sqrt{2}\right) &= \sqrt{6}\sqrt{3}-\sqrt{6}\sqrt{2}+\sqrt{12}\sqrt{3}-\sqrt{12}\sqrt{2}\\ &= \sqrt{18}-\sqrt{12}+\sqrt{36}-\sqrt{24}\\ &= 3\sqrt{2}-2\sqrt{3}+6-2\sqrt{6}\end{aligned}$$

When simplifying radicals it is useful to keep in mind that for any $x \geq 0$,

$$\sqrt{x}\cdot\sqrt{x} = x.$$

Example 8.2.23 Expand $\left(\sqrt{3}-\sqrt{2}\right)^2$.

Explanation. We will use the FOIL method to expand this expression:

$$\begin{aligned}\left(\sqrt{3}-\sqrt{2}\right)^2 &= \left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)\\ &= \left(\sqrt{3}\right)^2-\sqrt{3}\sqrt{2}-\sqrt{2}\sqrt{3}+\left(\sqrt{2}\right)^2\\ &= 3-\sqrt{6}-\sqrt{6}+2\\ &= 5-2\sqrt{6}\end{aligned}$$

Example 8.2.24 Multiply $\left(\sqrt{5}-\sqrt{7}\right)\left(\sqrt{5}+\sqrt{7}\right)$.

Explanation. We will again use the FOIL method to expand this expression, but will note that it is a special form $(a-b)(a+b)$ and will simplify to a^2-b^2:

$$\begin{aligned}\left(\sqrt{5}-\sqrt{7}\right)\left(\sqrt{5}+\sqrt{7}\right) &= \left(\sqrt{5}\right)^2+\sqrt{5}\sqrt{7}-\sqrt{7}\sqrt{5}-\left(\sqrt{7}\right)^2\\ &= 5+\sqrt{35}-\sqrt{35}-7\\ &= -2\end{aligned}$$

Exercises

Review and Warmup Which of the following are square numbers? There may be more than one correct answer.

1.

☐ 16 ☐ 86 ☐ 40 ☐ 9 ☐ 132 ☐ 64

2.

☐ 53 ☐ 121 ☐ 25 ☐ 84 ☐ 93 ☐ 1

Evaluate the following.

3.

a. $\sqrt{49} =$

b. $\sqrt{36} =$

c. $\sqrt{25} =$

4.

a. $\sqrt{64} =$

b. $\sqrt{4} =$

c. $\sqrt{121} =$

5.

a. $\sqrt{\frac{81}{121}} =$

b. $\sqrt{-\frac{9}{100}} =$

6.

a. $\sqrt{\frac{121}{36}} =$

b. $\sqrt{-\frac{64}{25}} =$

7. Do not use a calculator.

a. $\sqrt{144} =$

b. $\sqrt{1.44} =$

c. $\sqrt{14400} =$

8. Do not use a calculator.

a. $\sqrt{4} =$

b. $\sqrt{0.04} =$

c. $\sqrt{400} =$

9. Do not use a calculator.

a. $\sqrt{9} =$

b. $\sqrt{900} =$

c. $\sqrt{90000} =$

10. Do not use a calculator.

a. $\sqrt{16} =$

b. $\sqrt{1600} =$

c. $\sqrt{160000} =$

11. Do not use a calculator.

a. $\sqrt{25} =$

b. $\sqrt{0.25} =$

c. $\sqrt{0.0025} =$

12. Do not use a calculator.

a. $\sqrt{49} =$

b. $\sqrt{0.49} =$

c. $\sqrt{0.0049} =$

13. Without using a calculator, estimate the value of $\sqrt{65}$:

(☐ 7.94 ☐ 8.94 ☐ 8.06 ☐ 7.06)

14. Without using a calculator, estimate the value of $\sqrt{78}$:

(☐ 9.83 ☐ 8.83 ☐ 9.17 ☐ 8.17)

Simplify Radical Expressions Evaluate the following.

15. $\sqrt{\frac{100}{121}} = \boxed{}$.

16. $\sqrt{\frac{121}{144}} = \boxed{}$.

17. $-\sqrt{4} = \boxed{}$.

18. $-\sqrt{9} = \boxed{}$.

19. $\sqrt{-25} = \boxed{}$.

20. $\sqrt{-36} = \boxed{}$.

21. $\sqrt{-\frac{36}{49}} = \boxed{}$.

22. $\sqrt{-\frac{49}{144}} = \boxed{}$.

23. $-\sqrt{\frac{81}{100}} = \boxed{}$.

24. $-\sqrt{\frac{100}{121}} = \boxed{}$.

25. a. $\sqrt{169} - \sqrt{144} = \boxed{}$

b. $\sqrt{169 - 144} = \boxed{}$

26. a. $\sqrt{25} - \sqrt{9} = \boxed{}$

b. $\sqrt{25 - 9} = \boxed{}$

Simplify the radical expression or state that it is `not a real number`.

27. $\frac{\sqrt{8}}{\sqrt{2}} = \boxed{}$

28. $\frac{\sqrt{54}}{\sqrt{6}} = \boxed{}$

29. $\frac{\sqrt{4}}{\sqrt{36}} = \boxed{}$

30. $\frac{\sqrt{2}}{\sqrt{32}} = \boxed{}$

31. $\sqrt{343} = \boxed{}$

32. $\sqrt{90} = \boxed{}$

33. $\sqrt{360} = \boxed{}$

34. $\sqrt{2156} = \boxed{}$

35. $\sqrt{231} = \boxed{}$

36. $\sqrt{70} = \boxed{}$

Multiplying Square Root Expressions Simplify the expression.

37. $4\sqrt{7} \cdot 7\sqrt{5} = \boxed{}$

38. $5\sqrt{7} \cdot 4\sqrt{2} = \boxed{}$

39. $6\sqrt{13} \cdot 2\sqrt{25} = \boxed{}$

40. $6\sqrt{3} \cdot 7\sqrt{121} = \boxed{}$

41. $4\sqrt{6} \cdot 3\sqrt{12} = \boxed{}$

42. $5\sqrt{3} \cdot 5\sqrt{150} = \boxed{}$

43. $\sqrt{2} \cdot 4\sqrt{50} = \boxed{}$

44. $\sqrt{2} \cdot 6\sqrt{8} = \boxed{}$

45. $\sqrt{\frac{7}{3}} \cdot \sqrt{\frac{5}{3}} = \boxed{}$

46. $\sqrt{\frac{3}{4}} \cdot \sqrt{\frac{1}{4}} =$ ☐

47. $\sqrt{\frac{6}{19}} \cdot \sqrt{\frac{3}{19}} =$ ☐

48. $\sqrt{\frac{28}{11}} \cdot \sqrt{\frac{4}{11}} =$ ☐

Adding and Subtracting Square Root Expressions Simplify the expression.

49. $16\sqrt{10} - 17\sqrt{10} =$ ☐

50. $17\sqrt{3} - 18\sqrt{3} =$ ☐

51. $19\sqrt{2} - 19\sqrt{2} + 14\sqrt{2} =$ ☐

52. $20\sqrt{23} - 15\sqrt{23} + 19\sqrt{23} =$ ☐

53. $\sqrt{8} + \sqrt{18} =$ ☐

54. $\sqrt{50} + \sqrt{18} =$ ☐

55. $\sqrt{48} - \sqrt{12} =$ ☐

56. $\sqrt{12} - \sqrt{75} =$ ☐

57. $\sqrt{180} + \sqrt{125} + \sqrt{27} + \sqrt{75} =$ ☐

58. $\sqrt{54} + \sqrt{24} + \sqrt{108} + \sqrt{12} =$ ☐

59. $\sqrt{294} - \sqrt{54} - \sqrt{72} - \sqrt{50} =$ ☐

60. $\sqrt{175} - \sqrt{63} - \sqrt{180} - \sqrt{20} =$ ☐

Rationalizing the Denominator Evaluate the following.

61. $\frac{9}{\sqrt{25}} =$ ☐.

62. $\frac{1}{\sqrt{4}} =$ ☐.

Rationalize the denominator and simplify the expression.

63. $\frac{1}{\sqrt{2}} =$ ☐

64. $\frac{1}{\sqrt{3}} =$ ☐

65. $\frac{4}{\sqrt{5}} =$ ☐

66. $\frac{7}{\sqrt{5}} =$ ☐

67. $\frac{7}{8\sqrt{6}} =$ ☐

68. $\frac{10}{3\sqrt{7}} =$ ☐

69. $\frac{12}{\sqrt{30}} =$ ☐

70. $\frac{15}{\sqrt{35}} =$ ☐

71. $\frac{10}{\sqrt{2}} =$ ☐

72. $\frac{6}{\sqrt{2}} =$ ☐

73. $\frac{1}{\sqrt{18}} =$ ☐

74. $\frac{1}{\sqrt{63}} =$ ☐

75. $\frac{8}{\sqrt{180}} =$ ☐

76. $\frac{10}{\sqrt{252}} =$ ☐

77. $\sqrt{\frac{13}{64}} =$ ☐

78. $\sqrt{\frac{7}{81}} =$ ☐

79. $\sqrt{\frac{9}{2}} = \boxed{}$

80. $\sqrt{\frac{81}{2}} = \boxed{}$

81. $\sqrt{\frac{3}{7}} = \boxed{}$

82. $\sqrt{\frac{5}{2}} = \boxed{}$

83. $\sqrt{\frac{54}{7}} = \boxed{}$

84. $\sqrt{\frac{48}{11}} = \boxed{}$

More Complicated Square Root Operations Expand and simplify the expression.

85. $\sqrt{7}\left(\sqrt{3} + \sqrt{13}\right) = \boxed{}$

86. $\sqrt{7}\left(\sqrt{19} + \sqrt{5}\right) = \boxed{}$

87. $\left(5 + \sqrt{13}\right)\left(6 + \sqrt{13}\right) = \boxed{}$

88. $\left(10 + \sqrt{13}\right)\left(9 + \sqrt{13}\right) = \boxed{}$

89. $\left(8 - \sqrt{5}\right)\left(7 - 3\sqrt{5}\right) = \boxed{}$

90. $\left(5 - \sqrt{5}\right)\left(6 - 4\sqrt{5}\right) = \boxed{}$

91. $\left(3 + \sqrt{7}\right)^2 = \boxed{}$

92. $\left(4 + \sqrt{5}\right)^2 = \boxed{}$

93. $\left(\sqrt{3} - 5\right)^2 = \boxed{}$

94. $\left(\sqrt{7} - 6\right)^2 = \boxed{}$

95. $\left(\sqrt{35} - \sqrt{5}\right)^2 = \boxed{}$

96. $\left(\sqrt{14} + \sqrt{7}\right)^2 = \boxed{}$

97. $\left(9 - 5\sqrt{7}\right)^2 = \boxed{}$

98. $\left(6 - 3\sqrt{5}\right)^2 = \boxed{}$

99. $\left(4 - \sqrt{5}\right)\left(4 + \sqrt{5}\right) = \boxed{}$

100. $\left(9 - \sqrt{6}\right)\left(9 + \sqrt{6}\right) = \boxed{}$

101. $\left(\sqrt{6} + \sqrt{7}\right)\left(\sqrt{6} - \sqrt{7}\right) = \boxed{}$

102. $\left(\sqrt{7} + \sqrt{5}\right)\left(\sqrt{7} - \sqrt{5}\right) = \boxed{}$

103. $\left(5\sqrt{5} + 6\sqrt{7}\right)\left(5\sqrt{5} - 6\sqrt{7}\right) = \boxed{}$

104. $\left(6\sqrt{6} + 4\sqrt{5}\right)\left(6\sqrt{6} - 4\sqrt{5}\right) = \boxed{}$

8.3 Solving Quadratic Equations by Using a Square Root

In Section 8.1, we learned how to solve quadratic equations by factoring. In this section, we will learn how to solve some specific types of quadratic equations using the **square root property**. We will also learn how to use the **Pythagorean Theorem** to find the length of one side of a right triangle when the other two lengths are known.

8.3.1 Solving Quadratic Equations Using the Square Root Property

When we learned how to solve linear equations, we used inverse operations to isolate the variable. For example, we use subtraction to remove an unwanted term that is added to one side of a linear equation. We can't quite do the same thing with squaring and using square roots, but we can do something very similar. Taking the square root is the inverse of squaring *if you happen to know the original number was positive*. In general, we have to remember that the original number may have been negative, and that usually leads to *two* solutions to a quadratic equation.

For example, if $x^2 = 9$, we can think of undoing the square with a square root, and $\sqrt{9} = 3$. However, there are *two* numbers that we can square to get 9: -3 and 3. So we need to include both solutions. This brings us to the Square Root Property.

Fact 8.3.2 The Square Root Property. *If k is positive, and $x^2 = k$ then $x = -\sqrt{k}$ or $x = \sqrt{k}$. The positive solution, $\sqrt{k}$, is called the* ***principal root*** *of k.*

Example 8.3.3 Solve for y in $y^2 - 49 = 0$.

Explanation. While we could factor and use the Zero Product Property, here we are demonstrating The Square Root Property instead. We need to isolate the squared quantity.

$$\begin{aligned} y^2 - 49 &= 0 \\ y^2 - 49 + 49 &= 0 + 49 \\ y^2 &= 49 \end{aligned}$$

$$\begin{array}{ccc} y = -\sqrt{49} & \text{or} & y = \sqrt{49} \\ y = -7 & \text{or} & y = 7 \end{array}$$

To check these solutions, we will replace y with -7 and with 7:

$$\begin{array}{cc} y^2 - 49 = 0 & y^2 - 49 = 0 \\ (-7)^2 - 49 \stackrel{?}{=} 0 & (7)^2 - 49 \stackrel{?}{=} 0 \\ 49 - 49 \stackrel{\checkmark}{=} 0 & 49 - 49 \stackrel{\checkmark}{=} 0 \end{array}$$

The solution set is $\{-7, 7\}$.

Remark 8.3.4. Every solution to a quadratic equation can be checked, as shown in Example 8.3.3. In general, the process of checking is omitted from this section.

Remark 8.3.5. Factoring will generally be a possible approach to solving a quadratic equation when the solution(s) are rational, but won't be a possible approach when the solution(s) are irrational.

For example, we could have solved the quadratic equation in Example 8.3.3 by factoring in this way:

$$\begin{aligned} y^2 - 49 &= 0 \\ (y+7)(y-7) &= 0 \end{aligned}$$

$$\begin{array}{rcr} y+7=0 & \text{or} & y-7=0 \\ y=-7 & \text{or} & y=7 \end{array}$$

However, as we'll see in Example 8.3.9, we *cannot* solve $2n^2 - 3 = 0$ by factoring but we *can* use the square root property.

Checkpoint 8.3.6. Solve for z in $4z^2 - 81 = 0$.

Explanation. Before we use the square root property we need to isolate the squared quantity.

$$\begin{aligned} 4z^2 - 81 &= 0 \\ 4z^2 &= 81 \\ z^2 &= \frac{81}{4} \end{aligned}$$

$$\begin{array}{rcr} z = -\sqrt{\dfrac{81}{4}} & \text{or} & z = \sqrt{\dfrac{81}{4}} \\ z = -\dfrac{9}{2} & \text{or} & z = \dfrac{9}{2} \end{array}$$

The solution set is $\left\{-\frac{9}{2}, \frac{9}{2}\right\}$.

We can also use the square root property to solve an equation that has a squared expression.

> **Example 8.3.7** Solve for p in $50 = 2(p-1)^2$.
>
> **Explanation**. It's important here to suppress any urge you may have to expand the squared binomial. We begin by isolating the squared expression.
>
> $$\begin{aligned} 50 &= 2(p-1)^2 \\ \frac{50}{2} &= \frac{2(p-1)^2}{2} \\ 25 &= (p-1)^2 \end{aligned}$$
>
> Now that we have the squared expression isolated, we can use the square root property.
>
> $$\begin{array}{rcr} (p-1) = -\sqrt{25} & \text{or} & (p-1) = \sqrt{25} \\ p-1 = -5 & \text{or} & p-1 = 5 \\ p = -4 & \text{or} & p = 6 \end{array}$$
>
> The solution set is $\{-4, 6\}$.

This method of solving quadratic equations is not limited to equations that have rational solutions, or when the radicands are perfect squares. Here are a few examples where the solutions are irrational numbers.

Checkpoint 8.3.8. Solve for q in $(q+2)^2 - 12 = 0$.

Explanation. It's important here to suppress any urge you may have to expand the squared binomial.

$$\begin{aligned}(q+2)^2 - 12 &= 0\\ (q+2)^2 &= 12\end{aligned}$$

$$\begin{aligned}(q+2) &= -\sqrt{12} &&\text{or} & (q+2) &= \sqrt{12}\\ q+2 &= -2\sqrt{3} &&\text{or} & q+2 &= 2\sqrt{3}\\ q &= -2\sqrt{3}-2 &&\text{or} & q &= 2\sqrt{3}-2\end{aligned}$$

The solution set is $\left\{-2\sqrt{3}-2, 2\sqrt{3}-2\right\}$.

To check the solution, we would replace q with each of $-2\sqrt{3}-2$ and $2\sqrt{3}-2$ in the original equation, as shown here:

$$\begin{aligned}\left(\left(-2\sqrt{3}-2\right)+2\right)^2 - 12 &\stackrel{?}{=} 0 & \left(\left(2\sqrt{3}-2\right)+2\right)^2 - 12 &\stackrel{?}{=} 0\\ \left(-2\sqrt{3}\right)^2 - 12 &\stackrel{?}{=} 0 & \left(2\sqrt{3}\right)^2 - 12 &\stackrel{?}{=} 0\\ (-2)^2\left(\sqrt{3}\right)^2 - 12 &\stackrel{?}{=} 0 & (2)^2\left(\sqrt{3}\right)^2 - 12 &\stackrel{?}{=} 0\\ (4)(3) - 12 &\stackrel{?}{=} 0 & (4)(3) - 12 &\stackrel{?}{=} 0\\ 12 - 12 &\stackrel{\checkmark}{=} 0 & 12 - 12 &\stackrel{\checkmark}{=} 0\end{aligned}$$

Note that these simplifications relied on exponent rules and the multiplicative property of square roots.

Remember that if a square root is in the denominator then we need to rationalize it like we learned in Section 8.2. We will need to rationalize the denominator in the next example.

Example 8.3.9 Solve for n in $2n^2 - 3 = 0$.

Explanation.

$$\begin{aligned}2n^2 - 3 &= 0\\ 2n^2 &= 3\\ n^2 &= \frac{3}{2}\end{aligned}$$

$$\begin{aligned}n &= -\sqrt{\frac{3}{2}} &&\text{or} & n &= \sqrt{\frac{3}{2}}\\ n &= -\frac{\sqrt{3}}{\sqrt{2}} &&\text{or} & n &= \frac{\sqrt{3}}{\sqrt{2}}\\ n &= -\frac{\sqrt{3}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} &&\text{or} & n &= \frac{\sqrt{3}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}\end{aligned}$$

$$n = -\frac{\sqrt{6}}{2} \quad \text{or} \quad n = \frac{\sqrt{6}}{2}$$

The solution set is $\left\{-\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}\right\}$.

When the radicand is a negative number, there is no real solution. Here is an example of an equation with no real solution.

Example 8.3.10 Solve for x in $x^2 + 49 = 0$.

Explanation.

$$\begin{aligned} x^2 + 49 &= 0 \\ x^2 &= -49 \end{aligned}$$

Since $\sqrt{-49}$ is not a real number, we say the equation has no real solution.

8.3.2 The Pythagorean Theorem

Right triangles have an important property called the **Pythagorean Theorem**.

Theorem 8.3.11 The Pythagorean Theorem. *For any right triangle, the lengths of the three sides have the following relationship: $a^2 + b^2 = c^2$. The sides a and b are called **legs** and the longest side c is called the **hypotenuse**.*

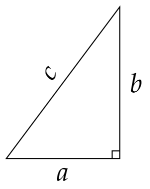


Figure 8.3.12: In a right triangle, the length of its three sides satisfy the equation $a^2 + b^2 = c^2$

Keisha is designing a wooden frame in the shape of a right triangle, as shown in Figure 8.3.14. The legs of the triangle are 3 ft and 4 ft. How long should she make the diagonal side? Use the Pythagorean Theorem to find the length of the hypotenuse.

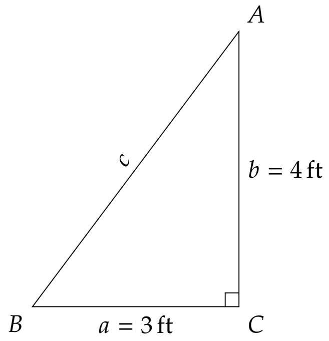


Figure 8.3.14

Example 8.3.13 According to Pythagorean Theorem, we have:

$$\begin{aligned} c^2 &= a^2 + b^2 \\ c^2 &= 3^2 + 4^2 \\ c^2 &= 9 + 16 \\ c^2 &= 25 \end{aligned}$$

Now we have a quadratic equation that we need to solve. We need to find the number that has a square of 25. That is what the square root operation does.

$$\begin{aligned} c &= \sqrt{25} \\ c &= 5 \end{aligned}$$

The diagonal side Keisha will cut is 5 ft long.

Note that -5 is also a solution of $c^2 = 25$ because $(-5)^2 = 25$ but a length cannot be a negative number. We will need to include both solutions when they are relevant.

Example 8.3.15 A 16.5ft ladder is leaning against a wall. The distance from the base of the ladder to the wall is 4.5 feet. How high on the wall can the ladder reach?

The Pythagorean Theorem says:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 4.5^2 + b^2 &= 16.5^2 \\ 20.25 + b^2 &= 272.25 \end{aligned}$$

Now we need to isolate b^2 in order to solve for b:

$$\begin{aligned} 20.25 + b^2 - 20.25 &= 272.25 - 20.25 \\ b^2 &= 252 \end{aligned}$$

To remove the square, we use the square root property. Because this is a geometric situation we only need to use the principal root:

$$b = \sqrt{252}$$

Now simplify this radical and then approximate it:

$$\begin{aligned} b &= \sqrt{36 \cdot 7} \\ b &= 6\sqrt{7} \\ b &\approx 15.87 \end{aligned}$$

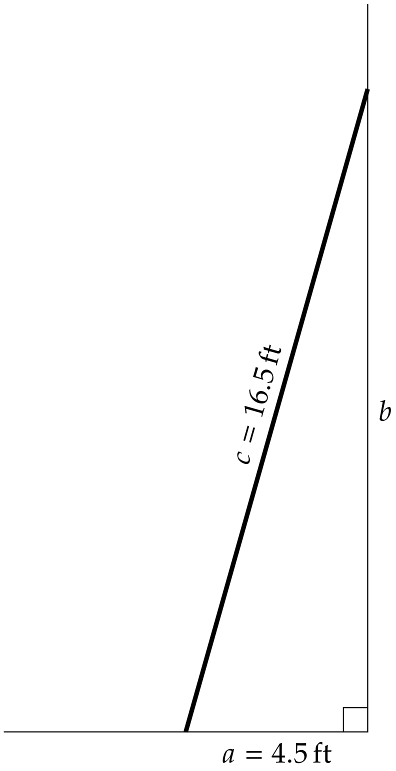


Figure 8.3.16: Leaning Ladder

The ladder can reach about 15.87 feet high on the wall.

Here are some more examples using the Pythagorean Theorem to find sides of triangles. Note that in many contexts, only the principal root will be relevant.

Example 8.3.17 Find the missing length in this right triangle.

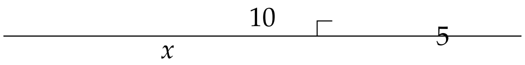


Figure 8.3.18: A Right Triangle

Explanation. We will use the Pythagorean Theorem to solve for x:

$$\begin{aligned} 5^2 + x^2 &= 10^2 \\ 25 + x^2 &= 100 \\ x^2 &= 75 \\ x &= \sqrt{75} && \text{(no need to consider } -\sqrt{75} \text{ in this context)} \\ x &= \sqrt{25 \cdot 3} \\ x &= 5\sqrt{3} \end{aligned}$$

The missing length is $x = 5\sqrt{3}$.

Example 8.3.19 Sergio is designing a 50-inch TV, which implies the diagonal of the TV's screen will be 50 inches long. He needs the screen's width to height ratio to be 4 : 3. Find the TV screen's width and height.

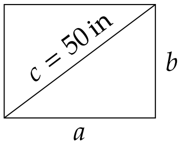


Figure 8.3.20: Pythagorean Theorem Problem

Explanation. Let's let x represent the height of the screen, in inches. Since the screen's width to height ratio will be 4 : 3, then the width is $\frac{4}{3}$ times as long as the height, or $\frac{4}{3}x$ inches. We will draw a diagram.

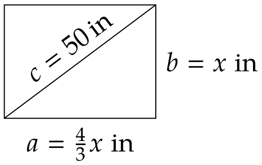


Figure 8.3.21: Pythagorean Theorem Problem

Now we can use the Pythagorean Theorem to write and solve an equation:

$$\begin{aligned} a^2+b^2&=c^2\\ \left(\frac{4}{3}x\right)^2+x^2&=50^2\\ \frac{16}{9}x^2+\frac{9}{9}x^2&=2500\\ \frac{25}{9}x^2&=2500\\ \frac{9}{25}\cdot\frac{25}{9}x^2&=\frac{9}{25}\cdot 2500\\ x^2&=900\\ x&=30 \end{aligned}$$

Since the screen's height is 30 inches, its width is $\frac{4}{3}x=\frac{4}{3}(30)=40$ inches.

Example 8.3.22 Luca wanted to make a bench.

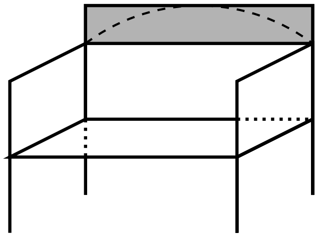


Figure 8.3.23: Sketch of a Bench with Highlighted Back

He wanted the top of the bench back to be a perfect portion of a circle, in the shape of an arc, as in Figure 8.3.24. (Note that this won't be a half-circle, just a small portion of a circular edge.) He started with a rectangular board 3 inches wide and 48 inches long, and a piece of string, like a compass, to draw a circular arc on the board. How long should the string be so that it can be swung round to draw the arc?

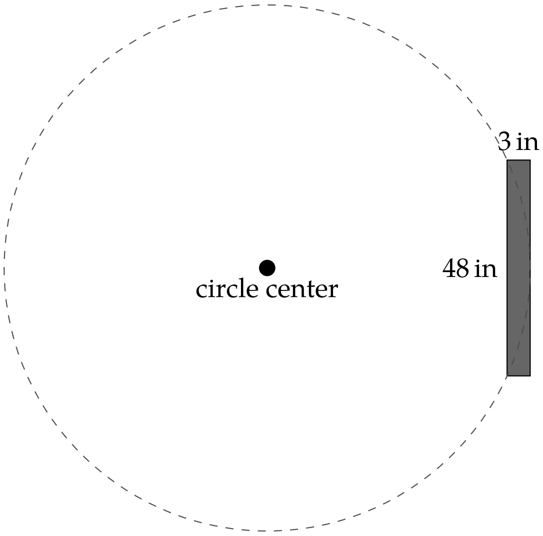


Figure 8.3.24: Bench Back Board

Explanation. Let's first define x to be the radius of the circle in question, in inches. The circle should go through the bottom corners of the board and just barely touch the top of the board. That means that the line from the middle of the bottom of the board to the center of the circle will be 3 inches shorter than the radius.

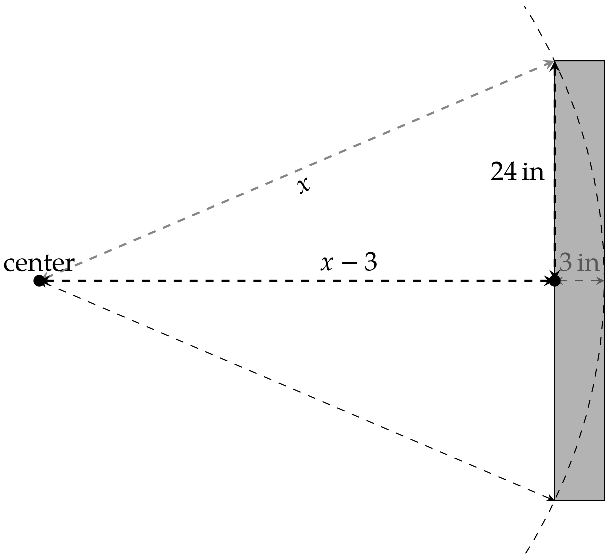


Figure 8.3.25: Bench Back Board Diagram

Now we can set up the Pythagorean Theorem based on the scenario. The equation $a^2 + b^2 = c^2$ turns into…

$$\begin{aligned}(x-3)^2 + 24^2 &= x^2\\ x^2 - 6x + 9 + 576 &= x^2\\ -6x + 585 &= 0\end{aligned}$$

Note that at this point the equation is no longer quadratic! Solve the linear equation by isolating x

$$\begin{aligned}6x &= 585\\ x &= 97.5\end{aligned}$$

So, the circle radius required is 97.5 inches. Luca found a friend to stand on the string end and drew a circular segment on the board to great effect.

Exercises

Solving Quadratic Equations with the Square Root Property Solve the equation.

1. $x^2 = 4$

2. $x^2 = 9$

3. $x^2 = \frac{1}{16}$

4. $x^2 = \frac{1}{25}$

5. $x^2 = 12$

6. $x^2 = 20$

7. $x^2 = 43$

8. $x^2 = 53$

9. $10x^2 = 160$

10. $2x^2 = 8$

11. $x^2 = \frac{144}{121}$

12. $x^2 = \frac{64}{25}$

13. $81x^2 = 16$

14. $121x^2 = 144$

15. $29x^2 - 37 = 0$

16. $5x^2 - 43 = 0$

17. $-5 - 7t^2 = -8$

18. $14 - 3t^2 = 7$

19. $5x^2 + 2 = 0$

20. $47x^2 + 5 = 0$

21. $(x - 5)^2 = 36$

22. $(x - 3)^2 = 4$

23. $(11x + 6)^2 = 100$

24. $(11x + 6)^2 = 16$

25. $26 - 2(r + 8)^2 = 8$

26. $8 - 5(t + 8)^2 = 3$

27. $(x - 4)^2 = 61$

28. $(x + 10)^2 = 2$

29. $(x + 6)^2 = 75$

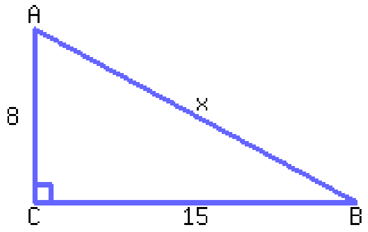
30. $(x + 7)^2 = 63$

31. $7 = 154 - (y - 6)^2$

32. $7 = 154 - (y - 8)^2$

Pythagorean Theorem Applications

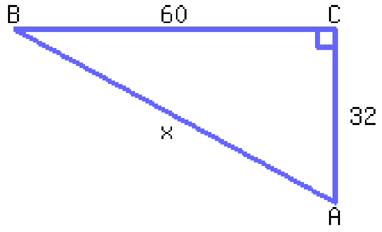
33.



Find the value of x.

$x =$

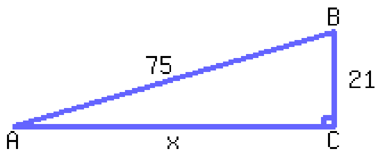
34.



Find the value of x.

$x =$

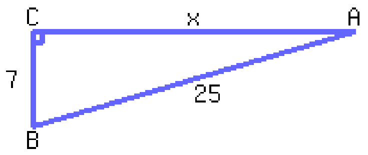
35.



Find the value of x.

$x =$ ☐

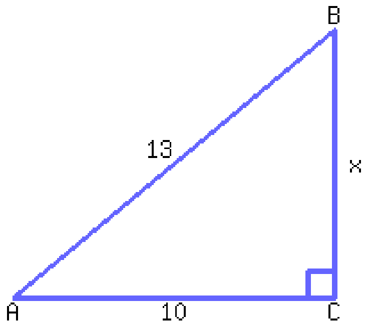
36.



Find the value of x.

$x =$ ☐

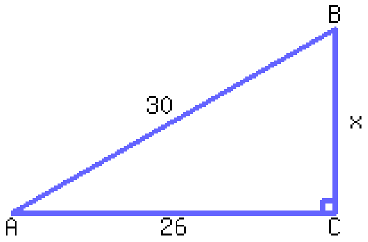
37.



Find the value of x, accurate to at least two decimal places.

$x \approx$ ☐

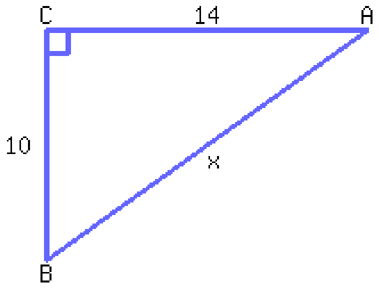
38.



Find the value of x, accurate to at least two decimal places.

$x \approx$ ☐

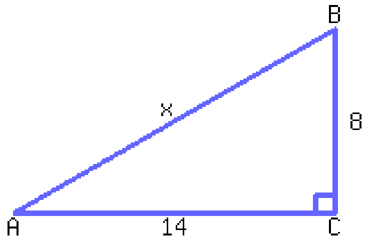
39.



Find the exact value of x.

$x =$ ☐

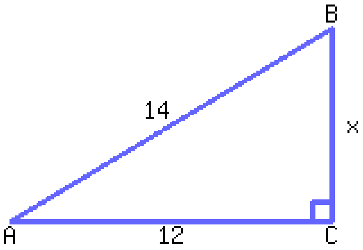
40.



Find the exact value of x.

$x =$ ☐

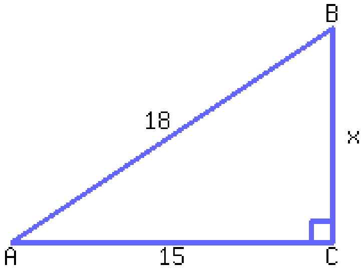
41.



Find the exact value of x.

$x =$ ☐

42.



Find the exact value of x.

$x =$ ☐

43. Rebecca is designing a rectangular garden. The garden's diagonal must be 18.5 feet, and the ratio between the garden's base and height must be 4 : 3. Find the length of the garden's base and height.

The garden's base is ☐ feet and its height is ☐.

44. Jessica is designing a rectangular garden. The garden's diagonal must be 44.2 feet, and the ratio between the garden's base and height must be 15 : 8. Find the length of the garden's base and height.

The garden's base is ☐ feet and its height is ☐.

45. Ashley is designing a rectangular garden. The garden's base must be 6 feet, and the ratio between the garden's hypotenuse and height must be 5 : 3. Find the length of the garden's hypotenuse and height.

The garden's hypotenuse is ☐ feet and its height is ☐.

46. Neil is designing a rectangular garden. The garden's base must be 4.8 feet, and the ratio between the garden's hypotenuse and height must be 13 : 5. Find the length of the garden's hypotenuse and height.

The garden's hypotenuse is ☐ feet and its height is ☐.

Challenge

47. Imagine that you are in Math Land, where roads are perfectly straight, and Mathlanders can walk along a perfectly straight line between any two points. One day, you bike 2 miles west, 4 miles north, and 8 miles east. Then, your bike gets a flat tire and you have to walk home. How far do you have to walk?You have to walk ☐ miles home.

8.4 The Quadratic Formula

We have learned how to solve quadratic equations using factoring and the square root property. In this section, we will learn a third method, the quadratic formula. We will also learn when to use each method and to distinguish between linear and quadratic equations.

8.4.1 Solving Quadratic Equations with the Quadratic Formula

The standard form of a quadratic equation is

$$ax^2 + bx + c = 0$$

Let's look at two examples as a review of solving quadratic equations.

First, let's look at when $b = 0$ and the equation looks like $ax^2 + c = 0$. One way to solve this type of equation is with the square root property.

> **Example 8.4.2** Solve for x in $x^2 - 4 = 0$.
>
> **Explanation**.
>
> $$\begin{aligned} x^2 - 4 &= 0 \\ x^2 &= 4 \end{aligned}$$
>
> $$x = -2 \qquad \text{or} \qquad x = 2$$
>
> The solution set is $\{-2, 2\}$.

Second, if we can factor the left side of the equation in $ax^2 + bx + c = 0$, then we can solve the equation by factoring.

> **Example 8.4.3** Solve for x in $x^2 - 4x - 12 = 0$.
>
> **Explanation**.
>
> $$\begin{aligned} x^2 - 4x - 12 &= 0 \\ (x-6)(x+2) &= 0 \end{aligned}$$
>
> $$\begin{aligned} x - 6 &= 0 & \text{or} \qquad x + 2 &= 0 \\ x &= 6 & \text{or} \qquad x &= -2 \end{aligned}$$
>
> The solution set is $\{-2, 6\}$.

A third method for solving a quadratic equation is to use what is known as the quadratic formula.

Fact 8.4.4 The Quadratic Formula. *For any quadratic equation* $ax^2 + bx + c = 0$, *the solutions are given by*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

As we have seen from solving quadratic equations, there can be at most two solutions. Both of the solutions are included in the quadratic formula with the ± symbol. We could write the two solutions separately as

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{or} \qquad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

but it is more efficient to simplify them together.

This method for solving quadratic equations will work to solve *every* quadratic equation. It is most helpful when $b \neq 0$ and when $ax^2 + bx + c$ cannot be factored. In this section, we will only focus on how to use the formula.

In Section 8.1, we saw an example where a physics class launched a tennis ball off the roof of a building. In that example, the numbers were simplified so we could solve it by factoring. Now we will solve a similar example with more realistic numbers.

Example 8.4.5 Linh is in another physics class that launches a tennis ball from a rooftop that is 90.2 feet above the ground. They fire it directly upward at a speed of 14.4 feet per second and measure the time it takes for the ball to hit the ground below. We can model the height of the tennis ball, h, in feet, with the quadratic equation $h = -16x^2 + 14.4x + 90.2$, where x represents the time in seconds after the launch. According to the model, when should the ball hit the ground? Round the time to one decimal place.

The ground has a height of 0 feet. Substituting 0 for h in the equation, we have this quadratic equation:

$$0 = -16x^2 + 14.4x + 90.2$$

We cannot solve this equation with factoring or the square root property, so we will use the quadratic formula. First we will identify that $a = -16$, $b = 14.4$ and $c = 90.2$, and substitute them into the formula:

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-(14.4) \pm \sqrt{(14.4)^2 - 4(-16)(90.2)}}{2(-16)} \\
x &= \frac{-14.4 \pm \sqrt{207.36 - (-5772.8)}}{-32} \\
x &= \frac{-14.4 \pm \sqrt{207.36 + 5772.8}}{-32} \\
x &= \frac{-14.4 \pm \sqrt{5980.16}}{-32}
\end{aligned}$$

These are the exact solutions but because we have a context we want to approximate the solutions with decimals.

$$x \approx -2.0 \text{ or } x \approx 2.9$$

We don't use the negative solution because a negative time does not make sense in this context. The ball will hit the ground approximately 2.9 seconds after it is launched.

The quadratic formula can be used to solve any quadratic equation, but it requires that you don't make *any* slip-up with remembering the formula, that you correctly identify a, b, and c, and that you don't make any arithmetic mistakes when you calculate and simplify. We recommend that you always check whether you can factor or use the square root property before using the quadratic formula. Here is another example.

Example 8.4.6 Solve for x in $2x^2 - 9x + 5 = 0$.

Explanation. First, we check and see that we cannot factor the left side (because we can't find two numbers that multiply to 10 and add to -9) or use the square root property (because $b \neq 0$) so we must use the quadratic formula. Next we identify that $a = 2$, $b = -9$ and $c = 5$. We will substitute them into the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-(-9) \pm \sqrt{(-9)^2 - 4(2)(5)}}{2(2)} \\ x &= \frac{-9 \pm \sqrt{81 - 40}}{4} \\ x &= \frac{-9 \pm \sqrt{41}}{4} \end{aligned}$$

This is fully simplified because we cannot simplify $\sqrt{41}$ or reduce the fraction. The solution set is $\left\{\frac{-9-\sqrt{22}}{4}, \frac{-9+\sqrt{22}}{4}\right\}$. We do not have a context here so we leave the solutions in their exact form.

When a quadratic equation is not in standard form we must convert it before we can identify the values of a, b and c. We will show that in the next example.

Example 8.4.7 Solve for x in $x^2 = -10x - 3$.

Explanation. First, we convert the equation into standard form by adding $10x$ and 3 to each side of the equation:

$$x^2 + 10x + 3 = 0$$

Next, we check and see that we cannot factor the left side or use the square root property so we must use the quadratic formula. We identify that $a = 1$, $b = 10$ and $c = 3$. We will substitute them into the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-10 \pm \sqrt{(10)^2 - 4(1)(3)}}{2(1)} \\ x &= \frac{-10 \pm \sqrt{100 - 12}}{2} \\ x &= \frac{-10 \pm \sqrt{88}}{2} \end{aligned}$$

Now we need to simplify the square root:

$$x = \frac{-10 \pm 2\sqrt{22}}{2}$$

Lastly we need to reduce the fractions, which can be done by separating terms:

$$\begin{aligned} x &= \frac{-10}{2} \pm \frac{2\sqrt{22}}{2} \\ x &= -5 \pm \sqrt{22} \end{aligned}$$

The solution set is $\{-5 - \sqrt{22}, -5 + \sqrt{22}\}$.

Remark 8.4.8. The irrational solutions to quadratic equations can be checked, although doing so can sometimes involve a lot of simplification and is not shown throughout this section. As an example, to check the solution of $-5 + \sqrt{22}$ from Example 8.4.7, we would replace x with $-5 + \sqrt{22}$ and check that the two sides of the equation are equal. This check is shown here:

$$\begin{aligned} x^2 &= -10x - 3 \\ (-5 + \sqrt{22})^2 &\overset{?}{=} -10(-5 + \sqrt{22}) - 3 \\ (-5)^2 + 2(-5)(\sqrt{22}) + (\sqrt{22})^2 &\overset{?}{=} -10(-5 + \sqrt{22}) - 3 \\ 25 - 10\sqrt{22} + 22 &\overset{?}{=} 50 - 10\sqrt{22} - 3 \\ 47 - 10\sqrt{22} &\overset{\checkmark}{=} 47 - 10\sqrt{22} \end{aligned}$$

When the radicand from the quadratic formula (which is called the **discriminant**) is a negative number, the quadratic equation has no real solution. Example 8.4.9 shows what happens in this case.

Example 8.4.9 Solve for y in $y^2 - 4y + 8 = 0$.

Explanation. Identify that $a = 1$, $b = -4$ and $c = 8$. We will substitute them into the quadratic formula:

$$\begin{aligned} y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(8)}}{2(1)} \\ &= \frac{4 \pm \sqrt{16 - 32}}{2} \\ &= \frac{4 \pm \sqrt{-16}}{2} \end{aligned}$$

The square root of a negative number is not a real number, so we will simply state that this equation has no real solutions.

8.4.2 Recognizing Linear and Quadratic Equations

Now that we have solved both linear and quadratic equations, it is important to identify each type of equation. Recall that a **linear equation** has a degree of one and a **quadratic equation** has a degree of two. If there is any other operation on the variable such as division or a square root then it is not linear or quadratic. We can have other operations on numbers, but not the variables. Let's look at an example.

Checkpoint 8.4.10. Identify whether each equation is linear, quadratic or neither.

a. The equation $3 = 7y^2 - 8y$ is (☐ linear ☐ quadratic ☐ neither) .

b. The equation $5x + 3 = 7x - 8$ is (☐ linear ☐ quadratic ☐ neither) .

c. The equation $r^3 - 7 = 4$ is (☐ linear ☐ quadratic ☐ neither) .

d. The equation $\sqrt{7}x + 4 = 10$ is (☐ linear ☐ quadratic ☐ neither) .

Explanation. We will check the degree of each equation:

a. This is a quadratic equation because there is a y^2 term.

b. This is a linear equation because the highest exponent is one.

c. This is neither linear nor quadratic because it has a degree of three.

d. This is a linear equation because it has a degree of one. The coefficient is irrational but the variable is not in the square root.

8.4.3 Solving Linear and Quadratic Equations

When an equation is **linear**, we move all variable terms to one side of the equation and all constant terms to the other side. Then we use division if needed to solve for the variable. This is outlined in List 3.1.4.

When an equation is **quadratic**, we have three different methods we can use. Here is an outline of the general process for determining which method to use.

Process 8.4.11 Solving Quadratic Equations.

1. *First, check whether there is a linear term, or whether there is only a squared expression and a constant. If there is only a squared expression and a constant, isolate the squared quantity and use the* ***square root method****.*
2. *If there* is *a linear term, put the equation in* ***standard form*** *with all of the terms on one side and zero on the other side.*
   a) *If the polynomial factors, solve the equation by* ***factoring****.*
   b) *If the polynomial does not factor, use the* ***quadratic formula****.*

Here are some examples:

Example 8.4.12 Solve for x in $x^2 = 7x^2 - 12$.

Explanation. This is a quadratic equation because there are x^2 terms. There are no x terms so we will use the square root method. We start by combining the x^2 terms on the left.

$$\begin{aligned} x^2 &= 7x^2 - 12 \\ -6x^2 &= -12 \\ x^2 &= 2 \end{aligned}$$

$$x = -\sqrt{2} \qquad \text{or} \qquad x = \sqrt{2}$$

The solution set is $\{-\sqrt{2}, \sqrt{2}\}$.

Example 8.4.13 Solve for p in $p^2 = -2p + 2$.

Explanation. This is a quadratic equation that also contains a linear term so we will put the equation in standard form. Then we see that the left side does not factor so we will use the quadratic formula.

$$\begin{aligned}
p^2 + 2p - 2 &= 0 \\
p &= \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-2)}}{2(1)} \\
p &= \frac{-2 \pm \sqrt{4+8}}{2} \\
p &= \frac{-2 \pm \sqrt{12}}{2} \\
p &= \frac{-2 \pm 2\sqrt{3}}{2} \\
p &= \frac{-2}{2} \pm \frac{2\sqrt{3}}{2} \\
p &= -1 \pm \sqrt{3}
\end{aligned}$$

The solution set is $\{1 - \sqrt{3}, 1 + \sqrt{3}\}$.

Example 8.4.14 Solve for t in $7 - t = 9t + 11$.

Explanation. This is a linear equation so we will combine the linear terms on one side and the constant terms on the other side.

$$\begin{aligned}
-10t &= 4 \\
t &= \frac{4}{-10} \\
t &= -\frac{2}{5}
\end{aligned}$$

The solution set is $\left\{-\frac{2}{5}\right\}$.

Example 8.4.15 Solve for z in $z^2 - 10 = 3z + 30$.

Explanation. This is a quadratic equation with a linear term so we will put it in standard form. Then we see that the left side factors so we solve by factoring.

$$\begin{aligned}
z^2 - 10 &= 3z + 30 \\
z^2 - 3z - 40 &= 0 \\
(z+5)(z-8) &= 0
\end{aligned}$$

$$\begin{aligned}
z + 5 &= 0 & \text{or} && z - 8 &= 0 \\
z &= -5 & \text{or} && z &= 8
\end{aligned}$$

The solution set is $\{-5, 8\}$.

Exercises

Review and Warmup

1. Evaluate $\dfrac{2a - 2B + 10}{-10a + 10B}$ for $a = 6$ and $B = -5$.

2. Evaluate $\dfrac{2c + 8b - 9}{-c - 5b}$ for $c = -6$ and $b = 10$.

3. Evaluate the expression $\frac{1}{3}(x + 3)^2 - 9$ when $x = -6$.

4. Evaluate the expression $\frac{1}{6}(x + 3)^2 - 7$ when $x = -9$.

5. Evaluate the expression $-16t^2 + 64t + 128$ when $t = 3$.

6. Evaluate the expression $-16t^2 + 64t + 128$ when $t = 5$.

7. Evaluate the expression x^2:

a. When $x = 6$, $x^2 =$

b. When $x = -4$, $x^2 =$

8. Evaluate the expression x^2:

a. When $x = 3$, $x^2 =$

b. When $x = -7$, $x^2 =$

9. Evaluate each algebraic expression for the given value(s):

$\dfrac{\sqrt{x}}{y} - \dfrac{y}{x}$, for $x = 16$ and $y = 3$:

10. Evaluate each algebraic expression for the given value(s):

$\dfrac{y}{4x} - \dfrac{\sqrt{x}}{2y}$, for $x = 9$ and $y = 8$:

Solve Quadratic Equations Using the Quadratic Formula Solve the equation.

11. $x^2 - 8x + 5 = 0$

12. $x^2 - 10x + 17 = 0$

13. $10x^2 + 17x - 20 = 0$

14. $15x^2 + 4x - 32 = 0$

15. $x^2 = -5x - 3$

16. $x^2 = x + 1$

17. $x^2 + 9x + 9 = 0$

18. $x^2 - 9x + 9 = 0$

19. $3x^2 + 5x + 1 = 0$

20. $2x^2 + 3x - 1 = 0$

21. $7x^2 - 2x - 1 = 0$

22. $4x^2 - 10x - 5 = 0$

23. $5x^2 - 10x + 10 = 0$

24. $5x^2 - 5x + 7 = 0$

Solve Quadratic Equations Using an Appropriate Method Solve the equation.

25. $2x^2 - 18 = 0$

26. $3x^2 - 75 = 0$

27. $9t + 7 = t + 23$

28. $7a + 2 = a + 50$

29. $9x^2 - 16 = 0$

30. $49x^2 - 64 = 0$

31. $-2 - 7r^2 = -4$

32. $8 - 7t^2 = 3$

33. $x^2 + 70x = 0$

34. $x^2 - 90x = 0$

35. $9y + 10 = -9y + 10 - 3y$

36. $6t + 5 = -6t + 5 - 3t$

37. $x^2 - 2x = 8$

38. $x^2 + 8x = 9$

39. $(x + 1)^2 = 49$

40. $(x + 3)^2 = 9$

41. $x^2 = -3x - 1$

42. $x^2 = -3x + 2$

43. $4x^2 = 9x - 4$

44. $3x^2 = 5x - 1$

45. $-6 - 3(x - 10)^2 = -9$

46. $14 - 2(y + 10)^2 = 6$

47. $-10 = -8c - 10 - c$

48. $31 = -5A - 5 - A$

Quadratic Formula Applications

49. Two numbers' sum is 16, and their product is 60. Find these two numbers.

These two numbers are ☐.

50. Two numbers' sum is 14, and their product is 48. Find these two numbers.

These two numbers are ☐.

51. Two numbers' sum is 11.7, and their product is 28.22. Find these two numbers.

These two numbers are ☐.
(Use a comma to separate your numbers.)

52. Two numbers' sum is 2.1, and their product is −95.92. Find these two numbers.

These two numbers are ☐.
(Use a comma to separate your numbers.)

53. A rectangle's base is 8 cm longer than its height. The rectangle's area is 65 cm^2. Find this rectangle's dimensions.

The rectangle's height is ☐.

The rectangle's base is ☐.

54. A rectangle's base is 6 cm longer than its height. The rectangle's area is 72 cm^2. Find this rectangle's dimensions.

The rectangle's height is ☐.

The rectangle's base is ☐.

55. A rectangle's base is 3 in shorter than five times its height. The rectangle's area is 36 in^2. Find this rectangle's dimensions.

The rectangle's height is .

The rectangle's base is .

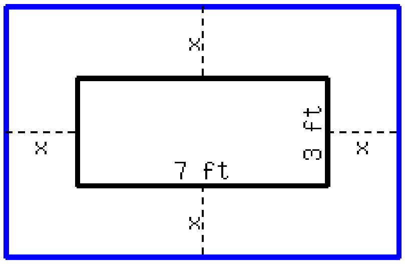
57. You will build a rectangular sheep pen next to a river. There is no need to build a fence along the river, so you only need to build three sides.

You have a total of 480 feet of fence to use, and the area of the pen must be 28600 square feet. Find the dimensions of the pen.

There should be two solutions:When the width is feet, the length is feet.

When the width is feet, the length is feet.

59. There is a rectangular lot in the garden, with 7 ft in length and 3 ft in width. You plan to expand the lot by an equal length around its four sides, and make the area of the expanded rectangle 77 ft^2. How long should you expand the original lot in four directions?



You should expand the original lot by in four directions.

56. A rectangle's base is 7 in shorter than four times its height. The rectangle's area is 15 in^2. Find this rectangle's dimensions.

The rectangle's height is .

The rectangle's base is .

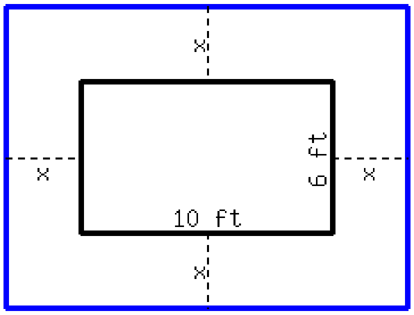
58. You will build a rectangular sheep pen next to a river. There is no need to build a fence along the river, so you only need to build three sides.

You have a total of 550 feet of fence to use, and the area of the pen must be 37700 square feet. Find the dimensions of the pen.

There should be two solutions:When the width is feet, the length is feet.

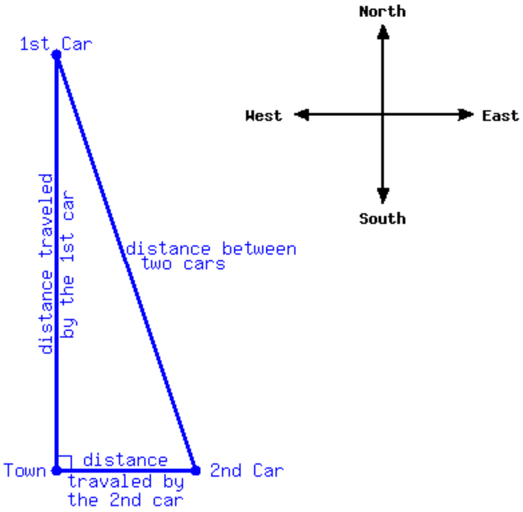
When the width is feet, the length is feet.

60. There is a rectangular lot in the garden, with 10 ft in length and 6 ft in width. You plan to expand the lot by an equal length around its four sides, and make the area of the expanded rectangle 192 ft^2. How long should you expand the original lot in four directions?



You should expand the original lot by in four directions.

61. One car started at Town A, and traveled due north at 70 miles per hour. 3.5 hours later, another car started at the same spot and traveled due east at 65 miles per hour. Assume both cars don't stop, after how many hours since the second car starts would the distance between them be 469 miles? Round your answer to two decimal places if needed.


North
1st Car
West
East
South
distance traveled by the 1st car
distance between two cars
Town
distance traveled by the 2nd car
2nd Car

Approximately ______ hours since the second car starts, the distance between those two cars would be 469 miles.

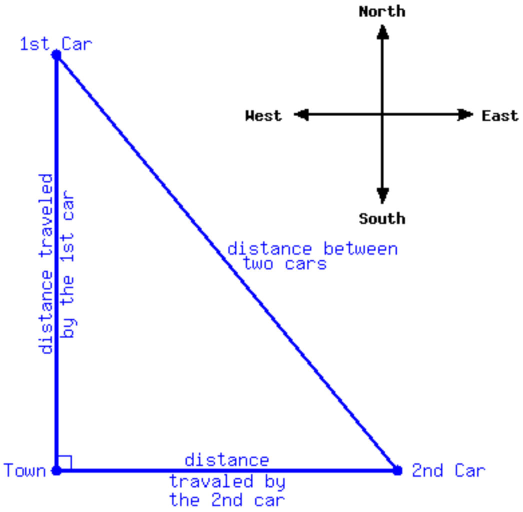
63. An object is launched upward at the height of 240 meters. It's height can be modeled by

$$h = -4.9t^2 + 50t + 240,$$

where h stands for the object's height in meters, and t stands for time passed in seconds since its launch. The object's height will be 280 meters twice before it hits the ground. Find how many seconds since the launch would the object's height be 280 meters. Round your answers to two decimal places if needed.

The object's height would be 280 meters the first time at ______ seconds, and then the second time at ______ seconds.

62. One car started at Town A, and traveled due north at 40 miles per hour. 1.5 hours later, another car started at the same spot and traveled due east at 55 miles per hour. Assume both cars don't stop, after how many hours since the second car starts would the distance between them be 229 miles? Round your answer to two decimal places if needed.


North
1st Car
West
East
South
distance traveled by the 1st car
distance between two cars
Town
distance traveled by the 2nd car
2nd Car

Approximately ______ hours since the second car starts, the distance between those two cars would be 229 miles.

64. An object is launched upward at the height of 260 meters. It's height can be modeled by

$$h = -4.9t^2 + 90t + 260,$$

where h stands for the object's height in meters, and t stands for time passed in seconds since its launch. The object's height will be 270 meters twice before it hits the ground. Find how many seconds since the launch would the object's height be 270 meters. Round your answers to two decimal places if needed.

The object's height would be 270 meters the first time at ______ seconds, and then the second time at ______ seconds.

65. Currently, an artist can sell 230 paintings every year at the price of \$60.00 per painting. Each time he raises the price per painting by \$5.00, he sells 10 fewer paintings every year.

Assume he will raise the price per painting x times, then he will sell $230 - 10x$ paintings every year at the price of $60 + 5x$ dollars. His yearly income can be modeled by the equation:

$$i = (60 + 5x)(230 - 10x)$$

where i stands for his yearly income in dollars. If the artist wants to earn \$14,700.00 per year from selling paintings, what new price should he set?

To earn \$14,700.00 per year, the artist could sell his paintings at two different prices. The lower price is ☐ per painting, and the higher price is ☐ per painting.

66. Currently, an artist can sell 250 paintings every year at the price of \$110.00 per painting. Each time he raises the price per painting by \$10.00, he sells 10 fewer paintings every year.

Assume he will raise the price per painting x times, then he will sell $250 - 10x$ paintings every year at the price of $110 + 10x$ dollars. His yearly income can be modeled by the equation:

$$i = (110 + 10x)(250 - 10x)$$

where i stands for his yearly income in dollars. If the artist wants to earn \$32,000.00 per year from selling paintings, what new price should he set?

To earn \$32,000.00 per year, the artist could sell his paintings at two different prices. The lower price is ☐ per painting, and the higher price is ☐ per painting.

67. Solve for x in the equation $mx^2 + nx + p = 0$.

8.5 Complex Solutions to Quadratic Equations

8.5.1 Imaginary Numbers

Let's look at how to simplify a square root that has a negative radicand. Remember that $\sqrt{16} = 4$ because $4^2 = 16$. So what could $\sqrt{-16}$ be equal to? There is no real number that we can square to get -16, because when you square a real number, the result is either positive or 0. You might think about 4 and -4, but:

$$4^2 = 16 \text{ and } (-4)^2 = 16$$

so neither of those could be $\sqrt{-16}$. To handle this situation, mathematicians separate a factor of $\sqrt{-1}$ and represent it with the letter i, which stands for **imaginary unit**.

Definition 8.5.2 Imaginary Numbers. The **imaginary unit**, i, is defined by $i = \sqrt{-1}$. The imaginary unit[1] satisfies the equation $i^2 = -1$. A real number times i, such as $4i$, is called an **imaginary number**.

Now we can simplify square roots with negative radicands like $\sqrt{-16}$.

$$\begin{aligned} \sqrt{-16} &= \sqrt{-1 \cdot 16} \\ &= \sqrt{-1} \cdot \sqrt{16} \\ &= i \cdot 4 \\ &= 4i \end{aligned}$$

Imaginary numbers are widely used in electrical engineering, physics, computer science and other fields. Let's look some more examples.

Example 8.5.3 Simplify $\sqrt{-2}$.

Explanation.

$$\begin{aligned} \sqrt{-2} &= \sqrt{-1 \cdot 2} \\ &= \sqrt{-1} \cdot \sqrt{2} \\ &= i\sqrt{2} \end{aligned}$$

We write the i first because it's difficult to tell the difference between $\sqrt{2}i$ and $\sqrt{2i}$.

Example 8.5.4 Simplify $\sqrt{-72}$.

Explanation.

$$\begin{aligned} \sqrt{-72} &= \sqrt{-1 \cdot 36 \cdot 2} \\ &= \sqrt{-1} \cdot \sqrt{36} \cdot \sqrt{2} \\ &= 6i\sqrt{2} \end{aligned}$$

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[1] en.wikipedia.org/wiki/Imaginary_number

8.5.2 Solving Quadratic Equations with Imaginary Solutions

Example 8.5.5 Solve for x in $x^2 + 49 = 0$, where x is an imaginary number.

Explanation. There is no x term so we will use the square root method.

$$\begin{aligned} x^2 + 49 &= 0 \\ x^2 &= -49 \end{aligned}$$

$$\begin{array}{lcl} x = -\sqrt{-49} & \text{or} & x = \sqrt{-49} \\ x = -\sqrt{-1} \cdot \sqrt{49} & \text{or} & x = \sqrt{-1} \cdot \sqrt{49} \\ x = -7i & \text{or} & x = 7i \end{array}$$

The solution set is $\{-7i, 7i\}$.

Example 8.5.6 Solve for p in $p^2 + 75 = 0$, where p is an imaginary number.

Explanation. There is no p term so we will use the square root method.

$$\begin{aligned} p^2 + 75 &= 0 \\ p^2 &= -75 \end{aligned}$$

$$\begin{array}{lcl} p = -\sqrt{-75} & \text{or} & p = \sqrt{-75} \\ p = -\sqrt{-1} \cdot \sqrt{25} \cdot \sqrt{3} & \text{or} & p = \sqrt{-1} \cdot \sqrt{25} \cdot \sqrt{3} \\ p = -5i\sqrt{3} & \text{or} & p = 5i\sqrt{3} \end{array}$$

The solution set is $\left\{-5i\sqrt{3}, 5i\sqrt{3}\right\}$.

8.5.3 Solving Quadratic Equations with Complex Solutions

A **complex number** is a combination of a real number and an imaginary number, like $3 + 2i$ or $-4 - 8i$.

Definition 8.5.7 Complex Number. A **complex number** is a number that can be expressed in the form $a+bi$, where a and b are real numbers and i is the imaginary unit. In this expression, a is the **real part** and b (not bi) is the **imaginary part** of the complex number[2].

Here are some examples of equations that have complex solutions.

Example 8.5.8 Solve for m in $(m - 1)^2 + 18 = 0$, where m is a complex number.

Explanation. This equation has a squared expression so we will use the square root method.

$$\begin{aligned} (m - 1)^2 + 18 &= 0 \\ (m - 1)^2 &= -18 \end{aligned}$$

---

[2] en.wikipedia.org/wiki/Complex_number

$$\begin{aligned} m-1 &= -\sqrt{-18} &\quad\text{or}\quad&& m-1 &= \sqrt{-18} \\ m-1 &= -\sqrt{-1}\cdot\sqrt{9}\cdot\sqrt{2} &\quad\text{or}\quad&& m-1 &= \sqrt{-1}\cdot\sqrt{9}\cdot\sqrt{2} \\ m-1 &= -3i\sqrt{2} &\quad\text{or}\quad&& m-1 &= 3i\sqrt{2} \\ m &= 1-3i\sqrt{2} &\quad\text{or}\quad&& m &= 1+3i\sqrt{2} \end{aligned}$$

The solution set is $\left\{1-3i\sqrt{2}, 1+3i\sqrt{2}\right\}$.

Example 8.5.9 Solve for y in $y^2 - 4y + 13 = 0$, where y is a complex number.

Explanation. Note that there is a y term, but the left side does not factor. We will use the quadratic formula. We identify that $a = 1$, $b = -4$ and $c = 13$ and substitute them into the quadratic formula.

$$\begin{aligned} y &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} \\ &= \frac{4 \pm \sqrt{16-52}}{2} \\ &= \frac{4 \pm \sqrt{-36}}{2} \\ &= \frac{4 \pm \sqrt{-1}\cdot\sqrt{36}}{2} \\ &= \frac{4 \pm 6i}{2} \\ &= 2 \pm 3i \end{aligned}$$

The solution set is $\{2-3i, 2+3i\}$.

Note that in Example 8.5.9, the expressions $2+3i$ and $2-3i$ are fully simplified. In the same way that the terms 2 and $3x$ cannot be combined, the terms 2 and $3i$ can not be combined.

Remark 8.5.10. Each complex solution can be checked, just as every real solution can be checked. For example, to check the solution of $2+3i$ from Example 8.5.9, we would replace y with $2+3i$ and check that the two sides of the equation are equal. In doing so, we will need to use the fact that $i^2 = -1$. This check is shown here:

$$\begin{aligned} y^2 - 4y + 13 &= 0 \\ (2+3i)^2 - 4(2+3i) + 13 &\overset{?}{=} 0 \\ (2^2 + 2(3i) + 2(3i) + (3i)^2) - 4\cdot 2 - 4\cdot(3i) + 13 &\overset{?}{=} 0 \\ 4 + 6i + 6i + 9i^2 - 8 - 12i + 13 &\overset{?}{=} 0 \\ 4 + 9(-1) - 8 + 13 &\overset{?}{=} 0 \\ 4 - 9 - 8 + 13 &\overset{?}{=} 0 \\ 0 &\overset{\checkmark}{=} 0 \end{aligned}$$

Exercises

Simplifying Square Roots with Negative Radicands Simplify the radical and write it into a complex number.

1. $\sqrt{-105} = \boxed{}$

2. $\sqrt{-42} = \boxed{}$

3. $\sqrt{-72} = \boxed{}$

4. $\sqrt{-56} = \boxed{}$

5. $\sqrt{-126} = \boxed{}$

6. $\sqrt{-200} = \boxed{}$

Quadratic Equations with Imaginary and Complex Solutions Solve the quadratic equation. Solutions could be complex numbers.

7. $y^2 = -16$

8. $r^2 = -100$

9. $-6r^2 + 2 = 296$

10. $-3t^2 + 2 = 50$

11. $2t^2 + 10 = 6$

12. $2t^2 - 2 = -8$

13. $-6x^2 + 4 = 124$

14. $-6x^2 - 7 = 293$

15. $9(y + 4)^2 + 1 = -323$

16. $6(y - 5)^2 - 7 = -61$

17. $r^2 + 2r + 5 = 0$

18. $r^2 + 6r + 10 = 0$

19. $t^2 + 4t + 9 = 0$

20. $t^2 + 4t + 11 = 0$

8.6 Strategies for Solving Quadratic Equations

In this section, we will review how to solve quadratic equations using three different methods: The square root method, factoring and the quadratic formula.

8.6.1 How to Choose a Method for Solving a Quadratic Equation

Process 8.6.2. *So far, we have learned three methods for solving quadratic equations in standard form, $ax^2+bx+c=0$:*

1. *When $b=0$, as in $x^2-4=0$, we can use the square root property, Property 8.3.2.*
2. *If we can easily factor the polynomial, as in $x^2-4x-12=0$, we will solve the equation by factoring and use the zero product property, Property 8.1.4.*
3. *If we cannot solve the equation by the first two methods, we must use the Quadratic Formula, Property 8.4.4. This formula works for any quadratic equation, but the first two methods are usually easier.*

Let's look at a few examples for how to choose which method to use.

> **Example 8.6.3** Solve for y in $y^2-49=0$.
>
> **Explanation**. In this equation, $b=0$, so it is easiest to use the square root method. We isolate the squared quantity and then use the square root property.
>
> $$\begin{aligned} y^2-49&=0\\ y^2&=49 \end{aligned}$$
>
> $$\begin{array}{ccc} y=-\sqrt{49} & \text{or} & y=\sqrt{49}\\ y=-7 & \text{or} & y=7 \end{array}$$
>
> The solution set is $\{-7,7\}$.
>
> Because 49 is a perfect square, we could also solve this equation by factoring.
>
> $$\begin{aligned} y^2-49&=0\\ (y+7)(y-7)&=0 \end{aligned}$$
>
> $$\begin{array}{ccc} y+7=0 & \text{or} & y-7=0\\ y=-7 & \text{or} & y=7 \end{array}$$
>
> We get the same solution set, $\{-7,7\}$.

We can also use the square root method when a binomial is squared, like $(p-1)^2$, as we will see in the next example.

> **Example 8.6.4** Solve for p in $-40=10-2(p-1)^2$.
>
> **Explanation**. We isolate the squared binomial and then use the square root property.
>
> $$-40=10-2(p-1)^2$$

$$\begin{aligned} -50 &= -2(p-1)^2 \\ 25 &= (p-1)^2 \end{aligned}$$

$$\begin{aligned} p-1 &= -5 & \text{or} && p-1 &= 5 \\ p &= -4 & \text{or} && p &= 6 \end{aligned}$$

The solution set is $\{-4, 6\}$

Let's check the solution $p = -4$:

$$\begin{aligned} -40 &= 10 - 2(p-1)^2 \\ -40 &\stackrel{?}{=} 10 - 2(-4-1)^2 \\ -40 &\stackrel{?}{=} 10 - 2(-5)^2 \\ -40 &\stackrel{?}{=} 10 - 2(25) \\ -40 &\stackrel{\checkmark}{=} 10 - 50 \end{aligned}$$

The solution $p = -4$ is verified. Checking $p = 6$ is left as an exercise.

When we have a middle term in $ax^2 + bx + c = 0$, we cannot use the square root property. We look first to see if we can solve the equation by factoring. Here are some examples.

Example 8.6.5 Solve for x in $x^2 - 4x - 12 = 0$.

Explanation. The equation is already in standard form and we can factor the polynomial on the left side of the equation. We will factor it and then use the zero product property to solve the equation.

$$\begin{aligned} x^2 - 4x - 12 &= 0 \\ (x-6)(x+2) &= 0 \end{aligned}$$

$$\begin{aligned} x-6 &= 0 & \text{or} && x+2 &= 0 \\ x &= 6 & \text{or} && x &= -2 \end{aligned}$$

The solution set is $\{-2, 6\}$.

Example 8.6.6 Solve for t in $2t^2 - 30t + 28 = 0$.

Explanation. First we factor out the common factor of 2. Then we can see that the polynomial is factorable. We solve it using the zero product property.

$$\begin{aligned} 2t^2 - 30t + 28 &= 0 \\ 2(t^2 - 15t + 14) &= 0 \\ 2(t-1)(t-14) &= 0 \end{aligned}$$

$$\begin{aligned} t-1 &= 0 & \text{or} && t-14 &= 0 \\ t &= 1 & \text{or} && t &= 14 \end{aligned}$$

The solution set is $\{1, 14\}$.

If the equation is not in standard form, we must rewrite it before we can solve it. Let's look at the next example.

Example 8.6.7 Solve for x in $(x+4)(x-3) = 18$.

Explanation. We need to have one side equal to 0 in order to use the zero product property, so we will multiply the left side and subtract 18 from both sides.

$$\begin{aligned}(x+4)(x-3) &= 18\\ x^2+x-12 &= 18\\ x^2+x-12-18 &= 18-18\\ x^2+x-30 &= 0\\ (x+6)(x-5) &= 0\end{aligned}$$

$$\begin{aligned}x+6 &= 0 &&\text{or} & x-5 &= 0\\ x &= -6 &&\text{or} & x &= 5\end{aligned}$$

The solution set is $\{-6, 5\}$.

When it's difficult or impossible to factor the trinomial in $ax^2+bx+c=0$, we have to resort to the Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Here is an example.

Example 8.6.8 Solve for x in $x^2-10x+3=0$.

Explanation. We identify that $a = 1$, $b = -10$ and $c = 3$ and substitute them into the Quadratic Formula:

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}\\ x &= \frac{-(-10) \pm \sqrt{(-10)^2-4(1)(3)}}{2(1)}\\ x &= \frac{10 \pm \sqrt{100-12}}{2}\\ x &= \frac{10 \pm \sqrt{88}}{2}\\ x &= \frac{10 \pm 2\sqrt{22}}{2}\\ x &= 5 \pm \sqrt{22}\end{aligned}$$

The solution set is $\left\{5-\sqrt{22}, 5+\sqrt{22}\right\}$.

If a quadratic equation is not in standard form we need to rewrite it to identify the values of a, b and c. Let's look at an example.

Example 8.6.9 Solve for x in $-3x^2 - 1 = -8x$.

Explanation. First, we convert the equation into standard form:

$$\begin{aligned} -3x^2 - 1 &= -8x \\ -3x^2 + 8x - 1 &= 0 \end{aligned}$$

Now we can identify that $a = -3$, $b = 8$ and $c = -1$. We will substitute them into the Quadratic Formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-8 \pm \sqrt{8^2 - 4(-3)(-1)}}{2(-3)} \\ x &= \frac{-8 \pm \sqrt{64 - 12}}{-6} \\ x &= \frac{-8 \pm \sqrt{52}}{-6} \\ x &= \frac{-8 \pm 2\sqrt{13}}{-6} \\ x &= \frac{4 \pm \sqrt{13}}{3} \end{aligned}$$

The solution set is $\left\{\frac{4-\sqrt{13}}{3}, \frac{4+\sqrt{13}}{3}\right\}$

Also recall that if the radicand is negative, there is no real solution to the equation.

This was a brief review of solving quadratic equations. If you would like the full explanation of solving using the square root method, you can go to Section 8.3. For solving by factoring, you can go to Section 8.1. If you want more on the quadratic formula, you can go to Section 8.4.

Exercises

Solving Quadratic Equations Using the Square Root Method Solve the equation.

1. $x^2 = 144$

2. $x^2 = 4$

3. $3x^2 = 48$

4. $4x^2 = 36$

5. $2x^2 + 23 = 0$

6. $43x^2 + 31 = 0$

7. $9 - 3(r + 3)^2 = -3$

8. $48 - 6(t - 3)^2 = -6$

Solving Quadratic Equations by Factoring Solve the equation.

9. $82(x + 8)(14x - 5) = 0$

10. $-82(x + 10)(7x - 9) = 0$

11. $x^2 - 11x + 10 = 0$

12. $x^2 - 13x + 36 = 0$

13. $x^2 - 15x = -56$

14. $x^2 - 7x = -6$

15. $x^2 + 14x + 49 = 0$

16. $x^2 + 16x + 64 = 0$

17. $x(5x + 59) = -90$

18. $x(5x + 51) = -10$

19. $(x + 3)(x - 5) = -7$

20. $(x + 6)(x + 3) = -2$

Solving Quadratic Equations Using the Quadratic Formula Solve the equation.

21. $x^2 - 2x - 11 = 0$

22. $x^2 + 6x - 2 = 0$

23. $x^2 = -3x - 1$

24. $x^2 = 5x - 2$

25. $5x^2 - 8x + 4 = 0$

26. $4x^2 + 10x + 7 = 0$

Choosing Which Method to Use Solve the equation.

27. $x^2 + 4x = 32$

28. $x^2 + 9x = 10$

29. $2x^2 = 32$

30. $3x^2 = 27$

31. $4x^2 - 8x + 5 = 0$

32. $3x^2 + 3x + 10 = 0$

33. $13x^2 + 29 = 0$

34. $67x^2 + 41 = 0$

35. $4x^2 = -29x - 7$

36. $5x^2 = -39x - 28$

37. $x^2 - 10x + 8 = 0$

38. $x^2 - 2x - 19 = 0$

39. $29 - 3(x + 10)^2 = 2$

40. $2 - 6(y + 9)^2 = -4$

41. $x^2 + 11x = -30$

42. $x^2 + 8x = -12$

8.7 Solving Quadratic Equations Chapter Review

8.7.1 Solving Quadratic Equations by Factoring

In Section 8.1 we covered the zero product property and learned an algorithm for solving quadratic equations by factoring.

Example 8.7.1 Solving Using Factoring. Solve the quadratic equations using factoring.

a. $x^2 - 2x - 15 = 0$

b. $4x^2 - 40x = -96$

c. $6x^2 + x - 12 = 0$

d. $(x-3)(x+2) = 14$

e. $x^3 - 64x = 0$

Explanation.

a. Use factor pairs.

$$\begin{aligned} x^2 - 2x - 15 &= 0 \\ (x-5)(x+3) &= 0 \end{aligned}$$

$$\begin{aligned} x - 5 &= 0 && \text{or} && x + 3 &= 0 \\ x &= 5 && \text{or} && x &= -3 \end{aligned}$$

So the solution set is $\{5, -3\}$.

b. Start by putting the equation in standard form and factoring out the greatest common factor.

$$\begin{aligned} 4x^2 - 40x &= -96 \\ 4x^2 - 40x + 96 &= 0 \\ 4\left(x^2 - 10x + 24\right) &= 0 \\ 4(x-6)(x-4) &= 0 \end{aligned}$$

$$\begin{aligned} x - 6 &= 0 && \text{or} && x - 4 &= 0 \\ x &= 6 && \text{or} && x &= 4 \end{aligned}$$

So the solution set is $\{4, 6\}$.

c. Use the AC method.

$$6x^2 + x - 12 = 0$$

Note that $a \cdot c = -72$ and that $9 \cdot -8 = -72$ and $9 - 8 = 1$

$$\begin{aligned} 6x^2 + 9x - 8x - 12 &= 0 \\ \left(6x^2 + 9x\right) + (-8x - 12) &= 0 \\ 3x\,(2x+3) - 4\,(2x+3) &= 0 \\ (2x+3)\,(3x-4) &= 0 \end{aligned}$$

$$2x+3=0 \quad \text{or} \quad 3x-4=0$$
$$x=-\frac{3}{2} \quad \text{or} \quad x=\frac{4}{3}$$

So the solution set is $\left\{-\frac{3}{2}, \frac{4}{3}\right\}$.

d. Start by putting the equation in standard form.

$$\begin{aligned}(x-3)(x+2)&=14\\ x^2-x-6&=14\\ x^2-x-20&=0\\ (x-5)(x+4)&=0\end{aligned}$$

$$x-5=0 \quad \text{or} \quad x+4=0$$
$$x=5 \quad \text{or} \quad x=-4$$

So the solution set is $\{5,-4\}$.

e. Even though this equation has a power higher than 2, we can still find all of its solutions by following the algorithm. Start by factoring out the greatest common factor.

$$\begin{aligned}x^3-64x&=0\\ x\left(x^2-64\right)&=0\\ x(x-8)(x+8)&=0\end{aligned}$$

$$x=0 \quad \text{or} \quad x-8=0 \quad \text{or} \quad x+8=0$$
$$x=0 \quad \text{or} \quad x=8 \quad \text{or} \quad x=-8$$

So the solution set is $\{0,8,-8\}$.

8.7.2 Square Root Properties

In Section 8.2 we covered the definition of a square root, how to estimate and simplify square roots, multiplication and division properties of square roots, and rationalizing the denominator.

Example 8.7.2 Estimating Square Roots. Estimate the value of $\sqrt{28}$ without a calculator.

Explanation. To estimate $\sqrt{28}$, we can find the nearest perfect squares that are whole numbers on either side of 28. Recall that the perfect squares are $1,4,9,16,25,36,49,64,\ldots$ The perfect square that is just below 28 is 25 and the perfect square just above 28 is 36. This tells us that $\sqrt{28}$ is between $\sqrt{25}$ and $\sqrt{36}$, or between 5 and 6. We can also say that $\sqrt{28}$ is closer to 5 than 6 because 28 is closer to 25, so we think 5.2 or 5.3 would be a good estimate.

On the calculator we can see that $\sqrt{28} \approx 5.29$, so our guess was very close to reality.

Example 8.7.3 Multiplication and Division Properties of Square Roots. Simplify the expressions using the multiplication and division properties of square roots.

a $\sqrt{18} \cdot \sqrt{2}$.

b $\frac{\sqrt{18}}{\sqrt{2}}$.

Explanation.

a
$$\begin{aligned}\sqrt{18} \cdot \sqrt{2} &= \sqrt{18 \cdot 2}\\ &= \sqrt{36}\\ &= 6\end{aligned}$$

b
$$\begin{aligned}\frac{\sqrt{18}}{\sqrt{2}} &= \sqrt{\frac{18}{2}}\\ &= \sqrt{9}\\ &= 3\end{aligned}$$

Example 8.7.4 Simplifying Square Roots. Simplify the expression $\sqrt{54}$.

Explanation. Recall that the perfect squares are $1, 4, 9, 16, 25, 36, 49, 64, \ldots$ To simplify the $\sqrt{54}$, we need to look at that list and find the largest perfect square the goes into 54 evenly. In this case, it is 9. We then break up 54 into two factors 9 and 6, and we have:

$$\begin{aligned}\sqrt{54} &= \sqrt{9 \cdot 6}\\ &= \sqrt{9} \cdot \sqrt{6}\\ &= 3\sqrt{6}\end{aligned}$$

Since 6 has no perfect square factors, we can stop.

Example 8.7.5 Multiplying Square Root Expressions. Simplify the expression $\sqrt{50} \cdot \sqrt{27}$.

Explanation. Note that 25 is a perfect-square factor of 50, and that 9 is a perfect-square factor of 27. Now we have:

$$\begin{aligned}\sqrt{50} \cdot \sqrt{27} &= \sqrt{25 \cdot 2} \cdot \sqrt{9 \cdot 3}\\ &= \sqrt{25} \cdot \sqrt{2} \cdot \sqrt{9} \cdot \sqrt{3}\\ &= 5 \cdot \sqrt{2} \cdot 3 \cdot \sqrt{3}\\ &= 15\sqrt{6}\end{aligned}$$

Example 8.7.6 Adding and Subtracting Square Root Expressions. Simplify the expression $\sqrt{32} + \sqrt{50}$.

Explanation. Recall that radicals can only be added if the radicands match identically, so we cannot initially combine these two terms. However, if we simplify first, we may be able to add terms later. Note that 16 is a perfect-square factor of 32, and that 25 is a perfect-square factor of 50.

$$\begin{aligned}\sqrt{32} + \sqrt{50} &= \sqrt{16 \cdot 2} + \sqrt{25 \cdot 2}\\ &= \sqrt{16} \cdot \sqrt{2} + \sqrt{25} \cdot \sqrt{2}\\ &= 4\sqrt{2} + 5\sqrt{2}\\ &= 9\sqrt{2}\end{aligned}$$

Example 8.7.7 Rationalizing the Denominator. Rationalize the denominator in the expression $\frac{2}{\sqrt{6}}$.

Explanation.

$$\begin{aligned}\frac{2}{\sqrt{6}} &= \frac{2}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}}\\ &= \frac{2 \cdot \sqrt{6}}{\sqrt{6} \cdot \sqrt{6}}\\ &= \frac{2\sqrt{6}}{6}\\ &= \frac{\sqrt{6}}{3}\end{aligned}$$

Example 8.7.8 More Complicated Square Roots. Expand $\left(\sqrt{5} + \sqrt{3}\right)^2$.

Explanation. We will use the FOIL method to expand this expression:

$$\begin{aligned}\left(\sqrt{5} + \sqrt{3}\right)^2 &= \left(\sqrt{5} + \sqrt{3}\right)\left(\sqrt{5} + \sqrt{3}\right)\\ &= \left(\sqrt{5}\right)^2 + \sqrt{5}\sqrt{3} + \sqrt{3}\sqrt{5} + \left(\sqrt{3}\right)^2\\ &= 5 + \sqrt{15} + \sqrt{15} + 3\\ &= 8 + 2\sqrt{15}\end{aligned}$$

8.7.3 Solving Quadratic Equations by Using a Square Root

In Section 8.3 we covered how to solve quadratic equations using the square root property and how to use the Pythagorean Theorem.

Example 8.7.9 Solving Quadratic Equations Using the Square Root Property. Solve for w in $3(2-w)^2 - 24 = 0$.

Explanation. It's important here to suppress any urge you may have to expand the squared binomial. We begin by isolating the squared expression.

$$\begin{aligned}3(2-w)^2 - 24 &= 0\\ 3(2-w)^2 &= 24\\ (2-w)^2 &= 8\end{aligned}$$

Now that we have the squared expression isolated, we can use the square root property.

$$\begin{aligned}2 - w &= -\sqrt{8} &\quad\text{or}\quad&& 2 - w &= \sqrt{8}\\ 2 - w &= -\sqrt{4 \cdot 2} &\quad\text{or}\quad&& 2 - w &= \sqrt{4 \cdot 2}\\ 2 - w &= -\sqrt{4} \cdot \sqrt{2} &\quad\text{or}\quad&& 2 - w &= \sqrt{4} \cdot \sqrt{2}\end{aligned}$$

$$\begin{aligned} 2-w &= -2\sqrt{2} && \text{or} && 2-w = 2\sqrt{2}\\ -w &= -2\sqrt{2}-2 && \text{or} && -w = 2\sqrt{2}-2\\ w &= 2\sqrt{2}+2 && \text{or} && w = -2\sqrt{2}+2 \end{aligned}$$

The solution set is $\left\{2\sqrt{2}+2, -2\sqrt{2}+2\right\}$.

Example 8.7.10 The Pythagorean Theorem. Faven was doing some wood working in her garage. She needed to cut a triangular piece of wood for her project that had a hypotenuse of 16 inches, and the sides of the triangle should be equal in length. How long should she make her sides?

Explanation.

Let's start by representing the length of the triangle, measured in inches, by the letter x. That would also make the other side x inches long.

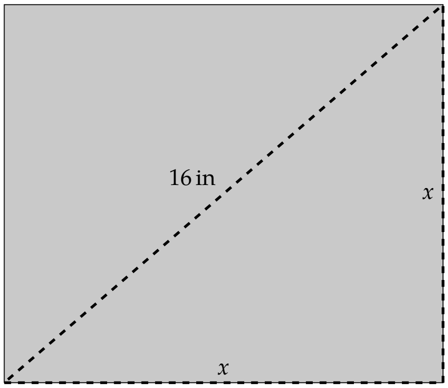


Figure 8.7.11: Piece of wood with labels for Faven

Faven should now set up the Pythagorean theorem regarding the picture. That would be

$$x^2 + x^2 = 16^2$$

Solving this equation, we have:

$$\begin{aligned} x^2 + x^2 &= 16^2\\ x^2 + x^2 &= 256\\ 2x^2 &= 256\\ x^2 &= 128\\ \sqrt{x^2} &= \sqrt{128}\\ x &= \sqrt{64 \cdot 2}\\ x &= \sqrt{64} \cdot \sqrt{2}\\ x &= 8\sqrt{2} \end{aligned}$$

$$x \approx 11.3$$

Faven should make the sides of her triangle about 11.3 inches long to force the hypotenuse to be 16 inches long.

8.7.4 The Quadratic Formula

In Section 8.4 we covered how to use the quadratic formula to solve any quadratic equation, as well as an algorithm to help solve linear and quadratic equations.

Example 8.7.12 Solving Quadratic Equations with the Quadratic Formula. Solve the equations using the quadratic formula.

a. $x^2 + 4x = 6$

b. $5x^2 - 2x + 1 = 0$

Explanation.

a. First we should change the equation into standard form.

$$\begin{aligned} x^2 + 4x &= 6 \\ x^2 + 4x - 6 &= 0 \end{aligned}$$

Next, we check and see that we cannot factor the left side or use the square root property so we must use the quadratic formula. We identify that $a = 1$, $b = 4$, and $c = -6$. We will substitute them into the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-6)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{16 + 24}}{2} \\ &= \frac{-4 \pm \sqrt{40}}{2} \\ &= \frac{-4 \pm \sqrt{4 \cdot 10}}{2} \\ &= \frac{-4 \pm \sqrt{4} \cdot \sqrt{10}}{2} \\ &= \frac{-4 \pm 2\sqrt{10}}{2} \\ &= -\frac{4}{2} \pm \frac{2\sqrt{10}}{2} \\ &= -2 \pm \sqrt{10} \end{aligned}$$

So the solution set is $\left\{-2 + \sqrt{10}, -2 - \sqrt{10}\right\}$.

b. Since the equation $5x^2 - 2x + 1 = 0$ is already in standard form, we check and see that we cannot factor the left side or use the square root property so we must use the quadratic formula. We identify that $a = 5$, $b = -2$, and $c = 1$. We will substitute them into the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(5)(1)}}{2(5)} \\ &= \frac{2 \pm \sqrt{4 - 20}}{10} \\ &= \frac{2 \pm \sqrt{-16}}{10} \end{aligned}$$

Since the solutions have square roots of negative numbers, we must conclude that there are no real solutions.

Example 8.7.13 Recognizing Linear and Quadratic Equations. Identify which equations are linear, which are quadratic, and which are neither.

a. $2(x-3)^2 - 5x = 6$

b. $2(x-3) - 5x = 6$

c. $2x - 6 = 7x^3$

d. $2x^2 - 6 = 7x^2$

e. $2\sqrt{x} - x - 6 = 0$

f. $2x - (x - 6) = 0$

Explanation.

a. $2(x-3)^2 - 5x = 6$ is quadratic.

b. $2(x-3) - 5x = 6$ is linear.

c. $2x - 6 = 7x^3$ is neither linear or quadratic.

d. $2x^2 - 6 = 7x^2$ is quadratic.

e. $2\sqrt{x} - x - 6 = 0$ is neither linear or quadratic.

f. $2x - (x - 6) = 0$ is linear.

Example 8.7.14 Solving Linear and Quadratic Equations. Use 8.4.11 to help solve the equations after deciding if they are linear or quadratic.

a. $4x^2 - 11x + 6 = 0$

b. $2(x-6)^2 - 16 = 0$

c. $2(x-6) - 16 = 0$

d. $3(x-4)^2 - 15x = 0$

Explanation.

a. To solve the equation $4x^2 - 11x + 6 = 0$ we first note that it is quadratic. Since there is a linear term ($-11x$), we must use either factoring or the quadratic formula, and we will try factoring first. Since the leading coefficient is 4, we will try the AC method. In this case, $ac = 24$: numbers that multiply to be 24 and add to be -11 are -8 and -3. So we split up th equation like this:

$$\begin{aligned} 4x^2 - 11x + 6 &= 0 \\ 4x^2 - 8x - 3x + 6 &= 0 \end{aligned}$$

$$\begin{aligned} \left(4x^2 - 8x\right) + (-3x + 6) &= 0 \\ 4x(x-2) - 3(x-2) &= 0 \\ (x-2)(4x-3) &= 0 \end{aligned}$$

$$\begin{array}{rcr} x - 2 = 0 & \text{or} & 4x - 3 = 0 \\ x = 2 & \text{or} & x = \dfrac{3}{4} \end{array}$$

So, the solution set is $\left\{2, \frac{3}{4}\right\}$.

b. To solve the equation $2(x-6)^2 - 16 = 0$ we first note that it is quadratic. Since there is no linear term, we should try using the square root method.

$$\begin{aligned} 2(x-6)^2 - 16 &= 0 \\ 2(x-6)^2 &= 16 \\ (x-6)^2 &= 8 \end{aligned}$$

$$\begin{array}{rcr} x - 6 = \sqrt{8} & \text{or} & x - 6 = -\sqrt{8} \\ x - 6 = \sqrt{4 \cdot 2} & \text{or} & x - 6 = -\sqrt{4 \cdot 2} \\ x - 6 = \sqrt{4} \cdot \sqrt{2} & \text{or} & x - 6 = -\sqrt{4} \cdot \sqrt{2} \\ x - 6 = 2\sqrt{2} & \text{or} & x - 6 = -2\sqrt{2} \\ x = 6 + 2\sqrt{2} & \text{or} & x = 6 - 2\sqrt{2} \end{array}$$

So, the solution set is $\left\{6 + 2\sqrt{2}, 6 - 2\sqrt{2}\right\}$.

c. To solve the equation $2(x-6) - 16 = 0$ we first we first note that it is linear. Since it is linear, we just need to isolate the terms with the variable on one side and all the other terms on the other side of the equals sign.

$$\begin{aligned} 2(x-6) - 16 &= 0 \\ 2x - 12 - 16 &= 0 \\ 2x - 28 &= 0 \\ 2x &= 28 \\ x &= 14 \end{aligned}$$

So, the solution set is $\{14\}$.

d. To solve the equation $2(x-6)^2 - 15x = 0$ we first note that it is quadratic. Since there is a linear term, we must use either factoring or the quadratic formula. Before we can decide which to use, we need to put the equation in standard form:

$$\begin{aligned} 3(x-4)^2 - 15x &= 0 \\ 3(x-4)(x-4) - 15x &= 0 \\ 3(x^2 - 8x + 16) - 15x &= 0 \\ 3x^2 - 24x + 48 - 15x &= 0 \end{aligned}$$

$$3x^2 - 39x + 48 = 0$$

Now we can see that the left hand side does not factor easily, so we will fall back on the quadratic formula. We identify that $a = 3$, $b = -39$, and $c = 48$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-(-39) \pm \sqrt{(-39)^2 - 4(3)(48)}}{2(3)} \\ &= \frac{39 \pm \sqrt{1521 - 576}}{6} \\ &= \frac{39 \pm \sqrt{945}}{6} \\ &= \frac{39 \pm \sqrt{9 \cdot 105}}{6} \\ &= \frac{39 \pm \sqrt{9} \cdot \sqrt{105}}{6} \\ &= \frac{39 \pm 3\sqrt{105}}{6} \end{aligned}$$

So the solution set is $\left\{\frac{39+3\sqrt{105}}{6}, \frac{39-3\sqrt{105}}{6}\right\}$.

8.7.5 Complex Solutions to Quadratic Equations

In Section 8.5 we covered what both imaginary numbers and complex numbers are, as well as how to solve quadratic equations where the solutions are imaginary numbers or complex numbers.

Example 8.7.15 Imaginary Numbers. Simplify the expression $\sqrt{-12}$ using the imaginary number, i.

Explanation. Start by splitting the -1 from the 12 and by looking for the largest perfect-square factor of -12, which happens to be 4.

$$\begin{aligned} \sqrt{-12} &= \sqrt{4 \cdot -1 \cdot 3} \\ &= \sqrt{4} \cdot \sqrt{-1} \cdot \sqrt{3} \\ &= 2i\sqrt{3} \end{aligned}$$

Example 8.7.16 Solving Quadratic Equations with Imaginary Solutions. Solve for m in $2m^2 + 16 = 0$, where p is an imaginary number.

Explanation. There is no m term so we will use the square root method.

$$\begin{aligned} 2m^2 + 16 &= 0 \\ 2m^2 &= -16 \\ m^2 &= -8 \end{aligned}$$

$$\begin{aligned} m &= -\sqrt{-8} & &\text{or} & m &= \sqrt{-8} \\ m &= -\sqrt{4} \cdot \sqrt{-1} \cdot \sqrt{2} & &\text{or} & m &= \sqrt{4} \cdot \sqrt{-1} \cdot \sqrt{2} \\ m &= -2i\sqrt{2} & &\text{or} & m &= 2i\sqrt{2} \end{aligned}$$

The solution set is $\left\{-2i\sqrt{2}, 2i\sqrt{2}\right\}$.

Example 8.7.17 Solving Quadratic Equations with Complex Solutions. Solve the equation $3(v-2)^2 + 54 = 0$, where v is a complex number.

Explanation.

$$\begin{aligned} 3(v-2)^2 + 54 &= 0 \\ 3(v-2)^2 &= -54 \\ (v-2)^2 &= -18 \end{aligned}$$

$$\begin{aligned} v-2 &= -\sqrt{-18} & &\text{or} & v-2 &= \sqrt{-18} \\ v-2 &= -\sqrt{9 \cdot -1 \cdot 2} & &\text{or} & v-2 &= \sqrt{9 \cdot -1 \cdot 2} \\ v-2 &= -\sqrt{9} \cdot \sqrt{-1} \cdot \sqrt{2} & &\text{or} & v-2 &= \sqrt{9} \cdot \sqrt{-1} \cdot \sqrt{2} \\ v-2 &= -3i\sqrt{2} & &\text{or} & v-2 &= 3i\sqrt{2} \\ v &= 2 - 3i\sqrt{2} & &\text{or} & v &= 2 + 3i\sqrt{2} \end{aligned}$$

So, the solution set is $\left\{2 + 3i\sqrt{2}, 2 - 3i\sqrt{2}\right\}$.

8.7.6 Strategies for Solving Quadratic Equations

In Section 8.6 we reviews all three methods for solving quadratic equations that we know. For the full explanation of solving using the factoring, visit Section 8.1, solving using the square root method, visit Section 8.3, and for more on the quadratic formula, visit Section 8.4.

Example 8.7.18 How to Choose a Method for Solving a Quadratic Equation. Solve the quadratic equations using an effective method.

a. $(x-4)^2 - 2 = 0$

b. $(x-4)^2 - 2x = 0$

c. $(x-4)^2 + 2x = 0$

Explanation. All three of the equations here are very similar, so we will need to examine them closely to choose the best method for solving them.

a. To solve the equation $(x-4)^2 - 2 = 0$, first note that there is no linear term: there is only a square and a constant. This leads us to consider the square root method. Before doing that, isolate the square:

$$(x-4)^2 - 2 = 0$$

$$(x-4)^2 = 2$$

Now we can apply the square root method to the equation.

$$\begin{aligned} x-4 &= \sqrt{2} & \text{or} \qquad x-4 &= -\sqrt{2} \\ x &= 4+\sqrt{2} & \text{or} \qquad x &= 4-\sqrt{2} \end{aligned}$$

So the solution set is $\left\{4+\sqrt{2}, 4-\sqrt{2}\right\}$

b. To solve the equation $(x-4)^2 - 2x = 0$, first note that there is a linear term $(-2x)$, so we must use either factoring or the quadratic formula. To use either, we must first put the equation in standard form.

$$\begin{aligned} (x-4)^2 - 2x &= 0 \\ (x-4)(x-4) - 2x &= 0 \\ x^2 - 8x + 16 - 2x &= 0 \\ x^2 - 10x + 16 &= 0 \end{aligned}$$

Now that the equation is in standard form, we can decide whether to use factoring or the quadratic formula. While the quadratic formula *always* works, it can take more time than factoring if factoring is possible. In this case, factoring entails answering the question "are there two integers that multiply to be 16 and add to be −10?" The answer is "yes": −8 and −2 are such numbers.

$$\begin{aligned} x^2 - 10x + 16 &= 0 \\ (x-8)(x-2) &= 0 \end{aligned}$$

$$\begin{aligned} x-8 &= 0 & \text{or} \qquad x-2 &= 0 \\ x &= 8 & \text{or} \qquad x &= 2 \end{aligned}$$

So the solution set is $\{2, 8\}$.

c. To solve the equation $(x-4)^2 + 2x = 0$, first note that there is a linear term $(+2x)$, so we must use either factoring or the quadratic formula. To use either, we must first put the equation in standard form.

$$\begin{aligned} (x-4)^2 + 2x &= 0 \\ (x-4)(x-4) + 2x &= 0 \\ x^2 - 8x + 16 + 2x &= 0 \\ x^2 - 6x + 16 &= 0 \end{aligned}$$

Now that the equation is in standard form, we can decide whether to use factoring or the quadratic formula. In this case, factoring entails answering the question "are there two integers that multiply to be 16 and add to be −6?" The answer is "no," so we must use the quadratic formula. First, identify that $a = 1$, $b = -6$, and $c = 16$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(16)}}{2(1)} \\ &= \frac{6 \pm \sqrt{36 - 48}}{2} \\ &= \frac{6 \pm \sqrt{-12}}{2} \end{aligned}$$

At this point, we notice that the solutions are complex. Continue to simplify until they are completely reduced.

$$\begin{aligned} x &= \frac{6 \pm \sqrt{4 \cdot -1 \cdot 3}}{2} \\ &= \frac{6 \pm \sqrt{4} \cdot \sqrt{-1} \cdot \sqrt{3}}{2} \\ &= \frac{6 \pm 2i\sqrt{3}}{2} \\ &= \frac{6}{2} \pm \frac{2i\sqrt{3}}{2} \\ &= 3 \pm i\sqrt{3} \end{aligned}$$

So the solution set is $\left\{3 + i\sqrt{3}, 3 - i\sqrt{3}\right\}$.

Exercises

Solving Quadratic Equations by Factoring Solve the equation.

1. $x^2 - 6x - 27 = 0$

2. $x^2 - 5x - 50 = 0$

3. $x^2 + 11x = -18$

4. $x^2 + 18x = -80$

5. $x^2 = 9x$

6. $x^2 = 7x$

7. $x^2 - 8x + 16 = 0$

8. $x^2 - 10x + 25 = 0$

9. $4x^2 = -41x - 10$

10. $4x^2 = -25x - 36$

11. $x(x + 20) = 5(2x - 5)$

12. $x(x + 36) = 9(2x - 9)$

13. A rectangle's base is 7 in shorter than four times its height. The rectangle's area is 65 in^2. Find this rectangle's dimensions.

The rectangle's height is ☐.

The rectangle's base is ☐.

14. A rectangle's base is 7 in shorter than four times its height. The rectangle's area is 2 in^2. Find this rectangle's dimensions.

The rectangle's height is ☐.

The rectangle's base is ☐.

Square Root Properties

15. Without using a calculator, estimate the value of $\sqrt{22}$:

(☐ 5.69 ☐ 4.69 ☐ 4.31 ☐ 5.31)

16. Without using a calculator, estimate the value of $\sqrt{38}$:

(☐ 6.16 ☐ 6.84 ☐ 5.84 ☐ 5.16)

Evaluate the following.

17. $\sqrt{\frac{25}{81}} = \boxed{}$.

18. $\sqrt{\frac{36}{49}} = \boxed{}$.

19. $\sqrt{-64} = \boxed{}$.

20. $\sqrt{-100} = \boxed{}$.

Simplify the radical expression or state that it is `not a real number`.

21. $\frac{\sqrt{6}}{\sqrt{216}} = \boxed{}$

22. $\frac{\sqrt{4}}{\sqrt{144}} = \boxed{}$

23. $\sqrt{8} = \boxed{}$

24. $\sqrt{147} = \boxed{}$

Simplify the expression.

25. $3\sqrt{5} \cdot 3\sqrt{40} = \boxed{}$

26. $3\sqrt{15} \cdot 2\sqrt{30} = \boxed{}$

27. $\sqrt{\frac{28}{13}} \cdot \sqrt{\frac{4}{13}} = \boxed{}$

28. $\sqrt{\frac{15}{19}} \cdot \sqrt{\frac{5}{19}} = \boxed{}$

29. $\sqrt{20} + \sqrt{45} = \boxed{}$

30. $\sqrt{175} + \sqrt{112} = \boxed{}$

31. $\sqrt{176} - \sqrt{44} = \boxed{}$

32. $\sqrt{8} - \sqrt{32} = \boxed{}$

Rationalize the denominator and simplify the expression.

33. $\frac{1}{\sqrt{3}} = \boxed{}$

34. $\frac{1}{\sqrt{3}} = \boxed{}$

35. $\frac{2}{5\sqrt{5}} = \boxed{}$

36. $\frac{7}{3\sqrt{6}} = \boxed{}$

37. $\frac{1}{\sqrt{75}} = \boxed{}$

38. $\frac{1}{\sqrt{175}} = \boxed{}$

Expand and simplify the expression.

39. $\left(7 - \sqrt{7}\right)\left(10 - 3\sqrt{7}\right) = \boxed{}$

40. $\left(5 - \sqrt{7}\right)\left(8 - 5\sqrt{7}\right) = \boxed{}$

41. $\left(\sqrt{5} + \sqrt{13}\right)\left(\sqrt{5} - \sqrt{13}\right) = \boxed{}$

42. $\left(\sqrt{5} + \sqrt{7}\right)\left(\sqrt{5} - \sqrt{7}\right) = \boxed{}$

Solving Quadratic Equations by Using a Square Root Solve the equation.

43. $x^2 = 20$

44. $x^2 = 99$

45. $144x^2 = 49$

46. $16x^2 = 9$

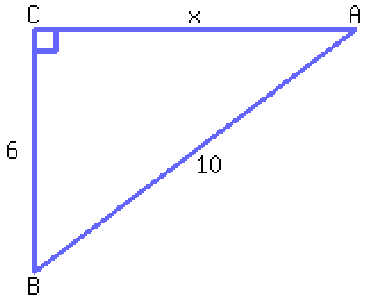
47. $(x+5)^2 = 121$

48. $(x+7)^2 = 49$

49. $2 - 2(t-5)^2 = -6$

50. $54 - 5(x-5)^2 = 9$

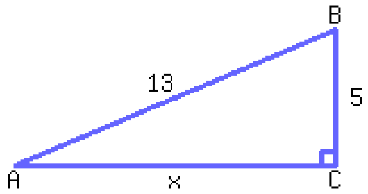
51.



Find the value of x.

$x =$

52.



Find the value of x.

$x =$

53. Parnell is designing a rectangular garden. The garden's diagonal must be 2 feet, and the ratio between the garden's base and height must be 4 : 3. Find the length of the garden's base and height.

The garden's base is

feet and its height is .

54. Gregory is designing a rectangular garden. The garden's diagonal must be 42.9 feet, and the ratio between the garden's base and height must be 12 : 5. Find the length of the garden's base and height.

The garden's base is

feet and its height is .

The Quadratic Formula Solve the equation.

55. $14x^2 - 11x - 9 = 0$

56. $18x^2 - 37x - 20 = 0$

57. $x^2 = -3x - 1$

58. $x^2 = 5x - 3$

59. $2x^2 + 4x + 5 = 0$

60. $2x^2 - x + 1 = 0$

61. $10t + 7 = t + 88$

62. $9b + 10 = b + 66$

63. $-1 - 3y^2 = -3$

64. $-8 - 7r^2 = -10$

65. $x^2 + 39x = 0$

66. $x^2 + 59x = 0$

67. $x^2 + 8x = 9$

68. $x^2 - 3x = 54$

69. $(x - 7)^2 = 16$

70. $(x - 5)^2 = 121$

71. $x^2 = -9x - 19$

72. $x^2 = -7x - 8$

73. An object is launched upward at the height of 320 meters. It's height can be modeled by

$$h = -4.9t^2 + 90t + 320,$$

where h stands for the object's height in meters, and t stands for time passed in seconds since its launch. The object's height will be 350 meters twice before it hits the ground. Find how many seconds since the launch would the object's height be 350 meters. Round your answers to two decimal places if needed.

The object's height would be 350 meters the first time at seconds, and then the second time at seconds.

74. An object is launched upward at the height of 340 meters. It's height can be modeled by

$$h = -4.9t^2 + 70t + 340,$$

where h stands for the object's height in meters, and t stands for time passed in seconds since its launch. The object's height will be 350 meters twice before it hits the ground. Find how many seconds since the launch would the object's height be 350 meters. Round your answers to two decimal places if needed.

The object's height would be 350 meters the first time at seconds, and then the second time at seconds.

Complex Solutions to Quadratic Equations Simplify the radical and write it into a complex number.

75. $\sqrt{-48} =$

76. $\sqrt{-98} =$

Solve the quadratic equation. Solutions could be complex numbers.

77. $-2x^2 - 4 = 8$

78. $4x^2 + 7 = -5$

79. $-9(y - 4)^2 + 8 = 584$

80. $7(y + 7)^2 + 8 = -167$

Strategies for Solving Quadratic Equations Solve the equation.

81. $2x^2 + 29 = 0$

82. $41x^2 + 37 = 0$

83. $5x^2 = -31x - 44$

84. $5x^2 = -27x - 36$

85. $x^2 + 4x + 1 = 0$

86. $x^2 + 10x + 7 = 0$

87. $28 - 6(x + 6)^2 = 4$

88. $34 - 4(y + 6)^2 = -2$

89. $x^2 + 7x = -12$

90. $x^2 + 14x = -45$

CHAPTER 9

Graphs of Quadratic Functions

9.1 Introduction to Functions

In mathematics, we use functions to model real-life data. In this section, we will learn the definition of a function and related concepts.

9.1.1 Introduction to Functions

When working with two variables, we are interested in the relationship between those two variables. For example, consider the two variables of hare population and lynx population in a Canadian forest. If we know the value of one variable, are we able to determine the value of the second variable? If we know that one variable is increasing over time, do we know if the other is increasing or decreasing?

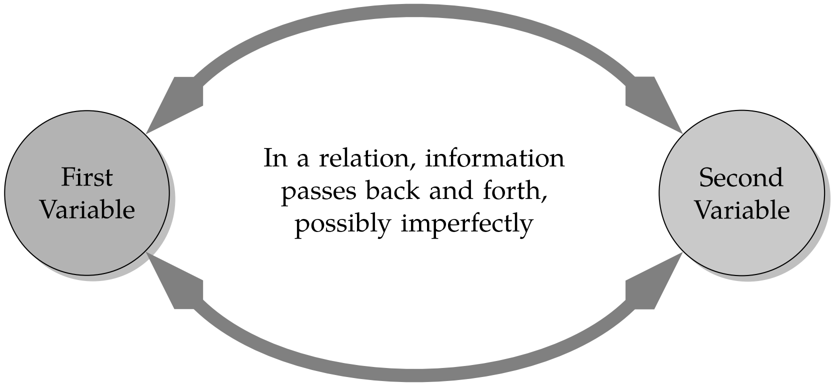

First Variable
In a relation, information passes back and forth, possibly imperfectly
Second Variable

Figure 9.1.2: In a relation, some knowledge of one variable implies some knowledge about the other

Definition 9.1.3 Relation. A **relation** is a very general situation between two variables, where having a little bit of information about one variable could tell you something about the other variable. For example, if you know the hare population is high this year, you can say the lynx population is probably increasing. So “hare population” and “lynx population” make a relation. If one of the variables is identified as the “first” variable, the relation’s **domain** is the set of all values that variable can take. Likewise, the relation’s **range** is the set of all values that the second variable can take.

We are not so much concerned with relations in this book. But we are interested in a special type of relation called a function. Informally, a **function** is a device that takes input values for one variable one by one, thinks about them, and gives respective output values one by one for the other variable.

Example 9.1.4 Mariana has 5 chickens: Hazel, Yvonne, Georgia, Isabella, and Emma. For the relation "Chicken to Egg Color," the first variable (the input) is a chicken's name and the second variable (the output) is the color of that chicken's eggs. The relation's domain is the set of all of Mariana's chicken's names, and its range is the set of colors of her chicken's eggs. Figure 9.1.5 shows two inputs and their corresponding outputs.

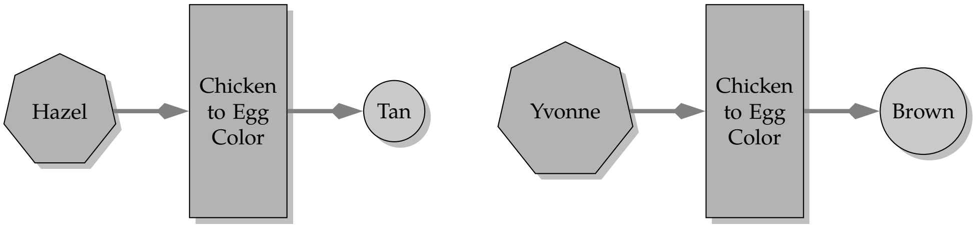


Figure 9.1.5: Two Pairs of Inputs and Outputs of the Relation "Chicken to Egg Color"

It would not be convenient to make diagrams like the ones in Figure 9.1.5 for all five chickens. There are too many inputs. Instead, Figure 9.1.6 represents the function graphically in a more concise way. The function's input variable is "chicken name," and its output variable is "egg color." Note that we are using the word "variable," because the chicken names and egg colors vary depending on which individual chicken you choose.

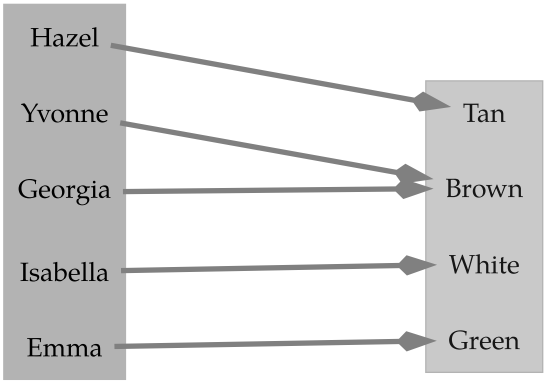


Figure 9.1.6: Diagram for the function "Chicken to Egg Color"

We can also use a set of ordered pairs to represent this function:

$$\{(\text{Hazel}, \text{Tan}), (\text{Yvonne}, \text{Brown}), (\text{Georgia}, \text{Brown}), (\text{Isabella}, \text{White}), (\text{Emma}, \text{Green})\}$$

where you read the ordered pair left to right, with the first value as an input and the second value as its output.

Definition 9.1.7 Function. In mathematics, a function is a relation between a set of inputs and a set of outputs with the property that each input is related to exactly one output.

In Figure 9.1.6, we can see each chicken's name (input) is related to exactly one output, so the relation "Chicken to Egg Color" qualifies as a function. Note that it is irrelevant that multiple inputs might related to the same output, like in (Yvonne, brown) and (Georgia, brown). The point is that whichever chicken you are thinking about, you know exactly which color egg it lays.

Example 9.1.8 Next, we will look at the "inverse" relation named "Egg Color to Chicken." Here we consider the color of an egg to be the input, and want the output to be the name of a chicken. If the egg is green, we know it is Emma's. But what if the egg is brown?

In Figure 9.1.9, we can see the color brown (an input) is related to *two* outputs, Yvonne and Georgia. This disqualifies the relation "Egg Color to Chicken" from being a function. (It is still a *relation*, because in general knowing the egg color does give you *some* information about which chicken it may have come from.)

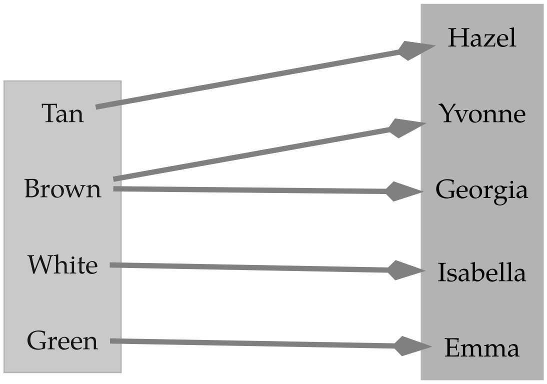


Figure 9.1.9: The relation "Egg Color to Chicken"

9.1.2 Functions as Predictors

Functions are useful because they describe our ability to accurately predict a result. If we can predict something precisely every time, then there is a function involved.

Example 9.1.10 If you go to the store and buy 5 two-dollar cans of soup, then you should predict that your total will be $10. No matter if you buy the soup in the morning, afternoon, or evening. If it doesn't total $10, then the cash register isn't *functioning*.

Example 9.1.11 A vending machine is like a function. You push a button and the item you desired pops out. In this case, the inputs are all of the buttons that you can press, and the outputs are the kinds of candy bars, chip bags, etc. that can come out. The mechanics and electronics that connect the buttons with the items represent the function.

Going in a little further, if the button `A1` delivers a bag of *M&M's*, then you would be surprised if you pressed `A1` and got anything other than *M&M's*. In this case, the machine wouldn't be functioning and you would get upset at your prediction ability being taken away.

What if buttons `A1` and `B3` both delivered *M&M's*? Would that violate the definition a function?

No, the vending machine is still a function even if two buttons generate the same output. Remember that to be a function, each *input* must generate a single output; that output doesn't have to be unique for each input. So as long as each button generates the same item each time you press it, there can be two buttons that deliver the same item.

Example 9.1.12 Some relations are not functions, because they can't be used to make 100% accurate predictions. For example, if you have a student's first name and want to determine their student ID number, you probably won't be able to look it up. For a common first name like "Michael," there will be many student ID number possibilities. Multiple outputs for a single input make "student ID number" not be a function of "first name."

On the other hand, if we exchange the variables and think of the ID number as the first variable, then there is only one student that ID number applies to, and there is only one official first name for that person. So "first name" *is* a function of "student ID number."

9.1.3 Algebraic Functions

Many functions have specific algebraic formulas to turn an input number into an output number. We explore some examples here.

Example 9.1.13 Aylen is an electrician who is hired to install a new circuit. She charges \$111 to come to your house and then, in addition, \$89 per hour to do the work. If x represents the total number of hours that the job takes and y represents the total cost of her work, in dollars, then the equation $y = 89x + 111$ relates the variables.

We know that at the end of the job, if Aylen worked x hours, you are going to have *one* bill that totals $89x + 111$. This must mean that the bill is a function of the number of hours of labor. For every possible number of hours that she could work, you would only get one bill for that cost.

Context aside, the expression $89x+111$ represents a function of x, because it can be used to turn an input number x into a specific output. In fact, every algebraic expression in one variable represents a function for similar reasons.

Example 9.1.14 The equation $x^2 + y^2 = 25$ represents a relation that is not a function, where we view x as the "first variable" as usual. Remember that to *not* be a function, all we have to do is find one input that has two outputs. Let's pick a nice easy input number to test: $x = 4$. Substituting this value gives us

$$\begin{aligned} x^2 + y^2 &= 25 \\ (4)^2 + y^2 &= 25 \\ 16 + y^2 &= 25 \\ y^2 &= 9 \end{aligned}$$

At this point, we see that y could be 3 *or* y could be -3. There are two y-values for the single x-value, so that must mean that $x^2 + y^2 = 25$ cannot represent a function.

Checkpoint 9.1.15. Identify which of the following represent functions and which do not.

a. The formula for the area of a circle is $A = \pi r^2$. With this equation, A (☐ is ☐ is not) a function of r.

b. A quadratic equation can be written as $y = ax^2 + bx + c$. With this equation, y (☐ is ☐ is not) a function of x.

c. With the equation $y^2 = x$, the variable y (☐ is ☐ is not) a function of x.

Explanation.

a. Since each circle with a given radius has only one area, this must be a function. Another way to look at it is that for any one r, the formula tells you exactly what A must be.

b. If you plug in any one x-value into $y = ax^2 + bx + c$, you will know exactly what the value of y is. So y is a function of x.

c. For example, if $x = 9$, then y could be 3 or -3. Since there are two y-values for the single x-value, that must mean that the equation $y^2 = x$ cannot represent y as a function of x.

9.1.4 Function Notation

We know that the equation $y = 5x + 3$ represents y as a function of x, because for each x-value (input), there is only one y-value (output). If we want to determine the value of the output when the input is 2, we'd replace x with 2 and find the value of y:

$$\begin{aligned} y &= 5(2) + 3 \\ &= 10 + 3 \\ &= 13 \end{aligned}$$

Our end result is that $y = 13$. Well, y *is* 13, but only in the situation when x is 2. In general, for other inputs, y is not going to be 13. So the equation $y = 13$ is lacking in the sense that it is not communicating everything we might want to say. It does not communicate the value of x that we used. **Function notation** will allow us to communicate *both* the input *and* the output at the same time. It will also allow us to give each function a name, which is helpful when we have multiple functions.

Functions can have names just like variables. The most common function name is f, since "f" stands for "function." A letter like f doesn't stand for a single number though. Instead, it represents an input-output relation like we've been discussing in this section.

We will write equations like $y = f(x)$, and what we mean is:

- "y equals f *of* x"
- the function's name is f
- the input variable is x
- the parentheses following the f surround the input; they do *not* indicate multiplication
- the output variable is y

Remark 9.1.16. Parentheses have a lot of uses in mathematics. Their use with functions is very specific, and it's important to note that f is *not* being multiplied by anything when we write $f(x)$. With function notation, the parentheses specifically are just meant to indicate the input by surrounding the input.

> **Example 9.1.17** The expression $f(x)$ is read as "f of x," and the expression $f(2)$ is read as "f of 2." Be sure to practice saying this correctly while reading.
>
> The expression $f(2)$ means that 2 is being treated as an input, and the function f is turning it into an output. And then $f(2)$ represents that actual output number.

Remark 9.1.18. The most common letters used to represent functions are f, g, and h. The most common variables we use are x, y, and z. But we can use any function name and any input and output variable. When dealing with functions in context, it often makes sense to use meaningful function names and variables. For example, if we are modeling temperature of a cup of coffee as a function of time with a function C, we could use $T = C(t)$, where T is the temperature (in degrees Fahrenheit) after t minutes.

9.1.5 Evaluating Functions

When we determine a function's value for a specific input, this is known as *evaluating a function*. To do so, we replace the input with the numerical value given and determine the associated output.

When using function notation, instead of writing $5x + 3$ or $y = 5x + 3$, we often write something like $f(x) = 5x + 3$. We are saying that the rule for function f is to use the expression $5x + 3$. To find $f(2)$, wherever you see x in the formula $f(x) = 5x + 3$, substitute in 2:

$$\begin{aligned} f(x) &= 5x + 3 \\ f(2) &= 5(2) + 3 \\ &= 10 + 3 \\ &= 13 \end{aligned}$$

Our end result is that $f(2) = 13$, which tells us that f turns 2 into 13. In other words, when the input is 2, the output will be 13.

Checkpoint 9.1.19 Functions with Algebraic Formulas. Find the given function values for a function g where $g(x) = 5x^2 - 3x + 4$.

a. $g(3) =$ ☐

b. $g(0) =$ ☐

c. $g(-2) =$ ☐

Explanation.

a. We will substitute $x = 3$ into $g(x)$:

$$\begin{aligned} g(3) &= 5(3)^2 - 3(3) + 4 \\ &= 5(9) - 9 + 4 \\ &= 40 \end{aligned}$$

b. We will substitute $x = 0$ into $g(x)$:

$$\begin{aligned} g(0) &= 5(0)^2 - 3(0) + 4 \\ &= 5(0) + 0 + 4 \\ &= 4 \end{aligned}$$

c. We will substitute $x = -2$ into $g(x)$. Especially when inputting negative numbers, be *certain* to put parentheses around the input values in the first step:

$$\begin{aligned} g(-2) &= 5(-2)^2 - 3(-2) + 4 \\ &= 5(4) + 6 + 4 \\ &= 30 \end{aligned}$$

A function may be described by explicitly listing many inputs and their corresponding outputs in a table.

> **Example 9.1.20 Functions given in Table Form.** Temperature readings for Portland, OR, on a given day are recorded in Table 9.1.21 . Let $f(x)$ be the temperature in degrees Fahrenheit x hours after midnight.
>
> | x, hours after midnight | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
> |---|---|---|---|---|---|---|---|---|---|---|---|
> | $f(x)$, temperature in °F | 45 | 44 | 42 | 42 | 43 | 44 | 45 | 48 | 49 | 50 | 53 |
>
> **Table 9.1.21:** Recorded Temperatures in Portland, OR, on a certain day

a. What was the temperature at midnight?

b. Find $f(9)$. Explain what this function value represents in the context of the problem.

Explanation.

a. To determine the temperature at midnight, we look in the table where $x = 0$ and see that the output is 45. Using function notation, we would write:

$$f(0) = 45.$$

Thus at midnight the temperature was 45 °F.

b. To determine the value of $f(9)$, we look in the table where $x = 9$ and read the output:

$$f(9) = 50.$$

In context, this means that at 9AM the temperature was 50 °F.

A function may be described using a graph, where the horizontal axis corresponds to possible input values (the domain), and the vertical axis corresponds to possible output values (the range).

Example 9.1.22 Functions in Graphical Form. A colony of bees settled in Zahid's backyard in 2012. Let $B(x)$ be the number of bees in the colony (in thousands) x months after April 1, 2012. A graph of $y = B(x)$ is shown in Figure 9.1.23.

a. Determine the number of bees in the colony on July 1, 2012.

b. Find $B(0)$. Explain what this function value represents in the context of the problem.

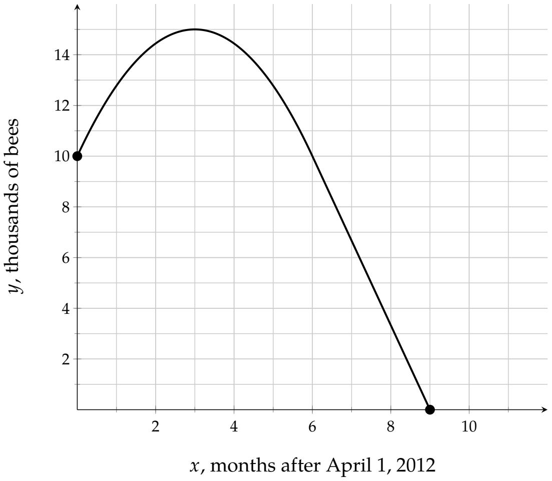


Figure 9.1.23: Bee Population

Explanation.

a. Since July 1 is 3 months after April 1, that means we need to use $x = 3$ as the input. On the horizontal axis, find 3, as in Figure 9.1.24. Looking straight up or down, we find the point $(3, 15)$ on the curve. That means that $B(3) = 15$. A value of $y = 15$ when $x = 3$ tells us that there were 15,000 bees 3 months after April 1, 2012 (on July 1, 2012).

b. To find $B(0)$, we recognize that this will be the output of the function when $x = 0$. The point $(0, 10)$

on the graph of $y = B(x)$ tells us that $B(0) = 10$. In the context of this problem, this means on April 1, 2012 there were 10,000 bees in the colony.

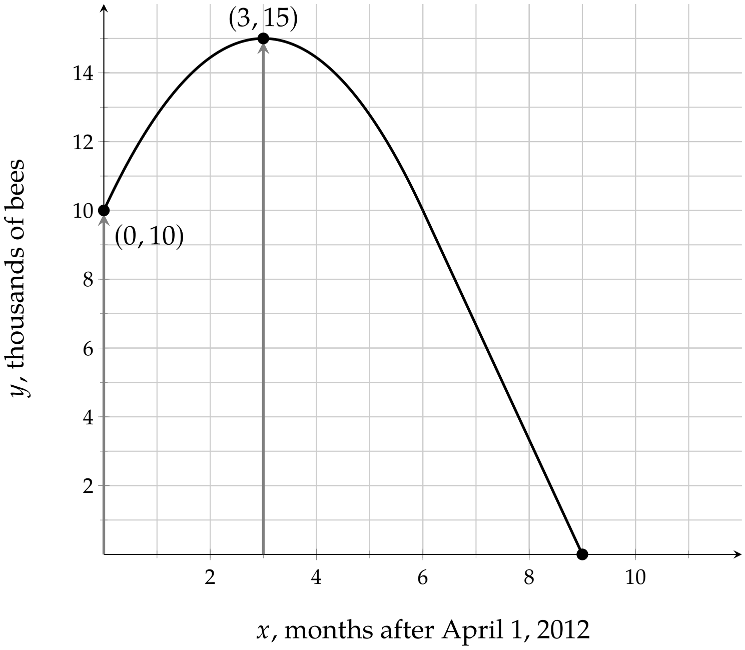


Figure 9.1.24: Bee Population

9.1.6 Solving Equations That Have Function Notation

Evaluating a function and *solving* an equation that has function notation in it are two separate things. Students understandably mix up these two tasks, because they are two sides of the same coin. Evaluating a function means that you know an input (typically, an x-value) and then you calculate an output (typically, a y-value). Solving an equation that has function notation in it is the opposite process. You know an output (typically, a y-value) and then you determine all the inputs that could have led to that output (typically, x-values).

Checkpoint 9.1.25 Functions with Algebraic Formulas. In Checkpoint 9.1.19, we found the function value when given the input, which we refer to as *evaluating* a function. To **solve** an equation that has function notation in it, we need to solve for the value of the variable that makes an equation true, just like in any other equation.

Solve the equations below for a function h where $h(x) = -4x + 7$. Check each answer and state the solution set.

a. $h(x) = -1$

The solution set is ☐.

b. $h(x) = 0$

The solution set is ☐.

c. $h(x) = 11$

The solution set is ☐.

Explanation.

a. To solve for x, we will substitute $h(x)$ with its formula, $-4x + 7$:

$$\begin{aligned} h(x) &= -1 \\ -4x + 7 &= -1 \\ -4x &= -8 \\ x &= 2 \end{aligned}$$

The solution is 2, and the solution set is $\{2\}$.

b. To solve for x, we will substitute $h(x)$ with its formula:

$$\begin{aligned} h(x) &= 0 \\ -4x + 7 &= 0 \\ -4x &= -7 \\ x &= \frac{7}{4} \end{aligned}$$

The solution is $\frac{7}{4}$, and the solution set is $\left\{\frac{7}{4}\right\}$.

c. To solve for x, we will substitute $h(x)$ with its formula:

$$\begin{aligned} h(x) &= 11 \\ -4x + 7 &= 11 \\ -4x &= 4 \\ x &= -1 \end{aligned}$$

The solution is -1, and the solution set is $\{-1\}$.

Example 9.1.26 Functions given in Table Form. In Example 9.1.20, we evaluated a function given in table form, using Table 9.1.21. Let's use that function to solve equations.

a. Solve $f(x) = 48$. Explain what this solution set represents in context.

b. When was the temperature 44 °F?

Explanation.

a. To solve $f(x) = 48$, we need to find the value of x that makes the equation true. Looking at the table, we look at the outputs and see that the output 48 occurs when $x = 7$. In abstract terms, the solution set is $\{7\}$. In context, this means that the temperature was 48 °F at 7AM.

b. To determine when the temperature was 44 °F, we look in the table to see where the output was 44 °F. This occurs when $x = 1$ and again when $x = 5$. In abstract terms, the solution set is $\{1, 5\}$. In context, this means that the temperature was 44 °F at 1AM and again at 5AM.

Example 9.1.27 Functions given in Graphical Form. In Example 9.1.22, we evaluated a function given in graphical form, using Figure 9.1.23. Now let's use that function to solve some equations.

a. Solve $B(x) = 0$. Explain what the solution set represents in the context of the problem.

b. When did the population reach 13,000 bees?

Explanation.

a. To solve $B(x) = 0$, we need to consider that number 0 as an output value, so it belongs on the vertical axis in Figure 9.1.28. Moving straight right or left, we find the point $(9, 0)$ is on the graph, and that tells us that $B(x) = 0$ when $x = 9$. Abstractly, the solution set is $\{9\}$. In the context of this problem, this means there were 0 bees in the colony 9 months after April 1, 2012 (on January 1, 2013).

b. To determine when the number of bees reached 13,000, we need to recognize that 13 is an output value and locate it on the vertical axis. Moving straight right or left, we find (approximately) that the points $(1, 13)$ and $(5, 13)$ are on the graph. This means $x \approx 1$ and $x \approx 5$ are solutions. Abstractly, the solution set is approximately $\{1, 5\}$. In context, there will be $13,000$ bees in the colony approximately 1 month after April 1, 2012 (on May 1, 2012) and again 5 months after April 1, 2012 (on September 1, 2012).

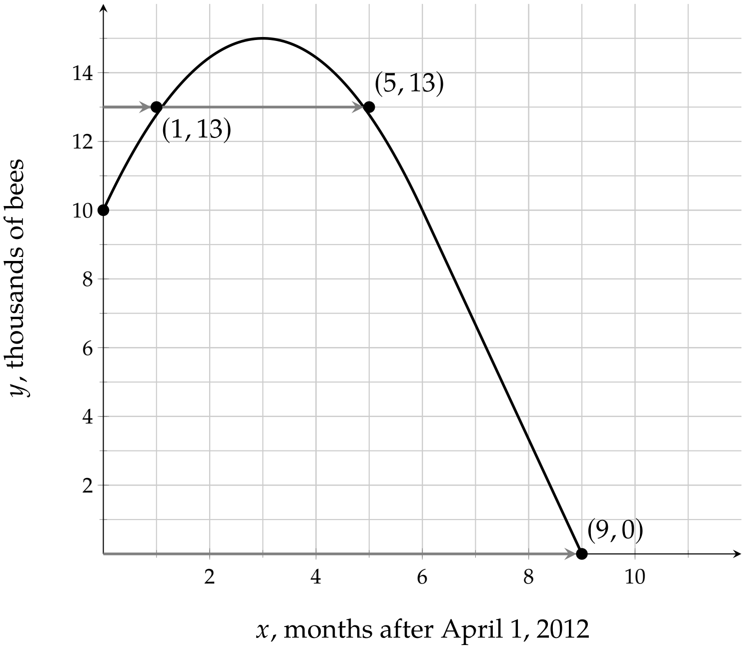


Figure 9.1.28: Bee Population

9.1.7 Domain and Range

Earlier we defined the domain and range of a relation. We repeat those definitions more formally here, specifically for functions.

Definition 9.1.29 Domain and Range. Given a function f, the **domain** of f is the collection of all valid input values for f. The **range** of f is the collection of all possible output values of f.

When working with functions, a common necessary task is to determine the function's domain and range. Also, the ability to identify domain and range is strong evidence that a person really understands the *concepts* of domain and range.

Example 9.1.30 Functions Defined by Ordered Pairs. The function f is defined by the ordered pairs

$$\{(1,2),(3,-2),(5,2),(7,-4),(9,6)\}.$$

Determine the domain and range of f.

Explanation. The ordered pairs tell us that $f(1) = 2$, $f(3) = -2$, etc. So the valid input values are 1, 3, 5, 7, and 9. This means the domain is the set $\{1,3,5,7,9\}$.

Similarly, the ordered pairs tell us that 2, −2, −4, and 6 are possible output values. Notice that the output 2 happened twice, but it only needs to be listed in this collection once. The range of f is $\{2,-2,-4,6\}$.

Example 9.1.31 Functions in Table Form. For each function defined using a table, state the domain and range.

a. The function g is defined by:

x	−2	−1	0	1	2
y	5	5	5	5	5

b. The function h is defined by:

x	0	1	2	3	4
y	8	6	4	2	0

Explanation.

a. The table tells us that $g(-2) = 5$, $g(-1) = 5$, etc. So the valid input values are −2, −1, 0, 1, and 2. This means the domain of g is the set $\{-2,-1,0,1,2\}$.

   The only output evident from this table is 5, so the range of g is the set $\{5\}$.

b. The table tells us that $h(0) = 8$, $h(1) = 6$, etc. So the valid input values are 0, 1, 2, 3, and 4. This means the domain of h is the set $\{0,1,2,3,4\}$.

   Similarly, the table shows us that the possible outputs are 8, 6, 4, 2, and 0. So the range of h is the set $\{8,6,4,2,0\}$.

Example 9.1.32 Functions in Graphical Form. Functions are graphed in Figure 9.1.33 through Figure 9.1.35. For each one, find its domain and range. In previous examples the domain and range were finite sets and we used set notation braces to communicate them. In these examples, the domain and range will be intervals, and so we use interval notation.

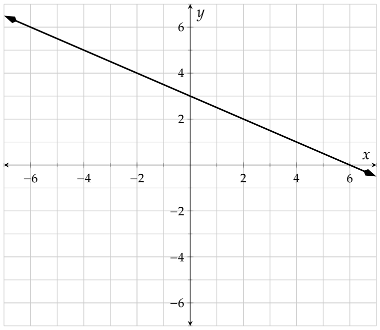


Figure 9.1.33: Function k

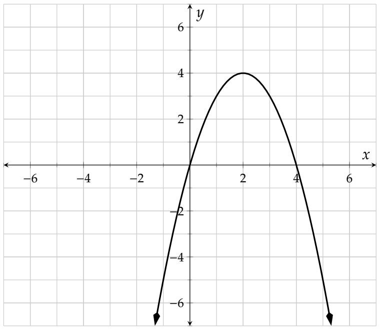


Figure 9.1.34: Function ℓ

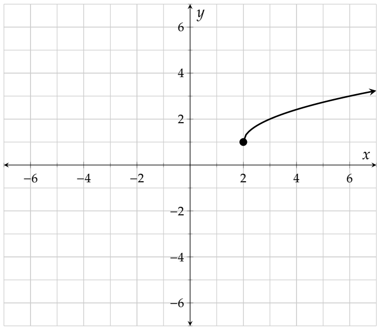


Figure 9.1.35: Function m

Explanation.

To find the domain of k, we look left and right across the x-axis. No matter where we look on the x-axis, we can look straight up or down and find a point on the graph. So no matter what input we imagine, there is always a corresponding output. Therefore the domain is all real numbers, which we write as $(-\infty, \infty)$.

Similarly, to find the range of k, we look up and down over the entire y-axis. No matter where we stop on the y-axis, we will be able to move straight left or right and find a point on the graph. So no matter what output we imagine, there is always an input that leads to that output. Therefore the range is also $(-\infty, \infty)$.

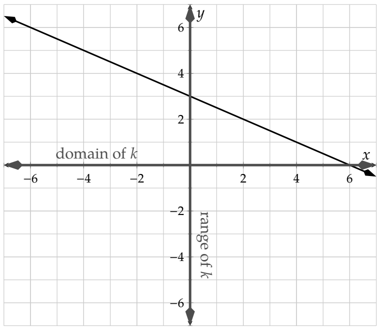


Figure 9.1.36: Function k

To find the domain of ℓ, we look left and right across the entire x-axis. No matter where we stop on the x-axis, we will be able to move straight up or down and find a point on the graph. (Although if we are far out left or right, we will have to look *very* far down to find that point.) So no matter what input we imagine, there is always an output to this function. Therefore the domain is $(-\infty, \infty)$.

To find the range of ℓ, we look up and down over the entire y-axis. For y-values larger than 4, if we look straight left or right, there is no point on the curve. Only with y-values 4 and under can we find an input that leads to such an output. So the range is all real numbers less than or equal to 4. In interval notation, that is $(-\infty, 4]$.

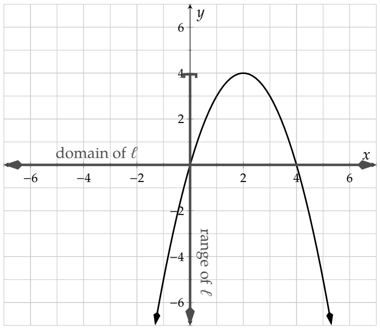


Figure 9.1.37: Function ℓ

To find the domain of m, we look left and right across the entire x-axis. For an x-value less than 2, there is no point on the curve directly above or below that x-value. So the graph is not telling you what $m(x)$ would be. Only for x-values 2 and greater do we get an output value. So the domain is $[2, \infty)$.

To find the range of m, we look up and down over the entire y-axis. For y-values less than 1, if we look straight left or right, there is no point on the curve. Only with y-values 1 and greater can we find an input that leads to such an output. So the range is all real numbers greater than or equal to 1. In interval notation, that is $[1, \infty)$.

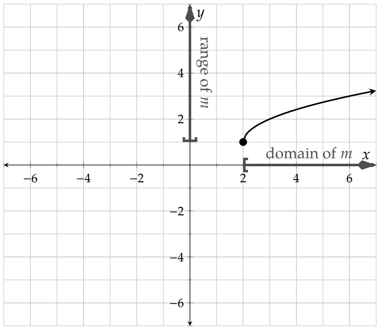


Figure 9.1.38: Function m

9.1.8 Determining if a Relation is a Function

We have seen functions that are defined using a verbal description, using an equation, using a list of ordered pairs, using a table, and using a graph. With all of these things, you might have a relation that is *not* actually a function. (We have seen a few examples of these as well.) Determining whether or not a given relation actually defines a function is another task that demonstrates actual understanding of the concept of a function.

Example 9.1.39 Verbally Described Relations. In the introduction, we discussed the relation between the variables of hare population and lynx population. These variables are related—for one thing, if the hare population is high, you will have information about the lynx population: it will probably be increasing because there is plenty of food.

But is this relation a function? Suppose that one year you know the hare population is 100,000 and the lynx population is 2000. Isn't it possible that some other year, the hare population is 100,000 again, but the lynx population is something different like 3000? Knowing one value of the first variable does not guarantee exactly one value of the second variable. So this relation is not a function.

Checkpoint 9.1.40 Sets of Ordered Pairs. The relations below are given in the form of ordered pairs. Determine if each is a function.

a. $\{(2,5),(3,6),(4,3),(4,-5),(8,0)\}$

(☐ yes ☐ no)

b. $\{(1,2),(3,-2),(5,2),(7,-4),(9,6)\}$

(☐ yes ☐ no)

Explanation.

a. The first relation *does not* represent a function because the input of 4 is associated with more than one output (3 and −5).

b. The second relation *does* represent a function as each input is associated with exactly one output. Note that it does not matter that both the inputs 1 and 5 lead to the same output. All that matters is that the input 1 has only one output. And the input 5 has only one output. And the same for each of the other inputs.

To determine if a relation that is represented graphically is a function, we need to visually determine if each input corresponds to exactly one output. How could that *not* happen? Somewhere on the horizontal axis, there would be a number, and looking straight up or straight down from there, you would find at least two points on the graph.

This thinking gives rise to the **vertical line test**.

Fact 9.1.41 Vertical Line Test. *If any vertical line passes through the graph of a relation more than once, then y is not a function of x in the graph.*

Example 9.1.42 Consider the relations given by Figure 9.1.43 and Figure 9.1.44.

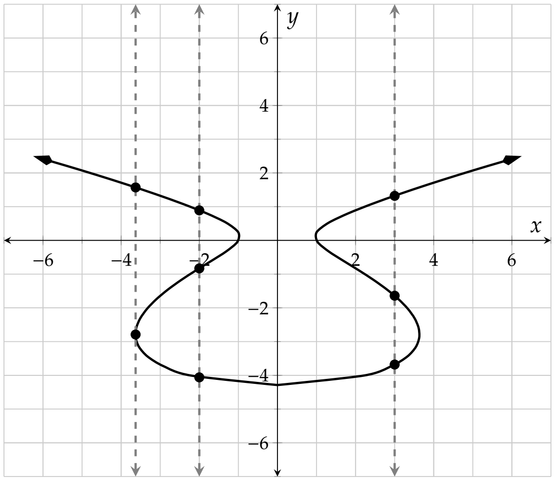


Figure 9.1.43: Fails the Vertical Line Test

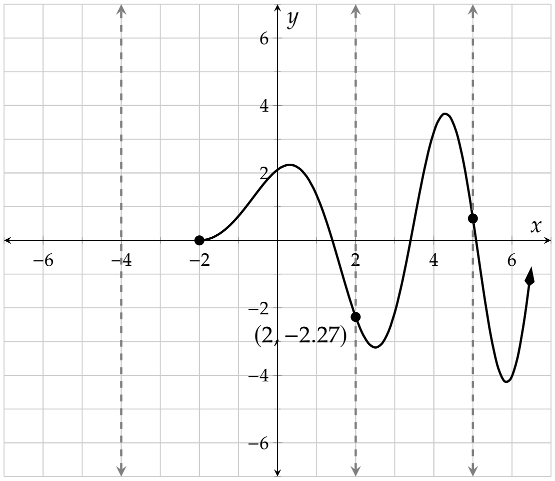


Figure 9.1.44: Passes the Vertical Line Test

In Figure 9.1.43, we can see that there are vertical lines that pass through the graph more than once. This means that this graph does *not* represent a function. The issue is that supposing it *were* a function f, then what would $f(3)$ be? The graph suggests it could be three different things: ≈ 1.3, ≈ -1.6, and ≈ -3.7. With no clear single output for the input 3, we don't have a function.

However, in Figure 9.1.44, we see that all vertical lines pass through the graph one time (or not at all). Therefore, this graph does represent a function. If we name the function g, the domain of g is only numbers greater than or equal to 2. And for such numbers, the graphs shows us exactly one output for each input. For example, the graph shows us that $g(2) \approx -2.27$.

For relations in table form, we can determine if it makes a function by again checking to see if multiple outputs are ever associated with a single input.

Example 9.1.45 Functions in Table Form. For each relation shown in Table 9.1.46 through Table 9.1.48, determine if y is a function of x.

This relation represents a function as each input corresponds to exactly one output. For instance, the input -2 only corresponds to the output 5 and no other output. (Note that it does not matter that multiple inputs correspond to the same output 5.)

x	-2	-1	0	1	2
y	5	5	5	5	5

Table 9.1.46: A function?

This relation represents a function as each input corresponds to exactly one output. For instance, the input 0 only corresponds to the output 8 and no other output.

x	0	1	2	3	4
y	8	6	4	2	0

Table 9.1.47

This relation does not represent a function as some of the inputs correspond to more than one output. In particular, the input of 2 corresponds to both 7 and 9 for outputs.

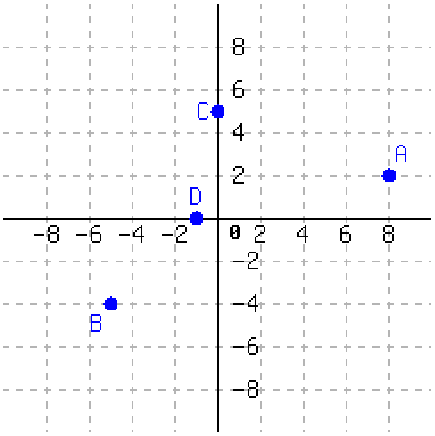
x	-2	-2	0	2	2
y	1	3	5	7	9

Table 9.1.48

Exercises

Review and Warmup

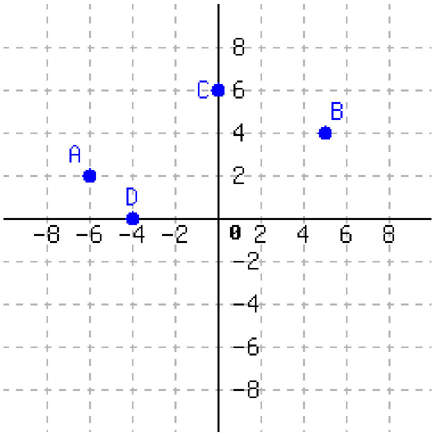
1.



Write each point's position as an ordered pair, like $(1, 2)$.

$A =$ ______ $B =$ ______
$C =$ ______ $D =$ ______

2.



Write each point's position as an ordered pair, like $(1, 2)$.

$A =$ ______ $B =$ ______
$C =$ ______ $D =$ ______

3. Sketch the points $(8, 2)$, $(5, 5)$, $(-3, 0)$, and $(2, -6)$ on a Cartesian plane.

4. Sketch the points $(1, -4)$, $(-3, 5)$, $(0, 4)$, and $(-2, -6)$ on a Cartesian plane.

Determining Whether a Relation Is a Function of Not

5. Do these sets of ordered pairs make functions of x? What are their domains and ranges?

a. $\left\{(9,8),(-10,4)\right\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ☐ and range ☐.

b. $\left\{(1,4),(0,6),(0,10)\right\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ☐ and range ☐.

c. $\left\{(3,0),(7,1),(-5,9),(-9,10)\right\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ☐ and range ☐.

d. $\left\{(5,8),(-3,8),(3,3),(9,9),(8,5)\right\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ☐ and range ☐.

6. Do these sets of ordered pairs make functions of x? What are their domains and ranges?

a. $\left\{(-3,1),(-10,6)\right\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ☐ and range ☐.

b. $\left\{(3,8),(-9,9),(2,5)\right\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ☐ and range ☐.

c. $\left\{(-2,7),(4,3),(-7,5),(-6,9)\right\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ☐ and range ☐.

d. $\left\{(-3,2),(-5,2),(-5,9),(10,4),(5,8)\right\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ☐ and range ☐.

7. Does the following set of ordered pairs make for a function of x?

$$\left\{(-8,8),(5,9),(-3,8),(7,10),(3,4)\right\}$$

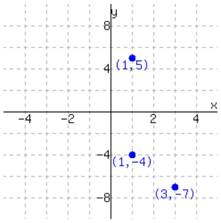
This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain and range .

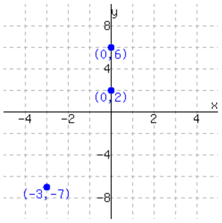
8. Does the following set of ordered pairs make for a function of x?

$$\left\{(8,3),(-6,5),(-1,6),(4,6),(1,10)\right\}$$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain and range .

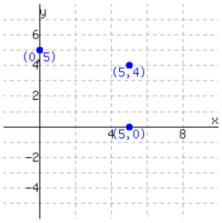
9. Decide whether each graph shows a relationship where y is a function of x.

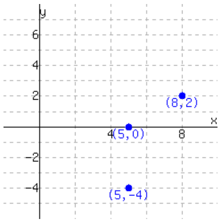




The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

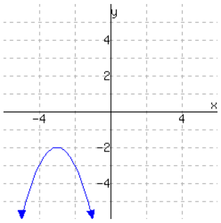
10. Decide whether each graph shows a relationship where y is a function of x.

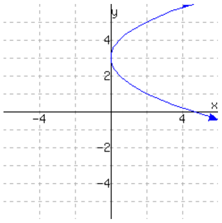




The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

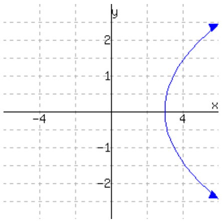
11. The following graphs show two relationships. Decide whether each graph shows a relationship where y is a function of x.

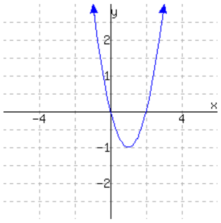




The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

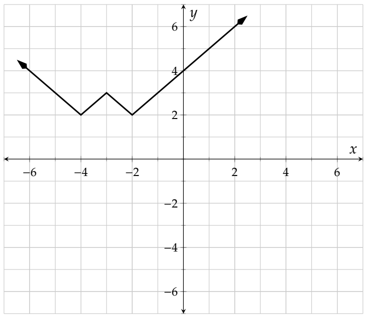
12. The following graphs show two relationships. Decide whether each graph shows a relationship where y is a function of x.

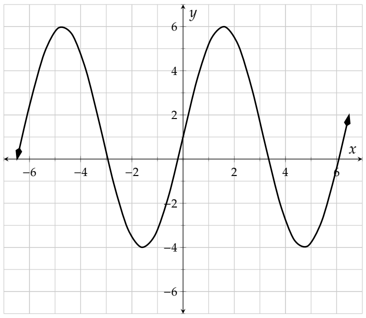


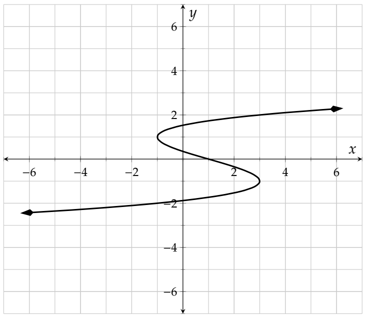


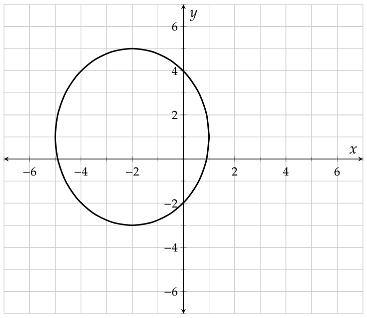
The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

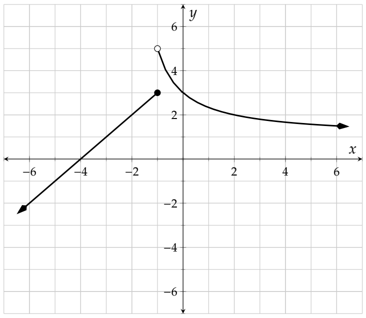
13. Which of the following graphs show y as a function of x?

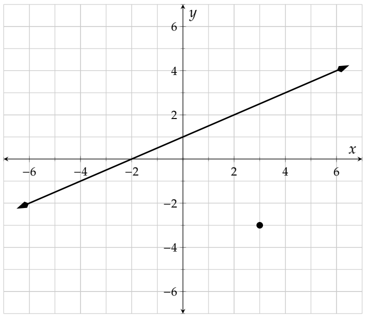




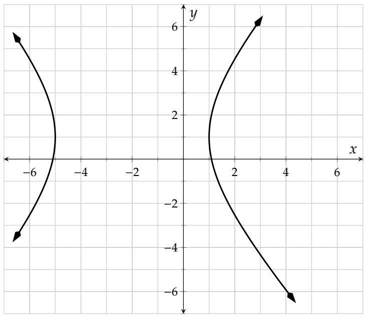


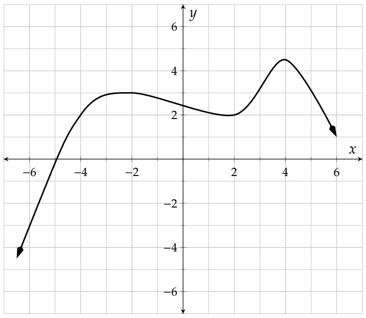


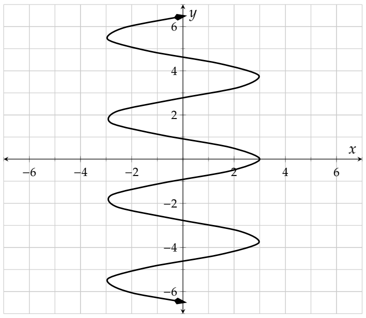


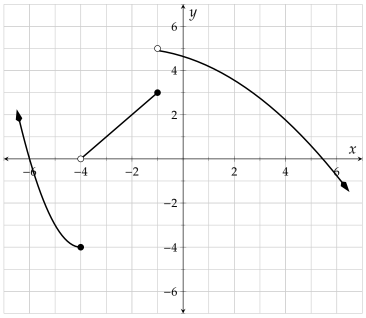


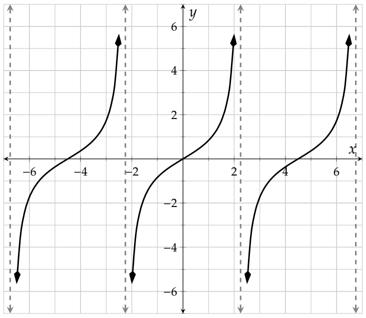
14. Which of the following graphs show y as a function of x?

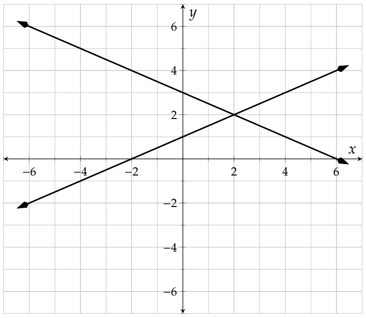












Evaluating Functions Algebraically Evaluate the function at the given values.

15. $G(x) = x - 7$

a. $G(4) = \boxed{}$

b. $G(-1) = \boxed{}$

c. $G(0) = \boxed{}$

16. $h(x) = x - 5$

a. $h(2) = \boxed{}$

b. $h(-4) = \boxed{}$

c. $h(0) = \boxed{}$

17. $h(x) = 10x$

a. $h(4) = \boxed{}$

b. $h(-5) = \boxed{}$

c. $h(0) = \boxed{}$

18. $F(x) = 7x$

a. $F(1) = \boxed{}$

b. $F(-1) = \boxed{}$

c. $F(0) = \boxed{}$

19. $G(x) = -4x + 6$

a. $G(3) = \boxed{}$

b. $G(-3) = \boxed{}$

c. $G(0) = \boxed{}$

20. $G(x) = -2x + 1$

a. $G(2) = \boxed{}$

b. $G(-3) = \boxed{}$

c. $G(0) = \boxed{}$

21. $H(x) = -x + 6$

a. $H(3) = \boxed{}$

b. $H(-5) = \boxed{}$

c. $H(0) = \boxed{}$

22. $K(x) = -x + 3$

a. $K(5) = \boxed{}$

b. $K(-2) = \boxed{}$

c. $K(0) = \boxed{}$

23. $f(t) = t^2 - 5$

a. $f(3) = \boxed{}$

b. $f(-3) = \boxed{}$

c. $f(0) = \boxed{}$

24. $f(r) = r^2 + 4$

a. $f(1) = \boxed{}$

b. $f(-3) = \boxed{}$

c. $f(0) = \boxed{}$

25. $g(x) = -x^2 - 9$

a. $g(5) = \boxed{}$

b. $g(-3) = \boxed{}$

c. $g(0) = \boxed{}$

26. $h(t) = -t^2 + 3$

a. $h(3) = \boxed{}$

b. $h(-2) = \boxed{}$

c. $h(0) = \boxed{}$

27. $F(r) = 9$

a. $F(2) = \boxed{}$

b. $F(9) = \boxed{}$

c. $F(0) = \boxed{}$

28. $G(x) = -3$

a. $G(1) = \boxed{}$

b. $G(-3) = \boxed{}$

c. $G(0) = \boxed{}$

29. $G(x) = \dfrac{7x}{5x + 6}$

a. $G(3) = \boxed{}$.

b. $G(-7) = \boxed{}$.

30. $H(x) = \dfrac{5x}{-7x+1}$

a. $H(3) = \boxed{}$.

b. $H(-2) = \boxed{}$.

31. $K(x) = -\dfrac{16}{x-5}$.

a. $K(7) = \boxed{}$.

b. $K(5) = \boxed{}$.

32. $f(x) = -\dfrac{60}{x+10}$.

a. $f(-20) = \boxed{}$.

b. $f(-10) = \boxed{}$.

33. $f(x) = x^2 - x - 1$

a. $f(4) = \boxed{}$

b. $f(-4) = \boxed{}$

34. $g(x) = x^2 - 5x + 4$

a. $g(2) = \boxed{}$

b. $g(-4) = \boxed{}$

35. $h(x) = -2x^2 - 3x - 5$

a. $h(1) = \boxed{}$

b. $h(-5) = \boxed{}$

36. $F(x) = -3x^2 + 2x + 5$

a. $F(1) = \boxed{}$

b. $F(-2) = \boxed{}$

37. $G(x) = \sqrt{x}$.

a. $G(4) = \boxed{}$

b. $G\left(\frac{1}{49}\right) = \boxed{}$

c. $G(-8) = \boxed{}$

38. $G(x) = \sqrt{x}$.

a. $G(64) = \boxed{}$

b. $G\left(\frac{36}{25}\right) = \boxed{}$

c. $G(-8) = \boxed{}$

39. $H(x) = \sqrt[3]{x}$

a. $H(-64) = \boxed{}$

b. $H\left(\frac{125}{8}\right) = \boxed{}$

40. $K(x) = \sqrt[3]{x}$

a. $K(-1) = \boxed{}$

b. $K\left(\frac{8}{125}\right) = \boxed{}$

Solving Equations with Function Notation

41. Solve for x, where $f(x) = -10x - 10$.

a. If $f(x) = -60$, then $x = \boxed{}$.

b. If $f(x) = -24$, then $x = \boxed{}$.

42. Solve for x, where $f(x) = 8x + 8$.

a. If $f(x) = 40$, then $x = \boxed{}$.

b. If $f(x) = 26$, then $x = \boxed{}$.

43. Solve for x, where $g(x) = x^2 + 9$.

a. If $g(x) = 18$, then $x = \boxed{}$.

b. If $g(x) = 8$, then $x = \boxed{}$.

44. Solve for x, where $h(x) = x^2 + 2$.

a. If $h(x) = 3$, then $x = \boxed{}$.

b. If $h(x) = -2$, then $x = \boxed{}$.

45. Solve for x, where $F(x) = x^2 - 6x + 2$.

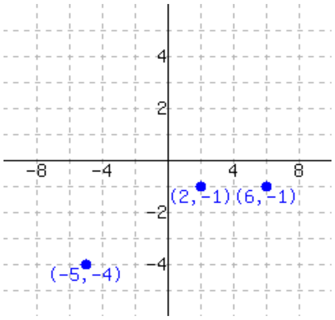
If $F(x) = -3$, then $x = \boxed{}$.

46. Solve for x, where $G(x) = x^2 + 15x + 47$.

If $G(x) = -9$, then $x = \boxed{}$.

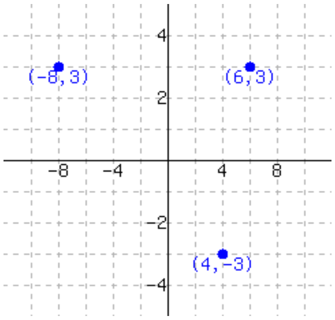
Functions Represented with Graphs

47. A function is graphed.



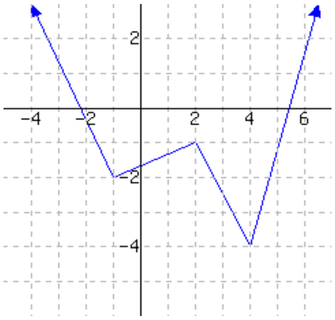
This function has domain and range .

48. A function is graphed.



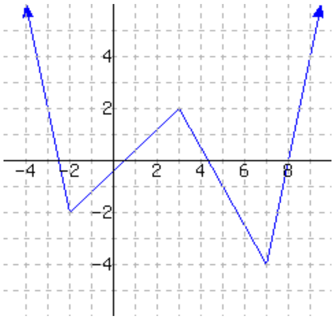
This function has domain and range .

49. A function is graphed.



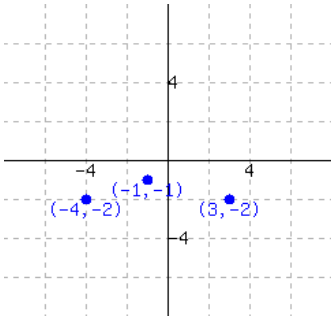
This function has domain and range .

50. A function is graphed.



This function has domain and range .

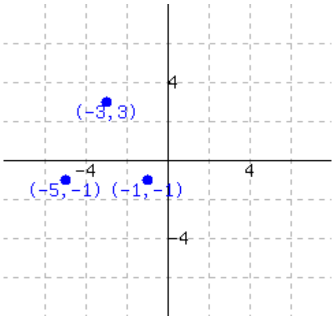
51. Function f is graphed.



a. $f(-1) =$

b. Solve $f(x) = -2$.

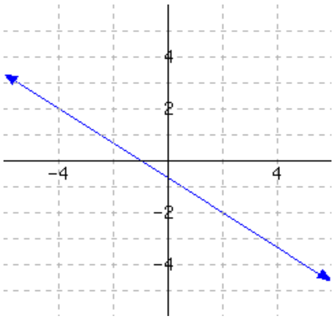
52. Function f is graphed.



a. $f(-3) =$

b. Solve $f(x) = -1$.

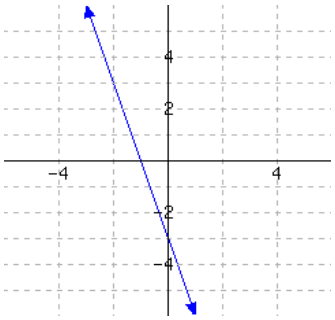
53. Function f is graphed.



a. $f(-1) =$

b. Solve $f(x) = -2$.

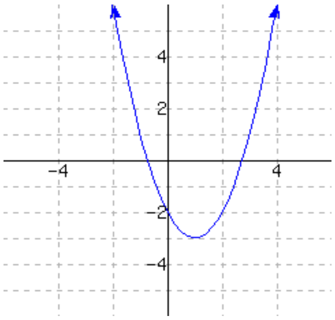
54. Function f is graphed.



a. $f(0) =$

b. Solve $f(x) = 3$.

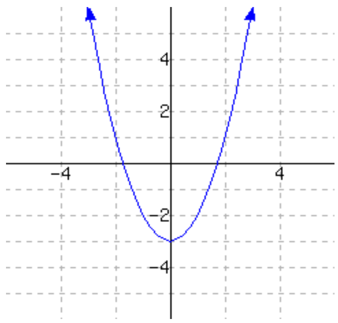
55. Function f is graphed.



a. $f(3) =$

b. Solve $f(x) = -2$.

56. Function f is graphed.



a. $f(-1) =$

b. Solve $f(x) = 1$.

57. If $H(1) = 0$, then the point is on the graph of H.

If $(1, 6)$ is on the graph of H, then $H(1) =$.

58. If $K(8) = 12$, then the point is on the graph of K.

If $(8, 13)$ is on the graph of K, then $K(8) =$.

59. If $f(y) = t$, then the point is on the graph of f.

The answer is not a specific numerical point, but one with variables for coordinates.

60. If $f(x) = r$, then the point is on the graph of f.

The answer is not a specific numerical point, but one with variables for coordinates.

61. If (r, x) is on the graph of g, then $g(r) =$.

62. If (y, r) is on the graph of h, then $h(y) =$.

Function Notation in Context

63. Suppose that M is the function that computes how many miles are in x feet. Find the algebraic rule for M. (If you do not know how many feet are in one mile, you can look it up on Google.)

$M(x) =$

Evaluate $M(19000)$ and interpret the result:

There are about miles in feet.

64. Suppose that K is the function that computes how many kilograms are in x pounds. Find the algebraic rule for K. (If you do not know how many pounds are in one kilogram, you can look it up on Google.)

$K(x) =$

Evaluate $K(214)$ and interpret the result.

Something that weighs pounds would weigh about kilograms.

65. Stephanie started saving in a piggy bank on her birthday. The function $f(x) = 3x + 2$ models the amount of money, in dollars, in Stephanie's piggy bank. The independent variable represents the number of days passed since her birthday.

Interpret the meaning of $f(4) = 14$.

⊙ *A*. Four days after Stephanie started her piggy bank, there were $14 in it.
⊙ *B*. Fourteen days after Stephanie started her piggy bank, there were $4 in it.
⊙ *C*. The piggy bank started with $14 in it, and Stephanie saves $4 each day.
⊙ *D*. The piggy bank started with $4 in it, and Stephanie saves $14 each day.

66. Sherial started saving in a piggy bank on her birthday. The function $f(x) = 5x + 3$ models the amount of money, in dollars, in Sherial's piggy bank. The independent variable represents the number of days passed since her birthday.

Interpret the meaning of $f(1) = 8$.

⊙ *A*. The piggy bank started with $1 in it, and Sherial saves $8 each day.
⊙ *B*. The piggy bank started with $8 in it, and Sherial saves $1 each day.
⊙ *C*. One days after Sherial started her piggy bank, there were $8 in it.
⊙ *D*. Eight days after Sherial started her piggy bank, there were $1 in it.

67. An arcade sells multi-day passes. The function $g(x) = \frac{1}{4}x$ models the number of days a pass will work, where x is the amount of money paid, in dollars.

Interpret the meaning of $g(12) = 3$.

⊙ *A*. If a pass costs $12, it will work for 3 days.
⊙ *B*. Each pass costs $12, and it works for 3 days.
⊙ *C*. If a pass costs $3, it will work for 12 days.
⊙ *D*. Each pass costs $3, and it works for 12 days.

68. An arcade sells multi-day passes. The function $g(x) = \frac{1}{2}x$ models the number of days a pass will work, where x is the amount of money paid, in dollars.

Interpret the meaning of $g(4) = 2$.

⊙ *A*. Each pass costs $2, and it works for 4 days.
⊙ *B*. Each pass costs $4, and it works for 2 days.
⊙ *C*. If a pass costs $4, it will work for 2 days.
⊙ *D*. If a pass costs $2, it will work for 4 days.

69. Bobbi will spend \$105 to purchase some bowls and some plates. Each bowl costs \$2, and each plate costs \$3. The function $p(b) = -\frac{2}{3}b + 35$ models the number of plates Bobbi will purchase, where b represents the number of bowls Bobbi will purchase.

Interpret the meaning of $p(39) = 9$.

- ⊙ *A*. \$9 will be used to purchase bowls, and \$39 will be used to purchase plates.
- ⊙ *B*. If 9 bowls are purchased, then 39 plates will be purchased.
- ⊙ *C*. If 39 bowls are purchased, then 9 plates will be purchased.
- ⊙ *D*. \$39 will be used to purchase bowls, and \$9 will be used to purchase plates.

70. Lisa will spend \$135 to purchase some bowls and some plates. Each bowl costs \$2, and each plate costs \$3. The function $p(b) = -\frac{2}{3}b + 45$ models the number of plates Lisa will purchase, where b represents the number of bowls Lisa will purchase.

Interpret the meaning of $p(30) = 25$.

- ⊙ *A*. \$25 will be used to purchase bowls, and \$30 will be used to purchase plates.
- ⊙ *B*. If 25 bowls are purchased, then 30 plates will be purchased.
- ⊙ *C*. If 30 bowls are purchased, then 25 plates will be purchased.
- ⊙ *D*. \$30 will be used to purchase bowls, and \$25 will be used to purchase plates.

71. Barbara will spend \$125 to purchase some bowls and some plates. Each plate costs \$3, and each bowl costs \$5. The function $q(x) = -\frac{3}{5}x + 25$ models the number of bowls Barbara will purchase, where x represents the number of plates to be purchased.

Interpret the meaning of $q(5) = 22$.

- ⊙ *A*. 5 plates and 22 bowls can be purchased.
- ⊙ *B*. 22 plates and 5 bowls can be purchased.
- ⊙ *C*. \$22 will be used to purchase bowls, and \$5 will be used to purchase plates.
- ⊙ *D*. \$5 will be used to purchase bowls, and \$22 will be used to purchase plates.

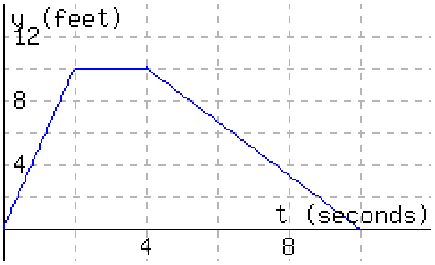
72. Sharell will spend \$360 to purchase some bowls and some plates. Each plate costs \$9, and each bowl costs \$8. The function $q(x) = -\frac{9}{8}x + 45$ models the number of bowls Sharell will purchase, where x represents the number of plates to be purchased.

Interpret the meaning of $q(24) = 18$.

- ⊙ *A*. \$18 will be used to purchase bowls, and \$24 will be used to purchase plates.
- ⊙ *B*. 18 plates and 24 bowls can be purchased.
- ⊙ *C*. 24 plates and 18 bowls can be purchased.
- ⊙ *D*. \$24 will be used to purchase bowls, and \$18 will be used to purchase plates.

The following figure has the graph $y = d(t)$, which models a particle's distance from the starting line in feet, where t stands for time in seconds since timing started.

73.



a. $d(8) =$

b. Interpret the meaning of $d(8)$:

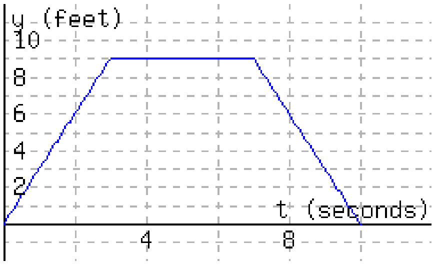
- ⊙ *A.* The particle was 3.33333 feet away from the starting line 8 seconds since timing started.
- ⊙ *B.* In the first 3.33333 seconds, the particle moved a total of 8 feet.
- ⊙ *C.* The particle was 8 feet away from the starting line 3.33333 seconds since timing started.
- ⊙ *D.* In the first 8 seconds, the particle moved a total of 3.33333 feet.

c. Solve $d(t) = 5$ for t. $t =$

d. Interpret the meaning of part c's solution(s):

- ⊙ *A.* The article was 5 feet from the starting line 7 seconds since timing started.
- ⊙ *B.* The article was 5 feet from the starting line 1 seconds since timing started, and again 7 seconds since timing started.
- ⊙ *C.* The article was 5 feet from the starting line 1 seconds since timing started, or 7 seconds since timing started.
- ⊙ *D.* The article was 5 feet from the starting line 1 seconds since timing started.

74.



a. $d(4) =$

b. Interpret the meaning of $d(4)$:

- ⊙ *A.* The particle was 9 feet away from the starting line 4 seconds since timing started.
- ⊙ *B.* In the first 4 seconds, the particle moved a total of 9 feet.
- ⊙ *C.* The particle was 4 feet away from the starting line 9 seconds since timing started.
- ⊙ *D.* In the first 9 seconds, the particle moved a total of 4 feet.

c. Solve $d(t) = 6$ for t. $t =$

d. Interpret the meaning of part c's solution(s):

- ⊙ *A.* The article was 6 feet from the starting line 2 seconds since timing started, or 8 seconds since timing started.
- ⊙ *B.* The article was 6 feet from the starting line 8 seconds since timing started.
- ⊙ *C.* The article was 6 feet from the starting line 2 seconds since timing started, and again 8 seconds since timing started.
- ⊙ *D.* The article was 6 feet from the starting line 2 seconds since timing started.

The function C models the the number of customers in a store t hours since the store opened.

75.

t	0	1	2	3	4	5	6	7
$C(t)$	0	48	78	92	100	78	38	0

a. $C(6) =$ ☐

b. Interpret the meaning of $C(6)$:

- ⊙ *A*. There were 38 customers in the store 6 hours after the store opened.
- ⊙ *B*. In 6 hours since the store opened, the store had an average of 38 customers per hour.
- ⊙ *C*. In 6 hours since the store opened, there were a total of 38 customers.
- ⊙ *D*. There were 6 customers in the store 38 hours after the store opened.

c. Solve $C(t) = 78$ for t. $t =$ ☐

d. Interpret the meaning of Part c's solution(s):

- ⊙ *A*. There were 78 customers in the store 2 hours after the store opened.
- ⊙ *B*. There were 78 customers in the store either 5 hours after the store opened, or 2 hours after the store opened.
- ⊙ *C*. There were 78 customers in the store 5 hours after the store opened.
- ⊙ *D*. There were 78 customers in the store 5 hours after the store opened, and again 2 hours after the store opened.

76.

t	0	1	2	3	4	5	6	7
$C(t)$	0	41	81	97	97	81	44	0

a. $C(7) =$ ☐

b. Interpret the meaning of $C(7)$:

- ⊙ *A*. There were 0 customers in the store 7 hours after the store opened.
- ⊙ *B*. There were 7 customers in the store 0 hours after the store opened.
- ⊙ *C*. In 7 hours since the store opened, there were a total of 0 customers.
- ⊙ *D*. In 7 hours since the store opened, the store had an average of 0 customers per hour.

c. Solve $C(t) = 97$ for t. $t =$ ☐

d. Interpret the meaning of Part c's solution(s):

- ⊙ *A*. There were 97 customers in the store 3 hours after the store opened.
- ⊙ *B*. There were 97 customers in the store either 3 hours after the store opened, or 4 hours after the store opened.
- ⊙ *C*. There were 97 customers in the store 3 hours after the store opened, and again 4 hours after the store opened.
- ⊙ *D*. There were 97 customers in the store 4 hours after the store opened.

9.2 Properties of Quadratic Functions

9.2.1 Introduction

In this section we will learn about quadratic functions and how to identify their key features on a graph. We will identify their direction, vertex, axis of symmetry and intercepts. We will also see how to graph a parabola by finding the vertex and making a table of function values. We will look at applications that involve the vertex of a quadratic function.

Definition 9.2.2. A **quadratic function** has the form $f(x) = ax^2 + bx + c$ where a, b, and c are real numbers, and $a \neq 0$. The graph of a quadratic function has the shape of a **parabola**.

Notice that a quadratic function has a squared term that linear functions do not have. If $a = 0$, the function is linear. To understand the shape and features of a quadratic function, let's look at an example.

9.2.2 Properties of Quadratic Functions

Hannah fired a toy rocket from the ground, which flew into the air at a speed of 64 feet per second. The path of the rocket can be modeled by the function f where $f(t) = -16t^2 + 64t$. To see the shape of the function we will make a table of values and plot the points. For the table we we will choose some values for t and then evaluate the function at each t-value:

t	$f(t) = -16t^2 + 64t$	Point
0	$f(0) = -16(0)^2 + 64(0) = 0$	$(0, 0)$
1	$f(1) = -16(1)^2 + 64(1) = 48$	$(1, 48)$
2	$f(2) = -16(2)^2 + 64(2) = 64$	$(2, 64)$
3	$f(3) = -16(3)^2 + 64(3) = 48$	$(3, 48)$
4	$f(4) = -16(4)^2 + 64(4) = 0$	$(4, 0)$

Table 9.2.3: Function values and points for $f(t) = -16t^2 + 64t$

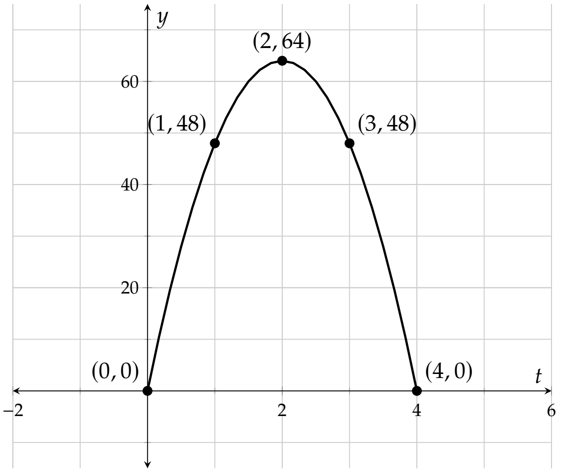


Figure 9.2.4: Graph of $f(t) = -16t^2 + 64t$

Now that we have Table 9.2.3 and Figure 9.2.4, we can see the features of this parabola. Notice the symmetry in the shape of the graph and the y-values in the table. Consecutive y-values do not increase by a constant amount in the way that linear functions do.

The first feature that we will talk about is the *direction* that a parabola opens. All parabolas open either upward or downward. This parabola in the rocket example opens downward because a is negative. Here are some more quadratic functions graphed so we can see which way they open.

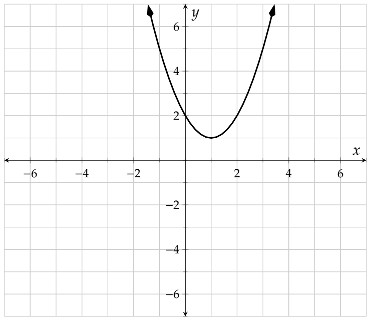


Figure 9.2.5: The graph of $y = x^2 - 2x + 2$ opens upward

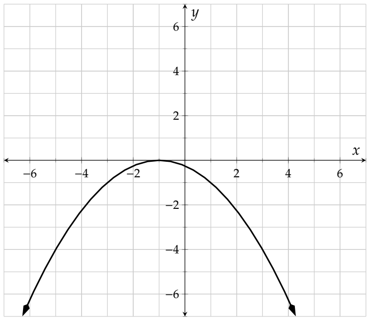


Figure 9.2.6: The graph of $y = -\frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{4}$ opens downward

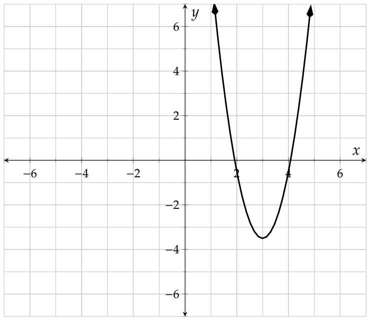


Figure 9.2.7: The graph of $y = 3x^2 - 18x + 23.5$ opens upward

Fact 9.2.8. *We only need to look at the sign of the leading coefficient to determine which way the graph opens. If the leading coefficient is positive, the parabola opens upward. If the leading coefficient is negative, the parabola opens downward.*

Checkpoint 9.2.9. Determine whether the graph of each quadratic function opens upward or downward.

a. The graph of the quadratic function $y = 3x^2 - 4x - 7$ opens (☐ upward ☐ downward) .
b. The graph of the quadratic function $y = -5x^2 + x$ opens (☐ upward ☐ downward) .
c. The graph of the quadratic function $y = 2 + 3x - x^2$ opens (☐ upward ☐ downward) .
d. The graph of the quadratic function $y = \frac{1}{3}x^2 - \frac{2}{5}x + \frac{1}{4}$ opens (☐ upward ☐ downward) .

Explanation.

a. The graph of the quadratic function $y = 3x^2 - 4x - 7$ opens upward as the leading coefficient is the positive number 3.
b. The graph of the quadratic function $y = -5x^2 + x$ opens downward as the leading coefficient is the negative number -5.
c. The graph of the quadratic function $y = 2 + 3x - x^2$ opens downward as the leading coefficient is -1. (Note that the leading coefficient is the coefficient on x^2.)
d. The graph of the quadratic function $y = \frac{1}{3}x^2 - \frac{2}{5}x + \frac{1}{4}$ opens upward as the leading coefficient is the positive number $\frac{1}{3}$.

The **vertex** is the highest or lowest point on the graph. In Figure 9.2.4, the vertex is $(2, 64)$. This tells us that Hannah's rocket reached its maximum height of 64 feet after 2 seconds. If the parabola opens downward, as in the rocket example, then the y-value of the vertex is the **maximum** y-value. If the parabola opens upward then the y-value of the vertex is the **minimum** y-value.

The **axis of symmetry** is a vertical line that passes through the vertex, dividing it in half. The vertex is the only point that does not have a symmetric point. We write the axis of symmetry as an equation of a vertical line so it always starts with "$x =$." In Figure 9.2.4, the equation for the axis of symmetry is $x = 2$.

The **vertical intercept** is the point where the parabola crosses the vertical axis. The vertical intercept is the y-intercept if the axes are labeled x and y. In Figure 9.2.4, the point $(0, 0)$ is the starting point of the rocket. The y-value of 0 means the rocket started on the ground.

The **horizontal intercept(s)** are the points where the parabola crosses the horizontal axis. They are the x-intercepts if the axes are labeled x and y. The point $(0, 0)$ on the path of the rocket is also a horizontal intercept. The t-value of 0 indicates the time when the rocket was launched from the ground. There is another horizontal intercept at the point $(4, 0)$, which means the rocket hit the ground after 4 seconds.

It is possible for a quadratic function to have 0, 1 or 2 horizontal intercepts. The figures below show an example of each.

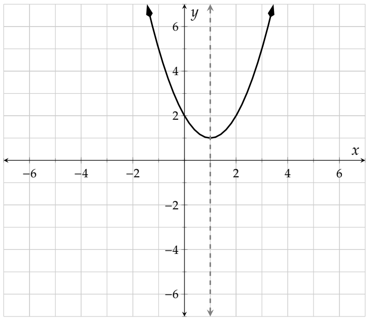


Figure 9.2.10: The graph of $y = x^2 - 2x + 2$ has no horizontal intercepts

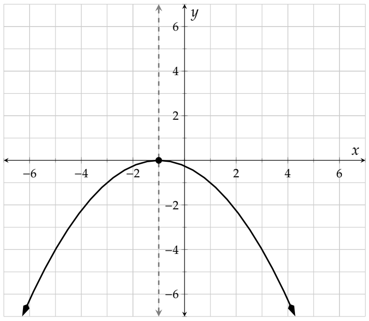


Figure 9.2.11: The graph of $y = -\frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{4}$ has one horizontal intercept

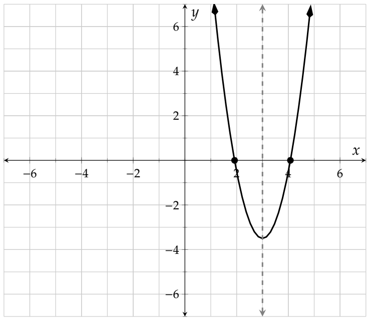


Figure 9.2.12: The graph of $y = 3x^2 - 18x + 23.5$ has two horizontal intercepts

Here is a summary of the properties of quadratic functions:

Direction A parabola opens upward if a is positive and opens downward of a is negative.

Vertex The vertex of a parabola is the maximum or minimum point on the graph.

Axis of Symmetry The axis of symmetry is the vertical line that passes through the vertex.

Vertical Intercept The vertical intercept is the point where the function intersects the vertical axis. There is exactly one vertical intercept.

Horizontal Intercept(s) The horizontal intercept(s) are the points where a function intersects the horizontal axis. The graph of a parabola can have $0, 1$, or 2 horizontal intercepts.

List 9.2.13: Summary of Properties of Quadratic Functions

Example 9.2.14 Identify the key features of the quadratic function $y = x^2 - 2x - 8$ shown in Figure 9.2.15.

Explanation.

First, we see that this parabola opens upward because the leading coefficient is positive.

Then we locate the vertex which is the point $(1, -9)$. The axis of symmetry is the vertical line $x = 1$.

The vertical intercept or y-intercept is the point $(0, -8)$.

The horizontal intercepts are the points $(-2, 0)$ and $(4, 0)$.

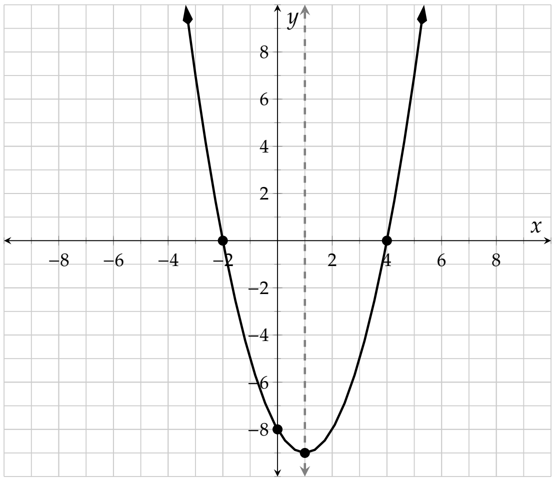
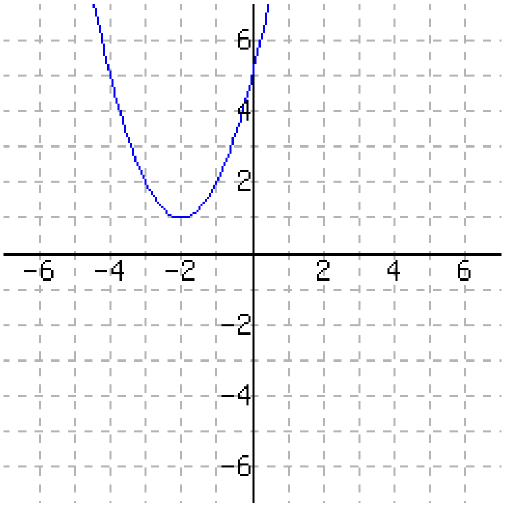


Figure 9.2.15: Graph of $y = x^2 - 2x - 8$

Checkpoint 9.2.16. Use the graph to answer the following questions.



a. What are the coordinates of the vertex?

b. What is the equation of the axis of symmetry?

c. What are the coordinates of the x-intercept(s)?

d. What are the coordinates of the y-intercept?

Explanation.

a. The vertex is at $(-2, 1)$.

b. The equation of the axis of symmetry is $x = -2$.

c. There are no x-intercepts. (Answer `None`.)

d. The y-intercept is at $(0, 5)$.

9.2.3 Finding the Vertex and Axis of Symmetry Algebraically

The coordinates of the vertex are not easy to identify on a graph if they are not integers. Another way to find it is by using a formula.

Fact 9.2.17. *If we denote (h, k) as the coordinates of the vertex of a quadratic function $f(x) = ax^2 + bx + c$, then $h = -\frac{b}{2a}$.*

To understand why, we can look at the quadratic formula. The vertex is on the axis of symmetry, so it will always occur halfway between the two x-intercepts (if there are any). The quadratic formula shows that the x-intercepts happen at $-\frac{b}{2a}$ minus some number and at $-\frac{b}{2a}$ plus that same number:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 9.2.18 Determine the vertex and axis of symmetry of the quadratic function $f(x) = x^2 - 4x - 12$.

We will find the x-value of the vertex using the formula $h = -\frac{b}{2a}$, for $a = 1$ and $b = -4$.

$$\begin{aligned} h &= -\frac{b}{2a} \\ h &= -\frac{(-4)}{2(1)} \\ h &= 2 \end{aligned}$$

Now we know the x-value of the vertex is 2, so we may evaluate $f(2)$ to determine the y-value of the vertex:

$$\begin{aligned} k &= f(2) = (2)^2 - 4(2) - 12 \\ k &= = 4 - 8 - 12 \\ k &= = -16 \end{aligned}$$

The vertex is the point $(2, -16)$ and the axis of symmetry is the line $x = 2$.

Example 9.2.19 Determine the vertex and axis of symmetry of the quadratic function $y = -3x^2 - 3x + 7$.

Explanation. Using the formula $h = -\frac{b}{2a}$ with $a = -3$ and $b = -3$, we have :

$$\begin{aligned} h &= -\frac{b}{2a} \\ h &= -\frac{(-3)}{2(-3)} \\ h &= -\frac{1}{2} \end{aligned}$$

Now that we've determined $h = -\frac{1}{2}$, we can substitute it for x to find the y-value of the vertex:

$$\begin{aligned}
y &= -3x^2 - 3x + 7\\
y &= -3\left(-\frac{1}{2}\right)^2 - 3\left(-\frac{1}{2}\right) + 7\\
y &= -3\left(\frac{1}{4}\right) + \frac{3}{2} + 7\\
y &= -\frac{3}{4} + \frac{3}{2} + 7\\
y &= -\frac{3}{4} + \frac{6}{4} + \frac{28}{4}\\
y &= \frac{31}{4}
\end{aligned}$$

The vertex is the point $\left(-\frac{1}{2}, \frac{31}{4}\right)$ and the axis of symmetry is the line $x = -\frac{1}{2}$.

9.2.4 Graphing Quadratic Functions by Making a Table

When we learned how to graph lines, we could choose any x-values. For quadratic functions, though, we want to find the vertex and choose our x-values around it. Then we can use the property of symmetry to help us. Let's look at an example.

Example 9.2.20 Determine the vertex and axis of symmetry for the quadratic function $y = -x^2 - 2x + 3$. Then make a table of values and sketch the graph of the function.

Explanation. To determine the vertex of $y = -x^2 - 2x + 3$, we want to find the x-value of the vertex first. We use $h = -\frac{b}{2a}$ with $a = -1$ and $b = -2$:

$$\begin{aligned}
h &= -\frac{(-2)}{2(-1)}\\
h &= \frac{2}{-2}\\
&= -1
\end{aligned}$$

To find the y-coordinate of the vertex, we substitute $x = -1$ into the equation for our parabola.

$$\begin{aligned}
y &= -x^2 - 2x + 3\\
y &= -(-1)^2 - 2(-1) + 3\\
&= -1 + 2 + 3\\
&= 4
\end{aligned}$$

Now we know that our axis of symmetry is the line $x = -1$ and the vertex is the point $(-1, 4)$. We will set up our table with two values on each side of $x = -1$. We choose $x = -3, -2, -1, 0$, and 1 as shown in Table 9.2.21.

Next, we'll determine the y-coordinates by replacing x with each value and we have the complete table as shown in Table 9.2.22. Notice that each pair of y-values on either side of the vertex match. This helps us to check that our vertex and y-values are correct.

x	$y = -x^2 - 2x + 3$	Point
-3		
-2		
-1		
0		
1		

Table 9.2.21: Setting up the table for $y = -x^2 - 2x + 3$

x	$y = -x^2 - 2x + 3$	Point
-3	$y = -(-3)^2 - 2(-3) + 3 = 0$	$(-3, 0)$
-2	$y = -(-2)^2 - 2(-2) + 3 = 3$	$(-2, 3)$
-1	$y = -(-1)^2 - 2(-1) + 3 = 4$	$(-1, 4)$
0	$y = -(0)^2 - 2(0) + 3 = 3$	$(0, 3)$
1	$y = -(1)^2 - 2(1) + 3 = 0$	$(1, 0)$

Table 9.2.22: Function values and points for $y = -x^2 - 2x + 3$

Now that we have our table, we will plot the points and draw in the axis of symmetry as shown in Figure 9.2.23. We complete the graph by drawing a smooth curve through the points and drawing an arrow on each end as shown in Figure 9.2.24

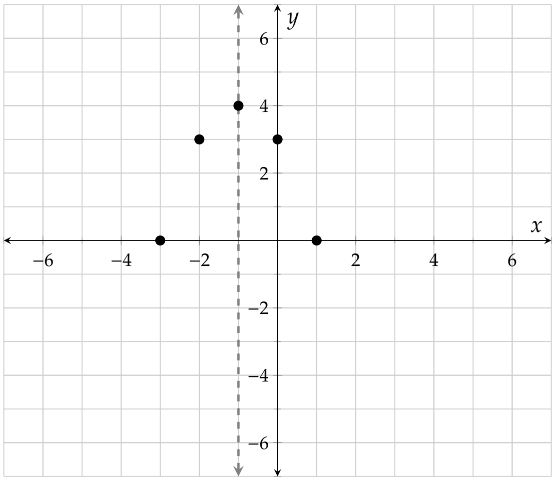


Figure 9.2.23: Plot of the points and axis of symmetry

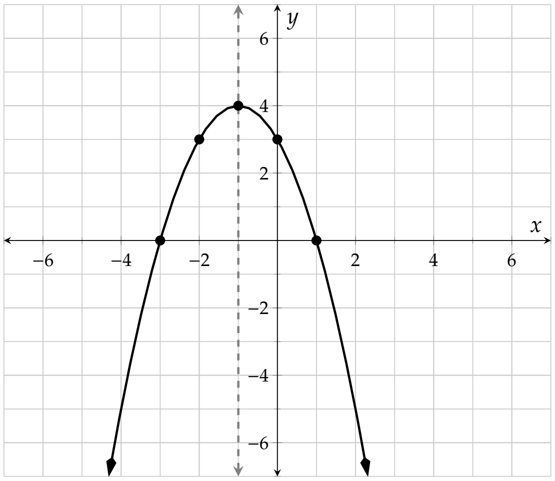


Figure 9.2.24: Graph of $y = -x^2 - 2x + 3$

The method we used works best when the x-value of the vertex is an integer. We can still make a graph if that is not the case as we will demonstrate in the next example.

Example 9.2.25 Determine the vertex and axis of symmetry for the quadratic function $y = 2x^2 - 3x - 4$. Use this to create a table of values and sketch the graph of this function.

Explanation. To determine the vertex of $y = 2x^2 - 3x - 4$, we'll find $h = -\frac{b}{2a}$ with $a = 2$ and $b = -3$:

$$h = -\frac{(-3)}{2(2)}$$
$$h = \frac{3}{4}$$

Next, we'll determine the y-coordinate by replacing x with $\frac{3}{4}$ in $y = 2x^2 - 3x - 4$:

$$\begin{aligned}
y &= 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) - 4\\
y &= 2\left(\frac{9}{16}\right) - \frac{9}{4} - 4\\
y &= \frac{9}{8} - \frac{18}{8} - \frac{32}{8}\\
y &= -\frac{41}{8}
\end{aligned}$$

Thus the vertex occurs at $\left(\frac{3}{4}, -\frac{41}{8}\right)$, or at $(0.75, -5.125)$. The axis of symmetry is then the line $x = \frac{3}{4}$, or $x = 0.75$.

Now that we know the x-value of the vertex, we will create a table. We will choose x-values on both sides of $x = 0.75$, but we will choose integers because it will be easier to find the function values.

x	$y = 2x^2 - 3x - 4$	Point
-1	1	$(-1, 1)$
0	-4	$(0, -4)$
0.75	-5.125	$(0.75, -5.125)$
1	-5	$(1, -5)$
2	-2	$(2, -2)$

Table 9.2.26: Function values and points for $y = 2x^2 - 3x - 4$

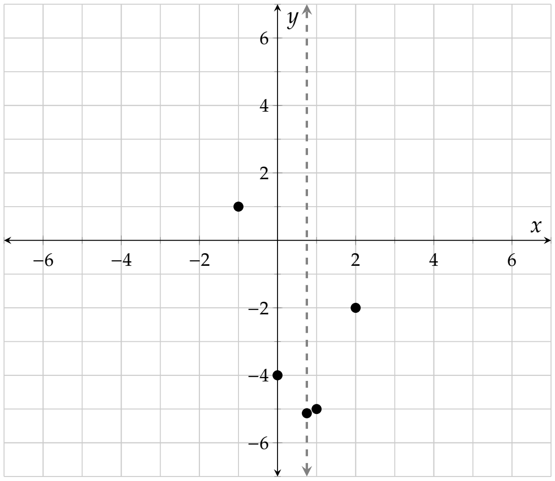


Figure 9.2.27: Plot of initial points

The points graphed in Figure 9.2.27 don't have the symmetry we'd expect from a parabola. This is because the vertex occurs at an x-value that is not an integer, and all of the chosen values in the table are integers. We can use the axis of symmetry to determine more points on the graph (as shown in Figure 9.2.28), which will give it the symmetry we expect. From there, we can complete the sketch of this graph.

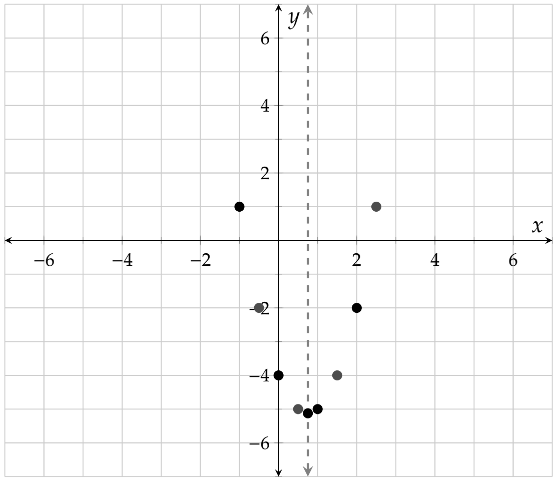


Figure 9.2.28: Plot of symmetric points

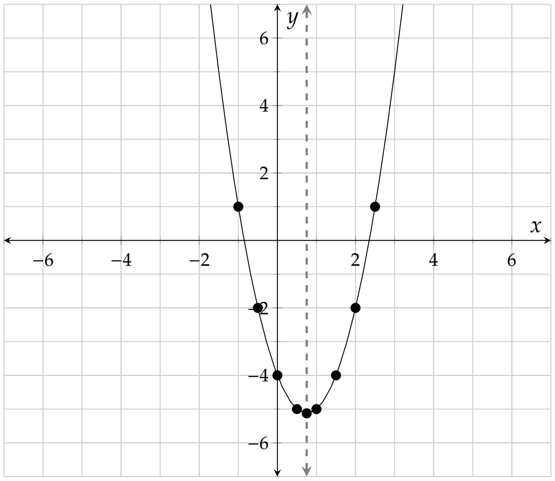


Figure 9.2.29: Graph of $y = 2x^2 - 3x - 4$

9.2.5 The Domain and Range of Quadratic Functions

In Example 9.1.32, we found the domain and range of different types of functions using their graphs. Now that we have graphed some quadratic functions, let's practice identifying the domain and range.

Example 9.2.30 We graphed the quadratic function $y = -x^2 - 2x + 3$ in Figure 9.2.24. The domain is the set of all possible inputs to the function. The function is a continuous curve and when we look horizontally, one arrow points to the left and the other arrow points to the right. This means all x-values can be used in the function. The domain is $\{x \mid x \text{ is a real number}\}$ which is equivalent to $(-\infty, \infty)$.

The range is the set of all outputs we can get from the function. For the range of this function we look vertically up and down the graph. This parabola opens downward, so both arrows point downward and the highest point along the graph is the vertex at $(-1, 4)$. The range is $\{y \mid y \leq 4\}$ which is equivalent to $(-\infty, 4]$.

Example 9.2.31 Use the graph of $y = 2x^2 - 3x - 4$ in Figure 9.2.29 and its vertex at $(0.75, -5.125)$ to identify the domain and range in set-builder and interval notation.

Explanation. For the domain, we look horizontally and see the graph is a continuous curve and one arrow points to the left and the other arrow points to the right. The domain is $\{x \mid x \text{ is a real number}\}$ which is equivalent to $(-\infty, \infty)$.

For the range we look vertically up and down the graph, which opens upward. Both arrows point upward and the lowest point on the graph is the vertex at $(0.75, -5.125)$. The range is $\{y \mid y \geq -5.125\}$ which is equivalent to $[-5.125, \infty)$.

Since all parabolas have the same shape, they all have the same domain of $\{x \mid x \text{ is a real number}\}$ which is equivalent to $(-\infty, \infty)$. The range depends on which way the parabola opens and the y-coordinate of the vertex. When we look at application problems, however, the domain and range will depend on the values that make sense in the given context. For example, times and lengths do not usually have negative values. We will revisit this after looking at some applications.

9.2.6 Applications of Quadratic Functions Involving the Vertex.

We looked at the height of Hannah's toy rocket with respect to time at the beginning of this section and saw that it reached a maximum height of 64 feet after 2 seconds. Let's look at some more applications that involve finding the **minimum** or **maximum** value of a quadratic function.

Example 9.2.32 Imagine that Jae got a new air rifle to shoot targets. The first thing they did with it was to sight the scope at a certain distance so the pellets consistently hit where the cross hairs are pointed. In Olympic 10-meter air rifle shooting[a], the bulls-eye is a 0.5 mm diameter dot, about the size of the head of a pin, so accuracy is key.

Jae would like to set up the air rifle scope to be accurate at a level distance of 35 yards (from the muzzle, which is the tip of the barrel), but they also need to know how much to correct for gravity at different distances. Since the projectile will be affected by gravity, knowing the distance that the target will be set up is essential to be accurate. After zeroing the scope reticule (cross-hairs) at 35 yards so that they can consistently hit the bulls-eye with the reticule directly over it, they set up targets at various distances to test the gun. Jae then shoots at the targets with the cross-hairs directly on the bulls-eye and measures the distance that the pellet hit above or below the bulls-eye when shot at those distances.

Distance to Target in Yards	5	10	20	30	35	40	50
Above/Below Bulls-eye	↓	↑	↑	↑	⊙	↓	↓
Distance Above/Below in Inches	0.1	0.6	1.1	0.6	0	0.8	3.2

Table 9.2.33: Shooting Distance vs Pellet Rise/Fall

Make a graph of the height above the bulls-eye that Jae shoots at the distances listed in the table and find the vertex. What does the vertex mean in this context?

Explanation.

(Note that values measured below the bulls-eye should be graphed as negative y-values. Keep in mind that the units on the axes are different: along the x-axis, the units are yards, whereas on the y-axis, the units are inches.)

Since the input values seem to be going up by 5s or 10s, we will scale the x-axis by 10s. The y-axis needs to be scaled by 1s.

From the graph we can see that the point $(20, 1.1)$ is our best guess for the real life vertex. This means the highest above the cross-hairs Jae hit was 1.1 inches when the target was 20 yards away.

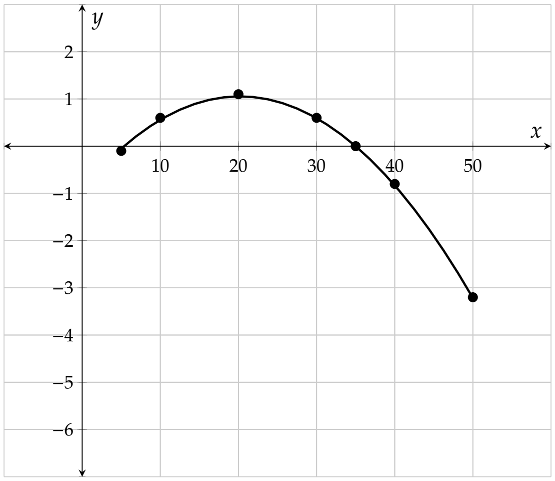


Figure 9.2.34: Graph of Target Data

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[a] en.wikipedia.org/wiki/ISSF_10_meter_air_rifle

Example 9.2.35 We looked at the quadratic function $R = (13 + 0.25x)(1500 - 50x)$ in Example 6.4.2 of Section 6.4, where R was the revenue (in dollars) for x 25-cent price increases. This function had each jar of Avery's jam priced at 13 dollars, and simplified to

$$R = -12.5x^2 - 275x + 19500.$$

Find the vertex of this quadratic function and explain what it means in the context of this model.

Explanation. Note that if we tried to use $R = (13 + 0.25x)(1500 - 50x)$, we would not be able to immediately identify the values of a and b needed to determine the vertex. Using the expanded form of $R = -12.5x^2 - 275x + 19500$, we see that $a = -12.5$ and $b = -275$, so the vertex occurs at:

$$\begin{aligned} h &= -\frac{b}{2a} \\ h &= -\frac{-275}{2(-12.5)} \\ h &= -11 \end{aligned}$$

We will now find the value of R for $x = -11$:

$$\begin{aligned} R &= -12.5(-11)^2 - 275(-11) + 19500 \\ R &= 21012.5 \end{aligned}$$

Thus the vertex occurs at $(-11, 21012.5)$.

Literally interpreting this, we can state that −11 of the 25-cent price increases result in a maximum revenue of \$21,012.50.

We can calculate "−11 of the 25-cent price increases" to be a decrease of \$2.75. The price was set at \$13 per jar, so the maximum revenue of \$21,012.50 would occur when Avery sets the price at \$10.25 per jar.

Example 9.2.36 Kali has 500 feet of fencing and she needs to build a rectangular pen for her goats. What are the dimensions of the rectangle that would give her goats the largest area?

Explanation. We will use ℓ for the length of the pen and w for the width, in feet. We know that the perimeter must be 500 feet so that gives us

$$2\ell + 2w = 500$$

First we will solve for the length:

$$\begin{aligned} 2\ell + 2w &= 500 \\ 2\ell &= 500 - 2w \\ \ell &= 250 - w \end{aligned}$$

Now we can build a function for the rectangle's area, using the formula for area:

$$\begin{aligned} A(w) &= \ell \cdot w \\ A(w) &= (250 - w) \cdot w \\ A(w) &= 250w - w^2 \\ A(w) &= -w^2 + 250w \end{aligned}$$

The area is a quadratic function so we can identify $a = -1$ and $b = 250$ and find the vertex:

$$\begin{aligned} w &= -\frac{(250)}{2(-1)} \\ w &= \frac{250}{2} \\ w &= 125 \end{aligned}$$

Since the width of the rectangle is 125 feet, we can find the length using our expression:

$$\begin{aligned} \ell &= 250 - w \\ \ell &= 250 - 125 \\ \ell &= 125 \end{aligned}$$

To find the maximum area we can either substitute the width into the area function or multiply the length by the width:

$$\begin{aligned} A &= \ell \cdot w \\ A &= 125 \cdot 125 \\ A &= 15{,}625 \end{aligned}$$

The maximum area that Kali can get is 15,625 square feet if she builds her pen to be a square with a length and width of 125 feet.

Returning to the domain and range, we will look at the path of Hannah's toy rocket in Graph 9.2.4. Looking horizontally, the t-values make sense from 0 seconds, when the rocket is fired, until 4 seconds, when it comes back to the ground. This give us a domain of $\{t \mid 0 \leq t \leq 4\}$ or $[0, 4]$. For the range, the height of the rocket goes from 0 feet on the ground and reaches a maximum height of 64 feet. The range is $\{f(t) \mid 0 \leq f(t) \leq 64\}$ or $[0, 64]$.

In the air-rifle application in Example 9.2.32, the x-values are connected from 5 to 50 yards. If we assume that Jae will never be competing in target shoots beyond 50 yards, the domain will be $[5, 50]$. The y-values go from -3.2 to 1.1 inches so the range is $[-3.2, 1.1]$.

In order to find the domain and range for many applications we need to know how to find the vertical and horizontal intercepts. We will look at that in the next section.

Exercises

Review and Warmup Make a table for the equation.

1. The first row is an example.

x	$y = -x + 3$	Points
-3	6	$(-3, 6)$
-2	______	____________
-1	______	____________
0	______	____________
1	______	____________
2	______	____________

2. The first row is an example.

x	$y = -x + 4$	Points
-3	7	$(-3, 7)$
-2	______	____________
-1	______	____________
0	______	____________
1	______	____________
2	______	____________

3. The first row is an example.

x	$y = \frac{5}{8}x - 1$	Points
-24	-16	$(-24, -16)$
-16	______	____________
-8	______	____________
0	______	____________
8	______	____________
16	______	____________

4. The first row is an example.

x	$y = \frac{5}{4}x + 7$	Points
-12	-8	$(-12, -8)$
-8	______	____________
-4	______	____________
0	______	____________
4	______	____________
8	______	____________

5. Evaluate the expression $\frac{1}{5}(x + 3)^2 - 2$ when $x = -8$.

6. Evaluate the expression $\frac{1}{2}(x + 4)^2 - 7$ when $x = -6$.

7. Evaluate the expression $-16t^2 + 64t + 128$ when $t = 3$.

8. Evaluate the expression $-16t^2 + 64t + 128$ when $t = -5$.

Algebraically Determining the Vertex and Axis of Symmetry of Quadratic Functions Find the axis of symmetry and vertex of the quadratic function.

9. $y = -4x^2 - 16x + 5$

Axis of symmetry:

Vertex:

10. $y = -3x^2 + 12x - 2$

Axis of symmetry:

Vertex:

11. $y = 3 - 40x - 4x^2$

Axis of symmetry:

Vertex:

12. $y = -3 - 6x - 3x^2$

Axis of symmetry: ☐

Vertex: ☐

13. $y = -3 - x^2 - 2x$

Axis of symmetry: ☐

Vertex: ☐

14. $y = 4 - x^2 - 4x$

Axis of symmetry: ☐

Vertex: ☐

15. $y = 3x^2 + 6x$

Axis of symmetry: ☐

Vertex: ☐

16. $y = 4x^2 - 24x$

Axis of symmetry: ☐

Vertex: ☐

17. $y = 4 - 5x^2$

Axis of symmetry: ☐

Vertex: ☐

18. $y = 2 - 4x^2$

Axis of symmetry: ☐

Vertex: ☐

19. $y = -3x^2 + 9x - 4$

Axis of symmetry: ☐

Vertex: ☐

20. $y = -2x^2 - 6x + 2$

Axis of symmetry: ☐

Vertex: ☐

21. $y = -2x^2 + 2x + 5$

Axis of symmetry: ☐

Vertex: ☐

22. $y = 2x^2 - 10x - 2$

Axis of symmetry: ☐

Vertex: ☐

23. $y = 3x^2$

Axis of symmetry: ☐

Vertex: ☐

24. $y = 0.4x^2$

Axis of symmetry: ☐

Vertex: ☐

25. $y = 0.5x^2 + 1$

Axis of symmetry: ☐

Vertex: ☐

26. $y = 5x^2 - 5$

Axis of symmetry: ☐

Vertex: ☐

27. $y = -0.5(x - 1)^2 + 3$

Axis of symmetry: ☐

Vertex: ☐

28. $y = -0.4(x + 4)^2 - 2$

Axis of symmetry: ☐

Vertex: ☐

Graphing Quadratic Functions Using the Vertex and a Table

29. For the given quadratic function, find the vertex. Then create a table of ordered pairs centered around the vertex and make a graph of the function.

$f(x) = x^2 + 2$

30. For the given quadratic function, find the vertex. Then create a table of ordered pairs centered around the vertex and make a graph of the function.

$f(x) = x^2 + 1$

31. For the given quadratic function, find the vertex. Then create a table of ordered pairs centered around the vertex and make a graph of the function.

$f(x) = x^2 - 5$

33. For the given quadratic function, find the vertex. Then create a table of ordered pairs centered around the vertex and make a graph of the function.

$f(x) = (x - 2)^2$

35. For the given quadratic function, find the vertex. Then create a table of ordered pairs centered around the vertex and make a graph of the function.

$f(x) = (x + 3)^2$

32. For the given quadratic function, find the vertex. Then create a table of ordered pairs centered around the vertex and make a graph of the function.

$f(x) = x^2 - 3$

34. For the given quadratic function, find the vertex. Then create a table of ordered pairs centered around the vertex and make a graph of the function.

$f(x) = (x - 4)^2$

36. For the given quadratic function, find the vertex. Then create a table of ordered pairs centered around the vertex and make a graph of the function.

$f(x) = (x + 2)^2$

Graphing Quadratic Functions Using the Vertex and a Table

37. For $f(x) = 4x^2 - 8x + 5$, determine the vertex, create a table of ordered pairs, and then graph the function.

38. For $f(x) = 2x^2 + 4x + 7$, determine the vertex, create a table of ordered pairs, and then graph the function.

39. For $f(x) = -x^2 + 4x + 2$, determine the vertex, create a table of ordered pairs, and then graph the function.

40. For $f(x) = -x^2 + 2x - 5$, determine the vertex, create a table of ordered pairs, and then graph the function.

41. For $f(x) = x^2 - 5x + 3$, determine the vertex, create a table of ordered pairs, and then graph the function.

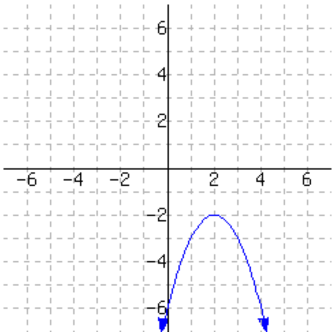
42. For $f(x) = x^2 + 7x - 1$, determine the vertex, create a table of ordered pairs, and then graph the function.

43. For $f(x) = -2x^2 - 5x + 6$, determine the vertex, create a table of ordered pairs, and then graph the function.

44. For $f(x) = 2x^2 - 9x$, determine the vertex, create a table of ordered pairs, and then graph the function.

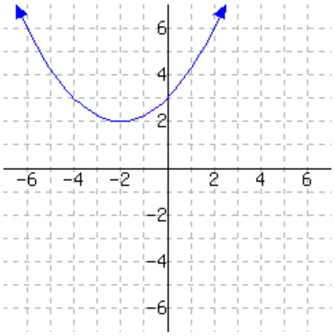
Domain and Range A function is graphed.

45.



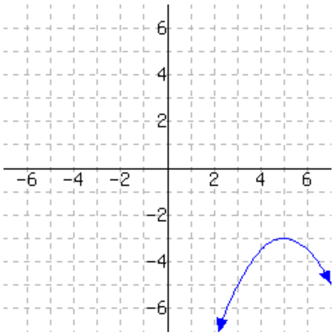
This function has domain
and range .

46.



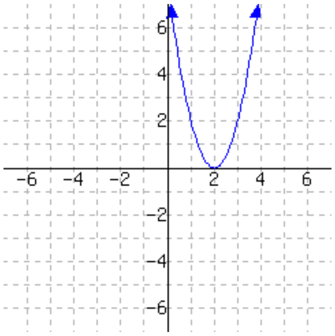
This function has domain
and range .

47.



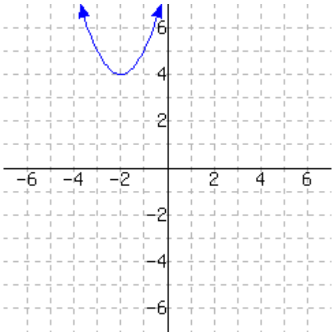
This function has domain
and range .

48.



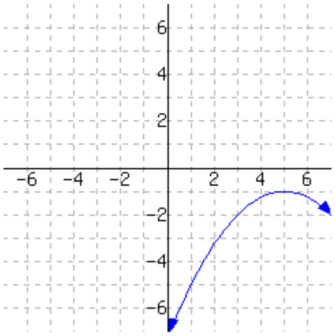
This function has domain
and range .

49.



This function has domain
and range .

50.



This function has domain
and range .

Finding Maximum and Minimum Values for Applications of Quadratic Functions

51. Consider two numbers where one number is 10 less than a second number. Find a pair of such numbers that has the least product possible. One approach is to let x represent the smaller number, and write a formula for a function of x that outputs the product of the two numbers. Then find its vertex and interpret it.

These two numbers are and the least possible product is .

52. Consider two numbers where one number is 4 less than a second number. Find a pair of such numbers that has the least product possible. One approach is to let x represent the smaller number, and write a formula for a function of x that outputs the product of the two numbers. Then find its vertex and interpret it.

These two numbers are and the least possible product is .

53. Consider two numbers where one number is 10 less than twice a second number. Find a pair of such numbers that has the least product possible. One approach is to let x represent the smaller number, and write a formula for a function of x that outputs the product of the two numbers. Then find its vertex and interpret it.

These two numbers are and the least possible product is .

54. Consider two numbers where one number is 7 less than twice a second number. Find a pair of such numbers that has the least product possible. One approach is to let x represent the smaller number, and write a formula for a function of x that outputs the product of the two numbers. Then find its vertex and interpret it.

These two numbers are and the least possible product is .

55. You will build a rectangular sheep enclosure next to a river. There is no need to build a fence along the river, so you only need to build on three sides. You have a total of 440 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that runs perpendicular to the river, and write a formula for a function of x that outputs the area of the enclosure. Then find its vertex and interpret it.

The length of the pen (parallel to the river) should be , the width (perpendicular to the river) should be , and the maximum possible area is .

56. You will build a rectangular sheep enclosure next to a river. There is no need to build a fence along the river, so you only need to build on three sides. You have a total of 460 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that runs perpendicular to the river, and write a formula for a function of x that outputs the area of the enclosure. Then find its vertex and interpret it.

The length of the pen (parallel to the river) should be , the width (perpendicular to the river) should be , and the maximum possible area is .

57. You will build a rectangular sheep enclosure next to a river. There is no need to build a fence along the river, so you only need to build on three sides. You have a total of 470 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that runs perpendicular to the river, and write a formula for a function of x that outputs the area of the enclosure. Then find its vertex and interpret it.

The length of the pen (parallel to the river) should be , the width (perpendicular to the river) should be , and the maximum possible area is .

58. You will build a rectangular sheep enclosure next to a river. There is no need to build a fence along the river, so you only need to build on three sides. You have a total of 480 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that runs perpendicular to the river, and write a formula for a function of x that outputs the area of the enclosure. Then find its vertex and interpret it.

The length of the pen (parallel to the river) should be , the width (perpendicular to the river) should be , and the maximum possible area is .

59. You will build two identical rectangular enclosures next to a each other, sharing a side. You have a total of 408 feet of fence to use. Find the dimensions of each pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that the two pens share, and write a formula for a function of x that outputs the total area of the enclosures. Then find its vertex and interpret it.

The length of each (along the wall that they share) should be , the width should be , and the maximum possible area of each pen is .

60. You will build two identical rectangular enclosures next to a each other, sharing a side. You have a total of 420 feet of fence to use. Find the dimensions of each pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that the two pens share, and write a formula for a function of x that outputs the total area of the enclosures. Then find its vertex and interpret it.

The length of each (along the wall that they share) should be , the width should be , and the maximum possible area of each pen is .

61. You plan to build four identical rectangular animal enclosures in a row. Each adjacent pair of pens share a fence between them. You have a total of 312 feet of fence to use. Find the dimensions of each pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that adjacent pens share, and write a formula for a function of x that outputs the total area. Then find its vertex and interpret it.

The length of each pen (along the walls that they share) should be , the width (perpendicular to the river) should be , and the maximum possible area of each pen is .

62. You plan to build four identical rectangular animal enclosures in a row. Each adjacent pair of pens share a fence between them. You have a total of 328 feet of fence to use. Find the dimensions of each pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that adjacent pens share, and write a formula for a function of x that outputs the total area. Then find its vertex and interpret it.

The length of each pen (along the walls that they share) should be , the width (perpendicular to the river) should be , and the maximum possible area of each pen is .

63. Currently, an artist can sell 240 paintings every year at the price of \$90.00 per painting. Each time he raises the price per painting by \$15.00, he sells 5 fewer paintings every year.

a. To obtain maximum income of , the artist should set the price per painting at .

b. To earn \$43,875.00 per year, the artist could sell his paintings at two different prices. The lower price is per painting, and the higher price is per painting.

64. Currently, an artist can sell 270 paintings every year at the price of \$150.00 per painting. Each time he raises the price per painting by \$5.00, he sells 5 fewer paintings every year.

a. To obtain maximum income of , the artist should set the price per painting at .

b. To earn \$43,700.00 per year, the artist could sell his paintings at two different prices. The lower price is per painting, and the higher price is per painting.

9.3 Graphing Quadratic Functions

We have learned how to locate the key features of quadratic functions on a graph and find the vertex algebraically. In this section we'll explore how to find the intercepts algebraically and use their coordinates to graph a quadratic function. Then we will see how to interpret the key features in context and distinguish between quadratic and other functions.

Let's start by looking at a quadratic function that models the path of a baseball after it is hit by Ignacio, the batter. The height of the baseball, $H(t)$, measured in feet, after t seconds is given by $H(t) = -16t^2 + 75t + 4.7$. We know this quadratic function has the shape of a parabola and we want to know the initial height, the maximum height, and the amount of time it takes for the ball to hit the ground if it is not caught. These key features correspond to the vertical intercept, the vertex, and one of the horizontal intercepts.

The graph of this function is shown in Figure 9.3.2. We cannot easily read where the intercepts occur from the graph because they are not integers. We previously covered how to determine the vertex algebraically. In this section, we'll learn how to find the intercepts algebraically. Then we'll come back to this example and find the intercepts for the path of the baseball.

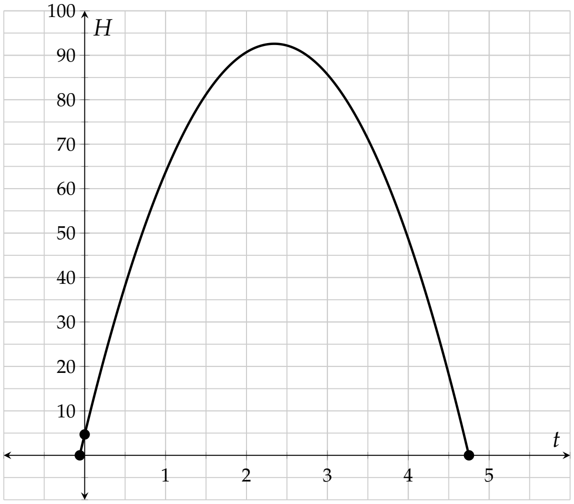


Figure 9.3.2: Graph of $H(t) = -16t^2 + 75t + 4.7$

9.3.1 Finding the Vertical and Horizontal Intercepts Algebraically

In List 9.2.13, we identified that the **vertical intercept** occurs where the graph of a function intersects the vertical axis. If we're using x and y as our variables, the x-value on the vertical axis is $x = 0$. We will substitute 0 for x to find the value of y. In function notation, we find $f(0)$.

The **horizontal intercepts** occur where the graph of a function intersects the horizontal axis. If we're using x and y as our variables, the y-value on the horizontal axis is $y = 0$, so we will substitute 0 for y and find the value(s) of x. In function notation, we solve the equation $f(x) = 0$.

Here is an example where we find the vertical and horizontal intercepts.

Example 9.3.3 Find the intercepts for the quadratic function $f(x) = x^2 - 4x - 12$ algebraically.

To determine the y-intercept, we find $f(0) = 0^2 - 4(0) - 12 = -12$. So the y-intercept occurs where $y = -12$. On a graph, this is the point $(0, -12)$.

To determine the x-intercept(s), we set $f(x) = 0$ and solve for x:

$$\begin{aligned} 0 &= x^2 - 4x - 12 \\ 0 &= (x-6)(x+2) \end{aligned}$$

$$\begin{array}{rcr} x-6=0 & \text{or} & x+2=0 \\ x=6 & \text{or} & x=-2 \end{array}$$

The x-intercepts occur where $x = 6$ and where $x = -2$. On a graph, these are the points $(6, 0)$ and $(-2, 0)$.

Notice in Example 9.3.3 that the y-intercept was $(0, -12)$ and the value of $c = -12$. When we substitute 0 for x we will always get the value of c.

Fact 9.3.4. *The vertical intercept of a quadratic function occurs at the point* $(0, c)$ *because* $f(0) = c$*.*

Example 9.3.5 Algebraically determine any horizontal and vertical intercepts of the quadratic function $f(x) = -x^2 + 5x - 7$.

Explanation. To determine the vertical intercept, we find $f(0)-(0)^2+5(0)-7 = -7$. Thus the y-intercept occurs at the point $(0, -7)$.

To determine the horizontal intercepts, we'll set $f(x) = 0$ and solve for x:

$$0 = -x^2 + 5x - 7$$

This equation cannot be solved using factoring so we'll use the quadratic formula:

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{5^2 - 4(-1)(-7)}}{2(-1)} \\ x &= \frac{-5 \pm \sqrt{-3}}{-2} \end{aligned}$$

The radicand is negative so there are no real solutions to the equation. This means there are no horizontal intercepts.

9.3.2 Graphing Quadratic Functions Using Their Key Features

To graph a quadratic function using its key features, we will algebraically determine the following: whether the function opens upward or downward, the vertical intercept, the horizontal intercepts and the vertex. Then we will graph the points and connect them with a smooth curve.

Example 9.3.6 Graph the function f where $f(x) = 2x^2 + 10x + 8$ by algebraically determining its key features.

To start, we'll note that this function will open upward, since the leading coefficient is positive.

To find the y-intercept, we evaluate $f(0) = 2(0)^2 + 10(0) + 8 = 8$. The y-intercept is $(0, 8)$.

Next, we'll find the horizontal intercepts by setting $f(x) = 0$ and solving for x:

$$\begin{aligned} 2x^2 + 10x + 8 &= 0 \\ 2(x^2 + 5x + 4) &= 0 \\ 2(x+4)(x+1) &= 0 \end{aligned}$$

$$\begin{array}{rcr} x+4=0 & \text{or} & x+1=0 \\ x=-4 & \text{or} & x=-1 \end{array}$$

The x-intercepts are $(-4, 0)$ and $(-1, 0)$.

Lastly, we'll determine the vertex. Noting that $a = 2$ and $b = 10$, we have:

$$\begin{aligned}h &= -\frac{b}{2a}\\ h &= -\frac{10}{2(2)}\\ h &= -2.5\end{aligned}$$

Using this x-value to find the y-coordinate, we have:

$$\begin{aligned}k &= f(-2.5) = 2(-2.5)^2 + 10(-2.5) + 8\\ k &= 12.5 - 25 + 8\\ k &= -4.5\end{aligned}$$

The vertex is the point $(-2.5, -4.5)$, and the axis of symmetry is the line $x = -2.5$.

We're now ready to graph this function. We'll start by drawing and scaling the axes so all of our key features will be displayed as shown in Figure 9.3.7. Next, we'll plot these key points as shown in Figure 9.3.8. Finally, we'll note that this parabola opens upward and connect these points with a smooth curve, as shown in Figure 9.3.9.

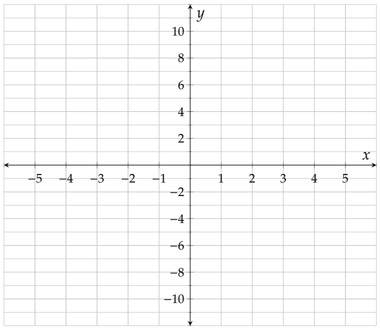


Figure 9.3.7: Setting up the grid.

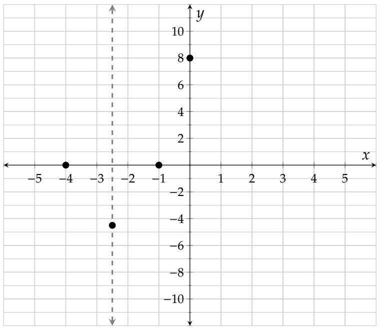


Figure 9.3.8: Marking key features.

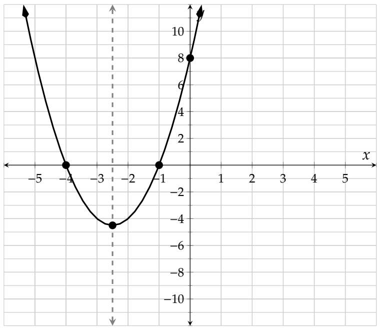


Figure 9.3.9: Completing the graph.

Example 9.3.10 Graph the function for which $y = -x^2 + 4x - 5$ by algebraically determining its key features.

To start, we'll note that this function will open downward, as the leading coefficient is negative.

To find the y-intercept, we'll substitute x with 0:

$$\begin{aligned}y &= -(0)^2 + 4(0) - 5\\ y &= -5\end{aligned}$$

The y-intercept is $(0, -5)$.

Next, we'll find the horizontal intercepts by setting $y = 0$ and solving for x. We cannot use factoring to solve this equation so we'll use the quadratic formula:

$$\begin{aligned}-x^2+4x-5&=0\\ x&=\frac{-4\pm\sqrt{(4)^2-4(-1)(-5)}}{2(-1)}\\ x&=\frac{-4\pm\sqrt{16-20}}{-2}\\ x&=\frac{-4\pm\sqrt{-8}}{-2}\end{aligned}$$

The radicand is negative, so there are no real solutions to the equation. This is a parabola that does not have any horizontal intercepts.

To determine the vertex, we'll use $a=-1$ and $b=4$:

$$\begin{aligned}x&=-\frac{4}{2(-1)}\\ x&=2\end{aligned}$$

Using this x-value to find the y-coordinate, we have:

$$\begin{aligned}y&=-(2)^2+4(2)-5\\ y&=-4+8-5\\ y&=-1\end{aligned}$$

The vertex is the point $(2,-1)$, and the axis of symmetry is the line $x=2$.

Plotting this information in an appropriate grid, we have:

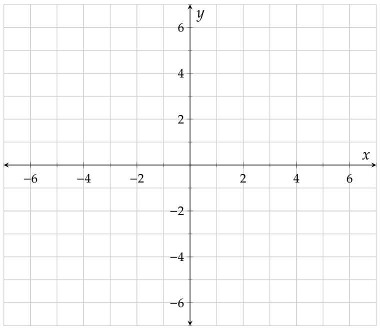


Figure 9.3.11: Setting up the grid.

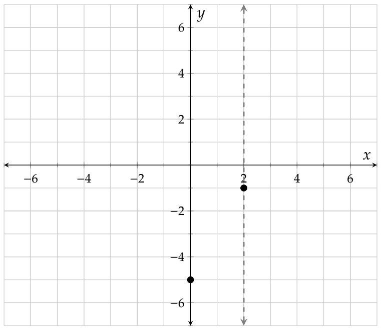


Figure 9.3.12: Marking key features.

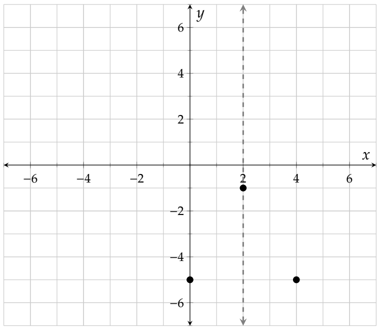


Figure 9.3.13: Using the axis of symmetry to determine one additional point.

Since we don't have any x-intercepts, we would like to have a few more points to graph. We will make a table with a few more values around the vertex, add these, and then draw a smooth curve. This is shown in Table 9.3.14 and Figure 9.3.15.

x	$y = -x^2 + 4x - 5$	Point
0	$-(0)^2 + 4(0) - 5 = -5$	$(0, -5)$
1	$-(1)^2 + 4(1) - 5 = -2$	$(1, -2)$
2	$-(2)^2 + 4(2) - 5 = -1$	$(2, -1)$
3	$-(3)^2 + 4(3) - 5 = -2$	$(3, -2)$
4	$-(4)^2 + 4(4) - 5 = -5$	$(4, -5)$

Table 9.3.14: Determine additional function values.

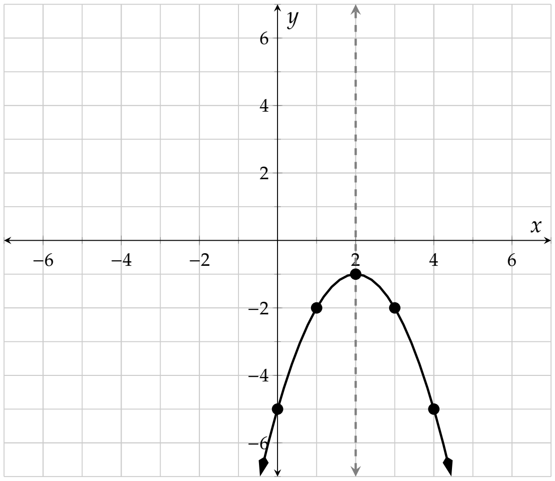


Figure 9.3.15: Completing the graph.

9.3.3 Applications of Quadratic Functions

Now we have learned how to find all the key features of a quadratic function algebraically. Here are some applications of quadratic functions so we can learn how to identify and interpret the vertex, intercepts and additional points in context. Let's look at a few examples.

Example 9.3.16 Returning to the path of the baseball in Figure 9.3.2, the function that represents the height of the baseball after Ignacio hit it, is $H(t) = -16t^2 + 75t + 4.7$. The height is is feet and the time, t, is in seconds. Find and interpret the following, in context.

a. The vertical intercept.
b. The horizontal intercept(s).
c. The vertex.
d. The height of the baseball 1 second after it was hit.
e. The time(s) when the baseball is 80 feet above the ground.

Explanation.

a. To determine the vertical intercept, we'll find $H(0) = -16(0)^2 + 75(0) + 4.7 = 4.7$. The vertical intercept occurs at $(0, 4.7)$. This is the height of the baseball at time $t = 0$, so the initial height of the baseball was 4.7 feet.

b. To determine the horizontal intercepts, we'll solve $H(t) = 0$. Since factoring is not a possibility to solve this equation, we'll use the quadratic formula:

$$\begin{aligned} H(t) &= 0 \\ -16t^2 + 75t + 4.7 &= 0 \\ t &= \frac{-75 \pm \sqrt{75^2 - 4(-16)(4.7)}}{2(-16)} \end{aligned}$$

$$t = \frac{-75 \pm \sqrt{5925.8}}{-32}$$

Rounding these two values with a calculator, we obtain:

$$t \approx -0.06185, \ t \approx 4.749$$

The horizontal intercepts occur at approximately $(-0.06185, 0)$ and $(4.749, 0)$. If we assume that the ball was hit when $t = 0$, a negative time does not make sense. The second horizontal intercept tells us that the ball hit the ground after approximately 4.75 seconds.

c. The vertex occurs at $t = h = -\frac{b}{2a}$, and for this function $a = -16$ and $b = 75$. So we have:

$$\begin{aligned} h &= -\frac{75}{2(-16)} \\ h &= 2.34375 \end{aligned}$$

We can now find the output for this input:

$$\begin{aligned} H(2.34375) &= -16(2.34375)^2 + 75(2.34375) + 4.7 \\ &\approx 92.59 \end{aligned}$$

Thus the vertex is $(2.344, 92.59)$.

The vertex tells us that the baseball reached a maximum height of approximately 92.6 feet about 2.3 seconds after Ignacio hit it.

d. To find the height of the baseball after 1 second, we can compute $H(1)$:

$$\begin{aligned} H(1) &= -16(1)^2 + 75(1) + 4.7 \\ &= 63.7 \end{aligned}$$

The height of the baseball was 63.7 feet after 1 second.

e. If we want to know when the baseball was 80 feet in the air, then we set $H(t) = 80$ and we have:

$$\begin{aligned} H(t) &= 80 \\ -16t^2 + 75t + 4.7 &= 80 \\ -16t^2 + 75t - 75.3 &= 0 \\ t &= \frac{-75 \pm \sqrt{75^2 - 4(-16)(-75.3)}}{2(-16)} \\ t &= \frac{-75 \pm \sqrt{805.8}}{-32} \end{aligned}$$

Rounding these two values with a calculator, we obtain:

$$t \approx 1.457, \ t \approx 3.231$$

The baseball was 80 feet above the ground at two times, at about 1.5 seconds on the way up and about 3.2 seconds on the way down.

Example 9.3.17 The profit that Keenan's manufacturing company makes for producing n refrigerators is given by $P = -0.01n^2 + 520n - 54000$, for $0 \leq n \leq 51{,}896$.

a. Determine the profit the company will make when they produce 1,000 refrigerators.

b. Determine the maximum profit and the number of refrigerators produced that yields this profit.

c. How many refrigerators need to be produced in order for the company to "break even?" (In other words, for their profit to be \$0.)

d. How many refrigerators need to be produced in order for the company to make a profit of \$1,000,000?

Explanation.

a. This question is giving us an input value and asking for the output value. We will substitute 1000 for n and we have:

$$\begin{aligned} P &= -0.01(1000)^2 + 520(1000) - 54000 \\ P &= 366000 \end{aligned}$$

If Keenan's company sells 1,000 refrigerators it will make a profit of \$366,000.

b. This question is asking for the maximum so we need to find the vertex. This parabola opens downward so the vertex will tell us the maximum profit and the corresponding number of refrigerators that need to be produced. Using $a = -0.01$ and $b = 520$, we have:

$$\begin{aligned} h &= -\frac{b}{2a} \\ h &= -\frac{520}{2(-0.01)} \\ h &= 26000 \end{aligned}$$

Now we will find the value of P when $n = 26000$:

$$\begin{aligned} P &= -0.01(26000)^2 + 520(26000) - 54000 \\ P &= 6706000 \end{aligned}$$

The maximum profit is \$6,706,000, which occurs if 26,000 units are produced.

c. This question is giving an output value of 0 and asking us to find the input(s) so we will be finding the horizontal intercept(s). We will set $P = 0$ and solve for n using the quadratic formula:

$$\begin{aligned} 0 &= -0.01n^2 + 520n - 54000 \\ n &= \frac{-520 \pm \sqrt{520^2 - 4(-0.01)(-54000)}}{2(-0.01)} \\ n &= \frac{-520 \pm \sqrt{268240}}{-0.02} \\ n &\approx 104, n \approx 51896 \end{aligned}$$

The company will break even if they produce about 104 refrigerators or 51,896 refrigerators. If the company produces more refrigerators than it can sell its profit will go down.

d. This question is giving an output value and asking us to find the input. To find the number of refrigerators that need to be produced for the company to make a profit of \$1,000,000, we will set

$P = 1000000$ and solve for n using the quadratic formula:

$$\begin{aligned}
1000000 &= -0.01n^2 + 520n - 54000 \\
0 &= -0.01n^2 + 520n - 1054000 \\
n &= \frac{-520 \pm \sqrt{520^2 - 4(-0.01)(-1054000)}}{2(-0.01)} \\
n &\frac{-520 \pm \sqrt{228240}}{-0.02} \\
n &\approx 2,113, n \approx 49,887
\end{aligned}$$

The company will make $1,000,000 in profit if they produce about 2,113 refrigerators or 49,887 refrigerators.

Example 9.3.18 Maia has a remote-controlled airplane and she is going to do a stunt dive where the plane dives toward the ground and back up along a parabolic path. The height of the plane is given by the function H where $H(t) = 0.7t^2 - 23t + 200$, for $0 \leq t \leq 30$. The height is measured in feet and the time, t, is measured in seconds.

a. Determine the starting height of the plane as the dive begins.

b. Determine the height of the plane after 5 seconds.

c. Will the plane hit the ground, and if so, at what time?

d. If the plane does not hit the ground, what is the closest it gets to the ground, and at what time?

e. At what time(s) will the plane have a height of 50 feet?

Explanation.

a. This question is asking for the starting height which is the vertical intercept. We will find $H(0)$:

$$\begin{aligned}
H(0) &= 0.7(0)^2 - 23(0) + 200 \\
H(0) &= 200
\end{aligned}$$

When Maia begins the stunt, the plane has a height of 200 feet. Recall that we can also look at the value of $c = 200$ to determine the vertical intercept.

b. This question is giving an input of 5 seconds and asking for the output so we will find $H(5)$:

$$\begin{aligned}
H(5) &= 0.7(5)^2 - 23(5) + 200 \\
H(5) &= 102.5
\end{aligned}$$

After 5 seconds, the plane is 102.5 feet above the ground.

c. The ground has a height of 0 feet, so it is asking us to find the horizontal intercept(s) if there are any. We will set $H(t) = 0$ and solve for t using the quadratic formula:

$$\begin{aligned}
H(t) &= 0.7t^2 - 23t + 200 \\
0 &= 0.7t^2 - 23t + 200 \\
t &= \frac{23 \pm \sqrt{(-23)^2 - 4(0.7)(200)}}{2(0.7)}
\end{aligned}$$

$$t = \frac{23 \pm \sqrt{-31}}{1.4}$$

The radicand is negative so there are no real solutions to the equation $H(t) = 0$. That means the plane did not hit the ground.

d. This question is asking for the lowest point of the plane so we will find the vertex. Using $a = 0.7$ and $b = -23$, we have:

$$\begin{aligned} h &= -\frac{b}{2a} \\ h &= -\frac{(-23)}{2(0.7)} \\ h &\approx 16.43 \end{aligned}$$

Now we will find the value of H when $t \approx 16.43$:

$$\begin{aligned} H(16.43) &= 0.7(16.43)^2 - 23(16.43) + 200 \\ H(16.43) &\approx 11.07 \end{aligned}$$

The minimum height of the plane is about 11 feet, which occurs after about 16 seconds.

e. This question is giving us a height and asking for the time(s) so we will set $H(t) = 50$ and solve for t using the quadratic formula:

$$\begin{aligned} H(t) &= 0.7t^2 - 23t + 200 \\ 50 &= 0.7t^2 - 23t + 200 \\ 0 &= 0.7t^2 - 23t + 150 \\ t &= \frac{23 \pm \sqrt{(-23)^2 - 4(0.7)(150)}}{2(0.7)} \\ t &= \frac{23 \pm \sqrt{109}}{1.4} \\ t &\approx 8.971, t \approx 23.89 \end{aligned}$$

Maia's plane will be 50 feet above the ground about 9 seconds and 24 seconds after the plane begins the stunt.

9.3.4 The Domain and Range of Quadratic Applications

Let's identify the domain and range in each of the applications of quadratic functions in this section.

Example 9.3.19 In the baseball example in Figure 9.3.2, Ignacio hit the ball at 0 seconds, and it lands on the ground at about 4.7 seconds. The domain is $[0, 4.7]$.

The baseball is at its lowest point when it hits the ground at 0 feet, and the vertex is its highest point at about 92.6 feet. The range is $[0, 92.6]$.

Example 9.3.20 Identify the domain and range in Keenan's refrigerator company application in Example 9.3.17. Write them in interval and set-builder notation.

Explanation. The domain is given in the model as $0 \leq n \leq 51{,}896$ refrigerators. Limits are often stated with a mathematical model because only part of the function fits the real-world situation. The domain is $[0, 51896]$ or $\{n \mid 0 \leq n \leq 51{,}896\}$.

When 0 units are produced, the profit is $-\$54{,}000$. The profit increases to a maximum value of \$6,706,000 at the vertex, and then goes back down to \$0 at 51,896 units produced. So the range is $[-54000, 6706000]$ or $\{P \mid -54{,}000 \leq P \leq 6{,}706{,}000\}$.

Example 9.3.21 Identify the domain and range of Maia's remote-controlled airplane application in Example 9.3.18. Write them in interval and set-builder notation.

Explanation. The domain is given in the model as $[0, 30]$ seconds, because this parabola opens upward and the plane cannot keep flying up forever. In set-builder notation the domain is $\{t \mid 0 \leq t \leq 30\}$.

When $t = 0$ seconds, the plane is 200 feet above the ground. It dives down to a height of about 11 feet and then flies up again. We need to know how high the plane is at 30 seconds to determine the range, so we find $H(30)$:

$$\begin{aligned} H(30) &= 0.7(30)^2 - 23(30) + 200 \\ H(30) &= 140 \end{aligned}$$

The plane has returned to a height of 140 feet after 30 seconds. The starting point of 200 feet is still the highest point, so the range is $[11, 200]$ or $\{H(t) \mid 11 \leq H(t) \leq 200\}$.

9.3.5 Distinguishing Quadratic Functions from Other Functions and Relations

So far, we've seen that the graphs of quadratic functions are parabolas and have a specific, curved shape. We've also seen that they have the algebraic form of $y = ax^2 + bx + c$. Here, we will learn to tell the difference between quadratic functions and other relations and functions.

Example 9.3.22 Determine if each relation represented algebraically is a quadratic function.

a. $y + 5x^2 - 4 = 0$
b. $x^2 + y^2 = 9$
c. $y = -5x + 1$
d. $y = (x - 6)^2 + 3$
e. $y = \sqrt{x + 1} + 5$

Explanation.

a. As $y + 5x^2 - 4 = 0$ can be re-written as $y = -5x^2 + 4$, this equation represents a quadratic function.

b. The equation $x^2 + y^2 = 9$ cannot be re-written in the form $y = ax^2 + bx + c$ (due to the y^2 term), so this equation does not represent a quadratic function.

c. The equation $y = -5x + 1$ represents a linear function, not a quadratic function.

d. The equation $y = (x - 6)^2 + 3$ can be re-written as $y = x^2 - 12x + 39$, so this does represent a quadratic function.

e. The equation $y = \sqrt{x + 1} + 5$ does not represent a quadratic function as x is inside a radical, not squared.

Example 9.3.23 Determine if each function represented graphically *could* represent a quadratic function.

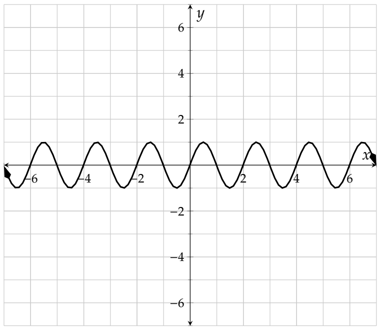


Figure 9.3.24: Function a.

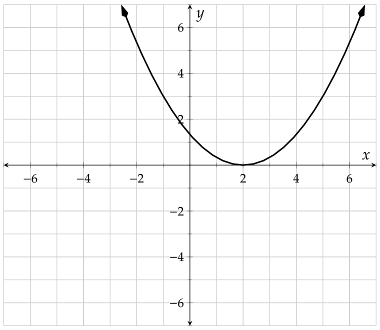


Figure 9.3.25: Function b.

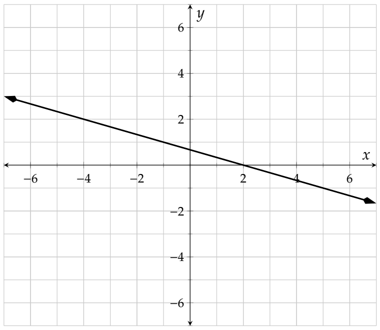


Figure 9.3.26: Function c.

Explanation.

a. Since this graph has multiple maximum points and minimum points, it is not a parabola and it is not possible that it represents a quadratic function.

b. This graph looks like a parabola, and it's possible that it represents a quadratic function.

c. This graph does not appear to be a parabola, but looks like a straight line. It's not likely that it represents a quadratic function.

Exercises

Review and Warmup Solve the equation.

1. $x^2 + 14x + 48 = 0$

2. $x^2 + 11x + 28 = 0$

3. $x^2 - 20x + 100 = 0$

4. $x^2 - 22x + 121 = 0$

5. $x^2 - 100 = 0$

6. $x^2 - 64 = 0$

7. $41x^2 - 11 = 0$

8. $13x^2 - 17 = 0$

9. $7x^2 - 10x + 1 = 0$

10. $7x^2 + 10x + 1 = 0$

11. $4x^2 - 5x + 6 = 0$

12. $3x^2 - 3x + 5 = 0$

Finding the Intercepts of Quadratic Functions Algebraically

13. Find the y-intercept and any x-intercept(s) of the quadratic function $y = x^2 - 5x + 4$.

y-intercept: ☐

x-intercept(s): ☐

14. Find the y-intercept and any x-intercept(s) of the quadratic function $y = -x^2 + 2x + 15$.

y-intercept: ☐

x-intercept(s): ☐

15. Find the y-intercept and any x-intercept(s) of the quadratic function $y = x^2 - 16$.

y-intercept: ☐

x-intercept(s): ☐

17. Find the y-intercept and any x-intercept(s) of the quadratic function $y = x^2 + 2x$.

y-intercept: ☐

x-intercept(s): ☐

19. Find the y-intercept and any x-intercept(s) of the quadratic function $y = x^2 + 8x + 16$.

y-intercept: ☐

x-intercept(s): ☐

21. Find the y-intercept and any x-intercept(s) of the quadratic function $y = x^2 + 4x + 8$.

y-intercept: ☐

x-intercept(s): ☐

23. Find the y-intercept and any x-intercept(s) of the quadratic function $y = x^2 + x + 10$.

y-intercept: ☐

x-intercept(s): ☐

25. Find the y-intercept and any x-intercept(s) of the quadratic function $y = x^2 - 2x - 6$.

y-intercept: ☐

x-intercept(s): ☐

16. Find the y-intercept and any x-intercept(s) of the quadratic function $y = -x^2 + 9$.

y-intercept: ☐

x-intercept(s): ☐

18. Find the y-intercept and any x-intercept(s) of the quadratic function $y = -x^2 - x$.

y-intercept: ☐

x-intercept(s): ☐

20. Find the y-intercept and any x-intercept(s) of the quadratic function $y = x^2 + x + 7$.

y-intercept: ☐

x-intercept(s): ☐

22. Find the y-intercept and any x-intercept(s) of the quadratic function $y = x^2 + 3x + 9$.

y-intercept: ☐

x-intercept(s): ☐

24. Find the y-intercept and any x-intercept(s) of the quadratic function $y = x^2 + 5x - 9$.

y-intercept: ☐

x-intercept(s): ☐

26. Find the y-intercept and any x-intercept(s) of the quadratic function $y = x^2 - 10x - 4$.

y-intercept: ☐

x-intercept(s): ☐

27. Find the y-intercept and any x-intercept(s) of the quadratic function $y = x^2 + 4x - 2$.

y-intercept:

x-intercept(s):

28. Find the y-intercept and any x-intercept(s) of the parabola with equation $y = 4x^2 - 17x + 18$.

y-intercept:

x-intercept(s):

29. Find the y-intercept and any x-intercept(s) of the parabola with equation $y = 16x^2 - 8x + 1$.

y-intercept:

x-intercept(s):

30. Find the y-intercept and any x-intercept(s) of the parabola with equation $y = 25x^2 - 49$.

y-intercept:

x-intercept(s):

31. Find the y-intercept and any x-intercept(s) of the parabola with equation $y = -x + 4 - 5x^2$.

y-intercept:

x-intercept(s):

32. Find the y-intercept and any x-intercept(s) of the parabola with equation $y = -6x - 5x^2$.

y-intercept:

x-intercept(s):

Sketching Graphs of Quadratic Functions Graph each curve by algebraically determining its key features.

33. $y = x^2 - 7x + 12$

34. $y = x^2 + 5x - 14$

35. $y = -x^2 - x + 20$

36. $y = -x^2 + 4x + 21$

37. $y = x^2 - 8x + 16$

38. $y = x^2 + 6x + 9$

39. $y = x^2 - 4$

40. $y = x^2 - 9$

41. $y = x^2 + 6x$

42. $y = x^2 - 8x$

43. $y = -x^2 + 5x$

44. $y = -x^2 + 16$

45. $y = x^2 + 4x + 7$

46. $y = x^2 - 2x + 6$

47. $y = x^2 + 2x - 5$

48. $y = x^2 - 6x + 2$

49. $y = -x^2 + 4x - 1$

50. $y = -x^2 - x + 3$

51. $y = 2x^2 - 4x - 30$

52. $y = 3x^2 + 21x + 36$

Applications of Quadratic Functions

53. An object was shot up into the air at an initial vertical speed of 320 feet per second. Its height as time passes can be modeled by the quadratic function f, where $f(t) = -16t^2 + 320t$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height in feet.

a. After ☐, this object reached its maximum height of ☐.

b. This object flew for ☐ before it landed on the ground.

c. This object was ☐ in the air 12 s after its release.

d. This object was 1584 ft high at two times: once ☐ after its release, and again later ☐ after its release.

54. An object was shot up into the air at an initial vertical speed of 384 feet per second. Its height as time passes can be modeled by the quadratic function f, where $f(t) = -16t^2 + 384t$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height in feet.

a. After ☐, this object reached its maximum height of ☐.

b. This object flew for ☐ before it landed on the ground.

c. This object was ☐ in the air 7 s after its release.

d. This object was 1520 ft high at two times: once ☐ after its release, and again later ☐ after its release.

55. From a clifftop over the ocean 200 m above sea level, an object was shot into the air with an initial vertical speed of 156.8 $\frac{\text{m}}{\text{s}}$. On its way down it fell into the ocean. Its height (above sea level) as time passes can be modeled by the quadratic function f, where $f(t) = -4.9t^2 + 156.8t + 200$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height (above sea level) in meters.

a. After ☐, this object reached its maximum height of ☐.

b. This object flew for ☐ before it landed in the ocean.

c. This object was ☐ above sea level 26 s after its release.

d. This object was 748.8 m above sea level twice: once ☐ after its release, and again later ☐ after its release.

56. From a clifftop over the ocean 160 m above sea level, an object was shot into the air with an initial vertical speed of 176.4 $\frac{\text{m}}{\text{s}}$. On its way down it fell into the ocean. Its height (above sea level) as time passes can be modeled by the quadratic function f, where $f(t) = -4.9t^2 + 176.4t + 160$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height (above sea level) in meters.

a. After ______, this object reached its maximum height of ______.

b. This object flew for ______ before it landed in the ocean.

c. This object was ______ above sea level 8 s after its release.

d. This object was 1742.7 m above sea level twice: once ______ after its release, and again later ______ after its release.

57. A remote control aircraft will perform a stunt by flying toward the ground and then up. Its height can be modeled by the function $h(t) = 1.2t^2 - 16.8t + 55.8$. The plane (☐ will ☐ will not) hit the ground during this stunt.

58. A remote control aircraft will perform a stunt by flying toward the ground and then up. Its height can be modeled by the function $h(t) = 0.1t^2 - 1.6t + 10.4$. The plane (☐ will ☐ will not) hit the ground during this stunt.

59. A submarine is traveling in the sea. Its depth can be modeled by $d(t) = -0.9t^2 + 16.2t - 72.9$, where t stands for time in seconds. The submarine (☐ will ☐ will not) hit the sea surface along this route.

60. A submarine is traveling in the sea. Its depth can be modeled by $d(t) = -1.6t^2 + 28.8t - 133.6$, where t stands for time in seconds. The submarine (☐ will ☐ will not) hit the sea surface along this route.

61. An object is launched upward at the height of 400 meters. It's height can be modeled by

$$h = -4.9t^2 + 90t + 400,$$

where h stands for the object's height in meters, and t stands for time passed in seconds since its launch. The object's height will be 420 meters twice before it hits the ground. Find how many seconds since the launch would the object's height be 420 meters. Round your answers to two decimal places if needed.

The object's height would be 420 meters the first time at ______ seconds, and then the second time at ______ seconds.

62. An object is launched upward at the height of 210 meters. It's height can be modeled by

$$h = -4.9t^2 + 70t + 210,$$

where h stands for the object's height in meters, and t stands for time passed in seconds since its launch. The object's height will be 250 meters twice before it hits the ground. Find how many seconds since the launch would the object's height be 250 meters. Round your answers to two decimal places if needed.

The object's height would be 250 meters the first time at ☐ seconds, and then the second time at ☐ seconds.

63. Currently, an artist can sell 220 paintings every year at the price of \$60.00 per painting. Each time he raises the price per painting by \$5.00, he sells 5 fewer paintings every year.

Assume he will raise the price per painting x times, then he will sell $220 - 5x$ paintings every year at the price of $60 + 5x$ dollars. His yearly income can be modeled by the equation:

$$i = (60 + 5x)(220 - 5x)$$

where i stands for his yearly income in dollars. If the artist wants to earn \$18,375.00 per year from selling paintings, what new price should he set?

To earn \$18,375.00 per year, the artist could sell his paintings at two different prices. The lower price is ☐ per painting, and the higher price is ☐ per painting.

64. Currently, an artist can sell 250 paintings every year at the price of \$90.00 per painting. Each time he raises the price per painting by \$15.00, he sells 5 fewer paintings every year.

Assume he will raise the price per painting x times, then he will sell $250 - 5x$ paintings every year at the price of $90 + 15x$ dollars. His yearly income can be modeled by the equation:

$$i = (90 + 15x)(250 - 5x)$$

where i stands for his yearly income in dollars. If the artist wants to earn \$37,125.00 per year from selling paintings, what new price should he set?

To earn \$37,125.00 per year, the artist could sell his paintings at two different prices. The lower price is ☐ per painting, and the higher price is ☐ per painting.

Challenge

65. Consider the function $f(x) = x^2 + nx + p$. Let n and p be real numbers. Give your answers as points.

a. Suppose the function has two real x-intercepts. What are they?

b. What is its y-intercept?

c. What is its vertex?

9.4 Graphs of Quadratic Functions Chapter Review

9.4.1 Introduction to Functions

In Section 9.1 we covered the definitions of a relation, a function, and domain and range. We then discussed function notation and how to evaluate functions at a particular value as well as how to solve an equation with function notation. Last we introduced the vertical line test

Example 9.4.1 Introduction to Functions. One week in Portland, it was rainy. Shocking, I know. Shown is a diagram for the day of the week and how much rain fell on those days. Convert this diagram to a set of ordered pairs where the first coordinate is the day of the week and the second coordinate is the amount of rain that fell in Portland that day, in inches.

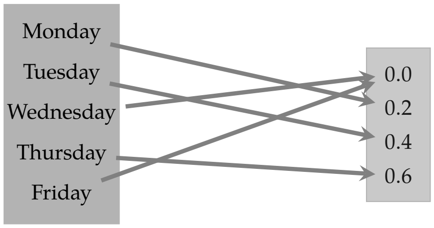


Figure 9.4.2: Diagram for the relation "Day of the week to rainfall in PDX"

Explanation. Since Monday has an arrow to 0.2, that must mean that it rained 0.2 inches of rain on Monday. That gives us the ordered pair (Monday, 0.2). Likewise, we also have (Tuesday, 0.4). To write down all of the ordered pairs, we will use a set:

$$\{(\text{Monday}, 0.2), (\text{Tuesday}, 0.4), (\text{Wednesday}, 0.0), (\text{Thursday}, 0.6), (\text{Friday}, 0.0)\}$$

Example 9.4.3 Functions as Predictors. Recall that a a function should predict the output perfectly if you know the input. In Example 9.4.1, we saw that the day of a particular week in Portland was related to the rainfall during that week. Was this relation a function?

Explanation. Yes, that relation was a function. If I tell you the day of the week, you can with certainty "predict" how much rain there was on that day by looking at the data. There was a single answer to the question "How much rain fell on Wednesday during that week in Portland?"

At this point, we should note that all historical weather data can be viewed as a function of some sort: On any particular day in the recent past, we can "predict" (by looking it up somewhere) how much rain fell in any city in the world.

However, the opposite question, "If it rained 0.0 inches, what day of the year was it?" is not going to represent a function because there will be more than one day of the year that it didn't rain at all. Even in Portland.

Example 9.4.4 Algebraic Functions.

a. For the equation $y = 2x^2 + x$, will y be a function of x?
b. For the equation $y^2 = x$, will y be a function of x?

Explanation.

a. This question is equivalent to the question “If you input any x value, will there be only one y value?” The answer is “yes.” If you input any number for x you first square that number and multiply it by two, then add the number. For example, if we substitute -3, we get:

$$\begin{aligned} y &= 2(x)^2 + (x) \\ y &= 2(-3)^2 + (-3) \\ &= 2(9) - 3 \\ &= 18 - 3 \\ &= 15 \end{aligned}$$

There is no way we could substitute a number for x and get more than one value for y.

b. This question is equivalent to the question “If you input any x value, will there be only one y value?” The answer is “no.” If you input any number for x and solve for y, you might actually get two solutions. For example, if we substitute 1, we get:

$$\begin{aligned} y^2 &= x \\ y^2 &= 1 \end{aligned}$$

$$\begin{array}{ccc} y = \sqrt{1} & \text{or} & y = -\sqrt{1} \\ y = 1 & \text{or} & y = -1 \end{array}$$

Since there is more than one y value for the x value 1, the relation $y^2 = x$ is *not* a function.

Example 9.4.5 Function Notation. For the function $V = g(x)$, what is the name of the function, what is the input variable, and what is the output variable?

Explanation. For the function $V = g(x)$, the name of the function is g, the input variable is x, and the output variable is V.

Example 9.4.6 Evaluating Functions.

Let $a(x) = x^2-2x$, let the function b be defined by the graph in Figure 9.4.7, and let the function c be defined by the table in Table 9.4.8.

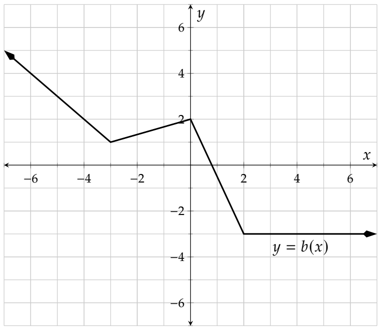


Figure 9.4.7: A graph of the function b

x	$c(x)$
-5	7
-3	-2
2	2
6	4
9	7

Table 9.4.8: The values of the function c

a. Evaluate $a(-3)$.

b. Evaluate $b(-3)$.

c. Evaluate $c(-3)$.

Explanation.

a. To evaluate $a(-3)$ we will substitute the value -3 in wherever we see x in the equation for $a(x)$.

$$\begin{aligned} a(x) &= x^2 - 2x \\ a(-3) &= (-3)^2 - 2(-3) \\ &= 9 + 6 \\ &= 15 \end{aligned}$$

b. To evaluate $b(-3)$, we need to look at the graph of $y = b(x)$ and look for the one place on the graph where the x-value is -3. The y-value at this point is 1, so we would say that

$$b(-3) = 1$$

c. To evaluate $c(-3)$ we examine the table and find the place where the x-value is -3. We conclude that $c(-3) = -2$.

Example 9.4.9 Solving Equations That Have Function Notation.

Still let $a(x) = x^2 - 2x$, let the function b be defined by the graph in Figure 9.4.10, and let the function c be defined by the table in Table 9.4.11.

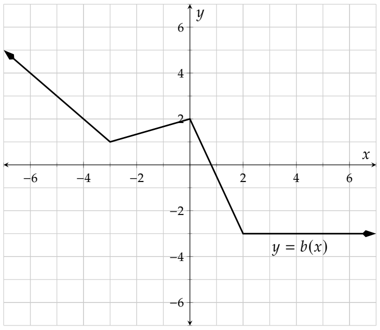


Figure 9.4.10: A graph of the function b

x	$c(x)$
−5	7
−3	−2
2	2
6	4
9	7

Table 9.4.11: The values of the function c

a. Solve the equation $a(x) = 0$.

b. Solve the equation $b(x) = 2$.

c. Solve the equation $c(x) = 7$.

Explanation.

a. To solve the equation $a(x) = 0$, we should set the formula for $a(x)$ equal to 0. In this case, factoring will then help.

$$\begin{aligned} x^2 - 2x &= 0 \\ x(x-2) &= 0 \end{aligned}$$

$$\begin{array}{ccc} x = 0 & \text{or} & x - 2 = 0 \\ x = 0 & \text{or} & x = 2 \end{array}$$

So, the solution set is $\{0, 2\}$.

b. To solve the equation $b(x) = 2$, we should examine the graph and find all of the locations where the y-value is 2. According to the graph, that happens twice, both when $x = -4$ and when $x = 0$. So the solution set is $\{-4, 0\}$.

c. To solve the equation $c(x) = 7$, we examine the table and find all the places where the function value is 7. According to the table, that happens twice: once when $x = -5$ and again when $x = 9$. So the solution set is $\{-5, 9\}$.

Example 9.4.12 Domain and Range.

Still let the function b be defined by the graph in Figure 9.4.13, and let the function c be defined by the table in Table 9.4.14.

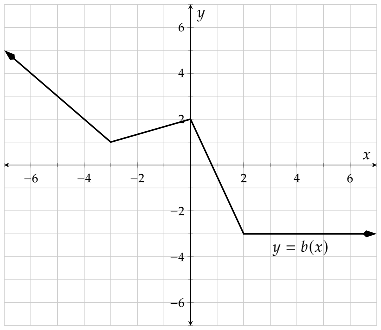


Figure 9.4.13: A graph of the function b

x	$c(x)$
−5	7
−3	−2
2	2
6	4
9	7

Table 9.4.14: The values of the function c

a. Write the domain of b in interval notation.

b. Write the domain of c in interval notation.

c. Write the range of b in interval notation.

d. Write the range of c in interval notation.

Explanation. Recall the definitions of domain and range.

a. To find the the domain of b, look at the graph of b and see what x-values are used for the graph. In other words, what x-values could you input and expect to get a y-value? It looks like the graph extends forever to the right as well as the left. Therefore, the domain is $(-\infty, \infty)$.

b. To find the the domain of c, we list the x-values in the table in a set. That would be $\{-5, -3, 2, 6, 9\}$.

c. To find the the range of b, look at the graph of b and see what y-values are on the graph. The lowest y-value is -3 and all higher y-values are possible outputs. Therefore, the range is $[-3, \infty)$.

d. To find the the range of c, we list the y-values in the table in a set. That would be $\{7, -2, 2, 4\}$.

Example 9.4.15 Determining if a Relation is a Function. Is y a function of x in the following graphs?

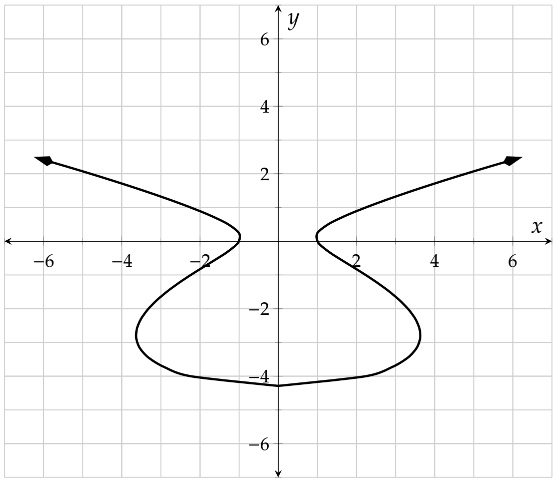


Figure 9.4.16: Is y a function of x?

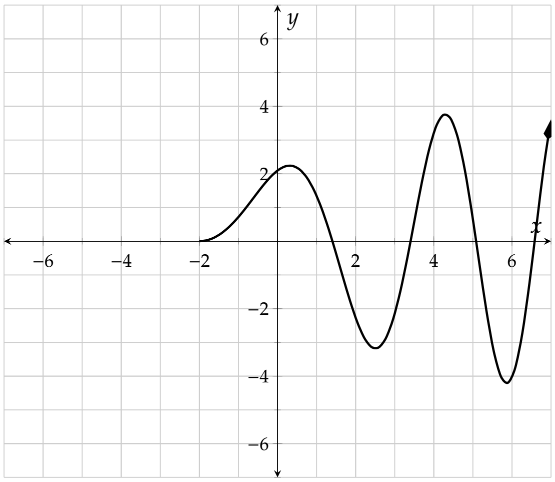


Figure 9.4.17: Is y a function of x?

Explanation. The graph in Figure 9.4.16 fails the vertical line test and so y is not a function of x. The graph in Figure 9.4.17 passes the vertical line test and so y is a function of x.

9.4.2 Properties of Quadratic Functions

In Section 9.2 we covered the definition of a quadratic function, and how to determine the direction, vertex, axis of symmetry, and vertical and horizontal intercepts. We then learned how to algebraically find the vertex, how to graph a parabola using a table, and how to find the domain and range of quadratic functions.

Example 9.4.18 Finding the Vertex and Axis of Symmetry. Algebraically find the vertex of the parabola described by the quadratic function $f(x) = 3x^2 + 8x - 7$.

Explanation. To find the vertex of a parabola algebraically, we use the formula $h = -\frac{b}{2a}$ to find the axis of symmetry first. For our equation, $a = 3$ and $b = 8$, so:

$$h = -\frac{8}{2(3)} = -\frac{4}{3}$$

Next, we evaluate $f\left(-\frac{4}{3}\right)$ to find the y-coordinate of the vertex.

$$f\left(-\frac{4}{3}\right) = 3\left(-\frac{4}{3}\right)^2 + 8\left(-\frac{4}{3}\right) - 7$$

$$
\begin{aligned}
&= 3\left(\frac{16}{9}\right) - \frac{32}{3} - 7\\
&= \frac{16}{3} - \frac{32}{3} - \frac{21}{3}\\
&= \frac{37}{3}
\end{aligned}
$$

So, the parabola has its axis of symmetry at $x = -\frac{4}{3}$ and has its vertex at the point $\left(-\frac{4}{3}, -\frac{37}{3}\right)$.

Example 9.4.19 Properties of Quadratic Functions.

Identify the key features of the quadratic function $y = -2x^2 - 4x + 6$ shown in Figure 9.4.20.

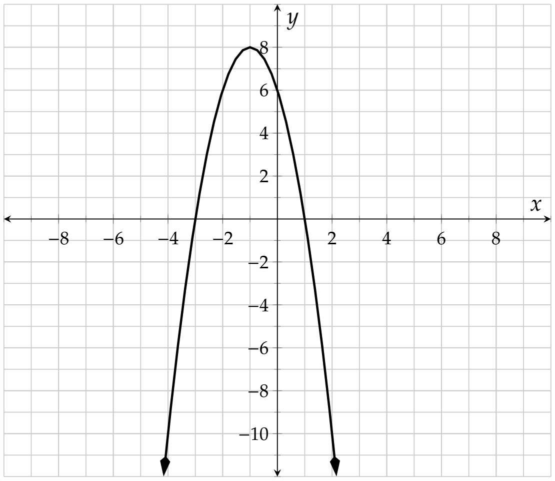


Figure 9.4.20: Graph of $y = -2x^2 - 4x + 6$

Explanation.

First, we see that this parabola opens downward because the leading coefficient is negative.

Then we locate the vertex which is the point $(-1, 8)$. The axis of symmetry is the vertical line $x = -1$.

The vertical intercept or y-intercept is the point $(0, 6)$.

The horizontal intercepts are the points $(-3, 0)$ and $(1, 0)$.

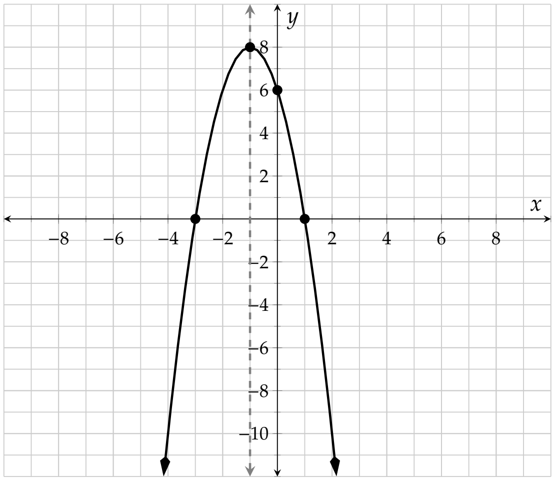


Figure 9.4.21: Graph of $y = -2x^2 - 4x + 6$

Example 9.4.22 Graphing Quadratic Functions by Making a Table. Determine the vertex and axis of symmetry for the quadratic function $g(x) = 2x^2 + 8x + 2$. Then make a table of values and sketch the graph of the function.

Explanation. To determine the vertex of $g(x) = 2x^2 + 8x + 2$, we want to find the x-value of the vertex first. We will use $h = -\frac{b}{2a}$ for $a = 2$ and $b = 8$:

$$\begin{aligned} h &= -\frac{b}{2a} \\ h &= -\frac{8}{2(2)} \\ h &= -2 \end{aligned}$$

To find the vertex, we evaluate $g(-2)$.

$$\begin{aligned} g(-2) &= 2(-2)^2 + 8(-2) + 2 \\ &= 2(4) - 16 + 2 \\ &= 8 - 16 + 2 \\ &= -6 \end{aligned}$$

Now we know that our axis of symmetry is the line $x = -2$ and the vertex is the point $(-2, -6)$. We will set up our table with two values on each side of $x = -2$. We choose $x = -4, -3, -2, -1$, and 0. Then, we'll determine the y-coordinates by replacing x with each value and we have the complete table as shown in Table 9.4.23. Notice that each pair of y-values on either side of the vertex match. This helps us to check that our vertex and y-values are correct.

x	$g(x) = 2x^2 + 8x + 2$	Point
-4	$g(-4) = 2(-4)^2 + 8(-4) + 2 = 2$	$(-4, 2)$
-3	$g(-3) = 2(-3)^2 + 8(-3) + 2 = -4$	$(-3, -4)$
-2	$g(-2) = 2(-2)^2 + 8(-2) + 2 = -6$	$(-2, -6)$
-1	$g(-1) = 2(-1)^2 + 8(-1) + 2 = -4$	$(-1, -4)$
0	$g(0) = 2(0)^2 + 8(0) + 2 = 2$	$(0, 2)$

Table 9.4.23: Function values and points for $g(x) = 2x^2 + 8x + 2$

Now that we have our table, we will plot the points and draw in the axis of symmetry as shown in Figure 9.4.24. We complete the graph by drawing a smooth curve through the points and drawing an arrow on each end as shown in Figure 9.4.25

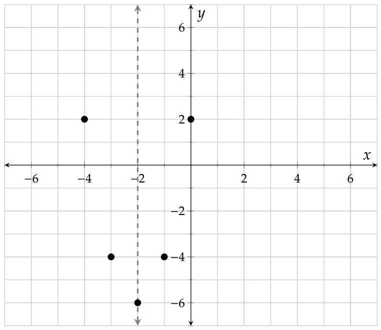


Figure 9.4.24: Plot of the points and axis of symmetry

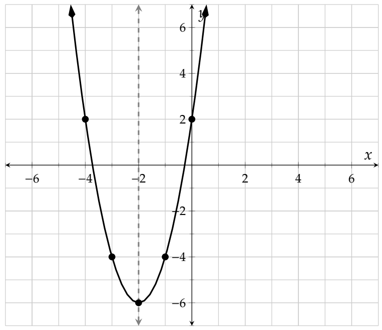


Figure 9.4.25: Graph of $g(x) = 2x^2 + 8x + 2$

Example 9.4.26 The Domain and Range of Quadratic Functions. Use the graph of $g(x) = 2x^2 + 8x + 2$ in from Example 9.4.22 shown in Figure 9.4.25 to identify the domain and range of g in set-builder and interval notation.

Explanation. For the domain, we look horizontally and see the graph is a continuous curve and one arrow points to the left and the other arrow points to the right. The domain is $\{x \mid x \text{ is a real number}\}$ which is equivalent to $(-\infty, \infty)$.

For the range we look up and down on the graph, which opens upward. Both arrows point upward and the lowest point on the graph is the vertex at the point $(-2, -6)$. The range is $\{y \mid y \geq -6\}$ which is equivalent to $[-6, \infty)$.

Example 9.4.27 Applications of Quadratic Functions Involving the Vertex. The value of FedEx stock, in dollars, between December 28, 2017 and February 9, 2018 can be very closely approximated by a quadratic function, $F(x) = -0.07x^2 + 2.7x + 248$. Find the vertex of this parabola and interpret it in the context of the situation.

Explanation. To find the vertex of $F(x) = -0.07x^2 + 2.7x + 248$, we first find the axis of symmetry using the formula $h = -\frac{b}{2a}$. For our equation, $a = -0.07$ and $b = 2.7$, so:

$$\begin{aligned} h &= -\frac{2.7}{2(-0.07)} \\ h &= -\frac{2.7}{-0.14} \\ h &\approx 19.3 \end{aligned}$$

Next, we substitute the value of the axis of symmetry, 19.3, into the formula for the parabola.

$$\begin{aligned} F(x) &= -0.07x^2 + 2.7x + 248 \\ F(19.3) &= -0.07\,(19.3)^2 + 2.7\,(19.3) + 248 \\ &\approx 274 \end{aligned}$$

So, the parabola has its axis of symmetry at $x = 19.3$ and has its vertex at about the point $(19.3, 274)$. In addition, since $a = -0.07$, the parabola will be opening downward. Given this information, the vertex of this parabola is its maximum.

In the reality of the situation, the 19.3 indicates the number of days after December 28, 2017 which would be sometime on January 16. The 274 indicates the stock price in dollars. In conclusion, FedEx's stock peaked (between December 28, 2017 and February 9, 2018) on January 16 at a value of \$274.

9.4.3 Graphing Quadratic Functions

In Section 9.3 we covered how to find the vertical and horizontal intercepts of a quadratic function algebraically, how to make a graph of a parabola using key features, applications of quadratic functions and what their real world domains and ranges mean, and how to tell a quadratic function from other types of functions.

Example 9.4.28 Finding the Vertical and Horizontal Intercepts Algebraically.

a. Algebraically determine the vertical and horizontal intercepts of the quadratic function $h(x) = 6x^2 - 13x + 6$.

b. Algebraically determine the vertical and horizontal intercepts of the quadratic function $k(x) = 2x^2 - 2x - 5$.

Explanation.

a. To find the vertical intercept, we evaluate the function when $x = 0$.

$$\begin{aligned} h(0) &= 6(0)^2 - 13(0) + 6 \\ &= 6 \end{aligned}$$

So, the vertical intercept of h is $(0, 6)$.

Next, to find the horizontal intercepts, we set the function equal to zero. To solve that equation we will use Algorithm 8.6.2. For this particular example, we will practice factoring using the AC method. Here, $ac = 36$ and factor pairs that add up to -13 are -9 and -4, as you will see.

$$\begin{aligned} h(x) &= 6x^2 - 13x + 6 \\ 0 &= 6x^2 - 13x + 6 \\ 0 &= 6x^2 - 9x - 4x + 6 \\ 0 &= 3x(2x - 3) - 2(2x - 3) \\ 0 &= (3x - 2)(2x - 3) \end{aligned}$$

$$\begin{aligned} 0 &= 3x - 2 && \text{or} && 0 = 2x - 3 \\ x &= \frac{2}{3} && \text{or} && x = \frac{3}{2} \end{aligned}$$

So, the horizontal intercepts are $\left(\frac{2}{3}, 0\right)$ and $\left(\frac{3}{2}, 0\right)$.

b. To find the vertical intercept, we have to evaluate the function when $x = 0$.

$$\begin{aligned} k(0) &= 2(0)^2 - 2(0) - 5 \\ &= -5 \end{aligned}$$

So, the vertical intercept of k is $(0, -5)$.

Next, to find the horizontal intercepts, we set the function equal to zero. To solve that equation we will use Algorithm 8.6.2. For this particular example, we will use the quadratic formula because the square root method will not work (because there is a linear term) and factoring fails.

$$\begin{aligned} k(x) &= 2x^2 - 2x - 5 \\ 0 &= 2x^2 - 2x - 5 \end{aligned}$$

We identify that $a = 2$, $b = -2$, and $c = -5$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-5)}}{2(2)} \\ x &= \frac{2 \pm \sqrt{4 + 40}}{4} \end{aligned}$$

$$\begin{aligned}x &= \frac{2 \pm \sqrt{44}}{4}\\ x &= \frac{2 \pm \sqrt{4 \cdot 11}}{4}\\ x &= \frac{2 \pm 2\sqrt{11}}{4}\\ x &= \frac{2}{4} \pm \frac{2\sqrt{11}}{4}\\ x &= \frac{1}{2} \pm \frac{\sqrt{11}}{2}\\ x &= \frac{1 \pm \sqrt{11}}{2}\end{aligned}$$

So, the horizontal intercepts are $\left(\frac{1+\sqrt{11}}{2}, 0\right)$ and $\left(\frac{1-\sqrt{11}}{2}, 0\right)$. If you wanted to graph these points, you would need to approximate them as $(2.16, 0)$ and $(-1.16, 0)$.

Example 9.4.29 Graphing Quadratic Functions Using their Key Features. Graph the function $j(x) = -2x^2 + 6x + 8$ by algebraically determining its key features.

Explanation. To start, we'll note that this function will open downward, as the leading coefficient is negative.

To find the y-intercept, we'll evaluate $j(0)$:

$$\begin{aligned}j(0) &= -2(0)^2 + 6(0) + 8\\ &= 8\end{aligned}$$

The y-intercept is $(0, 8)$.

Next, we'll find the horizontal intercepts by setting $j(x) = 0$ and solving for x:

$$\begin{aligned}-2x^2 + 6x + 8 &= 0\\ -2(x^2 - 3x - 4) &= 0\\ 2(x-4)(x+1) &= 0\end{aligned}$$

$$\begin{aligned}x - 4 &= 0 \qquad \text{or} \qquad & x + 1 &= 0\\ x &= 4 \qquad \text{or} \qquad & x &= -1\end{aligned}$$

The x-intercepts are $(4, 0)$ and $(-1, 0)$.

Lastly, we'll determine the vertex. Noting that $a = -2$ and $b = 6$, we have:

$$\begin{aligned}x &= -\frac{b}{2a}\\ x &= -\frac{6}{2(-2)}\\ &= 1.5\end{aligned}$$

Using the x-value 1.5 to find the y-coordinate, we have:

$$\begin{aligned} j(1.5) &= -2(1.5)^2 + 6(1.5) + 8 \\ &= 12.5 \end{aligned}$$

The vertex is the point $(1.5, 12.5)$, and the axis of symmetry is the line $x = 1.5$.

We're now ready to graph this function. We'll start by drawing and scaling the axes so all of our key features will be displayed as shown in Figure 9.4.30. Next, we'll plot these key points as shown in Figure 9.4.31. Finally, we'll note that this parabola opens downward and connect these points with a smooth curve, as shown in Figure 9.4.32.

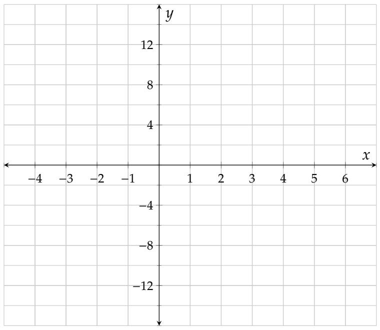


Figure 9.4.30: Setting up the grid.

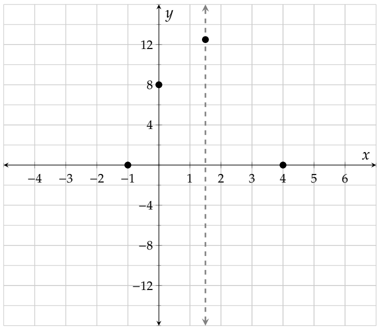


Figure 9.4.31: Marking key features.

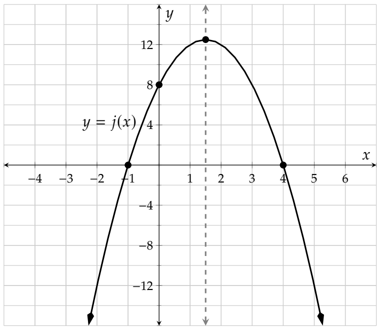


Figure 9.4.32: Completing the graph.

Example 9.4.33 Applications of Quadratic Functions. The Five-hundred-meter Aperture Spherical Telescope[a] (FAST) located in SW China is the worlds largest radio telescope. The name is actually incorrect because it is not spherical: it is parabolic! Parabolic telescopic dishes are very common because parabolas have the unique feature that they focus light coming in at a single point where a collecting instrument is placed.

If a scientist somehow managed to climb to the rim of the dish and roll a ball from the top down toward the center, her ball would travel along the parabola $h(x) = 0.0018x^2 - 0.9x$, where x is the horizontal distance from the rim of the dish where the ball was released, in meters, and $h(x)$ is the height of the ball above above the rim, in meters (so negative height would mean *below* the rim). Note that this is an approximation based on real data.

a. Find and interpret the vertical intercept of the graph.

b. Find and interpret the horizontal intercepts of the graph.

c. Find and interpret the vertex of the graph.

Explanation.

a. The vertical intercept is found when $x = 0$.

$$\begin{aligned} h(x) &= 0.0018x^2 - 0.9x \\ h(0) &= 0.0018(0)^2 - 0.9(0) \\ &= 0 \end{aligned}$$

So the vertical intercept is $(0,0)$ which means that the ball was released 0 meters from the rim (on the rim) at a height of 0 meters above the rim (again, on the rim).

b. The horizontal intercepts are found when $h(x) = 0$. This will be a quadratic equation that we can solve using factoring.

$$\begin{aligned} h(x) &= 0 \\ 0.0018x^2 - 0.9x &= 0 \\ x\,(0.0018x - 0.9) &= 0 \end{aligned}$$

$$\begin{array}{ccc} x = 0 & \text{or} & 0.0018x - 0.9 = 0 \\ x = 0 & \text{or} & 0.0018x = 0.9 \\ x = 0 & \text{or} & x = 500 \end{array}$$

The horizontal intercepts are $(0,0)$ and $(500,0)$. The interpretation of $(0,0)$ is the same as before. The interpretation of $(500,0)$ is that the other side of the rim of the dish is 500 meters away. If the ball were to somehow roll all the way down, and then back up the other side, it would pop up at the opposite rim exactly 500 meters horizontally away.

c. The vertex is found when the x-value is at the axis of symmetry. To find that, we use the formula $x = -\frac{b}{2a}$ where $a = 0.0018$ and $b = -0.9$

$$\begin{aligned} x &= -\frac{b}{2a} \\ x &= -\frac{-0.9}{2(0.0018)} \\ &= 250 \end{aligned}$$

Then, to find the y-value, we substitute that x-value, 250 into the original function.

$$\begin{aligned} h(x) &= 0.0018x^2 - 0.9x \\ h(250) &= 0.0018(250)^2 - 0.9(250) \\ &= -112.5 \end{aligned}$$

The vertex is the point $(250, -112.5)$. Since the vale of a is positive, we know that this parabola opens upward. This means that the vertex is the lowest point on the graph. So the lowest point of the telescope dish is 250 meters from the rim (horizontally) and 112.5 meters below the height of the rim. That is one big dish!

---

[a] en.wikipedia.org/wiki/Five_hundred_meter_Aperture_Spherical_Telescope

Example 9.4.34 The Domain and Range of Quadratic Applications. In the FAST telescope example, find the real world domain and range of the function h.

Explanation. Real world domain and range problems are sometimes subjective. The domain in this case would represent all possible x-values that make sense in the reality of the situation. Recall that x is the horizontal distance from the rim of the dish where the ball was released, in meters. Since we found out that the dish is 500 meters across we could say that the domain is $[0, 500]$.

The range will represent all possible y values, that make sense in reality. Recall that $h(x)$ is the height above above the rim, in meters. Since the ball won't roll any higher than the rim of the dish, 0 will be the largest y-value. We found out that the lowest point on the graph is -112.5 meters below the rim. So the range is [-112.5,0].

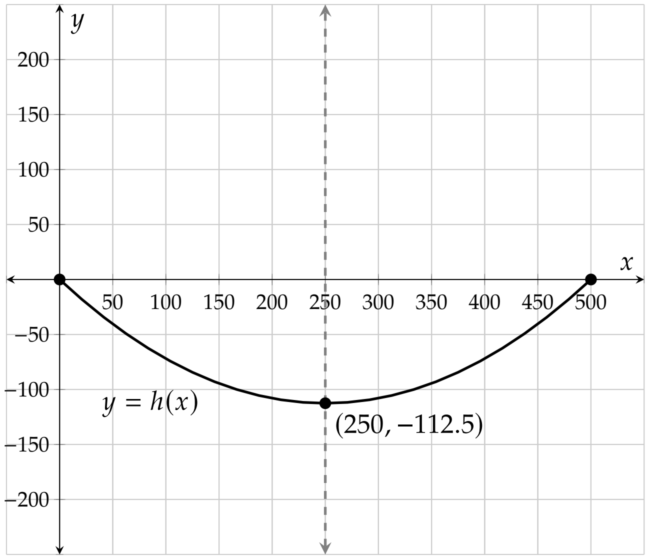


Figure 9.4.35: A diagram of the parabolic function representing the telescope's dish

Example 9.4.36 Distinguishing Quadratic Functions from Other Functions and Relations. Decide if the equations represent quadratic functions or something else.

a. $y-1=3(x-2)-5$ b. $y=3(x-2)^2-5$ c. $y=\sqrt{x-2}-5$ d. $y^2=x^2-5$

Explanation. Recall that Definition 9.2.2 says that a quadratic function has the form $f(x)=ax^2+bx+c$ where a, b, and c are real numbers, and $a \neq 0$.

a. The equation $y-1=3(x-2)-5$ *is not* quadratic. It is in fact a linear equation.

b. The equation $y=3(x-2)^2-5$ *is* quadratic. We could simplify the right hand side and would have something of the form $y=ax^2+bx+c$.

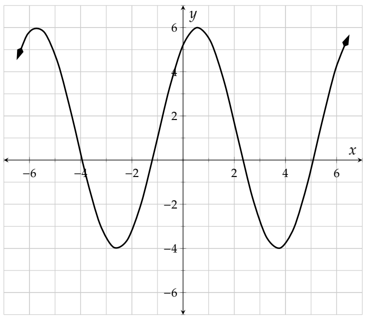
c. The equation $y=\sqrt{x-2}-5$ *is not* quadratic. The x is inside a radical, not squared, so it cannot be converted into the form $y=ax^2+bx+c$.

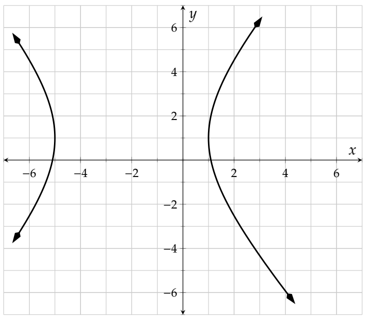
d. The equation $y^2=x^2-5$ *is not* quadratic. It cannot be re-written in the form $y=ax^2+bx+c$ (due to the y^2 term).

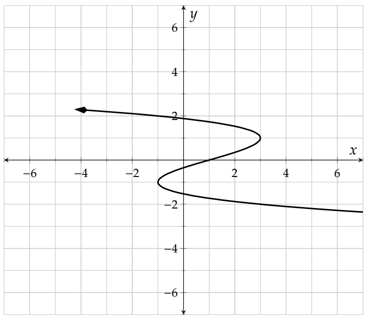
Exercises

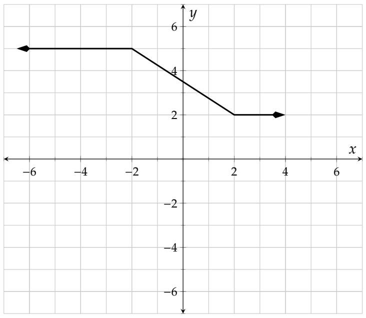
Introduction to Functions

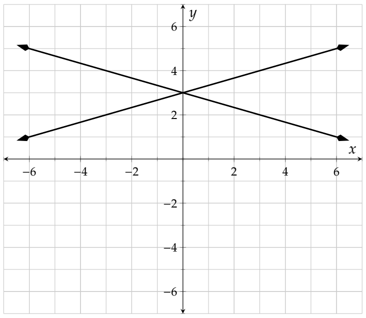
1. Which of the following graphs show y as a function of x?

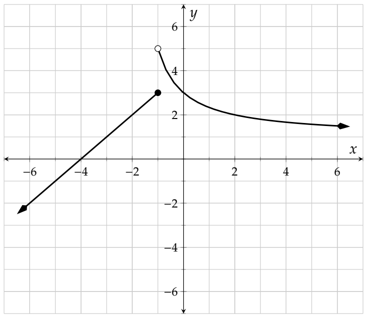




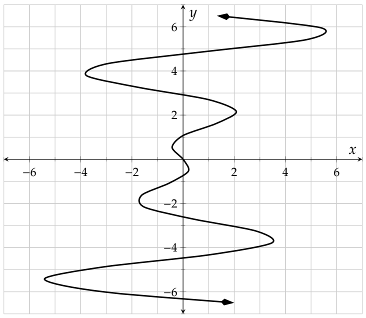


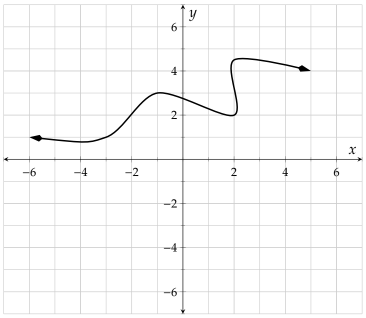


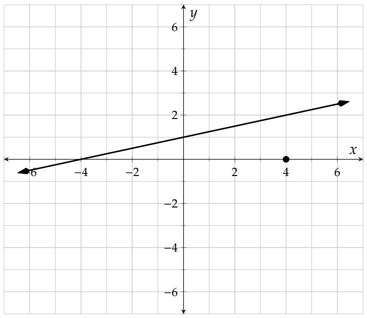


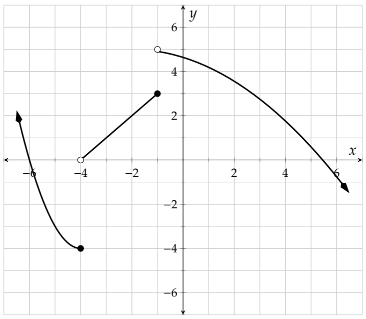


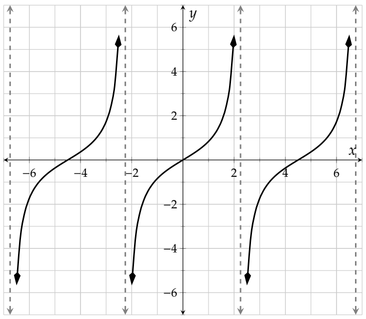
2. Which of the following graphs show y as a function of x?

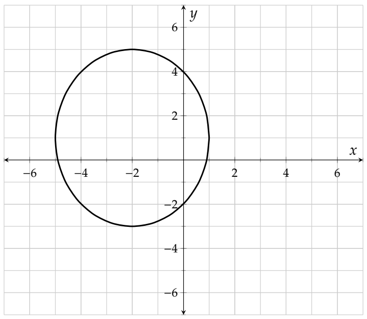












3. Evaluate the function at the given values.

$F(x) = -5x + 4$

a. $F(1) =$

b. $F(-3) =$

c. $F(0) =$

4. Evaluate the function at the given values.

$G(x) = -3x + 8$

a. $G(5) =$

b. $G(-3) =$

c. $G(0) =$

5. Evaluate the function at the given values.

$H(x) = \dfrac{7x}{-6x + 3}$

a. $H(4) =$.

b. $H(-6) =$.

6. Evaluate the function at the given values.

$K(x) = \dfrac{2x}{3x - 3}$

a. $K(4) =$.

b. $K(-1) =$.

7. Evaluate the function at the given values.

$K(x) = -2x^2 - 5x - 6$

a. $K(2) =$

b. $K(-4) =$

8. Evaluate the function at the given values.

$f(x) = -3x^2 + 3x - 1$

a. $f(4) =$

b. $f(-5) =$

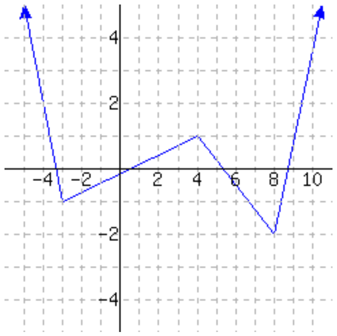
9. Solve for x, where $g(x) = x^2 - 2x - 82$.

If $g(x) = -2$, then $x =$.

10. Solve for x, where $h(x) = x^2 - x - 17$.

If $h(x) = -5$, then $x =$.

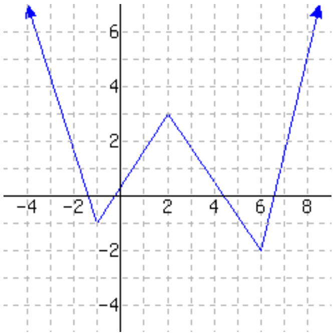
11. A function is graphed.



This function has domain

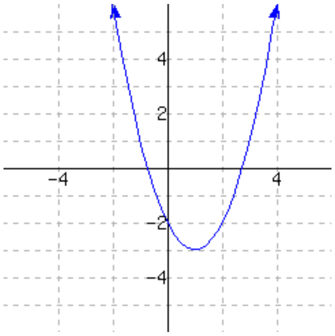
and range .

12. A function is graphed.



This function has domain ☐ and range ☐.

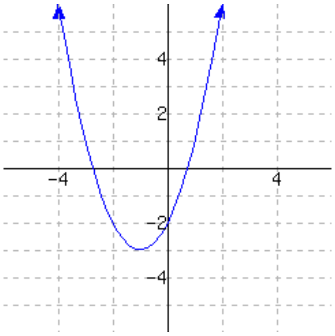
13. Function f is graphed.



a. $f(2) =$ ☐

b. Solve $f(x) = 1$.

14. Function f is graphed.



a. $f(0) =$ ☐

b. Solve $f(x) = 1$.

15. Heather started saving in a piggy bank on her birthday. The function $f(x) = 5x + 3$ models the amount of money, in dollars, in Heather's piggy bank. The independent variable represents the number of days passed since her birthday.

Interpret the meaning of $f(2) = 13$.

- ⊙ *A*. Two days after Heather started her piggy bank, there were $13 in it.
- ⊙ *B*. The piggy bank started with $2 in it, and Heather saves $13 each day.
- ⊙ *C*. Thirteen days after Heather started her piggy bank, there were $2 in it.
- ⊙ *D*. The piggy bank started with $13 in it, and Heather saves $2 each day.

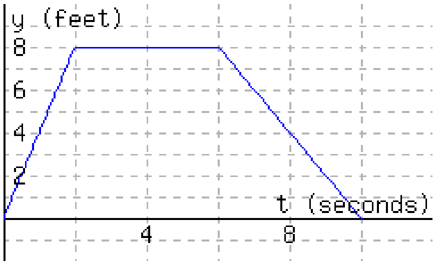
16. Haley started saving in a piggy bank on her birthday. The function $f(x) = 4x + 3$ models the amount of money, in dollars, in Haley's piggy bank. The independent variable represents the number of days passed since her birthday.

Interpret the meaning of $f(4) = 19$.

- ⊙ *A*. Nineteen days after Haley started her piggy bank, there were $4 in it.
- ⊙ *B*. The piggy bank started with $4 in it, and Haley saves $19 each day.
- ⊙ *C*. Four days after Haley started her piggy bank, there were $19 in it.
- ⊙ *D*. The piggy bank started with $19 in it, and Haley saves $4 each day.

The following figure has the graph $y = d(t)$, which models a particle's distance from the starting line in feet, where t stands for time in seconds since timing started.

17.



a. $d(2) =$

b. Interpret the meaning of $d(2)$:

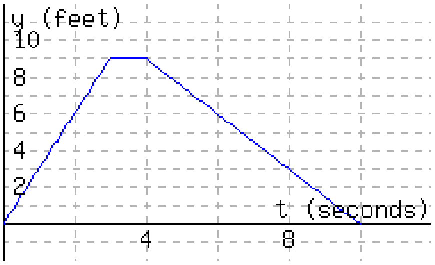
- ⊙ *A.* In the first 8 seconds, the particle moved a total of 2 feet.
- ⊙ *B.* In the first 2 seconds, the particle moved a total of 8 feet.
- ⊙ *C.* The particle was 8 feet away from the starting line 2 seconds since timing started.
- ⊙ *D.* The particle was 2 feet away from the starting line 8 seconds since timing started.

c. Solve $d(t) = 4$ for t. $t =$

d. Interpret the meaning of part c's solution(s):

- ⊙ *A.* The article was 4 feet from the starting line 1 seconds since timing started, or 8 seconds since timing started.
- ⊙ *B.* The article was 4 feet from the starting line 8 seconds since timing started.
- ⊙ *C.* The article was 4 feet from the starting line 1 seconds since timing started.
- ⊙ *D.* The article was 4 feet from the starting line 1 seconds since timing started, and again 8 seconds since timing started.

18.



a. $d(5) =$

b. Interpret the meaning of $d(5)$:

- ⊙ *A.* The particle was 5 feet away from the starting line 7.5 seconds since timing started.
- ⊙ *B.* The particle was 7.5 feet away from the starting line 5 seconds since timing started.
- ⊙ *C.* In the first 7.5 seconds, the particle moved a total of 5 feet.
- ⊙ *D.* In the first 5 seconds, the particle moved a total of 7.5 feet.

c. Solve $d(t) = 3$ for t. $t =$

d. Interpret the meaning of part c's solution(s):

- ⊙ *A.* The article was 3 feet from the starting line 8 seconds since timing started.
- ⊙ *B.* The article was 3 feet from the starting line 1 seconds since timing started, and again 8 seconds since timing started.
- ⊙ *C.* The article was 3 feet from the starting line 1 seconds since timing started, or 8 seconds since timing started.
- ⊙ *D.* The article was 3 feet from the starting line 1 seconds since timing started.

The function C models the the number of customers in a store t hours since the store opened.

19.

t	0	1	2	3	4	5	6	7
$C(t)$	0	41	75	97	97	73	48	0

a. $C(1) =$ ☐

b. Interpret the meaning of $C(1)$:

⊙ *A*. There were 41 customers in the store 1 hour after the store opened.

⊙ *B*. In 1 hour since the store opened, the store had an average of 41 customers per hour.

⊙ *C*. In 1 hour since the store opened, there were a total of 41 customers.

⊙ *D*. There was 1 customer in the store 41 hours after the store opened.

c. Solve $C(t) = 97$ for t. $t =$ ☐

d. Interpret the meaning of Part c's solution(s):

⊙ *A*. There were 97 customers in the store 3 hours after the store opened.

⊙ *B*. There were 97 customers in the store 4 hours after the store opened.

⊙ *C*. There were 97 customers in the store either 4 hours after the store opened, or 3 hours after the store opened.

⊙ *D*. There were 97 customers in the store 4 hours after the store opened, and again 3 hours after the store opened.

20.

t	0	1	2	3	4	5	6	7
$C(t)$	0	43	78	97	95	81	43	0

a. $C(3) =$ ☐

b. Interpret the meaning of $C(3)$:

⊙ *A*. There were 97 customers in the store 3 hours after the store opened.

⊙ *B*. There were 3 customers in the store 97 hours after the store opened.

⊙ *C*. In 3 hours since the store opened, there were a total of 97 customers.

⊙ *D*. In 3 hours since the store opened, the store had an average of 97 customers per hour.

c. Solve $C(t) = 43$ for t. $t =$ ☐

d. Interpret the meaning of Part c's solution(s):

⊙ *A*. There were 43 customers in the store 1 hours after the store opened, and again 6 hours after the store opened.

⊙ *B*. There were 43 customers in the store either 1 hours after the store opened, or 6 hours after the store opened.

⊙ *C*. There were 43 customers in the store 1 hours after the store opened.

⊙ *D*. There were 43 customers in the store 6 hours after the store opened.

Properties of Quadratic Functions

21. Find the axis of symmetry and vertex of the quadratic function.

$y = 4x^2 + 32x - 3$

Axis of symmetry: ☐

Vertex: ☐

22. Find the axis of symmetry and vertex of the quadratic function.

$y = x^2 - 4x + 5$

Axis of symmetry: ☐

Vertex: ☐

23. Find the axis of symmetry and vertex of the quadratic function.

$y = 3x^2 - 15x + 2$

Axis of symmetry: ☐

Vertex: ☐

24. Find the axis of symmetry and vertex of the quadratic function.

$y = 4x^2 + 12x - 4$

Axis of symmetry: ☐

Vertex: ☐

25. For $y = 4x^2 - 8x + 5$, determine the vertex, create a table of ordered pairs, and then graph the function.

26. For $y = 2x^2 + 4x + 7$, determine the vertex, create a table of ordered pairs, and then graph the function.

27. For $y = -x^2 + 4x + 2$, determine the vertex, create a table of ordered pairs, and then graph the function.

28. For $y = -x^2 + 2x - 5$, determine the vertex, create a table of ordered pairs, and then graph the function.

29. You will build a rectangular sheep enclosure next to a river. There is no need to build a fence along the river, so you only need to build on three sides. You have a total of 500 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that runs perpendicular to the river, and write a formula for a function of x that outputs the area of the enclosure. Then find its vertex and interpret it.

The length of the pen (parallel to the river) should be ☐, the width (perpendicular to the river) should be ☐, and the maximum possible area is ☐.

30. You will build a rectangular sheep enclosure next to a river. There is no need to build a fence along the river, so you only need to build on three sides. You have a total of 400 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that runs perpendicular to the river, and write a formula for a function of x that outputs the area of the enclosure. Then find its vertex and interpret it.

The length of the pen (parallel to the river) should be ☐, the width (perpendicular to the river) should be ☐, and the maximum possible area is ☐.

Graphing Quadratic Functions

31. Find the y-intercept and any x-intercept(s) of the quadratic function $y = x^2 + 4x + 3$.

y-intercept: ☐

x-intercept(s): ☐

32. Find the y-intercept and any x-intercept(s) of the quadratic function $y = -x^2 - 3x - 2$.

y-intercept: ☐

x-intercept(s): ☐

33. Find the y-intercept and any x-intercept(s) of the quadratic function $y = x^2 + x + 5$.

y-intercept: ☐

x-intercept(s): ☐

34. Find the y-intercept and any x-intercept(s) of the quadratic function $y = x^2 + 8x + 4$.

y-intercept: ☐

x-intercept(s): ☐

35. Find the y-intercept and any x-intercept(s) of the parabola with equation $y = 16x^2 - 8x + 1$.

y-intercept: ☐

x-intercept(s): ☐

36. Find the y-intercept and any x-intercept(s) of the parabola with equation $y = 25x^2 - 1$.

y-intercept: ☐

x-intercept(s): ☐

Graph each curve by algebraically determining its key features.

37. $y = x^2 - 7x + 12$

38. $y = x^2 + 5x - 14$

39. $y = -x^2 - x + 20$

40. $y = -x^2 + 4x + 21$

41. $y = x^2 + 6x$

42. $y = x^2 - 8x$

43. $y = x^2 + 4x + 7$

44. $y = x^2 - 2x + 6$

45. $y = 2x^2 - 4x - 30$

46. $y = 3x^2 + 21x + 36$

47. From a clifftop over the ocean 170 m above sea level, an object was shot into the air with an initial vertical speed of 264.6 $\frac{\text{m}}{\text{s}}$. On its way down it fell into the ocean. Its height (above sea level) as time passes can be modeled by the quadratic function f, where $f(t) = -4.9t^2 + 264.6t + 170$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height (above sea level) in meters.

a. After ☐, this object reached its maximum height of ☐.

b. This object flew for ☐ before it landed in the ocean.

c. This object was ☐ above sea level 47 s after its release.

d. This object was 3663.7 m above sea level twice: once ☐ after its release, and again later ☐ after its release.

48. From a clifftop over the ocean 130 m above sea level, an object was shot into the air with an initial vertical speed of 294 $\frac{\text{m}}{\text{s}}$. On its way down it fell into the ocean. Its height (above sea level) as time passes can be modeled by the quadratic function f, where $f(t) = -4.9t^2 + 294t + 130$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height (above sea level) in meters.

a. After ☐, this object reached its maximum height of ☐.

b. This object flew for ☐ before it landed in the ocean.

c. This object was ☐ above sea level 18 s after its release.

d. This object was 967.9 m above sea level twice: once ☐ after its release, and again later ☐ after its release.

CHAPTER 10

Functions and Their Representations

10.1 Function Basics

In Section 9.1 there is a light introduction to functions. This chapter introduces functions more thoroughly, and is independent from Section 9.1.

10.1.1 Informal Definition of a Function

We are familiar with the $\sqrt{}$ symbol. This symbol is used to turn numbers into their square roots. Sometimes it's simple to do this on paper or in our heads, and sometimes it helps to have a calculator. We can see some calculations in Table 10.1.2.

$$\begin{aligned}\sqrt{9} &= 3\\ \sqrt{1/4} &= 1/2\\ \sqrt{2} &\approx 1.41\end{aligned}$$

Table 10.1.2: Values of $\sqrt{x}$

The $\sqrt{}$ symbol represents a *process*; it's a way for us to turn numbers into other numbers. This idea of having a process for turning numbers into other numbers is the fundamental topic of this chapter.

Definition 10.1.3 Function (Informal Definition). A **function** is a process for turning numbers into (potentially) different numbers. The process must be *consistent*, in that whenever you apply it to some particular number, you always get the same result.

Section 10.5 covers a more technical definition for functions, and gets into some topics that are more appropriate when using that definition. Definition 10.1.3 is so broad that you probably use functions all the time.

Example 10.1.4 Think about each of these examples, where some process is used for turning one number into another.

- If you convert a person's birth year into their age, you are using a function.
- If you look up the Kelly Blue Book value of a Honda Odyssey based on how old it is, you are using a function.
- If you use the expected guest count for a party to determine how many pizzas you should order, you are using a function.

The process of using $\sqrt{\ }$ to change numbers might feel more "mathematical" than these examples. Let's continue thinking about $\sqrt{\ }$ for now, since it's a formula-like symbol that we are familiar with. One concern with $\sqrt{\ }$ is that although we live in the modern age of computers, this symbol is not found on most keyboards. And yet computers still tend to be capable of producing square roots. Computer technicians write sqrt( ) when they want to compute a square root, as we see in Table 10.1.5.

sqrt(9)	$= 3$
sqrt(1/4)	$= 1/2$
sqrt(2)	≈ 1.41

Table 10.1.5: Values of sqrt(x)

The parentheses are very important. To see why, try to put yourself in the "mind" of a computer, and look closely at `sqrt49`. The computer will recognize `sqrt` and know that it needs to compute a square root. But computers have myopic vision and they might not see the entire number 49. A computer might think that it needs to compute `sqrt4` and then append a `9` to the end, which would produce a final result of 29. This is probably not what was intended. And so the purpose of the parentheses in `sqrt(49)` is to denote exactly what number needs to be operated on.

This demonstrates the standard notation that is used worldwide to write down most functions. By having a standard notation for communicating about functions, people from all corners of the earth can all communicate mathematics with each other more easily, even when they don't speak the same language.

Functions have their own names. We've seen a function named sqrt, but any name you can imagine is allowable. In the sciences, it is common to name functions with whole words, like weight or health_index. In mathematics, we often abbreviate such function names to w or h. And of course, since the word "function" itself starts with "f," we will often name a function f.

It's crucial to continue reminding ourselves that functions are *processes* for changing numbers; they are not numbers themselves. And that means that we have a potential for confusion that we need to stay aware of. In some contexts, the symbol t might represent a variable—a number that is represented by a letter. But in other contexts, t might represent a function—a process for changing numbers into other numbers. By staying conscious of the context of an investigation, we avoid confusion.

Next we need to discuss how we go about using a function's name.

Definition 10.1.6 Function Notation. The standard notation for referring to functions involves giving the function itself a name, and then writing:

$$\begin{array}{c}\text{name}\\\text{of}\\\text{function}\end{array}\left(\text{input}\right)$$

> **Example 10.1.7** $f(13)$ is pronounced "f of 13." The word "of" is very important, because it reminds us that f is a process and we are about to apply that process to the input value 13. So f is the function, 13 is the input, and $f(13)$ is the output we'd get from using 13 as input.
>
> $f(x)$ is pronounced "f of x." This is just like the previous example, except that the input is not any specific number. The value of x could be 13 or any other number. Whatever x's value, $f(x)$ means the corresponding output from the function f.
>
> BudgetDeficit(2017) is pronounced "BudgetDeficit of 2017." This is probably about a function that takes a year as input, and gives that year's federal budget deficit as output. The process here of changing a year into a dollar amount might not involve any mathematical formula, but rather looking up information

> from the Congressional Budget Office's website.
>
> Celsius(F) is pronounced "Celsius of F." This is probably about a function that takes a Fahrenheit temperature as input and gives the corresponding Celsius temperature as output. Maybe a formula is used to do this; maybe a chart or some other tool is used to do this. Here, Celsius is the function, F is the input variable, and Celsius(F) is the output from the function.

Note 10.1.8. While a function has a name like f, and the input to that function often has a variable name like x, the expression $f(x)$ represents the output of the function. To be clear, $f(x)$ is *not* a function. Rather, f is a function, and $f(x)$ its output when the number x was used as input.

Checkpoint 10.1.9. Suppose you see the sentence, "If x is the number of software licenses you buy for your office staff, then $c(x)$ is the total cost of the licenses."

a. In the function notation, what represents input? ☐.

b. What is the function here? ☐.

c. What represents output? ☐.

Explanation. The input is x, the function is c, and $c(x)$ is the output from c when the input is x.

Warning 10.1.10 Notation Ambiguity. As mentioned earlier, we need to remain conscious of the context of any symbol we are using. It's possible for f to represent a function (a process), but it's also possible for f to represent a variable (a number). Similarly, parentheses might indicate the input of a function, or they might indicate that two numbers need to be multiplied. It's up to our judgment to interpret algebraic expressions in the right context. Consider the expression $a(b)$. This could easily mean the output of a function a with input b. It could also mean that two numbers a and b need to be multiplied. It all depends on the context in which these symbols are being used.

Sometimes it's helpful to think of a function as a machine, as in Figure 10.1.11. This illustrates how complicated functions can be. A number is just a number. But a *function* has the capacity to take in all kinds of different numbers into it's hopper (feeding tray) as inputs and transform them into their outputs.

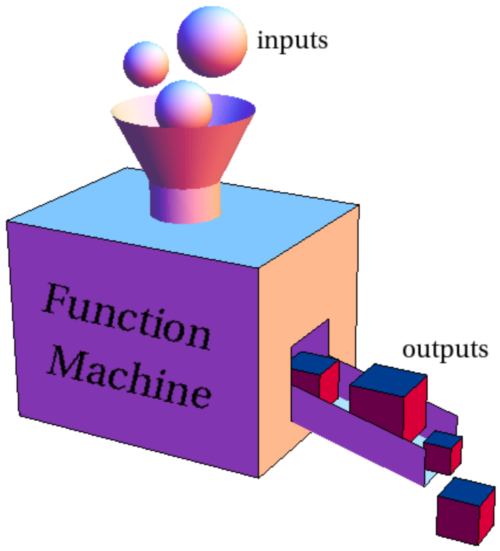


Figure 10.1.11: Imagining a function as a machine. (Image by Duane Nykamp using Mathematica.)

10.1.2 Tables and Graphs

Since functions are potentially complicated, we want ways to understand them more easily. Two basic tools for understanding a function better are tables and graphs.

Example 10.1.12 A Table for the Budget Deficit Function. Consider the function BudgetDeficit, that takes a year as its input and outputs the US federal budget deficit for that year. For example, the Congressional Budget Office's website tells us that BudgetDeficit(2009) is \$1.41 trillion. If we'd like to understand this function better, we might make a table of all the inputs and outputs we can find. Using the CBO's website (www.cbo.gov/topics/budget), we can put together Table 10.1.13.

input x (year)	output BudgetDeficit(x) (\$trillion)
2007	0.16
2008	0.46
2009	1.4
2010	1.3
2011	1.3
2012	1.1
2013	0.68
2014	0.48
2015	0.44
2016	0.59

Table 10.1.13

How is this table helpful? There are things about the function that we can see now by looking at the numbers in this table.

- We can see that the budget deficit had a spike between 2008 and 2009.
- And it fell again between 2012 and 2013.
- It appears to stay roughly steady for several years at a time, with occasional big jumps or drops.

These observations help us understand the function BudgetDeficit a little better.

Checkpoint 10.1.14. According to Table 10.1.13, what is the value of BudgetDeficit(2015)?

Explanation. Table 10.1.13 shows that when the input is 2015, the output is 0.44. So BudgetDeficit(2015) = 0.44. In context, that means that in 2015 the budget deficit was \$0.44 trillion.

Example 10.1.15 A Table for the Square Root Function. Let's return to our example of the function sqrt. Tabulating some inputs and outputs reveals 10.1.16

input, x	output, sqrt(x)
0	0
1	1
2	≈ 1.41
3	≈ 1.73
4	2
5	≈ 2.24
6	≈ 2.45
7	≈ 2.65
8	≈ 2.83
9	3

Table 10.1.16

How is this table helpful? Here are some observations that we can make now.

- We can see that when input numbers increase, so do output numbers.
- We can see even though outputs are increasing, they increase by less and less with each step forward in x.

These observations help us understand sqrt a little better. For instance, based on these observations which do you think is larger: the difference between sqrt(23) and sqrt(24), or the difference between sqrt(85) and sqrt(86)?

Checkpoint 10.1.17. According to Table 10.1.16, what is the value of sqrt(6)?

Explanation. Table 10.1.16 shows that when the input is 6, the output is about 2.45. So sqrt(6) $\approx$ 2.45.

Another powerful tool for understanding functions better is a graph. Given a function f, one way to make its graph is to take a table of input and output values, and read each row as the coordinates of a point in the xy-plane.

Example 10.1.18 A Graph for the Budget Deficit Function. Returning to the function BudgetDeficit that we studied in Example 10.1.12, in order to make a graph of this function we view Table 10.1.13 as a list of points with x and y coordinates, as in Table 10.1.19. We then plot these points on a set of coordinate axes, as in Figure 10.1.20. The points have been connected with a curve so that we can see the overall pattern given by the progression of points. Since there was not any actual data for inputs in between any two years, the curve is dashed. That is, this curve is dashed because it just represents someone's best guess as to how to connect the plotted points. Only the plotted points themselves are precise.

(input, output) $(x, \text{BudgetDeficit}(x))$
$(2007, 0.16)$
$(2008, 0.46)$
$(2009, 1.4)$
$(2010, 1.3)$
$(2011, 1.3)$
$(2012, 1.1)$
$(2013, 0.68)$
$(2014, 0.48)$
$(2015, 0.44)$
$(2016, 0.59)$

Table 10.1.19

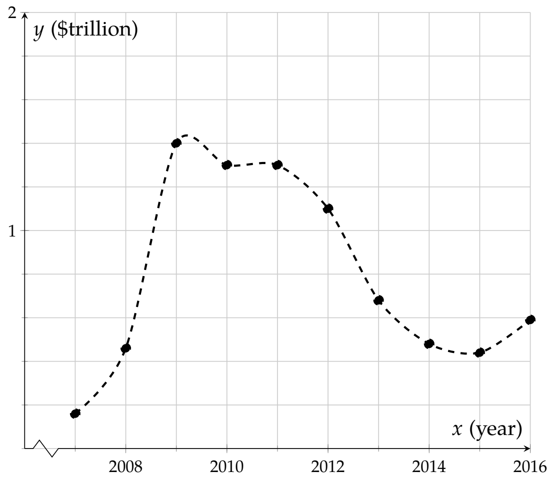


Figure 10.1.20: $y = \text{BudgetDeficit}(x)$

How has this graph helped us to understand the function better? All of the observations that we made in Example 10.1.12 are perhaps even more clear now. For instance, the spike in the deficit between 2008 and 2009 is now visually apparent. Seeking an explanation for this spike, we recall that there was a financial crisis in late 2008. Revenue from income taxes dropped at the same time that federal money was spent to prevent further losses.

Example 10.1.21 A Graph for the Square Root Function. Let's now construct a graph for sqrt. Tabulating inputs and outputs gives the points in Table 10.1.22, which in turn gives us the graph in Figure 10.1.23.

(input, output) $(x, \text{sqrt}(x))$
$(0, 0)$
$(1, 1)$
$\approx (2, 1.41)$
$\approx (3, 1.73)$
$(4, 2)$
$\approx (5, 2.24)$
$\approx (6, 2.45)$
$\approx (7, 2.65)$
$\approx (8, 2.83)$
$(9, 3)$

Table 10.1.22

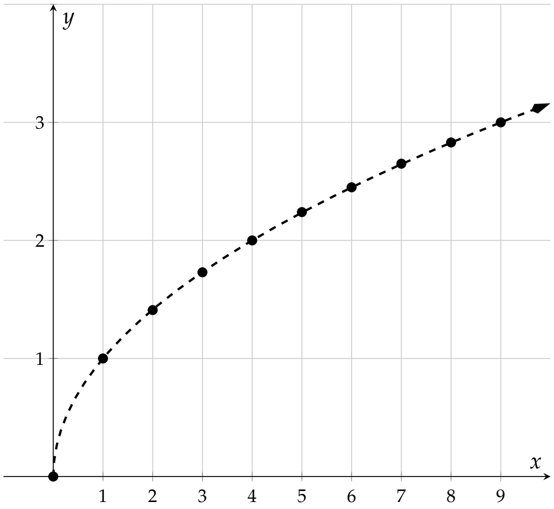


Figure 10.1.23: $y = \text{sqrt}(x)$

Just as in the previous example, we've plotted points where we have concrete coordinates, and then we have made our best attempt to connect those points with a curve. Unlike the previous example, here we believe that points will continue to follow the same pattern indefinitely to the right, and so we have added an arrowhead to the graph.

What has this graph done to improve our understanding of sqrt? As inputs (x-values) increase, the outputs (y-values) increase too, although not at the same rate. In fact we can see that our graph is steep on its left, and less steep as we move to the right. This confirms our earlier observation in Example 10.1.15 that outputs increase by smaller and smaller amounts as the input increases.

Note 10.1.24 Graph of a Function. Given a function f, when we refer to a **graph of** f we are *not* referring to an entire picture, like Figure 10.1.23. A graph of f is only *part* of that picture—the curve and the points that it connects. Everything else: axes, tick marks, the grid, labels, and the surrounding white space is just useful decoration, so that we can read the graph more easily.

It is also common to refer to the graph of f as the **graph of the equation** $y = f(x)$. However, we should avoid saying "the graph of $f(x)$." That would indicate a fundamental misunderstanding of our notation. We have decided that $f(x)$ is the output for a certain input x. That means that $f(x)$ is just a number; a relatively uninteresting thing compared to f the function, and not worthy of a two-dimensional picture.

While it is important to be able to make a graph of a function f, we also need to be capable of looking at a graph and reading it well. A graph of f provides us with helpful specific information about f; it tells us what f does to its input values. When we were making graphs, we plotted points of the form

$$(\text{input}, \text{output})$$

Now given a graph of f, we interpret coordinates in the same way.

Example 10.1.25 In Figure 10.1.26 we have a graph of a function f. If we wish to find $f(1)$, we recognize that 1 is being used as an input. So we would want to find a point of the form $(1, \quad)$. Seeking out x-coordinate 1 in Figure 10.1.26, we find that the only such point is $(1, 2)$. Therefore the output for 1 is 2; in other words $f(1) = 2$.

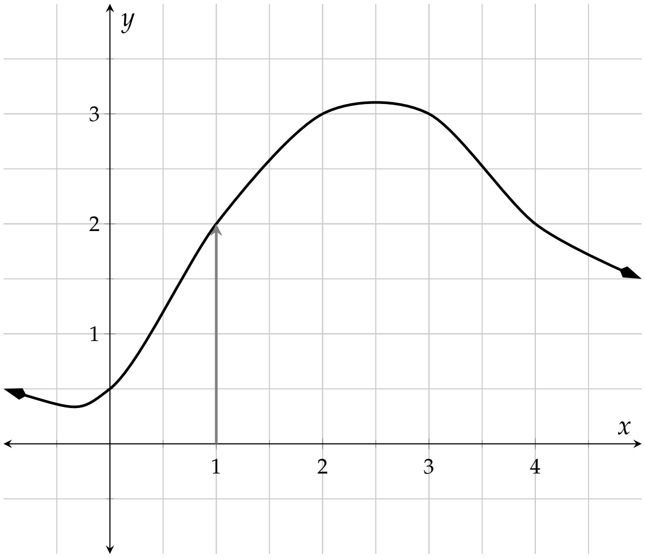


Figure 10.1.26: $y = f(x)$

Checkpoint 10.1.27. Use the graph of f in Figure 10.1.26 to find $f(0)$, $f(3)$, and $f(4)$.

a. $f(0) =$ ☐ b. $f(3) =$ ☐ c. $f(4) =$ ☐

Explanation.

a. $f(0) = 0.5$, since $(0, 0.5)$ is on the graph.

b. $f(3) = 3$, since $(3, 3)$ is on the graph.

c. $f(4) = 2$, since $(4, 2)$ is on the graph.

Suppose that u is the unemployment function of time. That is, $u(t)$ is the unemployment rate in the United States in year t. The graph of the equation $y = u(t)$ is given in Figure 10.1.29 (data.bls.gov/timeseries/LNS14000000).

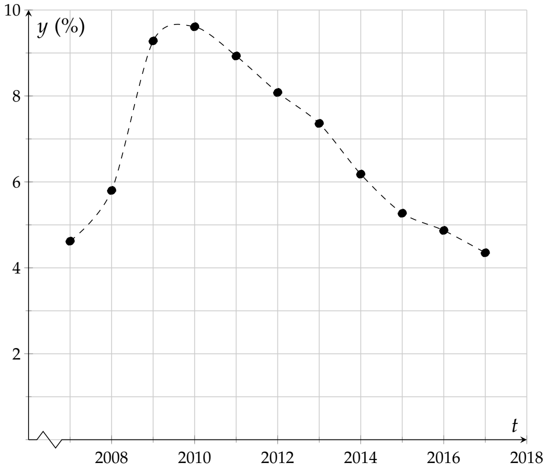


Figure 10.1.29: Unemployment in the United States

Example 10.1.28 Unemployment Rates. What was the unemployment in 2008? It is a straightforward matter to use Figure 10.1.29 to find that unemployment was about 6% in 2008. Asking this question is exactly the same thing as asking to find $u(2008)$. That is, we have one question that can either be asked in an everyday-English way or which can be asked in a terse, mathematical notation-heavy way:

"What was unemployment in 2008?"	"Find $u(2008)$."

If we use the table to establish that $u(2009) \approx 9.25$, then we should be prepared to translate that into everyday-English using the context of the function: In 2009, unemployment in the U.S. was about 9.25%.

If we ask the question "when was unemployment at 5%," we can read the graph and see that there were two such times: mid-2007 and about 2016. But there is again a more mathematical notation-heavy way to ask this question. Namely, since we are being told that the output of u is 5, we are being asked to solve the equation $u(t) = 5$. So the following communicate the same thing:

"When was unemployment at 5%?"	"Solve the equation $u(t) = 5$."

And our answer to this question is:

"Unemployment was at 5% in mid-2007 and about 2016."	"$t \approx 2007.5$ or $t \approx 2016$."

Checkpoint 10.1.30. Use the graph of u in Figure 10.1.29 to answer the following.

a. Find $u(2011)$ and interpret it.

   $u(2011) \approx$

   Interpretation:

b. Solve the equation $u(t) = 6$ and interpret your solution(s).

   $t \approx$ or $t \approx$

   Interpretation:

Explanation.

a. $u(2011) \approx 9$; In 2011 the US unemployment rate was about 9%.

b. $t \approx 2008$ or $t \approx 2014$; The points at which unemployment was 6% were in early 2008 and early 2014.

10.1.3 Translating Between Four Descriptions of the Same Function

We have noted that functions are complicated, and we want ways to make them easier to understand. It's common to find a problem involving a function and not know how to find a solution to that problem. Most functions have at least four standard ways to think about them, and if we learn how to translate between these four perspectives, we often find that one of them makes a given problem easier to solve.

The four modes for working with a given function are

- a verbal description
- a table of inputs and outputs
- a graph of the function
- a formula for the function

This has been visualized in Figure 10.1.31.

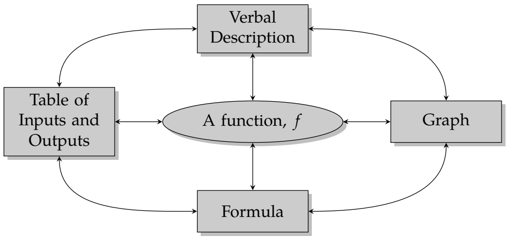


Figure 10.1.31: Function Perspectives

Example 10.1.32 Consider a function f that squares its input and then adds 1. Translate this verbal description of f into a table, a graph, and a formula.

Explanation.

To make a table for f, we'll have to select some input x-values. These choices are left entirely up to us, so we might as well choose small, easy-to-work-with values. However we shouldn't shy away from negative input values. Given the verbal description, we should be able to compute a column of output values. Table 10.1.33 is one possible table that we might end up with.

x	$f(x)$
-2	$(-2)^2+1=5$
-1	$(-1)^2+1=2$
0	$0^2+1=1$
1	$1^2+1=2$
2	5
3	10
4	17

Table 10.1.33

Once we have a table for f, we can make a graph for f as in Figure 10.1.34, using the table to plot points.

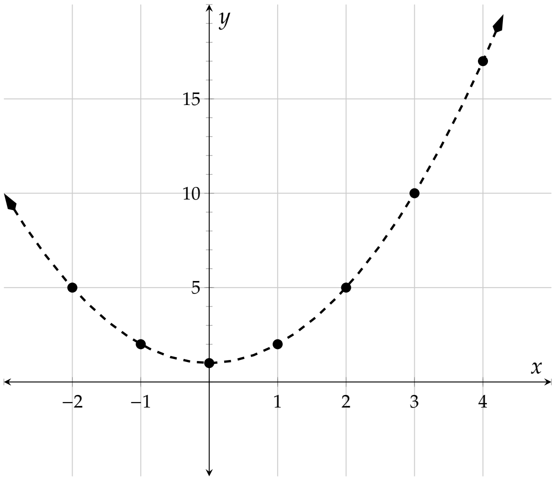


Figure 10.1.34: $y=f(x)$

Lastly, we must find a formula for f. This means we need to write an algebraic expression that says the same thing about f as the verbal description, the table, and the graph. For this example, we can focus on the verbal description. Since f takes its input, squares it, and adds 1, we have that

$$f(x) = x^2 + 1.$$

Example 10.1.35 Let F be the function that takes a Celsius temperature as input and outputs the corresponding Fahrenheit temperature. Translate this verbal description of F into a table, a graph, and a formula.

Explanation. To make a table for F, we will need to rely on what we know about Celsius and Fahrenheit temperatures. It is a fact that the freezing temperature of water at sea level is 0 °C, which equals 32 °F. Also, the boiling temperature of water at sea level is 100 °C, which is the same as 212 °F. One more piece of information we might have is that standard human body temperature is 37 °C, or 98.6 °F. All of this is compiled in Table 10.1.36. Note that we tabulated inputs and outputs by working with the context of the function, not with any computations.

C	$F(C)$
0	32
37	98.6
100	212

Table 10.1.36

Once a table is established, making a graph by plotting points is a simple matter, as in Figure 10.1.37. The three plotted points seem to be in a straight line, so we think it is reasonable to connect them in that way.

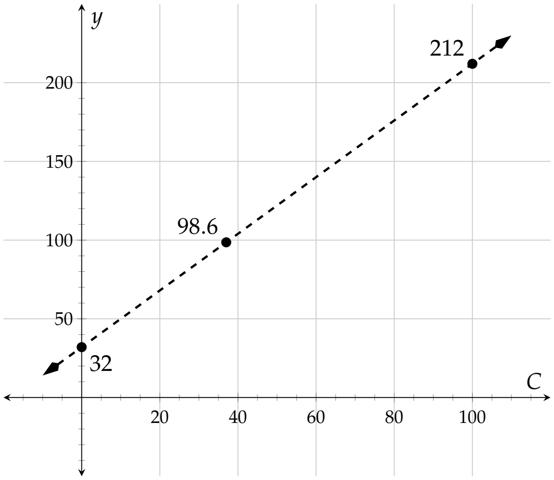


Figure 10.1.37: $y = F(C)$

To find a formula for F, the verbal definition is not of much direct help. But F's graph does seem to be a straight line. And linear equations are familiar to us. This line has a y-intercept at $(0, 32)$ and a slope we can calculate: $\frac{212-32}{100-0} = \frac{180}{100} = \frac{9}{5}$. So the equation of this line is $y = \frac{9}{5}C + 32$. On the other hand, the equation of this graph is $y = F(C)$, since it is a graph of the function F. So evidently,

$$F(C) = \frac{9}{5}C + 32.$$

Exercises

Review and Warmup

1. Evaluate $\dfrac{9r-1}{r}$ for $r = -6$.

2. Evaluate $\dfrac{3r-4}{r}$ for $r = 8$.

3. Evaluate the following expressions.

a. Evaluate $2t^2$ when $t = 3$. $2t^2 =$ ☐

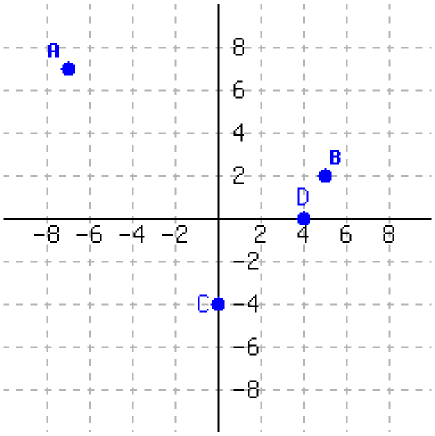
b. Evaluate $(2t)^2$ when $t = 3$. $(2t)^2 =$ ☐

4. Evaluate the following expressions.

a. Evaluate $2t^2$ when $t = 5$. $2t^2 =$ ☐

b. Evaluate $(2t)^2$ when $t = 5$. $(2t)^2 =$ ☐

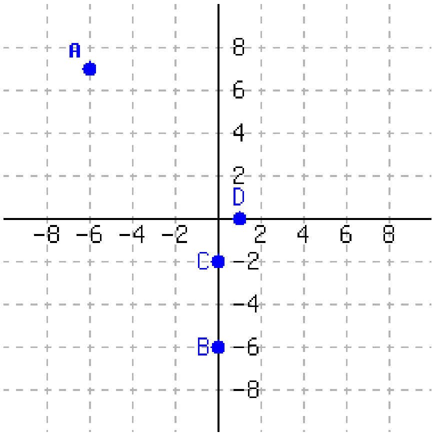
5. Locate each point in the graph:



Write each point's position as an ordered pair, like $(1, 2)$.

$A =$ ______ $B =$ ______
$C =$ ______ $D =$ ______

6. Locate each point in the graph:



Write each point's position as an ordered pair, like $(1, 2)$.

$A =$ ______ $B =$ ______
$C =$ ______ $D =$ ______

Function Formulas and Evaluation Evaluate the function at the given values.

7. $g(x) = x - 4$

a. $g(1) =$ ☐

b. $g(-4) =$ ☐

c. $g(0) =$ ☐

8. $K(x) = x - 2$

a. $K(4) =$ ☐

b. $K(-2) =$ ☐

c. $K(0) =$ ☐

9. $F(x) = 6x$

a. $F(1) =$ ☐

b. $F(-2) =$ ☐

c. $F(0) =$ ☐

10. $G(x) = 2x$

a. $G(3) =$ ☐

b. $G(-3) =$ ☐

c. $G(0) =$ ☐

11. $H(x) = -2x + 10$

a. $H(1) =$ ☐

b. $H(-1) =$ ☐

c. $H(0) =$ ☐

12. $H(x) = -4x + 4$

a. $H(5) =$ ☐

b. $H(-1) =$ ☐

c. $H(0) =$ ☐

13. $K(x) = -x + 1$

a. $K(5) =$ ☐

b. $K(-2) =$ ☐

c. $K(0) =$ ☐

14. $f(x) = -x + 8$

a. $f(2) =$ ☐

b. $f(-4) =$ ☐

c. $f(0) =$ ☐

15. $g(y) = y^2 + 3$

a. $g(1) = \boxed{}$

b. $g(-1) = \boxed{}$

c. $g(0) = \boxed{}$

16. $h(x) = x^2 - 9$

a. $h(4) = \boxed{}$

b. $h(-1) = \boxed{}$

c. $h(0) = \boxed{}$

17. $h(r) = -r^2 - 1$

a. $h(3) = \boxed{}$

b. $h(-1) = \boxed{}$

c. $h(0) = \boxed{}$

18. $F(y) = -y^2 + 8$

a. $F(2) = \boxed{}$

b. $F(-2) = \boxed{}$

c. $F(0) = \boxed{}$

19. $G(t) = -4$

a. $G(5) = \boxed{}$

b. $G(-4) = \boxed{}$

c. $G(0) = \boxed{}$

20. $H(r) = 5$

a. $H(4) = \boxed{}$

b. $H(5) = \boxed{}$

c. $H(0) = \boxed{}$

21. $H(x) = \dfrac{3x}{-8x - 2}$

a. $H(7) = \boxed{}.$

b. $H(-3) = \boxed{}.$

22. $K(x) = \dfrac{8x}{x - 8}$

a. $K(6) = \boxed{}.$

b. $K(-7) = \boxed{}.$

23. $f(x) = \dfrac{20}{x + 7}.$

a. $f(3) = \boxed{}.$

b. $f(-7) = \boxed{}.$

24. $g(x) = \dfrac{12}{x + 1}.$

a. $g(-3) = \boxed{}.$

b. $g(-1) = \boxed{}.$

25. $g(x) = -x + 5$

a. $g(3) = \boxed{}$

b. $g(-3) = \boxed{}$

26. $h(x) = -4x - 5$

a. $h(8) = \boxed{}$

b. $h(-3) = \boxed{}$

27. $F(x) = x^2 - 3x + 2$

a. $F(4) = \boxed{}$

b. $F(-2) = \boxed{}$

28. $G(x) = x^2 + 4x - 6$

a. $G(2) = \boxed{}$

b. $G(-2) = \boxed{}$

29. $H(x) = -3x^2 - 4x + 2$

a. $H(3) = \boxed{}$

b. $H(-3) = \boxed{}$

30. $H(x) = -3x^2 + 4x + 5$

a. $H(3) = \boxed{}$

b. $H(-5) = \boxed{}$

31. $K(x) = \sqrt{x}.$

a. $K(81) = \boxed{}$

b. $K\left(\frac{16}{49}\right) = \boxed{}$

c. $K(-4) = \boxed{}$

32. $f(x) = \sqrt{x}.$

a. $f(36) = \boxed{}$

b. $f\left(\frac{64}{25}\right) = \boxed{}$

c. $f(-4) = \boxed{}$

33. $g(x) = \sqrt[3]{x}$

a. $g(-125) = \boxed{}$

b. $g\left(\frac{1}{8}\right) = \boxed{}$

34. $g(x) = \sqrt[3]{x}$

a. $g(-8) = \boxed{}$

b. $g\left(\frac{27}{125}\right) = \boxed{}$

35. $h(x) = -4$

a. $h(8) = \boxed{}$

b. $h(-4) = \boxed{}$

36. $F(x) = -18$

a. $F(4) = \boxed{}$

b. $F(-7) = \boxed{}$

Function Formulas and Solving Equations

37. Solve for x, where $G(x) = 20x + 2$.

a. If $G(x) = -78$, then $x =$ ______.

b. If $G(x) = 7$, then $x =$ ______.

38. Solve for x, where $H(x) = 6x - 9$.

a. If $H(x) = -27$, then $x =$ ______.

b. If $H(x) = -7$, then $x =$ ______.

39. Solve for x, where $H(x) = x^2 - 10$.

a. If $H(x) = 6$, then $x =$ ______.

b. If $H(x) = -17$, then $x =$ ______.

40. Solve for x, where $K(x) = x^2 + 4$.

a. If $K(x) = 5$, then $x =$ ______.

b. If $K(x) = -5$, then $x =$ ______.

41. Solve for x, where $f(x) = x^2 + 3x - 26$.

If $f(x) = 2$, then $x =$ ______.

42. Solve for x, where $g(x) = x^2 + 18x + 81$.

If $g(x) = 1$, then $x =$ ______.

43. If G is a function defined by $G(y) = -3y - 5$,

Find $G(0)$. ______

Solve $G(y) = 0$. ______

44. If g is a function defined by $g(y) = 4y - 11$,

Find $g(0)$. ______

Solve $g(y) = 0$. ______

45. If K is a function defined by $K(y) = y^2 - 9$,

Find $K(0)$. ______

Solve $K(y) = 0$. ______

46. If G is a function defined by $G(r) = 3r^2 - 6$,

Find $G(0)$. ______

Solve $G(r) = 0$. ______

47. If g is a function defined by $g(r) = r^2 + 2r - 35$,

Find $g(0)$. ______

Solve $g(r) = 0$. ______

48. If K is a function defined by $K(t) = t^2 - 11t + 18$,

Find $K(0)$. ______

Solve $K(t) = 0$. ______

Functions and Points on a Graph

49. If $K(6) = 9$, then the point is on the graph of K.

If $(7, 6)$ is on the graph of K, then $K(7) =$.

50. If $f(2) = -1$, then the point is on the graph of f.

If $(4, 3)$ is on the graph of f, then $f(4) =$.

51. If $g(t) = r$, then the point is on the graph of g.

The answer is not a specific numerical point, but one with variables for coordinates.

52. If $g(y) = x$, then the point is on the graph of g.

The answer is not a specific numerical point, but one with variables for coordinates.

53. If (x, y) is on the graph of h, then $h(x) =$.

54. If (t, x) is on the graph of F, then $F(t) =$.

55. For the function $G(x)$, when $x = -2$, its y-value is 9.

Choose all true statements.

☐ The function's value is 9 at −2.
☐ The point $(9, -2)$ is on the graph of the function.
☐ $G(9) = -2$
☐ The point $(-2, 9)$ is on the graph of the function.
☐ $G(-2) = 9$
☐ The function's value is −2 at 9.

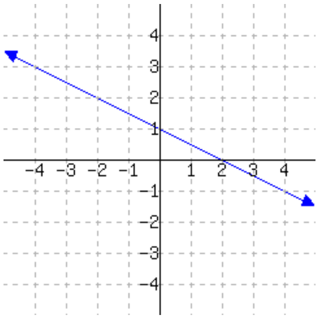
56. For the function $H(x)$, when $x = 2$, its y-value is −5.

Choose all true statements.

☐ $H(2) = -5$
☐ $H(-5) = 2$
☐ The function's value is 2 at −5.
☐ The function's value is −5 at 2.
☐ The point $(-5, 2)$ is on the graph of the function.
☐ The point $(2, -5)$ is on the graph of the function.

Function Graphs

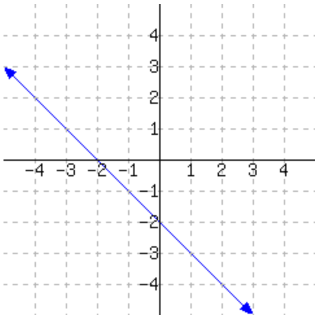
57. Use the graph of H below to evaluate the given expressions. (Estimates are OK.)



$H(-2) =$

$H(2) =$

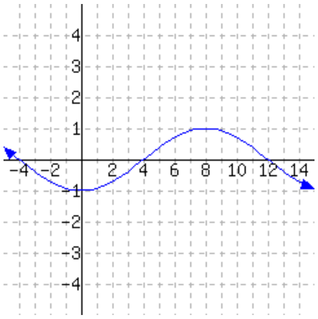
58. Use the graph of K below to evaluate the given expressions. (Estimates are OK.)



$K(-3) =$

$K(2) =$

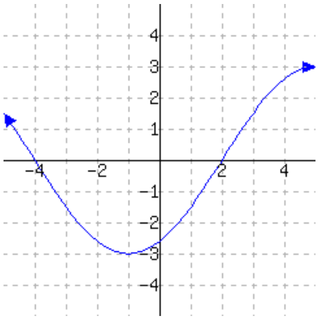
59. Use the graph of f below to evaluate the given expressions. (Estimates are OK.)



$f(0) =$

$f(12) =$

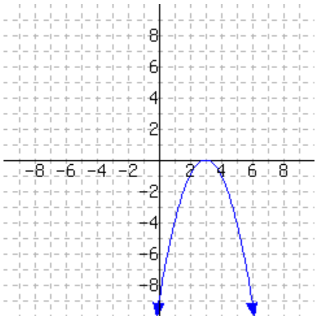
60. Use the graph of g below to evaluate the given expressions. (Estimates are OK.)



$g(-1) =$

$g(2) =$

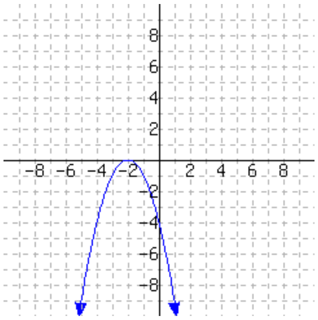
61. Use the graph of g below to evaluate the given expressions. (Estimates are OK.)



$g(3) =$

$g(5) =$

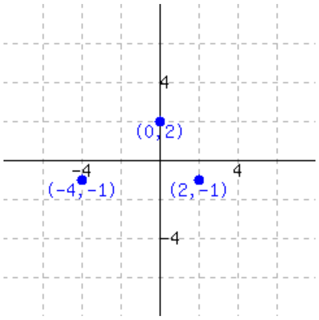
62. Use the graph of h below to evaluate the given expressions. (Estimates are OK.)



$h(-2) =$

$h(-1) =$

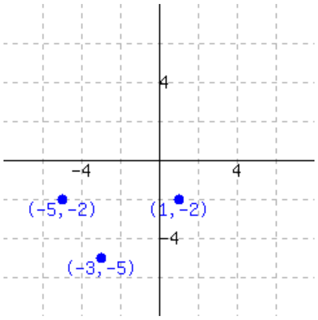
63. Function f is graphed.



a. $f(0) =$

b. Solve $f(x) = -1$.

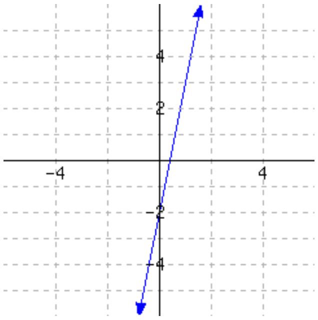
64. Function f is graphed.



a. $f(-3) =$

b. Solve $f(x) = -2$.

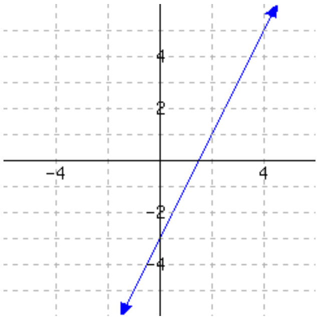
65. Function f is graphed.



a. $f(1) =$

b. Solve $f(x) = -2$.

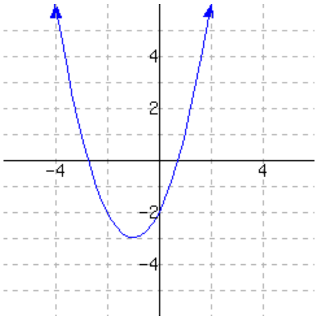
66. Function f is graphed.



a. $f(2) =$

b. Solve $f(x) = 3$.

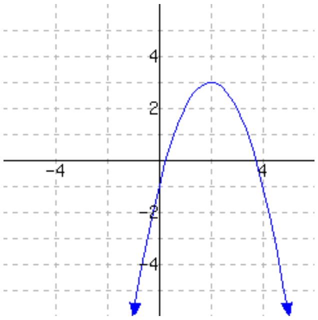
67. Function f is graphed.



a. $f(-3) =$

b. Solve $f(x) = -2$.

68. Function f is graphed.



a. $f(3) =$

b. Solve $f(x) = -1$.

Function Tables

69. Use the table of values for g below to evaluate the given expressions.

x	-4	-1	2	5	8
$g(x)$	-1	-0.5	4.1	9.9	-0.4

$g(-4) = \boxed{\qquad}$

$g(5) = \boxed{\qquad}$

70. Use the table of values for g below to evaluate the given expressions.

x	-1	1	3	5	7
$g(x)$	7.7	-0.6	-2	-1.1	4.5

$g(1) = \boxed{\qquad}$

$g(7) = \boxed{\qquad}$

71. Make a table of values for the function g, defined by $g(x) = -2x^2$. Based on values in the table, sketch a graph of g.

x	$g(x)$
________	________
________	________
________	________
________	________
________	________

72. Make a table of values for the function K, defined by $K(x) = \dfrac{2^x - 2}{x^2 + 3}$. Based on values in the table, sketch a graph of K.

x	$K(x)$
________	________
________	________
________	________
________	________
________	________

Translating Between Different Representations of a Function

73. Here is a verbal representation of a function G.

Square the input x to obtain the output y.Give a numeric representation of G:

x	0	1	2	3	4
$G(x)$	____	____	____	____	____

Give a symbolic representation of G:

$G(x) = \boxed{\qquad\qquad}$

74. Here is a verbal representation of a function G.

Cube the input x to obtain the output y.Give a numeric representation of G:

x	0	1	2	3	4
$G(x)$	____	____	____	____	____

Give a symbolic representation of G:

$G(x) = \boxed{\qquad\qquad}$

75. Here is a verbal representation of a function H.

Triple the input x and then subtract nine to obtain the output y.Give a numeric representation of H:

x	0	1	2	3	4
$H(x)$	____	____	____	____	____

Give a symbolic representation of H:

$H(x) = \boxed{}$

76. Here is a verbal representation of a function K.

Double the input x and then subtract two to obtain the output y.Give a numeric representation of K:

x	0	1	2	3	4
$K(x)$	____	____	____	____	____

Give a symbolic representation of K:

$K(x) = \boxed{}$

77. Express the function f numerically with the table.

$$f(x) = 2x^2 - \frac{1}{3}x$$

x	−3	−2	−1	0	1	2	3
$f(x)$	__	__	__	__	__	__	__

On graphing paper, you should be able to give a graphical representation of f too.

78. Express the function g numerically with the table.

$$g(x) = x^3 - \frac{1}{3}x^2$$

x	−3	−2	−1	0	1	2	3
$g(x)$	__	__	__	__	__	__	__

On graphing paper, you should be able to give a graphical representation of g too.

79. Express the function g numerically with the table.

$$g(x) = \frac{1 - x}{5 + x}$$

x	−3	−2	−1	0	1	2	3
$g(x)$	__	__	__	__	__	__	__

On graphing paper, you should be able to give a graphical representation of g too.

80. Express the function h numerically with the table.

$$h(x) = \frac{7 - x}{7 + x}$$

x	−3	−2	−1	0	1	2	3
$h(x)$	__	__	__	__	__	__	__

On graphing paper, you should be able to give a graphical representation of h too.

Functions in Context

81. Virginia started saving in a piggy bank on her birthday. The function $f(x) = 3x + 1$ models the amount of money, in dollars, in Virginia's piggy bank. The independent variable represents the number of days passed since her birthday.

Interpret the meaning of $f(1) = 4$.

- ⊙ *A*. The piggy bank started with \$4 in it, and Virginia saves \$1 each day.
- ⊙ *B*. Four days after Virginia started her piggy bank, there were \$1 in it.
- ⊙ *C*. The piggy bank started with \$1 in it, and Virginia saves \$4 each day.
- ⊙ *D*. One days after Virginia started her piggy bank, there were \$4 in it.

82. Timothy started saving in a piggy bank on his birthday. The function $f(x) = 2x + 2$ models the amount of money, in dollars, in Timothy's piggy bank. The independent variable represents the number of days passed since his birthday.

Interpret the meaning of $f(3) = 8$.

- ⊙ *A*. Eight days after Timothy started his piggy bank, there were \$3 in it.
- ⊙ *B*. Three days after Timothy started his piggy bank, there were \$8 in it.
- ⊙ *C*. The piggy bank started with \$8 in it, and Timothy saves \$3 each day.
- ⊙ *D*. The piggy bank started with \$3 in it, and Timothy saves \$8 each day.

83. An arcade sells multi-day passes. The function $g(x) = \frac{1}{4}x$ models the number of days a pass will work, where x is the amount of money paid, in dollars.

Interpret the meaning of $g(16) = 4$.

- ⊙ *A*. Each pass costs \$16, and it works for 4 days.
- ⊙ *B*. If a pass costs \$4, it will work for 16 days.
- ⊙ *C*. If a pass costs \$16, it will work for 4 days.
- ⊙ *D*. Each pass costs \$4, and it works for 16 days.

84. An arcade sells multi-day passes. The function $g(x) = \frac{1}{4}x$ models the number of days a pass will work, where x is the amount of money paid, in dollars.

Interpret the meaning of $g(12) = 3$.

- ⊙ *A*. If a pass costs \$12, it will work for 3 days.
- ⊙ *B*. Each pass costs \$3, and it works for 12 days.
- ⊙ *C*. Each pass costs \$12, and it works for 3 days.
- ⊙ *D*. If a pass costs \$3, it will work for 12 days.

85. Kayla will spend \$270 to purchase some bowls and some plates. Each bowl costs \$1, and each plate costs \$6. The function $p(b) = -\frac{1}{6}b + 45$ models the number of plates Kayla will purchase, where b represents the number of bowls Kayla will purchase.

Interpret the meaning of $p(180) = 15$.

- ⊙ *A*. If 180 bowls are purchased, then 15 plates will be purchased.
- ⊙ *B*. \$180 will be used to purchase bowls, and \$15 will be used to purchase plates.
- ⊙ *C*. If 15 bowls are purchased, then 180 plates will be purchased.
- ⊙ *D*. \$15 will be used to purchase bowls, and \$180 will be used to purchase plates.

86. Aleric will spend \$210 to purchase some bowls and some plates. Each bowl costs \$5, and each plate costs \$6. The function $p(b) = -\frac{5}{6}b + 35$ models the number of plates Aleric will purchase, where b represents the number of bowls Aleric will purchase.

Interpret the meaning of $p(6) = 30$.

- ⊙ *A*. If 30 bowls are purchased, then 6 plates will be purchased.
- ⊙ *B*. If 6 bowls are purchased, then 30 plates will be purchased.
- ⊙ *C*. \$6 will be used to purchase bowls, and \$30 will be used to purchase plates.
- ⊙ *D*. \$30 will be used to purchase bowls, and \$6 will be used to purchase plates.

87. Kara will spend \$400 to purchase some bowls and some plates. Each plate costs \$1, and each bowl costs \$8. The function $q(x) = -\frac{1}{8}x + 50$ models the number of bowls Kara will purchase, where x represents the number of plates to be purchased.

Interpret the meaning of $q(24) = 47$.

- ⊙ *A*. \$47 will be used to purchase bowls, and \$24 will be used to purchase plates.
- ⊙ *B*. 47 plates and 24 bowls can be purchased.
- ⊙ *C*. \$24 will be used to purchase bowls, and \$47 will be used to purchase plates.
- ⊙ *D*. 24 plates and 47 bowls can be purchased.

88. Cheryl will spend \$160 to purchase some bowls and some plates. Each plate costs \$9, and each bowl costs \$8. The function $q(x) = -\frac{9}{8}x + 20$ models the number of bowls Cheryl will purchase, where x represents the number of plates to be purchased.

Interpret the meaning of $q(16) = 2$.

- ⊙ *A*. 2 plates and 16 bowls can be purchased.
- ⊙ *B*. 16 plates and 2 bowls can be purchased.
- ⊙ *C*. \$2 will be used to purchase bowls, and \$16 will be used to purchase plates.
- ⊙ *D*. \$16 will be used to purchase bowls, and \$2 will be used to purchase plates.

89. Find the rule of the linear function f that gives the number of minutes in x weeks.

$f(x) =$

90. Find the rule of the linear function f that gives the number of seconds in x weeks.

$f(x) =$

91. Suppose that M is the function that computes how many miles are in x feet. Find the algebraic rule for M. (If you do not know how many feet are in one mile, you can look it up on Google.)

$M(x) =$

Evaluate $M(22000)$ and interpret the result:

There are about miles in feet.

92. Suppose that K is the function that computes how many kilograms are in x pounds. Find the algebraic rule for K. (If you do not know how many pounds are in one kilogram, you can look it up on Google.)

$K(x) =$

Evaluate $K(241)$ and interpret the result.

Something that weighs pounds would weigh about kilograms.

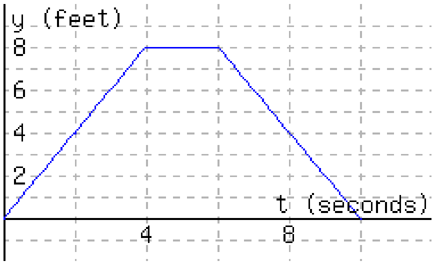
93. Suppose that f is the function that the phone company uses to determine what your bill will be (in dollars) for a long-distance phone call that lasts t minutes. Each call costs a fixed price of $4.95 plus 10 cents per minute. Write a formula for this linear function f.

94. Suppose that f is the function that gives the total cost (in dollars) of downhill skiing x times during a season with a $500 season pass. Write a formula for f.

95. Suppose that f is the function that tells you how many dimes are in x dollars. Write a formula for f.

The following figure has the graph $y = d(t)$, which models a particle's distance from the starting line in feet, where t stands for time in seconds since timing started.

96.



a. $d(3) =$

b. Interpret the meaning of $d(3)$:

⊙ *A*. The particle was 3 feet away from the starting line 6 seconds since timing started.

⊙ *B*. The particle was 6 feet away from the starting line 3 seconds since timing started.

⊙ *C*. In the first 6 seconds, the particle moved a total of 3 feet.

⊙ *D*. In the first 3 seconds, the particle moved a total of 6 feet.

c. Solve $d(t) = 2$ for t. $t =$

d. Interpret the meaning of part c's solution(s):

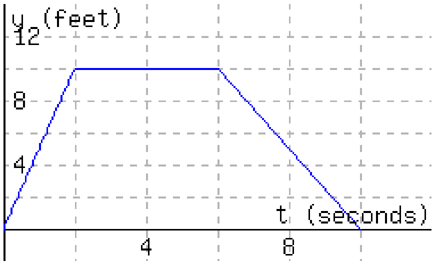
⊙ *A*. The article was 2 feet from the starting line 9 seconds since timing started.

⊙ *B*. The article was 2 feet from the starting line 1 seconds since timing started, and again 9 seconds since timing started.

⊙ *C*. The article was 2 feet from the starting line 1 seconds since timing started, or 9 seconds since timing started.

⊙ *D*. The article was 2 feet from the starting line 1 seconds since timing started.

97.



a. $d(4) =$

b. Interpret the meaning of $d(4)$:

⊙ *A*. The particle was 4 feet away from the starting line 10 seconds since timing started.

⊙ *B*. In the first 10 seconds, the particle moved a total of 4 feet.

⊙ *C*. The particle was 10 feet away from the starting line 4 seconds since timing started.

⊙ *D*. In the first 4 seconds, the particle moved a total of 10 feet.

c. Solve $d(t) = 5$ for t. $t =$

d. Interpret the meaning of part c's solution(s):

⊙ *A*. The article was 5 feet from the starting line 1 seconds since timing started.

⊙ *B*. The article was 5 feet from the starting line 1 seconds since timing started, and again 8 seconds since timing started.

⊙ *C*. The article was 5 feet from the starting line 1 seconds since timing started, or 8 seconds since timing started.

⊙ *D*. The article was 5 feet from the starting line 8 seconds since timing started.

98. The function C models the the number of customers in a store t hours since the store opened.

t	0	1	2	3	4	5	6	7
$C(t)$	0	43	81	102	98	83	43	0

a. $C(2) =$

b. Interpret the meaning of $C(2)$:

- ⊙ *A*. There were 81 customers in the store 2 hours after the store opened.
- ⊙ *B*. There were 2 customers in the store 81 hours after the store opened.
- ⊙ *C*. In 2 hours since the store opened, the store had an average of 81 customers per hour.
- ⊙ *D*. In 2 hours since the store opened, there were a total of 81 customers.

c. Solve $C(t) = 43$ for t. $t =$

d. Interpret the meaning of Part c's solution(s):

- ⊙ *A*. There were 43 customers in the store 1 hours after the store opened.
- ⊙ *B*. There were 43 customers in the store either 1 hours after the store opened, or 6 hours after the store opened.
- ⊙ *C*. There were 43 customers in the store 6 hours after the store opened.
- ⊙ *D*. There were 43 customers in the store 1 hours after the store opened, and again 6 hours after the store opened.

99. Chicago's average monthly rainfall, $R = f(t)$ inches, is given as a function of the month, t, where January is $t = 1$, in the table below.

t, month	1	2	3	4	5	6	7	8
R, inches	1.8	1.8	2.7	3.1	3.5	3.7	3.5	3.4

(a) Solve $f(t) = 3.1$.

$t =$

The solution(s) to $f(t) = 3.1$ can be interpreted as saying

- ⊙ Chicago's average rainfall in the month of April is 3.1 inches.
- ⊙ Chicago's average rainfall is greatest in the month of February.
- ⊙ Chicago's average rainfall increases by 3.1 inches in the month of February.
- ⊙ Chicago's average rainfall is least in the month of April.
- ⊙ None of the above

(b) Solve $f(t) = f(5)$.

$t =$

The solution(s) to $f(t) = f(5)$ can be interpreted as saying

- ⊙ Chicago's average rainfall is 3.5 inches in the months of May and July.
- ⊙ Chicago's average rainfall is 3.5 inches in the month of July.
- ⊙ Chicago's average rainfall is greatest in the month of May.
- ⊙ Chicago's average rainfall is 3.5 inches in the month of May.
- ⊙ None of the above

100. Let $f(t)$ denote the number of people eating in a restaurant t minutes after 5 PM. Answer the following questions:

a) Which of the following statements best describes the significance of the expression $f(4) = 19$?

⊙ There are 19 people eating at 9:00 PM

⊙ Every 4 minutes, 19 more people are eating

⊙ There are 19 people eating at 5:04 PM

⊙ There are 4 people eating at 5:19 PM

⊙ None of the above

b) Which of the following statements best describes the significance of the expression $f(a) = 30$?

⊙ At 5:30 PM there are a people eating

⊙ Every 30 minutes, the number of people eating has increased by a people

⊙ a hours after 5 PM there are 30 people eating

⊙ a minutes after 5 PM there are 30 people eating

⊙ None of the above

c) Which of the following statements best describes the significance of the expression $f(30) = b$?

⊙ b hours after 5 PM there are 30 people eating

⊙ Every 30 minutes, the number of people eating has increased by b people

⊙ At 5:30 PM there are b people eating

⊙ b minutes after 5 PM there are 30 people eating

⊙ None of the above

d) Which of the following statements best describes the significance of the expression $n = f(t)$?

⊙ n hours after 5 PM there are t people eating

⊙ t hours after 5 PM there are n people eating

⊙ Every t minutes, n more people have begun eating

⊙ n minutes after 5 PM there are t people eating

⊙ None of the above

101. Let $s(t) = 12t^2 - 2t + 300$, where s is the position (in mi) of a car driving on a straight road at time t (in hr). The car's velocity (in mi/hr) at time t is given by $v(t) = 24t - 2$.

a. *Using function notation*, express the car's position after 2.5 hours. The answer here is not a formula, it's just something using function notation like `f(8)`.

b. Where is the car then? The answer here is a number with units.

c. *Use function notation* to express the question, "When is the car going 65 $\frac{\text{mi}}{\text{hr}}$?" The answer is an equation that uses function notation; something like `f(x)=23`. You are not being asked to actually solve the equation, just to write down the equation.

d. Where is the car when it is going 22 $\frac{\text{mi}}{\text{hr}}$? The answer here is a number with units. You are being asked a question about its position, but have been given information about its speed.

102. Let $s(t) = 13t^2 - 3t + 300$, where s is the position (in mi) of a car driving on a straight road at time t (in hr). The car's velocity (in mi/hr) at time t is given by $v(t) = 26t - 3$.

a. *Using function notation*, express the car's position after 3.9 hours. The answer here is not a formula, it's just something using function notation like `f(8)`.

b. Where is the car then? The answer here is a number with units.

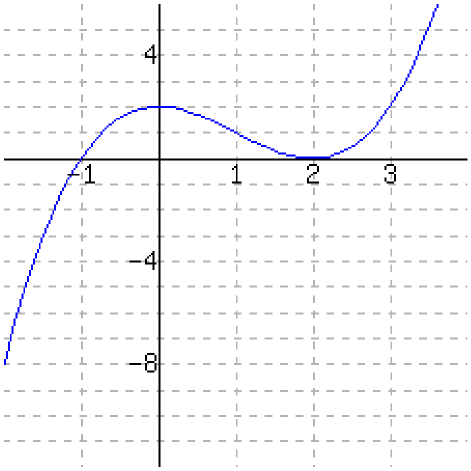
c. *Use function notation* to express the question, "When is the car going 61 $\frac{\text{mi}}{\text{hr}}$?" The answer is an equation that uses function notation; something like `f(x)=23`. You are not being asked to actually solve the equation, just to write down the equation.

d. Where is the car when it is going 75 $\frac{\text{mi}}{\text{hr}}$? The answer here is a number with units. You are being asked a question about its position, but have been given information about its speed.

103. Describe your own example of a function that has real context to it. You will need some kind of input variable, like "number of years since 2000" or "weight of the passengers in my car." You will need a process for using that number to bring about a different kind of number. The process does not need to involve a formula; a verbal description would be great, as would a formula.

Give your function a name. Write the symbol(s) that you would use to represent input. Write the symbol(s) that you would use to represent output.

104. Use the graph of h in the figure to fill in the table.



x	-2	-1	0	1	2
$h(x)$	________	________	________	________	________

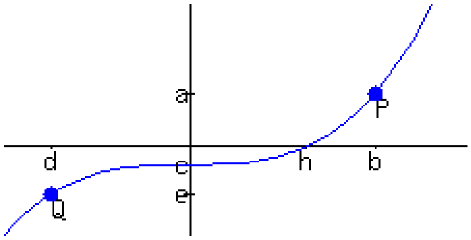
a. Evaluate $h(3) - h(0)$.

b. Evaluate $h(2) - h(-1)$.

c. Evaluate $2h(-1)$.

d. Evaluate $h(0) + 3$.

105. Use the given graph of a function f, along with a, b, c, d, e, and h to answer the following questions. Some answers are points, and should be entered as ordered pairs. Some answers ask you to solve for x, so the answer should be in the form x=...



a. What are the coordinates of the point P?

b. What are the coordinates of the point Q?

c. Evaluate $f(b)$. (The answer is symbolic, not a specific number.)

d. Solve $f(x) = e$ for x. (The answer is symbolic, not a specific number.)

e. Suppose $c = f(z)$. Solve the equation $z = f(x)$ for x.

10.2 Domain and Range

A function is a process for turning input values into output values. Occasionally a function f will have input values for which the process breaks down.

10.2.1 Domain

Example 10.2.2 Let P be the population of Portland as a function of the year. According to Google[a] we can say that:

$$P(2016) = 639863 \qquad P(1990) = 487849$$

But what if we asked to find $P(1600)$? The question doesn't really make sense anymore. The Multnomah tribe lived in villages in the area, but the city of Portland was not incorporated until 1851. We say that $P(1600)$ is *undefined*.

---

[a] https://www.google.com/publicdata/explore?ds=kf7tgg1uo9ude_&met_y=population&hl=en&dl=en#!ctype=l&strail=false&bcs=d&nselm=h&met_y=population&scale_y=lin&ind_y=false&rdim=country&idim=place:4159000&ifdim=country&hl=en_US&dl=en&ind=false

Example 10.2.3 If m is a person's mass in kg, let $w(m)$ be their weight in lb. There is an approximate formula for w:

$$w(m) \approx 2.2m$$

From this formula we can find:

$$w(50) \approx 110 \qquad w(80) \approx 176$$

which tells us that a 50-kg person weighs 110 lb, and an 80-kg person weighs 176 lb.

What if we asked for $w(-100)$? In the context of this example, we would be asking for the weight of a person whose mass is -100 kg. This is clearly nonsense. That means that $w(-100)$ is *undefined*. Note that the *context* of the example is telling us that $w(-100)$ is undefined even though the formula alone might suggest that $w(-100) = -220$.

Example 10.2.4 Let g have the formula

$$g(x) = \frac{x}{x-7}.$$

For most x-values, $g(x)$ is perfectly computable:

$$g(2) = -\frac{2}{5} \qquad g(14) = 2.$$

But if we try to compute $g(7)$, we run into an issue of arithmetic.

$$\begin{aligned} g(7) &= \frac{7}{7-7} \\ &= \frac{7}{0} \end{aligned}$$

The expression $\frac{7}{0}$ is *undefined*. There is no number that this could equal.

Checkpoint 10.2.5. If $f(x) = \frac{x+2}{x+8}$, find an input for f that would cause an undefined output.

The number ☐ would cause an undefined output.

Explanation. Trying -8 as an input value would not work out; it would lead to division by 0.

These examples should motivate the following definition.

Definition 10.2.6 Domain. The **domain** of a function f is the collection of all of its valid input values.

Example 10.2.7 Referring to the functions from Examples 10.2.2–10.2.4

- The domain of P is all years starting from 1851 and later. It would also be reasonable to say that the domain is actually all years from 1851 up to the current year, since we cannot guarantee that Portland will exist forever.
- The domain of w is all positive real numbers. It is nonsensical to have a person with negative mass or even one with zero mass. While there is some lower bound for the smallest mass a person could have, and also an upper bound for the largest mass a person could have, these boundaries are gray. We can say for sure that non-positive numbers should never be used as inputs for w.
- The domain of g is all real numbers except 7. This is the only number that causes a breakdown in g's formula.

10.2.2 Interval, Set, and Set-Builder Notation

Communicating the domain of a function can be wordy. In mathematics, we can communicate the same information using concise notation that is accepted for use almost everywhere. Table 10.2.8 contains example functions from this section and their domains, and demonstrates *interval notation* for these domains. Basic interval notation is covered in Section 1.6, but some of our examples here go beyond what that section covers.

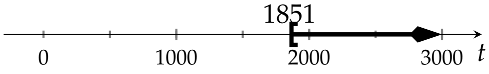
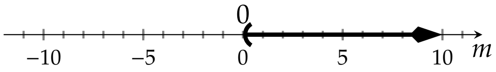
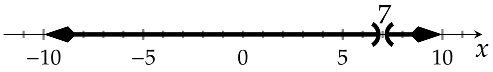
Function	Verbal Domain	Number Line Illustration	Interval Notation
P from Example 10.2.2	all years 1851 and greater		$[1851, \infty)$
w from Example 10.2.3	all real numbers greater than 0		$(0, \infty)$
g from Example 10.2.4	all real numbers except 7		$(-\infty, 7) \cup (7, \infty)$

Table 10.2.8: Domains from Earlier Examples

Again, basic interval notation is covered in Section 1.6, but one thing appears in Table 10.2.8 that is not explained in that earlier section: the $\cup$ symbol, which we see in the domain of g.

Occasionally there is a need to consider number line pictures such as Figure 10.2.9, where two or more intervals appear.

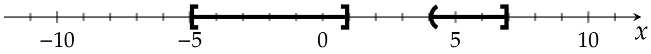


Figure 10.2.9: A number line with a union of two intervals

This picture is trying to tell you to consider numbers that are between −5 and 1, together with numbers that are between 4 and 7. That word "together" is related to the word "union," and in math the **union symbol**, ∪, captures this idea. So we can write the numbers in this picture as

$$[-5,1]\cup(4,7]$$

(which uses interval notation).

With the domain of g in Table 10.2.8, the number line picture shows us another "union" of two intervals. They are very close together, but there are still two separated intervals in that picture: $(-\infty,7)$ and $(7,\infty)$. Their union is represented by $(-\infty,7)\cup(7,\infty)$.

Checkpoint 10.2.10. What is the domain of the function sqrt, where $\operatorname{sqrt}(x)=\sqrt{x}$, using interval notation?

Explanation. The function sqrt cannot take a negative number as an input. It can however take any positive number as input, or the number 0 as input. Representing this on a number line, we find the domain is $[0,\infty)$ in interval notation.

Checkpoint 10.2.11. What is the domain of the function ℓ where $\ell(x)=\frac{2}{x-3}$, using interval notation?

Explanation. The function ℓ cannot take a 3 as an input. It can however take any other number as input. Representing this on a number line, we have an interval $(-\infty,3)$ to the left of 3, and $(3,\infty)$ to the right of 3. So we find the domain is $(-\infty,3)\cup(3,\infty)$.

Sometimes we will consider collections of only a short list of numbers. In those cases, we use **set notation** (first introduced in Section 1.5). With set notation, we have a list of numbers in mind, and we simply list all of those numbers. Curly braces are standard for surrounding the list. Table 10.2.12 illustrates set notation in use.

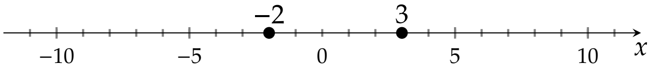
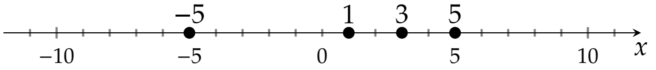
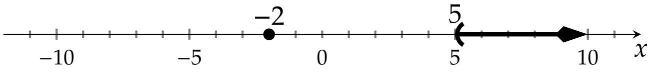
Picture of Set	Set Notation
	$\{-2,3\}$
	$\{-5,1,3,5\}$
	$\{-2\}\cup(5,\infty)$

Table 10.2.12: Set Notation

Checkpoint 10.2.13. A change machine lets you put in an x-dollar bill, and gives you $f(x)$ nickels in return equal in value to x dollars. Any current, legal denomination of US paper money can be fed to the change machine. What is the domain of f?

Explanation. The current, legal denominations of US paper money are \$1, \$2, \$5, \$10, \$20, \$50, and \$100. So the domain of f is the set $\{1,2,5,10,20,50,100\}$.

While most collections of numbers that we will encounter can be described using a combination of interval notation and set notation, there is another commonly used notation that is very useful in algebra: **set-builder notation**, which was introduced in Section 1.6. Set-builder notation also uses curly braces. Set-builder notation provides a template for what a number that is under consideration might look like, and then it gives you restrictions on how to use that template. A very basic example of set-builder notation is

$$\{x \mid x \geq 3\}.$$

Verbally, this is "the set of all x such that x is greater than or equal to 3." Table 10.2.14 gives more examples of set-builder notation in use.

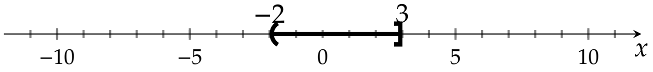
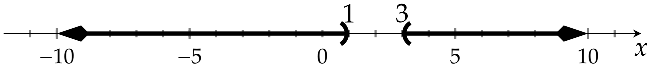
Picture of Set	Set Notation
	$\{x \mid -2 < x \leq 3\}$
	$\{x \mid x < 1 \text{ or } x > 3\}$

Table 10.2.14: Set-Builder Notation

Checkpoint 10.2.15. What is the domain of the function sqrt, where $\operatorname{sqrt}(x) = \sqrt{x}$, using set-builder notation?

Explanation. The function sqrt cannot take a negative number as an input. It can however take any positive number as input, or the number 0 as input. Representing this on a number line, we find the domain is $\{x \mid x \geq 0\}$ in set-builder notation.

Example 10.2.16 What is the domain of the function A, where $A(x) = \frac{2x+1}{x^2-2x-8}$?

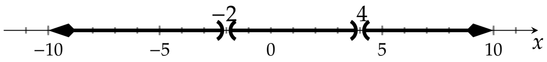
Note that if you plugged in some value for x, the only thing that might go wrong is if the denominator equals 0. So a *bad* value for x would be when

$$\begin{aligned} x^2 - 2x - 8 &= 0 \\ (x+2)(x-4) &= 0 \end{aligned}$$

Here, we used a basic factoring technique from Section 7.3. To continue, either

$$\begin{aligned} x + 2 &= 0 & &\text{or} & x - 4 &= 0 \\ x &= -2 & &\text{or} & x &= 4. \end{aligned}$$

These are the *bad* x-values because they lead to division by 0 in the formula for A. So on a number line, if we wanted to picture the domain of A, we would make a sketch like:



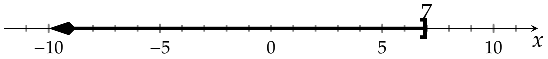
So the domain is the union of three intervals: $(-\infty, -2) \cup (-2, 4) \cup (4, \infty)$.

Example 10.2.17 What is the domain of the function B, where $B(x) = \sqrt{7-x} + 3$?

Note that if you plugged in some value for x, the only thing that might go wrong is if the value in the radical is negative. So the *good* values for x would be when

$$\begin{aligned} 7 - x &\geq 0 \\ 7 &\geq x \\ x &\leq 7 \end{aligned}$$

So on a number line, if we wanted to picture the domain of B, we would make a sketch like:



So the domain is the interval $(-\infty, 7]$.

There are three main properties of algebraic functions that cause numbers to be excluded from a domain, which are summarized here.

Denominators Division by zero is undefined. So if a function contains an expression in a denominator, it will only be defined where that expression is not equal to zero.

Example 10.2.16 demonstrates this.

Square Roots The square root of a negative number is undefined. So if a function contains a square root, it will only be defined when the expression inside that radical is greater than or equal to zero. (This is actually true for any even nth radical.)

Example 10.2.17 demonstrates this.

Context Some numbers are nonsensical in context. If a function has real-world context, then this may add additional restrictions on the input values.

Example 10.2.3 demonstrates this.

List 10.2.18: Summary of Algebraic Domain Restrictions

10.2.3 Range

The domain of a function is the collection of its valid inputs; there is a similar notion for *output*.

Definition 10.2.19 Range. The **range** of a function f is the collection of all of its possible output values.

Example 10.2.20 Let f be the function defined by the formula $f(x) = x^2$. Finding f's *domain* is straightforward. Any number anywhere can be squared to produce an output, so f has domain $(-\infty, \infty)$. What is the *range* of f?

Explanation. We would like to describe the collection of possible numbers that f can give as output. We will use a graphical approach. Figure 10.2.21 displays a graph of f, and the visualization that reveals f's range.

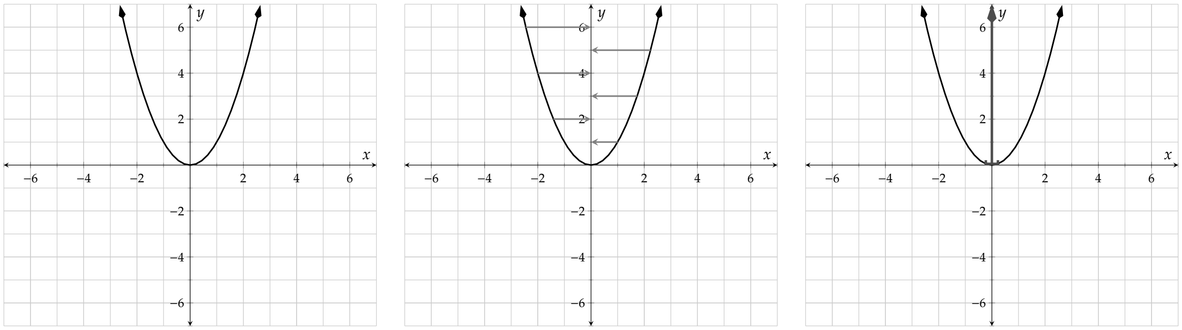


Figure 10.2.21: $y = f(x)$ where $f(x) = x^2$. The second graph illustrates how to visualize the range. In the third graph, the range is marked as an interval on the y-axis.

Output values are the y-coordinates in a graph. If we "slide the ink" left and right over to the y-axis to emphasize what the y-values in the graph are, we have y-values that start from 0 and continue upward forever. Therefore the range is $[0, \infty)$.

Warning 10.2.22 Finding range from a formula. Sometimes it is possible to compute a range without the aid of a graph. However, doing so can often require techniques covered in calculus. Therefore when you are asked to find the range of a function based on its formula, your approach will most often need to be a graphical one.

Example 10.2.23 Given the function g graphed in Figure 10.2.24, find the domain and range of g.

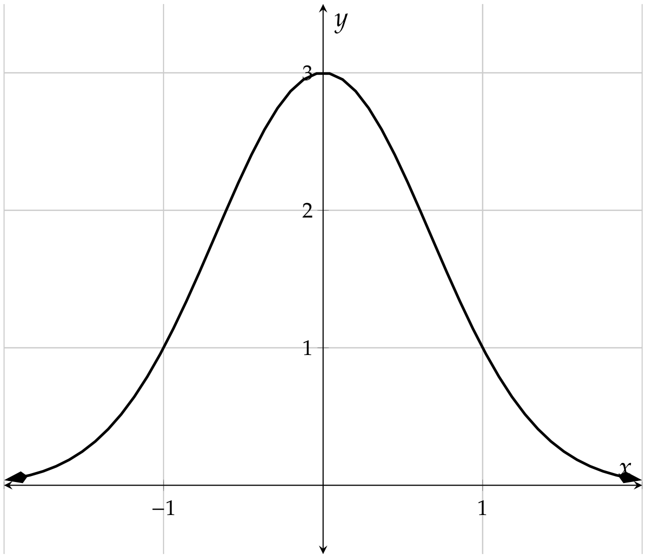


Figure 10.2.24: $y = g(x)$

Explanation. To find the domain, we can visualize all of the x-values that are valid inputs for this function by "sliding the ink" down onto the x-axis. The arrows at the far left and far right of the curve indicate that whatever pattern we see in the graph continues off to the left and right. Here, we see that the arms of the graph appear to be tapering down to the x-axis and extending left and right forever. Every x-value can be used to get an output for the function, so the domain is $(-\infty, \infty)$.

If we visualize the possible *outputs* by "sliding the ink" sideways onto the y-axis, we find that outputs as high as 3 are possible (including 3 itself). The outputs appear to get very close to 0 when x is large, but they aren't quite equal to 0. So the range is $(0, 3]$.

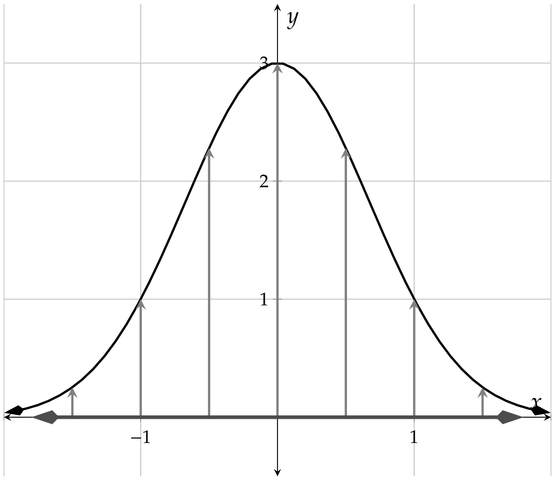


Figure 10.2.25: Domain of g

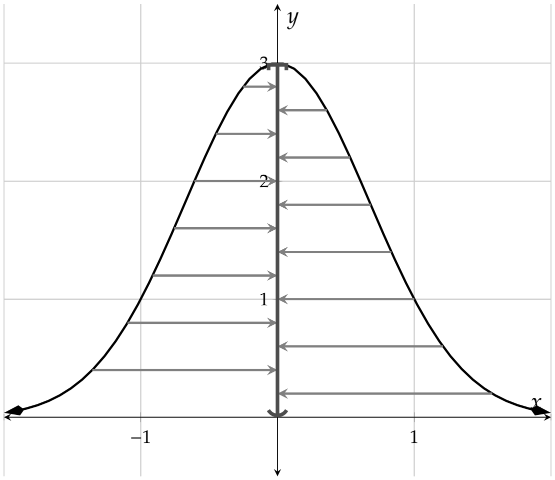
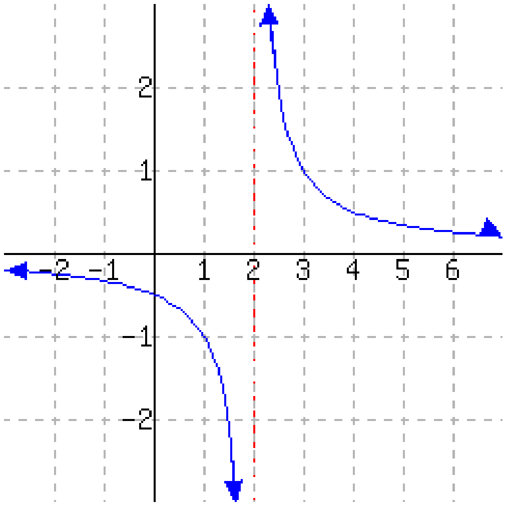


Figure 10.2.26: Range of g

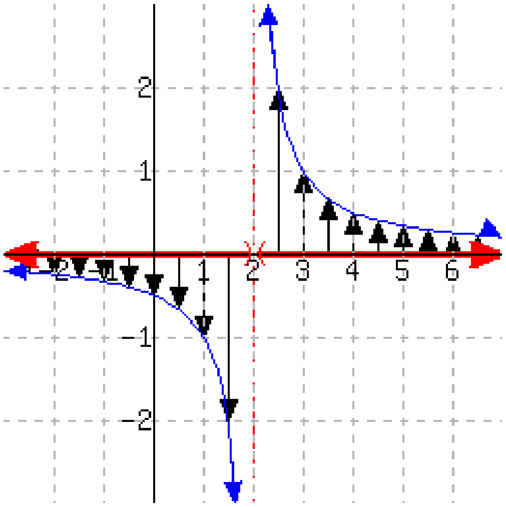
Checkpoint 10.2.27. Given the function h graphed below, find the domain and range of h. Note there is an invisible vertical line at $x = 2$, and the two arms of the graph are extending downward (and upward) forever, getting arbitrarily close to that vertical line, but never touching it. Also note that the two arms extend forever to the left and right, getting arbitrarily close to the y-axis, but never touching it.



The domain of h is and the range of h is .

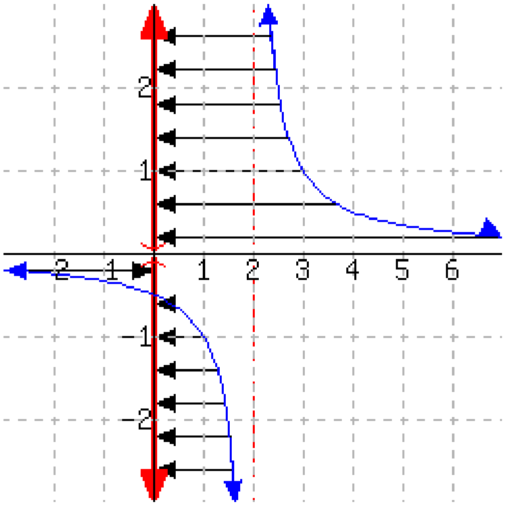
Explanation. To find the domain, we try to visualize all of the x-values that are valid inputs for this function. The arrows pointing left and right on the curve indicate that whatever pattern we see in the graph continues off to the left and right. So for x-values far to the right or left, we will be able to get an output for h.

The arrows pointing up and down are supposed to indicate that the curve will get closer and closer to the vertical line $x = 2$ after the curve leaves the viewing window we are using. So even when x is some number very close to 2, we will be able to get an output for h.



The one x-value that doesn't behave is $x = 2$. If we tried to use that as an input, there is no point on the graph directly above or below that on the x-axis. So the domain is $(-\infty, 2) \cup (2, \infty)$.

To find the range, we try to visualize all of the y-values that are possible outputs for this function. Sliding the ink of the curve left/right onto the y-axis reveals that $y = 0$ is the only y-value that we could never obtain as an output. So the range is $(-\infty, 0) \cup (0, \infty)$.



The examples of finding domain and range so far have all involved either a verbal description of a function, a formula for that function, or a graph of that function. Recall that there is a fourth perspective on functions:

a table. In the case of a table, we have very limited information about the function's inputs and outputs. If the table is all that we have, then there are a handful of input values listed in the table for which we know outputs. For any other input, the output is undefined.

Example 10.2.28 Consider the function k given in Table 10.2.29. What is the domain and range of k?

x	$k(x)$
3	4
8	5
10	5

Table 10.2.29

Explanation. All that we know about k is that $k(3) = 4$, $k(8) = 5$, and $k(10) = 5$. Without any other information such as a formula for k or a context for k that tells us its verbal description, we must assume that its domain is $\{3, 8, 10\}$; these are the only valid input for k. Similarly, k's range is $\{4, 5\}$.

Note that we have used set notation, not interval notation, since the answers here were *lists* of x-values (for the domain) and y-values (for the range). Also note that we could graph the information that we have about k in Figure 10.2.30, and the visualization of "sliding ink" to determine domain and range still works.

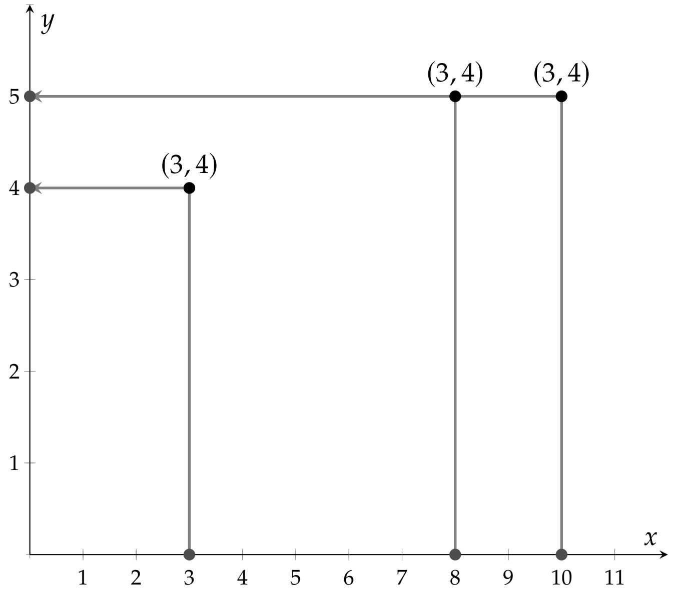
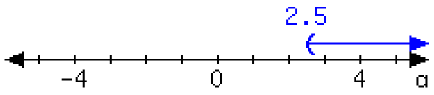


Figure 10.2.30

Exercises

Review and Warmup

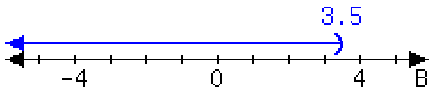
1. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

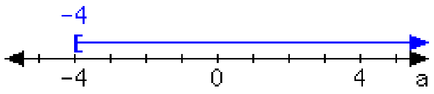
2. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

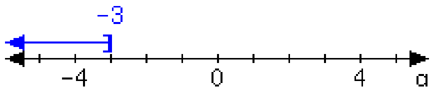
3. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

4. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

5. Solve this compound inequality, and write your answer in *interval notation*.

$x \geq 0$ and $x \leq 2$

6. Solve this compound inequality, and write your answer in *interval notation*.

$x \geq -3$ and $x < -2$

7. Solve this compound inequality, and write your answer in *interval notation*.

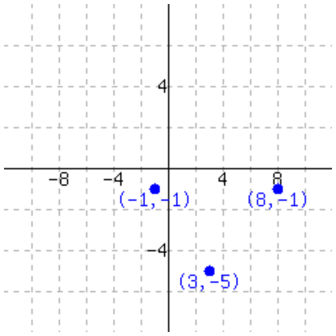
$x \geq 6$ or $x \leq 3$

8. Solve this compound inequality, and write your answer in *interval notation*.

$x > 2$ or $x < -2$

Domain and Range From a Graph A function is graphed.

9.



This function has domain and range .

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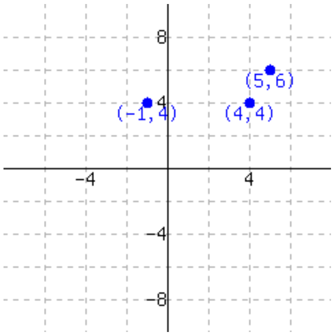

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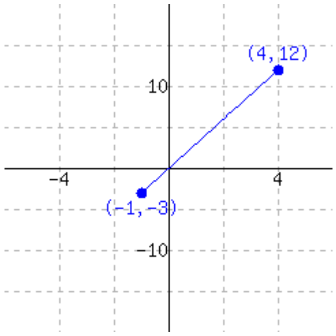

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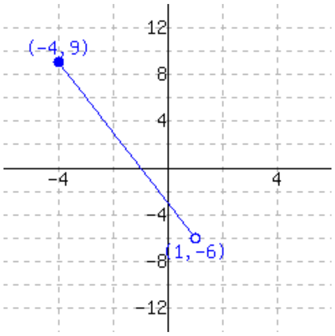

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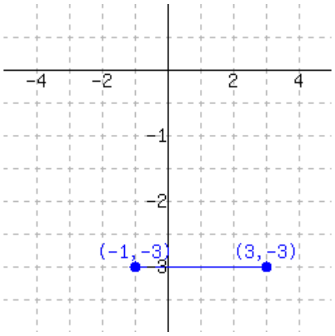

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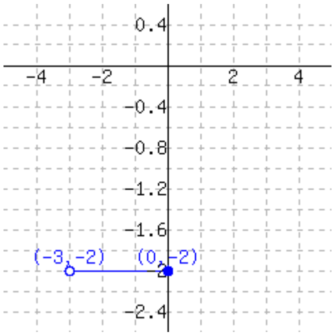

This function has domain and range .

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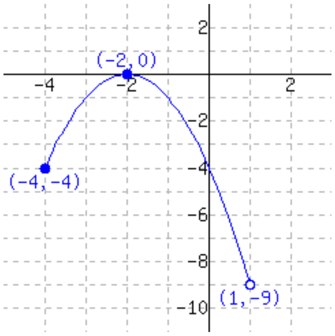

This function has domain and range .

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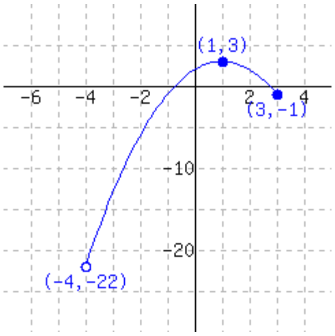

This function has domain and range .

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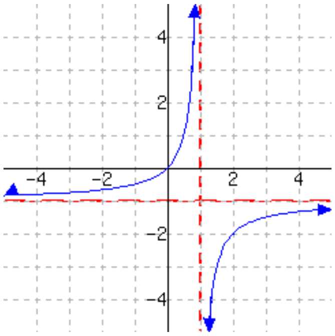

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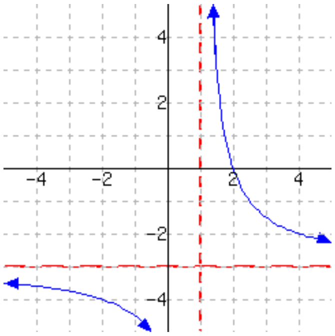

This function has domain and range .

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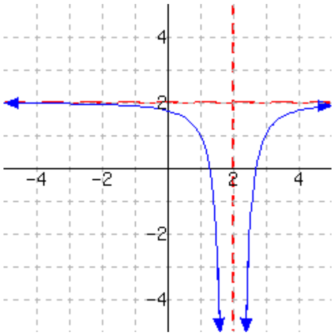

This function has domain and range .

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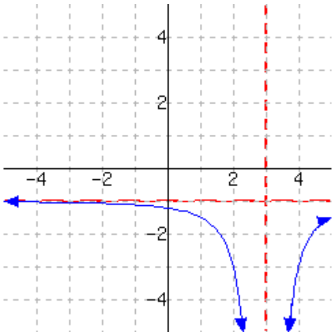

This function has domain and range .

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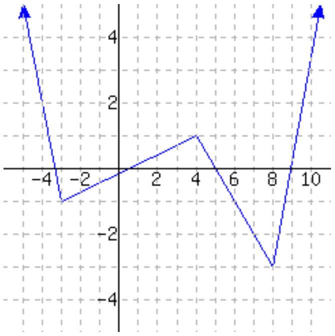
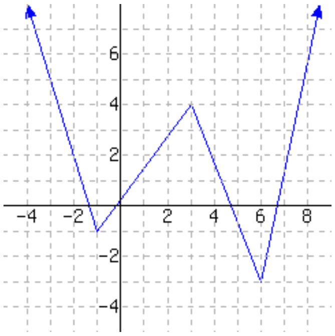


This function has domain ☐ and range ☐.

22.



This function has domain ☐ and range ☐.

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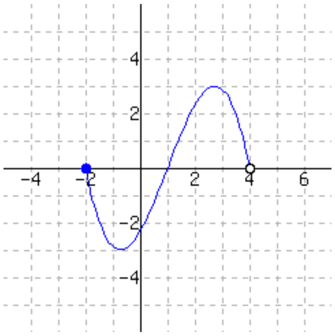

The function has domain ☐ and range ☐.

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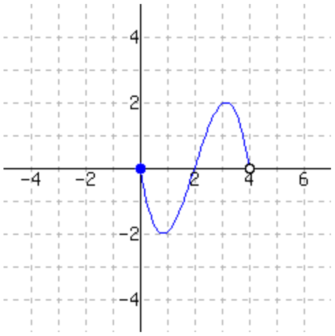

The function has domain ☐ and range ☐.

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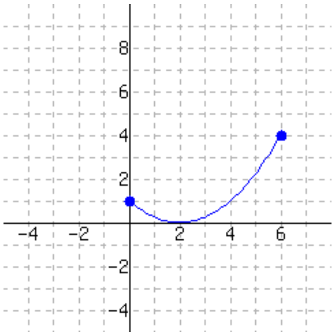
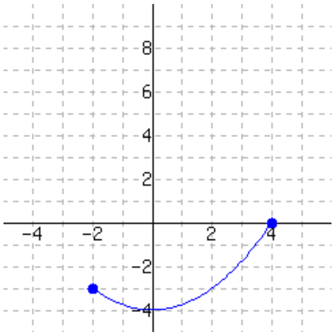


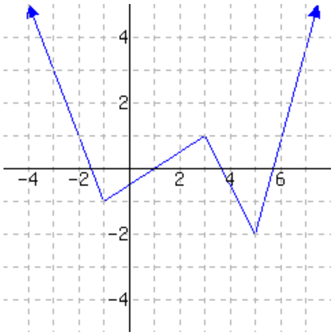
The function has domain ☐ and range ☐.

26.



The function has domain ☐ and range ☐.

27.



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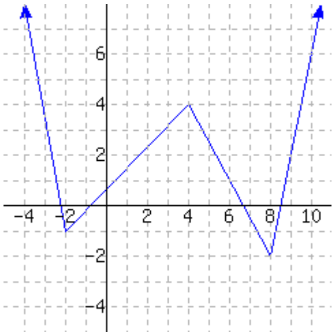
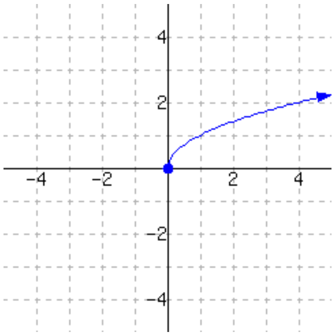


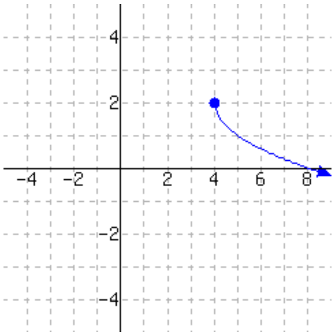
The function has domain ☐ and range ☐.

29.



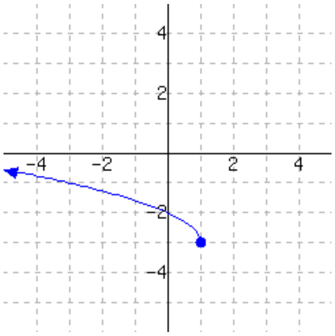
The function has domain ☐ and range ☐.

30.



The function has domain and range .

31.



The function has domain and range .

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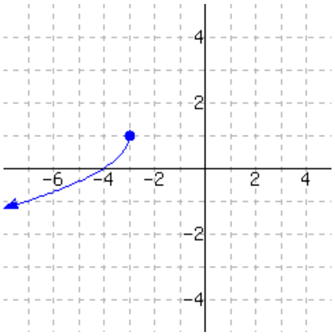
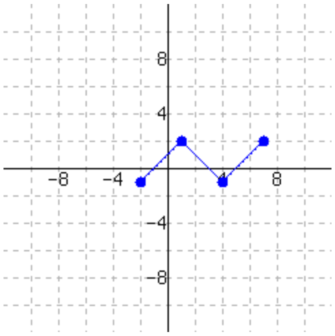


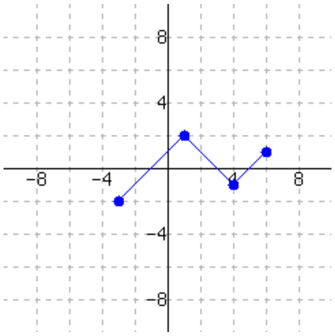
The function has domain and range .

33.



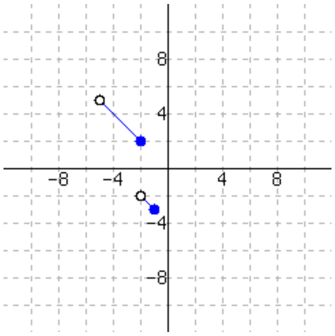
The function has domain and range .

34.



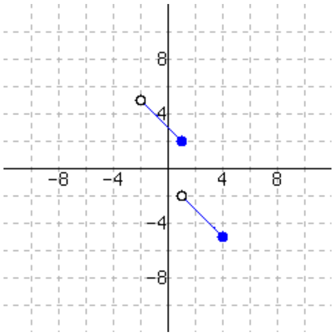
The function has domain and range .

35.



The function has domain and range .

36.



The function has domain and range .

Domain From a Formula

37. Find the domain of H where $H(x) = -x + 4$.

38. Find the domain of K where $K(x) = -8x - 9$.

39. Find the domain of f where $f(x) = \frac{8}{7}x^4$.

40. Find the domain of f where $f(x) = \frac{5}{7}x^2$.

41. Find the domain of g where $g(x) = |-9x - 3|$.

42. Find the domain of h where $h(x) = |5x + 5|$.

43. Find the domain of F where $F(x) = \frac{2x}{x+2}$.

44. Find the domain of F where $F(x) = \frac{5x}{x+10}$.

45. Find the domain of G where $G(x) = \frac{x}{7x-4}$.

46. Find the domain of H where $H(x) = \frac{4x}{7x+3}$.

47. Find the domain of K where $K(x) = \frac{10x+7}{x^2+14x+40}$.

48. Find the domain of f where $f(x) = \frac{3x-5}{x^2+8x-20}$.

49. Find the domain of f where $f(x) = \frac{4-4x}{x^2-7x}$.

50. Find the domain of g where $g(x) = \frac{9x-9}{x^2+8x}$.

51. Find the domain of h where $h(x) = \frac{2x+2}{x^2-1}$.

52. Find the domain of F where $F(x) = \frac{9-5x}{x^2-49}$.

53. Find the domain of F where $F(x) = \frac{9x-3}{16x^2-25}$.

54. Find the domain of G where $G(x) = \frac{x+5}{64x^2-25}$.

55. Find the domain of H where $H(x) = -\frac{6x+7}{x^2+5}$.

56. Find the domain of K where $K(x) = \frac{8x+2}{x^2+3}$.

57. Find the domain of the function.

$f(x) = -\frac{10}{\sqrt{x+9}}$

58. Find the domain of the function.

$f(x) = -\frac{7}{\sqrt{x-2}}$

59. Find the domain of the function.

$g(x) = \sqrt{9-x}$

60. Find the domain of the function.

$h(x) = \sqrt{6-x}$

61. Find the domain of the function.

$F(x) = \sqrt{3+14x}$

62. Find the domain of the function.

$F(x) = \sqrt{9+16x}$

63. Find the domain of A where $A(x) = \frac{x+14}{x^2-81}$.

64. Find the domain of p where $p(x) = \frac{x+16}{x^2-4}$.

65. Find the domain of a where $a(x) = \frac{16x+6}{x^2+7x-98}$.

66. Find the domain of m where $m(x) = \frac{16x-3}{x^2+2x-8}$.

67. Find the domain of r where $r(x) = \frac{\sqrt{3+x}}{1-x}$.

68. Find the domain of B where $B(x) = \frac{\sqrt{5+x}}{7-x}$.

Domain and Range Using Context

69. Ross bought a used car for \$9,000. The car's value decreases at a constant rate each year. After 7 years, the value decreased to \$6,900.

Use a function to model the car's value as the number of years increases. Find this function's domain and range in this context.

The function's domain in this context is ☐.

The function's range in this context is ☐.

70. Michael bought a used car for \$8,400. The car's value decreases at a constant rate each year. After 7 years, the value decreased to \$5,600.

Use a function to model the car's value as the number of years increases. Find this function's domain and range in this context.

The function's domain in this context is ☐.

The function's range in this context is ☐.

71. Assume a car uses gas at a constant rate. After driving 25 miles since a full tank of gas was purchased, there was 15.75 gallons of gas left; after driving 50 miles since a full tank of gas was purchased, there was 13.5 gallons of gas left.

Use a function to model the amount of gas in the tank (in gallons). Let the independent variable be the number of miles driven since a full tank of gas was purchased. Find this function's domain and range in this context.

The function's domain in this context is ☐.

The function's range in this context is ☐.

72. Assume a car uses gas at a constant rate. After driving 20 miles since a full tank of gas was purchased, there was 7.2 gallons of gas left; after driving 60 miles since a full tank of gas was purchased, there was 5.6 gallons of gas left.

Use a function to model the amount of gas in the tank (in gallons). Let the independent variable be the number of miles driven since a full tank of gas was purchased. Find this function's domain and range in this context.

The function's domain in this context is ☐.

The function's range in this context is ☐.

73. Henry inherited a collection of coins when he was 14 years old. Ever since, he has been adding into the collection the same number of coins each year. When he was 21 years old, there were 580 coins in the collection. When he was 29 years old, there were 900 coins in the collection. At the age of 51, Henry donated all his coins to a museum.

Use a function to model the number of coins in Henry's collection, starting in the year he inherited the collection, and ending in the year the collection was donated. Find this function's domain and range in this context.

The function's domain in this context is ☐.

The function's range in this context is ☐.

74. Scot inherited a collection of coins when he was 15 years old. Ever since, he has been adding into the collection the same number of coins each year. When he was 21 years old, there were 380 coins in the collection. When he was 29 years old, there were 540 coins in the collection. At the age of 57, Scot donated all his coins to a museum.

Use a function to model the number of coins in Scot's collection, starting in the year he inherited the collection, and ending in the year the collection was donated. Find this function's domain and range in this context.

The function's domain in this context is ☐.

The function's range in this context is ☐.

75. Assume a tree grows at a constant rate. When the tree was planted, it was 4 feet tall. After 8 years, the tree grew to 6.4 feet tall.

Use a function to model the tree's height as years go by. Assume the tree can live 200 years, find this function's domain and range in this context.

The function's domain in this context is ☐.

The function's range in this context is ☐.

77. An object was shot up into the air at an initial vertical speed of 384 feet per second. Its height as time passes can be modeled by the quadratic function f, where $f(t) = -16t^2 + 384t$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height in feet.

Find the function's domain and range in this context.

The function's domain in this context is ☐.

The function's range in this context is ☐.

79. From a clifftop over the ocean 421.89 m above sea level, an object was shot straight up into the air with an initial vertical speed of 125.93 $\frac{\text{m}}{\text{s}}$. On its way down it missed the cliff and fell into the ocean. Its height (above sea level) as time passes can be modeled by the quadratic function f, where $f(t) = -4.9t^2 + 125.93t + 421.89$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height (above sea level) in meters.

Find the function's domain and range in this context.

The function's domain in this context is ☐.

The function's range in this context is ☐.

76. Assume a tree grows at a constant rate. When the tree was planted, it was 2.2 feet tall. After 10 years, the tree grew to 10.2 feet tall.

Use a function to model the tree's height as years go by. Assume the tree can live 180 years, find this function's domain and range in this context.

The function's domain in this context is ☐.

The function's range in this context is ☐.

78. An object was shot up into the air at an initial vertical speed of 416 feet per second. Its height as time passes can be modeled by the quadratic function f, where $f(t) = -16t^2 + 416t$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height in feet.

Find the function's domain and range in this context.

The function's domain in this context is ☐.

The function's range in this context is ☐.

80. From a clifftop over the ocean 370.44 m above sea level, an object was shot straight up into the air with an initial vertical speed of 108.78 $\frac{\text{m}}{\text{s}}$. On its way down it missed the cliff and fell into the ocean. Its height (above sea level) as time passes can be modeled by the quadratic function f, where $f(t) = -4.9t^2 + 108.78t + 370.44$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height (above sea level) in meters.

Find the function's domain and range in this context.

The function's domain in this context is ☐.

The function's range in this context is ☐.

81. You will build a rectangular sheep pen next to a river. There is no need to build a fence along the river, so you only need to build three sides. You have a total of 470 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum area.

Use a function to model the area of the rectangular pen, with respect to the length of the width (the two sides perpendicular to the river). Find the function's domain and range in this context.

The function's domain is ______.

The function's range is ______.

82. You will build a rectangular sheep pen next to a river. There is no need to build a fence along the river, so you only need to build three sides. You have a total of 480 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum area.

Use a function to model the area of the rectangular pen, with respect to the length of the width (the two sides perpendicular to the river). Find the function's domain and range in this context.

The function's domain is ______.

The function's range is ______.

83. A student's first name is a function of their student identification number.

(a) Describe the domain for this function in a sentence. Specifics are not needed.

(b) Describe the range for this function in a sentence. Specifics are not needed.

84. The year a car was made is a function of its VIN (Vehicle Identification Number).

(a) Describe the domain for this function in a sentence. Specifics are not needed.

(b) Describe the range for this function in a sentence. Specifics are not needed.

Challenge

85. For each part, sketch the graph of a function with the given domain and range.

a. The domain is $(0, \infty)$ and the range is $(-\infty, 0)$.

b. The domain is $(1, 2)$ and the range is $(3, 4)$.

c. The domain is $(0, \infty)$ and the range is $[2, 3]$.

d. The domain is $(1, 2)$ and the range is $(-\infty, \infty)$.

e. The domain is $(-\infty, \infty)$ and the range is $(-1, 1)$.

f. The domain is $(0, \infty)$ and the range is $[0, \infty)$.

10.3 Using Technology to Explore Functions

Graphing technology allows us to explore the properties of functions more deeply than we can with only pencil and paper. It can quickly create a table of values, and quickly plot the graph of a function. Such technology can also evaluate functions, solve equations with functions, find maximum and minimum values, and explore other key features.

There are many graphing technologies currently available, including (but not limited to) physical (handheld) graphing calculators, *Desmos*, *GeoGebra*, *Sage*, and *WolframAlpha*.

This section will focus on *how* technology can be used to explore functions and their key features. Although the choice of particular graphing technology varies by each school and curriculum, the main ways in which technology is used to explore functions is the same and can be done with each of the technologies above.

10.3.1 Finding an Appropriate Window

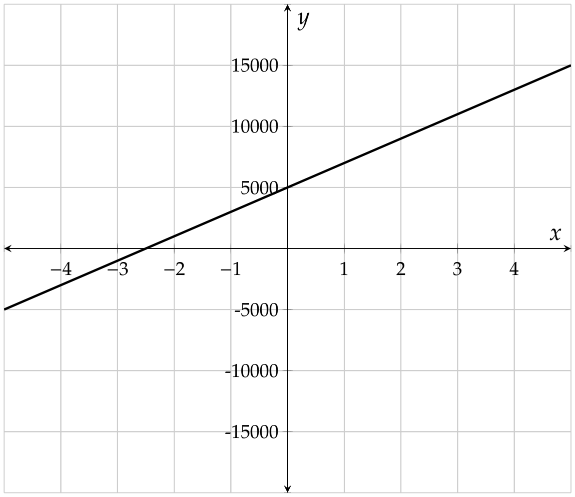
With a simple linear equation like $y = 2x + 5$, most graphing technologies will show this graph in a good window by default. A common default window goes from $x = -10$ to $x = 10$ and $y = -10$ to $y = 10$.

What if we wanted to graph something with a much larger magnitude though, such as $y = 2000x + 5000$? If we tried to view this for $x = -10$ to $x = 10$ and $y = -10$ to $y = 10$, the function would appear as an almost vertical line since it has such a steep slope.

Using technology, we will create a table of values for this function as shown in Figure 10.1a. Then we will to set the x-values for which we view the function to go from $x = -5$ to $x = 5$ and the y-values from $y = -20{,}000$ to $y = 20{,}000$. The graph is shown in Figure 10.1b.

x	$y = 2000x + 5000$
−5	−5000
−4	−3000
−3	−1000
−2	1000
−1	3000
0	5000
1	7000
2	9000
3	11000
4	13000
5	15000

(a) A table of values



(b) Graphed with an appropriate window

Figure 10.3.1: Creating a table of values to determine an appropriate graphing window

Now let's practice finding an appropriate viewing window with a less familiar function.

Example 10.3.2 Find an appropriate window for $q(x) = \frac{x^3}{100} - 2x + 1$.

Entering this function into graphing technology, we input `q(x)=(x^3)/100-2x+1`. A default window will generally give us something like this:

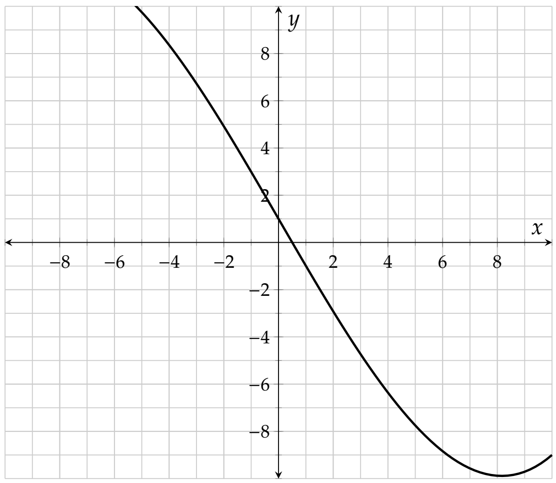


Figure 10.3.3: Function q graphed in the default window.

We can tell from the lower right corner of Figure 10.3.3 that we're not quite viewing all of the important details of this function. To determine a better window, we could use technology to make a table of values. Another more rudimentary option is to double the viewing constraints for x and y, as shown in Figure 10.3.4. Many graphing technologies have the ability to zoom in and out quickly.

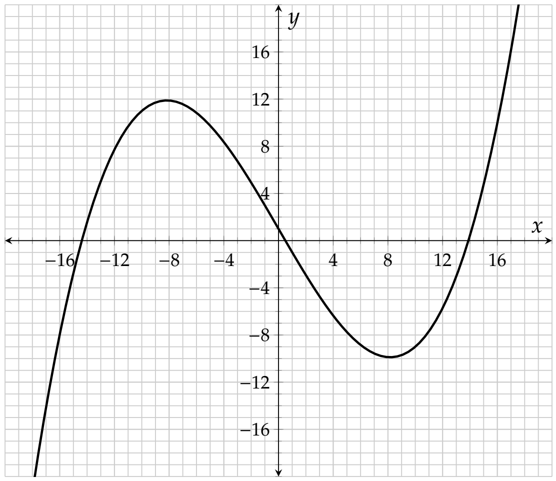


Figure 10.3.4: Function q graphed in an expanded window.

10.3.2 Using Technology to Determine Key Features of a Graph

The key features of a graph can be determined using graphing technology. Here, we'll show how to determine the x-intercepts, y-intercepts, maximum/minimum values, and the domain and range using technology.

Example 10.3.5 Graph the function given by $p(x) = -1000x^2 - 100x + 40$. Determine an appropriate viewing window, and then use graphing technology to determine the following:

a. Determine the x-intercepts of the function.

b. Determine the y-intercept of the function.

c. Determine the maximum function value and where it occurs.

d. State the domain and range of this function.

Explanation.

To start, we'll take a quick view of this function in a default window. We can see that we need to zoom in on the x-values, but we need to zoom out on the y-values.

From the graph we see that the x-values might as well run from about -0.5 to 0.5, so we will look at x-values in that window in increments of 0.1, as shown in Table 10.2a. This table allows us to determine an appropriate viewing window for $y = p(x)$ which is shown in Figure 10.2b. The table suggests we should go a little higher than 40 on the y-axis, and it would be OK to go the same distance in the negative direction to keep the x-axis centered.

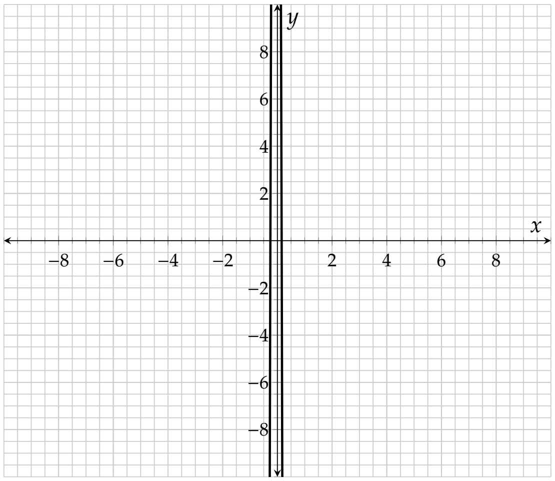
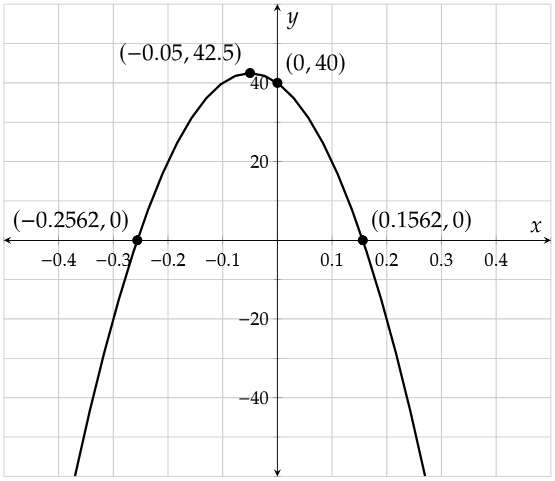


Figure 10.3.6: Graph of $y = p(x)$ in an inappropriate window

x	$p(x)$
-0.5	-160
-0.4	-80
-0.3	-20
-0.2	20
-0.1	40
0	40
0.1	20
0.2	-20
0.3	-80
0.4	-160
0.5	-260

(a) Function values for $y = p(x)$



(b) Graph of $y = p(x)$ in an appropriate window showing key features

Figure 10.3.6: Creating a table of values to determine an appropriate graphing window

We can now use Figure 10.2b to determine the x-intercepts, the y-intercept, the maximum function value, and the domain and range.

a. To determine the x-intercepts, we will find the points where y is zero. These are about $(-0.2562, 0)$ and $(0.1562, 0)$.

b. To determine the y-intercept, we need the point where x is zero. This point is $(0, 40)$.

c. The highest point on the graph is the vertex, which is about $(-0.05, 42.5)$. So the maximum function value is 42.5 and occurs at -0.05.

d. We can see that the function is defined for all x-values, so the domain is $(-\infty, \infty)$. The maximum function value is 42.5, and there is no minimum function value. Thus the range is $(-\infty, 42.5]$.

If we use graphing technology to graph the function g where $g(x) = 0.0002x^2 + 0.00146x + 0.00266$, we may be mislead by the way values are rounded. Without technology, we know that this function is a quadratic function and therefore has at most two x-intercepts and has a vertex that will determine the minimum function value. However, using technology we could obtain a graph with the following key points:

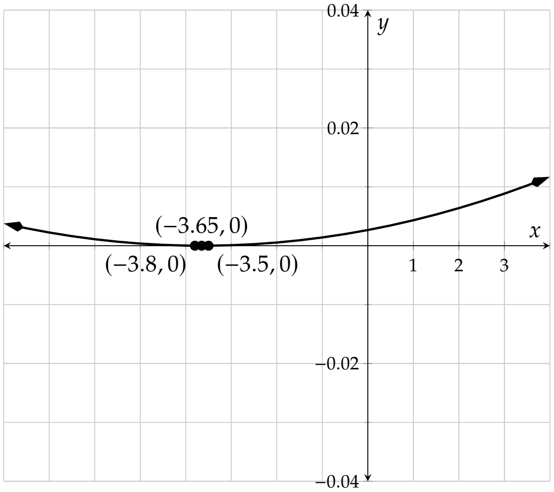


Figure 10.3.8: Misleading graph

Example 10.3.7 This *looks* like there are three x-intercepts, which we know is not possible for a quadratic function. We can evaluate g at $x = -3.65$ and determine that $g(-3.65) = -0.0000045$, which is *approximately* zero when rounded. So the true vertex of this function is $(-3.65, -0.0000045)$, and the minimum value of this function is -0.0000045 (not zero).

Every graphing tool generally has some type of limitation like this one, and it's good to be aware that these limitations exist.

10.3.3 Solving Equations and Inequalities Graphically Using Technology

To *algebraically* solve an equation like $h(x) = v(x)$ for

$$h(x) = -0.01(x - 90)(x + 20) \qquad \text{and} \qquad v(x) = -0.04(x - 10)(x - 80),$$

we'd start by setting up

$$-0.01(x - 90)(x + 20) = -0.04(x - 10)(x - 80)$$

To solve this, we'd then simplify each side of the equation, set it equal to zero, and finally use the quadratic formula.

An alternative is to *graphically* solve this equation, which is done by graphing

$$y = -0.01(x-90)(x+20) \qquad \text{and} \qquad y = -0.04(x-10)(x-80).$$

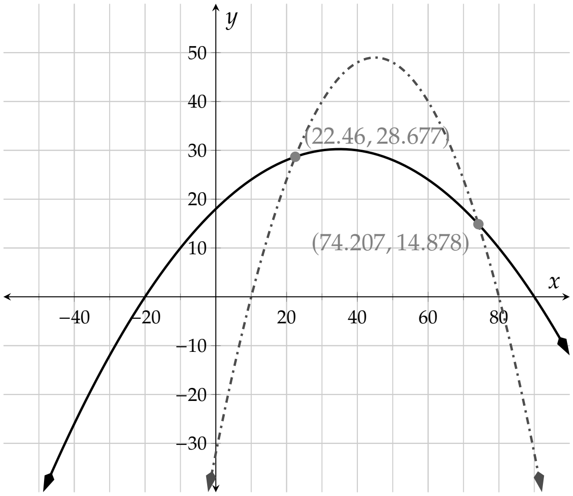


Figure 10.3.9: Points of intersection for $h(x) = v(x)$

The points of intersection, $(22.46, 28.677)$ and $(74.207, 14.878)$, show where these functions are equal. This means that the x-values give the solutions to the equation $-0.01(x-90)(x+20) = -0.04(x-10)(x-80)$. So the solutions are approximately 22.46 and 74.207, and the solution set is approximately $\{22.46, 74.207\}$.

Similarly, to *graphically* solve an equation like $h(x) = 25$ for

$$h(x) = -0.01(x-90)(x+20),$$

we can graph

$$y = -0.01(x-90)(x+20) \qquad \text{and} \qquad y = 25$$

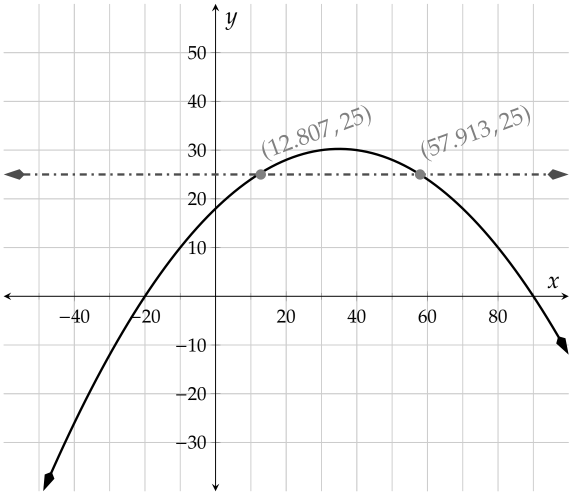


Figure 10.3.10: Points of intersection for $h(x) = 25$

The points of intersection are $(12.807, 25)$ and $(57.913, 25)$, which tells us that the solutions to $h(x) = 25$ are approximately 12.807 and 57.913. The solution set is approximately $\{12.807, 57.913\}$.

Example 10.3.11 Use graphing technology to solve the following inequalities:

a. $-20t^2 - 70t + 300 \geq -5t + 300$

b. $-20t^2 - 70t + 300 < -5t + 300$

Explanation. To solve these inequalities graphically, we will start by graphing the equations $y = -20t^2 - 70t + 300$ and $y = -5t + 300$ and determining the points of intersection:

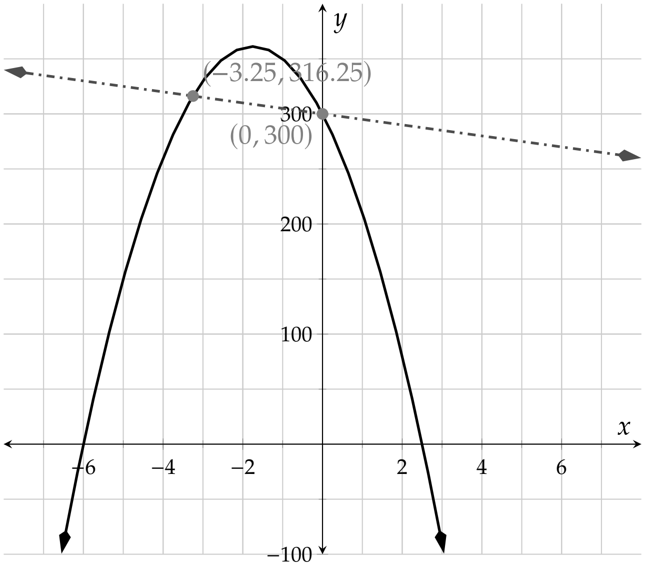


Figure 10.3.12: Points of intersection for $y = -20t^2 - 70t + 300$ and $y = -5t + 300$

To solve $-20t^2 - 70t + 300 \geq -5t + 300$, we need to determine where the y-values of the graph of $y = -20t^2 - 70t + 300$ are *greater* than the y-values of the graph of $y = -5t + 300$ in addition to the values where the y-values are equal. This region is highlighted in Figure 10.3.13.

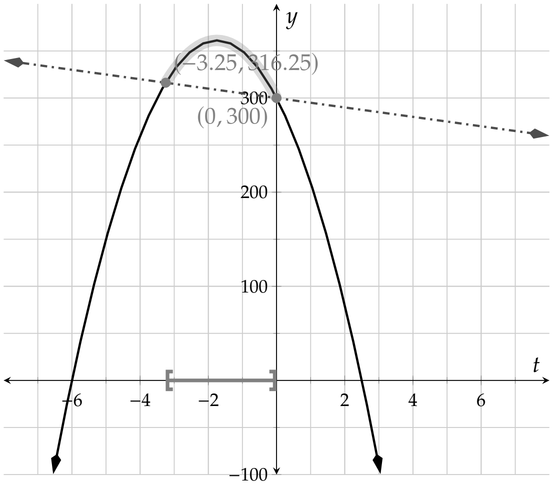


Figure 10.3.13

a. We can see that this region includes all values of t between, and including, $t = -3.25$ and $t = 0$. So the solutions to this inequality include all values of t for which $-3.25 \le t \le 0$. We can write this solution set in interval notation as $[-3.25, 0]$ or in set-builder notation as $\{t \mid -3.25 \le t \le 0\}$.

b. To now solve $-20t^2 - 70t + 300 < -5t + 300$, we will need to determine where the y-values of the graph of $y = -20t^2 - 70t + 300$ are *less than* the y-values of the graph of $y = -5t + 300$. This region is highlighted in Figure 10.3.14.

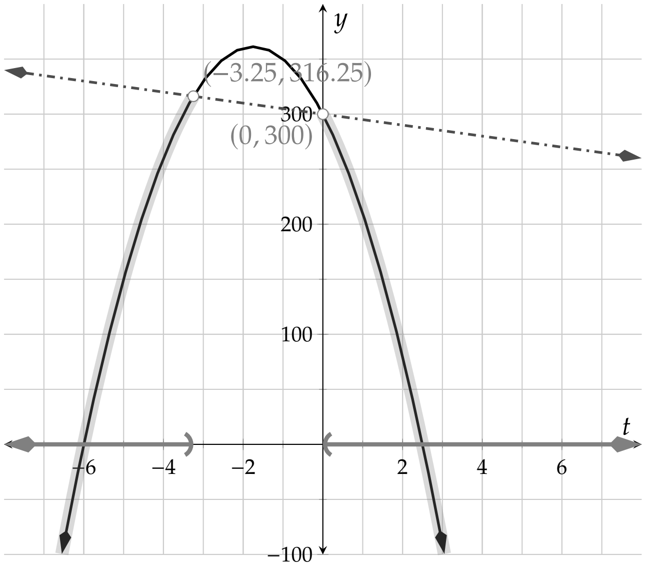


Figure 10.3.14

We can see that $-20t^2 - 70t + 300 < -5t + 300$ for all values of t where $t < -3.25$ or $t > 0$. We can write this solution set in interval notation as $(-\infty, -3.25) \cup (0, \infty)$ or in set-builder notation as $\{t \mid t < -3.25 \text{ or } t > 0\}$.

Exercises

Using Technology to Create a Table of Function Values Use technology to make a table of values for the function.

1. $K(x) = -4x^2 + 15x - 4$

x	$K(x)$

2. $K(x) = -3x^2 + 18x + 3$

x	$K(x)$

3. $f(x) = 2.25x^2 + 70x - 67$

x	$f(x)$

4. $g(x) = -0.5x^2 - 170x + 79$

x	$g(x)$

5. $h(x) = -10x^3 + 10x + 23$

x	$h(x)$

6. $F(x) = 6x^3 + 180x - 33$

x	$F(x)$

Determining Appropriate Windows

7. Let $f(x) = -5943x - 4132$. Choose an appropriate window for graphing f that shows its key features.

The x-interval could be ______ and the y-interval could be ______.

8. Let $f(x) = -663x + 767$. Choose an appropriate window for graphing f that shows its key features.

The x-interval could be ______ and the y-interval could be ______.

9. Let $f(x) = 772x^2 + 189x - 4162$. Choose an appropriate window for graphing f that shows its key features.

The x-interval could be ______ and the y-interval could be ______.

10. Let $f(x) = -882x^2 - 602x + 4033$. Choose an appropriate window for graphing f that shows its key features.

The x-interval could be ______ and the y-interval could be ______.

11. Let $f(x) = -0.0005x^2 + 0.001x - 0.41$. Choose an appropriate window for graphing f that shows its key features.

The x-interval could be ☐ and the y-interval could be ☐.

12. Let $f(x) = 0.00014x^2 + 0.0027x + 0.4$. Choose an appropriate window for graphing f that shows its key features.

The x-interval could be ☐ and the y-interval could be ☐.

Finding Points of Intersection

13. Use technology to determine how many times the equations $y = (350 - 12x)(-102 - 8x)$ and $y = -6000$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

14. Use technology to determine how many times the equations $y = (-234 - 7x)(-380 + 19x)$ and $y = -6000$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

15. Use technology to determine how many times the equations $y = -x^3 + x^2 + 3x$ and $y = 7x - 3$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

16. Use technology to determine how many times the equations $y = x^3 - 2x^2 - 8x$ and $y = x - 3$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

17. Use technology to determine how many times the equations $y = -0.7\left(6x^2 - 4\right)$ and $y = -0.11(9x - 4)$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

18. Use technology to determine how many times the equations $y = -0.4\left(7x^2 + 8\right)$ and $y = 0.19(6x - 9)$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

19. Use technology to determine how many times the equations $y = 1.55(x + 5)^2 - 2.7$ and $y = 0.95x - 1$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

20. Use technology to determine how many times the equations $y = 2(x - 7)^2 + 4.85$ and $y = -0.15x - 1$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

Using Technology to Find Key Features of a Graph

21. For the function j defined by

$$j(x) = -\frac{2}{5}(x-3)^2 + 6,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.
b. The vertex.
c. The domain.
d. The range.

22. For the function k defined by

$$k(x) = 2(x+1)^2 + 10,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.
b. The vertex.
c. The domain.
d. The range.

23. For the function L defined by

$$L(x) = 3000x^2 + 10x + 4,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.
b. The vertex.
c. The domain.
d. The range.

24. For the function M defined by

$$M(x) = -(300x - 2950)^2,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.
b. The vertex.
c. The domain.
d. The range.

25. For the function N defined by

$$N(x) = (300x - 1.05)^2,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.
b. The vertex.
c. The domain.
d. The range.

26. For the function B defined by

$$B(x) = x^2 - 0.05x + 0.0006,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.
b. The vertex.
c. The domain.
d. The range.

Solving Equations and Inequalities Graphically Using Technology

27. Let $s(x) = \frac{1}{5}x^2 - 2x + 10$ and $t(x) = -x + 40$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $s(x) = t(x)$.

c. Solve $s(x) > t(x)$.

d. Solve $s(x) \leq t(x)$.

28. Let $w(x) = \frac{1}{4}x^2 - 3x - 8$ and $m(x) = x + 12$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $w(x) = m(x)$.

c. Solve $w(x) > m(x)$.

d. Solve $w(x) \leq m(x)$.

29. Let $f(x) = 4x^2 + 5x - 1$ and $g(x) = 5$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $f(x) = g(x)$.

c. Solve $f(x) < g(x)$.

d. Solve $f(x) \geq g(x)$.

30. Let $p(x) = 6x^2 - 3x + 4$ and $k(x) = 7$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $p(x) = k(x)$.

c. Solve $p(x) < k(x)$.

d. Solve $p(x) \geq k(x)$.

31. Let $q(x) = -4x^2 - 24x + 10$ and $r(x) = 2x + 22$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $q(x) = r(x)$.

c. Solve $q(x) > r(x)$.

d. Solve $q(x) \leq r(x)$.

32. Let $h(x) = -10x^2 - 5x + 3$ and $j(x) = -3x - 9$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $h(x) = j(x)$.

c. Solve $h(x) > j(x)$.

d. Solve $h(x) \leq j(x)$.

33. Use graphing technology to solve the equation $0.4x^2 + 0.5x - 0.2 = 2.4$. Approximate the solution(s) if necessary

34. Use graphing technology to solve the equation $-0.25x^2 - 2x + 1.75 = 4.75$. Approximate the solution(s) if necessary

35. Use graphing technology to solve the equation $(200+5x)(100-2x) = 15000$. Approximate the solution(s) if necessary

36. Use graphing technology to solve the equation $(200 - 5x)(100 + 10x) = 20000$. Approximate the solution(s) if necessary.

37. Use graphing technology to solve the equation $2x^3 - 5x + 1 = -\frac{1}{2}x + 1$. Approximate the solution(s) if necessary.

39. Use graphing technology to solve the equation $-0.05x^2 - 2.03x - 19.6 = 0.05x^2 + 1.97x + 19.4$. Approximate the solution(s) if necessary.

41. Use graphing technology to solve the equation $-200x^2 + 60x - 55 = -20x - 40$. Approximate the solution(s) if necessary.

43. Use graphing technology to solve the inequality $2x^2 + 5x - 3 > -5$. State the solution set using interval notation, and approximate if necessary.

45. Use graphing technology to solve the inequality $10x^2 - 11x + 7 \le 7$. State the solution set using interval notation, and approximate if necessary.

47. Use graphing technology to solve the inequality $-x^2 - 6x + 1 > x + 5$. State the solution set using interval notation, and approximate if necessary.

49. Use graphing technology to solve the inequality $-10x + 4 \le 20x^2 - 34x + 6$. State the solution set using interval notation, and approximate if necessary.

51. Use graphing technology to solve the inequality $\frac{1}{2}x^2 + \frac{3}{2}x \ge \frac{1}{2}x - \frac{3}{2}$. State the solution set using interval notation, and approximate if necessary.

38. Use graphing technology to solve the equation $-x^3 + 8x = -4x + 16$. Approximate the solution(s) if necessary.

40. Use graphing technology to solve the equation $-0.02x^2 + 1.97x - 51.5 = 0.05\,(x - 50)^2 - 0.03\,(x - 50)$. Approximate the solution(s) if necessary.

42. Use graphing technology to solve the equation $150x^2 - 20x + 50 = 100x + 40$. Approximate the solution(s) if necessary.

44. Use graphing technology to solve the inequality $-x^2 + 4x - 7 > -12$. State the solution set using interval notation, and approximate if necessary.

46. Use graphing technology to solve the inequality $-10x^2 - 15x + 4 \le 9$. State the solution set using interval notation, and approximate if necessary.

48. Use graphing technology to solve the inequality $3x^2 + 5x - 4 > -2x + 1$. State the solution set using interval notation, and approximate if necessary.

50. Use graphing technology to solve the inequality $-15x^2 - 6 \le 10x - 4$. State the solution set using interval notation, and approximate if necessary.

52. Use graphing technology to solve the inequality $\frac{3}{4}x \ge \frac{1}{4}x^2 - 3x$. State the solution set using interval notation, and approximate if necessary.

10.4 Simplifying Expressions with Function Notation

In this section, we will discuss algebra simplification that will appear in many facets of education. Simplification is a skill, like cooking noodles or painting a wall. It may not always be exciting, but it does serve a purpose. Also like cooking noodles or painting a wall, it isn't usually difficult, and yet there are common avoidable mistakes that people make. With practice from this section, you'll have experience to prevent yourself from overcooking the noodles or ruining your paintbrush.

10.4.1 Negative Signs in and out of Function Notation

Let's start by reminding ourselves about the meaning of function notation. When we write $f(x)$, we have a process f that is doing something to an input value x. Whatever is inside those parentheses is the input to the function. What if we use something for input that is not quite as simple as "x?"

> **Example 10.4.2** Find and simplify a formula for $f(-x)$, where $f(x) = x^2 + 3x - 4$.
>
> **Explanation**. Those parentheses encase "$-x$," so we are meant to treat "$-x$" as the input. The rule that we have been given for f is
>
> $$f(x) = x^2 + 3x - 4.$$
>
> But the x's that are in this formula are just place-holders. What f does to a number can just as easily be communicated with
>
> $$f(\ \ ) = (\ \ )^2 + 3(\ \ ) - 4.$$
>
> So now that we are meant to treat "$-x$" as the input, we will insert "$-x$" into those slots, after which we can do more familiar algebraic simplification:
>
> $$\begin{aligned} f(\quad\ ) &= (\quad\ )^2 + 3(\quad\ ) - 4 \\ f(-x) &= (-x)^2 + 3(-x) - 4 \\ &= x^2 - 3x - 4 \end{aligned}$$

The previous example contrasts nicely with this one:

> **Example 10.4.3** Find and simplify a formula for $-f(x)$, where $f(x) = x^2 + 3x - 4$.
>
> **Explanation**. Here, the parentheses only encase "x." The negative sign is on the outside. So the way to see this expression is that first f will do what it does to x, and then that result will be negated:
>
> $$\begin{aligned} -f(x) &= -(x^2 + 3x - 4) \\ &= -x^2 - 3x + 4 \end{aligned}$$
>
> Note that the answer to this exercise, which was to simplify $-f(x)$, is different from the answer to Example 10.4.2, which was to simplify $f(-x)$. In general you cannot pass a negative sign in and out of function notation and still have the same quantity.

In Example 10.4.2 and Example 10.4.3, we are working with the expressions $f(-x)$ and $-f(x)$, and trying to find "simplified" formulas. If it seems strange to be doing these things, perhaps this applied example will help.

Checkpoint 10.4.4. The NASDAQ Composite Index measures how well a portion of the stock market is doing. Suppose $N(t)$ is the value of the index t days after January 1, 2018. A formula for N is $N(t) = 3.34t^2 + 26.2t + 6980$.

What if you wanted a new function, B, that gives the value of the NASDAQ index t days *before* January 1, 2018? Technically, t days *before* is the same as *negative* t days after. So $B(t)$ is the same as $N(-t)$, and now the expression $N(-t)$ means something. Find a simplified formula for $N(-t)$.

$N(-t) = \boxed{\qquad\qquad\qquad\qquad}$

Explanation.

$$\begin{aligned} N(\quad) &= 3.34(\quad)^2 + 26.2(\quad) + 6980 \\ N(-t) &= 3.34(-t)^2 + 26.2(-t) + 6980 \\ &= 3.34t^2 - 26.2t + 6980 \end{aligned}$$

10.4.2 Other Nontrivial Simplifications

Example 10.4.5 Find and simplify a formula for $h(5x)$, where $h(x) = \frac{x}{x-2}$.

Explanation. The parentheses encase "$5x$," so we are meant to treat "$5x$" as the input.

$$\begin{aligned} h(\quad) &= \frac{(\quad)}{(\quad) - 2} \\ h(5x) &= \frac{5x}{5x - 2} \\ &= \frac{5x}{5x - 2} \end{aligned}$$

Example 10.4.6 Find and simplify a formula for $\frac{1}{3}g(3x)$, where $g(x) = 2x^2 + 8$.

Explanation. Do the $\frac{1}{3}$ and the 3 cancel each other? No. The 3 is part of the input, affecting x right away. Then g does whatever it does to $3x$, and *then* we multiply the result by $\frac{1}{3}$. Since the function g acts "in between," we don't have the chance to cancel the 3 with the $\frac{1}{3}$. Let's see what actually happens:

Those parentheses encase "$3x$," so we are meant to treat "$3x$" as the input. We will keep the $\frac{1}{3}$ where it is until it is possible to simplify:

$$\begin{aligned} \frac{1}{3}g(\quad) &= \frac{1}{3}\left(2(\quad)^2 + 8\right) \\ \frac{1}{3}g(3x) &= \frac{1}{3}\left(2(3x)^2 + 8\right) \\ &= \frac{1}{3}\left(2\left(9x^2\right) + 8\right) \\ &= \frac{1}{3}\left(18x^2 + 8\right) \\ &= 6x^2 + \frac{8}{3} \end{aligned}$$

Example 10.4.7 If $k(x) = x^2 - 3x$, find and simplify a formula for $k(x - 4)$.

Explanation. This type of exercise is often challenging for algebra students. But let's focus on those parentheses one more time. They encase "$x - 4$," so we are meant to treat "$x - 4$" as the input.

$$\begin{aligned}k(\quad) &= (\quad)^2 - 3(\quad)\\ k(x-4) &= (x-4)^2 - 3(x-4)\\ &= x^2 - 8x + 16 - 3x + 12\\ &= x^2 - 11x + 28\end{aligned}$$

Checkpoint 10.4.8. If $q(x) = x + \sqrt{x+8}$, find and simplify a formula for $q(x+5)$.

$q(x+5) =$ ______

Explanation. Starting with the generic formula for q:

$$\begin{aligned}q(\quad) &= (\quad) + \sqrt{(\quad) + 8}\\ q(x+5) &= x + 5 + \sqrt{x+5+8}\\ &= x + 5 + \sqrt{x+13}\end{aligned}$$

Example 10.4.9 If $f(x) = \frac{1}{x}$, find and simplify a formula for $f(x+3) + 2$.

Explanation. Do not be tempted to add the 3 and the 2. The 3 is added to input *before* the function f does its work. The 2 is added to the result *after* f has done its work.

$$\begin{aligned}f(\quad) + 2 &= \frac{1}{(\quad)} + 2\\ f(x+3) + 2 &= \frac{1}{x+3} + 2\end{aligned}$$

This last expression is considered fully simplified. However you might combine the two terms using a technique from Section 13.3.

The tasks we have practiced in this section are the kind of tasks that will make it easier to understand interesting and useful material in college algebra and calculus.

Exercises

Review and Warmup

1. Use the distributive property to write an equivalent expression to $8(p+5)$ that has no grouping symbols.

2. Use the distributive property to write an equivalent expression to $5(q+8)$ that has no grouping symbols.

3. Use the distributive property to write an equivalent expression to $-10(y-6)$ that has no grouping symbols.

4. Use the distributive property to write an equivalent expression to $-5(r+2)$ that has no grouping symbols.

5. Multiply the polynomials.

$2(y+4)^2 =$ ______

6. Multiply the polynomials.

$4(y+10)^2 =$ ______

7. Expand the square of a *bi*nomial.

$(7r+9)^2 = \boxed{\qquad\qquad}$

8. Expand the square of a *bi*nomial.

$(4r+3)^2 = \boxed{\qquad\qquad}$

Simplifying Function Expressions

9. Simplify $K(r+7)$, where $K(r) = 4 + r$.

10. Simplify $G(t+2)$, where $G(t) = 3 - 4t$.

11. Simplify $g(-t)$, where $g(t) = 3 + 8t$.

12. Simplify $K(-x)$, where $K(x) = 4 + 8x$.

13. Simplify $F(x+4)$, where $F(x) = 3 - 1.1x$.

14. Simplify $g(y+8)$, where $g(y) = 2 - 5.5y$.

15. Simplify $H(y - \frac{2}{3})$, where $H(y) = -\frac{8}{3} + \frac{2}{9}y$.

16. Simplify $F(r + \frac{1}{3})$, where $F(r) = -\frac{7}{6} + \frac{2}{5}r$.

17. Simplify $f(r) + 1$, where $f(r) = -3r + 2$.

18. Simplify $H(r) + 5$, where $H(r) = -8r + 2$.

19. Simplify $F(t) + 8$, where $F(t) = 1 + 4.4t$.

20. Simplify $f(t) + 3$, where $f(t) = 1 - 0.1t$.

21. Simplify $H(7x)$, where $H(x) = -5x^2 + x + 8$.

22. Simplify $h(2x)$, where $h(x) = 7x^2 + x - 1$.

23. Simplify $f(-y)$, where $f(y) = y^2 + 3y + 7$.

24. Simplify $G(-y)$, where $G(y) = 8y^2 - 2y - 1$.

25. Simplify $4h(r)$, where $h(r) = -7r^2 + 7r + 8$.

26. Simplify $8f(r)$, where $f(r) = 6r^2 - r - 8$.

27. Simplify $G(r-6)$, where $G(r) = 0.9r^2 + 7r - 6$.

28. Simplify $h(t+2)$, where $h(t) = -3.6t^2 - t - 1$.

29. Simplify $K(t) + 2$, where $K(t) = -8t^2 - t + 7$.

30. Simplify $G(x) + 5$, where $G(x) = 4x^2 - x - 1$.

31. Simplify $g(x+3)$, where $g(x) = \sqrt{-1 - 7x}$.

32. Simplify $h(x+9)$, where $h(x) = \sqrt{-2 - 2x}$.

33. Simplify $h(x) + 6$, where $h(x) = \sqrt{-2 + 6x}$.

34. Simplify $F(x) + 3$, where $F(x) = \sqrt{-2 + x}$.

35. Simplify $G(x+8)$, where $G(x) = 8x + \sqrt{-2 - 5x}$.

36. Simplify $H(x+5)$, where $H(x) = -2x + \sqrt{-2 - 7x}$.

37. Simplify $g(t+4)$, where $g(t) = \frac{8}{5t-3}$.

38. Simplify $HK(t+8)$, where $HK(t) = -\frac{7}{-3t-2}$.

39. Simplify $F(-3x)$, where $F(x) = \frac{2x}{-3x^2+7}$.

40. Simplify $g(6x)$, where $g(x) = \frac{3x}{-3x^2-2}$.

41. Let f be a function given by $f(x) = 4x - 9$. Find and simplify the following:

a. $f(x)+7 =$

b. $f(x+7) =$

c. $7f(x) =$

d. $f(7x) =$

42. Let f be a function given by $f(x) = -5x - 1$. Find and simplify the following:

a. $f(x)+5 =$

b. $f(x+5) =$

c. $5f(x) =$

d. $f(5x) =$

43. Let f be a function given by $f(x) = -4x^2 + 4x$. Find and simplify the following:

a. $f(x)-5 =$

b. $f(x-5) =$

c. $-5f(x) =$

d. $f(-5x) =$

44. Let f be a function given by $f(x) = 4x^2 - 2x$. Find and simplify the following:

a. $f(x)-2 =$

b. $f(x-2) =$

c. $-2f(x) =$

d. $f(-2x) =$

Applications

45. A circular oil slick is expanding with radius, r in feet, at time t in hours given by $r = 18t - 0.3t^2$, for t in hours, $0 \le t \le 10$.

Find a formula for $A = f(t)$, the area of the oil slick as a function of time.

$A = f(t) =$

46. Suppose $T(t)$ represents the temperature outside, in Fahrenheit, at t hours past noon, and a formula for T is $T(t) = \frac{20t}{t^2+1} + 58$.

If we introduce $F(t)$ as the temperature outside, in Fahrenheit, at t hours past 1:00pm, then $F(t) = T(t+1)$. Find a simplified formula for $T(t+1)$.

$T(t+1) =$

47. Suppose $G(t)$ represents how many gigabytes of data has been downloaded t minutes after you started a download.

If we introduce $M(t)$ as how many megabytes of data has been downloaded t minutes after you started a download, then $M(t) = 1024G(t)$. Find a simplified formula for $1024G(t)$.

$1024G(t) =$

10.5 Technical Definition of a Function

In Section 10.1, we discussed a conceptual understanding of functions and Definition 10.1.3. In this section we'll start with a more technical definition of what is a function, consistent with the ideas from Section 10.1.

10.5.1 Formally Defining a Function

Definition 10.5.2 Function (Technical Definition). A **function** is a collection of ordered pairs (x, y) such that any particular value of x is paired with at most one value for y.

How is this definition consistent with the informal Definition 10.1.3, which describes a function as a *process*? Well, if you have a collection of ordered pairs (x, y), you can choose to view the left number as an input, and the right value as the output. If the function's name is f and you want to find $f(x)$ for a particular number x, look in the collection of ordered pairs to see if x appears among the first coordinates. If it does, then $f(x)$ is the (unique) y-value it was paired with. If it does not, then that x is just not in the domain of f, because you have no way to determine what $f(x)$ would be.

Example 10.5.3 Using Definition 10.5.1, a function f could be given by

$$\{(1,4),(2,3),(5,3),(6,1)\}.$$

a. What is $f(1)$? Since the ordered pair $(1,4)$ appears in the collection of ordered pairs, we would say that $f(1) = 4$.

b. What is $f(2)$? Since the ordered pair $(2,3)$ appears in the collection of ordered pairs, we would say that $f(2) = 3$.

c. What is $f(3)$? None of the ordered pairs in the collection start with 3, so $f(3)$ is undefined, and we would say that 3 is not in the domain of f.

Let's spend some time seeing how this new definition applies to things that we already understand as functions from Section 10.1.

Consider the function g expressed by Table 10.5.5. How is this "a collection of ordered pairs?" With tables the connection is most easily apparent. Pair off each x-value with its y-value.

x	$g(x)$
12	0.16
15	3.2
18	1.4
21	1.4
24	0.98

Table 10.5.5

Example 10.5.4 A Function Given as a Table. In this case, we can view this function as:

$$\{(12,0.16),(15,3.2),(18,3.2),(21,1.4),(24,0.98)\}.$$

Example 10.5.6 A Function Given as a Formula. Consider the function h expressed by the formula $h(x) = x^2$. How is this "a collection of ordered pairs?"

This time, the collection is *really big*. Imagine an x-value, like $x = 2$. We can calculate that $f(2) = 2^2 = 4$. So the input 2 pairs with the output 4 and the ordered pair $(2, 4)$ is part of the collection.

You could move on to *any* x-value, like say $x = 2.1$. We can calculate that $f(2.1) = 2.1^2 = 4.41$. So the input 2.1 pairs with the output 4.41 and the ordered pair $(2.1, 4.41)$ is part of the collection.

The collection is so large that we cannot literally list all the ordered pairs as was done in Example 10.5.3 and Example 10.5.4. We just have to imagine this giant collection of ordered pairs. And if it helps to conceptualize it, we know that the ordered pairs $(2, 4)$ and $(2.1, 4.41)$ are included.

Example 10.5.7 A Function Given as a Graph. Consider the functions p and q expressed in Figure 10.5.8 and Figure 10.5.9. How is each of these "a collection of ordered pairs?"

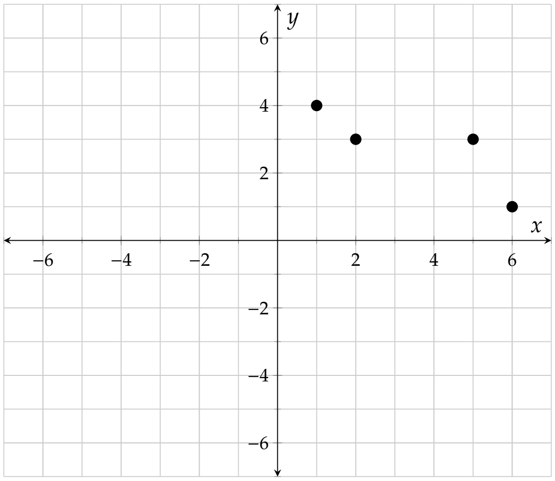


Figure 10.5.8: $y = p(x)$

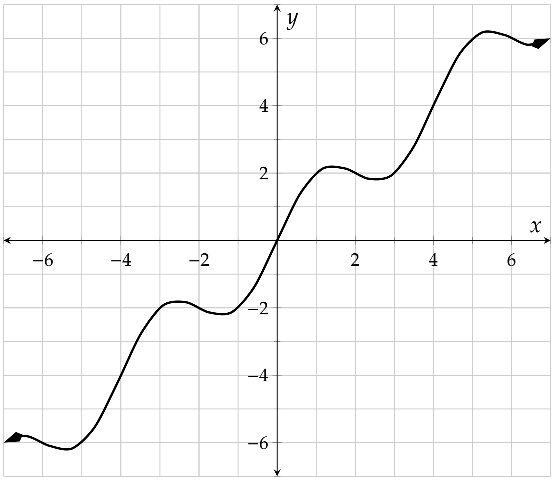


Figure 10.5.9: $y = q(x)$

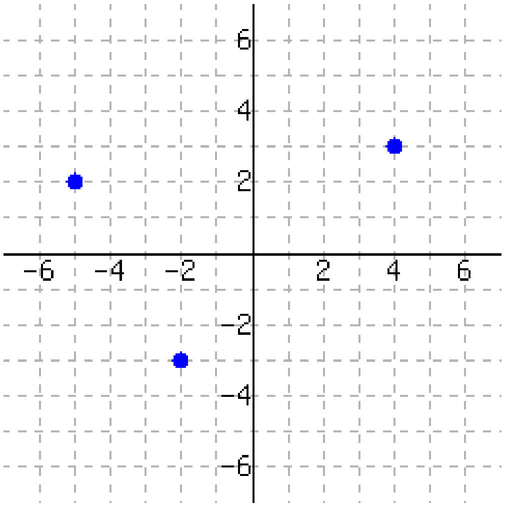
In Figure 10.5.8, we see that $p(1) = 4$, $p(2) = 3$, $p(5) = 3$, and $p(6) = 1$. The graph *literally is* the collection

$$\{(1, 4), (2, 3), (5, 3), (6, 1)\}.$$

In Figure 10.5.9, we can see a few whole number function values, like $q(0) = 0$ and $q(1) = 2$. But the entire curve has infinitely many points on it and we'd never be able to list them all. We just have to imagine the giant collection of ordered pairs. And if it helps to conceptualize it, we know that the ordered pairs $(0, 0)$ and $(1, 2)$ are included.

Try it yourself in the following exercise.

Checkpoint 10.5.10. The graph below is of $y = f(x)$.



Write the function f as a set of ordered pairs.

Explanation. The function can be expressed as the set $\{(-5,2),(-2,-3),(4,3)\}$.

10.5.2 Identifying What is *Not* a Function

Just because you have a set of order pairs, a table, a graph, or an equation, it does not necessarily mean that you have a function. Conceptually, whatever you have needs to give consistent outputs if you feed it the same input. More technically, the set of ordered pairs is not allowed to have two ordered pairs that have the same x-value but different y-values.

> **Example 10.5.11** Consider each set of ordered pairs. Does it make a function?
>
> a. $\left\{(5,9),(3,2),\left(\frac{1}{2},0.6\right),(5,1)\right\}$
>
> b. $\left\{(-5,12),(3,7),\left(\sqrt{2},1\right),(-0.9,4)\right\}$
>
> c. $\left\{(5,9),(3,9),\left(4.2,\sqrt{2}\right),\left(\frac{4}{3},\frac{1}{2}\right)\right\}$
>
> d. $\left\{(5,9),(0.7,2),\left(\sqrt{25},3\right),\left(\frac{2}{3},\frac{3}{2}\right)\right\}$
>
> **Explanation**.
>
> a. This set of ordered pairs is *not* a function. The problem is that it has both $(5,9)$ and $(5,1)$. It uses the same x-value paired with two different y-values. We have no clear way to turn the input 5 into an output.
>
> b. This set of ordered pairs *is* a function. It is a collection of ordered pairs, and the x-values are never reused.
>
> c. This set of ordered pairs *is* a function. It is a collection of ordered pairs, and the x-values are never reused. You might note that the *output* value 9 appears twice, but that doesn't matter. That just tells us that the function turns 5 into 9 and it also turns 3 into 9.
>
> d. This set of ordered pairs is *not* a function, but it's a little tricky. One of the ordered pairs uses $\sqrt{25}$

as an input value. But that is the same as 5, which is also used as an input value.

Now that we understand how some sets of ordered pairs might not be functions, what about tables, graphs, and equations? If we are handed one of these things, can we tell whether or not it is giving us a function?

Checkpoint 10.5.12 Does This Table Make a Function? Which of these tables make y a function of x?

x	y
2	1
3	1
4	2
5	2
6	2

x	y
8	3
9	2
5	1
2	0
8	1

x	y
5	9
5	9
6	2
6	2
6	2

a. This table (□ does □ does not) make y a function of x.

b. This table (□ does □ does not) make y a function of x.

c. This table (□ does □ does not) make y a function of x.

Explanation.

a. This table does make y a function of x. In the table, no x-value is repeated.

b. This table does not make y a function of x. In the table, the x-value 8 is repeated, and it is paired with two different y-values, 3 and 1.

c. This table does make y a function of x, but you have to think carefully. It's true that the x-value 5 is used more than once in the table. But in both places, the y-value is the same, 9. So there is no conceptual issue with asking for $f(5)$; it would definitely be 9. Similarly, the repeated use of 6 as an x-value is not a problem since it is always paired with output 2.

Example 10.5.13 Does This Graph Make a Function? Which of these graphs make y a function of x?

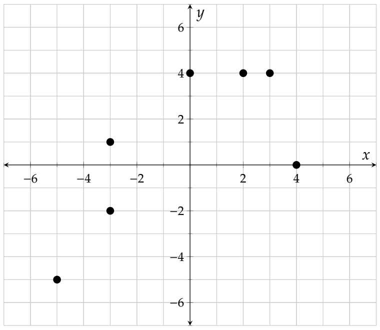


Figure 10.5.14

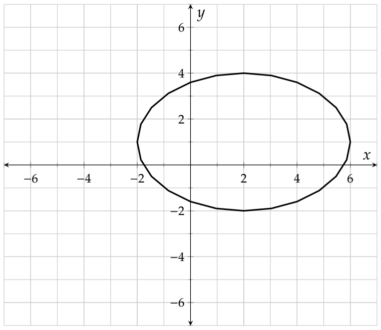


Figure 10.5.15

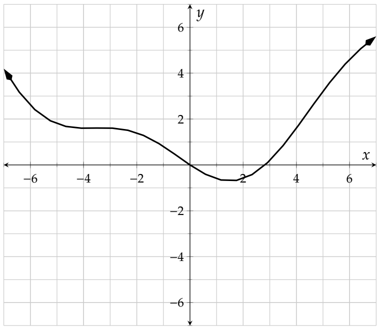


Figure 10.5.16

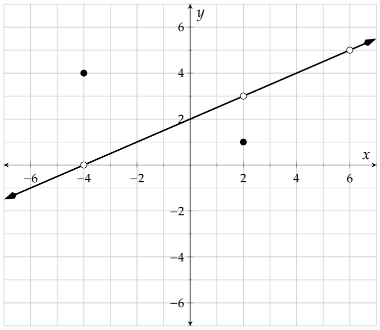


Figure 10.5.17

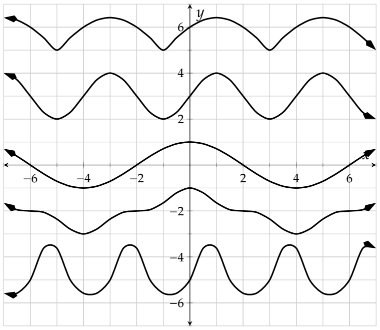


Figure 10.5.18

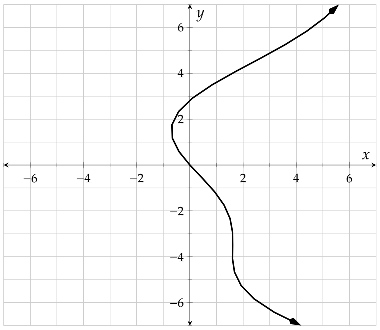


Figure 10.5.19

Explanation. The graph in Figure 10.5.14 does *not* make y a function of x. Two ordered pairs on that graph are $(-3, 1)$ and $(-3, -2)$, so an input value is used twice with different output values.

The graph in Figure 10.5.15 does *not* make y a function of x. There are many ordered pairs with the same input value but different output values. For example, $(2, -2)$ and $(2, 4)$.

The graph in Figure 10.5.16 *does* make y a function of x. It appears that no matter what x-value you choose on the x-axis, there is exactly one y-value paired up with it on the graph.

The graph in Figure 10.5.17 *does* make y a function of x, but we should discuss. The hollow dots on the line indicate that the line goes right up to that point, but never reaches it. We say there is a "hole" in the graph at these places. For two of these holes, there is a separate ordered pair immediately above or below the hole. The graph has the ordered pair $(-4, 4)$. It *also* has ordered pairs like (very close to -4, very close to 0), but it does not have $(-4, 0)$. Overall, there is no x-value that is used twice with different y-values, so this graph does make y a function of x

The graph in Figure 10.5.15 does *not* make y a function of x. There are many ordered pairs with the same input value but different output values. For example, $(0, 1)$, $(0, 3)$, $(0, -1)$, $(0, 5)$, and $(0, -6)$ all use $x = 0$.

The graph in Figure 10.5.15 does *not* make y a function of x. There are many ordered pairs with the same input value but different output values. For example at $x = 2$, there is both a positive and a negative associated y-value. It's hard to say exactly what these y-values are, but we don't have to.

This last set of examples might reveal something to you. For instance in Figure 10.5.15, the issue is that there are places on the graph with the same x-value, but different y-values. Visually, what that means is there are places on the graph that are directly above/below each other. Thinking about this leads to a quick visual "test" to determine if a graph gives y as a function of x.

Fact 10.5.20 Vertical Line Test. *Given a graph in the xy-plane, if a vertical line ever touches it in more than one place, the graph does* not *give y as a function of x. If vertical lines only ever touch the graph once or never at all, then the graph* does *give y as a function of x.*

Example 10.5.21 In each graph from Example 10.5.13, we can apply the Vertical Line Test.

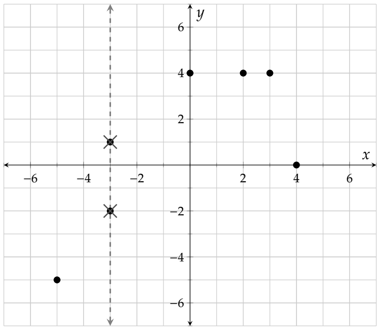


Figure 10.5.22: A vertical line touching the graph twice makes this graph not give y as a function of x.

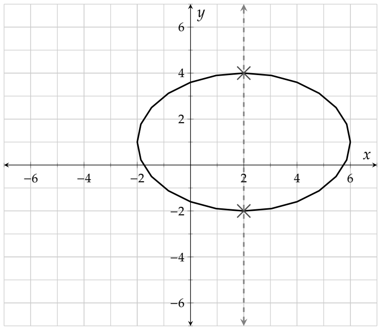


Figure 10.5.23: A vertical line touching the graph twice makes this graph not give y as a function of x.

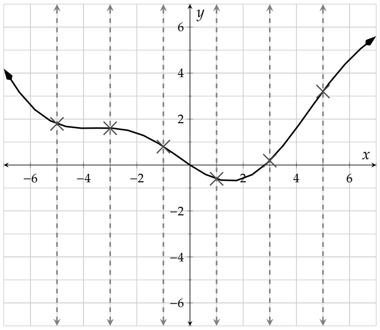


Figure 10.5.24: All vertical lines only touch the graph once, so this graph does give y as a function of x.

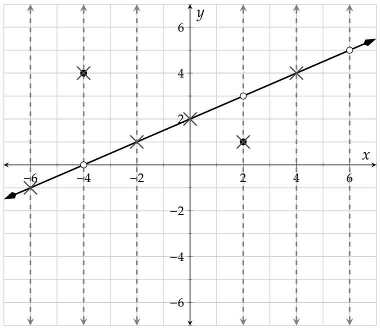


Figure 10.5.25: All vertical lines only touch the graph once, or not at all, so this graph does give y as a function of x.

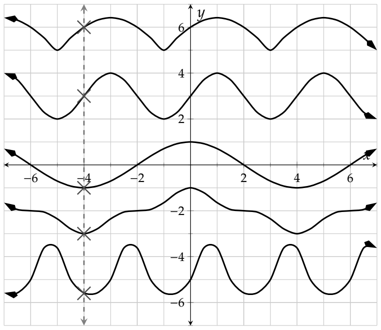


Figure 10.5.26: A vertical line touching the graph more than once makes this graph not give y as a function of x.

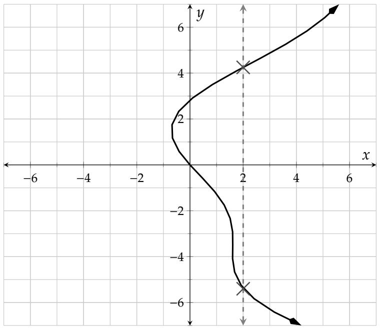


Figure 10.5.27: A vertical line touching the graph more than once makes this graph not give y as a function of x.

Lastly, we come to equations. Certain equations with variables x and y clearly make y a function of x. For example, $y = x^2 + 1$ says that if you have an x-value, all you have to do is substitute it into that equation and you will have determined an output y-value. You could then name the function f and give a formula for it: $f(x) = x^2 + 1$.

With other equations, it may not be immediately clear whether or not they make y a function of x.

Example 10.5.28 Do each of these equations make y a function of x?

a. $2x + 3y = 5$

b. $y = \pm\sqrt{x+4}$

c. $x^2 + y^2 = 9$

Explanation.

a. The equation $2x + 3y = 5$ *does* make y a function of x. Here are three possible explanations.

   i. You recognize that the graph of this equation would be a non-vertical line, and so it would pass the Vertical Line Test.

ii. Imagine that you have a specific value for x and you substitute it in to $2x + 3y = 5$. Will you be able to use algebra to solve for y? All you will need is to simplify, subtract from both sides, and divide on both sides, so you will be able to determine y.

iii. Can you just isolate y in terms of x? Yes, a few steps of algebra can turn $2x + 3y = 5$ into $y = \frac{5-2x}{3}$. Now you have an explicit formula for y in terms of x, so y is a function of x.

b. The equation $y = \pm\sqrt{x+4}$ does *not* make y a function of x. Just having the ± (plus *or* minus) in the equation immediately tells you that for almost any valid x-value, there would be *two* associated y-values.

c. The equation $x^2 + y^2 = 9$ does *not* make y a function of x. Here are three possible explanations.

   i. Imagine that you have a specific value for x and you substitute it in to $x^2 + y^2 = 9$. Will you be able to use algebra to solve for y? For example, if you substitute in $x = 1$, then you have $1 + y^2 = 9$, which simplifies to $y^2 = 8$. Can you really determine what y is? No, because it could be $\sqrt{8}$ or it could be $-\sqrt{8}$. So this equation does not provide you with a way to turn x-values into y-values.

   ii. Can you just isolate y in terms of x? You might get started and use algebra to convert $x^2+y^2 = 9$ into $y^2 = 9 - x^2$. But what now? The best you can do is acknowledge that y is either the positive or the negative square root of $9 - x^2$. You might write $y = \pm\sqrt{9-x^2}$. But now for almost any valid x-value, there are *two* associated y-values.

   iii. You recognize that the graph of this equation would be a circle with radius 3, and so it would not pass the Vertical Line Test.

Checkpoint 10.5.29. Do each of these equations make y a function of x?

a. $5x^2 - 4y = 12$

This equation (☐ does ☐ does not) make y a function of x.

b. $5x - 4y^2 = 12$

This equation (☐ does ☐ does not) make y a function of x.

c. $x = \sqrt{y}$

This equation (☐ does ☐ does not) make y a function of x.

Explanation.

a. The equation $5x^2 - 4y = 12$ *does* make y a function of x. You can isolate y in terms of x. A few steps of algebra can turn $5x^2 - 4y = 12$ into $y = \frac{5x^2-12}{4}$. Now you have an explicit formula for y in terms of x, so y is a function of x.

b. The equation $5x - 4y^2 = 12$ does *not* make y a function of x. You cannot isolate y in terms of x. You might get started and use algebra to convert $5x - 4y^2 = 12$ into $y^2 = \frac{5x-12}{4}$. But what now? The best you can do is acknowledge that y is either the positive or the negative square root of $\frac{5x-12}{4}$. You might write $y = \pm\sqrt{\frac{5x-12}{4}}$. But now for almost any valid x-value, there are *two* associated y-values.

c. The equation $x = \sqrt{y}$ *does* make y a function of x. If you try substituting a non-negative x-value, then you can square both sides and you know exactly what the value of y is.

   If you try substituting a negative x-value, then you are saying that $\sqrt{y}$ is negative which is impossible. So for negative x, there are no y-values. This is not a problem for the equation giving you a function. This just means that the domain of that function does not include negative numbers. Its domain would be $[0, \infty)$.

Exercises

Determining If Sets of Ordered Pairs Are Functions

1. Do these sets of ordered pairs make functions of x? What are their domains and ranges?

a. $\Big\{(-6,8),(2,9)\Big\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain and range .

b. $\Big\{(-8,8),(-8,3),(4,4)\Big\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain and range .

c. $\Big\{(9,7),(-6,7),(6,5),(2,4)\Big\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain and range .

d. $\Big\{(2,0),(-5,1),(-7,4),(3,1),(-3,1)\Big\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain and range .

2. Do these sets of ordered pairs make functions of x? What are their domains and ranges?

a. $\Big\{(-5,8),(-8,5)\Big\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain and range .

b. $\Big\{(-7,6),(2,9),(10,6)\Big\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain and range .

c. $\Big\{(0,2),(5,9),(-1,7),(5,5)\Big\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain and range .

d. $\Big\{(7,6),(-9,5),(-6,2),(3,0),(-10,2)\Big\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain and range .

3. Does the following set of ordered pairs make for a function of x?

$$\left\{(0,4),(-1,1),(-5,0),(-3,1),(-1,9)\right\}$$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ______ and range ______.

4. Does the following set of ordered pairs make for a function of x?

$$\left\{(-5,10),(9,8),(-3,10),(-5,8),(-3,5)\right\}$$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ______ and range ______.

Domain and Range

5. Below is all of the information that exists about a function H.

$H(3) = 4 \quad H(5) = -2 \quad H(8) = 4$

Write H as a set of ordered pairs.

H has domain ______

and range ______.

6. Below is all of the information about a function K.

$K(a) = 3 \quad K(b) = 1$

$K(c) = 0 \quad K(d) = 3$

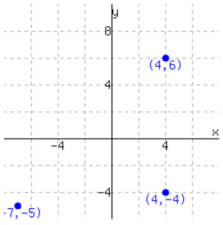
Write K as a set of ordered pairs.

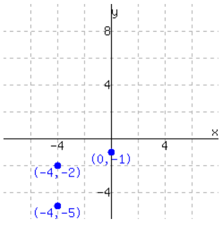
K has domain ______

and range ______.

Determining If Graphs Are Functions

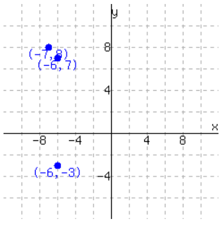
7. Decide whether each graph shows a relationship where y is a function of x.

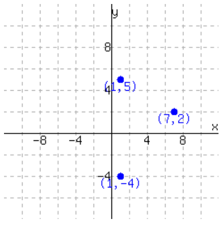




The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

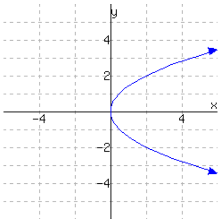
8. Decide whether each graph shows a relationship where y is a function of x.

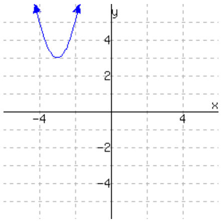




The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

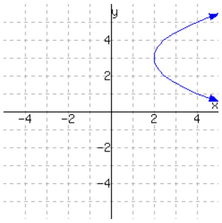
9. The following graphs show two relationships. Decide whether each graph shows a relationship where y is a function of x.

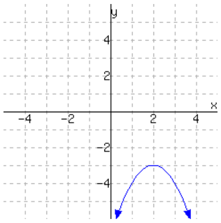




The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

10. The following graphs show two relationships. Decide whether each graph shows a relationship where y is a function of x.





The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

Determining If Tables Are Functions Determine whether or not the following table could be the table of values of a function. If the table can not be the table of values of a function, give an input that has more than one possible output.

11.

Input	Output
2	0
4	14
6	−5
8	−15
−2	−19

Could this be the table of values for a function? (☐ yes ☐ no)

If not, which input has more than one possible output? (☐ -2 ☐ 2 ☐ 4 ☐ 6 ☐ 8 ☐ None, the table represents a function.)

12.

Input	Output
2	5
4	0
6	12
8	15
−2	−20

Could this be the table of values for a function? (☐ yes ☐ no)

If not, which input has more than one possible output? (☐ -2 ☐ 2 ☐ 4 ☐ 6 ☐ 8 ☐ None, the table represents a function.)

13.

Input	Output
−4	13
−3	−4
−2	5
−3	13
−1	−19

Could this be the table of values for a function? (☐ yes ☐ no)

If not, which input has more than one possible output? (☐ -4 ☐ -3 ☐ -2 ☐ -1 ☐ None, the table represents a function.)

14.

Input	Output
−4	−7
−3	−2
−2	−19
−3	19
−1	0

Could this be the table of values for a function? (☐ yes ☐ no)

If not, which input has more than one possible output? (☐ -4 ☐ -3 ☐ -2 ☐ -1 ☐ None, the table represents a function.)

Determining If Equations Are Functions

15. Select all of the following relations that make y a function of x. There are several correct answers.

☐ $x^2 + y^2 = 81$
☐ $y = \sqrt[6]{x}$
☐ $y = x^3$
☐ $y = \pm\sqrt{1 - x^2}$
☐ $\left|y\right| = x$
☐ $4x+3y = 1$
☐ $y = \frac{1}{x^2}$
☐ $y = \sqrt{1 - x^2}$
☐ $x = y^9$
☐ $y = \frac{x+7}{8-x}$
☐ $x = y^8$
☐ $y = |x|$

17. Some equations involving x and y define y as a function of x, and others do not. For example, if $x + y = 1$, we can solve for y and obtain $y = 1 - x$. And we can then think of $y = f(x) = 1 - x$. On the other hand, if we have the equation $x = y^2$ then y is not a function of x, since for a given positive value of x, the value of y could equal $\sqrt{x}$ or it could equal $-\sqrt{x}$.

Select all of the following relations that make y a function of x. There are several correct answers.

☐ $y^2+x^2 = 1$
☐ $y-|x| = 0$
☐ $y^6+x = 1$
☐ $x+y = 1$
☐ $3x+8y+8 = 0$
☐ $y+x^2 = 1$
☐ $y^3 + x^4 = 1$
☐ $\left|y\right| - x = 0$

On the other hand, some equations involving x and y define x as a function of y (the other way round).

Select all of the following relations that make x a function of y. There are several correct answers.

☐ $\left|y\right|-x = 0$
☐ $y^4+x^5 = 1$
☐ $y-|x| = 0$
☐ $3x + 8y + 8 = 0$
☐ $y^2 + x^2 = 1$

16. Select all of the following relations that make y a function of x. There are several correct answers.

☐ $y = \sqrt[4]{x}$
☐ $x = y^9$
☐ $x^2 + y^2 = 16$
☐ $y = x^2$
☐ $\left|y\right| = x$
☐ $y = \pm\sqrt{64 - x^2}$
☐ $y = \frac{x+2}{4-x}$
☐ $4x + 5y = 1$
☐ $y = \frac{1}{x^3}$
☐ $y = |x|$
☐ $y = \sqrt{64 - x^2}$
☐ $x = y^8$

18. Some equations involving x and y define y as a function of x, and others do not. For example, if $x + y = 1$, we can solve for y and obtain $y = 1 - x$. And we can then think of $y = f(x) = 1 - x$. On the other hand, if we have the equation $x = y^2$ then y is not a function of x, since for a given positive value of x, the value of y could equal $\sqrt{x}$ or it could equal $-\sqrt{x}$.

Select all of the following relations that make y a function of x. There are several correct answers.

☐ $4x+5y+4 = 0$
☐ $y+x^2 = 1$
☐ $\left|y\right|-x = 0$
☐ $y-|x| = 0$
☐ $y^2+x^2 = 1$
☐ $x+y = 1$
☐ $y^3 + x^4 = 1$
☐ $y^6 + x = 1$

On the other hand, some equations involving x and y define x as a function of y (the other way round).

Select all of the following relations that make x a function of y. There are several correct answers.

☐ $4x+5y+4 = 0$
☐ $y-|x| = 0$
☐ $y^2+x^2 = 1$
☐ $\left|y\right| - x = 0$
☐ $y^4 + x^5 = 1$

10.6 Functions and their Representations Chapter Review

10.6.1 Function Basics

In Section 10.1 we defined functions 10.1.3 informally, as well as function notation 10.1.6. We saw functions in four forms 10.1.31: verbal descriptions, formulas, graphs and tables.

Example 10.6.1 Informal Definition of a Function. Determine whether each example below describes a function.

a. The area of a circle given its radius.

b. The number you square to get 9.

Explanation.

a. The area of a circle given its radius is a function because there is a set of steps or a formula that changes the radius into the area of the circle. We could write $A(r) = \pi r^2$.

b. The number you square to get 9 is not a function because the process we would apply to get the result does not give a single answer. There are two different answers, -3 and 3. A function must give a single output for a given input.

Example 10.6.2 Tables and Graphs. Make a table and a graph of the function f, where $f(x) = x^2$.

Explanation.

First we will set up a table with negative and positive inputs and calculate the function values. The values are shown in Table 10.6.3, which in turn gives us the graph in Figure 10.6.4.

input, x	output, f(x)
-3	9
-2	4
-1	1
0	0
1	2
2	4
3	9

Table 10.6.3

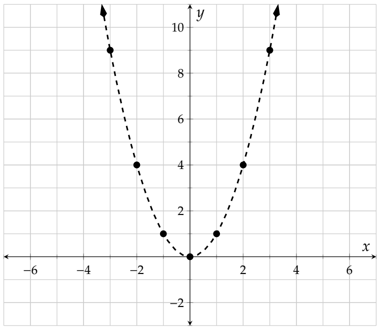


Figure 10.6.4: $y = f(x) = x^2$

Example 10.6.5 Translating between Four Descriptions of the Same Function. Consider a function f that triples its input and then adds 4. Translate this verbal description of f into a table, a graph, and a formula.

Explanation.

To make a table for f, we'll have to select some input x-values so we will choose some small negative and positive values that are easy to work with. Given the verbal description, we should be able to compute a column of output values. Table 10.6.6 is one possible table that we might end up with.

x	$f(x)$
-2	$3(-2)+4=-2$
-1	$3(-1)+4=1$
0	$3(0)+4=4$
1	$3(1)+4=7$
2	$3(2)+4=10$

Table 10.6.6

Once we have a table for f, we can make a graph for f as in Figure 10.6.7, using the table to plot points.

Lastly, we must find a formula for f. This means we need to write an algebraic expression that says the same thing about f as the verbal description, the table, and the graph. For this example, we can focus on the verbal description. Since f takes its input, triples it, and adds 4, we have the formula

$$f(x)=3x+4.$$

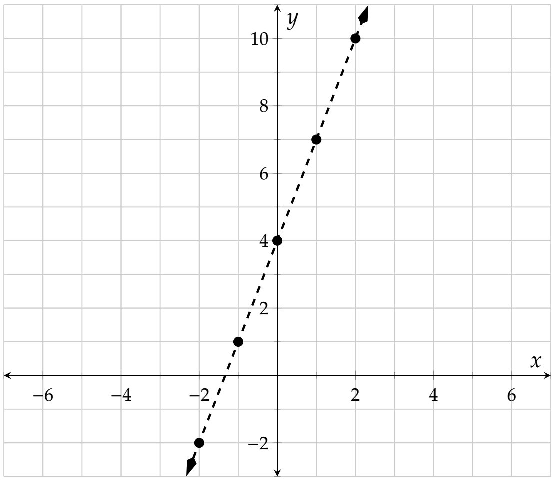


Figure 10.6.7: $y=f(x)$

10.6.2 Domain and Range

In Section 10.2 we saw the definition of domain 10.2.6 and range 10.2.19, and three types of domain restrictions 10.2.18. We also learned how to write the domain and range in interval and set-builder notation.

Example 10.6.8 Domain. Determine the domain of p, where $p(x)=\dfrac{x}{2x-1}$.

Explanation. This is an example of the first type of domain restriction, when you have a variable in the denominator. The denominator cannot equal 0 so a *bad* value for x would be when

$$\begin{aligned}2x-1&=0\\2x&=1\\x&=\frac{1}{2}\end{aligned}$$

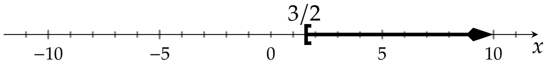
The domain is all real numbers except $\frac{1}{2}$.

Example 10.6.9 Interval, Set, and Set-Builder Notation. What is the domain of the function C, where $C(x) = \sqrt{2x-3} - 5$?

Explanation. This is an example of the second type of domain restriction where the value inside the radical cannot be negative. So the *good* values for x would be when

$$\begin{aligned} 2x - 3 &\geq 0 \\ 2x &\geq 3 \\ x &\geq \frac{3}{2} \end{aligned}$$

So on a number line, if we wanted to picture the domain of C, we would make a sketch like:



The domain is the interval $\left[\frac{3}{2}, \infty\right)$.

Find the range of the function q using its graph shown in Figure 10.6.11.

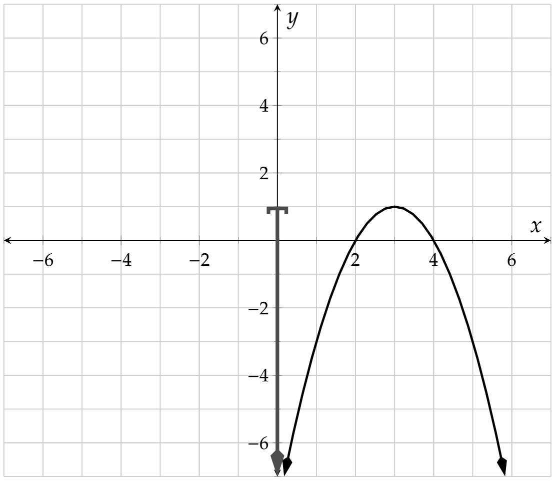


Figure 10.6.11: $y = q(x)$. The range is marked as an interval on the y-axis.

Example 10.6.10 Range. Explanation. The range is the collection of possible numbers that q can give for output. Figure 10.6.11 displays a graph of q, with the range shown as an interval on the y-axis.

The output values are the y-coordinates so we can see that the y-values start from 1 and continue downward forever. Therefore the range is $(-\infty, 1]$.

10.6.3 Using Technology to Explore Functions

In Section 10.3 we covered how to find a good graphing window and use it to identify all of the key features of a function. We also learned how to solve equations and inequalities using a graph. Here are some examples for review.

Example 10.6.12 Finding an Appropriate Window. Graph the function t, where $t(x) = (x+10)^2 - 15$, using technology and find a good viewing window.

Explanation.

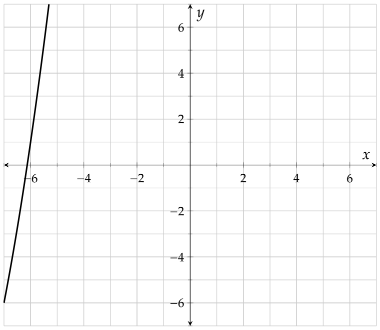


Figure 10.6.13: $y = t(x)$ in the viewing window of −7 to 7 on the x and y axes. We need to zoom out and move our window to the left.

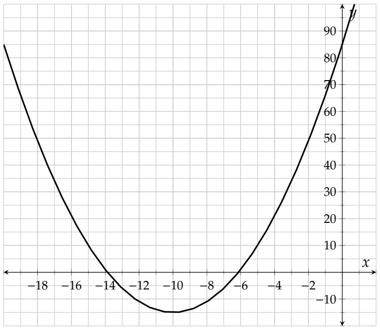


Figure 10.6.14: $y = t(x)$ in a good viewing window.

After some trial and error we found this window that goes from −20 to 2 on the x-axis and −20 to 100 on the y-axis.

Now we can see the vertex and all of the intercepts and we will identify them in the next example.

Example 10.6.15 Using Technology to Determine Key Features of a Graph. Use the previous graph in figure 10.6.14 to identify the intercepts, minimum or maximum function value, and the domain and range of the function t, where $t(x) = (x+10)^2 - 15$.

Explanation.

From our graph we can now identify the vertex at $(-10, -15)$, the y-intercept at $(0, 85)$, and the x-intercepts at approximately $(-13.9, 0)$ and $(-6.13, 0)$.

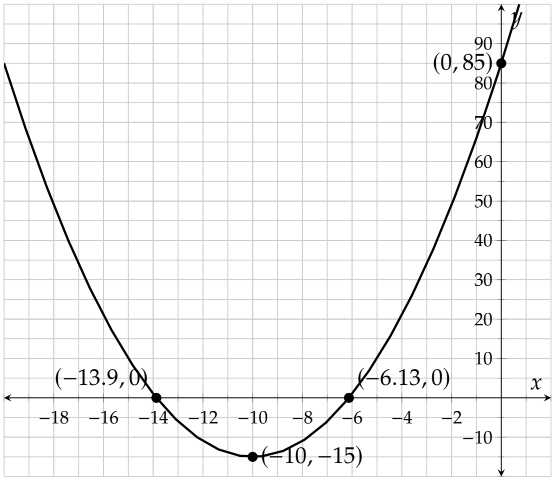


Figure 10.6.16: $y = t(x) = (x+10)^2 - 15$.

Example 10.6.17 Solving Equations and Inequalities Graphically Using Technology. Use graphing technology to solve the equation $t(x) = 40$, where $t(x) = (x + 10)^2 - 15$.

Explanation.

To solve the equation $t(x) = 40$, we need to graph $y = t(x)$ and $y = 40$ on the same axes and find the x-values where they intersect.

From the graph we can see that the intersection points are approximately $(-17.4, 40)$ and $(-2.58, 40)$. The solution set is $\{-17.4, -2.58\}$.

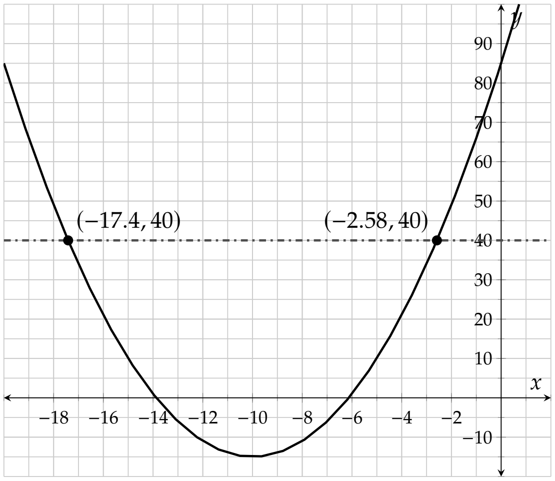


Figure 10.6.18: $y = t(x)$ where $t(x) = (x + 10)^2 - 15$ and $y = 40$.

10.6.4 Simplifying Expressions with Function Notation

In Section 10.4 we learned about the difference between $f(-x)$ and $-f(x)$ and how to simplify them. We also learned how to simplify other changes to the input and output like $f(3x)$ and $\frac{1}{3}f(x)$. Here are some examples.

Example 10.6.19 Negative Signs in and out of Function Notation. Find and simplify a formula for $f(-x)$ and $-f(x)$, where $f(x) = -3x^2 - 7x + 1$.

Explanation. To find $f(-x)$, we use an input of $-x$ in our function f and simplify to get:

$$\begin{aligned} f(-x) &= -3(-x)^2 - 7(-x) + 1 \\ &= -3x^2 + 7x + 1 \end{aligned}$$

To find $-f(x)$, we take the opposite of the function f and simplify to get:

$$\begin{aligned} -f(x) &= -(-3x^2 - 7x + 1) \\ &= 3x^2 + 7x - 1 \end{aligned}$$

Example 10.6.20 Other Nontrivial Simplifications. If $g(x) = 2x^2 - 3x - 5$, find and simplify a formula for $g(x - 1)$.

Explanation. To find $g(x - 1)$, we put in $(x - 1)$ for the input. It is important to keep the parentheses because we have exponents and negative signs in the function.

$$\begin{aligned} g(x-1) &= 2(x-1)^2 - 3(x-1) - 5 \\ &= 2(x^2 - 2x + 1) - 3x + 3 - 5 \\ &= 2x^2 - 4x + 2 - 3x - 2 \\ &= 2x^2 - 7x \end{aligned}$$

10.6.5 Technical Definition of a Function

In Section 10.5 we gave a formal definition of a function 10.5.2 and learned to identify what is and is not a function with sets or ordered pairs, tables and graphs. We also used the vertical line test 10.5.20.

Example 10.6.21 Formally Defining a Function. We learned that sets of ordered pairs, tables and graphs can meet the formal definition of a function. Here is an example that shows a function in all three forms. We can verify that each input has at most one output.

$\{(1,4),(2,4),(3,3),(4,6),(5,-2)\}$

Figure 10.6.22: The function f represented as a collection of ordered pairs.

x	$f(x)$
1	4
2	4
3	3
4	6
5	−2

Table 10.6.23: The function f represented as a table.

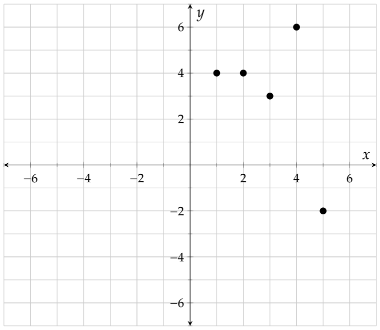


Figure 10.6.24: The function f represented as a graph.

Example 10.6.25 Identifying What is *Not* a Function. Identify whether each graph represents a function using the vertical line test 10.5.20.

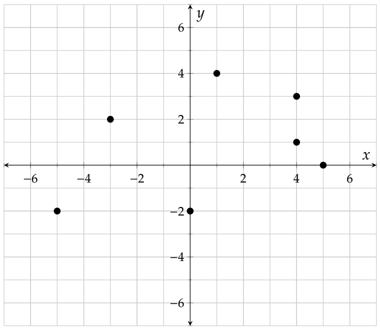


Figure 10.6.26

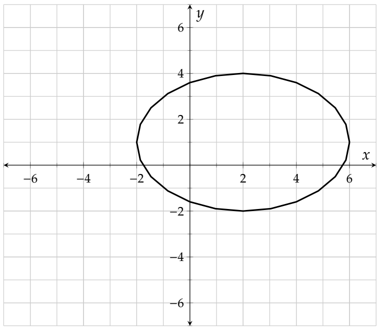


Figure 10.6.27

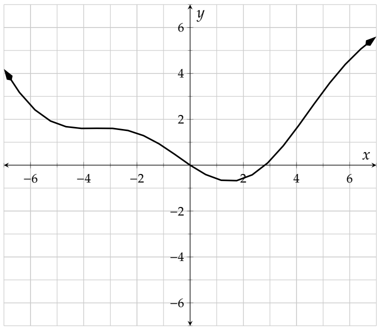


Figure 10.6.28

Explanation.

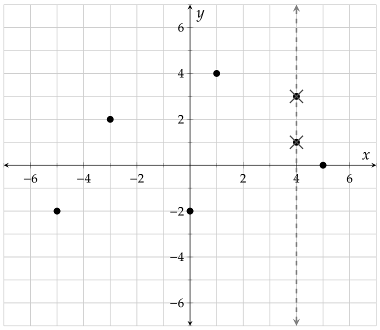


Figure 10.6.29: A vertical line touching the graph twice makes this graph not give y as a function of x.

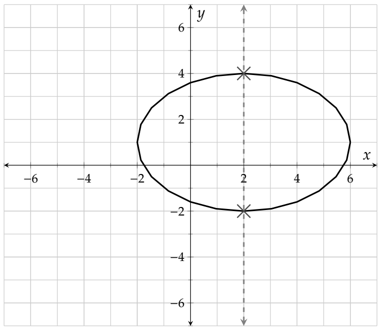


Figure 10.6.30: A vertical line touching the graph twice makes this graph not give y as a function of x.

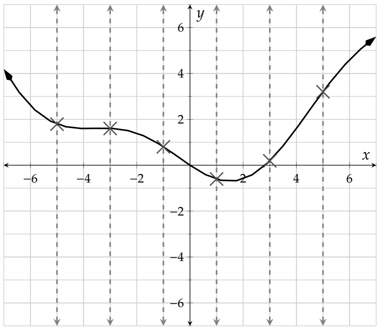


Figure 10.6.31: All vertical lines only touch the graph once, so this graph does give y as a function of x.

Exercises

Function Basics

1. Samantha will spend \$240 to purchase some bowls and some plates. Each plate costs \$1, and each bowl costs \$6. The function $q(x) = -\frac{1}{6}x + 40$ models the number of bowls Samantha will purchase, where x represents the number of plates to be purchased.

Interpret the meaning of $q(36) = 34$.

- ⊙ *A*. 34 plates and 36 bowls can be purchased.
- ⊙ *B*. 36 plates and 34 bowls can be purchased.
- ⊙ *C*. \$34 will be used to purchase bowls, and \$36 will be used to purchase plates.
- ⊙ *D*. \$36 will be used to purchase bowls, and \$34 will be used to purchase plates.

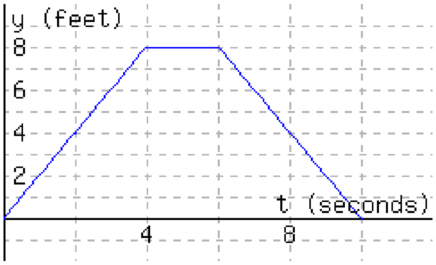
2. Fabrienne will spend \$140 to purchase some bowls and some plates. Each plate costs \$8, and each bowl costs \$7. The function $q(x) = -\frac{8}{7}x + 20$ models the number of bowls Fabrienne will purchase, where x represents the number of plates to be purchased.

Interpret the meaning of $q(14) = 4$.

- ⊙ *A*. 4 plates and 14 bowls can be purchased.
- ⊙ *B*. \$4 will be used to purchase bowls, and \$14 will be used to purchase plates.
- ⊙ *C*. \$14 will be used to purchase bowls, and \$4 will be used to purchase plates.
- ⊙ *D*. 14 plates and 4 bowls can be purchased.

The following figure has the graph $y = d(t)$, which models a particle's distance from the starting line in feet, where t stands for time in seconds since timing started.

3.



a. $d(8) =$

b. Interpret the meaning of $d(8)$:

⊙ *A.* The particle was 8 feet away from the starting line 4 seconds since timing started.

⊙ *B.* In the first 4 seconds, the particle moved a total of 8 feet.

⊙ *C.* The particle was 4 feet away from the starting line 8 seconds since timing started.

⊙ *D.* In the first 8 seconds, the particle moved a total of 4 feet.

c. Solve $d(t) = 6$ for t. $t =$

d. Interpret the meaning of part c's solution(s):

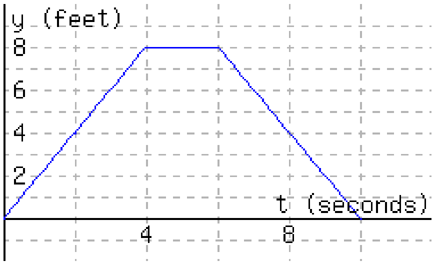
⊙ *A.* The article was 6 feet from the starting line 3 seconds since timing started.

⊙ *B.* The article was 6 feet from the starting line 7 seconds since timing started.

⊙ *C.* The article was 6 feet from the starting line 3 seconds since timing started, or 7 seconds since timing started.

⊙ *D.* The article was 6 feet from the starting line 3 seconds since timing started, and again 7 seconds since timing started.

4.



a. $d(4) =$

b. Interpret the meaning of $d(4)$:

⊙ *A.* The particle was 4 feet away from the starting line 8 seconds since timing started.

⊙ *B.* In the first 4 seconds, the particle moved a total of 8 feet.

⊙ *C.* In the first 8 seconds, the particle moved a total of 4 feet.

⊙ *D.* The particle was 8 feet away from the starting line 4 seconds since timing started.

c. Solve $d(t) = 4$ for t. $t =$

d. Interpret the meaning of part c's solution(s):

⊙ *A.* The article was 4 feet from the starting line 2 seconds since timing started, or 8 seconds since timing started.

⊙ *B.* The article was 4 feet from the starting line 2 seconds since timing started, and again 8 seconds since timing started.

⊙ *C.* The article was 4 feet from the starting line 8 seconds since timing started.

⊙ *D.* The article was 4 feet from the starting line 2 seconds since timing started.

5. Evaluate the function at the given values.

$H(x) = -\frac{12}{x+6}$.

a. $H(-9) =$.

b. $H(-6) =$.

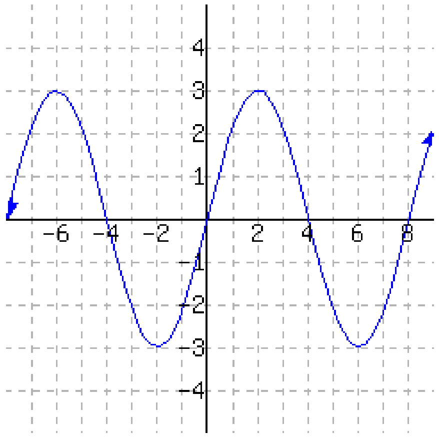
6. Evaluate the function at the given values.

$K(x) = -\frac{24}{x+1}$.

a. $K(5) =$.

b. $K(-1) =$.

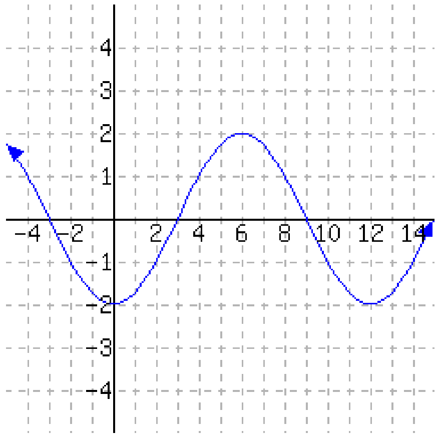
7. Use the graph of f below to evaluate the given expressions. (Estimates are OK.)



$f(-6) =$

$f(6) =$

8. Use the graph of g below to evaluate the given expressions. (Estimates are OK.)



$g(-3) =$

$g(12) =$

9. Use the table of values for g below to evaluate the given expressions.

x	−3	−1	1	3	5
$g(x)$	6.4	7.9	7.5	4.9	1.8

$g(-1) =$

$g(3) =$

10. Use the table of values for h below to evaluate the given expressions.

x	−4	1	6	11	16
$h(x)$	3.1	7.7	1.4	6	6.8

$h(1) =$

$h(16) =$

11. Make a table of values for the function h, defined by $h(x) = 2x^2$. Based on values in the table, sketch a graph of h.

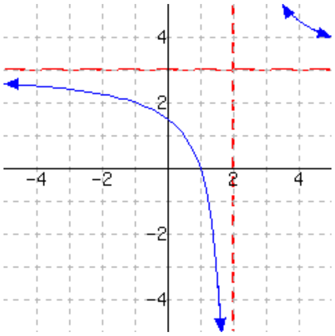
x	$h(x)$

12. Make a table of values for the function H, defined by $H(x) = \frac{2^x + 2}{x^2 + 2}$. Based on values in the table, sketch a graph of H.

x	$H(x)$

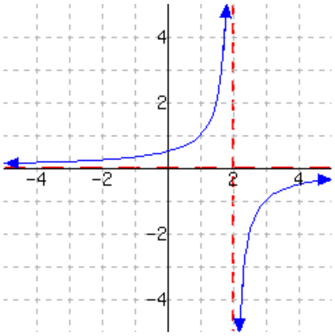
Domain and Range A function is graphed.

13.



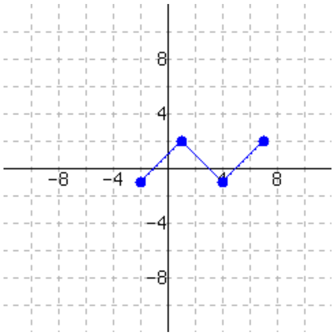
This function has domain ☐ and range ☐.

14.



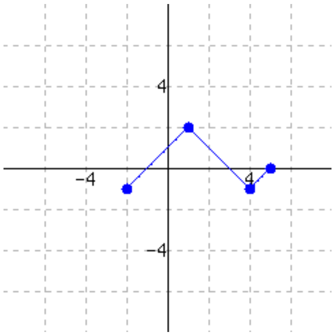
This function has domain ☐ and range ☐.

15.



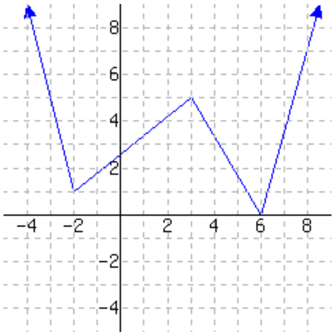
The function has domain ☐ and range ☐.

16.



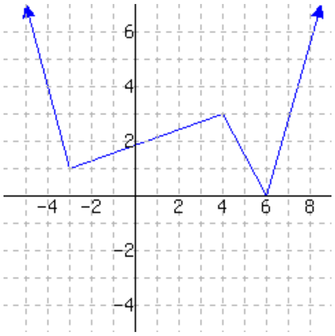
The function has domain ☐ and range ☐.

17.



The function has domain ☐ and range ☐.

18.



The function has domain ☐ and range ☐.

19. Find the domain of r where $r(x) = \dfrac{\sqrt{8+x}}{9-x}$.

20. Find the domain of B where $B(x) = \dfrac{\sqrt{10+x}}{5-x}$.

21. An object was shot up into the air at an initial vertical speed of 512 feet per second. Its height as time passes can be modeled by the quadratic function f, where $f(t) = -16t^2 + 512t$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height in feet.

Find the function's domain and range in this context.

The function's domain in this context is ____.

The function's range in this context is ____.

22. An object was shot up into the air at an initial vertical speed of 544 feet per second. Its height as time passes can be modeled by the quadratic function f, where $f(t) = -16t^2 + 544t$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height in feet.

Find the function's domain and range in this context.

The function's domain in this context is ____.

The function's range in this context is ____.

Using Technology to Explore Functions

23. Use technology to make a table of values for the function H defined by $H(x) = -4x^2 + 4x + 3$.

x	$H(x)$

24. Use technology to make a table of values for the function K defined by $K(x) = -2x^2 + 16x - 1$.

x	$K(x)$

25. Choose an appropriate window for graphing the function f defined by $f(x) = 1456x - 7423$ that shows its key features.

The x-interval could be ____ and the y-interval could be ____.

26. Choose an appropriate window for graphing the function f defined by $f(x) = -169x + 139$ that shows its key features.

The x-interval could be ____ and the y-interval could be ____.

27. Use technology to determine how many times the equations $y = -4x^3 - 3x^2 + x$ and $y = 4x + 2$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

28. Use technology to determine how many times the equations $y = -2x^3 + x^2 + 9x$ and $y = -x + 1$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

29. For the function L defined by

$$L(x) = 3000x^2 + 10x + 4,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.
b. The vertex.
c. The domain.
d. The range.

30. For the function M defined by

$$M(x) = -(300x - 2950)^2,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.
b. The vertex.
c. The domain.
d. The range.

31. Let $f(x) = 4x^2 + 5x - 1$ and $g(x) = 5$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?
b. Solve $f(x) = g(x)$.
c. Solve $f(x) < g(x)$.
d. Solve $f(x) \geq g(x)$.

32. Let $p(x) = 6x^2 - 3x + 4$ and $k(x) = 7$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?
b. Solve $p(x) = k(x)$.
c. Solve $p(x) < k(x)$.
d. Solve $p(x) \geq k(x)$.

33. Use graphing technology to solve the equation $-0.02x^2 + 1.97x - 51.5 = 0.05\,(x - 50)^2 - 0.03\,(x - 50)$. Approximate the solution(s) if necessary.

34. Use graphing technology to solve the equation $-200x^2 + 60x - 55 = -20x - 40$. Approximate the solution(s) if necessary.

35. Use graphing technology to solve the inequality $-15x^2 - 6 \leq 10x - 4$. State the solution set using interval notation, and approximate if necessary.

36. Use graphing technology to solve the inequality $\frac{1}{2}x^2 + \frac{3}{2}x \geq \frac{1}{2}x - \frac{3}{2}$. State the solution set using interval notation, and approximate if necessary.

Simplifying Expressions with Function Notation

37. Let f be a function given by $f(x) = 3x^2 + 2x$. Find and simplify the following:

a. $f(x) - 2 =$

b. $f(x - 2) =$

c. $-2f(x) =$

d. $f(-2x) =$

38. Let f be a function given by $f(x) = -3x^2 - 4x$. Find and simplify the following:

a. $f(x) - 3 =$

b. $f(x - 3) =$

c. $-3f(x) =$

d. $f(-3x) =$

39. Simplify $H(r) + 5$, where $H(r) = -1 - 1.8r$.

40. Simplify $F(r) + 9$, where $F(r) = -1 - 6.3r$.

Technical Definition of a Function

41. Does the following set of ordered pairs make for a function of x?

$\left\{(9,2),(5,8),(8,6),(-3,3),(-5,9)\right\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain and range .

42. Does the following set of ordered pairs make for a function of x?

$\left\{(5,8),(-6,5),(10,4),(-6,10),(-7,5)\right\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain and range .

43. Below is all of the information that exists about a function f.

$f(0) = 2 \quad f(2) = 2 \quad f(3) = 2$

Write f as a set of ordered pairs.

f has domain

and range .

44. Below is all of the information about a function g.

$g(a) = 1 \quad g(b) = 5$

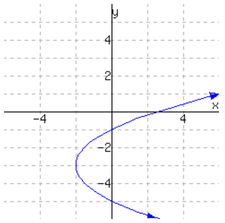
$g(c) = -5 \quad g(d) = 5$

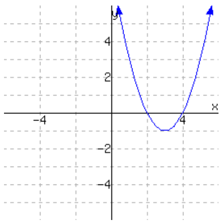
Write g as a set of ordered pairs.

g has domain

and range .

45. The following graphs show two relationships. Decide whether each graph shows a relationship where y is a function of x.





The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

47. Some equations involving x and y define y as a function of x, and others do not. For example, if $x + y = 1$, we can solve for y and obtain $y = 1 - x$. And we can then think of $y = f(x) = 1 - x$. On the other hand, if we have the equation $x = y^2$ then y is not a function of x, since for a given positive value of x, the value of y could equal $\sqrt{x}$ or it could equal $-\sqrt{x}$.

Select all of the following relations that make y a function of x. There are several correct answers.

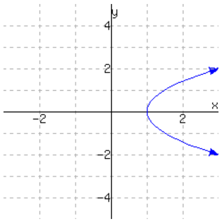
☐ $|y| - x = 0$ ☐ $y + x^2 = 1$ ☐ $y^2 + x^2 = 1$
☐ $y - |x| = 0$ ☐ $5x + 2y + 9 = 0$ ☐ $x + y = 1$
☐ $y^6 + x = 1$ ☐ $y^3 + x^4 = 1$

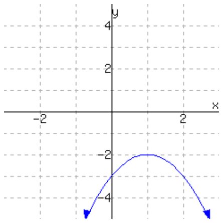
On the other hand, some equations involving x and y define x as a function of y (the other way round).

Select all of the following relations that make x a function of y. There are several correct answers.

☐ $y - |x| = 0$ ☐ $5x + 2y + 9 = 0$ ☐ $|y| - x = 0$
☐ $y^2 + x^2 = 1$ ☐ $y^4 + x^5 = 1$

46. The following graphs show two relationships. Decide whether each graph shows a relationship where y is a function of x.





The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

48. Some equations involving x and y define y as a function of x, and others do not. For example, if $x + y = 1$, we can solve for y and obtain $y = 1 - x$. And we can then think of $y = f(x) = 1 - x$. On the other hand, if we have the equation $x = y^2$ then y is not a function of x, since for a given positive value of x, the value of y could equal $\sqrt{x}$ or it could equal $-\sqrt{x}$.

Select all of the following relations that make y a function of x. There are several correct answers.

☐ $y^6 + x = 1$ ☐ $x + y = 1$ ☐ $6x + 7y + 4 = 0$
☐ $y^3 + x^4 = 1$ ☐ $y + x^2 = 1$ ☐ $y^2 + x^2 = 1$
☐ $y - |x| = 0$ ☐ $|y| - x = 0$

On the other hand, some equations involving x and y define x as a function of y (the other way round).

Select all of the following relations that make x a function of y. There are several correct answers.

☐ $y^4 + x^5 = 1$ ☐ $y - |x| = 0$ ☐ $y^2 + x^2 = 1$
☐ $|y| - x = 0$ ☐ $6x + 7y + 4 = 0$

49. Determine whether or not the following table could be the table of values of a function. If the table can not be the table of values of a function, give an input that has more than one possible output.

Input	Output
2	9
4	−5
6	9
8	5
−2	−8

Could this be the table of values for a function? (☐ yes ☐ no)

If not, which input has more than one possible output? (☐ -2 ☐ 2 ☐ 4 ☐ 6 ☐ 8 ☐ None, the table represents a function.)

50. Determine whether or not the following table could be the table of values of a function. If the table can not be the table of values of a function, give an input that has more than one possible output.

Input	Output
2	13
4	−19
6	−15
8	−7
−2	−9

Could this be the table of values for a function? (☐ yes ☐ no)

If not, which input has more than one possible output? (☐ -2 ☐ 2 ☐ 4 ☐ 6 ☐ 8 ☐ None, the table represents a function.)

51. Determine whether or not the following table could be the table of values of a function. If the table can not be the table of values of a function, give an input that has more than one possible output.

Input	Output
−4	7
−3	−4
−2	−5
−3	13
−1	−3

Could this be the table of values for a function? (☐ yes ☐ no)

If not, which input has more than one possible output? (☐ -4 ☐ -3 ☐ -2 ☐ -1 ☐ None, the table represents a function.)

52. Determine whether or not the following table could be the table of values of a function. If the table can not be the table of values of a function, give an input that has more than one possible output.

Input	Output
−4	−14
−3	−2
−2	12
−3	19
−1	15

Could this be the table of values for a function? (☐ yes ☐ no)

If not, which input has more than one possible output? (☐ -4 ☐ -3 ☐ -2 ☐ -1 ☐ None, the table represents a function.)

CHAPTER 11

Absolute Value Functions

11.1 Introduction to Absolute Value Functions

This section will introduce the basic concepts behind absolute value functions and their graphs. This information will also be useful at the end of this chapter when we solve absolute value equations and inequalities.

11.1.1 Definition of Absolute Value

Recall that in Section 1.3, we defined the **absolute value** of a number to be the distance between that number and 0 on a number line. Also recall that this causes the output of the absolute value function to never be a negative number since we are under the presumption that "distance" is always positive (or zero).

Example 11.1.2

a. Since the number 5 is 5 units from 0, then $|5| = 5$.

b. Since the number -3 is 3 units from 0, then $|-3| = 3$.

Example 11.1.3 Yonas takes a 5-block walk north from his home to a food cart. After enjoying dinner, he then walks 9 blocks south of the food cart to his favorite movie theater.

a. How many blocks has Yonas walked in total when he reaches the theater?

b. How many blocks is Yonas from home when he reaches the theater?

Explanation.

a. Since we only care about total distance, we can ignore the "signs" on the distances walked (either north or south) and simply add the two values together. Mathematically, if we think of north as positive values and south as having negative values, this situation is the same as

$$\begin{aligned} |5| + |-9| &= 5 + 9 \\ &= 14 \end{aligned}$$

Yonas has walked a total of 14 blocks when he reaches the theater.

b. When he reaches the theater, Yonas's actual position could be thought of as $5+(-9)$. But the actual distance from the theater to his home is better thought of as:

$$\begin{aligned} |5 + (-9)| &= |-4| \\ &= 4 \end{aligned}$$

> Yonas was 4 blocks from home when he reached the theater.

11.1.2 Evaluating Absolute Value Functions

The formula $f(x) = |x|$ *does* satisfy the requirements for f to be a function because no matter what number you put in for x, there is only one measured distance from 0 to that value x.

> **Example 11.1.4** Let $f(x) = |x|$ and $g(x) = |2x - 5|$. Evaluate the following expressions.
>
> a. $f(34)$
> b. $f(-63)$
> c. $f(0)$
> d. $g(13)$
> e. $g(1)$
>
> **Explanation**.
>
> a. $$\begin{aligned} f(34) &= |34| \\ &= 34 \end{aligned}$$
>
> b. $$\begin{aligned} f(-63) &= |-63| \\ &= 63 \end{aligned}$$
>
> c. $$\begin{aligned} f(0) &= |0| \\ &= 0 \end{aligned}$$
>
> d. $$\begin{aligned} g(13) &= |2 \cdot 13 - 5| \\ &= |21| \\ &= 21 \end{aligned}$$
>
> e. $$\begin{aligned} g(1) &= |2 \cdot 1 - 5| \\ &= |-3| \\ &= 3 \end{aligned}$$

Checkpoint 11.1.5. Mark each equation as True or False.

a. (☐ True ☐ False) $|10| = 10$.
b. (☐ True ☐ False) $|-3|$ is both 3 and -3.
c. (☐ True ☐ False) $|x + 4| = x + 4$.
d. (☐ True ☐ False) $|-6| = |6|$.
e. (☐ True ☐ False) $|x - 3| = |x + 3|$.

Explanation. Remember that to be "false" when there is a variable in the equation, all that has to occur is a single input number that makes the equation false.

a. True: $|10| = 10$.
b. False: $|-3|$ is only 3.
c. False: $|x + 4| \neq x + 4$ for many values of x. When you have to decide whether or not an equation is true, one good method to help you decide is to plug in a few numbers to see if each number makes the equation true or false. Be sure to pick a variety and input at least two numbers, if not three. In this case, we will choose 10 and -20.

   When $x = 10$:

$$\begin{aligned} |x + 4| &\overset{?}{=} x + 4 \\ |10 + 4| &\overset{?}{=} 10 + 4 \\ |14| &\overset{\checkmark}{=} 14 \end{aligned}$$

When $x = -20$:

$$\begin{aligned} |x+4| &\overset{?}{=} x+4 \\ |-20+4| &\overset{?}{=} -20+4 \\ |-16| &\overset{\text{no}}{=} -16 \end{aligned}$$

When we input -20, the equation was false, which indicates that the equation $|x+4| = x+4$ is false for general x.

d. True: $|-6| = |6|$. Both $|-6|$ and $|6|$ are equal to 6.

e. False: $|x-3| \neq |x+3|$. Again we should choose some numbers to check the validity of the equation. We will choose -12 and 15.

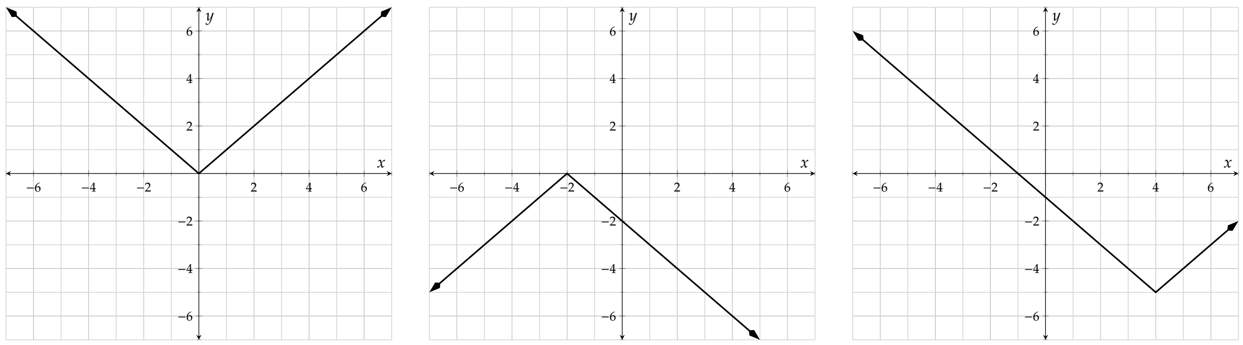
When $x = -12$:

$$\begin{aligned} |x-3| &\overset{?}{=} |x+3| \\ |-12-3| &\overset{?}{=} |-12+3| \\ |-15| &\overset{?}{=} |-9| \\ 15 &\overset{\text{no}}{=} 9 \end{aligned}$$

Since we had a false equation for our first value, we don't need to check the second input. The original equation is simply false.

11.1.3 Graphs of Absolute Value Functions

Absolute value functions have generally the same shape. They are usually described as "V"-shaped graphs and the tip of the "V" is called the **vertex**. A few graphs of various absolute value functions are shown in Figure 11.1.5. In general, the domain of an absolute value function (where there is a polynomial inside the absolute value) is $(-\infty, \infty)$.



(a) $y = |x|$ **(b)** $y = -|x+2|$ **(c)** $y = |x-4| - 5$

Figure 11.1.5

Example 11.1.6 Let $h(x) = -2|x-3|+5$. Using technology, create table of values with x-values from -3 to 3, using an increment of 1. Then sketch a graph of $y = h(x)$. State the domain and range of h.

Explanation.

x	y
-3	-7
-2	-5
-1	-3
0	-1
1	1
2	3
3	5

Table 11.1.7: Table for $y = h(x)$.

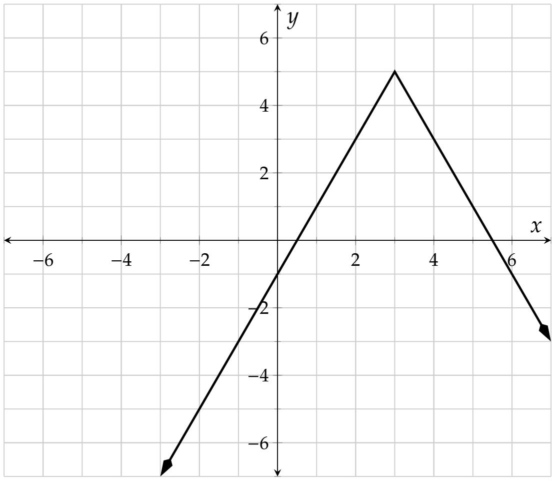


Figure 11.1.8: Graph of $y = h(x)$

The graph indicates that the domain is (∞, ∞) as it goes to the right and left indefinitely. The range is $(-\infty, 5]$.

Example 11.1.9 Let $j(x) = \big||x+1|-2\big|-1$. Using technology, create table of values with x-values from -5 to 5, using an increment of 1 and sketch a graph of $y = j(x)$. State the domain and range of j.

Explanation. This is a strange one because it has an absolute value within an absolute value.

x	y
-5	1
-4	0
-3	-1
-2	0
-1	1
0	0
1	-1
2	0
3	1
4	2
5	3

Table 11.1.10: A table of values for $y = j(x)$.

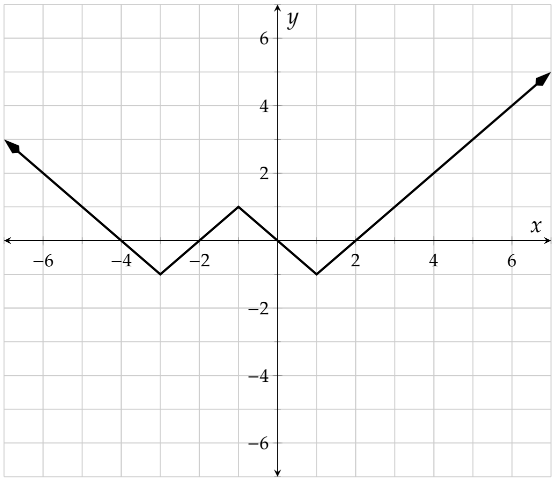


Figure 11.1.11: $y = \big||x+1|-2\big|-1$

The graph indicates that the domain is (∞, ∞) as it goes to the right and left indefinitely. The range is $[-1, \infty)$.

11.1.4 Another Definition for Absolute Value

How many definitions do we really need? Bear with us because this one is important.

Example 11.1.12 Consider the function f defined by $f(x) = \sqrt{x^2}$. First, we will evaluate this function at a few arbitrary values: 3, 0, and −5.

$$f(3) = \sqrt{3^2} = \sqrt{9} = 3 \qquad f(0) = \sqrt{0^2} = \sqrt{0} = 0 \qquad f(-5) = \sqrt{(-5)^2} = \sqrt{25} = 5$$

These results should seem familiar: $f(3) = 3$, $f(0) = 0$, and $f(-5) = 5$. The outputs are the same as the inputs, except for a missing negative sign on 5. It seems like we've seen a function that does that exact same thing already …

Make a quick graph using technology to see what the graph of $y = f(x)$ looks like.

Explanation. Figure 11.1.13 shows a graph of f where $f(x) = \sqrt{x^2}$. It looks just like that of $y = |x|$.

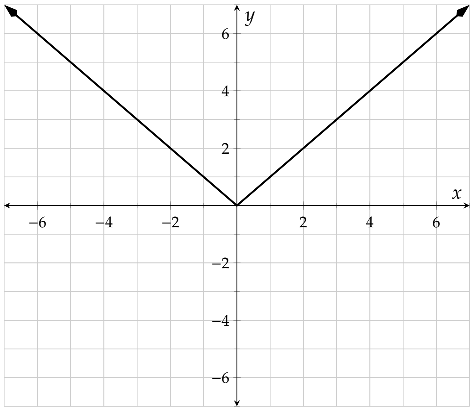


Figure 11.1.13: $y = \sqrt{x^2}$

Since the graphs of $y = \sqrt{x^2}$ and $y = |x|$ match up exactly, that must mean that

$$|x| = \sqrt{x^2}$$

This fact will be used later in this chapter and it will continue to pop up in subsequent math courses here and there as important.

Example 11.1.14 Simplify the following expressions using the fact that $|x| = \sqrt{x^2}$.

a. $\sqrt{(x-2)^2}$ b. $\sqrt{x^6}$ c. $\sqrt{x^2+10x+25}$ d. $\sqrt{x^4}$

Explanation.

a. $\sqrt{(x-2)^2} = |x-2|$. Note that $x - 2$ might be a negative number depending on the value of x, so the absolute value will change those negative numbers to be positive values.

b. $$\begin{aligned}\sqrt{x^6} &= \sqrt{(x^3)^2}\\ &= \left|x^3\right|\end{aligned}$$

We know from exponent rules that $x^6 = \left(x^3\right)^2$. Note that x^3 will be negative whenever x is a negative number, so the absolute value bars must remain.

c. $$\begin{aligned}\sqrt{x^2+10x+25} &= \sqrt{(x+5)^2}\\ &= |x+5|\end{aligned}$$

Note again that $x+5$ can be negative for certain values of x, so the absolute value bars must remain.

d. $$\begin{aligned}\sqrt{x^4} &= \sqrt{(x^2)^2}\\ &= \left|x^2\right|\\ &= x^2\end{aligned}$$

Note here that x^2 is never negative. No matter what number you substitute in for x in x^2, you always either get a positive result or zero. So the absolute value around x^2 doesn't have any effect. Absolute values change negative numbers to positive values but leave positive values alone. Thus, it is OK in this case to drop the absolute value bars.

11.1.5 Applications Involving Absolute Values

Absolute values are quite useful as models in a variety of real world applications. One example is the path of a billiards (pool) ball: when the ball bounces off one of the side rails, its path is mirrored and creates a "V" shape. The game gets more complicated when more than the rail is hit, but the fundamental mathematics doesn't change: absolute values model the bounces each time.

Here are some more examples. The first one we'll explore involves light reflecting off of a mirror.

When light reflects off of a mirror, the path it takes is in the shape of an absolute value graph. Khenbish was playing with a laser pointer in his bedroom mirror. He set up the laser pointer on his windowsill and the light hit the center of the mirror and reflected onto the corner of his room. He declared that the laser pointer is sitting at the origin, and x should stand for the horizontal distance from the left wall to the light beam. Shown is a birds-eye view of the situation.

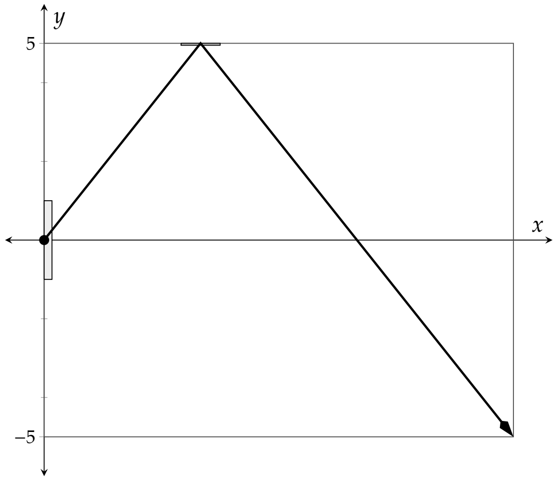


Figure 11.1.16: Birds-Eye View of Khenbish's Room with Laser

Example 11.1.15 After a little bit of work, Khenbish was able to come up with a formula for the light's path:

$$p(x) = 5 - \frac{5}{4}|x-4|$$

where $p(x)$ stands for the position, in ft, above (for positive values) or below (for negatives) the center line through his room that represents the x-axis, where x is also measured in ft. Use technology and a graph of this formula to answer the following questions.

a. Khenbish's room is 10 ft wide according to Figure 11.1.16 (in the vertical direction in the figure). What is the room's length (in the horizontal direction in the figure)?

b. How far along the wall is the mirror centered?

c. If you stood 9 ft from the left wall, how far above or below the room's center line (x-axis) should you stand to have the laser pointer hit you?

Explanation.

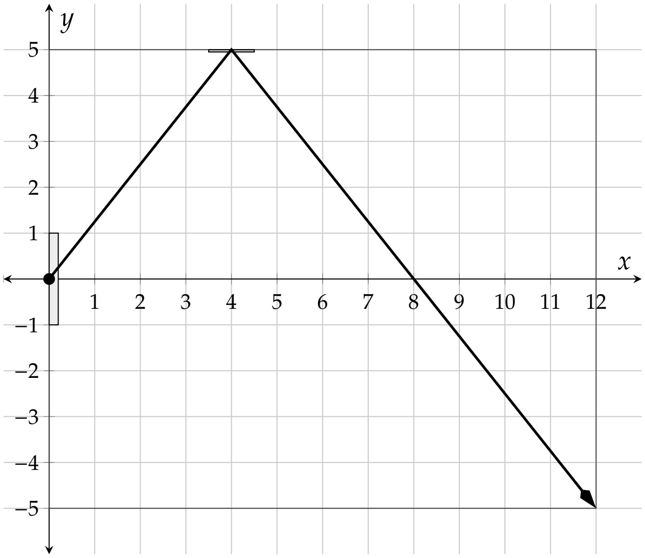


Figure 11.1.17: Detailed Birds-Eye View of Khenbish's Room

a. To find the room's length, first note that since the laser hits the corner of the room, the x coordinate of the lasers position would tell us the room's width. According to the detailed graph, the x-coordinate when $y = -5$ is 12. So the room must be 12 feet wide.

b. The mirror is centered exactly where the laser hits the wall. This is the vertex of the absolute value graph which, according to the graph, is at the point $(4, 5)$. This tells us that the mirror is centered 4 feet from the left wall.

c. If you are standing 9 feet from the left wall, the laser's position will be a bit more than one foot behind the rooms center line, by the diagram. While technology can tell us the exact answer, here is how to do this problem algebraically.

$$\begin{aligned} d(9) &= 5 - \frac{5}{4}\left|9-4\right| \\ &= 5 - \frac{5}{4}\left|5\right| \\ &= 5 - \frac{5}{4}\cdot 5 \\ &= \frac{20}{4} - \frac{25}{4} \\ &= -\frac{5}{4} \\ &= -1.25 \end{aligned}$$

So, it looks like if you stand 9 feet from the left wall, you need to stand 1.25 feet behind the center line (which would be 6.25 feet from the wall with the mirror on it) to be hit by the laser.

Absolute value functions are also used when a value must be within a certain distance or tolerance. For example, a person's body temperature is considered "normal" if it is within 0.5 degrees of 98.6 °F, so their temperature could be up to 0.5 degrees less than or greater than that temperature. To be within normal range, the difference between the two values must be less than or equal to 0.5, and it does not matter whether it is positive or negative. We will introduce a function for measuring this in the next example.

Example 11.1.18 The function D defined by $D(T) = \left|T - 98.6\right|$ represents the difference between a person's temperature, T, in Fahrenheit, and 98.6 °F. A person's temperature is considered "normal" if $D(T)$ is less than or equal to 0.5. Use $D(T)$ to determine whether each person's temperature is within the normal range.

a. LaShonda has a temperature of 98.3 °F.
b. Castel has a temperature of 99.3 °F.
c. Daniel has a temperature of 97.3 °F.

Explanation.

a. LaShonda has a temperature of 98.3 °F, so we have:

$$\begin{aligned} D(98.3) &= \left|98.3 - 98.6\right| \\ &= \left|-0.3\right| \\ &= 0.3 \end{aligned}$$

Since the value of $D(98.3)$ is a number smaller than 0.5, her temperature of 98.3 °F is within the normal range.

b. If Castel has a temperature of 99.3 °F, then we have:

$$\begin{aligned} D(99.3) &= \left|99.3 - 98.6\right| \\ &= \left|0.7\right| \\ &= 0.7 \end{aligned}$$

Since the value of $D(99.3)$ is a number bigger than 0.5, their temperature of 99.3 °F is *not* within the normal range.

c. Daniel's temperature is 97.3 °F, so we have:

$$\begin{aligned} D(97.3) &= |97.3 - 98.6| \\ &= |-1.3| \\ &= 1.3 \end{aligned}$$

Since the value of $D(97.3)$ is a number bigger than 0.5, his temperature of 97.3 °F is *not* within the normal range.

Example 11.1.19 The entryway to the Louvre Museum in Paris is through I. M. Pei's metal and glass Louvre Pyramid. This pyramid has a square base and is 71 feet high and 112 feet wide. The formula $h(x) = 71 - \frac{71}{56}|x - 56|$ gives the height above ground level of the pyramid at a distance of x from the left side of the pyramid base.

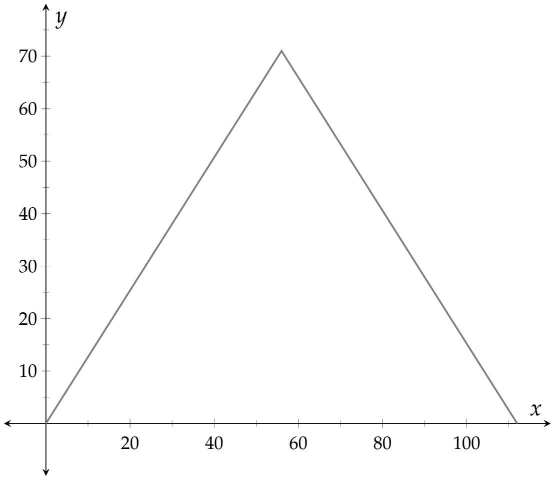


Figure 11.1.20: A Diagram of the Front of the Louvre Pyramid

a. If you are 20 feet from the left edge, how high will the pyramid rise in front of you? Round your result to the nearest tenth of an inch.

b. How far from the left edge is the center of the pyramid?

c. Using your previous answer, check that the formula gives you the correct height at the center.

Explanation.

a. If you are 20 feet from the left edge, then x is 20. Substituting 20 for x we have

$$\begin{aligned} h(20) &= 71 - \frac{71}{56}|20 - 56| \\ &= 71 - \frac{71}{56}|-36| \\ &= 71 - \frac{71}{56} \cdot 36 \\ &\approx 25.4 \end{aligned}$$

The pyramid is about 25.4 feet high at the position 20 feet from the left edge.

b. The center of the pyramid is 56 feet from the either edge since it's half of 112 feet.

c. Putting $x = 56$ into the formula for h gives us

$$\begin{aligned} h(56) &= 71 - \frac{71}{56}\,|56 - 56| \\ &= 71 - \frac{71}{56}\,|0| \\ &= 71 - \frac{71}{56} \cdot 0 \\ &= 71 \end{aligned}$$

And so the formula does give us the correct maximum height of 71 feet at the center of the pyramid.

Exercises

Review and Warmup Evaluate the following.

1. a. $|3| =$

b. $|-4| =$

c. $|0| =$

d. $|18 + (-8)| =$

e. $|-9 - (-4)| =$

2. a. $|4| =$

b. $|-8| =$

c. $|0| =$

d. $|12 + (-1)| =$

e. $|-9 - (-2)| =$

3. a. $-|5 - 9| =$

b. $|-5 - 9| =$

c. $-3|9 - 5| =$

4. a. $-|3 - 6| =$

b. $|-3 - 6| =$

c. $-3|6 - 3| =$

5. a. $-3 - 3\,|7 - 1| =$

b. $-3 - 3\,|1 - 7| =$

6. a. $-2 - 8\,|9 - 4| =$

b. $-2 - 8\,|4 - 9| =$

7. a. $-1 - 5\,|6 - 3| =$

b. $-1 - 5\,|3 - 6| =$

8. a. $-10 - 3\,|8 - 2| =$

b. $-10 - 3\,|2 - 8| =$

9. $1 - 8\,|1 - 2| + 2 =$

10. $2 - 6\,|3 - 8| + 2 =$

11. $4 - 2\left|-1 + (3 - 5)^3\right| =$

12. $5 - 8\left|-7 + (2 - 5)^3\right| =$

Function Notation with Absolute Value

13. Given $H(t) = |t - 296|$, find and simplify $H(159)$.

$H(159) =$

14. Given $g(r) = |r - 261|$, find and simplify $g(170)$.

$g(170) =$

15. Given $h(x) = |x + 16|$, find and simplify $h(18)$.

$h(18) =$

16. Given $f(x) = |-4x - 5|$, find and simplify $f(20)$.

$f(20) =$

17. Given $f(x) = 10 - |3x - 26|$, find and simplify $f(10)$.

$f(10) =$

18. Given $f(r) = 15 - |4r + 14|$, find and simplify $f(11)$.

$f(11) =$

19. Given $g(t) = t + |-2t - 7|$, find and simplify $g(12)$.

$g(12) =$

20. Given $K(t) = t + |2t - 29|$, find and simplify $K(14)$.

$K(14) =$

21. Given $G(t) = \left|t^2 - 16\right|$, find and simplify $G(-5)$.

$G(-5) =$

22. Given $h(t) = \left|t^2 - 25\right|$, find and simplify $h(1)$.

$h(1) =$

23. Given $f(t) = \left|t^2 - 2t - 15\right|$, find and simplify $f(7)$.

$f(7) =$

24. Given $H(t) = \left|t^2 - 2t - 24\right|$, find and simplify $H(-2)$.

$H(-2) =$

Domain

25. Find the domain of K where $K(x) = |-4x + 6|$.

26. Find the domain of f where $f(x) = |9x - 6|$.

27. Find the domain of f where $f(x) = 2x - |2x + 3|$.

28. Find the domain of g where $g(x) = 8x - |-5x - 9|$.

Tables

29. Make a table of values for the function h defined by $h(x) = |3x - 2|$.

x	$h(x)$

30. Make a table of values for the function F defined by $F(x) = |2x - 3|$.

x	$F(x)$

31. Make a table of values for the function G defined by $G(x) = \left|x^2 - 2x - 1\right|$.

x	$G(x)$

32. Make a table of values for the function G defined by $G(x) = \left|x^2 - x - 2\right|$.

x	$G(x)$

33. Make a table of values for the function H defined by $H(x) = -\left|2x - 3\right| + 1$.

x	$H(x)$

34. Make a table of values for the function K defined by $K(x) = -3|2x - 3| + 2$.

x	$K(x)$

35. Make a table of values for the function f defined by $f(x) = \Big|-2|2x - 3| - 1\Big|$.

x	$f(x)$

36. Make a table of values for the function f defined by $f(x) = \Big||-x + 2| + 2\Big|$.

x	$f(x)$

Graphs

37. Graph $y = f(x)$, where $f(x) = |2x - 1|$.

38. Graph $y = f(x)$, where $f(x) = |x - 2|$.

39. Graph $y = f(x)$, where $f(x) = \left|x^2 - 2x - 1\right|$.

40. Graph $y = f(x)$, where $f(x) = \left|x^2 + 3x - 2\right|$.

41. Graph $y = f(x)$, where $f(x) = \frac{1}{2}\,|4x - 5| - 3$.

42. Graph $y = f(x)$, where $f(x) = \frac{3}{4}\,|6 + x| + 2$.

43. Graph $y = f(x)$, where $f(x) = \left|2\,|3 - x| - 2\right|$.

44. Graph $y = f(x)$, where $f(x) = \left|3 - 2\,|2x - 3|\,\right|$.

Absolute Value and Square Roots

45. Simplify the expression. Do not assume the variables take only positive values.

$\sqrt{9z^2}$

46. Simplify the expression. Do not assume the variables take only positive values.

$\sqrt{64t^2}$

47. Simplify the expression.

$\sqrt{(r - 43)^2}$

48. Simplify the expression.

$\sqrt{(m - 8)^2}$

49. Simplify the expression.

$\sqrt{(-13698)^2}$

50. Simplify the expression.

$\sqrt{(-15696)^2}$

51. Simplify the expression.

$\sqrt{x^2 + 20x + 100}$

52. Simplify the expression.

$\sqrt{n^2 + 2n + 1}$

Applications

53. The height inside a camping tent when you are d feet from the edge of the tent is given by

$$h = -0.5|d - 4.4| + 5.5$$

where h stands for height in feet.

Determine the height when you are:

a. 5.9 ft from the edge.

The height inside a camping tent when you 5.9 ft from the edge of the tent is

b. 3.5 ft from the edge.

The height inside a camping tent when you 3.5 ft from the edge of the tent is

54. The height inside a camping tent when you are d feet from the edge of the tent is given by

$$h = -0.5|d - 6.6| + 6.5$$

where h stands for height in feet.

Determine the height when you are:

a. 11.5 ft from the edge.

The height inside a camping tent when you 11.5 ft from the edge of the tent is ☐

b. 1.2 ft from the edge.

The height inside a camping tent when you 1.2 ft from the edge of the tent is ☐

Challenge

55. Write two numbers so that

- The first number is less than the second number, and
- The absolute value of the first number is greater than the absolute value of the second number

☐ and ☐

11.2 Compound Inequalities

On the newest version of the SAT (an exam that often qualifies students for colleges) the minimum score that you can earn is 400 and the maximum score that you can earn is 1600. This means that only numbers between 400 and 1600, including these endpoints, are possible scores. To plot all of these values on a number line would look something like:

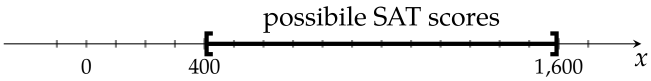


Figure 11.2.1: Possible SAT Scores

Going back to the original statement, "the minimum score that you can earn is 400 and the maximum score that you can earn is 1600," this really says two things. First, it says that (a SAT score) ≥ 400, and second, that (a SAT score) ≤ 1600. When we combine two inequalities like this into a single problem, it becomes a **compound inequality**.

Our lives are often constrained by the compound inequalities of reality: you need to buy enough materials to complete your project, but you can only fit so much into your vehicle; you would like to finish your degree early, but only have so much money and time to put toward your courses; you would like a vegetable garden big enough to supply you with veggies all summer long, but your yard or balcony only gets so much sun. In the rest of the section we hope to illuminate how to think mathematically about problems like these.

Before continuing, a review on how notation for intervals works may be useful, and you may benefit from revisiting Section 1.6. Then a refresher on solving linear inequalities may also benefit you, which you can revisit in Section 3.2 and Section 3.3.

11.2.1 Unions of Intervals

Definition 11.2.3. The **union** of two sets, A and B, is the set of all elements contained in either A or B (or both). We write $A \cup B$ to indicate the union of the two sets.

In other words, the union of two sets is what you get if you toss every number in both sets into a bigger set.

Example 11.2.4 The union of sets $\{1, 2, 3, 4\}$ and $\{3, 4, 5, 6\}$ is the set of all elements from either set. So $\{1, 2, 3, 4\} \cup \{3, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$. Note that we don't write duplicates.

Example 11.2.5 Let's visualize the union of the sets $(-\infty, 4)$ and $[7, \infty)$. First we make a number line with both intervals drawn to understand what both sets mean.

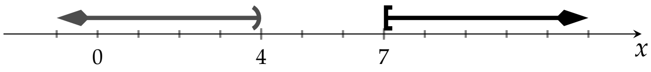


Figure 11.2.6: A number line sketch of $(-\infty, 4)$ as well as $[7, \infty)$

The two intervals should be viewed as a single object when stating the union, so here is the picture of the union. It looks the same, but now it is a graph of a single set.

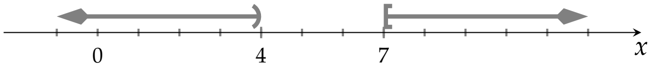


Figure 11.2.7: A number line sketch of $(-\infty, 4) \cup [7, \infty)$

11.2.2 "Or" Compound Inequalities

Definition 11.2.8. A **compound inequality** is a grouping of two or more inequalities into a larger inequality statement. These usually come in two flavors: "or" and "and" inequalities. For an example of an "or" compound inequality, you might get a discount at the movie theater if your age is less than 13 *or* greater than 64. In general, compound inequalities of the "and" variety currently are beyond the scope of this book. However a special type of the "and" variety is covered later in Subsection 11.2.3.

In math, the technical term **or** means "either or both." So, mathematically, if we asked if you would like "chocolate cake *or* apple pie" for dessert, your choices are either "chocolate cake," "apple pie," or "both chocolate cake and apple pie." This is slightly different than the English "or" which usually means "one or the other but not both."

"Or" shows up in math between equations (as in when solving a quadratic equation, you might end up with "$x = 2$ or $x = -3$") or between inequalities (which is what we're about to discuss).

Remark 11.2.9. The definition of "or" is very close to the definition of a union where you combine elements from either or both sets together. In fact, when you have an "or" between inequalities in a compound inequality, to find the solution set of the compound inequality, you find the union of the the solutions sets of each of the pieces.

Example 11.2.10 Solve the compound inequality.

$$x \leq 1 \text{ or } x > 4$$

Explanation.

Writing the solution set to this compound inequality doesn't require any algebra beforehand because each of the inequalities is already solved for x. The first thing we should do is understand what each inequality is saying using a graph.

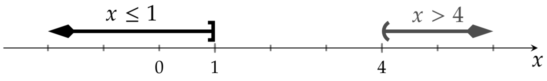


Figure 11.2.11: A number line sketch of solutions to $x \leq 1$ as well as to $x > 4$

An "or" statement becomes a union of solution sets, so the solution set to the compound inequality must be:

$$(-\infty, 1] \cup (4, \infty).$$

Example 11.2.12 Solve the compound inequality.

$$3 - 5x > -7 \text{ or } 2 - x \leq -3$$

Explanation. First we need to do some algebra to isolate x in each piece. Note that we are going to do algebra on both pieces simultaneously. Also note that the mathematical symbol "or" should be written on each line.

$$\begin{array}{ccc}
3-5x > -7 & \text{or} & 2-x \leq -3 \\
3-5x-3 > -7-3 & \text{or} & 2-x-2 \leq -3-2 \\
-5x > -10 & \text{or} & -x \leq -5 \\
\dfrac{-5x}{-5} < \dfrac{-10}{-5} & \text{or} & \dfrac{-x}{-1} \geq \dfrac{-5}{-1} \\
x < 2 & \text{or} & x \geq 5
\end{array}$$

The solution set for the compound inequality $x < 2$ is $(-\infty, 2)$ and the solution set to $x \geq 5$ is $[5, \infty)$. To do the "or" portion of the problem, we need to take the union of these two sets. Let's first make a graph of the solution sets to visualize the problem.

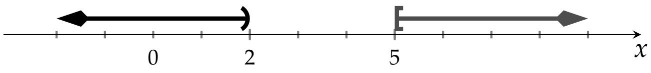


Figure 11.2.13: A number line sketch of $(-\infty, 2)$ as well as $[5, \infty)$

The union combines both solution sets into one, and so

$$(-\infty, 2) \cup [5, \infty)$$

We have finished the problem, but for the sake of completeness, let's try to verify that our answer is reasonable.

- First, let's choose a number that is *not* in our proposed solution set. We will arbitrarily choose 3.

  $$\begin{array}{ccc}
  3-5x > -7 & \text{or} & 2-x \leq -3 \\
  3-5(3) \overset{?}{>} -7 & \text{or} & 2-(3) \overset{?}{\leq} -3 \\
  -9 \overset{\text{no}}{>} -7 & \text{or} & -1 \overset{\text{no}}{\leq} -3
  \end{array}$$

  This value made *both* inequalities false which is why 3 isn't in our solution set.

- Next, let's choose a number that *is* in our solution region. We will arbitrarily choose 1.

  $$\begin{array}{ccc}
  3-5x > -7 & \text{or} & 2-x \leq -3 \\
  3-5(1) \overset{?}{>} -7 & \text{or} & 2-(1) \overset{?}{\leq} -3 \\
  -12 \overset{\checkmark}{<} -7 & \text{or} & -1 \overset{\text{no}}{\leq} -3
  \end{array}$$

  This value made *one* of the inequalities true. Since this is an "or" statement, only one *or* the other piece has to be true to make the compound inequality true.

- Last, what will happen if we choose a value that was in the other solution region in Figure 11.2.13, like the number 6?

$$\begin{array}{rcr} 3-5x > -7 & \text{or} & 2-x \leq -3 \\ 3-5(6) \overset{?}{>} -7 & \text{or} & 2-(6) \overset{?}{\leq} -3 \\ -27 \overset{\text{no}}{>} -7 & \text{or} & -4 \overset{\checkmark}{\leq} -3 \end{array}$$

  This solution made the *other* inequality piece true.

This completes the check. Numbers from within the solution region make the compound inequality true and numbers outside the solution region make the compound inequality false.

Example 11.2.14 Solve the compound inequality.

$$\frac{3}{4}t+2 \leq \frac{5}{2} \text{ or } -\frac{1}{2}(t-3) < -2$$

Explanation. First we will solve each inequality for t. Recall that we usually try to clear denominators by multiplying both sides by the least common denominator.

$$\begin{array}{rcr} \frac{3}{4}t+2 \leq \frac{5}{2} & \text{or} & -\frac{1}{2}(t-3) < -2 \\ 4\cdot\left(\frac{3}{4}t+2\right) \leq 4\cdot\frac{5}{2} & \text{or} & 2\cdot\left(-\frac{1}{2}(t-3)\right) < 2\cdot(-2) \\ 3t+8 \leq 10 & \text{or} & -t+3 < -4 \\ 3t+8-8 \leq 10-8 & \text{or} & -t+3-3 < -4-3 \\ 3t \leq 2 & \text{or} & -t < -7 \\ \frac{3t}{3} \leq \frac{2}{3} & \text{or} & \frac{-t}{-1} > \frac{-7}{-1} \\ t \leq \frac{2}{3} & \text{or} & t > 7 \end{array}$$

The solution set to $t \leq \frac{2}{3}$ is $\left(-\infty, \frac{2}{3}\right]$ and the solution set to $t > 7$ is $(7, \infty)$. Figure 11.2.15 shows these two sets.

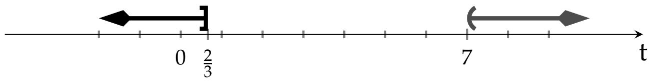


Figure 11.2.15: A number line sketch of $\left(-\infty, \frac{2}{3}\right]$ and also $(7, \infty)$

Note that the two sets do not overlap so there will be no way to simplify the union. Thus the solution set to the compound inequality is:

$$\left(-\infty, \frac{2}{3}\right] \cup (7, \infty)$$

Example 11.2.16 Solve the compound inequality.

$$3y - 15 > 6 \text{ or } 7 - 4y \geq y - 3$$

Explanation. First we solve each inequality for y.

$$\begin{array}{ccc}
3y - 15 > 6 & \text{or} & 7 - 4y \geq y - 3 \\
3y - 15 + 15 > 6 + 15 & \text{or} & 7 - 4y - 7 - y \geq y - 3 - 7 - y \\
3y > 21 & \text{or} & -5y \geq -10 \\
\frac{3y}{3} > \frac{21}{3} & \text{or} & \frac{-5y}{-5} \leq \frac{-10}{-5} \\
y > 7 & \text{or} & y \leq 2
\end{array}$$

The solution set to $y > 7$ is $(7, \infty)$ and the solution set to $y \leq 2$ is $(-\infty, 2]$. Figure 11.2.17 shows these two sets.

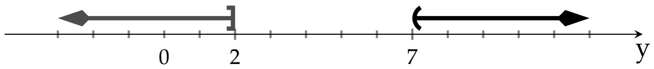


Figure 11.2.17: A number line sketch of $(7, \infty)$ as well as $(-\infty, 2]$

So the solution set to the compound inequality is:

$$(-\infty, 2] \cup (7, \infty)$$

11.2.3 Three-Part Inequalities

The inequality $1 \leq 2 < 3$ says a lot more than you might think. It actually says four different single inequalities which are highlighted for you to see.

$$1 \leq 2 < 3 \quad 1 \leq 2 < 3 \quad 1 \leq 2 < 3 \quad 1 \leq 2 < 3$$

This might seem trivial at first, but if you are presented with an inequality like $-1 < 3 \geq 2$, at first it might look sensible; however, in reality, you need to check that *all four* linear inequalities make sense. Those are highlighted here.

$$-1 < 3 \geq 2 \quad -1 < 3 \geq 2 \quad -1 < 3 \geq 2 \quad -1 < 3 \geq 2$$

One of these inequalities is false: $-1 \ngeq 2$. This implies that the entire original inequality, $-1 < 3 \geq 2$, is nonsense.

Example 11.2.18 Decide whether or not the following inequalities are true or false.

a. True or False: $-5 < 7 \leq 12$?

b. True or False: $-7 \leq -10 < 4$?

c. True or False: $-2 \leq 0 \geq 1$?

d. True or False: $5 > -3 \geq -9$?

e. True or False: $3 < 3 \leq 5$?

f. True or False: $9 > 1 < 5$?

g. True or False: $3 < 8 \leq -2$?

h. True or False: $-9 < -4 \leq -2$?

Explanation. We need to go through all four single inequalities for each. If the inequality is false, for simplicity's sake, we will only highlight the one single inequality that makes the inequality false.

a. True: $-5 < 7 \leq 12$.
b. False: $-7 \overset{\text{no}}{\leq} -10 < 4$.
c. False: $-2 \leq 0 \overset{\text{no}}{\geq} 1$.
d. True: $5 > -3 \geq -9$.
e. False: $3 \overset{\text{no}}{<} 3 \leq 5$.
f. False: $9 > 1 \overset{\text{no}}{<} 5$.
g. False: $3 < 8 \overset{\text{no}}{\leq} -2$.
h. True: $-9 < -4 \leq -2$.

As a general hint, no (nontrivial) three-part inequality can ever be true if the inequality signs are not pointing in the same direction. So no matter what numbers a, b, and c are, both $a < b \geq c$ and $a \geq b < c$ cannot be true! Soon you will be writing inequalities like $2 < x \leq 4$ and you need to be sure to check that your answer is feasible. You will know that if you get $2 > x \leq 4$ or $2 < x \geq 4$ that something went wrong in the solving process. The only exception is that something like $1 \leq 1 \geq 1$ is true because $1 = 1 = 1$, although this shouldn't come up very often!

Example 11.2.19 Write the solution set to the compound inequality.

$$-7 < x \leq 5$$

Explanation. The solutions to the three-part inequality $-7 < x \leq 5$ are those numbers that are trapped between -7 and 5, including 5 but not -7. Keep in mind that there are infinitely many decimal numbers and irrational numbers that satisfy this inequality like -2.781828 and π. We will write these numbers in interval notation as $(-7, 5]$ or in set builder notation as $\{x \mid -7 < x \leq 5\}$.

Example 11.2.20 Solve the compound inequality.

$$4 \leq 9x + 13 < 20$$

Explanation.

This is a three-part inequality which we can treat just as a regular inequality with three "sides." The goal is to isolate x in the middle and whatever you do to one "side," you have to do to the other two "sides."

The solutions to the three-part inequality $-1 \leq x < \frac{7}{9}$ are those numbers that are trapped between -1 and $\frac{7}{9}$, including -1 but not $\frac{7}{9}$. The solution set in interval notation is $\left[-1, \frac{7}{9}\right)$.

$$\begin{aligned}
4 &\leq 9x + 13 < 20\\
4 - 13 &\leq 9x + 13 - 13 < 20 - 13\\
-9 &\leq 9x < 7\\
\frac{-9}{9} &\leq \frac{9x}{9} < \frac{7}{9}\\
-1 &\leq x < \frac{7}{9}
\end{aligned}$$

Example 11.2.21 Solve the compound inequality.

$$-13 < 7 - \frac{4}{3}x \leq 15$$

Explanation.

This is a three-part inequality which we can treat just as a regular inequality with three "sides." The goal is to isolate x in the middle and whatever you do to one "side," you have to do to the other two "sides." We will begin by canceling the fraction by multiplying each part by the least common denominator.

At the end we reverse the entire statement to go from smallest to largest. The solution set is $(-6, 15]$.

$$\begin{gathered}
-13 < 7 - \frac{4}{3}x \leq 15 \\
-13 \cdot 3 < \left(7 - \frac{4}{3}x\right) \cdot 3 \leq 15 \cdot 3 \\
-39 < 21 - 4x \leq 45 \\
-39 - 21 < 21 - 4x - 21 \leq 45 - 21 \\
-60 < -4x \leq 24 \\
\frac{-60}{-4} > \frac{-4x}{-4} \geq \frac{24}{-4} \\
15 > x \geq -6 \\
-6 \leq x < 15
\end{gathered}$$

11.2.4 Solving Compound Inequalities Graphically

So far we have focused on solving inequalities algebraically. Next, we will describe how to solve compound inequalities graphically.

Example 11.2.22 Figure 11.2.23 shows a graph of $y = f(x)$. Use the graph to solve the inequality $2 \leq f(x) < 6$.

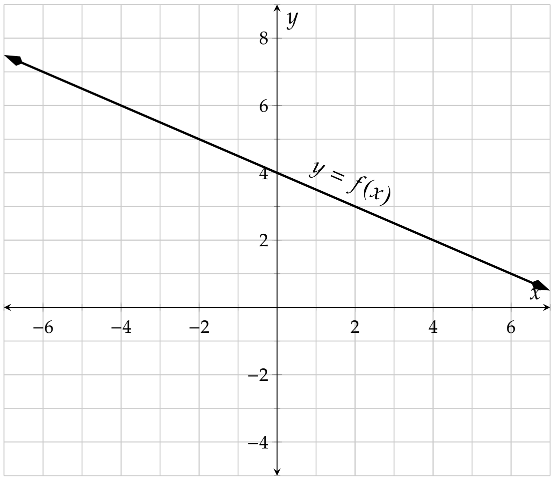


Figure 11.2.23: Graph of $y = f(x)$

To solve the inequality $2 \leq f(x) < 6$ means to find the x-values that give function values between 2 and 6, not including 6. We draw the horizontal lines $y = 2$ and $y = 6$. Then we look for the points of intersection and find their x-values. We see that when x is between -4 and 4, not including -4, the inequality will be true.

We have drawn the interval $(-4, 4]$ along the x-axis, which is the solution set.

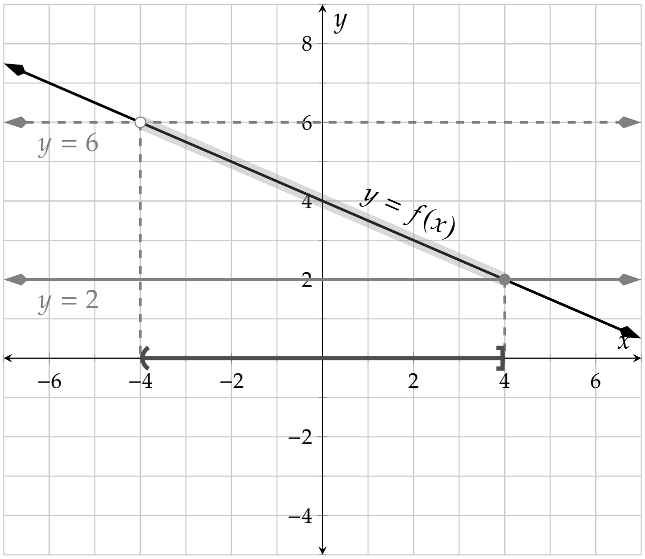


Figure 11.2.24: Graph of $y = f(x)$ and the solution set to $2 \leq f(x) < 6$

Example 11.2.25 Figure 11.2.26 shows a graph of $y = g(x)$. Use the graph to solve the inequality $-4 < g(x) \leq 3$.

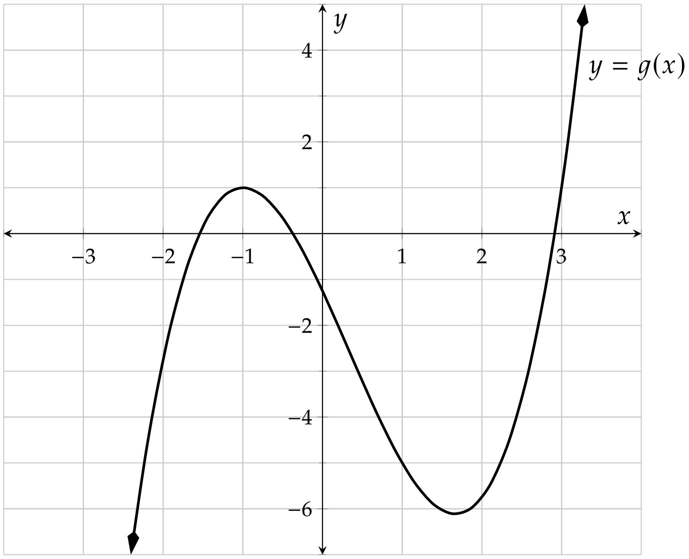


Figure 11.2.26: Graph of $y = g(x)$

Explanation. To solve $-4 < g(x) \leq 3$, we first draw the horizontal lines $y = -4$ and $y = 3$. To solve this inequality we notice that there are two pieces of the function g that are trapped between the y-values -4 and 3.

The solution set is the compound inequality $(-2.1, 0.7) \cup (2.4, 3.2]$.

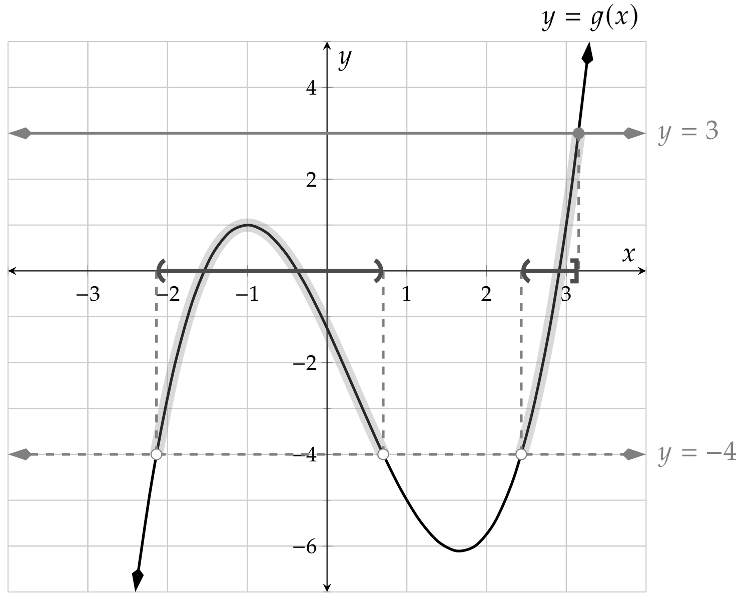


Figure 11.2.27: Graph of $y = g(x)$ and solution set to $-4 < g(x) \leq 3$

11.2.5 Applications of Compound inequalities

Example 11.2.28 Raphael's friend is getting married and he's decided to give them some dishes from their registry. Raphael doesn't want to seem cheap but isn't a wealthy man either, so he wants to buy "enough" but not "too many." He's decided that he definitely wants to spend at least $150 on his friend, but less than $250. Each dish is $21.70 and shipping on an order of any size is going to be $19.99. Given his budget, set up and algebraically solve a compound inequality to find out what his different options are for the number of dishes that he can buy.

Explanation. First, we should define our variable. Let x represent the number of dishes that Raphael can afford. Next we should write a compound inequality that describes this situation. In this case, Raphael wants to spend between $150 and $250 and, since he's buying x dishes, the price that he will pay is $21.70x + 19.99$. All of this translates to a triple inequality

$$150 < 21.70x + 19.99 < 250$$

Now we have to solve this inequality in the usual way.

$$\begin{aligned}
150 &< 21.70x + 19.99 < 250\\
150 - 19.99 &< 21.70x + 19.99 - 19.99 < 250 - 19.99\\
130.01 &< 21.70x < 230.01\\
\frac{130.01}{21.70} &< \frac{21.70x}{21.70} < \frac{230.01}{21.70}\\
5.991 &< x < 10.6 && \text{(note: these values are approximate)}
\end{aligned}$$

The interpretation of this inequality is a little tricky. Remember that x represents the number of dishes Raphael can afford. Since you cannot buy 5.991 dishes (manufacturers will typically only ship whole number amounts of tableware) his minimum purchase must be 6 dishes. We have a similar problem with his maximum purchase: clearly he cannot buy 10.6 dishes. So, should we round up or down? If we rounded up, that would be 11 dishes and that would cost $\$21.70 \cdot 11 + \$19.99 = \$258.69$, which is outside his price range. Therefore, we should actually round *down* in this case.

In conclusion, Raphael should buy somewhere between 6 and 10 dishes for his friend to stay within his budget.

Example 11.2.29 Oak Ridge National Laboratory, a renowned scientific research facility, compiled some data in table 4.28[a] on fuel efficiency of a mid-size hybrid car versus the speed that the car was driven. A model for the fuel efficiency $e(x)$ (in miles per gallon, mpg) at a speed x (in miles per hour, mph) is $e(x) = 88 - 0.7x$.

a. Evaluate and interpret $e(60)$ in the context of the problem.

b. Note that this model only applies between certain speeds. The maximum fuel efficiency for which this formula applies is 55 mpg and the minimum fuel efficiency for which it applies is 33 mpg. Set up and algebraically solve a compound inequality to find the range of speeds for which this model applies.

Explanation.

a. Let's evaluate $e(60)$ first.

$$\begin{aligned} e(x) &= 88 - 0.7x \\ e(60) &= 88 - 0.7(60) \\ &= 46 \end{aligned}$$

So, when the hybrid car travels at a speed of 60 mph, it has a fuel efficiency of 46 mpg.

b. In this case, the minimum efficiency is 33 mpg and the maximum efficiency is 55 mpg. We need to trap our formula between these two values to solve for the respective speeds.

$$\begin{aligned} 33 &< 88 - 0.7x < 55 \\ 33 - 88 &< 88 - 0.7x - 88 < 55 - 88 \\ -55 &< -0.7x < -33 \\ \frac{-55}{-0.7} &> \frac{-0.7x}{-0.7} > \frac{-33}{-0.7} \\ 78.57 &> x > 47.14 && \text{(note: these values are approximate)} \end{aligned}$$

This inequality says that our model is applicable when the car's speed is between about 47 mph and about 79 mph.

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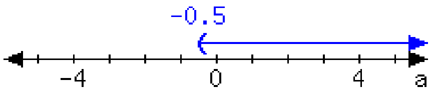
[a] cta.ornl.gov/data/chapter4.shtml

Exercises

Review and Warmup

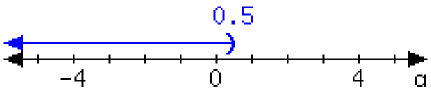
1. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

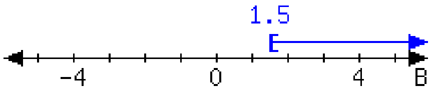
2. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

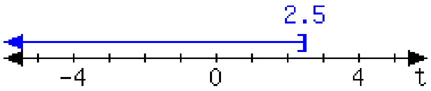
3. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

4. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

5. Solve this inequality.

$5 > x + 10$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

6. Solve this inequality.

$1 > x + 8$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

7. Solve this inequality.

$-2x \geq 4$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

8. Solve this inequality.

$-2x \geq 8$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

9. Solve this inequality.

$4 \geq -5x + 4$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

10. Solve this inequality.

$2 \geq -6x + 2$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

11. Solve this inequality.

$8t + 9 < 3t + 34$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

12. Solve this inequality.

$9t + 6 < 5t + 18$

In set-builder notation, the solution set is ☐.

In interval notation, the solution set is ☐.

Check Solutions Decide whether the given value for the variable is a solution.

13.

a. $x > 8$ and $x \leq 2$ $x = 4$

The given value (☐ is ☐ is not) a solution.

b. $x < 8$ or $x \geq 5$ $x = 4$

The given value (☐ is ☐ is not) a solution.

c. $x \geq -2$ and $x \leq 6$ $x = 8$

The given value (☐ is ☐ is not) a solution.

d. $-2 \leq x \leq 3$ $x = 1$

The given value (☐ is ☐ is not) a solution.

14.

a. $x > 9$ and $x \leq 7$ $x = 8$

The given value (☐ is ☐ is not) a solution.

b. $x < 6$ or $x \geq 5$ $x = 9$

The given value (☐ is ☐ is not) a solution.

c. $x \geq -1$ and $x \leq 9$ $x = -5$

The given value (☐ is ☐ is not) a solution.

d. $-1 \leq x \leq 2$ $x = 1$

The given value (☐ is ☐ is not) a solution.

Compound Inequalities and Interval Notation

15. Solve the compound inequality. Write the solution set in interval notation.

$-10 < x \leq 5$

x is in ☐

16. Solve the compound inequality. Write the solution set in interval notation.

$-9 < x \leq 1$

x is in ☐

17. Solve the compound inequality. Write the solution set in interval notation.

$-8 > x$ or $x \geq 8$

x is in ☐

18. Solve the compound inequality. Write the solution set in interval notation.

$-7 > x$ or $x \geq 4$

x is in ☐

19. Express the following inequality using interval notation.

$x < -6$ or $x \leq 1$

x is in ☐

20. Express the following inequality using interval notation.

$x < -5$ or $x \leq 7$

x is in ☐

Solving a Compound Inequality Algebraically Solve the compound inequality algebraically.

21. $-14 < 7 - x \leq -9$

x is in

22. $-16 < 20 - x \leq -11$

x is in

23. $19 \leq x + 13 < 24$

x is in

24. $1 \leq x + 7 < 6$

x is in

25. $4 \leq \frac{5}{9}(F - 32) \leq 50$

F is in

26. $8 \leq \frac{5}{9}(F - 32) \leq 43$

F is in

27. $-10x - 11 \leq -20$ and $-2x - 18 < -10$

28. $17x + 6 \leq 9$ and $18x - 14 \leq 7$

29. $-9x - 8 \leq -5$ or $-4x - 15 \geq -13$

30. $-12x - 13 \geq -11$ or $-7x - 1 < -17$

31. $13x + 13 \leq -13$ or $-5x + 11 \leq -4$

32. $-9x + 13 < 2$ or $16x - 2 < -1$

33. $-13x + 10 < -7$ and $-3x + 4 < 8$

34. $13x - 14 < -18$ and $18x + 8 \leq -16$

35. $6 < \frac{2}{5}x < 20$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

36. $15 < \frac{3}{2}x < 54$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

37. $5 > -1 - \frac{3}{7}x \geq -10$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

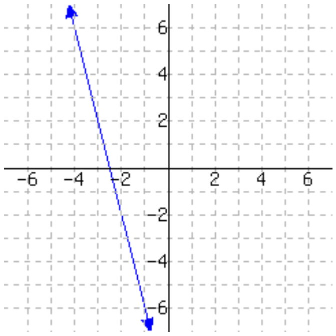
38. $20 > 4 - \frac{4}{5}x \geq -8$

In set-builder notation, the solution set is .

In interval notation, the solution set is .

Solving a Compound Inequality Graphically A graph of f is given. Use the graph alone to solve the compound inequalities.

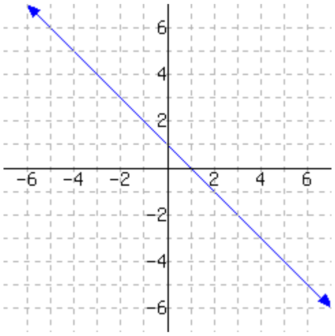
39.



a. $f(x) > 2$

b. $-2 < f(x) \leq 2$

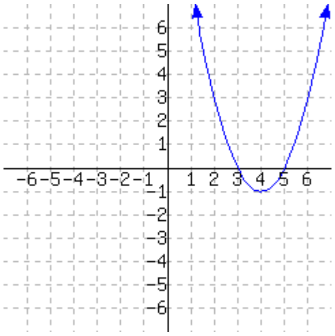
40.



a. $f(x) > -1$

b. $-1 < f(x) \leq 3$

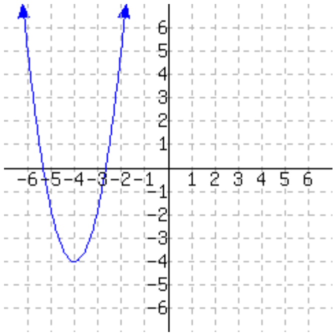
41.



a. $f(x) > 0$

b. $f(x) \leq 0$

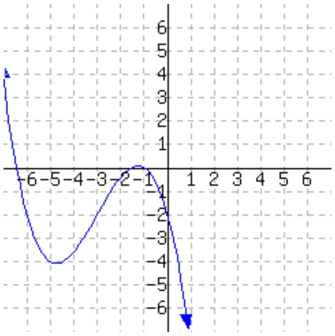
42.



a. $f(x) > 5$

b. $f(x) \leq 5$

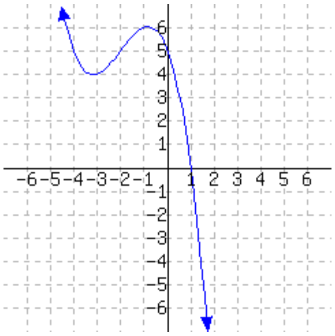
43.



a. $f(x) > -2$

b. $f(x) \leq -2$

44.



a. $f(x) > 5$

b. $f(x) \leq 5$

Applications

45. As dry air moves upward, it expands. In so doing, it cools at a rate of about 1°C for every 100 m rise, up to about 12 km.

a. If the ground temperature is 18°C, write a formula for the temperature at height x km. $T(x) =$ ☐

b. What range of temperature will a plane be exposed to if it takes off and reaches a maximum height of 5 km? Write answer in interval notation.

The range is ☐

11.3 Absolute Value Equations and Inequalities

Whether it's a washer, nut, bolt, or gear, when a machine part is made, it must be made to fit with all of the other parts of the system. Since no manufacturing process is perfect, there are small deviations from the norm when each piece is made. In fact, manufacturers have a *range* of acceptable values for each measurement of every screw, bolt, etc.

Let's say we were examining some new bolts just out of the factory. The manufacturer specifies that each bolt should be within a *tolerance* of 0.04 mm to 10 mm in diameter. So the lowest diameter that the bolt could be to make it through quality assurance is 0.04 mm smaller than 10 mm, which is 9.96 mm. Similarly, the largest diameter that the bolt could be is 0.04 mm larger than 10 mm, which is 10.04 mm.

Summarizing, we want the difference between the actual diameter and the specification to be less than or equal to 0.04 mm. Since absolute values are used to describe distances, we can summarize our thoughts mathematically as $|x - 10| \leq 0.04$, where x represents the diameter of an acceptably sized bolt, in millimeters. Since the minimum value is 9.96 mm and the maximum value is 10.04 mm, our range of acceptable values should be $9.96 \leq x \leq 10.04$.

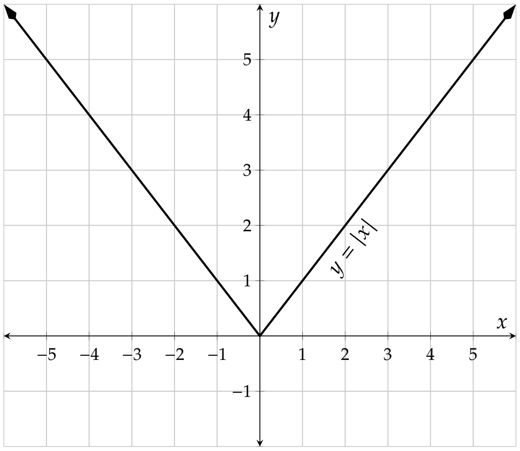
In this section we will examine a variety of problems and applications that relate to this sort of math with absolute values.

11.3.1 Solving Absolute Value Equations

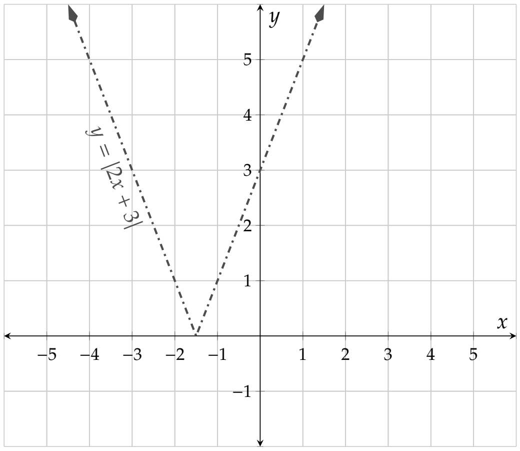
Recall in Section 11.1 that we learned that graphs of absolute value function are in general shaped like "V"s. We can now solve some absolute value equations graphically.

Example 11.3.2 Solve the equations graphically using the graphs provided.

a. $|x| = 3$

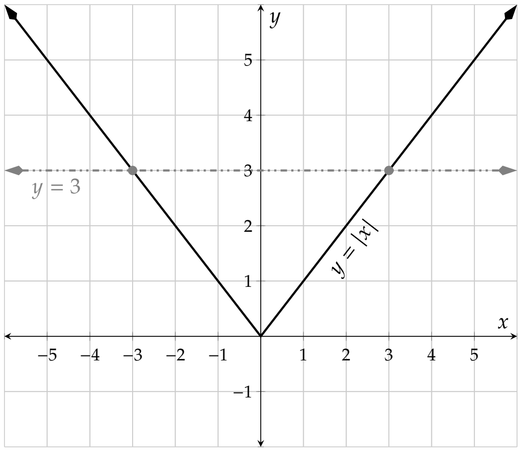


b. $|2x + 3| = 5$



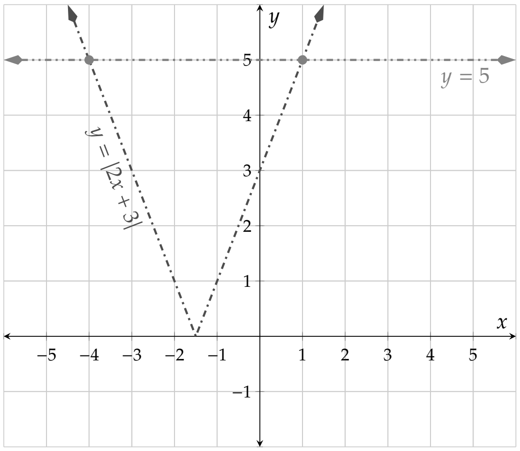
Explanation. To solve the equations graphically, first we need to graph the right sides of the equations also.

a. $|x| = 3$



Since the graph of $y = |x|$ crosses $y = 3$ at the x-values -3 and 3, the solution set to the equation $|x| = 3$ must be $\{-3, 3\}$.

b. $|2x + 3| = 5$



Since the graph of $y = |2x + 3|$ crosses $y = 5$ at the x-values -4 and 1, the solution set to the equation $|2x + 3| = 5$ must be $\{-4, 1\}$.

Remark 11.3.3. At this point, please note that there is a big difference between the expression $|3|$ and the equation $|x| = 3$.

1. The expression $|3|$ is describing the distance from 0 to the number 3. The distance is just 3. So $|3| = 3$.
2. The equation $|x| = 3$ is asking you to find the numbers that are a distance of 3 from 0. We saw in Explanation 11.3.2.1 that these two numbers are 3 and -3.

Example 11.3.4

a. Verify that the value 4 is a solution to the absolute value equation $|2x - 3| = 5$.

b. Verify that the value $\frac{3}{2}$ is a solution to the absolute value equation $\left|\frac{1}{6}x - \frac{1}{2}\right| = \frac{1}{4}$.

Explanation.

a. We will substitute the value 4 into the absolute value equation $|2x - 3| = 5$. We get:

$$\begin{aligned} |2x - 3| &= 5 \\ |2 \cdot 4 - 3| &\overset{?}{=} 5 \\ |8 - 3| &\overset{?}{=} 5 \\ |5| &\overset{\checkmark}{=} 5 \end{aligned}$$

b. We will substitute the value $\frac{3}{2}$ into the absolute value equation $\left|\frac{1}{6}x - \frac{1}{2}\right| = \frac{1}{4}$. We get:

$$\begin{aligned} \left|\frac{1}{6}x - \frac{1}{2}\right| &= \frac{1}{4} \\ \left|\frac{1}{6} \cdot \frac{3}{2} - \frac{1}{2}\right| &\overset{?}{=} \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \left|\frac{1}{4}-\frac{1}{2}\right| &\stackrel{?}{=} \frac{1}{4} \\ \left|-\frac{1}{4}\right| &\stackrel{\checkmark}{=} \frac{1}{4} \end{aligned}$$

Now we will learn to solve absolute value equations algebraically. To motivate this, we will think about what an absolute value equation means in terms of the "distance from zero" definition of absolute value. If

$$|X| = n,$$

where $n \geq 0$, then this means that we want all of the numbers, X, that are a distance n from 0. Since we can only go left or right along the number line, this is describing both $X = n$ as well as $X = -n$.

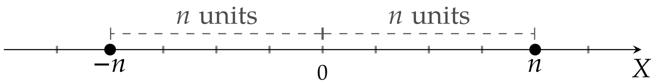


Figure 11.3.5: A Numberline with Points a Distance n from 0

Let's summarize this with a fact.

Fact 11.3.6 Equations with an Absolute Value Expression. *Let n be a non-negative number and X be an algebraic expression. Then the equation*

$$|X| = n$$

has the same solutions as

$$X = n \text{ or } X = -n.$$

Example 11.3.7 Solve the absolute value equations using Fact 11.3.6. Write solutions in a solution set.

a. $|x| = 6$

b. $|x| = -4$

c. $|5x - 7| = 23$

d. $|14 - 3x| = 8$

e. $|3 - 4x| = 0$

Explanation.

a. Fact 11.3.6 says that the equation $|x| = 6$ is the same as

$$x = 6 \text{ or } x = -6.$$

Thus the solution set is $\{6, -6\}$.

b. Fact 11.3.6 doesn't actually apply to the equation $|x| = -4$ because the value on the right side is *negative*. How often is an absolute value of a number negative? Never! Thus, there are no solutions and the solution set is the empty set, denoted $\emptyset$.

c. The equation $|5x - 7| = 23$ breaks into two pieces, each of which needs to be solved independently.

$$\begin{aligned} 5x - 7 &= 23 & &\text{or} & 5x - 7 &= -23 \\ 5x &= 30 & &\text{or} & 5x &= -16 \\ x &= 6 & &\text{or} & x &= -\frac{16}{5} \end{aligned}$$

Thus the solution set is $\left\{6, -\frac{16}{5}\right\}$.

d. The equation $|14 - 3x| = 8$ breaks into two pieces, each of which needs to be solved independently.

$$\begin{aligned} 14 - 3x &= 8 & \text{or} \quad && 14 - 3x &= -8 \\ -3x &= -6 & \text{or} \quad && -3x &= -22 \\ x &= 2 & \text{or} \quad && x &= \frac{22}{3} \end{aligned}$$

Thus the solution set is $\left\{2, \frac{22}{3}\right\}$.

e. The equation $|3 - 4x| = 0$ breaks into two pieces, each of which needs to be solved independently.

$$3 - 4x = 0 \qquad \text{or} \qquad 3 - 4x = -0$$

Since these are identical equations, all we have to do is solve one equation.

$$\begin{aligned} 3 - 4x &= 0 \\ -4x &= -3 \\ x &= \frac{3}{4} \end{aligned}$$

Thus, the equation $|3 - 4x| = 0$ only has one solution, and the solution set is $\left\{\frac{3}{4}\right\}$.

Now we will look at an equation with an absolute value expression on each side, such as $|x| = |2x + 6|$. Since $|x| = 5$ has two solutions, you might be wondering how many solutions $|x| = |2x + 6|$ will have. Let's look at a graph to find out.

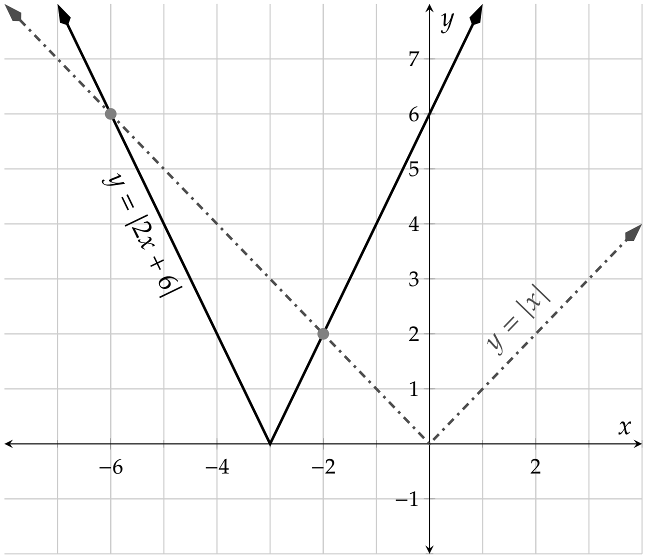


Figure 11.3.8: $y = |x|$ and $y = |2x + 6|$

Figure 11.3.8 shows that there are also two points of intersection between the graphs of $y = |x|$ and $y = |2x + 6|$. The solutions to the equation $|x| = |2x + 6|$ are the x-values where the graphs cross. So, the solution set is $\{-6, -2\}$.

Example 11.3.9 Solve the equation $|x+1| = |2x-4|$ graphically.

Explanation.

First break up the equation into the left side and the right side and graph each separately, as in $y = |x+1|$ and $y = |2x-4|$. We can see in the graph that the graphs intersect twice. The x-values of those intersections are 1 and 5 so the solution set to the equation $|x+1| = |2x-4|$ is $\{1, 5\}$.

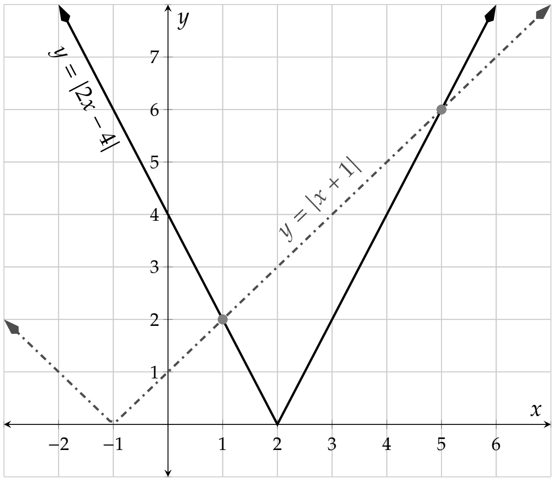


Figure 11.3.10: $y = |x+1|$ and $y = |2x-4|$

Fact 11.3.11 Equations with Two Absolute Value Expressions. *Let X and Y be linear algebraic expressions. Then, the equation*

$$|X| = |Y|$$

has the same solutions as

$$X = Y \text{ or } X = -Y.$$

Remark 11.3.12. You might be confused as to why the negative sign *has* to go on the right side of the equation in $X = -Y$. Well, it doesn't: it can go on either side of the equation. The equations $X = -Y$ and $-X = Y$ are equivalent. Similarly, $-X = -Y$ is equivalent to $X = Y$. That's why we only need to solve two of the four possible equations.

Example 11.3.13 Solve the equations using Fact 11.3.11.

a. $|x-4| = |3x-2|$

b. $\left|\frac{1}{2}x+1\right| = \left|\frac{1}{3}x+2\right|$

c. $|x-2| = |x+1|$

d. $|x-1| = |1-x|$

Explanation.

a. The equation $|x-4| = |3x-2|$ breaks down into two pieces:

$$\begin{aligned}
x-4 &= 3x-2 & &\text{or} & x-4 &= -(3x-2)\\
x-4 &= 3x-2 & &\text{or} & x-4 &= -3x+2\\
-2 &= 2x & &\text{or} & 4x &= 6\\
\frac{-2}{2} &= \frac{2x}{2} & &\text{or} & \frac{4x}{4} &= \frac{6}{4}\\
-1 &= x & &\text{or} & x &= \frac{3}{2}
\end{aligned}$$

So, the solution set is $\left\{-1, \frac{3}{2}\right\}$.

b. The equation $\left|\frac{1}{2}x+1\right| = \left|\frac{1}{3}x+2\right|$ breaks down into two pieces:

$$\begin{aligned}
\frac{1}{2}x+1 &= \frac{1}{3}x+2 &\quad\text{or}\quad&& \frac{1}{2}x+1 &= -\left(\frac{1}{3}x+2\right)\\
\frac{1}{2}x+1 &= \frac{1}{3}x+2 &\quad\text{or}\quad&& \frac{1}{2}x+1 &= -\frac{1}{3}x-2\\
6\cdot\left(\frac{1}{2}x+1\right) &= 6\cdot\left(\frac{1}{3}x+2\right) &\quad\text{or}\quad&& 6\cdot\left(\frac{1}{2}x+1\right) &= 6\cdot\left(-\frac{1}{3}x-2\right)\\
3x+6 &= 2x+12 &\quad\text{or}\quad&& 3x+6 &= -2x-12\\
x &= 6 &\quad\text{or}\quad&& 5x &= -18\\
x &= 6 &\quad\text{or}\quad&& x &= -\frac{18}{5}
\end{aligned}$$

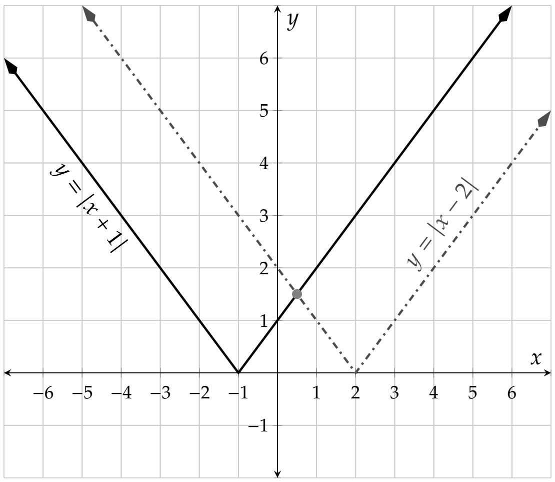
So, the solution set is $\left\{6, -\frac{18}{5}\right\}$.

c. The equation $|x-2| = |x+1|$ breaks down into two pieces:

$$\begin{aligned}
x-2 &= x+1 &\quad\text{or}\quad&& x-2 &= -(x+1)\\
x-2 &= x+1 &\quad\text{or}\quad&& x-2 &= -x-1\\
x &= x+3 &\quad\text{or}\quad&& 2x &= 1\\
0 &= 3 &\quad\text{or}\quad&& x &= \frac{1}{2}
\end{aligned}$$

Note that one of the two pieces gives us an equation with no solutions. Since $0 \neq 3$, we can safely ignore this piece. Thus the only solution is $\frac{1}{2}$.

We should visualize this equation graphically because our previous assumption was that two absolute value graphs would cross twice. The graph shows why there is only one crossing: the left and right sides of each "V" are parallel.



d. The equation $|x-1| = |1-x|$ breaks down into two pieces:

$$\begin{aligned} x-1 &= 1-x & \text{or} \quad && x-1 &= -(1-x) \\ x-1 &= 1-x & \text{or} \quad && x-1 &= -1+x \\ 2x &= 2 & \text{or} \quad && x &= 0+x \\ x &= 1 & \text{or} \quad && 0 &= 0 \end{aligned}$$

Note that our second equation is an identity so recall from Section 3.6 that the solution set is "all real numbers."

So, our two pieces have solutions 1 and "all real numbers." Since 1 *is* a real number and we have an *or* statement, our overall solution set is $(-\infty, \infty)$. The graph confirms our answer since the two "V" graphs are coinciding.

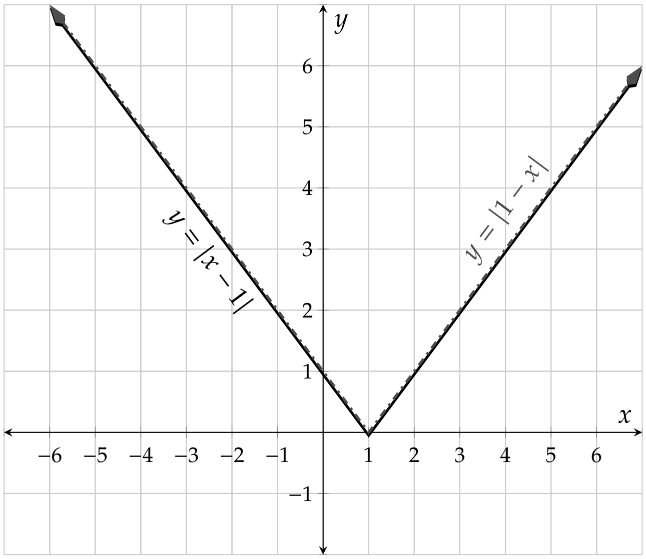


Figure 11.3.14: $y = |x-1|$ and $y = |1-x|$

11.3.2 Solving Absolute Value Inequalities

Now we turn our attention away from equations and onto absolute value inequalities. Don't dismiss this topic as it will actually be used in some capacity in many subsequent math courses. So let's give these the full treatment. We start with a graphical interpretation of what $|2x-1| \leq 5$ means.

Graphically solving the inequality $|2x-1| \leq 5$ means looking for the x-values where the graph of $y = |2x-1|$ is below (or touching) the line $y = 5$. On the graph the highlighted region of $y = |2x-1|$ is the portion that is below the line $y = 5$, and the x-values in that region are $[-2, 5]$.

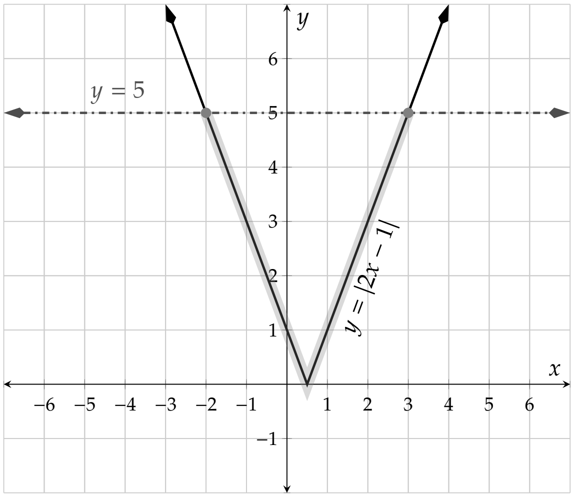


Figure 11.3.15: $y = |2x-1|$ and $y = 5$

Example 11.3.16 Solve the inequality $\left|\frac{2}{3}x+1\right| < 3$ graphically.

Explanation. To solve the inequality $\left|\frac{2}{3}x+1\right| < 3$, we will start by making a graph with both $y = \left|\frac{2}{3}x+1\right|$ and $y = 3$.

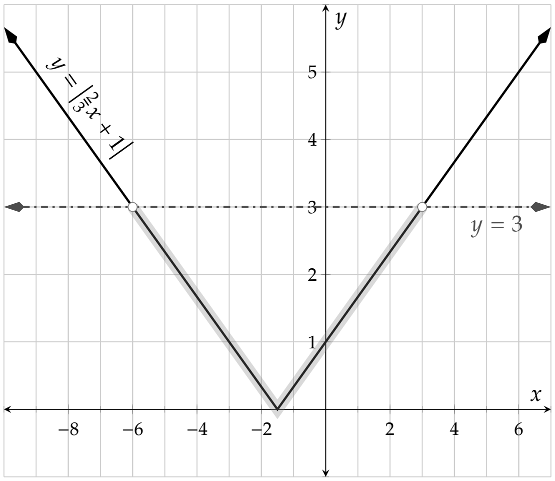


Figure 11.3.17: $y = \left|\frac{2}{3}x+1\right|$ and $y = 3$

The portion of the graph of $y = \left|\frac{2}{3}x+1\right|$ that is below $y = 3$ is highlighted and the x-values of that highlighted region are trapped between -6 and 3: $-6 < x < 3$. That means that the solution set is $(-6, 3)$. Note that we shouldn't include the endpoints of the interval because at those values, the two graphs are *equal* whereas the original inequality was only *less than* and not equal.

For a more verbal approach to understanding the concept, let's try to describe "values that are less than 4 units from 0." We would say that those are "numbers between -4 and 4." Let's translate each sentence into math. "Values that are less than 4 units from 0" translates to "$|x| < 4$," and the piece "numbers between -4 and 4" translates to be "$-4 < x < 4$."

For a graphical interpretation, let's think in terms of the "distance from zero" definition of absolute value. If

$$|X| \leq n,$$

where $n \geq 0$, then we want all of the numbers, X, that are a distance n or less from 0. Since we can only go left or right along the number line, this is describing all numbers from $-n$ to n.

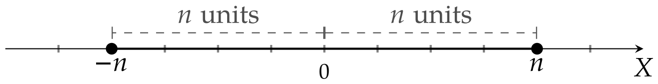


Figure 11.3.18: A Numberline with Points a Distance n or less from 0

Fact 11.3.19 An Absolute Value Expression Less Than a Value. *Let n be a non-negative number and X be a linear algebraic expression.*

Then, the inequality $|X| < n$ has the same solutions as the compound inequality $-n < X < n$.

Likewise, the inequality $|X| \leq n$ has the same solutions as the compound inequality $-n \leq X \leq n$.

Example 11.3.20 Solve the absolute value inequalities using Fact 11.3.19.

a. $|x| \leq 9$
b. $|x| < -6$
c. $|4x+3| < 9$
d. $3 \cdot |3-x| + 1 \leq 13$

Explanation.

a. The inequality $|x| < 9$ breaks down into a triple inequality:

$$-9 \leq x \leq 9$$

This inequality is already written in simplest form and all that remains for us to do is to write the solution set in interval notation: $[-9, 9]$.

b. Fact 11.3.19 doesn't apply to the inequality $|x| < -6$ because the right side is a negative number. Let's translate the meaning of the inequality into English. It says, "The distance from 0 to what numbers is less than -6?" Since we define distance to be non-negative, there are no possible numbers that are less than -6 units distance from 0. Thus, the solution set is the empty set, denoted $\emptyset$.

c. The inequality $|4x+3| < 9$ breaks down into a triple inequality that we can then solve:

$$\begin{gathered} -9 < 4x+3 < 9 \\ -9-3 < 4x+3-3 < 9-3 \\ -12 < 4x < 6 \\ \frac{-12}{4} < \frac{4x}{4} < \frac{6}{4} \\ -3 < x < \frac{3}{2} \end{gathered}$$

So, the solution set to the inequality is $\left(-3, \frac{3}{2}\right)$.

d. The inequality $3 \cdot |3-x| + 1 \leq 13$ *must* be simplified into the form that matches Fact 11.3.19, so we will first isolate the absolute value expression on the left side of the inequality:

$$\begin{aligned} 3 \cdot |3-x| + 1 &\leq 13 \\ 3 \cdot |3-x| &\leq 12 \end{aligned}$$

$$|3-x| \leq 4$$

Now that we have the absolute value isolated, we can split it into a triple inequality that we can finish solving:

$$\begin{aligned}
-4 &\leq 3-x \leq 4\\
-4-3 &\leq 3-x-3 \leq 4-3\\
-7 &\leq -x \leq 1\\
\frac{-7}{-1} &\geq \frac{-x}{-1} \geq \frac{1}{-1}\\
7 &\geq x \geq -1
\end{aligned}$$

So, the solution set to the inequality is $[-1,7]$.

Example 11.3.21 If a machined circular washer must have a circumference that is within 0.2 mm of 36 mm, then what is the acceptable range for the radius of the washer? Round your answers to the nearest hundredth of a millimeter.

Explanation. We will first define the radius of the washer to be r, measured in millimeters. The formula $C = 2\pi r$ gives us the circumference, C, of a circle with radius r. Now we know that "distance" between the circumference and our preferred circumference of 36 mm must be less than or equal to 0.2 mm. In math, this translates to

$$|C-36| \leq 0.2$$

Now we can substitute our formula for circumference and solve for r.

$$\begin{aligned}
|C-36| &\leq 0.2\\
|2\pi r-36| &\leq 0.2
\end{aligned}$$

To solve this we will use Fact 11.3.19 to break the absolute value inequality into a triple inequality:

$$\begin{aligned}
-0.2 &\leq 2\pi r-36 \leq 0.2\\
-0.2+36 &\leq 2\pi r-36+36 \leq 0.2+36\\
35.8 &\leq 2\pi r \leq 36.2\\
\frac{35.8}{2\pi} &\leq \frac{2\pi r}{2\pi} \leq \frac{36.2}{2\pi}\\
5.70 &\leq r \leq 5.76 && \text{(note: these values are rounded)}
\end{aligned}$$

This shows that the radius must be somewhere between 5.70 mm and 5.76 mm, inclusive.

The last few examples have all revolved around absolute values being *less than* some value. We now need to investigate what happens when we have an absolute value that is *greater than* a value. We will again start with a graphical interpretation.

Example 11.3.22 To graphically solve the inequality $|x-1| > 3$ would mean looking for the x-values where the graph of $y = |x-1|$ is *above* the line $y = 3$.

On the graph the highlighted region of $y = |x - 1|$ is the portion that is above the line $y = 3$ and the x-values in that region can be represented by $(-\infty, -2) \cup (4, \infty)$.

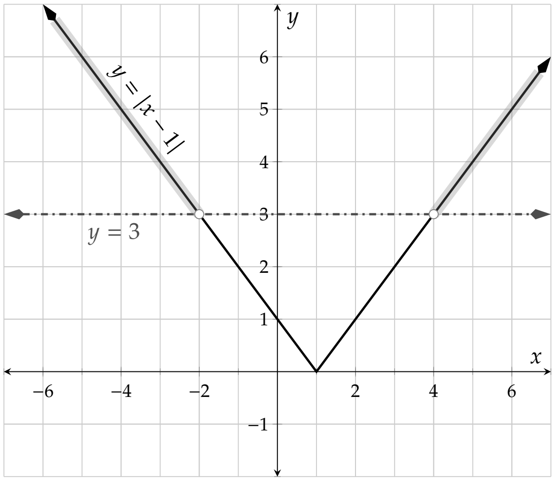


Figure 11.3.23: $y = |x - 1|$ and $y = 3$

Example 11.3.24 Solve the inequality $\left|\frac{1}{3}x + 2\right| \geq 6$ graphically.

Explanation. To solve the inequality $\left|\frac{1}{3}x + 2\right| \geq 6$, we will start by making a graph with both $y = \left|\frac{1}{3}x + 2\right|$ and $y = 6$.

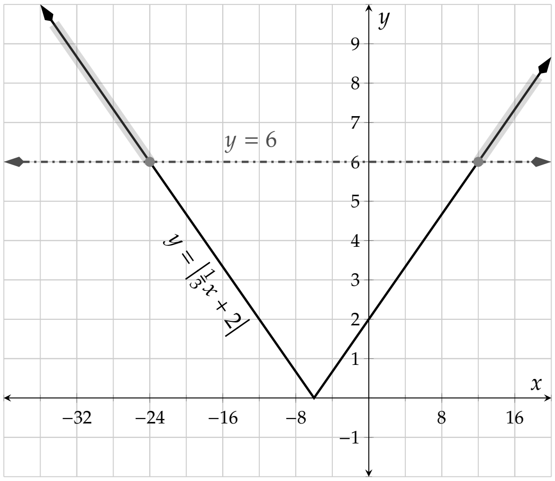


Figure 11.3.25: $y = \left|\frac{1}{3}x + 2\right|$ and $y = 3$

The portion of the graph of $y = \left|\frac{1}{3}x + 2\right|$ that is above $y = 6$ is highlighted and the x-values of that highlighted region are those below (or equal to) -24 and those above (or equal to) 12: $x \leq -24$ or $x \geq 12$. That means that the solution set is $(-\infty, -24) \cup (12, \infty)$.

Again, for a more verbal approach to understanding the concept, lets try to describe “values that are more than 4 units from 0.” We would say that those are “numbers below −4 as well as numbers above 4.” We will again translate each sentence into math. “Values that are more than 4 units from 0” translates to “$|x| > 4$,” and the piece “numbers below −4 as well as numbers above 4” translates to be “$x < -4$ or $x > 4$.”

For a graphical interpretation, let’s think in terms of the “distance from zero” definition of absolute value. If

$$|X| \geq n,$$

where $n \geq 0$, then we want all of the numbers, X, that are a distance n or more from 0. Since we can only go left or right along the number line, this is describing all numbers below $-n$ as well as those above n.

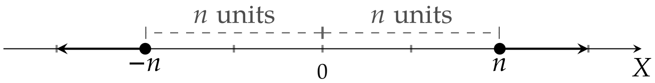


Figure 11.3.26: A Numberline with Points a Distance n or less from 0

Fact 11.3.27 An Absolute Value Expression Greater Than a Value. *Let n be a non-negative number and X be a linear algebraic expression.*

Then, the inequality $|X| > n$ has the same solutions as the compound inequality $X < -n$ or $X > n$.

Likewise, the inequality $|X| \geq n$ has the same solutions as the compound inequality $X \leq -n$ or $X \geq n$.

Remark 11.3.28. Since Fact 11.3.27 specifies that an "absolute value greater than a number"-type inequality breaks down into an *or* statement, we will therefore need to find the *union* of the solution sets of the pieces.

Example 11.3.29 Solve the absolute value inequalities using Fact 11.3.27.

a. $|x| \geq 4$

b. $|x| > -2$

c. $|5x - 7| > 7$

d. $2 \cdot |3 - 2x| - 5 \geq 13$

Explanation.

a. The inequality $|x| \geq 4$ breaks down into a compound inequality:

$$x \leq -4 \qquad \text{or} \qquad x \geq 4$$

So, the solution set is $(-\infty, -4] \cup [4, \infty)$.

b. Fact 11.3.27 doesn't apply to the inequality $|x| > -2$ because the right side is negative. Instead, we will make sense of it logically. This is asking, "When is an absolute value greater than a negative number?" The answer is that absolute values are *always* bigger than negative numbers! So, our solution set is $(-\infty, \infty)$.

c. The inequality $|5x - 7| > 7$ breaks down into a compound inequality:

$$\begin{aligned} 5x - 7 &< -7 &\qquad \text{or} \qquad&& 5x - 7 &> 7 \\ 5x &< 0 &\qquad \text{or} \qquad&& 5x &> 14 \\ x &< 0 &\qquad \text{or} \qquad&& x &> \frac{14}{5} \end{aligned}$$

We will write the solution set as $(-\infty, 0) \cup \left(\frac{14}{5}, \infty\right)$.

d. Before we break up the inequality $2 \cdot |3 - 2x| - 5 \geq 13$ into an "or" statement, we *must* isolate the absolute value expression:

$$\begin{aligned} 2 \cdot |3 - 2x| - 5 &\geq 13 \\ 2 \cdot |3 - 2x| &\geq 18 \\ |3 - 2x| &\geq 9 \end{aligned}$$

Now that the absolute value expression has been isolated on the left side, we can use Fact 11.3.27 to break it into an "or" statement:

$$\begin{array}{rcr} 3-2x \leq -9 & \text{or} & 3-2x \geq 9 \\ -2x \leq -12 & \text{or} & -2x \geq 6 \\ x \geq 6 & \text{or} & x \leq -3 \end{array}$$

Our final simplified solution set is $(-\infty, 3] \cup [6, \infty)$.

Example 11.3.30 Phuong is taking the standard climbing route on Mount Hood from Timberline Lodge up the Southside Hogsback and back down. Her altitude can be very closely modeled by an absolute value function since the angle of ascent is nearly constant. Let x represent the number of miles walked from Timberline Lodge, and let $f(x)$ represent the altitude, in miles, after walking for a distance x. The altitude can be modeled by $f(x) = 2.1 - 0.3077 \cdot |x - 3.25|$. Note that below Timberline Lodge this model fails to be accurate.

a. Solve the equation $f(x) = 1.1$ and interpret the results in the context of the problem.

b. Altitude sickness can occur at altitudes above 1.5 miles. Set up and solve an inequality to find out how far Phuong can walk the trail and still be under 1.5 miles of elevation.

Explanation.

a. First, we substitute the formula for $f(x)$ and simplify the equation.

$$\begin{aligned} f(x) &= 1.1 \\ 2.1 - 0.3077 \cdot |x-3.25| &= 1.1 \\ -0.3077 \cdot |x-3.25| &= -1 \\ \frac{-0.3077 \cdot |x-3.25|}{-0.3077} &= \frac{-1}{-0.3077} \\ |x-3.25| &\approx 3.25 \end{aligned}$$

At this point, we can use Fact 11.3.6 to split apart the equation:

$$\begin{array}{rcr} x-3.25 \approx 3.25 & \text{or} & x-3.25 \approx -3.25 \\ x \approx 6.5 & \text{or} & x \approx 0 \end{array}$$

According to the model, Phuong will be at 1.1 miles of elevation after walking about 0 miles and about 6.5 miles along the trail. This seems to imply that Timberline Lodge is very close to 1.1 miles of elevation. In addition, it implies that the entire hike is 6.5 miles round trip, ending at Timberline Lodge again.

b. The inequality we are looking for will describe when the altitude is below 1.5 miles. Since $f(x)$ is the altitude, the inequality we need is:

$$f(x) < 1.5$$

To solve this, we need to input the formula and simplify before using one of the absolute value inequality rules.

$$\begin{aligned} f(x) &< 1.5 \\ 2.1 - 0.3077 \cdot |x-3.25| &< 1.5 \\ -0.3077 \cdot |x-3.25| &< -0.6 \end{aligned}$$

$$\frac{-0.3077 \cdot |x-3.25|}{-0.3077} > \frac{-0.6}{-0.3077}$$
$$|x-3.25| > 1.95 \qquad \text{(note: this value is rounded)}$$

At this point, we can use Fact 11.3.27 to split apart the inequality:

$$\begin{aligned} x-3.25 &< -1.95 &\quad \text{or} \quad & x-3.25 > 1.95 \\ x &< 1.3 &\quad \text{or} \quad & x > 5.2 \end{aligned}$$

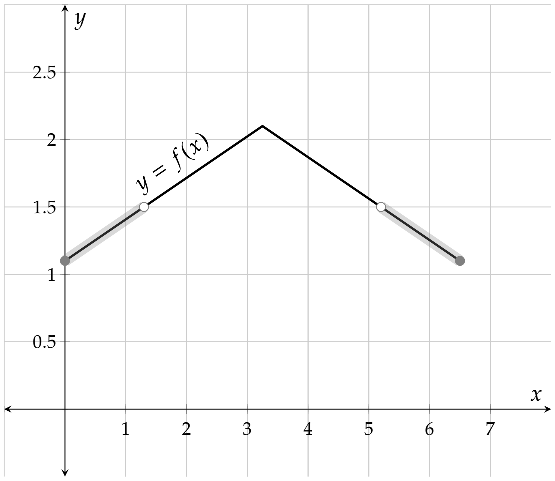


Figure 11.3.31: $y = f(x)$, the Graph of the Mt Hood Ascent and Descent

The image only shows the portion of the graph that is above Timberline Lodge, which we learned was at 1.1 miles in elevation in the previous part. The highlighted portions of the graph are those indicated by $x > 5.2$ or $x < 1.3$.

In conclusion, based both on our math and the reality of the situation, regions of the trail that are below 1.5 miles are those that are from Timberline Lodge (at 0 miles on the trail), to 1.3 miles along the trail and then also from 5.2 miles along the trail (and by now we are on our way back down) to 6.5 miles along the trail (back at Timberline Lodge). If we wanted to write this in interval notation, we might write $[0, 1.3) \cup (5.2, 6.5]$. There is a big portion along the trail (from 1.3 miles to 5.2 miles) that Phuong will be above the 1.5 mile altitude and should watch for signs of altitude sickness.

Exercises

Review and Warmup

1. Solve the equation.

$$\frac{c}{5} - 6 = \frac{c}{7}$$

2. Solve the equation.

$$\frac{A}{3} - 10 = \frac{A}{8}$$

3. Solve the equation.

$$-30 = -10(C+10)$$

4. Solve the equation.

$$-98 = -7(m+5)$$

5. Solve the equation.

$$4p + 9 = 7p + 10$$

6. Solve the equation.

$$10x + 4 = 5x + 10$$

7. Solve this inequality.

$17 \geq 3x - 4$

In set-builder notation, the solution set is $\boxed{}$.

In interval notation, the solution set is $\boxed{}$.

8. Solve this inequality.

$6 \geq 4x - 2$

In set-builder notation, the solution set is $\boxed{}$.

In interval notation, the solution set is $\boxed{}$.

9. Solve this inequality.

$-5x - 9 < -39$

In set-builder notation, the solution set is $\boxed{}$.

In interval notation, the solution set is $\boxed{}$.

10. Solve this inequality.

$-6x - 6 < -66$

In set-builder notation, the solution set is $\boxed{}$.

In interval notation, the solution set is $\boxed{}$.

11. Solve this inequality.

$-3 > 3 - x$

In set-builder notation, the solution set is $\boxed{}$.

In interval notation, the solution set is $\boxed{}$.

12. Solve this inequality.

$-6 > 4 - x$

In set-builder notation, the solution set is $\boxed{}$.

In interval notation, the solution set is $\boxed{}$.

Solving Absolute Value Equations Algebraically

13. a. Write the equation $6 = |3x| - 5$ as two separate equations. Neither of your equations should use absolute value.

$\boxed{}$

$\boxed{}$

b. Solve both equations above.

14. a. Write the equation $7 = |5x| - 3$ as two separate equations. Neither of your equations should use absolute value.

$\boxed{}$

$\boxed{}$

b. Solve both equations above.

15. a. Write the equation $\left|8 - \frac{r}{7}\right| = 3$ as two separate equations. Neither of your equations should use absolute value.

$\boxed{}$

$\boxed{}$

b. Solve both equations above.

16. a. Write the equation $\left|2 - \frac{r}{5}\right| = 7$ as two separate equations. Neither of your equations should use absolute value.

$\boxed{}$

$\boxed{}$

b. Solve both equations above.

17. (a) Verify that the value -1 is a solution to the absolute value equation $\left|\frac{x-3}{2}\right| = 2$.

(b) Verify that the value $\frac{2}{3}$ is a solution to the absolute value equation $|6x - 5| < 4$.

18. (a) Verify that the value 8 is a solution to the absolute value equation $\left|\frac{1}{2}x - 2\right| = 2$.

(b) Verify that the value 6 is a solution to the absolute value equation $|7 - 2x| \geq 5$.

19. Solve the following equation.

$|3x - 9| = 9$

20. Solve the following equation.

$|4x + 5| = 3$

21. Solve the equation $|3x - 1| = 17$.

22. Solve the equation $|4x - 4| = 10$.

23. Solve: $|x| = 9$

24. Solve: $|x| = 5$

25. Solve: $|y - 1| = 11$

26. Solve: $|y - 5| = 15$

27. Solve: $|2a + 3| = 9$

28. Solve: $|2b + 7| = 13$

29. Solve: $\left|\dfrac{2b - 5}{9}\right| = 3$

30. Solve: $\left|\dfrac{2t - 3}{5}\right| = 1$

31. Solve: $|t| = -4$

32. Solve: $|x| = -6$

33. Solve: $|x + 2| = 0$

34. Solve: $|y + 4| = 0$

35. Solve: $|4 - 3y| = 9$

36. Solve: $|2 - 3a| = 14$

37. Solve: $\left|\frac{1}{4}b + 3\right| = 1$

38. Solve: $\left|\frac{1}{2}b + 5\right| = 1$

39. Solve: $|0.2 - 0.1t| = 4$

40. Solve: $|0.8 - 0.4t| = 3$

41. Solve: $|x + 5| - 2 = 2$

42. Solve: $|x + 1| - 4 = 6$

43. Solve: $|4y - 20| + 6 = 6$

44. Solve: $|3y - 6| + 4 = 4$

45. Solve: $|a + 1| + 7 = 6$

46. Solve: $|b + 7| + 7 = 2$

47. Solve: $|4b + 3| + 7 = 4$

48. Solve: $|4t + 1| + 8 = 6$

49. Solve the equation *by inspection* (meaning in your head).

$|6x + 18| = 0$

50. Solve the equation *by inspection* (meaning in your head).

$|6x + 12| = 0$

51. The equation $|x| = |y|$ is satisfied if $x = y$ or $x = -y$. Use this fact to solve the following equation.

$|2x + 4| = |-3x - 1|$

52. The equation $|x| = |y|$ is satisfied if $x = y$ or $x = -y$. Use this fact to solve the following equation.

$|3x + 1| = |x - 3|$

53. The equation $|x| = |y|$ is satisfied if $x = y$ or $x = -y$. Use this fact to solve the following equation.

$|x + 6| = |x - 5|$

54. The equation $|x| = |y|$ is satisfied if $x = y$ or $x = -y$. Use this fact to solve the following equation.

$|x + 2| = |x - 1|$

55. Solve the equation: $|2x - 6| = |9x + 4|$

56. Solve the equation: $|4x - 3| = |5x + 2|$

57. Solve the following equation.

$|3x + 8| = |9x + 6|$

58. Solve the following equation.

$|3x + 1| = |6x - 4|$

Testing Possible Solutions Decide whether the given value for the variable is a solution.

59. a. $|x - 6| \le 2 \qquad x = 5$

The given value (☐ is ☐ is not) a solution.

b. $\left|\frac{2}{3}x - 1\right| \ge 7 \qquad x = 6$

The given value (☐ is ☐ is not) a solution.

c. $|8t - 5| > 6 \qquad t = 7$

The given value (☐ is ☐ is not) a solution.

d. $|3(z - 3)| < 8 \qquad z = \pi$

The given value (☐ is ☐ is not) a solution.

60. a. $|x - 7| \le 8 \qquad x = 9$

The given value (☐ is ☐ is not) a solution.

b. $\left|3x - \frac{3}{8}\right| \ge 2 \qquad x = -4$

The given value (☐ is ☐ is not) a solution.

c. $|4t - 5| > 5 \qquad t = 6$

The given value (☐ is ☐ is not) a solution.

d. $|8(z - 6)| < 6 \qquad z = \pi$

The given value (☐ is ☐ is not) a solution.

Solving Absolute Value Equations Graphically

61. Solve the equations and inequalities graphically. Use interval notation when applicable.

a. $\left|\frac{2}{3}x + 2\right| = 4$ b. $\left|\frac{2}{3}x + 2\right| > 4$ c. $\left|\frac{2}{3}x + 2\right| \le 4$

62. Solve the equations and inequalities graphically. Use interval notation when applicable.

a. $\left|\frac{11-2x}{5}\right| = 4$ b. $\left|\frac{11-2x}{5}\right| > 4$ c. $\left|\frac{11-2x}{5}\right| \le 4$

Solving Absolute Value Inequalities Algebraically Solve the inequality.

63. $\left|\dfrac{7-x}{6}\right| \geq 7$

64. $\left|\dfrac{8-x}{3}\right| \geq 12$

65. $|9-x| \geq 5$

66. $|6-2x| \geq 10$

67. $|3x-2| < 3$

68. $|4x-8| < 8$

69. $\left|\dfrac{x+5}{5}\right| \leq 13$

70. $\left|\dfrac{x+6}{2}\right| \leq 6$

71. $|x-7| > 13$

72. $|x-7| > 10$

73. $|2-8x| < 9$

74. $|7-x| < 15$

75. $20-|3x+1| \leq 6$

76. $11-|4x+7| \leq 1$

Challenge

77. Algebraically, solve for x in the equation:

$$5 = |x-5| + |x-10|$$

11.4 Absolute Value Functions Chapter Review

11.4.1 Introduction to Absolute Value Functions

In Section 11.1 we covered the definition of absolute value, what the graphs of absolute value functions look like, the fact that $\sqrt{x^2} = |x|$, and applications of absolute values.

Example 11.4.1 Evaluating Absolute Value Functions. Given that $h(x) = |9 - 4x|$, evaluate the following expressions.

a. $h(-1)$.

b. $h(4)$.

Explanation.

a. $$\begin{aligned} h(-1) &= |9 - 4(-1)| \\ &= |9 + 4| \\ &= |13| \\ &= 13 \end{aligned}$$

b. $$\begin{aligned} h(4) &= |9 - 4(4)| \\ &= |9 - 16| \\ &= |-7| \\ &= 7 \end{aligned}$$

Example 11.4.2 Graphing Absolute Value Functions. Absolute value functions always make "V" shaped graphs. We usually use technology to make graphs to help speed up the process. Use technology to make a graph of $y = |2x - 6| - 4$.

Explanation. To make a graph of a function, we often use technology to generate a table of values for that function. Then we use the graph that the technology creates to thoughtfully connect the points.

x	$y = \|2x - 6\| - 4$
-1	4
0	2
1	0
2	-2
2	-4
2	-2
2	0

Table 11.4.3: A table of values for $y = |2x - 6| - 4$

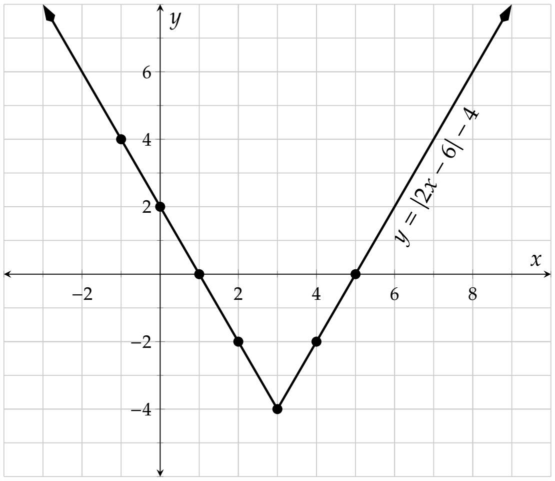


Figure 11.4.4: A graph of $y = |2x - 6| - 4$

Example 11.4.5 The Alternate Definition of Absolute Value: $|x| = \sqrt{x^2}$. Simplify the following expressions using the fact that $|x| = \sqrt{x^2}$.

a. $\sqrt{x^{14}}$

b. $\sqrt{x^2 - 12x + 36}$

Explanation.

a. $$\begin{aligned}\sqrt{x^{14}} &= \sqrt{\left(x^7\right)^2}\\ &= \left|x^7\right|\end{aligned}$$

We know from exponent rules that $\left(x^7\right)^2 = x^{14}$. Note that x^7 will be negative whenever x is a negative number, so the absolute value is meaningful.

b. $$\begin{aligned}\sqrt{x^2 - 12x + 36} &= \sqrt{(x-6)^2}\\ &= |x-6|\end{aligned}$$

Note that $x - 6$ can be negative for certain values of x, so the absolute value is meaningful.

Example 11.4.6 An Application of Absolute Value. Mariam arrived at school one day only to realize that she had left her favorite pencil on her porch at home. She hopped on her bicycle and headed back to get it. Her distance from her home, $d(t)$ in yards, can be modeled as a function of the time, t in seconds, since she left school:

$$d(t) = |5t - 300|$$

Use this function to answer the following questions.

a. Find and interpret the meaning of $d(0)$.

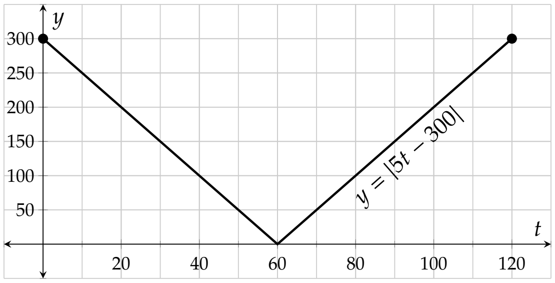
b. Using technology, make a graph of $y = d(t)$.

c. Using your graph, find out how long it took Mariam to get to her home to get her pencil and get back to school.

Explanation.

a. $$\begin{aligned}d(0) &= |5(0) - 300|\\ &= |-300|\\ &= 300\end{aligned}$$

This means that just as Mariam was leaving her school, she was 300 yards from her home.



b.

c. Mariam was back at a y-value of 300 at $t = 120$. We should assume that she is back at her school again here. So it took her 120 seconds, which is 2 minutes.

11.4.2 Compound Inequalities

In Section 11.2 we defined the union of intervals, what compound inequalities are, and how to solve both "or" inequalities and triple inequalities.

Example 11.4.7 Unions of Intervals. Draw a representation of the union of the sets $(-\infty, -1]$ and $(2, \infty)$.

Explanation. First we make a number line with both intervals drawn to understand what both sets mean.

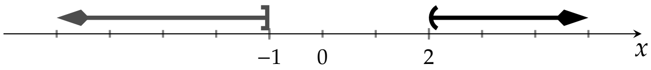


Figure 11.4.8: A number line sketch of $(-\infty, -1]$ as well as $(2, \infty)$

The two intervals should be viewed as a single object when stating the union, so here is the picture of the union. It looks the same, but now it is a graph of a single set.

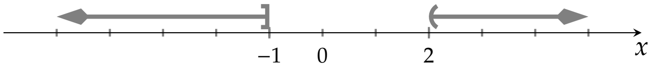


Figure 11.4.9: A number line sketch of $(-\infty, -1] \cup (2, \infty)$

Example 11.4.10 "Or" Compound Inequalities. Solve the compound inequality.

$$5z + 12 \le 7 \text{ or } 3 - 9z < -2$$

Explanation. First we will solve each inequality for z.

$$\begin{aligned} 5z + 12 &\le 7 & &\text{or} & 3 - 9z &< -2 \\ 5z &\le -5 & &\text{or} & -9z &< -5 \\ z &\le -1 & &\text{or} & z &> \frac{5}{9} \end{aligned}$$

The solution set to the compound inequality is:

$$(-\infty, -1] \cup \left(\frac{5}{9}, \infty\right)$$

Example 11.4.11 Three-Part Inequalities. Solve the three-part inequality $-4 \le 20 - 6x < 32$.

Explanation. This is a three-part inequality. The goal is to isolate x in the middle and whatever you do to one "side," you have to do to the other two "sides."

$$-4 \le 20 - 6x < 32$$

$$\begin{aligned} -4-20 \leq 20-6x-20 &< 32-20 \\ -24 \leq -6x &< 12 \\ \frac{-24}{-6} \geq \frac{-6x}{-6} &> \frac{12}{-6} \\ 4 \geq x &> -2 \end{aligned}$$

The solutions to the three-part inequality $4 \geq x > -2$ are those numbers that are trapped between -2 and 4, including 4 but not -2. The solution set in interval notation is $(-2, 4]$.

Example 11.4.12 Solving Compound Inequalities Graphically. Figure 11.4.13 shows a graph of $y = G(x)$. Use the graph do the following.

a. Solve $G(x) < -2$.

b. Solve $G(x) \geq 1$.

c. Solve $-1 \leq G(x) < 1$.

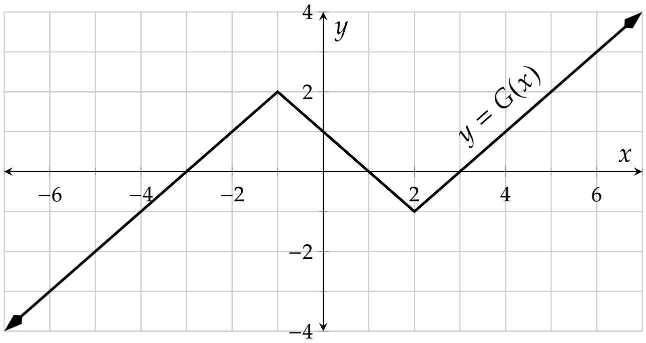


Figure 11.4.13: Graph of $y = G(x)$

Explanation.

a. To solve $G(x) < -2$, we first draw a dotted line (since it's a less-than, not a less-than-or-equal) at $y = -2$. Then we examine the graph to find out where the graph of $y = G(x)$ is underneath the line $y = -2$. Our graph is below the line $y = -2$ for x-values less than -5. So the solution set is $(-\infty, -5)$.

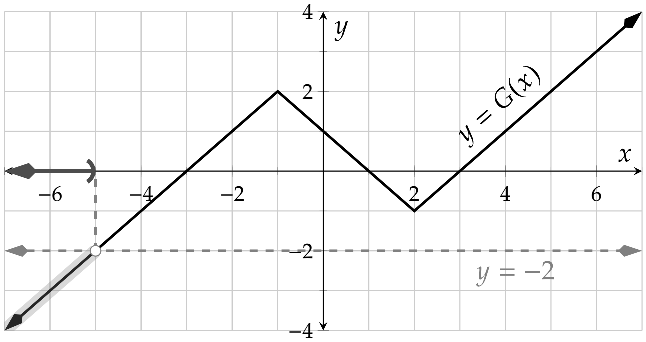


Figure 11.4.14: Graph of $y = G(x)$ and solution set to $G(x) < -2$

b. To solve $G(x) \geq 1$, we first draw a solid line (since it's a greater-than-or-equal) at $y = 1$. Then we examine the graph to find out what parts of the graph of $y = G(x)$ are above the line $y = 1$. Our graph is above (or on) the line $y = 1$ for x-values between -2 and 0 as well as x-values bigger than 4. So the solution set is $[-2, 0] \cup [4, \infty)$.

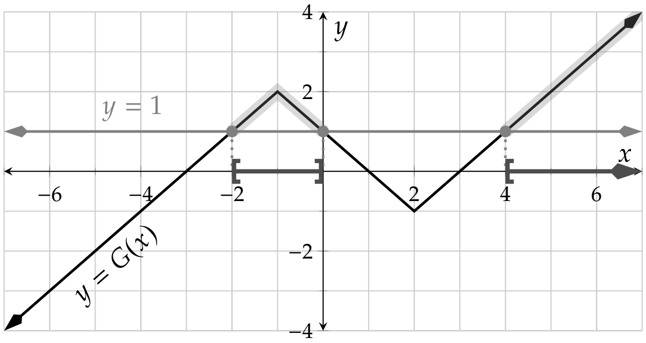


Figure 11.4.15: Graph of $y = G(x)$ and solution set to $G(x) \geq 1$

c. To solve $-1 < G(x) \leq 1$, we first draw a solid line at $y = 1$ and dotted line at $y = -1$. Then we examine the graph to find out what parts of the graph of $y = G(x)$ are trapped between the two lines we just drew. Our graph is between those values for x-values between -4 and -2 as well as x-values between 0 and 2 as well as as well as x-values between 2 and 4. We use the solid and hollow dots on the graph to decide whether or not to include those values. So the solution set is $(-4, -2] \cup [0, 2) \cup (2, 4]$.

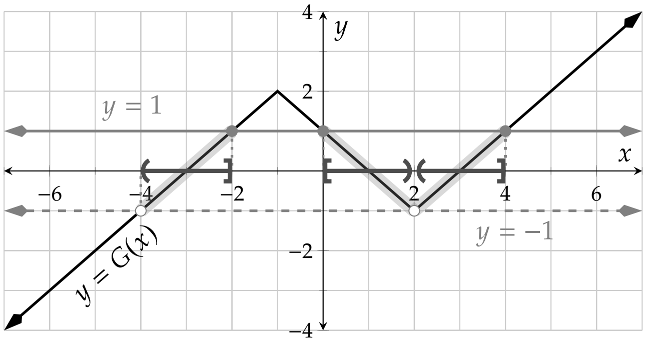


Figure 11.4.16: Graph of $y = G(x)$ and solution set to $-1 < G(x) \leq 1$

Example 11.4.17 Application of Compound Inequalities. Mishel wanted to buy some mulch for their spring garden. Each cubic yard of mulch cost \$27 and delivery for any size load was \$40. If they wanted to spend between \$200 and \$300, set up and solve a compound inequality to solve for the number of cubic yards, x, that they could buy.

Explanation. Since the mulch costs \$27 per cubic yard and delivery is \$40, the formula for the cost of x yards of mulch is $27x + 40$. Since Mishel wants to spend between \$200 and \$300, we just trap their cost between these two values.

$$200 < 27x + 40 < 300$$
$$200 - 40 < 27x + 40 - 40 < 300 - 40$$

$$\begin{aligned} 160 &< 27x < 260 \\ \frac{160}{27} &< \frac{27x}{27} < \frac{260}{27} \\ 5.93 &< x < 9.63 \\ &\quad \text{Note: these values are approximate} \end{aligned}$$

Most companies will only sell whole number cubic yards of mulch, so we have to round appropriately. Since Mishel wants to spend more than \$200, we have to round our lower value from 5.93 up to 6 cubic yards.

If we round the 9.63 up to 10, then the total cost will be $27 \cdot 10 + 40 = 310$ (which represents \$310), which is more than Mishel wanted to spend. So we actually have to round down to 9cubic yards to stay below the \$300 maximum.

In conclusion, Mishel could buy 6, 7, 8, or 9 cubic yards of mulch to stay between \$200 and \$300.

11.4.3 Absolute Value Equations and Inequalities

In Section 11.3 we covered how to solve equations when an absolute value is equal to a number and when an absolute value is equal to an absolute value. We also covered how to solve inequalities when an absolute value is less than a number and when an absolute value is greater than a number.

Example 11.4.18 Solving an Equation with an Absolute Value. Solve the absolute value equation $|9 - 4x| = 17$ using Fact 11.3.6.

Explanation. The equation $|9 - 4x| = 17$ breaks into two pieces, each of which needs to be solved independently.

$$\begin{aligned} 9 - 4x &= 17 &\quad \text{or} \quad&& 9 - 4x &= -17 \\ -4x &= 8 &\quad \text{or} \quad&& -4x &= -26 \\ \frac{-4x}{-4} &= \frac{8}{-4} &\quad \text{or} \quad&& \frac{-4x}{-4} &= \frac{-26}{-4} \\ x &= -2 &\quad \text{or} \quad&& x &= \frac{13}{2} \end{aligned}$$

The solution set is $\left\{-2, \frac{13}{2}\right\}$.

Example 11.4.19 Solving an Equation with Two Absolute Values. Solve the absolute value equation $|7 - 3x| = |6x - 5|$ using Fact 11.3.11.

Explanation. The equation $|7 - 3x| = |6x - 5|$ breaks into two pieces, each of which needs to be solved independently.

$$\begin{aligned} 7 - 3x &= 6x - 5 &\quad \text{or} \quad&& 7 - 3x &= -(6x - 5) \\ 7 - 3x &= 6x - 5 &\quad \text{or} \quad&& 7 - 3x &= -6x + 5 \\ 12 - 3x &= 6x &\quad \text{or} \quad&& 2 - 3x &= -6x \\ 12 &= 9x &\quad \text{or} \quad&& 2 &= -3x \\ \frac{12}{9} &= \frac{9x}{9} &\quad \text{or} \quad&& \frac{2}{-3} &= \frac{-3x}{-3} \end{aligned}$$

$$\frac{4}{3} = x \qquad \text{or} \qquad -\frac{2}{3} = x$$

The solution set is $\left\{\frac{4}{3}, -\frac{2}{3}\right\}$.

Example 11.4.20 Solving an Absolute Value Less-Than Inequality. Solve the absolute value inequality $4 \cdot |7 - 2x| + 1 < 25$ using Fact 11.3.19.

Explanation. The inequality $4 \cdot |7 - 2x| + 1 < 25$ *must* be simplified into the form that matches Fact 11.3.19, so we will first isolate the absolute value expression on the left side of the equation:

$$\begin{aligned} 4 \cdot |7 - 2x| + 1 &< 25 \\ 4 \cdot |7 - 2x| &< 24 \\ |7 - 2x| &< 6 \end{aligned}$$

Now that we have the absolute value isolated, we can us Fact 11.3.19 to split it into a triple inequality that we can finish solving:

$$\begin{gathered} -6 < 7 - 2x < 6 \\ -6 - 7 < 7 - 2x - 7 < 6 - 7 \\ -13 < -2x < -1 \\ \frac{-13}{-2} > \frac{-2x}{-2} > \frac{-1}{-2} \\ \frac{13}{2} > x > \frac{1}{2} \end{gathered}$$

So, the solution set to the inequality is $\left(\frac{1}{2}, \frac{13}{2}\right)$.

Example 11.4.21 Solving an Absolute Value Greater-Than Inequality. To solve the absolute value inequality $\left|13 - \frac{3}{2}x\right| \geq 15$ using Fact 11.3.27.

Explanation. Using Fact 11.3.27, the inequality $\left|13 - \frac{3}{2}x\right| \geq 15$ breaks down into a compound inequality:

$$\begin{aligned} 13 - \frac{3}{2}x &\leq -15 & &\text{or} & 13 - \frac{3}{2}x &\geq 15 \\ -\frac{3}{2}x &\leq -28 & &\text{or} & -\frac{3}{2}x &\geq 2 \\ -\frac{2}{3} \cdot \left(-\frac{3}{2}x\right) &\geq -\frac{2}{3} \cdot (-28) & &\text{or} & -\frac{2}{3} \cdot \left(-\frac{3}{2}x\right) &\leq -\frac{2}{3} \cdot (2) \\ x &\geq \frac{56}{3} & &\text{or} & x &\leq -\frac{4}{3} \end{aligned}$$

We will write the solution set as $\left(-\infty, -\frac{4}{3}\right] \cup \left[\frac{56}{3}, \infty\right)$.

Exercises

Introduction to Absolute Value Functions

1. Evaluate the following.

$3 - 5\,|3 - 7| + 1 =$ ☐

2. Evaluate the following.

$4 - 9\,|1 - 3| + 1 =$ ☐

3. Given $f(x) = 20 - |-x + 5|$, find and simplify $f(18)$.

$f(18) =$ ☐

4. Given $f(r) = 17 - |3r - 16|$, find and simplify $f(19)$.

$f(19) =$ ☐

5. Find the domain of K where $K(x) = |8x - 5|$.

6. Find the domain of f where $f(x) = |x + 4|$.

7. Make a table of values for the function g defined by $g(x) = |2x - 3|$.

x	$g(x)$

8. Make a table of values for the function h defined by $h(x) = |-2x + 1|$.

x	$h(x)$

9. Graph $y = f(x)$, where $f(x) = \frac{1}{2}\,|4x - 5| - 3$.

10. Graph $y = f(x)$, where $f(x) = \frac{3}{4}\,|6 + x| + 2$.

11. Simplify the expression. Do not assume the variables take only positive values.

$\sqrt{36r^2}$

12. Simplify the expression. Do not assume the variables take only positive values.

$\sqrt{9m^2}$

13. Simplify the expression.

$\sqrt{a^2 + 14a + 49}$

14. Simplify the expression.

$\sqrt{r^2 + 16r + 64}$

15. The height inside a camping tent when you are d feet from the edge of the tent is given by

$$h = -0.7|d - 6.6| + 7$$

where h stands for height in feet.

Determine the height when you are:

a. 7.6 ft from the edge.

The height inside a camping tent when you 7.6 ft from the edge of the tent is $\boxed{\quad}$

b. 3.1 ft from the edge.

The height inside a camping tent when you 3.1 ft from the edge of the tent is $\boxed{\quad}$

16. The height inside a camping tent when you are d feet from the edge of the tent is given by

$$h = -0.5|d - 6| + 4.5$$

where h stands for height in feet.

Determine the height when you are:

a. 8.3 ft from the edge.

The height inside a camping tent when you 8.3 ft from the edge of the tent is $\boxed{\quad}$

b. 1.3 ft from the edge.

The height inside a camping tent when you 1.3 ft from the edge of the tent is $\boxed{\quad}$

Compound Inequalities Solve the compound inequality algebraically.

17. $-4x - 3 \geq 5$ and $-14x - 7 \geq -5$

18. $-18x + 14 \geq -6$ and $6x + 4 > 9$

19. $9x - 10 \geq -17$ and $-15x + 7 \geq -15$

20. $-6x + 7 \geq 12$ and $6x + 11 \leq 2$

21. $7x + 15 > 11$ or $x + 15 < -20$

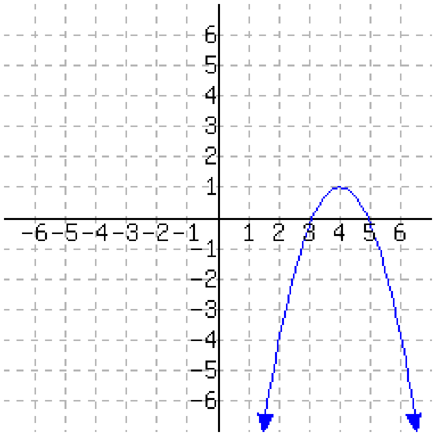
22. $12x - 2 \geq 10$ or $6x + 6 < 6$

23. $13x + 3 < -20$ or $10x + 1 \geq 8$

24. $19x - 6 \geq 8$ or $5x - 16 \leq -14$

A graph of f is given. Use the graph alone to solve the compound inequalities.

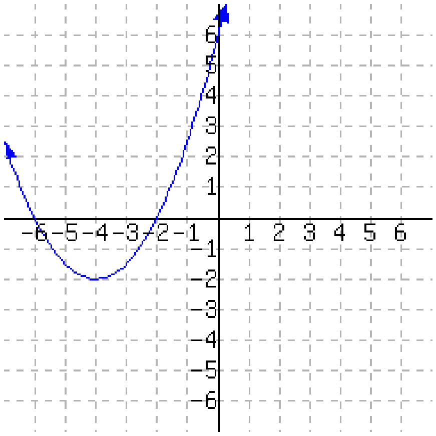
25.



a. $f(x) > -4$

b. $f(x) \leq -4$

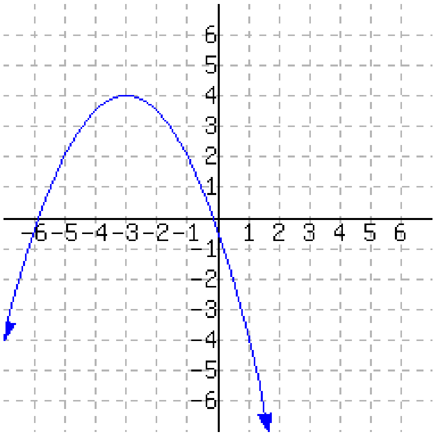
26.



a. $f(x) > 0$

b. $f(x) \leq 0$

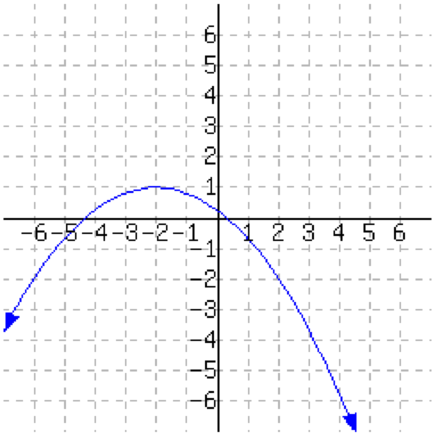
27.



a. $f(x) > 2$

b. $f(x) \leq 2$

28.



a. $f(x) > -2$

b. $f(x) \leq -2$

Absolute Value Equations and Inequalities

29. Solve the following equation.

$|5x - 6| = 5$

30. Solve the following equation.

$|6x + 8| = 9$

31. Solve the equation $|4x - 2| = 18$.

32. Solve the equation $|4x - 3| = 17$.

33. Solve: $\left|\dfrac{2y - 7}{3}\right| = 1$

34. Solve: $\left|\dfrac{2y - 3}{7}\right| = 3$

35. Solve: $\left|\frac{1}{2}a + 7\right| = 3$

36. Solve: $\left|\frac{1}{4}a + 5\right| = 3$

37. Solve: $|b + 5| - 8 = 2$

38. Solve: $|t + 1| - 2 = 6$

39. Solve: $|5t - 20| + 5 = 5$

40. Solve: $|3x - 3| + 3 = 3$

The equation $|x| = |y|$ is satisfied if $x = y$ or $x = -y$. Use this fact to solve the following equation.

41. $|x + 6| = |x - 5|$

42. $|x + 6| = |x - 1|$

43. Solve the equation: $|8x - 4| = |5x + 5|$

44. Solve the equation: $|2x - 2| = |7x + 3|$

45. Solve the following equation.

$|x + 4| = |6x + 8|$

46. Solve the following equation.

$|2x - 4| = |8x + 2|$

Solve the inequality.

47. $|10 - 4x| \geq 15$

48. $|7 - 5x| \geq 7$

49. $|5x - 4| < 12$

50. $|6x - 10| < 5$

CHAPTER 12

More on Quadratic Functions

12.1 Graphs and Vertex Form

In this section, we will explore quadratic functions using graphing technology and learn the vertex and factored forms of a quadratic function's formula. We will also see how parabola graphs can be shifted.

12.1.1 Exploring Quadratic Functions with Graphing Technology

Graphing technology is very important and useful for applications and for finding points quickly. Let's explore some quadratic functions with graphing technology.

Example 12.1.2 Use technology to graph and make a table of the quadratic function f defined by $f(x) = 2x^2 + 4x - 3$ and find each of the key points or features.

a. Find the vertex.

b. Find the vertical intercept (i.e. y-intercept).

c. Find the horizontal or (i.e. x-intercept(s)).

d. Find $f(-2)$.

e. Solve $f(x) = 3$ using the graph.

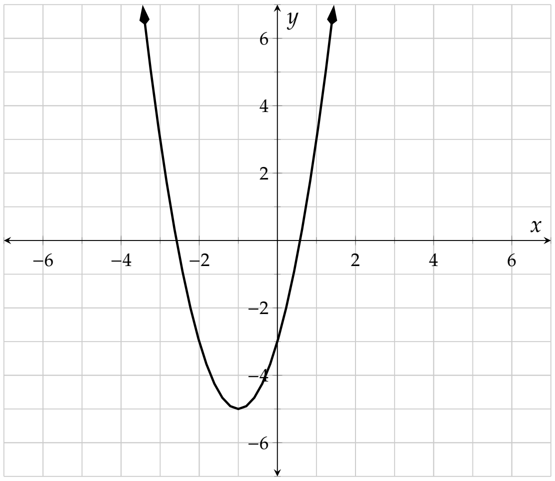
f. Solve $f(x) \leq 3$ using the graph.

g. State the domain and range of the function.

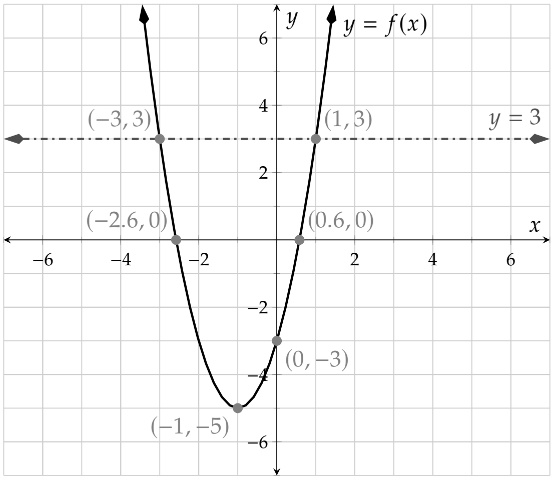
Explanation.

The specifics of how to use any one particular technology tool vary. Whether you use an app, a physical calculator, or something else, a table and graph should look like:

x	$f(x)$
-2	-3
-1	-5
0	-3
1	3
2	13



Additional features of your technology tool can enhance the graph to help answer these questions. You may be able to make the graph appear like:



a. The vertex is $(-1, -5)$.

b. The vertical intercept is $(0, -3)$.

c. The horizontal intercepts are approximately $(-2.6, 0)$ and $(0.6, 0)$.

d. When $x = -2$, $y = -3$, so $f(-2) = -3$.

e. The solutions to $f(x) = 3$ are the x-values where $y = 3$. We graph the horizontal line $y = 3$ and find the x-values where the graphs intersect. The solution set is $\{-3, 1\}$.

f. The solutions are all of the x-values where the function's graph is below (or touching) the line $y = 3$. The interval is $[-3, 1]$.

g. The domain is $(-\infty, \infty)$ and the range is $[-5, \infty)$.

Now we will look at an application with graphing technology and put the points of interest in context.

Example 12.1.3 A reduced-gravity aircraft[a] is a fixed-wing airplane that astronauts use for training. The airplane flies up and then down in a parabolic path to simulate the feeling of weightlessness. In one training flight, the pilot will fly 40 to 60 parabolic maneuvers.

For the first parabolic maneuver, the altitude of the plane, in feet, at time t, in seconds since the maneuver began, is given by $H(t) = -16t^2 + 400t + 30500$.

a. Determine the starting altitude of the plane for the first maneuver.

b. What is the altitude of the plane 10 seconds into the maneuver?

c. Determine the maximum altitude of the plane and how long it takes to reach that altitude.

d. The zero-gravity effect is experienced when the plane begins the parabolic path until it gets back down to 30,500 feet. Write an inequality to express this and solve it using the graph. Write the times of the zero-gravity effect as an interval and determine how long the astronauts experience weightlessness during each cycle.

e. Use technology to make a table for H with t-values from 0 to 25 seconds. Use an increment of 5 seconds and then use the table to solve $H(t) = 32100$.

f. State the domain and range for this context.

Explanation. We can answer the questions based on the information in the graph.

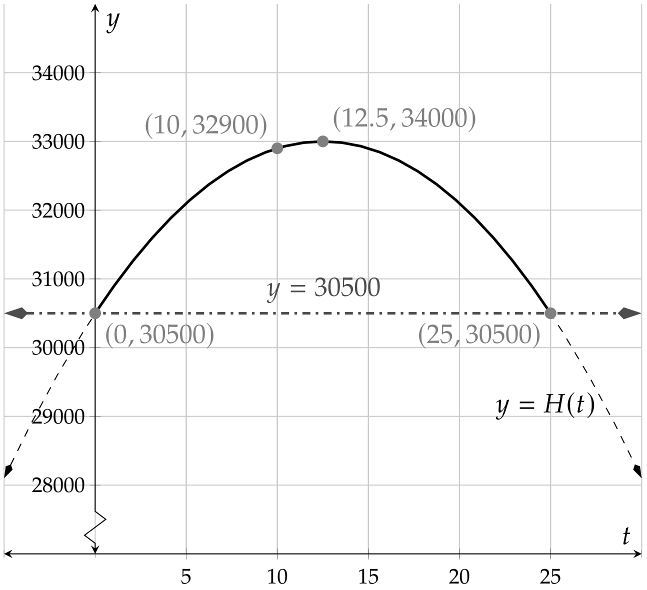


Figure 12.1.4: Graph of $H(t) = -16t^2 + 400t + 30500$ with $y = 30500$

a. The starting altitude can be read from the vertical intercept, which is $(0, 30500)$. The feeling of weightlessness begins at 30,500 feet.

b. After 10 seconds, the altitude of the plane is 32,900 feet.

c. For the maximum altitude of the plane we look at the vertex, which is approximately $(12.5, 33000)$. This tells us that after 12.5 seconds the plane will be at its maximum altitude of 33,000 feet.

d. We can write an inequality to describe when the plane is at or above 30,500 feet and solve it graphically.

$$\begin{aligned} H(t) &\geq 30500 \\ -16t^2 + 400t + 30500 &\geq 30500. \end{aligned}$$

We graph the line $y = 30500$ and find the points of intersection with the parabola. The astronauts experience weightlessness from 0 seconds to 25 seconds into the maneuver, or $[0, 25]$ seconds. They experience weightlessness for 25 seconds in each cycle.

e. To solve $H(t) = 32100$ using the table, we look for where the H-values are equal to 32100.

t	0	5	10	15	20	25
$H(t)$	30500	32100	32900	32900	32100	30500

There are two solutions, 5 seconds and 20 seconds. The solution set is $\{5, 20\}$.

f. When we use technology we see the entire function but in this context the plane is only on a parabolic path from $t = 0$ to $t = 25$ seconds. So the domain is $[0, 25]$, and the range is the set of corresponding y-values which is $[30500, 33000]$ feet.

---

[a] en.wikipedia.org/wiki/Reduced-gravity_aircraft

Let's look at the remote-controlled airplane dive from Example 9.3.18. This time we will use technology to answer the questions.

Example 12.1.5 Maia has a remote-controlled airplane and she is going to do a stunt dive where the plane dives toward the ground and back up along a parabolic path. The altitude or height of the plane is given by the function H where $H(t) = 0.7t^2 - 23t + 200$, for $0 \leq t \leq 30$. The height is measured in feet and the time, t, is measured in seconds since the stunt began.

a. Determine the starting height of the plane as the dive begins.

b. Determine the height of the plane after 5 seconds.

c. Will the plane hit the ground, and if so, at what time?

d. If the plane does not hit the ground, what is the closest it gets to the ground, and at what time?

e. At what time(s) will the plane have an altitude of 50 feet?

f. State the domain and the range of the function (in context).

Explanation. We have graphed the function and we will find the key information and put it in context.

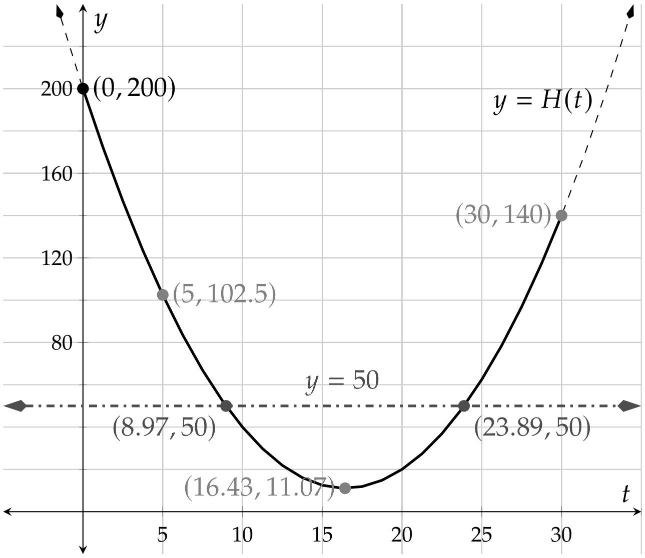


Figure 12.1.6: Graph of $H(t) = 0.7t^2 - 23t + 200$

a. The starting altitude can be read from the vertical intercept, which is $(0, 200)$. When the stunt begins, the plane has a altitude of 200 feet.

b. When $x = 5$, the y-value is 102.5. So $H(5) = 102.5$. This means that after 5 seconds, the plane is 102.5 feet above the ground.

c. From the graph we can see that the parabola does not touch or cross the x-axis, which represents the ground. This means the plane does not hit the ground and there are no real solutions to the equation $H(t) = 0$.

d. The lowest point is the vertex, which is approximately $(16.43, 11.07)$. The minimum altitude of the plane is about 11 feet, which occurs after about 16.4 seconds.

e. We graph the horizontal line $y = 50$ and look for the points of intersection. The plane will be 50 feet above the ground about 9 seconds after the plane begins the stunt, and again at about 24 seconds.

f. The domain for this function is given in the problem statement because only part of the parabola represents the path of the plane. The domain is $[0, 30]$. For the range we look at the possible altitudes of the plane and see that it is $[11.07\ldots, 200]$. The plane is doing this stunt from 0 to 30 seconds and its height ranges from about 11 to 200 feet above the ground.

12.1.2 The Vertex Form of a Parabola

We have learned the standard form of a quadratic function's formula, which is $f(x) = ax^2 + bx + c$. In this subsection, we will learn another form called the **vertex form**.

Using graphing technology, consider the graphs of $f(x) = x^2 - 6x + 7$ and $g(x) = (x - 3)^2 - 2$ on the same axes.

We see only one parabola because these are two different forms of the same function. Indeed, if we convert $g(x)$ into standard form:

$$\begin{aligned} g(x) &= (x-3)^2 - 2 \\ g(x) &= x^2 - 6x + 9 - 2 \\ g(x) &= x^2 - 6x + 7 \end{aligned}$$

it is clear that f and g are the same function.

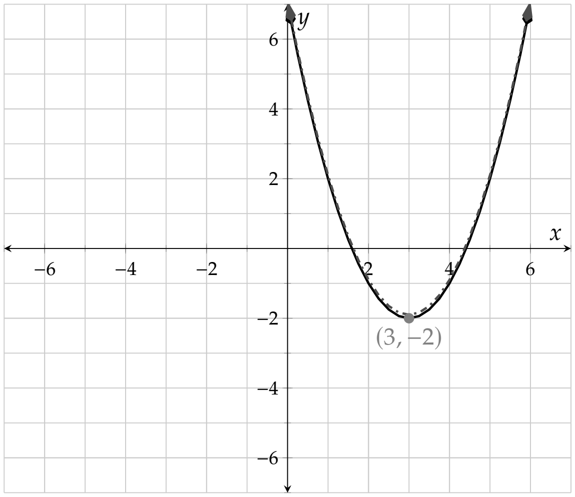


Figure 12.1.7: Graph of $f(x) = x^2 - 6x + 7$ and $g(x) = (x-3)^2 - 2$

The formula given for g is written in vertex form which is very useful because it allows us to read the vertex without doing any calculations. The vertex of the parabola is $(3, -2)$. We can see those numbers in $g(x) = (x-3)^2 - 2$. The x-value is the solution to $(x - 3) = 0$, and the y-value is the constant *added* at the end.

Here are the graphs of three more functions with formulas in vertex form. Compare each function with the vertex of its graph.

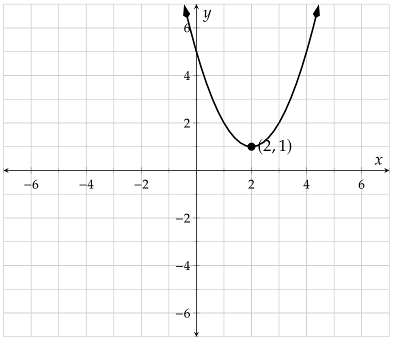


Figure 12.1.8: $r(x) = (x - 2)^2 + 1$

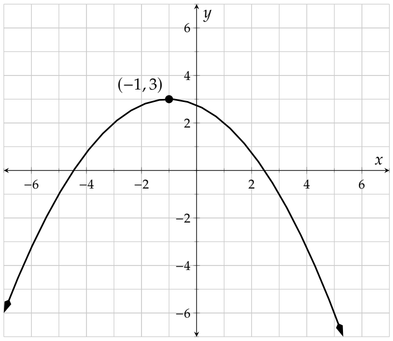


Figure 12.1.9: $s(x) = -\frac{1}{4}(x+1)^2+3$

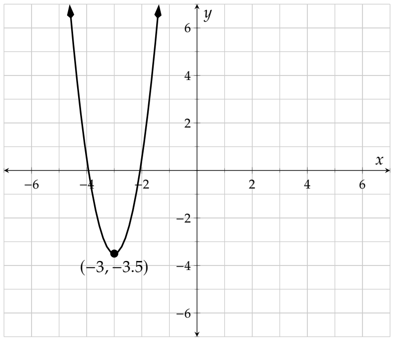


Figure 12.1.10: $t(x) = 4(x + 3)^2 - 3.5$

Notice that the x-coordinate of the vertex has the opposite sign as the value in the function formula. On the other hand, the y-coordinate of the vertex has the same sign as the value in the function formula. Let's look at an example to understand why. We will evaluate $r(2)$.

$$\begin{aligned} r(2) &= (2-2)^2 + 1 \\ &= 1 \end{aligned}$$

The x-value is the solution to $(x - 2) = 0$, which is positive 2. When we substitute 2 for x we get the value $y = 1$. Note that these coordinates create the vertex at $(2, 1)$. Now we can define the vertex form of a quadratic function.

Fact 12.1.11 Vertex Form of a Quadratic Function. *A quadratic function with the vertex at the point* (h, k) *is given by* $f(x) = a(x - h)^2 + k$.

Checkpoint 12.1.12. Find the vertex of each quadratic function.

a. $r(x) = -2(x + 4)^2 + 10$
b. $s(x) = 5(x - 1)^2 + 2$
c. $t(x) = (x - 10)^2 - 5$
d. $u(x) = 3(x + 7)^2 - 13$

Explanation.

a. The vertex of $r(x) = -2(x + 4)^2 + 10$ is $(-4, 10)$.
b. The vertex of $s(x) = 5(x - 1)^2 + 2$ is $(1, 2)$.
c. The vertex of $t(x) = (x - 10)^2 - 5$ is $(10, -5)$.
d. The vertex of $u(x) = 3(x + 7)^2 - 13$ is $(-7, -13)$.

Now let's do the reverse. When given the vertex and the value of a, we can write the function in vertex form.

Example 12.1.13 Write a formula for the quadratic function f with the given vertex and value of a.

a. Vertex $(-2, 8)$, $a = 1$
b. Vertex $(4, -9)$, $a = -4$
c. Vertex $(-3, -1)$, $a = 2$
d. Vertex $(5, 12)$, $a = -3$

Explanation.

a. The vertex form is $f(x) = (x + 2)^2 + 8$.

b. The vertex form is $f(x) = -4(x - 4)^2 - 9$.

c. The vertex form is $f(x) = 2(x + 3)^2 - 1$.

d. The vertex form is $f(x) = -3(x - 5)^2 + 12$.

Once we read the vertex we can also state the domain and range. All quadratic functions have a domain of $(-\infty, \infty)$ because we can put in any value to a quadratic function. The range, however, depends on the y-value of the vertex and whether the parabola opens upward or downward. When we have a quadratic function in vertex form we can read the range from the formula. Let's look at the graph of f, where $f(x) = 2(x - 3)^2 - 5$, as an example.

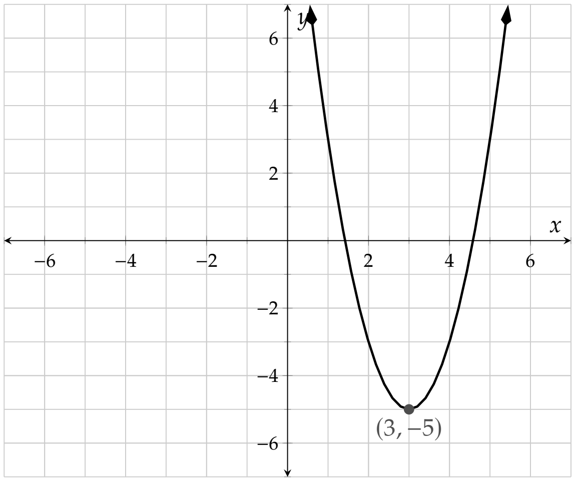


Figure 12.1.14: The graph of $f(x) = 2(x - 3)^2 - 5$

The domain is $(-\infty, \infty)$. The graph of f opens upward (which we know because $a = 2 > 0$) so the vertex is the minimum point. The y-value of -5 is the minimum. The range is $[-5, \infty)$.

Example 12.1.15 Identify the domain and range of g, where $g(x) = -3(x + 1)^2 + 6$.

Explanation.

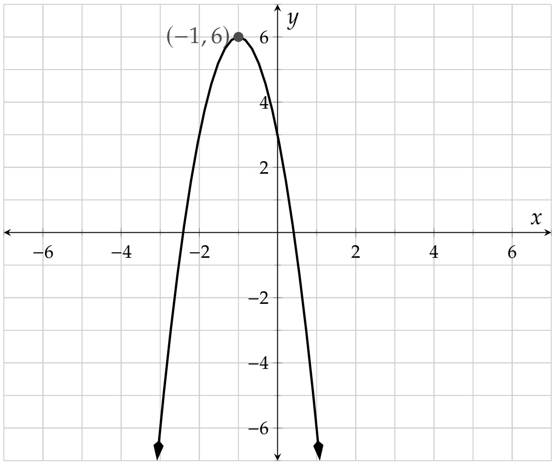


Figure 12.1.16: $g(x) = -3(x + 1)^2 + 6$

The domain is $(-\infty, \infty)$. The graph of g opens downward (which we know because $a = -3 < 0$) so the vertex is the maximum point. The y-value of 6 is the maximum. The range is $(-\infty, 6]$.

Checkpoint 12.1.17. Identify the domain and range of each quadratic function.

a. $w(x) = -3(x+10)^2 - 11$

The domain is ____ and the range is ____.

b. $u(x) = 4(x-7)^2 + 20$

The domain is ____ and the range is ____.

c. $y(x) = -(x-1)^2$

The domain is ____ and the range is ____.

d. $z(x) = 3(x+9)^2 - 4$

The domain is ____ and the range is ____.

Explanation.

a. The domain of w is $(-\infty, \infty)$. The parabola opens downward so the range is $(-\infty, -11]$.
b. The domain of u is $(-\infty, \infty)$. The parabola opens upward so the range is $[20, \infty)$.
c. The domain of y is $(-\infty, \infty)$. The parabola opens downward so the range is $(-\infty, 0]$.
d. The domain of z is $(-\infty, \infty)$. The parabola opens upward so the range is $[-4, \infty)$.

12.1.3 Horizontal and Vertical Shifts

Let $f(x) = x^2$ and $g(x) = (x-4)^2 + 1$. The graph of $y = f(x)$ has its vertex at the point $(0, 0)$. Now we will compare this with the graph of $y = g(x)$ on the same axes.

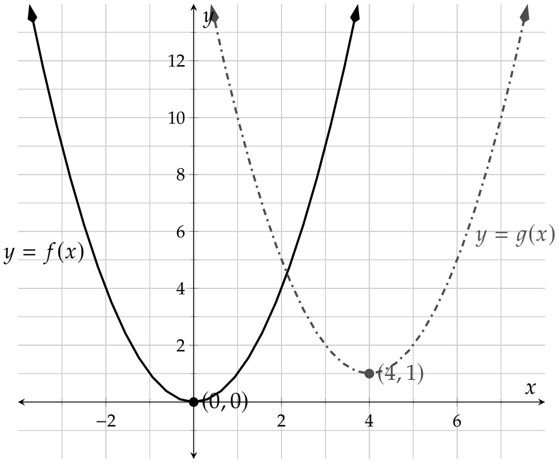


Figure 12.1.18: The graph of f and g

Both graphs open upward and have the same shape. Notice that the graph of g is the same as the graph of f but is shifted to the right by 4 units and up by 1 units because its vertex is $(4, 1)$.

Let's look at another graph. Let $h(x) = -x^2$ and let $j(x) = -(x+3)^2 + 4$.

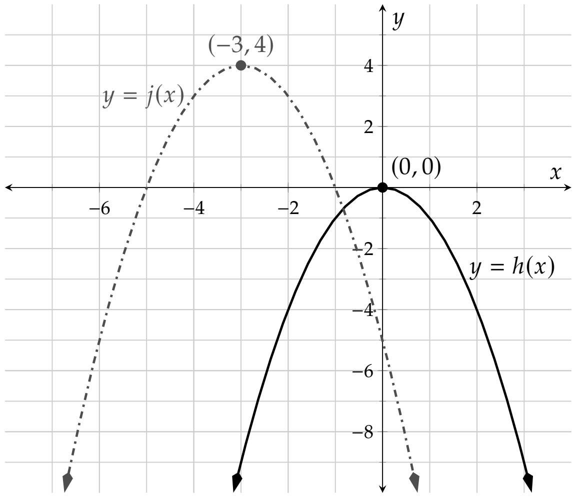


Figure 12.1.19: The graph of h and j

Both parabolas open downward and have the same shape. The graph of j is the same as the graph of h but it has been shifted to the left by 3 units and up by 4 units making its vertex $(-3, 4)$.

To summarize this, when a quadratic function is written in vertex form, the h-value is the horizontal shift and the k-value is the vertical shift.

Example 12.1.20 Identify the horizontal and vertical shifts compared with $f(x) = x^2$.

a. $m(x) = (x + 7)^2 + 3$

b. $n(x) = (x - 1)^2 + 6$

c. $o(x) = (x - 5)^2 - 1$

d. $p(x) = (x + 3)^2 - 11$

Explanation.

a. The graph of $y = m(x)$ has vertex at $(-7, 3)$. Therefore the graph is the same as $y = f(x)$ shifted to the left 7 units and up 3 units.

b. The graph of $y = n(x)$ has vertex at $(1, 6)$. Therefore the graph is the same as $y = f(x)$ shifted to the right 1 unit and up 6 units.

c. The graph of $y = o(x)$ has vertex at $(5, -1)$. Therefore the graph is the same as $y = f(x)$ shifted to the right 5 units and down 1 unit.

d. The graph of $y = p(x)$ has vertex at $(-3, -11)$. Therefore the graph is the same as $y = f(x)$ shifted to the left 3 units and down 11 units.

12.1.4 The Factored Form of a Parabola

There is another form of a quadratic function's formula, called **factored form**, which we will explore next. Let's consider the two functions $q(x) = -x^2 + 3x + 4$ and $s(x) = -(x - 4)(x + 1)$. Using graphing technology, we will graph $y = q(x)$ and $y = s(x)$ on the same axes.

These graphs coincide because the functions are actually the same. We can tell by multiplying out the formula for g to get back to the formula for f.

$$\begin{aligned} g(x) &= -(x-4)(x+1) \\ g(x) &= -(x^2-3x-4) \\ g(x) &= -x^2+3x+4 \end{aligned}$$

Now we can see that f and g are really the same function.

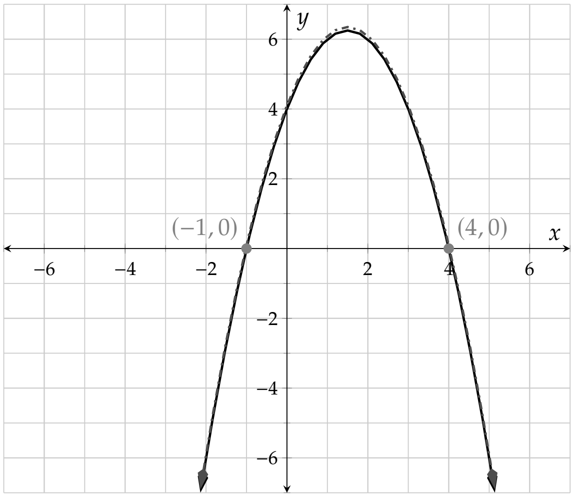


Figure 12.1.21: Graph of q and s

Factored form is very useful because we can read the x-intercepts directly from the function, which in this case are $(4,0)$ and $(-1,0)$. We find these by looking for the values that make the factors equal to 0, so the x-values have the opposite signs as are shown in the formula. To demonstrate this, we will find the roots by solving $g(x)=0$.

$$\begin{aligned} g(x) &= -(x-4)(x+1) \\ 0 &= -(x-4)(x+1) \end{aligned}$$

$$\begin{array}{ccc} x-4=0 & \text{or} & x+1=0 \\ x=4 & \text{or} & x=-1 \end{array}$$

This shows us that the x-intercepts are $(4,0)$ and $(-1,0)$.

The x-values of the x-intercepts are also called **zeros** or **roots**. The zeros or roots of the function g are -1 and 4.

Fact 12.1.22 Factored Form of a Quadratic Function. *A quadratic function with horizontal intercepts at* $(r,0)$ *and* $(s,0)$ *has the formula* $f(x)=a(x-r)(x-s)$.

Checkpoint 12.1.23. Write the horizontal intercepts of each function.

a. $t(x)=-(x+2)(x-4)$

b. $u(x)=6(x-7)(x-5)$

c. $v(x)=-2(x+1)(x+4)$

d. $w(x)=10(x-8)(x+3)$

Explanation.

a. The horizontal intercepts of t are $(-2,0)$ and $(4,0)$.

b. The horizontal intercepts of u are $(7,0)$ and $(5,0)$.

c. The horizontal intercepts of v are $(-1,0)$ and $(-4,0)$.

d. The horizontal intercepts of w are $(8,0)$ and $(-3,0)$.

Let's summarize the three forms of a quadratic function's formula:

Standard Form $f(x) = ax^2 + bx + c$, with y-intercept $(0, c)$.

Vertex Form $f(x) = a(x - h)^2 + k$, with vertex (h, k).

Factored Form $f(x) = a(x - r)(x - s)$, with x-intercepts $(m, 0)$ and $(n, 0)$.

Exercises

Review and Warmup

1. Multiply the polynomials.

$(x + 6)(x - 2) =$

2. Multiply the polynomials.

$(y + 2)(y - 8) =$

3. Multiply the polynomials.

$(9y - 4)(y + 9) =$

4. Multiply the polynomials.

$(6r - 9)(r + 6) =$

5. Factor the given polynomial.

$r^2 + 8r + 12 =$

6. Factor the given polynomial.

$r^2 + 19r + 90 =$

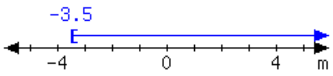
7. Factor the given polynomial.

$2t^2 + 20t + 18 =$

8. Factor the given polynomial.

$5t^2 + 25t + 20 =$

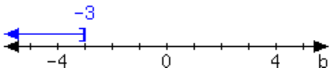
9. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

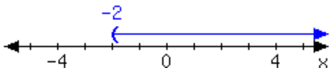
10. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

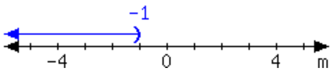
11. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

12. Here is an interval:



Write the interval using set-builder notation.

Write the interval using interval notation.

Technology and Tables

13. Let $F(x) = x^2 + 3x - 2$. Use technology to make a table of values F.

x	$F(x)$

14. Let $G(x) = x^2 + 2x - 2$. Use technology to make a table of values G.

x	$G(x)$

15. Let $H(x) = -x^2 + 2x + 1$. Use technology to make a table of values H.

x	$H(x)$

16. Let $H(x) = -x^2 - 3x + 1$. Use technology to make a table of values H.

x	$H(x)$

17. Let $K(x) = 3x^2 + 8x - 4$. Use technology to make a table of values K.

x	$K(x)$

18. Let $f(x) = 3x^2 + 3x - 4$. Use technology to make a table of values f.

x	$f(x)$

19. Let $g(x) = 3x^2 - 8x + 35$. Use technology to make a table of values g.

x	$g(x)$

20. Let $g(x) = 3x^2 - 7x + 46$. Use technology to make a table of values g.

x	$g(x)$

Technology and Graphs

21. Use technology to make a graph of f where $f(x) = x^2 + 3x - 2$.

22. Use technology to make a graph of f where $f(x) = x^2 - 2x - 1$.

23. Use technology to make a graph of f where $f(x) = -x^2 + 3x + 2$.

24. Use technology to make a graph of f where $f(x) = -x^2 + x + 2$.

25. Use technology to make a graph of f where $f(x) = 3x^2 - 6x - 5$.

26. Use technology to make a graph of f where $f(x) = -3x^2 - 8x + 3$.

27. Use technology to make a graph of f where $f(x) = -3x^2 + 4x + 49$.

28. Use technology to make a graph of f where $f(x) = 2x^2 - 2x + 41$.

Technology and Features of Quadratic Function Graphs Use technology to find features of a quadratic function and its graph.

29. Let $h(x) = -2x^2 + 3x - 4$. Use technology to find the following.

a. The vertex is ☐.

b. The y-intercept is ☐.

c. The x-intercept(s) is/are ☐.

d. The domain of h is ☐.

e. The range of h is ☐.

f. Calculate $h(1)$. ☐.

g. Solve $h(x) = -9$. ☐

h. Solve $h(x) \leq -9$. ☐

30. Let $F(x) = -x^2 + 3x - 1$. Use technology to find the following.

a. The vertex is ☐.

b. The y-intercept is ☐.

c. The x-intercept(s) is/are ☐.

d. The domain of F is ☐.

e. The range of F is ☐.

f. Calculate $F(-1)$. ☐.

g. Solve $F(x) = -4$. ☐

h. Solve $F(x) \leq -4$. ☐

31. Let $G(x) = -1.4x^2 + 1.1x + 0.3$. Use technology to find the following.

a. The vertex is ☐.

b. The y-intercept is ☐.

c. The x-intercept(s) is/are ☐.

d. The domain of G is ☐.

e. The range of G is ☐.

f. Calculate $G(2)$. ☐.

g. Solve $G(x) = -10$. ☐

h. Solve $G(x) \geq -10$. ☐

32. Let $H(x) = 0.3x^2 - 0.7x - 5$. Use technology to find the following.

a. The vertex is ☐.

b. The y-intercept is ☐.

c. The x-intercept(s) is/are ☐.

d. The domain of H is ☐.

e. The range of H is ☐.

f. Calculate $H(-7)$. ☐.

g. Solve $H(x) = 5$. ☐

h. Solve $H(x) \geq 5$. ☐

33. Let $H(x) = \frac{x^2}{2} + 4.3x + 2.3$. Use technology to find the following.

a. The vertex is ☐.

b. The y-intercept is ☐.

c. The x-intercept(s) is/are ☐.

d. The domain of H is ☐.

e. The range of H is ☐.

f. Calculate $H(-7)$. ☐.

g. Solve $H(x) = -1$. ☐

h. Solve $H(x) \geq -1$. ☐

34. Let $K(x) = \frac{x^2}{4} - 3.8x + 4.6$. Use technology to find the following.

a. The vertex is ☐.

b. The y-intercept is ☐.

c. The x-intercept(s) is/are ☐.

d. The domain of K is ☐.

e. The range of K is ☐.

f. Calculate $K(-1)$. ☐.

g. Solve $K(x) = -4$. ☐

h. Solve $K(x) > -4$. ☐

Applications

35. An object was launched from the top of a hill with an upward vertical velocity of 50 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 50t + 200$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Find the answer using graphing technology.

The object's height was ☐ feet when it was launched.

36. An object was launched from the top of a hill with an upward vertical velocity of 70 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 70t + 100$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Find the answer using graphing technology.

Use a table to list the object's height within the first second after it was launched, at an increment of 0.1 second. Fill in the blanks. Round your answers to two decimal places when needed.

Time in Seconds	Height in Feet
0.1	____________
0.2	____________
0.3	____________

37. An object was launched from the top of a hill with an upward vertical velocity of 90 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 90t + 300$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Use technology to find the answer.

The object was ☐ feet in the air 4 seconds after it was launched.

38. An object was launched from the top of a hill with an upward vertical velocity of 110 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 110t + 200$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Find the answer using technology.

☐ seconds after its launch, the object reached its maximum height of ☐ feet.

39. An object was launched from the top of a hill with an upward vertical velocity of 120 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 120t + 100$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Find the answer using technology.

☐ seconds after its launch, the object fell to the ground at sea level.

40. An object was launched from the top of a hill with an upward vertical velocity of 140 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 140t + 250$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Find the answer using technology. Round your answers to two decimal places. If there is more than one answer, use a comma to separate them.

The object was 483 feet high at the following number of seconds after it was launched: ☐.

41. In a race, a car drove through the starting line at the speed of 6 meters per second. It was accelerating at 3.9 meters per second squared. Its distance from the starting position can be modeled by the function $d(t) = 1.95t^2 + 6t$. Find the answer using technology.

After ☐ seconds, the car was 172.8 meters away from the starting position.

42. In a race, a car drove through the starting line at the speed of 3 meters per second. It was accelerating at 4.4 meters per second squared. Its distance from the starting position can be modeled by the function $d(t) = 2.2t^2 + 3t$. Find the answer using technology.

After ☐ seconds, the car was 473.2 meters away from the starting position.

43. A farmer purchased 800 meters of fencing, and will build a rectangular pen with it. To enclose the largest possible area, what should the pen's length and width be? Model the pen's area with a function, and then find its maximum value.

Use a comma to separate your answers.

To enclose the largest possible area, the pen's length and width should be ☐ meters.

44. A farmer purchased 210 meters of fencing, and will build a rectangular pen along a river. This implies the pen has only 3 fenced sides. To enclose the largest possible area, what should the pen's length and width be? Model the pen's area with a function, and then find its maximum value.

To enclose the largest possible area, the pen's length and width should be ☐ meters.

Quadratic Functions in Vertex Form

45. Find the vertex of the graph of

$$y = -7(x + 10)^2 + 8$$

46. Find the vertex of the graph of

$$y = -5(x - 4)^2 - 5$$

47. Find the vertex of the graph of

$$y = -2(x + 4)^2 + 4$$

48. Find the vertex of the graph of

$$y = 10(x + 8)^2 + 1$$

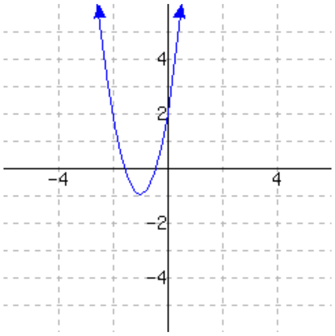
49. Find the vertex of the graph of

$$y = 2.2(x - 2.7)^2 + 0.6$$

50. Find the vertex of the graph of

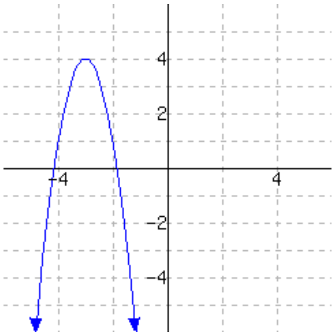
$$y = 4.4(x + 4.3)^2 + 9$$

51. A graph of a function f is given. Use the graph to write a formula for f in vertex form. You will need to identify the vertex and also one more point on the graph to find the leading coefficient a.



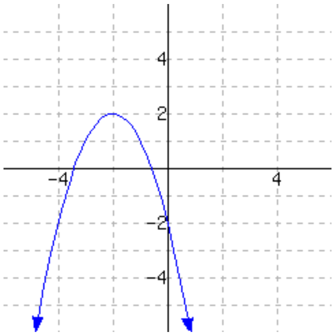
$f(x) =$ ☐

52. A graph of a function f is given. Use the graph to write a formula for f in vertex form. You will need to identify the vertex and also one more point on the graph to find the leading coefficient a.



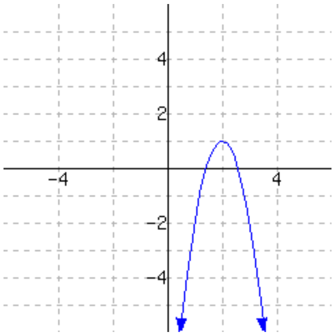
$f(x) =$ ☐

53. A graph of a function f is given. Use the graph to write a formula for f in vertex form. You will need to identify the vertex and also one more point on the graph to find the leading coefficient a.



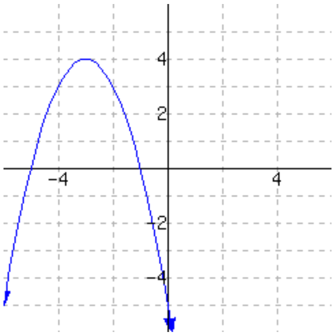
$f(x) =$ ☐

54. A graph of a function f is given. Use the graph to write a formula for f in vertex form. You will need to identify the vertex and also one more point on the graph to find the leading coefficient a.



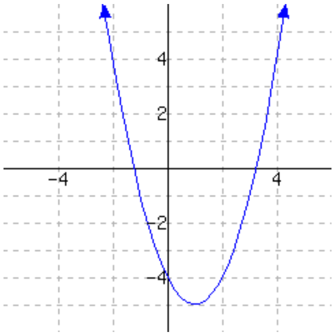
$f(x) =$ ☐

55. A graph of a function f is given. Use the graph to write a formula for f in vertex form. You will need to identify the vertex and also one more point on the graph to find the leading coefficient a.



$f(x) =$ ☐

56. A graph of a function f is given. Use the graph to write a formula for f in vertex form. You will need to identify the vertex and also one more point on the graph to find the leading coefficient a.



$f(x) =$ ☐

57. Write the vertex form for the quadratic function f, whose vertex is $(7, -8)$ and has leading coefficient $a = 7$.

$f(x) =$

58. Write the vertex form for the quadratic function f, whose vertex is $(-4, 6)$ and has leading coefficient $a = 2$.

$f(x) =$

59. Write the vertex form for the quadratic function f, whose vertex is $(-6, 4)$ and has leading coefficient $a = 4$.

$f(x) =$

60. Write the vertex form for the quadratic function f, whose vertex is $(6, -7)$ and has leading coefficient $a = 6$.

$f(x) =$

61. Let K be defined by $K(x) = (x + 1)^2 + 1$.

a. What is the domain of K?

b. What is the range of K?

62. Let f be defined by $f(x) = (x + 7)^2 + 9$.

a. What is the domain of f?

b. What is the range of f?

63. Let g be defined by $g(x) = 7.7(x + 3)^2 - 2$.

a. What is the domain of g?

b. What is the range of g?

64. Let g be defined by $g(x) = 4.3(x - 5)^2 - 7$.

a. What is the domain of g?

b. What is the range of g?

65. Let h be defined by $h(x) = -8(x + 6)^2 + 7$.

a. What is the domain of h?

b. What is the range of h?

66. Let F be defined by $F(x) = 4(x - 2)^2 + 2$.

a. What is the domain of F?

b. What is the range of F?

67. Let G be defined by $G(x) = -5\left(x + \frac{4}{3}\right)^2 - \frac{1}{4}$.

a. What is the domain of G?

b. What is the range of G?

68. Let G be defined by $G(x) = -2(x - (-3))^2 - \frac{7}{9}$.

a. What is the domain of G?

b. What is the range of G?

69. Let H be defined by $H(x) = 7\left(x - \frac{5}{3}\right)^2 + \frac{2}{7}$.

a. What is the domain of H?

b. What is the range of H?

70. Let K be defined by $K(x) = -4\left(x - \frac{2}{7}\right)^2 + \frac{5}{7}$.

a. What is the domain of K?

b. What is the range of K?

71. Consider the graph of the equation $y = (x-4)^2 - 6$.

Compared to the graph of $y = x^2$, the vertex has been shifted $\boxed{}$ units (☐ left ☐ right) and $\boxed{}$ units (☐ down ☐ up) .

72. Consider the graph of the equation $y = (x+7)^2 + 3$.

Compared to the graph of $y = x^2$, the vertex has been shifted $\boxed{}$ units (☐ left ☐ right) and $\boxed{}$ units (☐ down ☐ up) .

73. Consider the graph of the equation $y = (x+46.6)^2 - 41.2$.

Compared to the graph of $y = x^2$, the vertex has been shifted $\boxed{}$ units (☐ left ☐ right) and $\boxed{}$ units (☐ down ☐ up) .

74. Consider the graph of the equation $y = (x+24.7)^2 + 88.1$.

Compared to the graph of $y = x^2$, the vertex has been shifted $\boxed{}$ units (☐ left ☐ right) and $\boxed{}$ units (☐ down ☐ up) .

75. Consider the graph of the equation $y = \left(x + \frac{1}{4}\right)^2 + \frac{1}{5}$.

Compared to the graph of $y = x^2$, the vertex has been shifted $\boxed{}$ units (☐ left ☐ right) and $\boxed{}$ units (☐ down ☐ up) .

76. Consider the graph of the equation $y = \left(x + \frac{2}{5}\right)^2 + \frac{5}{6}$.

Compared to the graph of $y = x^2$, the vertex has been shifted $\boxed{}$ units (☐ left ☐ right) and $\boxed{}$ units (☐ down ☐ up) .

Three Forms of Quadratic Functions

77. The quadratic expression $(x-3)^2 - 49$ is written in vertex form.

a. Write the expression in standard form.

b. Write the expression in factored form.

78. The quadratic expression $(x-3)^2 - 36$ is written in vertex form.

a. Write the expression in standard form.

b. Write the expression in factored form.

79. The quadratic expression $(x-3)^2-25$ is written in vertex form.

a. Write the expression in standard form.

b. Write the expression in factored form.

80. The quadratic expression $(x-2)^2-36$ is written in vertex form.

a. Write the expression in standard form.

b. Write the expression in factored form.

Factored Form and Intercepts

81. The formula for a quadratic function h is $h(x)=(x+7)(x-7)$.

a. The y-intercept is ☐.

b. The x-intercept(s) is/are ☐.

82. The formula for a quadratic function h is $h(x)=(x+5)(x+6)$.

a. The y-intercept is ☐.

b. The x-intercept(s) is/are ☐.

83. The formula for a quadratic function K is $K(x)=-2(x-6)(x-4)$.

a. The y-intercept is ☐.

b. The x-intercept(s) is/are ☐.

84. The formula for a quadratic function F is $F(x)=-(x+7)(x-2)$.

a. The y-intercept is ☐.

b. The x-intercept(s) is/are ☐.

85. The formula for a quadratic function F is $F(x)=2x(x+7)$.

a. The y-intercept is ☐.

b. The x-intercept(s) is/are ☐.

86. The formula for a quadratic function K is $K(x)=4(x-5)\,x$.

a. The y-intercept is ☐.

b. The x-intercept(s) is/are ☐.

87. The formula for a quadratic function h is $h(x)=6(x+1)(x+1)$.

a. The y-intercept is ☐.

b. The x-intercept(s) is/are ☐.

88. The formula for a quadratic function H is $H(x)=8(x+8)(x+8)$.

a. The y-intercept is ☐.

b. The x-intercept(s) is/are ☐.

89. The formula for a quadratic function h is $h(x)=-9(7x+6)(3x+2)$.

a. The y-intercept is ☐.

b. The x-intercept(s) is/are ☐.

90. The formula for a quadratic function g is $g(x)=-7(5x-2)(8x-9)$.

a. The y-intercept is ☐.

b. The x-intercept(s) is/are ☐.

12.2 Completing the Square

In this section, we will learn how to "complete the square" with a quadratic expression. This topic is very useful for solving quadratic equations and putting quadratic functions in vertex form.

12.2.1 Solving Quadratic Equations by Completing the Square

When we have an equation like $(x+5)^2 = 4$, we can solve it quickly using the square root property:

$$(x+5)^2 = 4$$

$$\begin{aligned} x+5 &= -2 & \text{or} \qquad x+5 &= 2 \\ x &= -7 & \text{or} \qquad x &= -3 \end{aligned}$$

The method of **completing the square** allows us to solve *any* quadratic equation using the square root property. The challenge is that most quadratic equations don't come with a perfect square already on one side. Let's explore how to do this by looking at some perfect square trinomials to see the pattern.

$$\begin{aligned} (x+1)^2 &= x^2+2x+1 \\ (x+2)^2 &= x^2+4x+4 \\ (x+3)^2 &= x^2+6x+9 \\ (x+4)^2 &= x^2+8x+16 \\ (x+5)^2 &= x^2+10x+25 \\ &\vdots \end{aligned}$$

There is an important pattern here. Notice that with each middle coefficient on the right, you may cut it in half to get the constant term in the binomial on the left side. And then you may square that number to get the constant term back on the right side. Mathematically, this says:

$$\left(x+\frac{b}{2}\right)^2 = x^2+bx+\left(\frac{b}{2}\right)^2$$

We will use this fact to make perfect square trinomials.

Fact 12.2.2 The Term that Completes the Square. *For a polynomial* x^2+bx*, the constant term needed to make a perfect square trinomial is* $\left(\frac{b}{2}\right)^2$.

Example 12.2.3 Solve the quadratic equation $x^2+6x=16$ by completing the square.

Explanation. To solve the quadratic equation $x^2+6x=16$, on the left side we can complete the square by adding $\left(\frac{b}{2}\right)^2$; note that $b=6$ in this case, which makes $\left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 3^2 = 9$. We add it to both sides to maintain equality.

$$\begin{aligned} x^2+6x+9 &= 16+9 \\ x^2+6x+9 &= 25 \\ (x+3)^2 &= 25 \end{aligned}$$

Now that we have completed the square, we can solve the equation using the square root property.

$$\begin{aligned} x+3&=-5 &\quad\text{or}\quad&& x+3&=5\\ x&=-8 &\quad\text{or}\quad&& x&=2 \end{aligned}$$

The solution set is $\{-8,2\}$.

Now let's see the process for completing the square when the quadratic equation is given in standard form.

Example 12.2.4 Solve $x^2-14x+11=0$ by completing the square.

Explanation. We will solve $x^2-14x+11=0$. We see that the polynomial on the left side is not a perfect square trinomial, so we need to complete the square. We subtract 11 from both sides so we can add the missing term on the left.

$$\begin{aligned} x^2-14x+11&=0\\ x^2-14x&=-11 \end{aligned}$$

Next comes the completing-the-square step. We need to add the correct number to both sides of the equation to make the left side a perfect square. Remember that Fact 12.2.2 states that we need to use $\left(\frac{b}{2}\right)^2$ for this. In our case, $b=-14$, so $\left(\frac{b}{2}\right)^2=\left(\frac{-14}{2}\right)^2=49$

$$\begin{aligned} x^2-14x+49&=-11+49\\ (x-7)^2&=38 \end{aligned}$$

$$\begin{aligned} x-7&=-\sqrt{38} &\quad\text{or}\quad&& x-7&=\sqrt{38}\\ x&=7-\sqrt{38} &\quad\text{or}\quad&& x&=7+\sqrt{38} \end{aligned}$$

The solution set is $\{7-\sqrt{38},7+\sqrt{38}\}$.

Here are some more examples.

Example 12.2.5 Complete the square to solve for y in $y^2-20y-21=0$.

Explanation. To complete the square, we will first move the constant term to the right side of the equation. Then we will use Fact 12.2.2 to find $\left(\frac{b}{2}\right)^2$ to add to both sides.

$$\begin{aligned} y^2-20y-21&=0\\ y^2-20y&=21 \end{aligned}$$

In our case, $b=-20$, so $\left(\frac{b}{2}\right)^2=\left(\frac{-20}{2}\right)^2=100$

$$\begin{aligned} y^2-20y+100&=21+100\\ (y-10)^2&=121 \end{aligned}$$

$$\begin{aligned} y-10&=-11 &\quad\text{or}\quad&& y-10&=11\\ y&=-1 &\quad\text{or}\quad&& y&=21 \end{aligned}$$

The solution set is $\{-1, 21\}$.

So far, the value of b has been even each time, which makes $\frac{b}{2}$ a whole number. When b is odd, we will end up adding a fraction to both sides. Here is an example.

Example 12.2.6 Complete the square to solve for z in $z^2 - 3z - 10 = 0$.

Explanation. We will first move the constant term to the right side of the equation:

$$\begin{aligned} z^2-3z-10&=0\\ z^2-3z&=10 \end{aligned}$$

Next, to complete the square, we will need to find the right number to add to both sides. According to Fact 12.2.2, we need to divide the value of b by 2 and then square the result to find the right number. First, divide by 2:

$$\frac{b}{2}=\frac{-3}{2}=-\frac{3}{2} \tag{12.2.1}$$

and then we square that result:

$$\left(-\frac{3}{2}\right)^2=\frac{9}{4} \tag{12.2.2}$$

Now we can add the $\frac{9}{4}$ from Equation (12.2.2) to both sides of the equation to complete the square.

$$z^2-3z+\frac{9}{4}=10+\frac{9}{4}$$

Now, to factor the seemingly complicated expression on the left, just know that it should always factor using the number from the first step in the completing the square process, Equation (12.2.1).

$$\left(z-\frac{3}{2}\right)^2=\frac{49}{4}$$

$$\begin{aligned} z-\frac{3}{2}&=-\frac{7}{2} &\quad\text{or}\quad&& z-\frac{3}{2}&=\frac{7}{2}\\ z&=\frac{3}{2}-\frac{7}{2} &\quad\text{or}\quad&& z&=\frac{3}{2}+\frac{7}{2}\\ z&=-\frac{4}{2} &\quad\text{or}\quad&& z&=\frac{10}{2}\\ z&=-2 &\quad\text{or}\quad&& z&=5 \end{aligned}$$

The solution set is $\{-2, 5\}$.

In each of the previous examples, the value of a was equal to 1. This is necessary for our missing term formula to work. When a is not equal to 1 we will divide both sides by a. Let's look at an example of that.

Example 12.2.7 Solve for r in $2r^2 + 2r = 3$ by completing the square.

Explanation. Because there is a leading coefficient of 2, we will divide both sides by 2.

$$\begin{aligned} 2r^2+2r&=3\\ \frac{2r^2}{2}+\frac{2r}{2}&=\frac{3}{2}\\ r^2+r&=\frac{3}{2} \end{aligned}$$

Next, we will complete the square. Since $b=1$, first,

$$\frac{b}{2}=\frac{1}{2} \tag{12.2.3}$$

and next, squaring that, we have

$$\left(\frac{1}{2}\right)^2=\frac{1}{4}. \tag{12.2.4}$$

So we will add $\frac{1}{4}$ from Equation (12.2.4) to both sides of the equation:

$$\begin{aligned} r^2+r+\frac{1}{4}&=\frac{3}{2}+\frac{1}{4}\\ r^2+r+\frac{1}{4}&=\frac{6}{4}+\frac{1}{4} \end{aligned}$$

Here, remember that we always factor with the number found in the first step of completing the square, Equation (12.2.3).

$$\left(r+\frac{1}{2}\right)^2=\frac{7}{4}$$

$$\begin{aligned} r+\frac{1}{2}&=-\frac{\sqrt{7}}{2} &&\text{or}&& r+\frac{1}{2}=\frac{\sqrt{7}}{2}\\ r&=-\frac{1}{2}-\frac{\sqrt{7}}{2} &&\text{or}&& r=-\frac{1}{2}+\frac{\sqrt{7}}{2}\\ r&=\frac{-1-\sqrt{7}}{2} &&\text{or}&& r=\frac{-1+\sqrt{7}}{2} \end{aligned}$$

The solution set is $\left\{\frac{-1-\sqrt{7}}{2}, \frac{-1+\sqrt{7}}{2}\right\}$.

12.2.2 Deriving the Vertex Formula and the Quadratic Formula by Completing the Square

In Section 9.2, we learned a formula to find the vertex. In Section 8.4, we learned the Quadratic Formula. You may have wondered where they came from, and now that we know how to complete the square, we can derive them. We will solve the standard form equation $ax^2+bx+c=0$ for x.

First, we subtract c from both sides and divide both sides by a.

$$\begin{aligned} ax^2+bx+c&=0\\ ax^2+bx&=-c \end{aligned}$$

$$\begin{aligned} \frac{ax^2}{a} + \frac{bx}{a} &= -\frac{c}{a} \\ x^2 + \frac{b}{a}x &= -\frac{c}{a} \end{aligned}$$

Next, we will complete the square by taking half of the middle coefficient and squaring it. First,

$$\frac{\frac{b}{a}}{2} = \frac{b}{2a} \tag{12.2.5}$$

and then squaring that we have

$$\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} \tag{12.2.6}$$

We add the $\frac{b^2}{4a^2}$ from Equation (12.2.6) to both sides of the equation:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = +\frac{b^2}{4a^2} - \frac{c}{a}$$

Remember that the left side always factors with the value we found in the first step of the completing the square process from Equation (12.2.5). So we have:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

To find a common denominator on the right, we multiply by $4a$ in the numerator and denominator on the second term.

$$\begin{aligned} \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} \cdot \frac{4a}{4a} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \end{aligned}$$

Now that we have completed the square, we can see that the x-value of the vertex is $-\frac{b}{2a}$. That is the vertex formula. Next, we will solve the equation using the square root property to find the Quadratic Formula.

$$\begin{aligned} x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\ x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\ x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Note on the ± Form. Because of the complexity of the formula we choose to use the ± symbol rather than write out each solution separately. An expression of the form $x = A \pm B$ really means "either $x = A - B$ or $x = A + B$."

This shows us that the solutions to the equation $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

12.2.3 Putting Quadratic Functions in Vertex Form

In Section 12.1, we learned about the vertex form of a parabola, which allows us to quickly read the coordinates of the vertex. We can now use the method of completing the square to put a quadratic function in vertex form. Completing the square with a function is a little different than with an equation so we will start with an example.

Example 12.2.8 Write a formula in vertex form for the function q defined by $q(x) = x^2 + 8x$

Explanation. The formula is in the form $x^2 + bx$, so we will need to add $\left(\frac{b}{2}\right)^2$ to complete the square by Fact 12.2.2. When we had an equation, we could add the same quantity to both sides. But now we do not wish to change the left side, since we are trying to end up with a formula that still says $q(x) = \ldots$. Instead, we will add *and subtract* the term from the right side in order to maintain equality. In this case,

$$\begin{aligned}\left(\frac{b}{2}\right)^2 &= \left(\frac{8}{2}\right)^2\\ &= 4^2\\ &= 16\end{aligned}$$

To maintain equality, we will both add *and* subtract 16 on the same side of the equation. It is functionally the same as adding 0 on the right, but the 16 makes it possible to factor the expression in a particular way:

$$\begin{aligned}q(x) &= x^2 + 8x + 16 - 16\\ &= \left(x^2 + 8x + 16\right) - 16\\ &= (x+4)^2 - 16\end{aligned}$$

Now that we have completed the square, our function is in vertex form. The vertex is $(-4, -16)$. One way to verify that our work is correct is to graph the original version of the function and check that the vertex is where it should be.

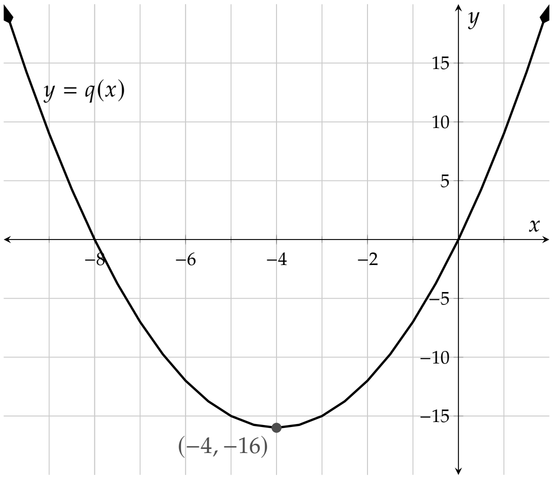


Figure 12.2.9: Graph of $y = x^2 + 8x$

Let's look at a function that has a constant term and see how to complete the square.

Example 12.2.10 Write a formula in vertex form for the function f defined by $f(x) = x^2 - 12x + 3$

Explanation. To complete the square, we need to add and subtract $\left(-\frac{12}{2}\right)^2 = (-6)^2 = 36$ on the right side.

$$\begin{aligned} f(x) &= x^2 - 12x + 36 - 36 + 3 \\ &= \left(x^2 - 12x + 36\right) - 36 + 3 \\ &= (x-6)^2 - 33 \end{aligned}$$

The vertex is $(6, -33)$.

In the first two examples, a was equal to 1. When a is not equal to one, we have an additional step. Since we are working with an expression where we intend to preserve the left side as $f(x) = \ldots$, we cannot divide both sides by a. Instead we will factor a out of the first two terms. Let's look at an example of that.

Example 12.2.11 Write a formula in vertex form for the function g defined by $g(x) = 5x^2 + 20x + 25$

Explanation. Before we can complete the square, we will factor the 5 out of the first two terms.

$$g(x) = 5\left(x^2 + 4x\right) + 25$$

Now we will complete the square inside the parentheses by adding and subtracting $\left(\frac{4}{2}\right)^2 = 2^2 = 4$.

$$g(x) = 5\left(x^2 + 4x + 4 - 4\right) + 25$$

Notice that the constant that we subtracted is inside the parentheses, but it will not be part of our perfect square trinomial. In order to bring it outside, we need to multiply it by 5. We are distributing the 5 to that term so we can combine it with the outside term.

$$\begin{aligned} g(x) &= 5\left(\left(x^2 + 4x + 4\right) - 4\right) + 25 \\ &= 5\left(x^2 + 4x + 4\right) - 5 \cdot 4 + 25 \\ &= 5(x+2)^2 - 20 + 25 \\ &= 5(x+2)^2 + 5 \end{aligned}$$

The vertex is $(-2, 5)$.

Here is an example that includes fractions.

Example 12.2.12 Write a formula in vertex form for the function h defined by $h(x) = -3x^2 - 4x - \frac{7}{4}$

Explanation. First, we will factor the leading coefficient out of the first two terms.

$$\begin{aligned} h(x) &= -3x^2 - 4x - \frac{7}{4} \\ &= -3\left(x^2 + \frac{4}{3}x\right) - \frac{7}{4} \end{aligned}$$

Next, we will complete the square for $x^2 + \frac{4}{3}x$ inside the grouping symbols by adding and subtracting the right number. To find that number, we divide the value of b by two and square the result. That looks like:

$$\frac{b}{2} = \frac{\frac{4}{3}}{2} = \frac{4}{3} \cdot \frac{1}{2} = \frac{2}{3} \tag{12.2.7}$$

and then,

$$\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9} \tag{12.2.8}$$

Adding and subtracting the value from Equation (12.2.8), we have:

$$\begin{aligned} h(x) &= -3\left(x^2 + \frac{4}{3}x + \frac{4}{9} - \frac{4}{9}\right) - \frac{7}{4} \\ &= -3\left(\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) - \frac{4}{9}\right) - \frac{7}{4} \\ &= -3\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) - \left(3 \cdot -\frac{4}{9}\right) - \frac{7}{4} \end{aligned}$$

Remember that when completing the square, the expression should always factor with the number found in the first step of the completing-the-square process, Equation (12.2.7).

$$\begin{aligned} &= -3\left(x + \frac{2}{3}\right)^2 + \frac{4}{3} - \frac{7}{4} \\ &= -3\left(x + \frac{2}{3}\right)^2 + \frac{16}{12} - \frac{21}{12} \\ &= -3\left(x + \frac{2}{3}\right)^2 - \frac{5}{12} \end{aligned}$$

The vertex is $\left(-\frac{2}{3}, -\frac{5}{12}\right)$.

Completing the square can also be used to find a minimum or maximum in an application.

Example 12.2.13 In Example 6.4.19, we learned that artist Tyrone's annual income from paintings can be modeled by $I(x) = -100x^2 + 1000x + 20000$, where x is the number of times he will raise the price per painting by \$20.00. To maximize his income, how should Tyrone set his price per painting? Find the maximum by completing the square.

Explanation. To find the maximum is essentially the same as finding the vertex, which we can find by completing the square. To complete the square for $I(x) = -100x^2 + 1000x + 20000$, we start by factoring out the -100 from the first two terms:

$$\begin{aligned} I(x) &= -100x^2 + 1000x + 20000 \\ &= -100\left(x^2 - 10x\right) + 20000 \end{aligned}$$

Next, we will complete the square for $x^2 - 10x$ by adding and subtracting $\left(-\frac{10}{2}\right)^2 = (-5)^2 = 25$.

$$\begin{aligned} I(x) &= -100\left(x^2 - 10x + 25 - 25\right) + 20000 \\ &= -100\left(\left(x^2 - 10x + 25\right) - 25\right) + 20000 \\ &= -100\left(x^2 - 10x + 25\right) - (100 \cdot -25) + 20000 \\ &= -100(x - 5)^2 + 2500 + 20000 \end{aligned}$$

$$= -100(x-5)^2 + 22500$$

The vertex is the point $(5, 22500)$. This implies Tyrone should raise the price per painting 5 times, which is $5 \cdot 20 = 100$ dollars. He would sell $100 - 5(5) = 75$ paintings. This would make the price per painting $200 + 100 = 300$ dollars, and his annual income from paintings would become \$22500 by this model.

12.2.4 Graphing Quadratic Functions by Hand

Now that we know how to put a quadratic function in vertex form, let's review how to graph a parabola by hand.

Example 12.2.14 Graph the function h defined by $h(x) = 2x^2 + 4x - 6$ by determining its key features algebraically.

Explanation. To start, we'll note that this function opens upward because the leading coefficient, 2, is positive.

Now we will complete the square to find the vertex. We will factor the 2 out of the first two terms, and then add and subtract $\left(\frac{2}{2}\right)^2 = 1^2 = 1$ on the right side.

$$\begin{aligned} h(x) &= 2\left(x^2 + 2x\right) - 6 \\ &= 2\left[x^2 + 2x + 1 - 1\right] - 6 \\ &= 2\left[\left(x^2 + 2x + 1\right) - 1\right] - 6 \\ &= 2\left(x^2 + 2x + 1\right) - (2 \cdot 1) - 6 \\ &= 2\left(x+1\right)^2 - 2 - 6 \\ &= 2\left(x+1\right)^2 - 8 \end{aligned}$$

The vertex is $(-1, -8)$ so the axis of symmetry is the line $x = -1$.

To find the y-intercept, we'll replace x with 0 or read the value of c from the function in standard form:

$$\begin{aligned} h(0) &= 2(0)^2 + 2(0) - 6 \\ &= -6 \end{aligned}$$

The y-intercept is $(0, -6)$ and we will find its symmetric point on the graph, which is $(-2, -6)$.

Next, we'll find the horizontal intercepts. We see this function factors so we will write the factored form to get the horizontal intercepts.

$$\begin{aligned} h(x) &= 2x^2 + 4x - 6 \\ &= 2\left(x^2 + 2x - 3\right) \\ &= 2(x-1)(x+3) \end{aligned}$$

The x-intercepts are $(1, 0)$ and $(-3, 0)$.

Now we will plot all of the key points and draw the parabola.

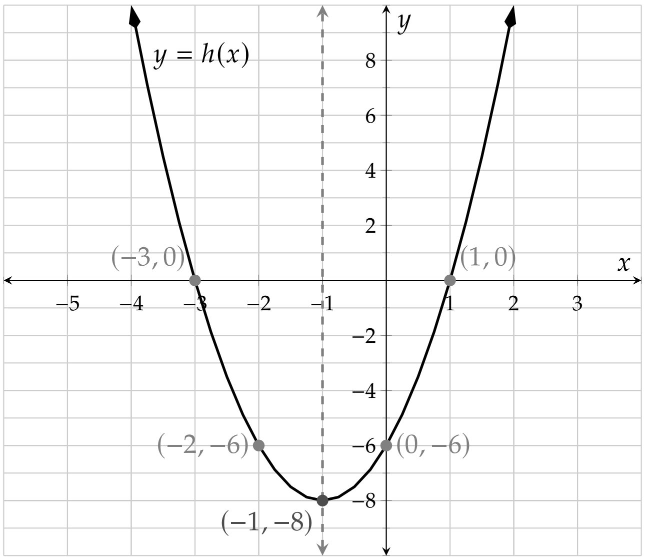


Figure 12.2.15: The graph of $y = 2x^2 + 4x - 6$.

Example 12.2.16 Write a formula in vertex form for the function p defined by $p(x) = -x^2 - 4x - 1$, and find the graph's key features algebraically. Then sketch the graph.

Explanation. In this function, the leading coefficient is negative so it will open downward. To complete the square we first factor −1 out of the first two terms.

$$\begin{aligned} p(x) &= -x^2 - 4x - 1 \\ &= -\left(x^2 + 4x\right) - 1 \end{aligned}$$

Now, we add and subtract the correct number on the right side of the function: $\left(\frac{b}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 2^2 = 4$.

$$\begin{aligned} p(x) &= -\left(x^2 + 4x + 4 - 4\right) - 1 \\ &= -\left(\left(x^2 + 4x + 4\right) - 4\right) - 1 \\ &= -\left(x^2 + 4x + 4\right) - (-4) - 1 \\ &= -(x + 2)^2 + 4 - 1 \\ &= -(x + 2)^2 + 3 \end{aligned}$$

The vertex is $(-2, 3)$ so the axis of symmetry is the line $x = -2$.

We find the y-intercept by looking at the value of c, which is −1. So, the y-intercept is $(0, -1)$ and we will find its symmetric point on the graph, $(-4, -1)$.

The original expression, $-x^2 - 4x - 1$, does not factor so to find the x-intercepts we need to set $p(x) = 0$ and complete the square or use the quadratic formula. Since we just went through the process of completing the square above, we can use that result to save several repetitive steps.

$$\begin{aligned} p(x) &= 0 \\ -(x + 2)^2 + 3 &= 0 \end{aligned}$$

$$-(x+2)^2 = -3$$
$$(x+2)^2 = 3$$

$$\begin{aligned} x+2 &= -\sqrt{3} &\quad \text{or} \quad && x+2 &= \sqrt{3} \\ x &= -2-\sqrt{3} &\quad \text{or} \quad && x &= -2+\sqrt{3} \\ x &\approx -3.73 &\quad \text{or} \quad && x &\approx -0.268 \end{aligned}$$

The x-intercepts are approximately $(-3.7, 0)$ and $(-0.3, 0)$. Now we can plot all of the points and draw the parabola.

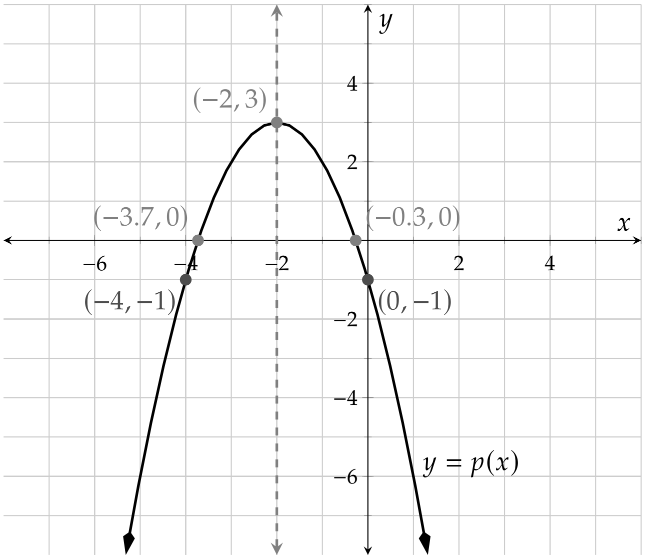


Figure 12.2.17: The graph of $y = -x^2 - 4x - 1$.

Exercises

Review and Warmup

1. Use a square root to solve $(y-9)^2 = 4$.
2. Use a square root to solve $(y+4)^2 = 25$.
3. Use a square root to solve $(4y+7)^2 = 4$.
4. Use a square root to solve $(9r-4)^2 = 64$.
5. Use a square root to solve $(r+3)^2 = 14$.
6. Use a square root to solve $(t-4)^2 = 3$.
7. Use a square root to solve $t^2 + 18t + 81 = 36$.
8. Use a square root to solve $x^2 + 4x + 4 = 81$.

9. Use a square root to solve $16x^2 - 16x + 4 = 25$.

10. Use a square root to solve $81y^2 + 108y + 36 = 9$.

11. Use a square root to solve $36y^2 - 72y + 36 = 17$.

12. Use a square root to solve $9y^2 + 12y + 4 = 13$.

Completing the Square to Solve Equations Solve the equation by completing the square.

13. $r^2 + 2r = 63$

14. $r^2 + 10r = -9$

15. $t^2 - t = 42$

16. $t^2 - 3t = 10$

17. $x^2 + 4x = 6$

18. $x^2 - 8x = 8$

19. $y^2 - 6y + 5 = 0$

20. $y^2 - 8y - 9 = 0$

21. $y^2 + 15y + 56 = 0$

22. $r^2 - r - 42 = 0$

23. $r^2 - 2r - 6 = 0$

24. $t^2 - 10t + 3 = 0$

25. $12t^2 + 28t + 15 = 0$

26. $12x^2 + 20x + 7 = 0$

27. $2x^2 - x - 5 = 0$

28. $2x^2 + 5x + 1 = 0$

Converting to Vertex Form

29. Consider $h(y) = y^2 + 4y + 4$.

a. Give the formula for h in vertex form.

b. What is the vertex of the parabola graph of h?

30. Consider $F(t) = t^2 - 6t + 1$.

a. Give the formula for F in vertex form.

b. What is the vertex of the parabola graph of F?

31. Consider $G(r) = r^2 + r - 2$.

a. Give the formula for G in vertex form.

b. What is the vertex of the parabola graph of G?

32. Consider $G(y) = y^2 + 9y - 5$.

a. Give the formula for G in vertex form.

b. What is the vertex of the parabola graph of G?

33. Consider $H(t) = 5t^2 + 25t + 5$.

a. Give the formula for H in vertex form.

b. What is the vertex of the parabola graph of H?

34. Consider $K(r) = 8r^2 + 64r + 2$.

a. Give the formula for K in vertex form.

b. What is the vertex of the parabola graph of K?

Domain and Range Complete the square to convert the quadratic function from standard form to vertex form, and use the result to find the fnction's domain and range.

35. $f(x) = x^2 + 20x + 94$

The domain of f is ☐

The range of f is ☐

36. $f(x) = x^2 + 16x + 72$

The domain of f is ☐

The range of f is ☐

37. $f(x) = -x^2 - 10x - 34$

The domain of f is ☐

The range of f is ☐

38. $f(x) = -x^2 - 6x - 16$

The domain of f is ☐

The range of f is ☐

39. $f(x) = 5x^2 + 10x + 12$

The domain of f is ☐

The range of f is ☐

40. $f(x) = 3x^2 - 12x + 11$

The domain of f is ☐

The range of f is ☐

41. $f(x) = -4x^2 + 32x - 72$

The domain of f is ☐

The range of f is ☐

42. $f(x) = -2x^2 + 24x - 66$

The domain of f is ☐

The range of f is ☐

Sketching Graphs of Quadratic Functions Graph each function by algebraically determining its key features. Then state the domain and range of the function.

43. $f(x) = x^2 - 7x + 12$

44. $f(x) = x^2 + 5x - 14$

45. $f(x) = -x^2 - x + 20$

46. $f(x) = -x^2 + 4x + 21$

47. $f(x) = x^2 - 8x + 16$

48. $f(x) = x^2 + 6x + 9$

49. $f(x) = x^2 - 4$

50. $f(x) = x^2 - 9$

51. $f(x) = x^2 + 6x$

52. $f(x) = x^2 - 8x$

53. $f(x) = -x^2 + 5x$

54. $f(x) = -x^2 + 16$

55. $f(x) = x^2 + 4x + 7$

56. $f(x) = x^2 - 2x + 6$

57. $f(x) = x^2 + 2x - 5$

58. $f(x) = x^2 - 6x + 2$

59. $f(x) = -x^2 + 4x - 1$

60. $f(x) = -x^2 - x + 3$

61. $f(x) = 2x^2 - 4x - 30$

62. $f(x) = 3x^2 + 21x + 36$

Information from Vertex Form

63. Find the minimum value of the function

$$f(x) = 10x^2 - x + 1$$

64. Find the minimum value of the function

$$f(x) = x^2 - 9x + 10$$

65. Find the maximum value of the function

$$f(x) = 5x - 2x^2 - 2$$

66. Find the maximum value of the function

$$f(x) = 6 - \left(3x^2 + 2x\right)$$

67. Find the range of the function

$$f(x) = -\left(4x^2 + 10x + 6\right)$$

68. Find the range of the function

$$f(x) = 4x - 5x^2 + 3$$

69. Find the range of the function

$$f(x) = 6x^2 - 3x - 9$$

70. Find the range of the function

$$f(x) = 7x^2 + 10x - 1$$

71. If a ball is throw straight up with a speed of $66\,\frac{\text{ft}}{\text{s}}$, its height at time t (in seconds) is given by

$$h(t) = -8t^2 + 66t + 2$$

Find the maximum height the ball reaches.

72. If a ball is throw straight up with a speed of $68\,\frac{\text{ft}}{\text{s}}$, its height at time t (in seconds) is given by

$$h(t) = -8t^2 + 68t + 2$$

Find the maximum height the ball reaches.

Challenge

73. Let $f(x) = x^2 + bx + c$. Let b and c be real numbers. Complete the square to find the vertex of $f(x) = x^2 + bx + c$. Write $f(x)$ in vertex form and then state the vertex.

12.3 More on Complex Solutions to Quadratic Equations

When we solve a quadratic equation, sometimes there are no real solutions. In this section we will explore when that happens and what it means on a graph. We will also learn how to handle complex solutions algebraically.

12.3.1 Applications with Real or Complex Solutions

Let's look at an application where we will determine whether the solutions are real or complex. Iman is a pilot and in a stunt plane performance, she plans to dive the plane toward the ground and then back up. The plane's height can be modeled by a quadratic function. If one possible function is h, where $h(t) = \frac{1}{2}t^2 - 5t + 12$, with t standing for time in seconds after the stunt begins, determine whether the plane would hit the ground during the stunt.

To check whether the plane on that flight path would hit the ground, we will solve the equation $h(t) = 0$. We will solve this equation with the quadratic formula. First, we identify that $a = \frac{1}{2}$, $b = -5$ and $c = 12$.

$$\begin{aligned}
t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1/2)(12)}}{2(1/2)} \\
&= \frac{5 \pm \sqrt{25 - 24}}{1} \\
&= 5 \pm \sqrt{1} \\
&= 5 \pm 1
\end{aligned}$$

So, either:

$$t = 6 \qquad \text{or} \qquad t = 4$$

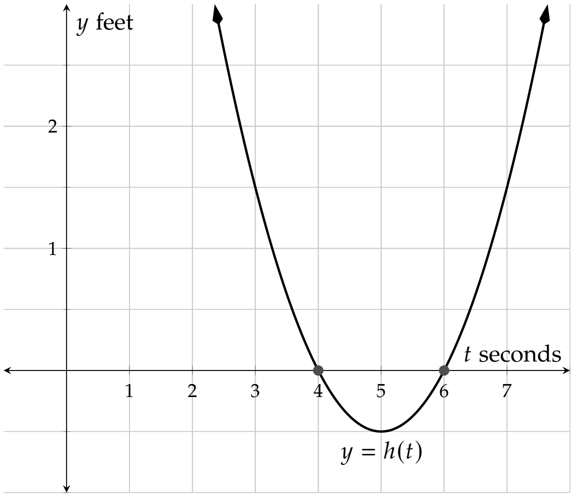


Figure 12.3.2: Graph of $y = h(t)$

This equation has two real solutions and we can see from the graph that the real solutions are the zeros of h. The solution 4 shows that the plane would hit the ground 4 seconds into the stunt, so this is not a good flight path.

To avoid hitting the ground, Iman adjusted the function to p, where $p(t) = \frac{1}{2}t^2 - 5t + 12.5$. To see whether the plane on this flight path would hit the ground, we will solve the equation $p(t) = 0$. We will again use the quadratic formula to solve this equation. We identify that $a = \frac{1}{2}$, $b = -5$ and $c = 12.5$.

$$\begin{aligned}
t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\
&= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1/2)(12.5)}}{2(1/2)}\\
&= \frac{5 \pm \sqrt{25 - 25}}{1}\\
&= 5 \pm \sqrt{0}\\
&= 5 \pm 0\\
&= 5
\end{aligned}$$

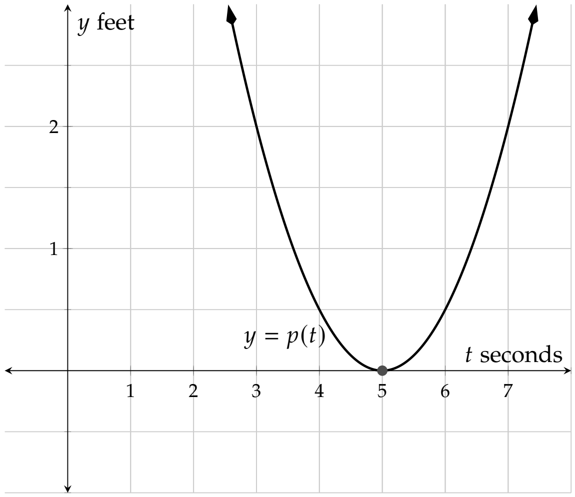


Figure 12.3.3: Graph of $y = p(t)$

This equation has one real solution because p has one zero. This time the plane would hit the ground 5 seconds into the stunt. This is also not a good flight path.

Iman again adjusted the flight path to q, where $q(t) = \frac{1}{2}t^2 - 5t + 13$. We will solve the equation $q(t) = 0$ using the quadratic formula. Identify that $a = \frac{1}{2}$, $b = -5$ and $c = 13$.

$$\begin{aligned}
t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\
&= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1/2)(13)}}{2(1/2)}\\
&= \frac{5 \pm \sqrt{25 - 26}}{1}\\
&= 5 \pm \sqrt{-1}
\end{aligned}$$

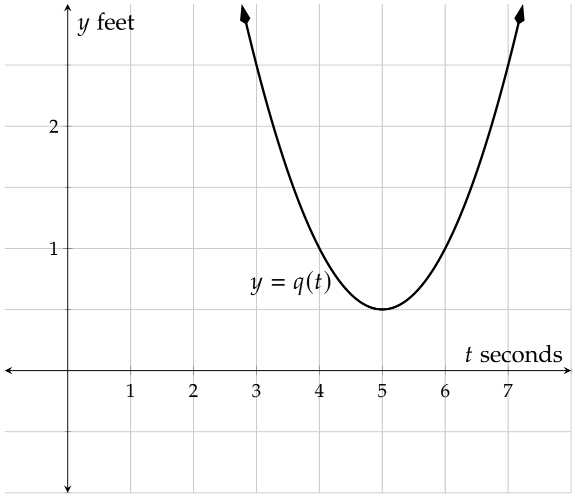


Figure 12.3.4: Graph of $y = q(t)$

Because the radicand is negative, there are no real solutions and the function has no horizontal intercepts. This means the plane will not touch the ground and Iman can complete her stunt using this path.

In general, the radicand of the quadratic formula, $b^2 - 4ac$ is called the **discriminant**. The sign of the discriminant will tell us how many horizontal intercepts a quadratic function will have

- When a quadratic function h has two horizontal intercepts, the equation $h(t) = 0$ has two real solutions. The discriminant will be a positive number so that the ± from the quadratic formula will provide two solutions.

- When a quadratic function p has one horizontal intercept, the equation $p(t) = 0$ has one real solution. The discriminant will be zero so that the ± from the quadratic formula will provide only one solution.
- When a quadratic function q has no horizontal intercepts, the equation $q(t) = 0$ has no real solutions, but it has two complex solutions. The discriminant will be a negative number so that the $\sqrt{}$ from the quadratic formula will provide imaginary numbers, and then the ± will provide two complex solutions.

Example 12.3.5 Futsal[a] is a form of what is usually called soccer in the United States. The game is played on a hard court surface and is usually indoors. The ceiling is out of bounds, so if the ball hits the ceiling it goes to the opposing team.

Borna kicks the ball from the ground with an upward velocity of 8 meters per second. The ball's height in meters can be modeled by the quadratic function h, where $h(t) = -4.9t^2 + 8t$, with t standing for time in seconds after the ball was kicked. If the ceiling height is 4 meters, the minimum height allowed by regulation, determine whether the ball will hit the ceiling.

Explanation. To see whether their ball will hit the ceiling, we will solve the equation $h(t) = 4$. We could complete the square or use the quadratic formula. Because this equation has decimal coefficients we will use the quadratic formula. We put the equation in standard form and identify that $a = -4.9$, $b = 8$ and $c = -4$.

$$\begin{aligned} -4.9t^2 + 8t &= 4 \\ -4.9t^2 + 8t - 4 &= 0 \end{aligned}$$

$$\begin{aligned} t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-8 \pm \sqrt{8^2 - 4(-4.9)(-4)}}{2(-4.9)} \\ &= \frac{-8 \pm \sqrt{64 - 78.4}}{-9.8} \\ &= \frac{-8 \pm \sqrt{-14.4}}{-9.8} \end{aligned}$$

The radicand is negative so we can conclude that there are no real solutions to the equation $h(t) = 4$. That means the parabola will not cross the line $y = 4$ and the ball will not hit the ceiling.

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[a] en.wikipedia.org/wiki/Futsal

Example 12.3.6 Emma kicks the ball from the ground with an upward velocity of 10 meters per second. This gives us the quadratic function for the height of the ball $h(t) = -4.9t^2 + 10t$, with t standing for time in seconds after the ball was kicked. If the ceiling height is 4.5 meters, determine whether the ball will hit the ceiling.

Explanation. To see whether her ball will hit the ceiling, we will solve the equation $h(t) = 4.5$. We will use the quadratic formula because this equation has decimal coefficients. We put the equation in standard form and identify that $a = -4.9$, $b = 10$ and $c = -4.5$.

$$\begin{aligned} -4.9t^2 + 10t &= 4.5 \\ -4.9t^2 + 10t - 4.5 &= 0 \end{aligned}$$

$$\begin{aligned} t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-10 \pm \sqrt{10^2 - 4(-4.9)(-4.5)}}{2(-4.9)} \\ &= \frac{-10 \pm \sqrt{100 - 88.2}}{-9.8} \\ &= \frac{-10 \pm \sqrt{11.8}}{-9.8} \end{aligned}$$

The radicand is positive so there are two real solutions to the equation $h(t) = 4.5$. That means the parabola will cross the line $y = 4.5$ and the ball will hit the ceiling.

12.3.2 Solving Equations with Complex Solutions

In a physical context we may only want to know whether solutions are real or complex. Or we may want to find the solutions. When the radicand is negative, we need to go into the complex number system. First we will revisit the definition of complex numbers. Recall that i is defined as $\sqrt{-1}$.

Definition 12.3.7 Complex Number. A **complex number**[1] is a number that can be expressed in the form $a + bi$, where a and b are real numbers and i is the imaginary unit. In this expression, a is the **real part** and b (not bi) is the **imaginary part**.

Here are some examples of solving equations that have complex solutions.

Example 12.3.8 Solve for s in $s^2 - 10s = -34$.

Explanation. We will use the method of completing the square. To do so, we need to add $\left(\frac{b}{2}\right)^2 = (-5)^2 = 25$ to both sides to complete the square.

$$\begin{aligned} s^2 - 10s &= -34 \\ s^2 - 10s + 25 &= -34 + 25 \\ (s - 5)^2 &= -9 \end{aligned}$$

$$\begin{aligned} s - 5 &= -\sqrt{-9} & &\text{or} & s - 5 &= \sqrt{-9} \\ s - 5 &= -\sqrt{9} \cdot \sqrt{-1} & &\text{or} & s - 5 &= \sqrt{9} \cdot \sqrt{-1} \\ s - 5 &= -3i & &\text{or} & s - 5 &= 3i \\ s &= 5 - 3i & &\text{or} & s &= 5 + 3i \end{aligned}$$

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[1] en.wikipedia.org/wiki/Complex_number

> The solution set is $\{5-3i, 5+3i\}$.

Checkpoint 12.3.9. Solve for x in $2x^2+12x+26=0$.

Explanation. We will use the completing-the-square method again. To do so, we first need to divide both sides by the leading coefficient, 2.

$$\begin{aligned}2x^2+12x&=-26\\ \frac{2x^2}{2}+\frac{12x}{2}&=\frac{-26}{2}\\ x^2+6x&=-13\end{aligned}$$

Now we can add $\left(\frac{b}{2}\right)^2=(3)^2=9$ to both sides to complete the square.

$$\begin{aligned}x^2+6x+9&=-13+9\\ (x+3)^2&=-4\end{aligned}$$

$$\begin{aligned}x+3&=-\sqrt{-4} &&\text{or}\quad & x+3&=\sqrt{-4}\\ x+3&=-\sqrt{4}\cdot\sqrt{-1} &&\text{or}\quad & x+3&=\sqrt{4}\cdot\sqrt{-1}\\ x+3&=-2i &&\text{or}\quad & x+3&=2i\\ x&=-3-2i &&\text{or}\quad & x&=-3+2i\end{aligned}$$

The solution set is $\{-3-2i, -3+2i\}$.

The quadratic formula can also be used to solve for complex solutions. Here is an example where it makes more sense to use the quadratic formula.

> **Example 12.3.10** Solve for x in $5x^2-2x=-3$.
>
> **Explanation**. If we were to complete the square, we would divide both sides by 5 and have lots of fractions in our equation. Instead, we will put the equation in standard form and use the quadratic formula.
>
> $$\begin{aligned}5x^2-2x&=-3\\ 5x^2-2x+3&=0\end{aligned}$$
>
> We identify that $a=5$, $b=-2$ and $c=3$ and substitute them into the Quadratic Formula:
>
> $$\begin{aligned}x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\ &=\frac{-(-2)\pm\sqrt{(-2)^2-4(5)(3)}}{2(5)}\\ &=\frac{2\pm\sqrt{4-60}}{10}\\ &=\frac{2\pm\sqrt{-56}}{10}\\ &=\frac{2\pm\sqrt{-1\cdot 4\cdot 14}}{10}\end{aligned}$$

$$= \frac{2 \pm \sqrt{-1} \cdot \sqrt{4} \cdot \sqrt{14}}{10}$$
$$= \frac{2 \pm i \cdot 2 \cdot \sqrt{14}}{10}$$

Now we need to put the solutions in standard form which is $a + bi$.

$$x = \frac{2}{10} \pm \frac{2i\sqrt{14}}{10}$$
$$x = \frac{1}{5} \pm \frac{\sqrt{14}}{5}i$$

The solution set is $\left\{\frac{1}{5} - \frac{\sqrt{14}}{5}i, \frac{1}{5} + \frac{\sqrt{14}}{5}i\right\}$.

Exercises

Review and Warmup Simplify the radical and write it into a complex number.

1. $\sqrt{-30} =$ ☐

2. $\sqrt{-105} =$ ☐

3. $\sqrt{-56} =$ ☐

4. $\sqrt{-28} =$ ☐

5. $\sqrt{-252} =$ ☐

6. $\sqrt{-112} =$ ☐

Real Versus Complex Solutions Determine the nature of the solutions to this quadratic equation.

7. $-7r^2 - 18r - 10 = 0$

(☐ two real solutions ☐ two non-real solutions ☐ one doubled real solution ☐ none of these)

8. $7x^2 - x + 5 = 0$

(☐ two real solutions ☐ two non-real solutions ☐ one doubled real solution ☐ none of these)

9. $8x^2 - x + 1 = 0$

(☐ two real solutions ☐ two non-real solutions ☐ one doubled real solution ☐ none of these)

10. $-8y^2 - 8y - 6 = 0$

(☐ two real solutions ☐ two non-real solutions ☐ one doubled real solution ☐ none of these)

11. $6y^2 + 2y + 9 = 0$

(☐ two real solutions ☐ two non-real solutions ☐ one doubled real solution ☐ none of these)

12. $-z^2 - 4z + 3 = 0$

(☐ two real solutions ☐ two non-real solutions ☐ one doubled real solution ☐ none of these)

13. $-9z^2 + z - 3 = 0$

(☐ two real solutions ☐ two non-real solutions ☐ one doubled real solution ☐ none of these)

14. $5t^2 + 5t - 9 = 0$

(☐ two real solutions ☐ two non-real solutions ☐ one doubled real solution ☐ none of these)

Solving Equations with Complex Solutions Solve the quadratic equation. Solutions could be complex numbers.

15. $t^2 = -25$

16. $t^2 = -4$

17. $-4x^2 - 1 = 255$

18. $10x^2 - 2 = -252$

19. $-x^2 - 8 = 9$

20. $-2y^2 + 3 = 7$

21. $3(y - 8)^2 - 2 = -50$

22. $-9(r + 2)^2 - 3 = 897$

23. $5r^2 - 8 = -233$

24. $-7t^2 + 9 = 93$

25. $t^2 - 6t + 10 = 0$

26. $x^2 + 2x + 5 = 0$

27. $x^2 - 4x + 7 = 0$

28. $x^2 + 6x + 16 = 0$

Applications

29. A remote control aircraft will perform a stunt by flying toward the ground and then up. Its height can be modeled by the function $h(t) = 1.7t^2 - 17t + 42.5$. The plane (☐ will ☐ will not) hit the ground during this stunt.

30. A remote control aircraft will perform a stunt by flying toward the ground and then up. Its height can be modeled by the function $h(t) = 1.2t^2 - 14.4t + 39.2$. The plane (☐ will ☐ will not) hit the ground during this stunt.

31. A submarine is traveling in the sea. Its depth can be modeled by $d(t) = -0.2t^2 + 2.8t - 5.8$, where t stands for time in seconds. The submarine (☐ will ☐ will not) hit the sea surface along this route.

32. A submarine is traveling in the sea. Its depth can be modeled by $d(t) = -0.9t^2 + 14.4t - 57.6$, where t stands for time in seconds. The submarine (☐ will ☐ will not) hit the sea surface along this route.

12.4 Complex Number Operations

Complex numbers[1] are used in many math, science and engineering applications. In this section, we will learn the basics of complex number operations.

12.4.1 Adding and Subtracting Complex Numbers

Adding and subtracting complex numbers is just like combining like terms. We combine the terms that are real and the terms that are imaginary. Here are some examples

> **Example 12.4.2** Simplify the expression $(1-7i)+(5+4i)$.
>
> $$\begin{aligned}(1-7i)+(5+4i) &= 1+5-7i+4i\\ &= 6-3i\end{aligned}$$

> **Example 12.4.3** Simplify the expression $(3-10i)-(4-6i)$.
>
> $$\begin{aligned}(3-10i)-(4-6i) &= 3-10i-4+6i\\ &= -1-4i\end{aligned}$$

Checkpoint 12.4.4. Simplify the expression $(8+2i)-(5+3i)$.

Explanation.

$$\begin{aligned}(8+2i)-(5+3i) &= 8+2i-5-3i\\ &= 3-i\end{aligned}$$

12.4.2 Multiplying Complex Numbers

Now let's learn how to multiply complex numbers. It is very similar to multiplying polynomials.

> **Example 12.4.5** Simplify the expression $2i(3-2i)$.
>
> We distribute the $2i$ to both terms, then we simplify any powers of i.
>
> $$\begin{aligned}2i(3-2i) &= 2i\cdot 3-2i\cdot 2i\\ &= 6i-4i^2\\ &= 6i-4(-1)\\ &= 6i+4\\ &= 4+6i\end{aligned}$$
>
> Note that we always write a complex number in standard form, which is $a+bi$.

When we multiply two complex numbers we can use the distributive method, FOIL method, or generic rectangles. Here is an example of each method.

---

[1] en.wikipedia.org/wiki/Complex_number#Applications

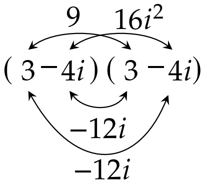
Example 12.4.6 Multiply $(1+5i)(2-7i)$.

We will use the distributive method to multiply the two binomials.

$$\begin{aligned}(1+5i)(2-7i) &= 2(1+5i)-7i(1+5i)\\ &= 2+10i-7i-35i^2\\ &= 2+10i-7i-35(-1)\\ &= 2+3i+35\\ &= 37+3i\end{aligned}$$

Example 12.4.7 Expand and simplify the expression $(3-4i)^2$.

Explanation. We will use the FOIL method to expand this perfect square.



$$\begin{aligned}(3-4i)^2 &= (3-4i)(3-4i)\\ &= 9-12i-12i+16i^2\\ &= 9-24i+16(-1)\\ &= 9-24i-16\\ &= -7-24i\end{aligned}$$

Figure 12.4.8: Using the FOIL method to expand $(3-4i)^2$.

Example 12.4.9 Multiply $(3+4i)(3-4i)$.

Explanation. We will use the Generic Rectangle Method to multiply those two binomials.

	3	$4i$
3	9	$12i$
$-4i$	$-12i$	$-16i^2$

$$\begin{aligned}(3+4i)(3-4i) &= 9+12i-12i-16i^2\\ &= 9-16(-1)\\ &= 9+16\\ &= 25\end{aligned}$$

Figure 12.4.10: Using the Generic Rectangle Method to multiply $(3+4i)(3-4i)$.

As the last example shows, it is possible to multiply two complex numbers and get a real number result. Notice that the middle terms, $12i$ and $-12i$, are opposites, which makes the result a real number. This happens when we multiply a sum and difference of the same real and imaginary parts, called **complex conjugates**. This pair of factors results in the difference of squares:

$$(a+b)(a-b)=a^2-b^2.$$

Example 12.4.11 Here is an example using the sum and difference formula to multiply $(5+2i)(5-2i)$:

$$\begin{aligned}(5+2i)(5-2i)&=5^2-(2i)^2\\&=25-4i^2\\&=25-4(-1)\\&=25+4\\&=29\end{aligned}$$

Checkpoint 12.4.12. Multiply $(7-9i)(7+9i)$.

Explanation.

$$\begin{aligned}(7-9i)(7+9i)&=(7)^2-(9i)^2\\&=49-81i^2\\&=49-81(-1)\\&=49+81\\&=130\end{aligned}$$

12.4.3 Dividing Complex Numbers

When we divide by i we use a process that is similar to rationalizing the denominator. We use the property $\sqrt{x}\cdot\sqrt{x}=x$ when we rationalize the denominator, and we use the property $i\cdot i=-1$ when we have complex numbers. Let's compare these two problems $\frac{2}{\sqrt{2}}$ and $\frac{2}{i}$:

$$\begin{aligned}\frac{2}{\sqrt{2}}&=\frac{2\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}\\&=\frac{2\sqrt{2}}{2}\\&=\sqrt{2}\end{aligned}$$

$$\begin{aligned}\frac{2}{i}&=\frac{2\cdot i}{i\cdot i}\\&=\frac{2i}{-1}\\&=-2i\end{aligned}$$

Example 12.4.13 Rationalize the denominator in the expression $-\frac{7}{4i}$.

$$\begin{aligned}-\frac{7}{4i}&=-\frac{7\cdot i}{4i\cdot i}\\&=-\frac{7i}{4(-1)}\\&=\frac{7i}{4}\end{aligned}$$

$$= \frac{7}{4}i$$

Checkpoint 12.4.14. Rationalize the denominator in the expression $\frac{5}{3i}$.

Explanation.

$$\begin{aligned}\frac{5}{3i} &= \frac{5\cdot i}{3i\cdot i}\\ &= \frac{5i}{3(-1)}\\ &= -\frac{5i}{3}\\ &= -\frac{5}{3}i\end{aligned}$$

When the denominator is in the form $a+bi$, we need to use the complex conjugate to remove the imaginary terms from the denominator. Here is an example.

Example 12.4.15 Simplify the expression $\frac{1}{4+3i}$.

Explanation. To get a real result in the denominator we multiply the numerator and denominator by $4-3i$, and we have:

$$\begin{aligned}\frac{1}{4+3i} &= \frac{1}{4+3i}\cdot\frac{(4-3i)}{(4-3i)}\\ &= \frac{4-3i}{16-12i+12i-9i^2}\\ &= \frac{4-3i}{16-9(-1)}\\ &= \frac{4-3i}{16+9}\\ &= \frac{4-3i}{25}\\ &= \frac{4}{25}-\frac{3}{25}i\end{aligned}$$

Note that we always write complex numbers in standard form which is $a + bi$.

Now we can divide two complex numbers as in the next example.

Example 12.4.16 Simplify the expression $\frac{1+2i}{2-4i}$.

Explanation. To divide complex numbers, we rationalize the denominator using the conjugate $2+4i$:

$$\begin{aligned}\frac{1+2i}{2-4i} &= \frac{(1+2i)}{(2-4i)}\cdot\frac{(2+4i)}{(2+4i)}\\ &= \frac{2+4i+4i+8i^2}{4+8i-8i-16i^2}\\ &= \frac{2+8i+8(-1)}{4-16(-1)}\\ &= \frac{2+8i-8}{4+16}\end{aligned}$$

$$
\begin{aligned}
&= \frac{-6+8i}{20}\\
&= \frac{-6}{20} + \frac{8i}{20}\\
&= -\frac{3}{10} + \frac{2}{5}i
\end{aligned}
$$

Checkpoint 12.4.17. Simplify the expression $\frac{4-7i}{5+i}$.

Explanation. To divide, we rationalize the denominator using the conjugate $5-i$:

$$
\begin{aligned}
\frac{4-7i}{5+i} &= \frac{(4-7i)}{(5+i)} \cdot \frac{(5-i)}{(5-i)}\\
&= \frac{20-4i-35i+7i^2}{25-5i+5i-i^2}\\
&= \frac{20-39i+7(-1)}{25-1(-1)}\\
&= \frac{20-39i-7}{25+1}\\
&= \frac{13-39i}{26}\\
&= \frac{13}{26} - \frac{39i}{26}\\
&= \frac{1}{2} - \frac{3}{2}i
\end{aligned}
$$

Exercises

Adding and Subtracting Complex Numbers

1. Add up the following complex numbers:

$(-7+6i)+(2+5i) = \boxed{}$

2. Add up the following complex numbers:

$(-4-3i)+(12-2i) = \boxed{}$

3. Subtract the following complex numbers:

$(-1-11i)-(-3-8i) = \boxed{}$

4. Subtract the following complex numbers:

$(1+5i)-(8+10i) = \boxed{}$

5. Write the complex number in standard form.

$$(3-3i)+(-6+2i)$$

6. Write the complex number in standard form.

$$(6-10i)+(3-4i)$$

7. Write the complex number in standard form.

$$(8+3i)+(-9-9i)$$

8. Write the complex number in standard form.

$$(10-4i)+(6i)$$

9. Write the complex number in standard form.

$$(-8+10i)-(8)$$

10. Write the complex number in standard form.

$$(-6+2i)-(-4-6i)$$

11. Write the complex number in standard form.

$$(-4-5i)-(5+9i)$$

12. Write the complex number in standard form.

$$(-1+9i)-(-7+4i)$$

Multiplying Complex Numbers

13. Multiply the following complex numbers:

$i(1+2i) = \boxed{}$

14. Multiply the following complex numbers:

$i(4-7i) = \boxed{}$

15. Multiply the following complex numbers:

$(7+9i)(-3+8i) = \boxed{}$

16. Multiply the following complex numbers:

$(10+i)(8+i) = \boxed{}$

17. Multiply the following complex numbers:

$(12-8i)^2 = \boxed{}$

18. Multiply the following complex numbers:

$(-10+8i)^2 = \boxed{}$

19. Multiply the following complex numbers:

$(-7-11i)(-7+11i) = \boxed{}$

20. Multiply the following complex numbers:

$(-4-9i)(-4+9i) = \boxed{}$

21. Write the complex number in standard form.

$$(-1+6i)(8-7i)$$

22. Write the complex number in standard form.

$$(1-i)(-4+8i)$$

23. Write the complex number in standard form.

$$(3-8i)(5+3i)$$

24. Write the complex number in standard form.

$$(6+5i)(-7-3i)$$

Dividing Complex Numbers

25. Rewrite the following expression into the form of a+b*i*:

$\frac{6}{i} =$

26. Rewrite the following expression into the form of a+b*i*:

$\frac{2}{i} =$

27. Rewrite the following expression into the form of a+b*i*:

$\frac{-8+4i}{-2+6i} =$

28. Rewrite the following expression into the form of a+b*i*:

$\frac{-2+4i}{-4-2i} =$

29. Rewrite the following expression into the form of a+b*i*:

$\frac{-3-8i}{-5+8i} =$

30. Rewrite the following expression into the form of a+b*i*:

$\frac{3-8i}{2+3i} =$

31. Write the complex number in standard form.

$$\frac{1-4i}{-9-2i}$$

32. Write the complex number in standard form.

$$\frac{3+10i}{-8i}$$

33. Write the complex number in standard form.

$$\frac{6+3i}{8+7i}$$

34. Write the complex number in standard form.

$$\frac{8-5i}{-4+i}$$

12.5 More on Quadratic Functions Chapter Review

12.5.1 Graphs and Vertex Form

In Section 12.1 we covered the use of technology in analyzing quadratic functions, the vertex form of a quadratic function and how it affects horizontal and vertical shifts of the graph of a parabola, and the factored form of a quadratic function.

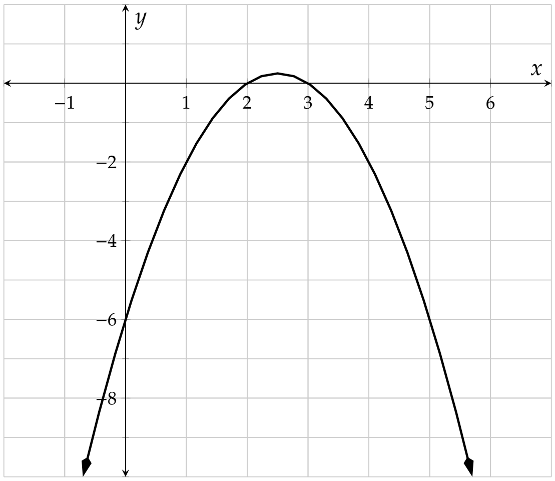
Example 12.5.1 Exploring Quadratic Functions with Graphing Technology. Use technology to graph and make a table of the quadratic function g defined by $g(x) = -x^2 + 5x - 6$ and find each of the key points or features.

a. Find the vertex.
b. Find the vertical intercept.
c. Find the horizontal intercept(s).
d. Find $g(-1)$.
e. Solve $g(x) = -6$ using the graph.
f. Solve $g(x) \leq -6$ using the graph.
g. State the domain and range of the function.

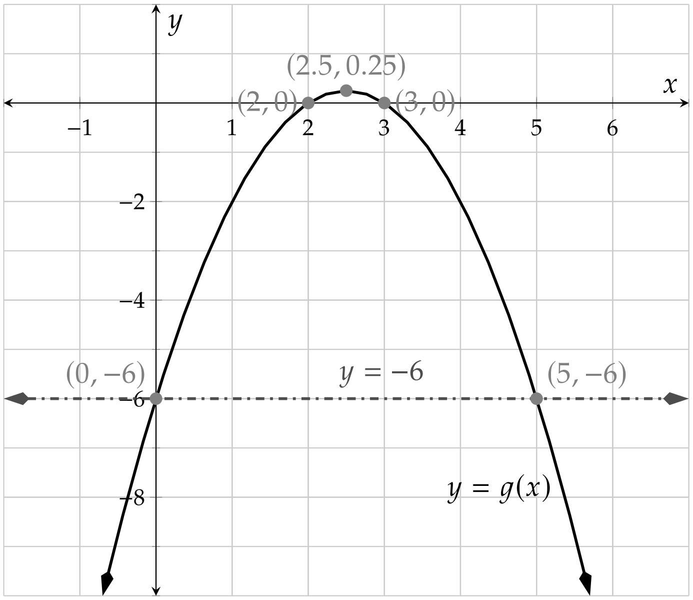
Explanation.

The specifics of how to use any one particular technology tool vary. Whether you use an app, a physical calculator, or something else, a table and graph should look like:

x	$g(x)$
-1	-12
0	-6
1	-2
2	0
2	0
3	0
4	-2



Additional features of your technology tool can enhance the graph to help answer these questions. You may be able to make the graph appear like:



a. The vertex is $(2.5, 0.25)$.

b. The vertical intercept is $(0, -6)$.

c. The horizontal intercepts are $(2, 0)$ and $(3, 0)$.

d. $g(-1) = -2$.

e. The solutions to $g(x) = -6$ are the x-values where $y = 6$. We graph the horizontal line $y = -6$ and find the x-values where the graphs intersect. The solution set is $\{0, 5\}$.

f. The solutions are all x-values where the function below (or touching) the line $y = -6$. The solution set is $(-\infty, 0] \cup [5, \infty)$.

g. The domain is $(-\infty, \infty)$ and the range is $(-\infty, 0.25]$.

Example 12.5.2 The Vertex Form of a Parabola. Recall that the vertex form of a quadratic function tells us the location of the vertex of a parabola.

a. State the vertex of the quadratic function $r(x) = -8(x + 1)^2 + 7$.

b. State the vertex of the quadratic function $u(x) = 5(x - 7)^2 - 3$.

c. Write the formula for a parabola with vertex $(-5, 3)$ and $a = 2$.

d. Write the formula for a parabola with vertex $(1, -17)$ and $a = -4$.

Explanation.

a. The vertex of the quadratic function $r(x) = -8(x+1)^2 + 7$ is $(-1, 7)$.

b. The vertex of the quadratic function $u(x) = 5(x-7)^2 - 3$ is $(7, -3)$.

c. The formula for a parabola with vertex $(-5, 3)$ and $a = 2$ is $y = 2(x+5)^2 + 3$.

d. The formula for a parabola with vertex $(1, -17)$ and $a = -4$ is $y = 4(x-1)^2 - 17$.

Example 12.5.3 Horizontal and Vertical Shifts. Identify the horizontal and vertical shifts compared with $f(x) = x^2$.

a. $s(x) = (x+1)^2 + 7$.

b. $v(x) = (x-7)^2 - 3$.

Explanation.

a. The graph of the quadratic function $s(x) = -8(x+1)^2 + 7$ is the same as the graph of $f(x) = x^2$ shifted to the left 1 unit and up 7 units.

b. The graph of the quadratic function $v(x) = 5(x-7)^2 - 3$ is the same as the graph of $f(x) = x^2$ shifted to the right 7 units and down 3 units.

Example 12.5.4 The Factored Form of a Parabola. Recall that the factored form of a quadratic function tells us the horizontal intercepts very quickly.

a. $n(x) = 13(x-1)(x+6)$.

b. $p(x) = -6(x - \frac{2}{3})(x + \frac{1}{2})$.

Explanation.

a. The horizontal intercepts of n are $(1, 0)$ and $(-6, 0)$.

b. The horizontal intercepts of p are $(\frac{2}{3}, 0)$ and $(-\frac{1}{2}, 0)$.

12.5.2 Completing the Square

In Section 12.2 we covered how to complete the square to both solve quadratic equations in one variable and to put quadratic functions into vertex form.

Example 12.5.5 Solving Quadratic Equations by Completing the Square. Solve the equations by completing the square.

a. $k^2 - 18k + 1 = 0$

b. $4p^2 - 3p = 2$

Explanation.

a. To complete the square in the equation $k^2 - 18k + 1 = 0$, we first we will first move the constant term to the right side of the equation. Then we will use Fact 12.2.2 to find $\left(\frac{b}{2}\right)^2$ to add to both sides.

$$k^2 - 18k + 1 = 0$$

$$k^2 - 18k = -1$$

In our case, $b = -18$, so $\left(\frac{b}{2}\right)^2 = \left(\frac{-18}{2}\right)^2 = 81$

$$\begin{aligned} k^2 - 18k + 81 &= -1 + 81 \\ (k-9)^2 &= 80 \end{aligned}$$

$$\begin{aligned} k - 9 &= -\sqrt{80} & \text{or} \quad && k - 9 &= \sqrt{80} \\ k - 9 &= -4\sqrt{5} & \text{or} \quad && k - 9 &= 4\sqrt{5} \\ k &= 9 - 4\sqrt{5} & \text{or} \quad && k &= 9 + 4\sqrt{5} \end{aligned}$$

The solution set is $\{9 + 4\sqrt{5}, 9 - 4\sqrt{5}\}$.

b. To complete the square in the equation $4p^2 - 3p = 2$, we first divide both sides by 4 since the leading coefficient is 4.

$$\begin{aligned} \frac{4p^2}{4} - \frac{3p}{4} &= \frac{2}{4} \\ p^2 - \frac{3}{4}p &= \frac{1}{2} \\ p^2 - \frac{3}{4}p &= \frac{1}{2} \end{aligned}$$

Next, we will complete the square. Since $b = -\frac{3}{4}$, first,

$$\frac{b}{2} = \frac{-\frac{3}{4}}{2} = -\frac{3}{8} \tag{12.5.1}$$

and next, squaring that, we have

$$\left(-\frac{3}{8}\right)^2 = \frac{9}{64}. \tag{12.5.2}$$

So we will add $\frac{9}{64}$ from Equation (12.5.2) to both sides of the equation:

$$\begin{aligned} p^2 - \frac{3}{4}p + \frac{9}{64} &= \frac{1}{2} + \frac{9}{64} \\ p^2 - \frac{3}{4}p + \frac{9}{64} &= \frac{32}{64} + \frac{9}{64} \\ p^2 - \frac{3}{4}p + \frac{9}{64} &= \frac{41}{64} \end{aligned}$$

Here, remember that we always factor with the number found in the first step of completing the square, Equation (12.5.1).

$$\left(p - \frac{3}{8}\right)^2 = \frac{41}{64}$$

$$p - \frac{3}{8} = -\frac{\sqrt{41}}{8} \quad \text{or} \quad p - \frac{3}{8} = \frac{\sqrt{41}}{8}$$

$$p = \frac{3}{8} - \frac{\sqrt{41}}{8} \qquad \text{or} \qquad p = \frac{3}{8} + \frac{\sqrt{41}}{8}$$
$$p = \frac{3 - \sqrt{41}}{8} \qquad \text{or} \qquad p = \frac{3 + \sqrt{41}}{8}$$

The solution set is $\left\{\frac{3-\sqrt{41}}{8}, \frac{3+\sqrt{41}}{8}\right\}$.

Example 12.5.6 Putting Quadratic Functions in Vertex Form. Write a formula in vertex form for the function T defined by $T(x) = 4x^2 + 20x + 24$.

Explanation. Before we can complete the square, we will factor the 4 out of the first two terms. Don't be tempted to factor the 4 out of the constant term.

$$T(x) = 4\left(x^2 + 5x\right) + 24$$

Now we will complete the square inside the parentheses by adding and subtracting $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$.

$$T(x) = 4\left(x^2 + 5x + \frac{25}{4} - \frac{25}{4}\right) + 24$$

Notice that the constant that we subtracted is inside the parentheses, but it will not be part of our perfect square trinomial. In order to bring it outside, we need to multiply it by 4. We are distributing the 4 to that term so we can combine it with the outside term.

$$\begin{aligned} T(x) &= 4\left(\left(x^2 + 5x + \frac{25}{4}\right) - \frac{25}{4}\right) + 24 \\ &= 4\left(x^2 + 5x + \frac{25}{4}\right) - 4 \cdot \frac{25}{4} + 24 \\ &= 4\left(x + \frac{5}{2}\right)^2 - 25 + 24 \\ &= 4\left(x + \frac{5}{2}\right)^2 - 1 \end{aligned}$$

Note that The vertex is $\left(-\frac{5}{2}, -1\right)$.

Example 12.5.7 Graphing Quadratic Functions by Hand. Graph the function H defined by $H(x) = -x^2 - 8x - 15$ by determining its key features algebraically.

Explanation. To start, we'll note that this function opens downward because the leading coefficient, -1, is negative.

Now we will complete the square to find the vertex. We will factor the -1 out of the first two terms, and then add and subtract $\left(\frac{8}{2}\right)^2 = 4^2 = 16$ on the right side.

$$\begin{aligned} H(x) &= -\left[x^2 + 8x\right] - 15 \\ &= -\left[x^2 + 8x + 16 - 16\right] - 15 \\ &= -\left[\left(x^2 + 8x + 16\right) - 16\right] - 15 \end{aligned}$$

$$
\begin{aligned}
&= -\left(x^2+8x+16\right)-(-1\cdot 16)-15\\
&= -\left(x+4\right)^2+16-15\\
&= -\left(x+4\right)^2+1
\end{aligned}
$$

The vertex is $(-4,1)$ so the axis of symmetry is the line $x=-4$.

To find the y-intercept, we'll replace x with 0 or read the value of c from the function in standard form:

$$
\begin{aligned}
H(0)&=-(0)^2-8(0)-15\\
&=-15
\end{aligned}
$$

The y-intercept is $(0,-15)$ and we will find its symmetric point on the graph, which is $(-8,-15)$.

Next, we'll find the horizontal intercepts. We see this function factors so we will write the factored form to get the horizontal intercepts.

$$
\begin{aligned}
H(x)&=-x^2-8x-15\\
&=-\left(x^2+8x+15\right)\\
&=-(x+3)(x+5)
\end{aligned}
$$

The x-intercepts are $(-3,0)$ and $(-5,0)$.

Now we will plot all of the key points and draw the parabola.

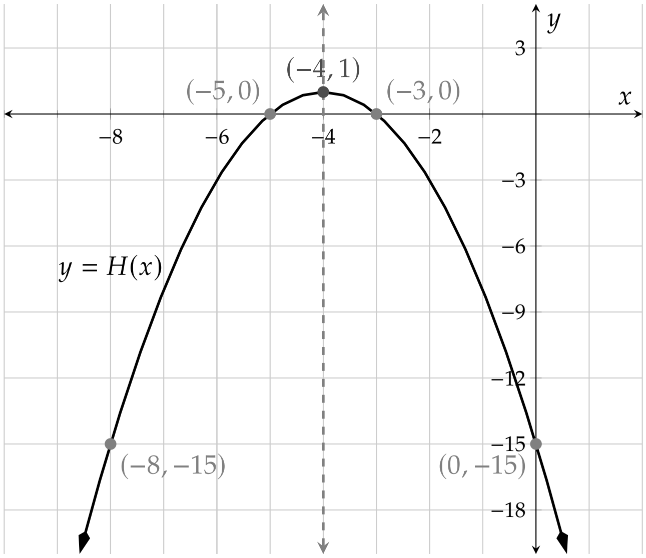


Figure 12.5.8: The graph of $y=-x^2-8x-15$.

12.5.3 More on Complex Solutions to Quadratic Equations

In Section 12.3 we covered the definition of a complex number, and discussed both quadratic applications and equations where complex numbers appear as solutions.

Example 12.5.9 Applications with Real or Complex Solutions. One day, Samar was bouncing a ball inside the house. The trajectory of his bounce followed the quadratic function $H(t) = -16t^2 + 24t$, where $H(t)$ describes the height of the ball, in feet, at time t seconds after it bounced off the ground. If the ceilings in Samar's house were 10 feet tall, find out if the ball will hit the ceiling.

Explanation. To find out if the ball will hit the ceiling, we need to set the formula for the function equal to 10 and solve.

$$\begin{aligned} H(t) &= -16t^2 + 24t \\ 10 &= -16t^2 + 24t \\ 0 &= -16t^2 + 24t - 10 \end{aligned}$$

This is a quadratic equation where verything is divisible by 2. We will divide every term by 2 which can simplify the process.

$$\begin{aligned} \frac{0}{2} &= \frac{-16t^2}{2} + \frac{24t}{2} - \frac{10}{2} \\ 0 &= -8t^2 + 12t - 5 \end{aligned}$$

Since the equation doesn't seem to factor easily, we will use the quadratic formula to solve it. Note that $a = -8$, $b = 12$, and $c = -5$.

$$\begin{aligned} t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ t &= \frac{-(12) \pm \sqrt{(12)^2 - 4(-8)(-5)}}{2(-8)} \\ t &= \frac{-12 \pm \sqrt{-16}}{-16} \end{aligned}$$

Note that the discriminant is negative, which means that the equation has no real solutions. Just for practice, we will finish the simplification process, but we are ready to make our conclusion here.

$$\begin{aligned} t &= \frac{-12 \pm \sqrt{16 \cdot -1}}{-16} \\ t &= \frac{-12 \pm 4i}{-16} \\ t &= \frac{-12}{-16} \pm \frac{4i}{-16} \\ t &= \frac{3}{4} \mp \frac{i}{4} \end{aligned}$$

Since the solutions to the equation are complex numbers, the reality of the situation must be that the ball never does hit the ceiling. Samar's ceiling lights are safe for now.

Example 12.5.10 Solving Equations with Complex Solutions. Solve for x in $3x^2 - 12x + 36 = 0$.

Explanation. We will use the completing-the-square method. To do so, we first need to divide both sides by the leading coefficient, 3.

$$\begin{aligned} 3x^2 - 12x + 36 &= 0 \\ 3x^2 - 12x &= -36 \end{aligned}$$

$$\begin{aligned}\frac{3x^2}{3} - \frac{12x}{3} &= \frac{-36}{3}\\ x^2 - 4x &= -12\end{aligned}$$

Now we can add $\left(\frac{b}{2}\right)^2 = (-2)^2 = 4$ to both sides to complete the square.

$$\begin{aligned}x^2 - 4x + 4 &= -12 + 4\\ (x-2)^2 &= -8\end{aligned}$$

$$\begin{aligned}x - 2 &= -\sqrt{-8} &\quad\text{or}\quad&& x - 2 &= \sqrt{-8}\\ x - 2 &= -\sqrt{4 \cdot -1 \cdot 2} &\quad\text{or}\quad&& x - 2 &= \sqrt{4 \cdot -1 \cdot 2}\\ x - 2 &= -2i\sqrt{2} &\quad\text{or}\quad&& x - 2 &= 2i\sqrt{2}\\ x &= 2 - 2i\sqrt{2} &\quad\text{or}\quad&& x &= 2 + 2i\sqrt{2}\end{aligned}$$

The solution set is $\{2 - 2i\sqrt{2}, 2 + 2i\sqrt{2}\}$.

12.5.4 Complex Number Operations

In Section 12.4 we covered the essential algebra of complex numbers.

Example 12.5.11 Adding and Subtracting Complex Numbers. Simplify the expression $(5-3i)-(1-7i)$.

Explanation.

$$\begin{aligned}(5-3i)-(1-7i) &= 5 - 3i - 1 + 7i\\ &= 4 + 4i\end{aligned}$$

Example 12.5.12 Multiplying Complex Numbers. Multiply $(3+2i)(5-6i)$.

Explanation. We will use the FOIL method to multiply the two binomials.

$$\begin{aligned}(1+5i)(2-7i) &= 15 - 18i + 10i - 12i^2\\ &= 15 - 8i - 12(-1)\\ &= 15 - 8i + 12\\ &= 27 - 8i\end{aligned}$$

Example 12.5.13 Dividing Complex Numbers. Simplify the expression $\frac{3+5i}{5-6i}$.

Explanation. To divide complex numbers, we rationalize the denominator using the conjugate $2+4i$:

$$\begin{aligned}\frac{3+5i}{5-6i} &= \frac{(3+5i)}{(5-6i)} \cdot \frac{(5+6i)}{(5+6i)}\\ &= \frac{15 + 18i + 25i + 30i^2}{25 + 30i - 30i - 36i^2}\end{aligned}$$

$$\begin{aligned}
&= \frac{15 + 43i + 30(-1)}{25 - 36(-1)} \\
&= \frac{15 + 43i - 30}{25 + 36} \\
&= \frac{-15 + 43i}{61} \\
&= -\frac{15}{61} + \frac{43i}{61}
\end{aligned}$$

Exercises

Graphs and Vertex Form

1. Use technology to make a table of values for the function K defined by $K(x) = 3x^2 - 2x - 3$.

x	$K(x)$

2. Use technology to make a table of values for the function f defined by $f(x) = -3x^2 - 8x + 37$.

x	$f(x)$

3. Use technology to make a graph of f where $f(x) = 3x^2 - 6x - 5$.

4. Use technology to make a graph of f where $f(x) = -3x^2 - 8x + 3$.

5. Let $g(x) = -x^2 + x + 3$. Use technology to find the following.

a. The vertex is ______.

b. The y-intercept is ______.

c. The x-intercept(s) is/are ______.

d. The domain of g is ______.

e. The range of g is ______.

f. Calculate $g(3)$. ______.

g. Solve $g(x) = 2$. ______

h. Solve $g(x) > 2$. ______

6. Let $h(x) = -x^2 - 4x - 2$. Use technology to find the following.

a. The vertex is ______.

b. The y-intercept is ______.

c. The x-intercept(s) is/are ______.

d. The domain of h is ______.

e. The range of h is ______.

f. Calculate $h(-1)$. ______.

g. Solve $h(x) = -5$. ______

h. Solve $h(x) \geq -5$. ______

7. An object was launched from the top of a hill with an upward vertical velocity of 110 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 110t + 200$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Find the answer using technology.

seconds after its launch, the object reached its maximum height of feet.

8. An object was launched from the top of a hill with an upward vertical velocity of 130 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 130t + 100$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Find the answer using technology.

seconds after its launch, the object fell to the ground at sea level.

9. Find the vertex of the graph of

$$y = 3(x-7)^2 - 6$$

10. Find the vertex of the graph of

$$y = 6(x-3)^2 - 3$$

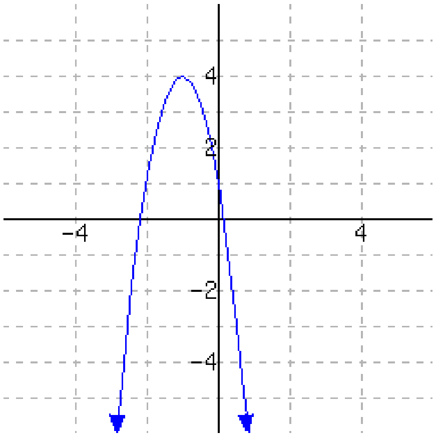
11. Write the vertex form for the quadratic function f, whose vertex is $(-7,-8)$ and has leading coefficient $a = 7$.

$f(x) =$

12. Write the vertex form for the quadratic function f, whose vertex is $(6,6)$ and has leading coefficient $a = 9$.

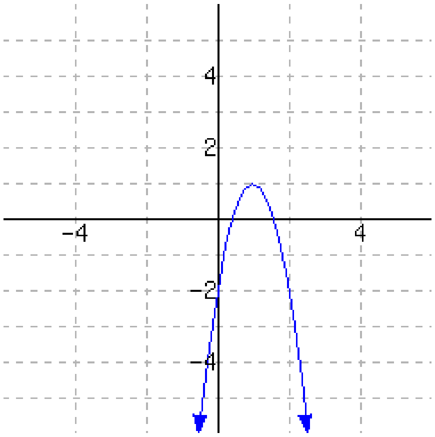
$f(x) =$

13. A graph of a function f is given. Use the graph to write a formula for f in vertex form. You will need to identify the vertex and also one more point on the graph to find the leading coefficient a.



$f(x) =$

14. A graph of a function f is given. Use the graph to write a formula for f in vertex form. You will need to identify the vertex and also one more point on the graph to find the leading coefficient a.



$f(x) =$

15. Let h be defined by $h(x) = (x - 5)^2 + 5$.

a. What is the domain of h?

b. What is the range of h?

16. Let h be defined by $h(x) = (x + 2)^2 - 7$.

a. What is the domain of h?

b. What is the range of h?

17. Consider the graph of the equation $y = (x - 1)^2 - 8$.

Compared to the graph of $y = x^2$, the vertex has been shifted $\boxed{}$ units (☐ left ☐ right) and $\boxed{}$ units (☐ down ☐ up) .

18. Consider the graph of the equation $y = (x - 3)^2 + 4$.

Compared to the graph of $y = x^2$, the vertex has been shifted $\boxed{}$ units (☐ left ☐ right) and $\boxed{}$ units (☐ down ☐ up) .

19. The quadratic expression $(x - 3)^2 - 36$ is written in vertex form.

a. Write the expression in standard form.

b. Write the expression in factored form.

20. The quadratic expression $(x - 3)^2 - 25$ is written in vertex form.

a. Write the expression in standard form.

b. Write the expression in factored form.

21. The formula for a quadratic function g is $g(x) = (x - 9)(x - 3)$.

a. The y-intercept is $\boxed{}$.

b. The x-intercept(s) is/are $\boxed{}$.

22. The formula for a quadratic function G is $G(x) = (x + 8)(x + 3)$.

a. The y-intercept is $\boxed{}$.

b. The x-intercept(s) is/are $\boxed{}$.

Completing the Square

23. Solve $x^2 - 4x = 5$ by completing the square.

24. Solve $y^2 + 8y = -12$ by completing the square.

25. Solve $y^2 + 7y + 6 = 0$ by completing the square.

26. Solve $r^2 - 5r - 24 = 0$ by completing the square.

27. Solve $12r^2 - 4r - 5 = 0$ by completing the square.

28. Solve $3t^2 + 8t + 5 = 0$ by completing the square.

29. Complete the square to convert the quadratic function from standard form to vertex form, and use the result to find the fnction's domain and range.

$f(x) = -4x^2 + 64x - 248$

The domain of f is

The range of f is

31. Graph $f(x) = x^2 - 7x + 12$ by algebraically determining its key features. Then state the domain and range of the function.

33. Graph $f(x) = x^2 - 8x + 16$ by algebraically determining its key features. Then state the domain and range of the function.

35. Graph $f(x) = x^2+4x+7$ by algebraically determining its key features. Then state the domain and range of the function.

37. Graph $f(x) = 2x^2 - 4x - 30$ by algebraically determining its key features. Then state the domain and range of the function.

39. Find the minimum value of the function

$$f(x) = x^2 - 7x - 2$$

30. Complete the square to convert the quadratic function from standard form to vertex form, and use the result to find the fnction's domain and range.

$f(x) = -5x^2 + 100x - 499$

The domain of f is

The range of f is

32. Graph $f(x) = -x^2 + 4x + 21$ by algebraically determining its key features. Then state the domain and range of the function.

34. Graph $f(x) = x^2+6x+9$ by algebraically determining its key features. Then state the domain and range of the function.

36. Graph $f(x) = x^2-2x+6$ by algebraically determining its key features. Then state the domain and range of the function.

38. Graph $f(x) = 3x^2 + 21x + 36$ by algebraically determining its key features. Then state the domain and range of the function.

40. Find the minimum value of the function

$$f(x) = 2x^2 + 7x + 7$$

More on Complex Solutions to Quadratic Equations

41. Solve the quadratic equation. Solutions could be complex numbers.

$-10(y + 8)^2 - 3 = 357$

43. Solve the quadratic equation. Solutions could be complex numbers.

$r^2 - 10r + 32 = 0$

42. Solve the quadratic equation. Solutions could be complex numbers.

$8(y - 3)^2 - 4 = -76$

44. Solve the quadratic equation. Solutions could be complex numbers.

$r^2 - 8r + 19 = 0$

45. A remote control aircraft will perform a stunt by flying toward the ground and then up. Its height can be modeled by the function $h(t) = 1.7t^2 - 30.6t + 133.7$. The plane (☐ will ☐ will not) hit the ground during this stunt.

46. A remote control aircraft will perform a stunt by flying toward the ground and then up. Its height can be modeled by the function $h(t) = 0.6t^2 - 10.8t + 51.6$. The plane (☐ will ☐ will not) hit the ground during this stunt.

Complex Number Operations

47. Write the complex number in standard form.

$$(10 - 2i) - (5 - 4i)$$

48. Write the complex number in standard form.

$$(-9 - 9i) - (-7 - 10i)$$

49. Write the complex number in standard form.

$$(-6 + 4i)(2 + 6i)$$

50. Write the complex number in standard form.

$$(-4 - 3i)(10)$$

51. Rewrite the following expression into the form of a+bi:

$\dfrac{-1 - 8i}{-2 - 5i} = \boxed{}$

52. Rewrite the following expression into the form of a+bi:

$\dfrac{3 + 6i}{7 - 5i} = \boxed{}$

CHAPTER 13

Rational Functions and Equations

13.1 Introduction to Rational Functions

In this chapter we will learn about rational functions, which are ratios of two polynomial functions. Creating this ratio inherently requires division, and we'll explore the effect this has on the graphs of rational functions and their domain and range.

13.1.1 Graphs of Rational Functions

When a drug is injected into a patient, the drug's concentration in the patient's bloodstream can be modeled by the function C, with formula

$$C(t) = \frac{3t}{t^2 + 8}$$

where $C(t)$ gives the drug's concentration, in milligrams per liter, t hours since the injection. A new injection is needed when the concentration falls to 0.35 milligrams per liter. Let's use graphing technology to explore this situation.

a. What is the concentration after 10 hours?
b. After how many hours since the first injection is the drug concentration greatest?
c. After how many hours since the first injection should the next injection be given?
d. What happens to the drug concentration if no further injections are given?

Using graphing technology, we will graph $y = \frac{3t}{t^2+8}$ and $y = 0.35$.

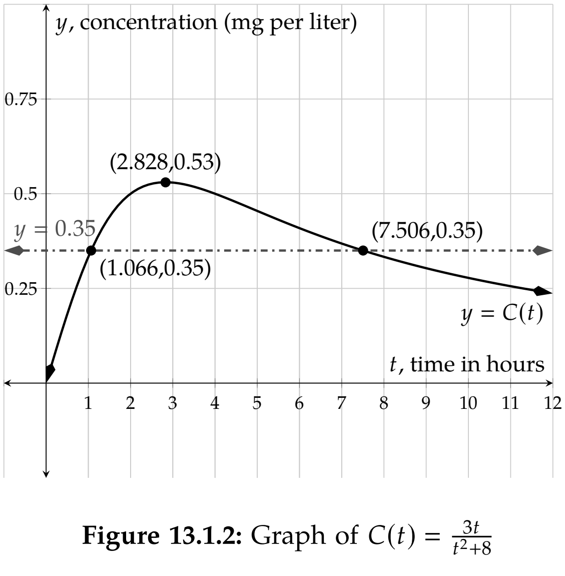


Figure 13.1.2: Graph of $C(t) = \frac{3t}{t^2+8}$

To determine the concentration after 10 hours, we will evaluate C at $t = 10$. After 10 hours, the concentration will be about 0.2777 $\frac{\text{mg}}{\text{L}}$.

$$\begin{aligned} C(10) &= \frac{3(10)}{10^2+8} \\ &= \frac{30}{108} \\ &= \frac{5}{18} \approx 0.2777 \end{aligned}$$

Using the graph, we can see that the maximum concentration of the drug will be 0.53 $\frac{\text{mg}}{\text{L}}$ and will occur after about 2.828 hours.

The approximate points of intersection $(1.066, 0.35)$ and $(7.506, 0.35)$ tell us that the concentration of the drug will reach 0.35 $\frac{\text{mg}}{\text{L}}$ after about 1.066 hours and again after about 7.506 hours. Given the rising, then falling shape of the graph, this means that another dose will need to be administered after about 7.506 hours.

From the initial graph, it appears that the concentration of the drug will diminish to zero with enough time passing. Exploring further, we can see both numerically and graphically that for larger and larger values of t, the function values get closer and closer to zero. This is shown in Table 13.1.3 and Figure 13.1.4.

t	$C(t)$
24	0.123 ...
48	0.062 ...
72	0.041 ...
96	0.031 ...
120	0.020 ...

Table 13.1.3: Numerical Values for $C(t) = \frac{3t}{t^2+8}$

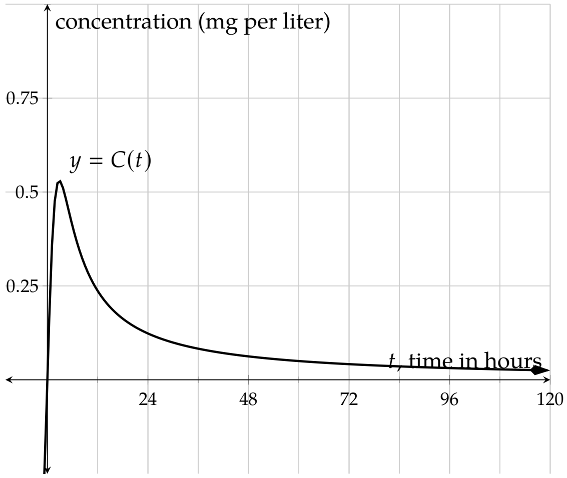


Figure 13.1.4: Graph of $C(t) = \frac{3t}{t^2+8}$

In Section 13.5, we'll explore how to algebraically solve $C(t) = 0.35$. For now, we just relied on technology to make the graph and determine intersection points.

The function C, where $C(t) = \frac{3t}{t^2+8}$, is a *rational function*, which is a type of function defined as follows.

Definition 13.1.5 Rational Function. A **rational function** f is a function in the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomial functions, but Q is not the constant zero function.

Checkpoint 13.1.6. Identify which of the following are rational functions and which are not.

a. f defined by $f(x) = \frac{25x^2+3}{25x^2+3}$ (☐ is ☐ is not) a rational function.

b. Q defined by $Q(x) = \frac{5x^2+3\sqrt{x}}{2x}$ (□ is □ is not) a rational function.

c. g defined by $g(t) = \frac{t\sqrt{5}-t^3}{2t+1}$ (□ is □ is not) a rational function.

d. P defined by $P(x) = \frac{5x+3}{|2x+1|}$ (□ is □ is not) a rational function.

e. h defined by $h(x) = \frac{3^x+1}{x^2+1}$ (□ is □ is not) a rational function.

Explanation.

a. f defined by $f(x) = \frac{25x^2+3}{25x^2+3}$ is a rational function as its formula is a polynomial divided by another polynomial.

b. Q defined by $Q(x) = \frac{5x^2+3\sqrt{x}}{2x}$ is not a rational function because the numerator contains $\sqrt{x}$ and is therefore not a polynomial.

c. g defined by $g(t) = \frac{t\sqrt{5}-t^3}{2t+1}$ is a rational function as its formula is a polynomial divided by another polynomial.

d. P defined by $P(x) = \frac{5x+3}{|2x+1|}$ is not a rational function because the denominator contains the absolute value of an expression with variables in it.

e. h defined by $h(x) = \frac{3^x+1}{x^2+1}$ is not a rational function because the numerator contains 3^x, which has a variable in the exponent.

A rational function's graph is not always smooth like the one shown in Example 13.1.2. It could have breaks, as we'll see now.

Example 13.1.7 Build a table and sketch the graph of the function f where $f(x) = \frac{1}{x-2}$. Find the function's domain and range.

Since $x = 2$ makes the denominator 0, the function will be undefined for $x = 2$. We'll start by choosing various x-values and plotting the associated points.

x	$f(x)$	Point
-6	$\frac{1}{-6-2} = -0.125$	$(-6, -0.125)$
-4	$\frac{1}{-4-2} \approx -0.167$	$\left(-4, -\frac{1}{6}\right)$
-2	$\frac{1}{-2-2} \approx -0.25$	$(-2, -0.25)$
0	$\frac{1}{0-2} = -0.5$	$(0, -0.5)$
1	$\frac{1}{1-2} = -1$	$(1, -1)$
2	undefined	
3	$\frac{1}{3-2} = 1$	$(3, 1)$
4	$\frac{1}{4-2} = 0.5$	$(4, 0.5)$

Table 13.1.8: Initial Values of $f(x) = \frac{1}{x-2}$

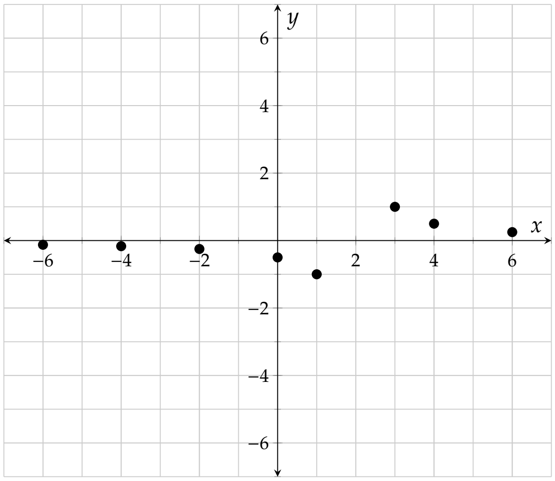


Figure 13.1.9: Initial Points for $f(x) = \frac{1}{x-2}$

Note that extra points were chosen near $x = 2$ in the Table 13.1.8, but it's still not clear on the graph what happens really close to $x = 2$. It will be essential that we include at least one x-value between 1 and 2 and also between 2 and 3.

Further, we'll note that dividing one number by a number that is close to 0 yields a large number. For example, $\frac{1}{0.0005} = 2000$. In fact, the smaller the number is that we divide by, the larger our result becomes. So when x gets closer and closer to 2, then $x-2$ gets closer and closer to 0. And then $\frac{1}{x-2}$ takes very large values.

When we plot additional points closer and closer to 2, we get larger and larger results. To the left of 2, the results are negative, so the connected curve has an arrow pointing downward there. The opposite happens to the right of $x = 2$, and an arrow points upward. We'll also draw the vertical line $x = 2$ as a dashed line to indicate that the graph never actually touches it.

x	$f(x)$	Point
-6	$\frac{1}{-6-2} = -0.125$	$(-6, -0.125)$
-4	$\frac{1}{-4-2} \approx -0.167$	$\left(-4, -\frac{1}{6}\right)$
-2	$\frac{1}{-2-2} \approx -0.25$	$(-2, -0.25)$
0	$\frac{1}{0-2} = -0.5$	$(0, -0.5)$
1	$\frac{1}{1-2} = -1$	$(1, -1)$
1.5	$\frac{1}{1.5-2} = -2$	$(1.5, -2)$
1.9	$\frac{1}{1.9-2} = -10$	$(1, -10)$
2	undefined	
2.1	$\frac{1}{2.1-2} = 10$	$(2.1, 10)$
2.5	$\frac{1}{2.5-2} = 2$	$(2.5, 2)$
3	$\frac{1}{3-2} = 1$	$(3, 1)$
4	$\frac{1}{4-2} = 0.5$	$(4, 0.5)$

Table 13.1.10: Values of $f(x) = \frac{1}{x-2}$

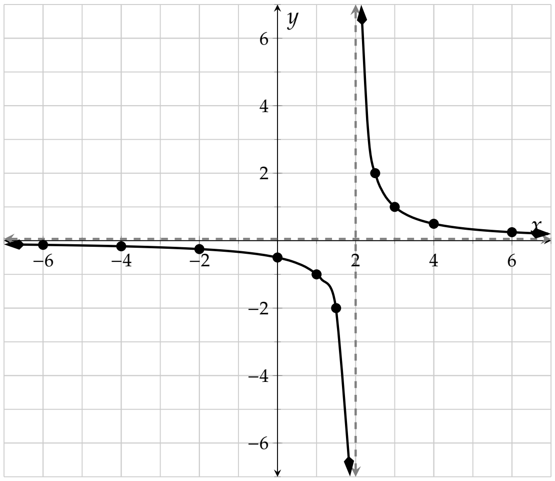


Figure 13.1.11: Full Graph of $f(x) = \frac{1}{x-2}$

Note that in Figure 13.1.11, the line $y = 0$ was also drawn as a dashed line. This is because the values of $y = f(x)$ will get closer and closer to zero as the inputs become more and more positive (or negative).

We know that the domain of this function is $(-\infty, 2) \cup (2, \infty)$ as the function is undefined at 2. We can determine this algebraically, and it is also evident in the graph.

We can see from the graph that the range of the function is $(-\infty, 0) \cup (0, \infty)$. See Checkpoint 10.2.27 for a discussion of how to see the range using a graph like this one.

Remark 13.1.12. The line $x = 2$ in Example 13.1.7 is referred to as a **vertical asymptote**. The line $y = 0$ is referred to as a **horizontal asymptote**. We'll use this vocabulary when referencing such lines, but the classification of vertical asymptotes and horizontal asymptotes is beyond the scope of this book.

Example 13.1.13 Algebraically find the domain of $g(x) = \frac{3x^2}{x^2-2x-24}$. Use technology to sketch a graph of this function.

Explanation. To find a rational function's domain, we set the denominator equal to 0 and solve:

$$\begin{aligned} x^2 - 2x - 24 &= 0 \\ (x-6)(x+4) &= 0 \end{aligned}$$

$$\begin{aligned} x-6&=0 &\quad\text{or}\quad&& x+4&=0\\ x&=6 &\quad\text{or}\quad&& x&=-4 \end{aligned}$$

Since $x = 6$ and $x = -4$ will cause the denominator to be 0, they are excluded from the domain. The function's domain is $\{x \mid x \neq 6, x \neq -4\}$. In interval notation, the domain is $(-\infty, -4) \cup (-4, 6) \cup (6, \infty)$.

To begin creating this graph, we'll use technology to create a table of function values, making sure to include values near both -4 and 6. We'll sketch an initial plot of these.

x	$g(x) = \frac{3x^2}{x^2-2x-24}$
-10	3.125
-9	3.24
-8	3.428 . . .
-7	3.769 . . .
-6	4.5
-5	6.818 . . .
-4	undefined
-3	-3
-2	-0.75
-1	$-0.142\ldots$
0	0 . . .
1	-0.12
2	-0.5
3	-1.285
4	-3
5	$-8.333\ldots$
6	undefined
7	13.363 . . .
8	8
9	6.230 . . .
10	5.357 . . .

Table 13.1.14: Numerical Values for g

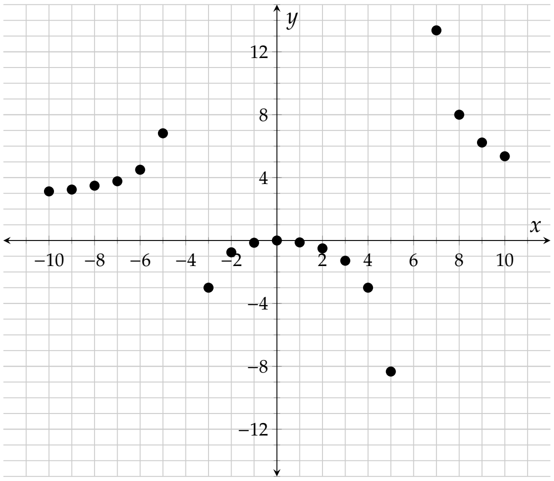


Figure 13.1.15: Initial Set-Up to Graph g

We can now begin to see what happens near $x = -4$ and $x = 6$. These are referred to as vertical asymptotes and will be graphed as dashed vertical lines as they are features of the graph but do not include function values.

The last thing we need to consider is what happens for large positive values of x and large negative values of x. Choosing a few values, we find:

x	$g(x)$
1000	3.0060...
2000	3.0030...
3000	3.0020...
4000	3.0015...

Table 13.1.16: Values for Large Positive x

x	$g(x)$
−1000	2.9940...
−2000	2.9970...
−3000	2.9980...
−4000	2.9985...

Table 13.1.17: Values for Large Negative x

Thus for really large positive x and for really large negative x, we see that the function values get closer and closer to $y = 3$. This is referred to as the horizontal asymptote, and will be graphed as a dashed horizontal line on the graph.

Putting all of this together, we can sketch a graph of this function.

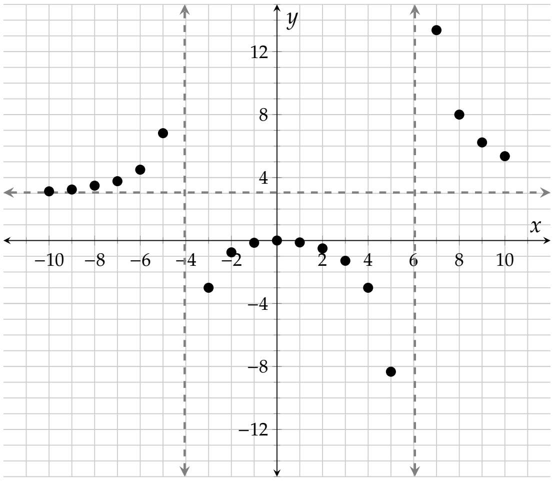


Figure 13.1.18: Asymptotes Added for Graphing $g(x) = \frac{3x^2}{x^2-2x-24}$

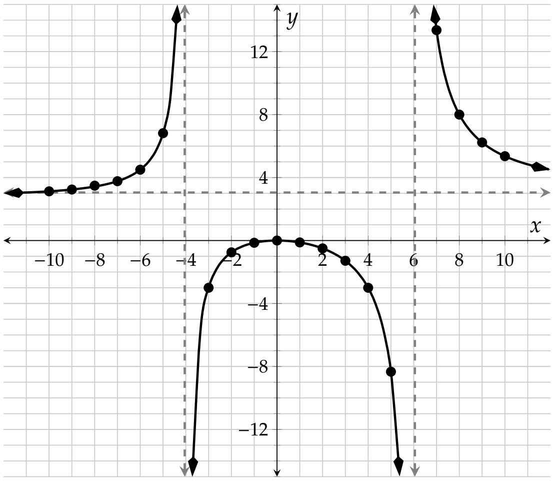


Figure 13.1.19: Full Graph of $g(x) = \frac{3x^2}{x^2-2x-24}$

Let's look at another example where a rational function is used to model real life data.

Example 13.1.20 The monthly operation cost of Saqui's shoe company is approximately $300,000.00. The cost of producing each pair of shoes is $30.00. As a result, the cost of producing x pairs of shoes is $30x + 300000$ dollars, and the average cost of producing each pair of shoes can be modeled by

$$\bar{C}(x) = \frac{30x + 300000}{x}.$$

Answer the following questions with technology.

a. What's the average cost of producing 100 pairs of shoes? Of 1,000 pairs? Of 10,000 pairs? What's the pattern?

b. To make the average cost of producing each pair of shoes cheaper than $50.00, at least how many pairs of shoes must Saqui's company produce?

c. Assume that her company's shoes are very popular. What happens to the average cost of producing shoes if more and more people keep buying them?

Explanation. We will graph the function with technology. After adjusting window settings, we have:

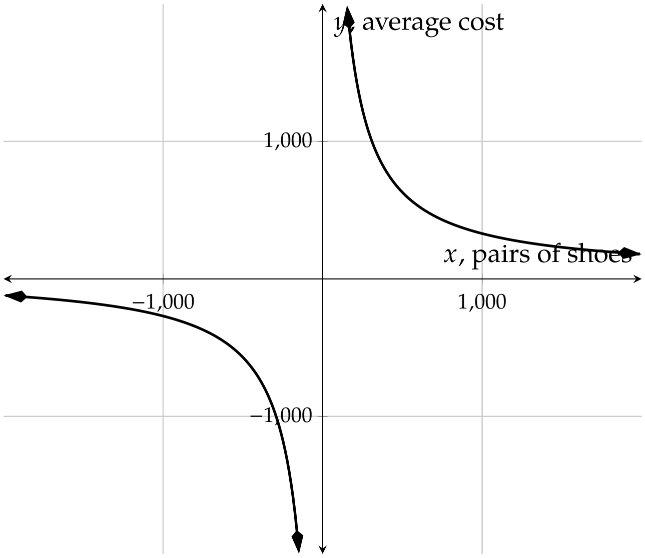


Figure 13.1.21: Graph of $\bar{C}(x) = \frac{30x+300000}{x}$

a. What's the average cost of producing 100 pairs of shoes? 1,000 pairs? 10,000 pairs? What's the pattern?

To answer this question, we locate the points where x values are 100, 1,000 and 10,000. They are $(100, 3030), (1000, 330)$ and $(10000, 60)$. They imply:

- If the company produces 100 pairs of shoes, the average cost of producing one pair is \$3,030.00.
- If the company produces 1,000 pairs of shoes, the average cost of producing one pair is \$330.00.
- If the company produces 10,000 pairs of shoes, the average cost of producing one pair is \$60.00.

We can see the more shoes her company produces, the lower the average cost.

b. To make the average cost of producing each pair of shoes cheaper than \$50.00, at least how many pairs of shoes must Saqui's company produce?

To answer this question, we locate the point where its y-value is 50. With technology, we graph both $y = \bar{C}(x)$ and $y = 50$, and locate their intersection.

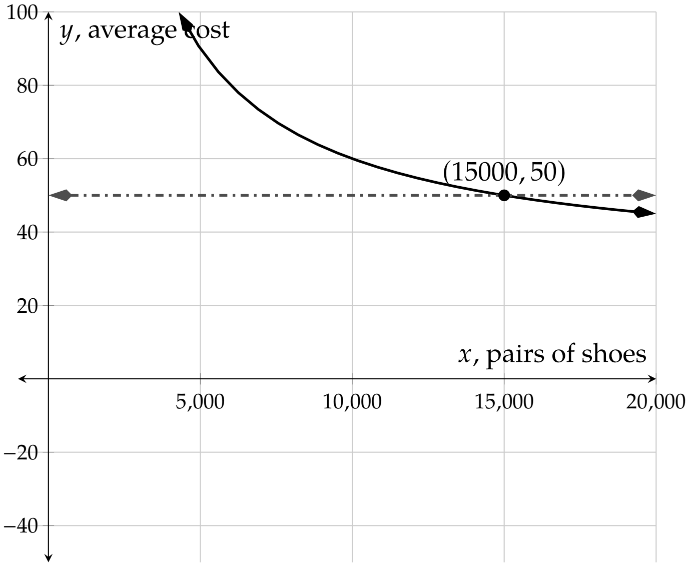


Figure 13.1.22: Intersection of $\bar{C}(x) = \frac{30x+300000}{x}$ and $y = 50$

The intersection $(15000, 50)$ implies the average cost of producing one pair is \$50.00 if her company produces 15,000 pairs of shoes.

c. Assume her company's shoes are very popular. What happens to the average cost of producing shoes if more and more people keep buying them?

To answer this question, we substitute x with some large numbers, and use technology to create a table of values:

x	$g(x)$
100000	33
1000000	31
10000000	30.03
100000000	30.003

Table 13.1.23: Values for Large Positive x

We can estimate that the average cost of producing one pair is getting closer and closer to \$30.00 as her company produces more and more pairs of shoes.

Note that the cost of producing each pair is \$30.00. This implies, for big companies whose products are very popular, the cost of operations can be ignored when calculating the average cost of producing each unit of product.

Exercises

Rational Functions in Context

1. The population of deer in a forest can be modeled by

$$P(x) = \frac{3920x + 1540}{8x + 7}$$

where x is the number of years in the future. Answer the following questions.

a. How many deer live in this forest this year?

b. How many deer will live in this forest 9 years later? Round your answer to an integer.

c. After how many years, the deer population will be 472? Round your answer to an integer.

d. Use a calculator to answer this question: As time goes on, the population levels off at about how many deer?

2. The population of deer in a forest can be modeled by

$$P(x) = \frac{2970x + 180}{9x + 2}$$

where x is the number of years in the future. Answer the following questions.

a. How many deer live in this forest this year?

b. How many deer will live in this forest 9 years later? Round your answer to an integer.

c. After how many years, the deer population will be 327? Round your answer to an integer.

d. Use a calculator to answer this question: As time goes on, the population levels off at about how many deer?

3. In a certain store, cashiers can serve 60 customers per hour on average. If x customers arrive at the store in a given hour, then the average number of customers C waiting in line can be modeled by the function

$$C(x) = \frac{x^2}{3600 - 60x}$$

where $x < 60$.

Answer the following questions with a graphing calculator. Round your answers to integers.

a. If 44 customers arrived in the store in the past hour, there are approximately ☐ customers waiting in line.

b. If there are 7 customers waiting in line, approximately ☐ customers arrived in the past hour.

4. In a certain store, cashiers can serve 55 customers per hour on average. If x customers arrive at the store in a given hour, then the average number of customers C waiting in line can be modeled by the function

$$C(x) = \frac{x^2}{3025 - 55x}$$

where $x < 55$.

Answer the following questions with a graphing calculator. Round your answers to integers.

a. If 48 customers arrived in the store in the past hour, there are approximately ☐ customers waiting in line.

b. If there are 2 customers waiting in line, approximately ☐ customers arrived in the past hour.

Identify Rational Functions Select all rational functions. There are several correct answers.

5.

☐ $n(x) = \frac{3x^2+6\sqrt{x}-4}{3-2x^3}$ ☐ $m(x) = \frac{3x+6}{3x+6}$ ☐ $r(x) = \frac{3x^2+6x-4}{3-2x^{-3}}$ ☐ $t(x) = \frac{3-2x^3}{3x^{0.7}+6x-4}$
☐ $b(x) = \frac{3x^2+6x-4}{3}$ ☐ $s(x) = \frac{\sqrt{3}x^2+6x-4}{3-2x^3}$ ☐ $c(x) = \frac{3x^2+6x-4}{3+|x|}$ ☐ $a(x) = \frac{3x^2+6x-4}{3-2x^3}$
☐ $h(x) = \frac{3}{3x^2+6x-4}$

6.

☐ $t(x) = \frac{8-2x^3}{3x^{0.7}+3x-7}$ ☐ $s(x) = \frac{\sqrt{3}x^2+3x-7}{8-2x^3}$ ☐ $m(x) = \frac{3x+3}{3x+3}$ ☐ $r(x) = \frac{3x^2+3x-7}{8-2x^{-3}}$
☐ $a(x) = \frac{3x^2+3x-7}{8-2x^3}$ ☐ $c(x) = \frac{3x^2+3x-7}{8+|x|}$ ☐ $h(x) = \frac{8}{3x^2+3x-7}$ ☐ $b(x) = \frac{3x^2+3x-7}{8}$
☐ $n(x) = \frac{3x^2+3\sqrt{x}-7}{8-2x^3}$

Domain

7. Find the domain of h where $h(x) = \dfrac{2x}{x-8}$.

8. Find the domain of F where $F(x) = \dfrac{5x}{x-1}$.

9. Find the domain of G where $G(x) = \dfrac{10-6x}{x^2-16x+63}$.

10. Find the domain of H where $H(x) = \dfrac{7x-2}{x^2-x-72}$.

11. Find the domain of H where $H(x) = \dfrac{7x+2}{x^2-10x}$.

12. Find the domain of K where $K(x) = -\dfrac{7x+5}{x^2-4x}$.

13. Find the domain of f where $f(x) = \dfrac{6x+3}{x^2-100}$.

14. Find the domain of g where $g(x) = -\dfrac{x+9}{x^2-9}$.

15. Find the domain of the function c defined by $c(x) = \dfrac{x-4}{x^2}$

16. Find the domain of the function p defined by $p(x) = \dfrac{x-2}{x^5}$

17. Find the domain of the function q defined by $q(x) = \dfrac{x-2}{x^2+16}$

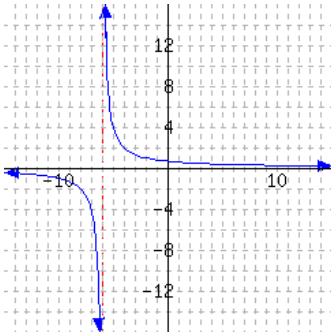
18. Find the domain of the function m defined by $m(x) = \dfrac{x+3}{x^2+81}$

19. Find the domain of the function m defined by $m(x) = \dfrac{x+5}{x+5}$

20. Find the domain of the function b defined by $b(x) = \dfrac{x+7}{x+7}$

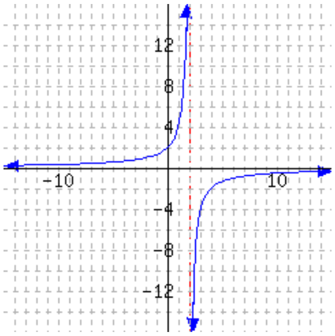
A function is graphed.

21.



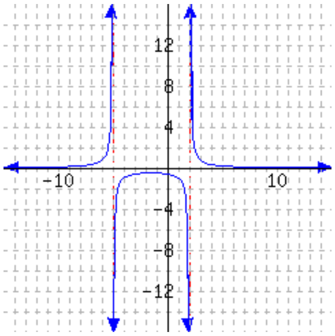
What is its domain?

22.



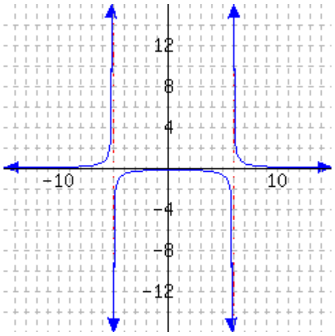
What is its domain?

23.



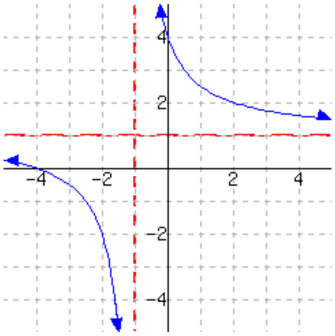
What is its domain?

24.



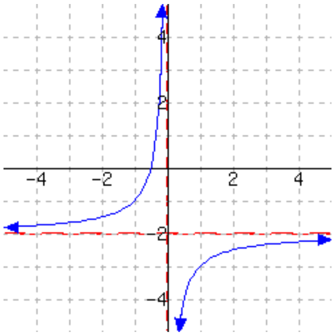
What is its domain?

25.



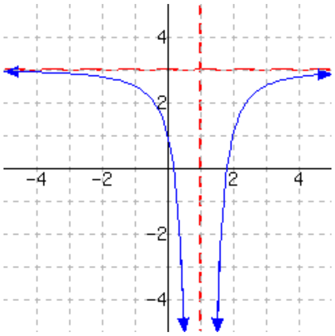
This function has domain ☐ and range ☐.

26.



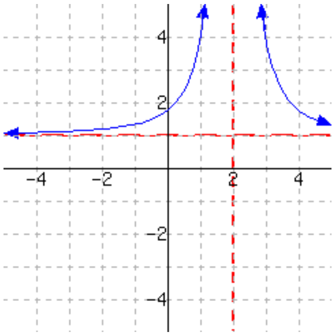
This function has domain ☐ and range ☐.

27.



This function has domain ☐ and range ☐.

28.



This function has domain ☐ and range ☐.

Graphing Technology

29. In a forest, the number of deer can be modeled by the function $f(x) = \frac{180t+270}{0.6t+3}$, where t stands for the number of years from now. Answer the question with technology. Round your answer to a whole number.

After 30 years, there would be approximately deer living in the forest.

30. In a forest, the number of deer can be modeled by the function $f(x) = \frac{180t+900}{0.4t+9}$, where t stands for the number of years from now. Answer the question with technology. Round your answer to one decimal place.

After years, there would be approximately 350 deer living in the forest.

31. In a forest, the number of deer can be modeled by the function $f(x) = \frac{25t+180}{0.1t+6}$, where t stands for the number of years from now. Answer the question with technology. Round your answer to one decimal place.

As time goes on, the population levels off at approximately deer living in the forest.

32. The concentration of a drug in a patient's blood stream, in milligrams per liter, can be modeled by the function $C(t) = \frac{3t}{t^2+6}$, where t is the number of hours since the drug is injected. Answer the following question with technology. Round your answer to two decimal places if needed.

The drug's concentration after 8 hours is milligrams per liter.

33. The concentration of a drug in a patient's blood stream, in milligrams per liter, can be modeled by the function $C(t) = \frac{4t}{t^2+9}$, where t is the number of hours since the drug is injected. Answer the following question with technology. Round your answer to two decimal places if needed. If there are more than one answer, use commas to separate them.

hours since injection, the drug's concentration is 0.22 milligrams per liter.

34. The concentration of a drug in a patient's blood stream, in milligrams per liter, can be modeled by the function $C(t) = \frac{5t}{t^2+7}$, where t is the number of hours since the drug is injected. Answer the following question with technology. Round your answer to two decimal places if needed.

hours since injection, the drug's concentration is at the maximum value of milligrams per liter.

35. The concentration of a drug in a patient's blood stream, in milligrams per liter, can be modeled by the function $C(t) = \frac{7t}{t^2+4}$, where t is the number of hours since the drug is injected. Answer the following question with technology. Round your answer to two decimal places if needed.

As time goes on, the drug's concentration in the patient's blood stream levels off at approximately milligrams per liter.

13.2 Multiplication and Division of Rational Expressions

In the last section, we learned some rational function applications. In this section, we will learn how to simplify rational expressions, and how to multiply and divide them.

13.2.1 Simplifying Rational Expressions

Consider the two rational functions below. At first glance, which function *looks* simpler?

$$f(x) = \frac{8x^3 - 12x^2 + 8x - 12}{2x^3 - 3x^2 + 10x - 15} \qquad g(x) = \frac{4(x^2+1)}{x^2+5}, \text{ for } x \neq \frac{3}{2}$$

It can be argued that the function g is simpler, at least with regard to the ease with which we can determine its domain, quickly evaluate it, and also determine where its function value is zero. All of these things are considerably more difficult with the function f.

These two functions are actually the *same* function. Using factoring and the same process of canceling that's used with numerical ratios, we will learn how to simplify the function f into the function g. (The full process for simplifying $f(x) = \frac{8x^3-12x^2+8x-12}{2x^3-3x^2+10x-15}$ will be shown in Example 13.2.8.)

To see a simple example of the process for simplifying a rational function or expression, let's look at simplifying $\frac{14}{21}$ and $\frac{(x+2)(x+7)}{(x+3)(x+7)}$ by canceling common factors:

$$\begin{aligned} \frac{14}{21} &= \frac{2 \cdot \cancel{7}}{3 \cdot \cancel{7}} \\ &= \frac{2}{3} \end{aligned} \qquad \begin{aligned} \frac{(x+2)(x+7)}{(x+3)(x+7)} &= \frac{(x+2)\cancel{(x+7)}}{(x+3)\cancel{(x+7)}} \\ &= \frac{x+2}{x+3}, \text{ for } x \neq -7 \end{aligned}$$

The statement "for $x \neq -7$" was added when the factors of $x+7$ were canceled. This is because $\frac{(x+2)(x+7)}{(x+3)(x+7)}$ was undefined where $x = -7$, so the simplified version must also be undefined for $x = -7$.

Warning 13.2.2 Cancel Factors, not Terms. It may be tempting to want to try to simplify $\frac{x+2}{x+3}$ into $\frac{2}{3}$ by canceling each x that appears. But these x's are *terms* (pieces that are added with other pieces), not *factors*. Canceling (an act of division) is only possible with *factors*.

The process of canceling factors is key to simplifying rational expressions. If the expression is not given in factored form, then this will be our first step. We'll now look at a few more examples.

Example 13.2.3 Simplify the rational function formula $Q(x) = \frac{3x-12}{x^2+x-20}$ and state the domain of Q.

Explanation.

To start, we'll factor the numerator and denominator. We'll then cancel any factors common to both the numerator and denominator.

$$\begin{aligned} Q(x) &= \frac{3x-12}{x^2+x-20} \\ Q(x) &= \frac{3\cancel{(x-4)}}{(x+5)\cancel{(x-4)}} \\ Q(x) &= \frac{3}{x+5}, \text{ for } x \neq 4 \end{aligned}$$

The domain of this function will incorporate the *explicit* domain restriction $x \neq 4$ that was stated when the factor of $x-4$ was canceled from both the numerator and denominator. We will also exclude -5 from the domain as this value would make the denominator zero. Thus the domain of Q is $\{x \mid x \neq -5, 4\}$.

Warning 13.2.4. When simplifying the function Q in Example 13.2.3, we cannot simply write $Q(x) = \frac{3}{x+5}$. The reason is that this would result in our simplified version of the function Q having a different domain than the original Q. More specifically, for our original function Q it held that $Q(4)$ was undefined, and this still needs to be true for the simplified form of Q.

Example 13.2.5 Simplify the rational function formula $R(y) = \frac{-y-2y^2}{2y^3-y^2-y}$ and state the domain of R.

Explanation.

$$\begin{aligned}
R(y) &= \frac{-y-2y^2}{2y^3-y^2-y} \\
R(y) &= \frac{-2y^2-y}{y(2y^2-y-1)} \\
R(y) &= \frac{-\cancel{y(2y+1)}}{\cancel{y(2y+1)}(y-1)} \\
R(y) &= -\frac{1}{y-1}, \text{ for } y \neq 0, y \neq -\frac{1}{2}
\end{aligned}$$

The domain of this function will incorporate the explicit restrictions $y \neq 0, y \neq -\frac{1}{2}$ that were stated when the factors of y and $2y+1$ were canceled from both the numerator and denominator. Since the factor $y-1$ is still in the denominator, we also need the restriction that $y \neq 1$. Therefore the domain of R is $\left\{y \mid y \neq -\frac{1}{2}, 0, 1\right\}$.

Example 13.2.6 Simplify the expression $\frac{9y+2y^2-5}{y^2-25}$.

Explanation.

To start, we need to recognize that $9y+2y^2-5$ is not written in standard form (where terms are written from highest degree to lowest degree). Before attempting to factor this expression, we'll re-write it as $2y^2+9y-5$.

$$\begin{aligned}
\frac{9y+2y^2-5}{y^2-25} &= \frac{2y^2+9y-5}{y^2-25} \\
&= \frac{(2y-1)\cancel{(y+5)}}{\cancel{(y+5)}(y-5)} \\
&= \frac{2y-1}{y-5}, \text{ for } y \neq -5
\end{aligned}$$

Example 13.2.7 Simplify the expression $\frac{-48z+24z^2-3z^3}{4-z}$.

Explanation. To begin simplifying this expression, we will rewrite each polynomial in descending order. Then we'll factor out the GCF, including the constant -1 from both the numerator and denominator because their leading terms are negative.

$$\frac{-48z+24z^2-3z^3}{4-z} = \frac{-3z^3+24z^2-48z}{-z+4}$$

$$\begin{aligned}
&= \frac{-3z(z^2-8z+16)}{-(z-4)}\\
&= \frac{-3z(z-4)^2}{-(z-4)}\\
&= \frac{-3z(z-4)\cancel{(z-4)}}{-\cancel{(z-4)}}\\
&= \frac{3z(z-4)}{1},\ \text{for } z \neq 4\\
&= 3z(z-4),\ \text{for } z \neq 4
\end{aligned}$$

Example 13.2.8 Simplify the rational function formula $f(x) = \frac{8x^3-12x^2+8x-12}{2x^3-3x^2+10x-15}$ and state the domain of f.

Explanation.

To simplify this rational function, we'll first note that both the numerator and denominator have four terms. To factor them we'll need to use factoring by grouping. (Note that if this technique didn't work, very few other approaches would be possible.) Once we've used factoring by grouping, we'll cancel any factors common to both the numerator and denominator and state the associated restrictions.

$$\begin{aligned}
f(x) &= \frac{8x^3-12x^2+8x-12}{2x^3-3x^2+10x-15}\\
f(x) &= \frac{4(2x^3-3x^2+2x-3)}{2x^3-3x^2+10x-15}\\
f(x) &= \frac{4(x^2(2x-3)+(2x-3))}{x^2(2x-3)+5(2x-3)}\\
f(x) &= \frac{4(x^2+1)\cancel{(2x-3)}}{(x^2+5)\cancel{(2x-3)}}\\
f(x) &= \frac{4(x^2+1)}{x^2+5},\ \text{for } x \neq \frac{3}{2}
\end{aligned}$$

In determining the domain of this function, we'll need to account for any implicit and explicit restrictions. When the factor $2x-3$ was canceled, the explicit statement of $x \neq \frac{3}{2}$ was given. The denominator in the final simplified form of this function has x^2+5. There is no value of x for which $x^2+5=0$, so the only restriction is that $x \neq \frac{3}{2}$. Therefore the domain is $\left\{x \mid x \neq \frac{3}{2}\right\}$.

Example 13.2.9 Simplify the expression $\frac{3y-x}{x^2-xy-6y^2}$. In this example, there are two variables. It is still possible that in examples like this, there can be domain restrictions when simplifying rational expressions. However since we are not studying *functions* of more than one variable, this textbook ignores domain restrictions with examples like this one.

Explanation.

$$\begin{aligned}
\frac{3y-x}{x^2-xy-6y^2} &= \frac{-\cancel{(x-3y)}}{\cancel{(x-3y)}(x+2y)}\\
&= \frac{-1}{x+2y}
\end{aligned}$$

13.2.2 Multiplication and Division of Rational Functions and Expressions

Recall the property for multiplying fractions 1.2.16, which states that the product of two fractions is equal to the product of their numerators divided by the product of their denominators. We will use this same property for multiplying rational expressions.

When multiplying fractions, one approach is to multiply the numerator and denominator, and then simplify the fraction that results by determining the greatest common factor in both the numerator and denominator, like this:

$$\begin{aligned}\frac{14}{9}\cdot\frac{3}{10}&=\frac{14\cdot 3}{9\cdot 10}\\&=\frac{42}{90}\\&=\frac{7\cdot\cancel{6}}{15\cdot\cancel{6}}\\&=\frac{7}{15}\end{aligned}$$

This approach works great when we can easily identify that 6 is the greatest common factor in both 42 and 90. But in more complicated instances, it isn't always an easy approach. It also won't work particularly well when we have $(x+2)$ instead of 2 as a factor, as we'll see shortly.

Another approach to multiplying and simplifying fractions involves utilizing the prime factorization of each the numerator and denominator, like this:

$$\begin{aligned}\frac{14}{9}\cdot\frac{3}{10}&=\frac{2\cdot 7}{3^2}\cdot\frac{3}{2\cdot 5}\\&=\frac{\cancel{2}\cdot 7\cdot\cancel{3}}{\cancel{3}\cdot 3\cdot\cancel{2}\cdot 5}\\&=\frac{7}{15}\end{aligned}$$

The method for multiplying and simplifying rational expressions is nearly identical, as shown here:

$$\begin{aligned}\frac{x^2+9x+14}{x^2+6x+9}\cdot\frac{x+3}{x^2+7x+10}&=\frac{(x+2)(x+7)}{(x+3)^2}\cdot\frac{x+3}{(x+2)(x+5)}\\&=\frac{\cancel{(x+2)}(x+7)\cancel{(x+3)}}{\cancel{(x+3)}(x+3)\cancel{(x+2)}(x+5)}\\&=\frac{(x+7)}{(x+3)(x+5)},\ \text{for } x\neq -2\end{aligned}$$

This process will be used for both multiplying and dividing rational expressions. The main distinctions in various examples will be in the factoring methods required.

Example 13.2.10 Multiply the rational expressions: $\frac{x^2-4x}{x^2-4}\cdot\frac{4-4x+x^2}{20-x-x^2}$.

Explanation. Note that to factor the second rational expression, we'll want to re-write the terms in descending order for both the numerator and denominator. In the denominator, we'll first factor out -1 as the leading term is $-x^2$.

$$\frac{x^2-4x}{x^2-4}\cdot\frac{4-4x+x^2}{20-x-x^2}=\frac{x^2-4x}{x^2-4}\cdot\frac{x^2-4x+4}{-x^2-x+20}$$

$$\begin{aligned}
&= \frac{x^2-4x}{x^2-4} \cdot \frac{x^2-4x+4}{-(x^2+x-20)} \\
&= \frac{x\cancel{(x-4)}}{(x+2)\cancel{(x-2)}} \cdot \frac{(x-2)\cancel{(x-2)}}{-(x+5)\cancel{(x-4)}} \\
&= -\frac{x(x-2)}{(x+2)(x+5)}, \text{ for } x \neq 2, x \neq 4
\end{aligned}$$

Example 13.2.11 Multiply the rational expressions: $\frac{p^2q^4}{3r} \cdot \frac{9r^2}{pq^2}$. Note this book ignores domain restrictions on multivariable expressions.

Explanation. We won't need to factor anything in this example, and can simply multiply across and then simplify.

$$\begin{aligned}
\frac{p^2q^4}{3r} \cdot \frac{9r^2}{pq^2} &= \frac{p^2q^2 \cdot 9r^2}{3r \cdot pq^2} \\
&= \frac{pq^2 \cdot 3r}{1} \\
&= 3pq^2r
\end{aligned}$$

We can divide rational expressions using the property for dividing fractions 1.2.18, which simply requires that we change dividing by an expression to multiplying by its reciprocal. Let's look at a few examples.

Example 13.2.12 Divide the rational expressions: $\frac{x+2}{x+5} \div \frac{x+2}{x-3}$.

Explanation.

$$\begin{aligned}
\frac{x+2}{x+5} \div \frac{x+2}{x-3} &= \frac{\cancel{x+2}}{x+5} \cdot \frac{x-3}{\cancel{x+2}}, \text{ for } x \neq 3 \\
&= \frac{x-3}{x+5}, \text{ for } x \neq -2, x \neq 3
\end{aligned}$$

Example 13.2.13 Simplify the rational expression using division: $\frac{\frac{3x-6}{2x+10}}{\frac{x^2-4}{3x+15}}$.

Explanation. To begin, we'll note that the larger fraction bar is denoting division, so we will use multiplication by the reciprocal. After that, we'll factor each expression and cancel any common factors.

$$\begin{aligned}
\frac{\frac{3x-6}{2x+10}}{\frac{x^2-4}{3x+15}} &= \frac{3x-6}{2x+10} \div \frac{x^2-4}{3x+15} \\
&= \frac{3x-6}{2x+10} \cdot \frac{3x+15}{x^2-4} \\
&= \frac{3\cancel{(x-2)}}{2\cancel{(x+5)}} \cdot \frac{3\cancel{(x+5)}}{(x+2)\cancel{(x-2)}} \\
&= \frac{3 \cdot 3}{2(x+2)}, \text{ for } x \neq -5, x \neq 2 \\
&= \frac{9}{2x+4}, \text{ for } x \neq -5, x \neq 2
\end{aligned}$$

Example 13.2.14 Divide the rational expressions: $\frac{x^2-5x-14}{x^2+7x+10} \div \frac{x-7}{x+4}$.

Explanation.

$$\begin{aligned}\frac{x^2-5x-14}{x^2+7x+10} \div \frac{x-7}{x+4} &= \frac{x^2-5x-14}{x^2+7x+10} \cdot \frac{x+4}{x-7}, \text{ for } x \neq -4\\ &= \frac{\cancel{(x-7)}\cancel{(x+2)}}{(x+5)\cancel{(x+2)}} \cdot \frac{x+4}{\cancel{x-7}}, \text{ for } x \neq -4\\ &= \frac{x+4}{x+5}, \text{ for } x \neq -4, x \neq -2, x \neq 7\end{aligned}$$

Example 13.2.15 Divide the rational expressions: $(p^4-16) \div \frac{p^4-2p^3}{2p}$.

Explanation.

$$\begin{aligned}(p^4-16) \div \frac{p^4-2p^3}{2p} &= \frac{p^4-16}{1} \cdot \frac{2p}{p^4-2p^3}\\ &= \frac{(p^2+4)(p+2)\cancel{(p-2)}}{1} \cdot \frac{2p}{p^3\cancel{(p-2)}}\\ &= \frac{2(p^2+4)(p+2)}{p^2}, \text{ for } p \neq 2\end{aligned}$$

Example 13.2.16 Divide the rational expressions: $\frac{3x^2}{x^2-9y^2} \div \frac{6x^3}{x^2-2xy-15y^2}$. Note this book ignores domain restrictions on multivariable expressions.

Explanation.

$$\begin{aligned}\frac{3x^2}{x^2-9y^2} \div \frac{6x^3}{x^2-2xy-15y^2} &= \frac{3x^2}{x^2-9y^2} \cdot \frac{x^2-2xy-15y^2}{6x^3}\\ &= \frac{3x^2}{\cancel{(x+3y)}(x-3y)} \cdot \frac{\cancel{(x+3y)}(x-5y)}{6x^3}\\ &= \frac{1}{x-3y} \cdot \frac{x-5y}{2x}\\ &= \frac{x-5y}{2x(x-3y)}\end{aligned}$$

Example 13.2.17 Divide the rational expressions: $\frac{m^2n^2-3mn-4}{2mn} \div (m^2n^2-16)$. Note this book ignores domain restrictions on multivariable expressions.

Explanation.

$$\begin{aligned}\frac{m^2n^2-3mn-4}{2mn} \div (m^2n^2-16) &= \frac{m^2n^2-3mn-4}{2mn} \cdot \frac{1}{m^2n^2-16}\\ &= \frac{\cancel{(mn-4)}(mn+1)}{2mn} \cdot \frac{1}{(mn+4)\cancel{(mn-4)}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{mn+1}{2mn} \cdot \frac{1}{mn+4} \\
&= \frac{mn+1}{2mn(mn+4)}
\end{aligned}
$$

Exercises

Review and Warmup

1. Multiply: $-\frac{6}{11} \cdot \frac{11}{21}$

2. Multiply: $-\frac{15}{11} \cdot \frac{11}{15}$

3. Multiply: $-\frac{20}{17} \cdot \left(-\frac{13}{18}\right)$

4. Multiply: $-\frac{20}{11} \cdot \left(-\frac{7}{24}\right)$

5. Divide: $\frac{1}{5} \div \frac{9}{4}$

6. Divide: $\frac{1}{2} \div \frac{8}{3}$

7. Divide: $\frac{8}{15} \div \left(-\frac{7}{20}\right)$

8. Divide: $\frac{5}{12} \div \left(-\frac{8}{15}\right)$

Factor the given polynomial.

9. $r^2 - 1 =$

10. $r^2 - 81 =$

11. $r^2 + 9r + 20 =$

12. $t^2 + 19t + 90 =$

13. $t^2 - 10t + 21 =$

14. $x^2 - 10x + 21 =$

15. $2x^2 - 14x + 20 =$

16. $2y^2 - 18y + 28 =$

17. $2y^4 + 16y^3 + 30y^2 =$

18. $3r^7 + 12r^6 + 9r^5 =$

19. $64r^2 - 16r + 1 =$

20. $16r^2 - 8r + 1 =$

Simplifying Rational Expressions with One Variable

21. Select all correct simplifications, ignoring possible domain restrictions.

☐ $\frac{7x+9}{x+9} = 7$ ☐ $\frac{x}{7x} = \frac{1}{7}$ ☐ $\frac{7x+9}{7} = x+9$ ☐ $\frac{9}{x+9} = \frac{1}{x+1}$ ☐ $\frac{x+9}{x} = 9$ ☐ $\frac{9x}{x} = 9$
☐ $\frac{x+9}{x+7} = \frac{9}{7}$ ☐ $\frac{x+9}{9} = x$ ☐ $\frac{x+9}{x+9} = 1$ ☐ $\frac{7(x-9)}{x-9} = 7$ ☐ $\frac{9}{x+9} = \frac{1}{x}$

22. Select all correct simplifications, ignoring possible domain restrictions.

☐ $\frac{9(x-10)}{x-10} = 9$ ☐ $\frac{x+10}{x+9} = \frac{10}{9}$ ☐ $\frac{x+10}{x} = 10$ ☐ $\frac{x+10}{10} = x$ ☐ $\frac{9x+10}{9} = x+10$
☐ $\frac{x+10}{x+10} = 1$ ☐ $\frac{10x}{x} = 10$ ☐ $\frac{9x+10}{x+10} = 9$ ☐ $\frac{10}{x+10} = \frac{1}{x}$ ☐ $\frac{10}{x+10} = \frac{1}{x+1}$ ☐ $\frac{x}{9x} = \frac{1}{9}$

23. Simplify the following expressions, and if applicable, write the restricted domain on the simplified expression.

a. $\frac{x+3}{x+3} =$ ☐

b. $\frac{x+3}{3+x} =$ ☐

c. $\frac{x-3}{x-3} =$ ☐

d. $\frac{x-3}{3-x} =$ ☐

24. Simplify the following expressions, and if applicable, write the restricted domain on the simplified expression.

a. $\frac{x+9}{x+9} =$ ☐

b. $\frac{x+9}{9+x} =$ ☐

c. $\frac{x-9}{x-9} =$ ☐

d. $\frac{x-9}{9-x} =$ ☐

Simplify the following expression, and if applicable, write the restricted domain on the simplified expression.

25. $\frac{y-6}{(y-2)(y-6)} =$ ☐

26. $\frac{y+3}{(y-8)(y+3)} =$ ☐

27. $\frac{-8(r-8)}{(r-6)(r-8)} =$ ☐

28. $\frac{-3(r+5)}{(r-3)(r+5)} =$ ☐

29. $\frac{(r+2)(r-10)}{10-r} =$ ☐

30. $\frac{(t-8)(t-7)}{7-t} =$ ☐

31. $\frac{5t-25}{t-5} =$ ☐

32. $\frac{-2x+4}{x-2} =$ ☐

33. $\frac{-9x}{x^2+8x} =$ ☐

34. $\frac{-6y}{y^2-5y} =$ ☐

35. $\frac{6y-y^2}{y^2-9y+18} =$ ☐

36. $\frac{4y-y^2}{y^2+2y-24} =$ ☐

37. $\frac{r^2+2r}{4-r^2} =$ ☐

38. $\frac{r^2-6r}{36-r^2} =$ ☐

39. $\frac{-t^2+4t}{-4+5t-t^2} =$ ☐

40. $\frac{-t^2-2t}{12+4t-t^2} =$ ☐

41. $\dfrac{5x^2+11x+6}{-7x-5-2x^2} = \boxed{\quad}$

42. $\dfrac{3x^2+x-2}{-2x+3-5x^2} = \boxed{\quad}$

43. $\dfrac{y^2+y-2}{2y-y^2-1} = \boxed{\quad}$

44. $\dfrac{y^2-11y+30}{10y-y^2-25} = \boxed{\quad}$

45. $\dfrac{-y^2-y+12}{y^2-9} = \boxed{\quad}$

46. $\dfrac{-r^2+3r-2}{r^2-4} = \boxed{\quad}$

47. $\dfrac{5r^2+9r+4}{-7r-5-2r^2} = \boxed{\quad}$

48. $\dfrac{3t^2+t-2}{-11t-5-6t^2} = \boxed{\quad}$

49. $\dfrac{3t^2-t^3}{t^2-5t+6} = \boxed{\quad}$

50. $\dfrac{-2x^4-x^5}{x^2-3x-10} = \boxed{\quad}$

51. $\dfrac{x^5-x^4-20x^3}{x^5-11x^4+30x^3} = \boxed{\quad}$

52. $\dfrac{y^4-5y^3-6y^2}{y^4-4y^3-5y^2} = \boxed{\quad}$

53. $\dfrac{y^3+64}{y^2-16} = \boxed{\quad}$

54. $\dfrac{y^3-27}{y^2-9} = \boxed{\quad}$

Simplifying Rational Expressions with More Than One Variable Simplify this expression.

55. $\dfrac{5rx-r^2x^2}{r^2x^2-10rx+25} = \boxed{\quad}$

56. $\dfrac{3rx-r^2x^2}{r^2x^2-9rx+18} = \boxed{\quad}$

57. $\dfrac{4t+8y}{t^2-2ty-8y^2} = \boxed{\quad}$

58. $\dfrac{6t-36x}{t^2-5tx-6x^2} = \boxed{\quad}$

59. $\dfrac{-x^2-9xr-18r^2}{x^2-36r^2} = \boxed{\quad}$

60. $\dfrac{-x^2+4xy+12y^2}{x^2-4y^2} = \boxed{\quad}$

61. $\dfrac{2y^2t^2-yt-3}{-11yt-6-5y^2t^2} = \boxed{\quad}$

62. $\dfrac{5y^2r^2+3yr-2}{-7yr-3-4y^2r^2} = \boxed{\quad}$

Simplifying Rational Functions Simplify the function formula, and if applicable, write the restricted domain.

63. $g(y) = \dfrac{y+2}{y^2+12y+20}$

Reduced $g(y) =$

64. $K(r) = \dfrac{r-6}{r^2-15r+54}$

Reduced $K(r) =$

65. $F(r) = \dfrac{r^3-100r}{r^3+16r^2+60r}$

Reduced $F(r) =$

66. $g(t) = \dfrac{t^3-16t}{t^3+11t^2+28t}$

Reduced $g(t) =$

67. $K(t) = \dfrac{t^4+6t^3+9t^2}{3t^4+11t^3+6t^2}$

Reduced $K(t) =$

68. $F(x) = \dfrac{x^4-6x^3+9x^2}{2x^4-9x^3+9x^2}$

Reduced $F(x) =$

69. $g(x) = \dfrac{3x^3+5x^2}{3x^3+8x^2+5x}$

Reduced $g(x) =$

70. $H(y) = \dfrac{2y^3+3y^2}{2y^3-7y^2-15y}$

Reduced $H(y) =$

Multiplying and Dividing Rational Expressions with One Variable

71. Select all correct equations:

- ☐ $5\cdot\frac{x}{y}=\frac{5x}{5y}$
- ☐ $5\cdot\frac{x}{y}=\frac{x}{5y}$
- ☐ $5\cdot\frac{x}{y}=\frac{5x}{y}$
- ☐ $-\frac{x}{y}=\frac{-x}{-y}$
- ☐ $-\frac{x}{y}=\frac{-x}{y}$
- ☐ $-\frac{x}{y}=\frac{x}{-y}$

72. Select all correct equations:

- ☐ $6\cdot\frac{x}{y}=\frac{6x}{y}$
- ☐ $6\cdot\frac{x}{y}=\frac{6x}{6y}$
- ☐ $6\cdot\frac{x}{y}=\frac{x}{6y}$
- ☐ $-\frac{x}{y}=\frac{-x}{y}$
- ☐ $-\frac{x}{y}=\frac{x}{-y}$
- ☐ $-\frac{x}{y}=\frac{-x}{-y}$

73. Simplify the following expressions, and if applicable, write the restricted domain.

$-\dfrac{r^5}{r+5}\cdot r^2 =$

$-\dfrac{r^5}{r+5}\cdot\dfrac{1}{r^2} =$

74. Simplify the following expressions, and if applicable, write the restricted domain.

$-\dfrac{y^4}{y+4}\cdot y^2 =$

$-\dfrac{y^4}{y+4}\cdot\dfrac{1}{y^2} =$

Simplify this expression, and if applicable, write the restricted domain.

75. $\dfrac{t^2+3t+2}{t+5} \cdot \dfrac{3t+15}{t+1} =$ ☐

76. $\dfrac{t^2+2t-24}{t-5} \cdot \dfrac{4t-20}{t-4} =$ ☐

77. $\dfrac{x^2-16x}{x^2-16} \cdot \dfrac{x^2-4x}{x^2-15x-16} =$ ☐

78. $\dfrac{x^2-9x}{x^2-9} \cdot \dfrac{x^2-3x}{x^2-5x-36} =$ ☐

79. $\dfrac{25y+25}{24-2y-2y^2} \cdot \dfrac{y^2-6y+9}{5y^2+5y} =$ ☐

80. $\dfrac{20y+20}{-135-72y-9y^2} \cdot \dfrac{y^2+6y+9}{5y^2+5y} =$ ☐

81. $\dfrac{6y^2-13y+7}{36y^6-24y^5} \cdot \dfrac{4y^5-6y^6}{36y^2-49} =$ ☐

82. $\dfrac{3r^2+4r-7}{80r^5-120r^4} \cdot \dfrac{15r^4-10r^5}{9r^2-49} =$ ☐

83. $\dfrac{r}{r+15} \div 3r^4 =$ ☐

84. $\dfrac{t}{t-4} \div 4t^3 =$ ☐

85. $9t \div \dfrac{3}{t^4} =$ ☐

86. $8x \div \dfrac{4}{x^4} =$ ☐

87. $(2x+2) \div (12x+12) =$ ☐

88. $(4y+4) \div (16y+16) =$ ☐

89. $\dfrac{25y^2-16}{5y^2+9y+4} \div (4-5y) =$ ☐

90. $\dfrac{9y^2-4}{3y^2+5y+2} \div (2-3y) =$ ☐

91. $\dfrac{r^5}{r^2-5r} \div \dfrac{1}{r^2+r+(-30)} =$ ☐

92. $\dfrac{r^5}{r^2+2r} \div \dfrac{1}{r^2+(-1)\,r+(-6)} =$ ☐

93. $\dfrac{\frac{9m+2}{m}}{\frac{m+7}{m}} =$ ☐

94. $\dfrac{\frac{6m-10}{m}}{\frac{m-9}{m}} =$ ☐

95. $\dfrac{\frac{z}{(z-2)^2}}{\frac{5z}{z^2-4}} =$ ☐

96. $\dfrac{\frac{z}{(z-8)^2}}{\frac{9z}{z^2-64}} =$ ☐

97. $\dfrac{x^2-3x}{x^2-1} \div \dfrac{x^2-9}{x^2-3x+2} =$ ☐

98. $\dfrac{x^2-3x}{x^2-25} \div \dfrac{x^2-9}{x^2-3x-10} =$ ☐

Multiplying and Dividing Rational Expressions with More Than One Variable Simplify this expression.

99. $\frac{6(y+r)}{y-r}\cdot\frac{y-r}{2(3y+r)}=$

100. $\frac{8(r+y)}{r-y}\cdot\frac{r-y}{4(3r+y)}=$

101. $\frac{5r^3x}{2r}\cdot\frac{8r^3x^3}{15x^5}=$

102. $\frac{2tr}{5t^3}\cdot\frac{5t^4r^3}{4r^5}=$

103. $\frac{t^2-8ty+15y^2}{t-y}\cdot\frac{6t-6y}{t-5y}=$

104. $\frac{x^2+xt-2t^2}{x+3t}\cdot\frac{2x+6t}{x+2t}=$

105. $\frac{x^2r^5}{5}\div\frac{x^2r^3}{30}=$

106. $\frac{y^5x^4}{4}\div\frac{y^5x^2}{12}=$

107. $(y^3-2y^2t+yt^2)\div(y^5-y^4t)=$

108. $(y^3+10y^2r+25yr^2)\div(y^5+5y^4r)=$

109. $\frac{1}{r^2-4ry+3y^2}\div\frac{r^4}{r^2-3ry}=$

110. $\frac{1}{r^2+9rt+18t^2}\div\frac{r^4}{r^2+3rt}=$

111. $\frac{t^3}{t^2r-5t}\div\frac{1}{t^2r^2-9tr+20}=$

112. $\frac{t^5}{t^2x+5t}\div\frac{1}{t^2x^2+2tx-15}=$

113. $\frac{30x^3t^3}{x-5t}\div\frac{6x^4t}{x^2-25t^2}=$

114. $\frac{9x^5r^5}{x-5r}\div\frac{3x^6r}{x^2-25r^2}=$

115. $\frac{\frac{a}{b}}{\frac{3a}{2b^2}}=$

116. $\frac{\frac{a}{b}}{\frac{6a}{5b^2}}=$

117. $\frac{\frac{st^2}{6u}}{\frac{s}{7tu}}=$

118. $\frac{\frac{st^2}{4u}}{\frac{s}{2tu}}=$

Challenge

119. Simplify the following: $\frac{1}{x+1}\div\frac{x+2}{x+1}\div\frac{x+3}{x+2}\div\frac{x+4}{x+3}\div\cdots\div\frac{x+75}{x+74}$. For this exercise, you do not have to write the restricted domain of the simplified expression.

13.3 Addition and Subtraction of Rational Expressions

In the last section, we learned how to multiply and divide rational expressions. In this section, we will learn how to add and subtract rational expressions.

13.3.1 Introduction

Example 13.3.2 Julia is taking her family on a boat trip 12 miles down the river and back. The river flows at a speed of 2 miles per hour and she wants to drive the boat at a constant speed, v miles per hour downstream and back upstream. Due to the current of the river, the actual speed of travel is $v+2$ miles per hour going downstream, and $v-2$ miles per hour going upstream. If Julia plans to spend 8 hours for the whole trip, how fast should she drive the boat?

We need to review three forms of the formula for movement at a constant rate:

$$d = vt \qquad v = \frac{d}{t} \qquad t = \frac{d}{v}$$

where d stands for distance, v represents speed, and t stands for time. According to the third form, the time it takes the boat to travel downstream is $\frac{12}{v+2}$, and the time it takes to get back upstream is $\frac{12}{v-2}$.

The function to model the time of the whole trip is

$$t(v) = \frac{12}{v-2} + \frac{12}{v+2}$$

where t stands for time in hours, and v is the boat's speed in miles per hour. Let's look at the graph of this function in Figure 13.3.3. Note that since the speed v and the time $t(v)$ should be positive in context, it's only the first quadrant of Figure 13.3.3 that matters.

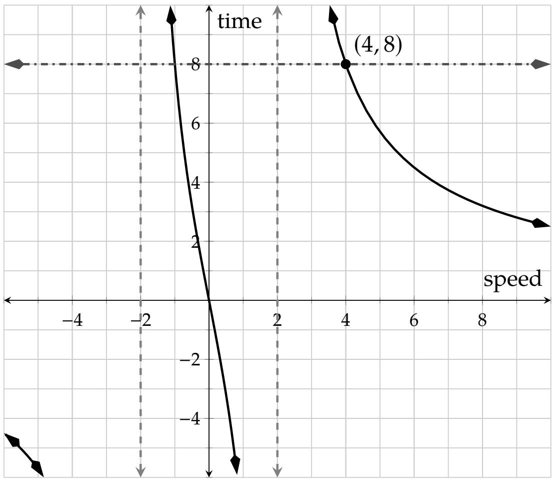


Figure 13.3.3: Graph of $t(v) = \frac{12}{v-2} + \frac{12}{v+2}$ and $t = 8$

To find the speed that Julia should drive the boat to make the round trip last 8 hours we can use graphing technology to solve the equation

$$\frac{12}{v-2} + \frac{12}{v+2} = 8$$

graphically and we see that $v = 4$. This tells us that a speed of 4 miles per hour will give a total time of 8 hours to complete the trip. To go downstream it would take $\frac{12}{v+2} = \frac{12}{4+2} = 2$ hours; and to go upstream it would take $\frac{12}{v-2} = \frac{12}{4-2} = 6$ hours.

The point of this section is to work with expressions like $\frac{12}{v-2} + \frac{12}{v+2}$, where two rational expressions are added (or subtracted). There are times when it is useful to combine them into a single fraction. We will learn that the expression $\frac{12}{v-2}+\frac{12}{v+2}$ is equal to the expression $\frac{24v}{v^2-4}$, and we will learn how to make that simplification.

13.3.2 Addition and Subtraction of Rational Expressions with the Same Denominator

The process of adding and subtracting rational expressions will be very similar to the process of adding and subtracting purely numerical fractions.

If the two expressions have the same denominator, then we can rely on the property of adding and subtracting fractions and simplify that result.

Let's review how to add fractions with the same denominator:

$$\begin{aligned}\frac{1}{10}+\frac{3}{10}&=\frac{1+3}{10}\\&=\frac{4}{10}\\&=\frac{2}{5}\end{aligned}$$

We can add and subtract rational expressions in the same way:

$$\begin{aligned}\frac{2}{3x}-\frac{5}{3x}&=\frac{2-5}{3x}\\&=\frac{-3}{3x}\\&=-\frac{1}{x}\end{aligned}$$

Identify the LCD Determine the least common denominator of all of the denominators.

Build If necessary, build each expression so that the denominators are the same.

Add/Subtract Combine the numerators using the properties of adding and subtracting fractions.

Simplify Simplify the resulting rational expression as much as possible. This may require factoring the numerator.

List 13.3.4: Steps to Adding/Subtracting Rational Expressions

Example 13.3.5 Add the rational expressions: $\frac{2x}{x+y}+\frac{2y}{x+y}$.

Explanation. These expressions already have a common denominator:

$$\begin{aligned}\frac{2x}{x+y}+\frac{2y}{x+y}&=\frac{2x+2y}{x+y}\\&=\frac{2\cancel{(x+y)}}{\cancel{x+y}}\\&=\frac{2}{1}\\&=2\end{aligned}$$

> Note that we didn't stop at $\frac{2x+2y}{x+y}$. If possible, we must simplify the numerator and denominator. Since this is a multivariable expression, this textbook ignores domain restrictions while canceling.

13.3.3 Addition and Subtraction of Rational Expressions with Different Denominators

To add rational expressions with different denominators, we'll need to build each fraction to the least common denominator, in the same way we do with numerical fractions. Let's briefly review this process by adding $\frac{3}{5}$ and $\frac{1}{6}$:

$$\begin{aligned}\frac{3}{5}+\frac{1}{6}&=\frac{3}{5}\cdot\frac{6}{6}+\frac{1}{6}\cdot\frac{5}{5}\\&=\frac{18}{30}+\frac{5}{30}\\&=\frac{18+5}{30}\\&=\frac{23}{30}\end{aligned}$$

This exact method can be used when adding rational expressions containing variables. The key is that the expressions *must* have the same denominator before they can be added or subtracted. If they don't have this initially, then we'll identify the least common denominator and build each expression so that it has that denominator.

Let's apply this to adding the two expressions with denominators that are $v-2$ and $v+2$ from Example 13.3.2.

> **Example 13.3.6** Add the rational expressions and fully simplify the function given by $t(v)=\frac{12}{v-2}+\frac{12}{v+2}$.
>
> **Explanation**.
>
> $$\begin{aligned}t(v)&=\frac{12}{v-2}+\frac{12}{v+2}\\t(v)&=\frac{12}{v-2}\cdot\frac{v+2}{v+2}+\frac{12}{v+2}\cdot\frac{v-2}{v-2}\\t(v)&=\frac{12v+24}{(v-2)(v+2)}+\frac{12v-24}{(v+2)(v-2)}\\t(v)&=\frac{(12v+24)+(12v-24)}{(v+2)(v-2)}\\t(v)&=\frac{24v}{(v+2)(v-2)}\end{aligned}$$

> **Example 13.3.7** Add the rational expressions: $\frac{2}{5x^2y}+\frac{3}{20xy^2}$
>
> **Explanation**. The least common denominator of $5x^2y$ and $20xy^2$ must include two x's and two y's, as well as 20. Thus it is $20x^2y^2$. We will build both denominators to $20x^2y^2$ before doing addition.
>
> $$\frac{2}{5x^2y}+\frac{3}{20xy^2}=\frac{2}{5x^2y}\cdot\frac{4y}{4y}+\frac{3}{20xy^2}\cdot\frac{x}{x}$$

$$\begin{aligned}
&= \frac{8y}{20x^2y^2} + \frac{3x}{20x^2y^2} \\
&= \frac{8y+3x}{20x^2y^2}
\end{aligned}$$

Let’s look at a few more complicated examples.

Example 13.3.8 Subtract the rational expressions: $\dfrac{y}{y-2} - \dfrac{8y-8}{y^2-4}$

Explanation. To start, we’ll make sure each denominator is factored. Then we’ll find the least common denominator and build each expression to that denominator. Then we will be able to combine the numerators and simplify the expression.

$$\begin{aligned}
\frac{y}{y-2} - \frac{8y-8}{y^2-4} &= \frac{y}{y-2} - \frac{8y-8}{(y+2)(y-2)} \\
&= \frac{y}{y-2}\cdot\frac{y+2}{y+2} - \frac{8y-8}{(y+2)(y-2)} \\
&= \frac{y^2+2y}{(y+2)(y-2)} - \frac{8y-8}{(y+2)(y-2)} \\
&= \frac{y^2+2y-\overset{\downarrow}{(}8y-8\overset{\downarrow}{)}}{(y+2)(y-2)} \\
&= \frac{y^2+2y-8y+8}{(y+2)(y-2)} \\
&= \frac{y^2-6y+8}{(y+2)(y-2)} \\
&= \frac{\cancel{(y-2)}(y-4)}{(y+2)\cancel{(y-2)}} \\
&= \frac{y-4}{y+2}, \text{ for } y \neq 2
\end{aligned}$$

Note that we must factor the numerator in $\frac{y^2-6y+8}{(y+2)(y-2)}$ and try to reduce the fraction (which we did).

Warning 13.3.9. In Example 13.3.8, be careful to subtract the entire numerator of $8y-8$. When this expression is in the numerator of $\frac{8y-8}{(y+2)(y-2)}$, it’s implicitly grouped and doesn’t need parentheses. But once $8y-8$ is subtracted from y^2+2y, we need to add parentheses so the entire expression is subtracted.

In the next example, we’ll look at adding a rational expression to a polynomial. Much like adding a fraction and an integer, we’ll rely on writing that expression as itself over one in order to build its denominator.

Example 13.3.10 Add the expressions: $-\dfrac{2}{r-1} + r$

Explanation.

$$-\frac{2}{r-1} + r = -\frac{2}{r-1} + \frac{r}{1}$$

$$\begin{aligned}
&= -\frac{2}{r-1} + \frac{r}{1} \cdot \frac{r-1}{r-1} \\
&= \frac{-2}{r-1} + \frac{r^2 - r}{r-1} \\
&= \frac{-2 + r^2 - r}{r-1} \\
&= \frac{r^2 - r - 2}{r-1} \\
&= \frac{(r-2)(r+1)}{r-1}
\end{aligned}$$

Note that we factored the numerator to reduce the fraction if possible. Even though it was not possible in this case, leaving it in factored form makes it easier to see that it is reduced.

Example 13.3.11 Subtract the expressions: $\frac{6}{x^2 - 2x - 8} - \frac{1}{x^2 + 3x + 2}$

Explanation. To start, we'll need to factor each of the denominators. After that, we'll identify the LCD and build each denominator accordingly. Then we can combine the numerators and simplify the resulting expression.

$$\begin{aligned}
\frac{6}{x^2 - 2x - 8} - \frac{1}{x^2 + 3x + 2} &= \frac{6}{(x-4)(x+2)} - \frac{1}{(x+2)(x+1)} \\
&= \frac{6}{(x-4)(x+2)} \cdot \frac{x+1}{x+1} - \frac{1}{(x+2)(x+1)} \cdot \frac{x-4}{x-4} \\
&= \frac{6x+6}{(x-4)(x+2)(x+1)} - \frac{x-4}{(x+2)(x+1)(x-4)} \\
&= \frac{6x+6-(x-4)}{(x-4)(x+2)(x+1)} \\
&= \frac{6x+6-x+4}{(x-4)(x+2)(x+1)} \\
&= \frac{5x+10}{(x-4)(x+2)(x+1)} \\
&= \frac{5\cancel{(x+2)}}{(x-4)\cancel{(x+2)}(x+1)} \\
&= \frac{5}{(x-4)(x+1)}, \text{ for } x \neq -2
\end{aligned}$$

Exercises

Review and Warmup

1. Add: $\frac{17}{24} + \frac{11}{24}$

2. Add: $\frac{7}{12} + \frac{7}{12}$

3. Add: $\frac{7}{10} + \frac{5}{6}$

4. Add: $\frac{9}{10} + \frac{5}{6}$

5. Subtract: $\frac{23}{21} - \frac{20}{21}$

6. Subtract: $\frac{11}{14} - \frac{3}{14}$

7. Subtract: $\frac{3}{7} - \frac{1}{21}$

8. Subtract: $\frac{3}{7} - \frac{19}{21}$

Factor the given polynomial.

9. $r^2 - 64 =$

10. $t^2 - 16 =$

11. $t^2 + 11t + 10 =$

12. $x^2 + 12x + 35 =$

13. $x^2 - 12x + 27 =$

14. $x^2 - 13x + 30 =$

15. $9y^2 - 27y + 18 =$

16. $8y^2 - 24y + 16 =$

Addition and Subtraction of Rational Expressions with One Variable Add or subtract the rational expressions to a single rational expression and then simplify. If applicable, state the restricted domain.

17. $\frac{4r}{r+2} + \frac{8}{r+2} =$

18. $\frac{6r}{r+6} + \frac{36}{r+6} =$

19. $\frac{3t}{t+5} + \frac{15}{t+5} =$

20. $\frac{6t}{t+3} + \frac{18}{t+3} =$

21. $\frac{1}{x^2 - x - 30} - \frac{x-5}{x^2 - x - 30} =$

22. $\frac{6}{x^2 - 12x + 20} - \frac{x-4}{x^2 - 12x + 20} =$

23. $\frac{4}{x^2 - 25} - \frac{x-1}{x^2 - 25} =$

24. $\frac{5}{y^2 - 8y + 7} - \frac{y-2}{y^2 - 8y + 7} =$

25. $\frac{4y}{5} + \frac{y}{30} =$

26. $\frac{5r}{6} + \frac{r}{18} =$

27. $\frac{6}{r+2} - \frac{5}{r+4} =$

28. $\frac{5}{t-5} - \frac{2}{t-4} =$

29. $\frac{3}{t-1} - \frac{2}{t+5} =$

30. $\frac{5}{x+6} - \frac{2}{x+2} =$

31. $\frac{1}{x+2} + \frac{4}{x^2 - 4} =$

32. $\frac{1}{x+3} + \frac{6}{x^2 - 9} =$

33. $\frac{1}{y-1} - \frac{2}{y^2 - 1} =$

34. $\frac{1}{y-3} - \frac{6}{y^2 - 9} =$

35. $\frac{5}{r+2}-\frac{10r}{r^2-4}=$ ☐

36. $\frac{4}{r-3}-\frac{8r}{r^2-9}=$ ☐

37. $\frac{5}{t+5}-\frac{10t}{t^2-25}=$ ☐

38. $\frac{4}{t-2}-\frac{8t}{t^2-4}=$ ☐

39. $\frac{x}{x+2}-\frac{6x+16}{x^2+2x}=$ ☐

40. $\frac{x}{x+2}-\frac{4x+12}{x^2+2x}=$ ☐

41. $\frac{x}{x-2}-\frac{3x-2}{x^2-2x}=$ ☐

42. $\frac{y}{y-9}-\frac{15y-54}{y^2-9y}=$ ☐

43. $\frac{6}{y^2-9}+\frac{3}{y+3}-\frac{1}{y-3}=$ ☐

44. $\frac{6}{r^2-1}+\frac{5}{r+1}-\frac{3}{r-1}=$ ☐

45. $\frac{4r}{r^2-7r+12}+\frac{4r}{r-3}=$ ☐

46. $-\frac{3t}{t^2-5t+6}-\frac{3t}{t-2}=$ ☐

47. $\frac{4t}{t^2-11t+30}+\frac{4t}{t-5}=$ ☐

48. $-\frac{3x}{x^2+5x+6}-\frac{3x}{x+3}=$ ☐

49. $\frac{x^2-5}{x^2+5x}-\frac{x-1}{x}=$ ☐

50. $\frac{x^2+15}{x^2-3x}-\frac{x-5}{x}=$ ☐

51. $\frac{3}{y+5}+4=$ ☐

52. $-\frac{5}{y+1}-4=$ ☐

53. $\frac{4r}{r+2}+\frac{r}{r-2}-5=$ ☐

54. $\frac{6r}{r+1}+\frac{r}{r-1}-7=$ ☐

Addition and Subtraction of Rational Expressions with More Than Variable Add or subtract the rational expressions to a single rational expression and then simplify.

55. $\frac{25t^2}{5t-8r}-\frac{64r^2}{5t-8r}=$ ☐

56. $\frac{81t^2}{9t+4y}-\frac{16y^2}{9t+4y}=$ ☐

57. $\frac{5x}{18t}+\frac{x}{6t}=$ ☐

58. $\frac{4x}{15r}+\frac{4x}{3r}=$ ☐

59. $\frac{2x}{3y^3} - \frac{2}{5xy} = \boxed{\quad}$

60. $-\frac{3y}{2t^4} + \frac{6}{5yt} = \boxed{\quad}$

61. $\frac{3}{yr-6} - \frac{6yr}{y^2r^2-36} = \boxed{\quad}$

62. $\frac{3}{rx+2} - \frac{6rx}{r^2x^2-4} = \boxed{\quad}$

63. $-\frac{6rt}{r^2+4rt+3t^2} + \frac{3r}{r+t} = \boxed{\quad}$

64. $\frac{6tr}{t^2-13tr+40r^2} - \frac{2t}{t-8r} = \boxed{\quad}$

13.4 Complex Fractions

In this section, we will learn how to simplify complex fractions, which have fractions in the numerator and/or denominator of another fraction.

13.4.1 Simplifying Complex Fractions

Consider the rational expression

$$\frac{\frac{6}{x-4}}{\frac{6}{x-4}+3}.$$

It's difficult to quickly evaluate this expression, or determine the important information such as its domain. This type of rational expression, which contains a "fraction within a fraction," is referred to as a **complex fraction**. Our goal is to simplify such a fraction so that it has a *single* numerator and a *single* denominator, neither of which contain any fractions themselves.

A complex fraction may have fractions in its numerator and/or denominator. Here is an example to show how we use division to simplify a complex fraction.

$$\begin{aligned}\frac{\frac{1}{2}}{3} &= \frac{1}{2} \div 3\\ &= \frac{1}{2} \div \frac{3}{1}\\ &= \frac{1}{2} \cdot \frac{1}{3}\\ &= \frac{1}{6}\end{aligned}$$

What if the expression had something more complicated in the denominator, like $\frac{\frac{1}{2}}{\frac{1}{3}+\frac{1}{4}}$? We would no longer be able to simply multiply by the reciprocal of the denominator, since we don't immediately know the reciprocal of that denominator. Instead, we could multiply the "main" numerator and denominator by something that eliminates all of the "internal" denominators. (We'll use the LCD to determine this). For example, with $\frac{\frac{1}{2}}{3}$, we can multiply by $\frac{2}{2}$:

$$\begin{aligned}\frac{\frac{1}{2}}{3} &= \frac{\frac{1}{2}}{3} \cdot \frac{2}{2}\\ &= \frac{1}{6}\end{aligned}$$

Remark 13.4.2. In the last example, it's important to identify which fraction bar is the "main" fraction bar, and which fractions are "internal." Comparing the two expressions below, both of which are "one over two over three", we see that they are not equivalent.

$$\begin{aligned}\frac{\frac{1}{2}}{3} &= \frac{\frac{1}{2}}{3} \cdot \frac{2}{2}\\ &= \frac{1}{6}\end{aligned} \qquad \text{versus} \qquad \begin{aligned}\frac{1}{\frac{2}{3}} &= \frac{1}{\frac{2}{3}} \cdot \frac{3}{3}\\ &= \frac{3}{2}\end{aligned}$$

For the first of these, the "main" fraction bar is above the 3, but for the second of these, the "main" fraction bar is above the $\frac{2}{3}$.

To attack multiple fractions in a complex fraction, we need to multiply the numerator and denominator by the LCD of all the internal fractions, as we will show in the next example.

Example 13.4.3 Simplify the complex fraction $\dfrac{\frac{1}{2}}{\frac{1}{3}+\frac{1}{4}}$.

Explanation.

The internal denominators are 2, 3, and 4, so the LCD is 12. We will thus multiply the main numerator and denominator by 12 and simplify the result:

$$\begin{aligned}
\frac{\frac{1}{2}}{\frac{1}{3}+\frac{1}{4}} &= \frac{\frac{1}{2}}{\frac{1}{3}+\frac{1}{4}} \cdot \frac{12}{12} \\
&= \frac{\frac{1}{2}\cdot 12}{\left(\frac{1}{3}+\frac{1}{4}\right)\cdot 12} \\
&= \frac{6}{4+3} \\
&= \frac{6}{7}
\end{aligned}$$

Next we will evaluate a function whose formula is a complex fraction and then simplify the result.

Example 13.4.4 Find each function value for $f(x) = \dfrac{\frac{x+2}{x+3}}{\frac{2}{x+3}-\frac{3}{x-1}}$.

a. $f(4)$

b. $f(0)$

c. $f(-3)$

d. $f(-11)$

Explanation. We will determine each function value by replacing x with the specified number and then simplify the complex fraction:

a.
$$\begin{aligned}
f(4) &= \frac{\frac{4+2}{4+3}}{\frac{2}{4+3}-\frac{3}{4-1}} \\
&= \frac{\frac{6}{7}}{\frac{2}{7}-\frac{3}{3}} \\
&= \frac{\frac{6}{7}}{\frac{2}{7}-1}\cdot\frac{7}{7} \\
&= \frac{6}{2-7} \\
&= -\frac{6}{5}
\end{aligned}$$

b.
$$\begin{aligned}
f(0) &= \frac{\frac{0+2}{0+3}}{\frac{2}{0+3}-\frac{3}{0-1}} \\
&= \frac{\frac{2}{3}}{\frac{2}{3}-\frac{3}{-1}} \\
&= \frac{\frac{2}{3}}{\frac{2}{3}+3}\cdot\frac{3}{3} \\
&= \frac{2}{2+9} \\
&= \frac{2}{11}
\end{aligned}$$

c. When evaluating f at -3, we can quickly see that this results in division by zero:

$$\begin{aligned}
f(-3) &= \frac{\frac{-3+2}{-3+3}}{\frac{2}{-3+3}-\frac{3}{-3-1}} \\
&= \frac{\frac{2}{0}}{\frac{2}{0}-\frac{3}{-4}}
\end{aligned}$$

Thus $f(-3)$ is undefined.

d.
$$\begin{aligned}
f(-11) &= \frac{\frac{-11+2}{-11+3}}{\frac{2}{-11+3}-\frac{3}{-11-1}} \\
&= \frac{\frac{-8}{-9}}{\frac{2}{-8}-\frac{3}{-12}} \\
&= \frac{\frac{8}{9}}{-\frac{1}{4}+\frac{1}{4}} \\
&= \frac{\frac{8}{9}}{0}
\end{aligned}$$

Therefore $f(-11)$ is undefined.

We have simplified complex fractions involving numbers and now we will apply the same concept to complex fractions with variables.

Example 13.4.5 Simplify the complex fraction $\dfrac{3}{\frac{1}{y}+\frac{5}{y^2}}$.

Explanation.

To start, we look at the internal denominators and identify the LCD as y^2. We'll multiply the main numerator and denominator by the LCD, and then simplify. Since we are multiplying by $\frac{y^2}{y^2}$, it is important to note that y cannot be 0, since $\frac{0}{0}$ is undefined.

$$\begin{aligned}\frac{3}{\frac{1}{y}+\frac{5}{y^2}} &= \frac{3}{\frac{1}{y}+\frac{5}{y^2}}\cdot\frac{y^2}{y^2}\\ &= \frac{3\cdot y^2}{\frac{1}{y}\cdot y^2+\frac{5}{y^2}\cdot y^2}\\ &= \frac{3y^2}{y+5},\text{ for } y\neq 0\end{aligned}$$

Example 13.4.6 Simplify the complex fraction $\frac{\frac{5x-6}{2x+1}}{\frac{3x+2}{2x+1}}$.

Explanation.

The internal denominators are both $2x+1$, so this is the LCD and we will multiply the main numerator and denominator by this expression. Since we are multiplying by $\frac{2x+1}{2x+1}$, what x-value would cause $2x+1$ to equal 0? Solving $2x+1=0$ leads to $x=-\frac{1}{2}$. So x cannot be $-\frac{1}{2}$, since $\frac{0}{0}$ is undefined.

$$\begin{aligned}\frac{\frac{5x-6}{2x+1}}{\frac{3x+2}{2x+1}} &= \frac{\frac{5x-6}{2x+1}}{\frac{3x+2}{2x+1}}\cdot\frac{2x+1}{2x+1}\\ &= \frac{5x-6}{3x+2},\text{ for } x\neq -\frac{1}{2}\end{aligned}$$

Example 13.4.7 Completely simplify the function defined by $f(x)=\frac{\frac{x+2}{x+3}}{\frac{2}{x+3}-\frac{3}{x-1}}$. Then determine the domain of this function.

Explanation. The LCD of the internal denominators is $(x+3)(x-1)$. We will thus multiply the main numerator and denominator by the expression $(x+3)(x-1)$ and then simplify the resulting expression.

$$\begin{aligned}f(x) &= \frac{\frac{x+2}{x+3}}{\frac{2}{x+3}-\frac{3}{x-1}}\\ f(x) &= \frac{\frac{x+2}{x+3}}{\frac{2}{x+3}-\frac{3}{x-1}}\cdot\frac{(x+3)(x-1)}{(x+3)(x-1)}\\ f(x) &= \frac{\frac{x+2}{x+3}\cdot(x+3)(x-1)}{\left(\frac{2}{x+3}-\frac{3}{x-1}\right)\cdot(x+3)(x-1)}\\ f(x) &= \frac{\frac{x+2}{x+3}\cdot(x+3)(x-1)}{\frac{2}{x+3}\cdot(x+3)(x-1)-\frac{3}{x-1}\cdot(x+3)(x-1)}\\ f(x) &= \frac{(x+2)(x-1)}{2(x-1)-3(x+3)},\text{ for } x\neq -3, x\neq 1\\ f(x) &= \frac{(x+2)(x-1)}{2x-2-3x-9},\text{ for } x\neq -3, x\neq 1\\ f(x) &= \frac{(x+2)(x-1)}{-x-11},\text{ for } x\neq -3, x\neq 1\end{aligned}$$

$$f(x) = \frac{(x+2)(x-1)}{-(x+11)}, \text{ for } x \neq -3, x \neq 1$$

In the original (unsimplified) function, we could see that $x \neq -3$ and $x \neq 1$. In the simplified function, we need $x + 11 \neq 0$, so we can also see that $x \neq -11$. Therefore the domain of the function f is $\{x \mid x \neq -11, -3, 1\}$.

Example 13.4.8 Simplify the complex fraction $\frac{2\left(\frac{-4x+3}{x-2}\right)+3}{\frac{-4x+3}{x-2}+4}$.

Explanation. The only internal denominator is $x-2$, so we will begin by multiplying the main numerator and denominator by this. Then we'll simplify the resulting expression.

$$\begin{aligned}
\frac{2\left(\frac{-4x+3}{x-2}\right)+3}{\frac{-4x+3}{x-2}+4} &= \frac{2\left(\frac{-4x+3}{x-2}\right)+3}{\frac{-4x+3}{x-2}+4} \cdot \frac{x-2}{x-2} \\
&= \frac{2\left(\frac{-4x+3}{x-2}\right)\cdot(x-2)+3\cdot(x-2)}{\left(\frac{-4x+3}{x-2}\right)\cdot(x-2)+4\cdot(x-2)} \\
&= \frac{2(-4x+3)+3(x-2)}{(-4x+3)+4(x-2)}, \text{ for } x \neq 2 \\
&= \frac{-8x+6+3x-6}{-4x+3+4x-8}, \text{ for } x \neq 2 \\
&= \frac{-5x}{-5}, \text{ for } x \neq 2 \\
&= x, \text{ for } x \neq 2
\end{aligned}$$

Example 13.4.9 Simplify the complex fraction $\frac{\frac{5}{x}+\frac{4}{y}}{\frac{3}{x}-\frac{2}{y}}$. Recall that with a multivariable expression, this textbook ignores domain restrictions.

Explanation.

We multiply the numerator and denominator by the common denominator of x and y, which is xy:

$$\begin{aligned}
\frac{\frac{5}{x}+\frac{4}{y}}{\frac{3}{x}-\frac{2}{y}} &= \frac{\frac{5}{x}+\frac{4}{y}}{\frac{3}{x}-\frac{2}{y}} \cdot \frac{xy}{xy} \\
&= \frac{\left(\frac{5}{x}+\frac{4}{y}\right)\cdot xy}{\left(\frac{3}{x}-\frac{2}{y}\right)\cdot xy} \\
&= \frac{\frac{5}{x}\cdot xy+\frac{4}{y}\cdot xy}{\frac{3}{x}\cdot xy-\frac{2}{y}\cdot xy} \\
&= \frac{5y+4x}{3y-2x}
\end{aligned}$$

Example 13.4.10 Simplify the complex fraction $\dfrac{\frac{t}{t+3}+\frac{2}{t-3}}{1-\frac{t}{t^2-9}}$.

Explanation. First, we check all quadratic polynomials to see if they can be factored and factor them:

$$\frac{\frac{t}{t+3}+\frac{2}{t-3}}{1-\frac{t}{t^2-9}} = \frac{\frac{t}{t+3}+\frac{2}{t-3}}{1-\frac{t}{(t-3)(t+3)}}$$

Next, we identify the common denominator of the three fractions, which is $(t+3)(t-3)$. We then multiply the main numerator and denominator by that expression:

$$\begin{aligned}
\frac{\frac{t}{t+3}+\frac{2}{t-3}}{1-\frac{t}{t^2-9}} &= \frac{\frac{t}{t+3}+\frac{2}{t-3}}{1-\frac{t}{(t-3)(t+3)}}\cdot\frac{(t+3)(t-3)}{(t+3)(t-3)}\\
&= \frac{\frac{t}{t+3}\cdot(t+3)(t-3)+\frac{2}{t-3}\cdot(t+3)(t-3)}{1\cdot(t+3)(t-3)-\frac{t}{(t-3)(t+3)}\cdot(t+3)(t-3)}\\
&= \frac{t(t-3)+2(t+3)}{(t+3)(t-3)-t}\text{ for } t\neq -3, t\neq 3\\
&= \frac{t^2-3t+2t+6}{t^2-9-t}\text{ for } t\neq -3, t\neq 3\\
&= \frac{t^2-t+6}{t^2-t-9}\text{ for } t\neq -3, t\neq 3
\end{aligned}$$

Note that since both the numerator and denominator are prime trinomials, this expression can neither factor nor simplify any further.

Exercises

Review and Warmup Calculate the following. Use an improper fraction in your answer.

1. a. $\dfrac{\frac{3}{7}}{\frac{3}{4}} = \boxed{}$ b. $\dfrac{\frac{t}{x}}{\frac{r}{y}} = \boxed{}$

2. a. $\dfrac{\frac{16}{5}}{\frac{4}{7}} = \boxed{}$ b. $\dfrac{\frac{t}{r}}{\frac{y}{x}} = \boxed{}$

3. a. $\dfrac{3}{\frac{4}{7}} = \boxed{}$ b. $\dfrac{\frac{3}{4}}{7} = \boxed{}$

4. a. $\dfrac{2}{\frac{3}{8}} = \boxed{}$ b. $\dfrac{\frac{2}{3}}{8} = \boxed{}$

5. $\dfrac{\frac{2}{3}+\frac{4}{5}}{\frac{1}{6}} = \boxed{}$

6. $\dfrac{\frac{5}{4}+\frac{1}{5}}{\frac{2}{3}} = \boxed{}$

7. $\dfrac{4}{\frac{1}{2}-\frac{2}{3}} = \boxed{}$

8. $\dfrac{4}{\frac{5}{4}-\frac{6}{5}} = \boxed{}$

Simplifying Complex Fractions with One Variable Simplify this expression, and if applicable, write the restricted domain.

9. $\dfrac{\frac{6p+8}{p}}{\frac{p-5}{p}} = \boxed{}$

10. $\dfrac{\frac{3m-4}{m}}{\frac{m+6}{m}} = \boxed{}$

11. $\dfrac{\frac{k}{(k-8)^2}}{\frac{8k}{k^2-64}} = \boxed{}$

12. $\dfrac{\frac{z}{(z-4)^2}}{\frac{3z}{z^2-16}} = \boxed{}$

13. $\dfrac{2+\frac{1}{a}}{a+6} = \boxed{}$

14. $\dfrac{7+\frac{1}{a}}{a+10} = \boxed{}$

15. $\dfrac{2}{\frac{4}{y}+\frac{3}{y+2}} = \boxed{}$

16. $\dfrac{5}{\frac{1}{r}-\frac{4}{r+4}} = \boxed{}$

17. $\dfrac{7+\frac{1}{q-3}}{\frac{1}{q-3}-\frac{1}{6}} = \boxed{}$

18. $\dfrac{4+\frac{1}{q-7}}{\frac{1}{q-7}-\frac{1}{3}} = \boxed{}$

19. $\dfrac{\frac{1}{k+10}+\frac{1}{k-10}}{10-\frac{1}{k-10}} = \boxed{}$

20. $\dfrac{\frac{1}{k+6}+\frac{5}{k-6}}{8-\frac{1}{k-6}} = \boxed{}$

21. $\dfrac{\frac{1}{x-3}+\frac{9}{x-3}}{5-\frac{1}{x+3}} = \boxed{}$

22. $\dfrac{\frac{1}{a-9}+\frac{4}{a-9}}{3-\frac{1}{a+9}} = \boxed{}$

23. $\dfrac{\frac{6}{b-1}-8}{\frac{1}{b-1}+\frac{1}{b-9}} = \boxed{}$

24. $\dfrac{\frac{2}{t-1}-3}{\frac{1}{t-1}+\frac{1}{t-7}} = \boxed{}$

25. $\dfrac{\frac{4r}{r^2-25}-1}{\frac{5}{r+5}-\frac{6}{r-5}} = \boxed{}$

26. $\dfrac{\frac{6r}{r^2-4}+1}{\frac{6}{r+2}-\frac{5}{r-2}} = \boxed{}$

27. $\dfrac{\frac{r}{r^2-4}-\frac{1}{r^2-4}}{\frac{1}{r+4}} = \boxed{}$

28. $\dfrac{\frac{k}{k^2-49}-\frac{1}{k^2-49}}{\frac{1}{k+49}} = \boxed{}$

Simplifying Complex Fractions with More Than One Variable Simplify this expression.

29. $\dfrac{\frac{m}{n}}{\frac{4m}{3n^2}} = \boxed{}$

30. $\dfrac{\frac{x}{y}}{\frac{3x}{2y^2}} = \boxed{}$

31. $\dfrac{\frac{ab^2}{9c}}{\frac{a}{2bc}} = \boxed{}$

32. $\dfrac{\frac{ab^2}{6c}}{\frac{a}{6bc}} = \boxed{}$

33.

a. $\dfrac{\frac{y}{x}}{t} = \boxed{}$

b. $\dfrac{y}{\frac{x}{t}} = \boxed{}$

34.

a. $\dfrac{\frac{r}{y}}{t} = \boxed{}$

b. $\dfrac{r}{\frac{y}{t}} = \boxed{}$

35. $\dfrac{\frac{4}{r}}{16+\frac{4y}{5}} = \boxed{}$

36. $\dfrac{\frac{2}{t}}{4+\frac{2x}{5}} = \boxed{}$

37. $\dfrac{\frac{4}{t}+\frac{12}{r}}{\frac{8}{t}+\frac{12}{r}} = \boxed{}$

38. $\dfrac{\frac{3}{t}+\frac{6}{y}}{\frac{9}{t}+\frac{6}{y}} = \boxed{}$

13.5 Solving Rational Equations

13.5.1 Solving Rational Equations

To start this section, we will use a scenario we have seen before in Example 13.3.2:

Julia is taking her family on a boat trip 12 miles down the river and back. The river flows at a speed of 2 miles per hour and she wants to drive the boat at a constant speed, v miles per hour downstream and back upstream. Due to the current of the river, the actual speed of travel is $v+2$ miles per hour going downstream, and $v-2$ miles per hour going upstream. If Julia plans to spend 8 hours for the whole trip, how fast should she drive the boat?

The time it takes Julia to drive the boat downstream is $\frac{12}{v+2}$ hours, and upstream is $\frac{12}{v-2}$ hours. The function to model the whole trip's time is

$$t(v) = \frac{12}{v-2} + \frac{12}{v+2}$$

where t stands for time in hours. The trip will take 8 hours, so we substitute $t(v)$ with 8, and we have:

$$\frac{12}{v-2} + \frac{12}{v+2} = 8.$$

Instead of using the function's graph, we will solve this equation algebraically. You may wish to review the technique of eliminating denominators discussed in Subsection 3.3.2. We can use the same technique with variable expressions in the denominators. To remove the fractions in this equation, we will multiply both sides of the equation by the least common denominator $(v-2)(v+2)$, and we have:

$$\begin{aligned}
\frac{12}{v-2} + \frac{12}{v+2} &= 8 \\
(v+2)(v-2)\cdot\left(\frac{12}{v-2} + \frac{12}{v+2}\right) &= (v+2)(v-2)\cdot 8 \\
(v+2)\cancel{(v-2)}\cdot\frac{12}{\cancel{v-2}} + \cancel{(v+2)}(v-2)\cdot\frac{12}{\cancel{v+2}} &= (v+2)(v-2)\cdot 8 \\
12(v+2) + 12(v-2) &= 8(v^2-4) \\
12v + 24 + 12v - 24 &= 8v^2 - 32 \\
24v &= 8v^2 - 32 \\
0 &= 8v^2 - 24v - 32 \\
0 &= 8(v^2 - 3v - 4) \\
0 &= 8(v-4)(v+1)
\end{aligned}$$

$$\begin{aligned}
v-4 &= 0 & &\text{or} & v+1 &= 0 \\
v &= 4 & &\text{or} & v &= -1
\end{aligned}$$

Remark 13.5.2. At this point, logically all that we know is that the only *possible* solutions are -1 and 4. Because of the step where factors were canceled, it's possible that these might not actually be solutions to the original equation. They each might be what is called an **extraneous solution**. An extraneous solution is a number that would appear to be a solution based on the solving process, but actually does not make the original equation true. Because of this, it is important that these proposed solutions be checked. Note that we're not checking to see if we made a calculation error, but are instead checking to see if the proposed solutions actually solve the original equation.

We check these values.

$$\begin{aligned}\frac{12}{-1-2}+\frac{12}{-1+2}&\stackrel{?}{=}8\\ \frac{12}{-3}+\frac{12}{1}&\stackrel{?}{=}8\\ -4+12&\stackrel{\checkmark}{=}8\end{aligned}\qquad\qquad\begin{aligned}\frac{12}{4-2}+\frac{12}{4+2}&\stackrel{?}{=}8\\ \frac{12}{2}+\frac{12}{6}&\stackrel{?}{=}8\\ 6+2&\stackrel{\checkmark}{=}8\end{aligned}$$

Algebraically, both values do check out to be solutions. In the context of this scenario, the boat's speed can't be negative, so we only take the solution 4. If Julia drives at 4 miles per hour, the whole trip would take 8 hours. This result matches the solution in Example 13.3.2.

Let's look at another application problem.

> **Example 13.5.3** It takes Ku 3 hours to paint a room and it takes Jacob 6 hours to paint the same room. If they work together, how long would it take them to paint the room?
>
> **Explanation**. Since it takes Ku 3 hours to paint the room, he paints $\frac{1}{3}$ of the room each hour. Similarly, Jacob paints $\frac{1}{6}$ of the room each hour. If they work together, they paint $\frac{1}{3}+\frac{1}{6}$ of the room each hour.
>
> Assume it takes x hours to paint the room if Ku and Jacob work together. This implies they paint $\frac{1}{x}$ of the room together each hour. Now we can write this equation:
>
> $$\frac{1}{3}+\frac{1}{6}=\frac{1}{x}.$$
>
> To clear away denominators, we multiply both sides of the equation by the common denominator of 3, 6 and x, which is $6x$:
>
> $$\begin{aligned}\frac{1}{3}+\frac{1}{6}&=\frac{1}{x}\\ 6x\cdot\left(\frac{1}{3}+\frac{1}{6}\right)&=6\cancel{x}\cdot\frac{1}{\cancel{x}}\\ 6x\cdot\frac{1}{3}+6x\cdot\frac{1}{6}&=6\\ 2x+x&=6\\ 3x&=6\\ x&=2\end{aligned}$$
>
> Does the possible solution $x=2$ check as an actual solution?
>
> $$\begin{aligned}\frac{1}{3}+\frac{1}{6}&\stackrel{?}{=}\frac{1}{2}\\ \frac{2}{6}+\frac{1}{6}&\stackrel{?}{=}\frac{1}{2}\\ \frac{3}{6}&\stackrel{\checkmark}{=}\frac{1}{2}\end{aligned}$$
>
> It does, so it is a solution. If Ku and Jacob work together, it would take them 2 hours to paint the room.

Let's look at a few more examples of solving rational equations.

Example 13.5.4 Solve for y in $\frac{2}{y+1} = \frac{3}{y}$.

Explanation. The common denominator is $y(y+1)$. We will multiply both sides of the equation by $y(y+1)$:

$$\begin{aligned}
\frac{2}{y+1} &= \frac{3}{y}\\
y\cancel{(y+1)}\cdot\frac{2}{\cancel{y+1}} &= \cancel{y}(y+1)\cdot\frac{3}{\cancel{y}}\\
2y &= 3(y+1)\\
2y &= 3y+3\\
0 &= y+3\\
-3 &= y
\end{aligned}$$

Does the possible solution $y = -3$ check as an actual solution?

$$\begin{aligned}
\frac{2}{-3+1} &\overset{?}{=} \frac{3}{-3}\\
\frac{2}{-2} &\overset{\checkmark}{=} -1
\end{aligned}$$

It checks, so -3 is a solution. We write the solution set as $\{-3\}$.

Example 13.5.5 Solve for z in $z + \frac{1}{z-4} = \frac{z-3}{z-4}$.

Explanation. The common denominator is $z-4$. We will multiply both sides of the equation by $z-4$:

$$\begin{aligned}
z + \frac{1}{z-4} &= \frac{z-3}{z-4}\\
(z-4)\cdot\left(z + \frac{1}{z-4}\right) &= \cancel{(z-4)}\cdot\frac{z-3}{\cancel{z-4}}\\
(z-4)\cdot z + \cancel{(z-4)}\cdot\frac{1}{\cancel{z-4}} &= z-3\\
(z-4)\cdot z + 1 &= z-3\\
z^2 - 4z + 1 &= z-3\\
z^2 - 5z + 4 &= 0\\
(z-1)(z-4) &= 0
\end{aligned}$$

$$\begin{aligned}
z-1 &= 0 &\quad&\text{or}&\quad z-4 &= 0\\
z &= 1 &&\text{or}& z &= 4
\end{aligned}$$

Do the possible solutions $z = 1$ and $z = 4$ check as actual solutions?

$$\begin{aligned}
1 + \frac{1}{1-4} &\overset{?}{=} \frac{1-3}{1-4}\\
1 - \frac{1}{3} &\overset{\checkmark}{=} \frac{-2}{-3}
\end{aligned}$$

$$\begin{aligned}
4 + \frac{1}{4-4} &\overset{?}{=} \frac{4-3}{4-4}\\
4 + \frac{1}{0} &\overset{\text{no}}{=} \frac{1}{0}
\end{aligned}$$

The possible solution $z = 4$ does not actually work, since it leads to division by 0 in the equation. It is an extraneous solution. However, $z = 1$ is a valid solution. The only solution to the equation is 1, and thus we can write the solution set as $\{1\}$.

Example 13.5.6 Solve for p in $\frac{3}{p-2} + \frac{5}{p+2} = \frac{12}{p^2-4}$.

Explanation. To find the common denominator, we need to factor all denominators if possible:

$$\frac{3}{p-2} + \frac{5}{p+2} = \frac{12}{(p+2)(p-2)}$$

Now we can see the common denominator is $(p+2)(p-2)$. We will multiply both sides of the equation by $(p+2)(p-2)$:

$$\begin{aligned}
\frac{3}{p-2} + \frac{5}{p+2} &= \frac{12}{p^2-4}\\
\frac{3}{p-2} + \frac{5}{p+2} &= \frac{12}{(p+2)(p-2)}\\
(p+2)(p-2)\cdot\left(\frac{3}{p-2} + \frac{5}{p+2}\right) &= (p+2)(p-2)\cdot\frac{12}{(p+2)(p-2)}\\
(p+2)\cancel{(p-2)}\cdot\frac{3}{\cancel{p-2}} + \cancel{(p+2)}(p-2)\cdot\frac{5}{\cancel{p+2}} &= \cancel{(p+2)(p-2)}\cdot\frac{12}{\cancel{(p+2)(p-2)}}\\
3(p+2)+5(p-2) &= 12\\
3p+6+5p-10 &= 12\\
8p-4 &= 12\\
8p &= 16\\
p &= 2
\end{aligned}$$

Does the possible solution $p = 2$ check as an actual solution?

$$\begin{aligned}
\frac{3}{2-2} + \frac{5}{2+2} &\overset{?}{=} \frac{12}{2^2-4}\\
\frac{3}{0} + \frac{5}{4} &\overset{\text{no}}{=} \frac{12}{0}
\end{aligned}$$

The possible solution $p = 2$ does not actually work, since it leads to division by 0 in the equation. So this is an extraneous solution, and the equation actually has no solution. We could say that its solution set is the empty set, $\emptyset$.

Example 13.5.7 Solve $C(t) = 0.35$, where $C(t) = \frac{3t}{t^2+8}$ gives a drug's concentration in milligrams per liter t hours since an injection. (This function was explored in the introduction of Section 13.1.)

Explanation. To solve $C(t) = 0.35$, we'll begin by setting up $\frac{3t}{t^2+8} = 0.35$. We'll begin by identifying that the LCD is $t^2 + 8$, and multiply each side of the equation by this:

$$\begin{aligned}
\frac{3t}{t^2+8} &= 0.35\\
\frac{3t}{\cancel{t^2+8}}\cdot\cancel{\left(t^2+8\right)} &= 0.35\cdot\left(t^2+8\right)
\end{aligned}$$

$$\begin{aligned}3t &= 0.35\left(t^2+8\right)\\3t &= 0.35t^2+2.8\end{aligned}$$

This results in a quadratic equation so we will put it in standard form and use the quadratic formula:

$$\begin{aligned}0 &= 0.35t^2-3t+2.8\\t &= \frac{-(-3)\pm\sqrt{(-3)^2-4(0.35)(2.8)}}{2(0.35)}\\t &= \frac{3\pm\sqrt{5.08}}{0.7}\\t &\approx 1.066 \text{ or } t\approx 7.506\end{aligned}$$

Each of these answers should be checked in the original equation; they both work. In context, this means that the drug concentration will reach 0.35 milligrams per liter about 1.066 hours after the injection was given, and again 7.506 hours after the injection was given.

13.5.2 Solving Rational Equations for a Specific Variable

Rational equations can contain many variables and constants and we can solve for any one of them. The process for solving still involves multiplying each side of the equation by the LCD. Instead of having a numerical answer though, our final result will contain other variables and constants.

Example 13.5.8 In physics, when two resistances, R_1 and R_2, are connected in a parallel circuit, the combined resistance, R, can be calculated by the formula

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}.$$

Solve for R in this formula.

Explanation. The common denominator is RR_1R_2. We will multiply both sides of the equation by RR_1R_2:

$$\begin{aligned}\frac{1}{R} &= \frac{1}{R_1}+\frac{1}{R_2}\\\cancel{R}R_1R_2\cdot\frac{1}{\cancel{R}} &= RR_1R_2\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\\R_1R_2 &= R\cancel{R_1}R_2\cdot\frac{1}{\cancel{R_1}}+RR_1\cancel{R_2}\cdot\frac{1}{\cancel{R_2}}\\R_1R_2 &= RR_2+RR_1\\R_1R_2 &= R\left(R_2+R_1\right)\\\frac{R_1R_2}{R_2+R_1} &= R\\R &= \frac{R_1R_2}{R_1+R_2}\end{aligned}$$

Example 13.5.9 Here is the slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Solve for x_1 in this formula.

Explanation. The common denominator is $x_2 - x_1$. We will multiply both sides of the equation by $x_2 - x_1$:

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
(x_2 - x_1) \cdot m &= \cancel{(x_2 - x_1)} \cdot \frac{y_2 - y_1}{\cancel{x_2 - x_1}} \\
mx_2 - mx_1 &= y_2 - y_1 \\
-mx_1 &= y_2 - y_1 - mx_2 \\
\frac{-mx_1}{-m} &= \frac{y_2 - y_1 - mx_2}{-m} \\
x_1 &= -\frac{y_2 - y_1 - mx_2}{m}
\end{aligned}$$

Example 13.5.10 Solve the rational equation $x = \frac{4y-1}{2y-3}$ for y.

Explanation. Our first step will be to multiply each side by the LCD, which is simply $2y - 3$. After that, we'll isolate all terms containing y, factor out y, and then finish solving for that variable.

$$\begin{aligned}
x &= \frac{4y - 1}{2y - 3} \\
x \cdot (2y - 3) &= \frac{4y - 1}{\cancel{2y - 3}} \cdot \cancel{(2y - 3)} \\
2xy - 3x &= 4y - 1 \\
2xy &= 4y - 1 + 3x \\
2xy - 4y &= -1 + 3x \\
y(2x - 4) &= 3x - 1 \\
\frac{y\cancel{(2x - 4)}}{\cancel{2x - 4}} &= \frac{3x - 1}{2x - 4} \\
y &= \frac{3x - 1}{2x - 4}
\end{aligned}$$

13.5.3 Solving Rational Equations Using Technology

In some instances, it may be difficult to solve rational equations algebraically. We can instead use graphing technology to obtain approximate solutions. Let's look at one such example.

Example 13.5.11 Solve the equation $\frac{2}{x-3} = \frac{x^3}{8}$ using graphing technology.

Explanation.

We will define $f(x) = \frac{2}{x-3}$ and $g(x) = \frac{x^3}{8}$, and then look for the points of intersection.

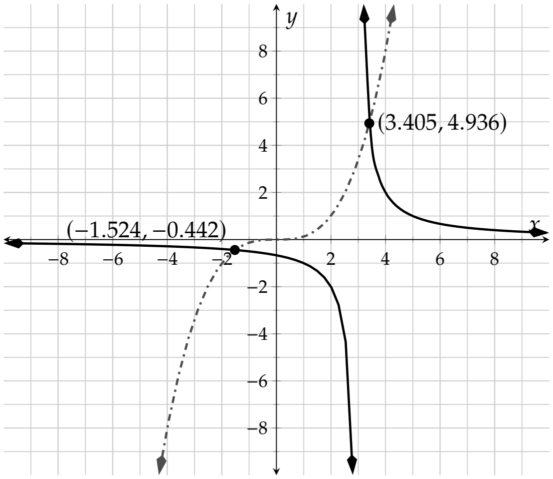


Figure 13.5.12: Graph of $f(x) = \frac{2}{x-3}$ and $g(x) = \frac{x^3}{8}$

Since the two functions intersect at approximately $(-1.524, -0.442)$ and $(3.405, 4.936)$, the solutions to $\frac{2}{x-3} = \frac{x^3}{8}$ are approximately -1.524 and 3.405. We can write the solution set as $\{-1.524\ldots, 3.405\ldots\}$ or in several other forms. It may be important to do *something* to commuincate that these solutions are approximations. Here we used . . ., but you could also just say in words that the solutions are approximate.

Exercises

Review and Warmup Solve the equation.

1. $10y + 5 = y + 50$
2. $9r + 9 = r + 33$
3. $6 = 3 - 3(a - 9)$
4. $52 = 10 - 7(b - 8)$
5. $4(A + 1) - 7(A - 7) = 35$
6. $2(C + 5) - 6(C - 2) = -2$
7. $(x + 5)^2 = 121$
8. $(x + 8)^2 = 49$
9. $x^2 + 4x - 96 = 0$
10. $x^2 - x - 90 = 0$
11. $x^2 - 6x + 1 = 8$
12. $x^2 - 14x + 64 = 19$

Solving Rational Equations Solve the equation.

13. $\dfrac{-12}{y} = 6$
14. $\dfrac{-6}{r} = -1$
15. $\dfrac{r}{r + 3} = 4$
16. $\dfrac{t}{t - 4} = -3$
17. $\dfrac{t + 7}{5t + 7} = \dfrac{1}{19}$
18. $\dfrac{t - 2}{4t - 1} = \dfrac{1}{3}$

19. $\dfrac{-x-2}{x-1} = -\dfrac{x}{x-6}$

20. $\dfrac{-x+10}{x-6} = -\dfrac{x}{x-1}$

21. $\dfrac{6}{y} = 2 - \dfrac{4}{y}$

22. $\dfrac{6}{y} = -5 + \dfrac{16}{y}$

23. $\dfrac{4}{5A} - \dfrac{1}{4A} = -2$

24. $\dfrac{1}{5B} + \dfrac{5}{6B} = 4$

25. $\dfrac{t}{2t-10} + \dfrac{1}{t-5} = 4$

26. $\dfrac{t}{6t+24} - \dfrac{3}{t+4} = 2$

27. $\dfrac{t+8}{t^2+3} = 0$

28. $\dfrac{x+2}{x^2+6} = 0$

29. $-\dfrac{5}{x} = 0$

30. $\dfrac{7}{y} = 0$

31. $\dfrac{y+1}{y^2+8y+7} = 0$

32. $\dfrac{r-6}{r^2-10r+24} = 0$

33. $-\dfrac{9}{r} - \dfrac{6}{r+6} = 1$

34. $\dfrac{6}{r} + \dfrac{6}{r-9} = 1$

35. $\dfrac{1}{t-7} - \dfrac{7}{t^2-7t} = \dfrac{1}{6}$

36. $\dfrac{1}{t+6} + \dfrac{6}{t^2+6t} = -\dfrac{1}{9}$

37. $\dfrac{1}{x-8} - \dfrac{6}{x^2-8x} = \dfrac{1}{8}$

38. $\dfrac{1}{x-5} - \dfrac{4}{x^2-5x} = \dfrac{1}{6}$

39. $\dfrac{y+3}{y+7} - \dfrac{2}{y+5} = -1$

40. $\dfrac{y+1}{y+3} - \dfrac{8}{y+6} = -1$

Solve the equation.

41. $-\dfrac{3}{r+5} = -\left(\dfrac{6}{r-5} + \dfrac{3}{r^2-25}\right)$

42. $-\dfrac{6}{r+3} = -\left(\dfrac{4}{r-3} + \dfrac{2}{r^2-9}\right)$

43. $\dfrac{7}{r+4} - \dfrac{8}{r+1} = -\dfrac{9}{r^2+5r+4}$

44. $\dfrac{9}{t+7} - \dfrac{7}{t+3} = \dfrac{4}{t^2+10t+21}$

45. $-\dfrac{2}{t-8} + \dfrac{2t}{t-6} = -\dfrac{4}{t^2-14t+48}$

46. $\dfrac{2}{x-7} + \dfrac{2x}{x-5} = \dfrac{4}{x^2-12x+35}$

47. $\dfrac{4}{x-4} + \dfrac{8x}{x+1} = -\dfrac{8}{x^2-3x-4}$

48. $\dfrac{2}{y+5} + \dfrac{2y}{y+7} = -\dfrac{2}{y^2+12y+35}$

Solving Rational Equations for a Specific Variable

49. Solve this equation for y:

$$b = \frac{a}{y}$$

50. Solve this equation for n:

$$A = \frac{q}{n}$$

51. Solve this equation for t:

$$B = \frac{t}{r}$$

52. Solve this equation for t:

$$m = \frac{t}{C}$$

53. Solve this equation for n:

$$\frac{1}{5n} = \frac{1}{p}$$

54. Solve this equation for q:

$$\frac{1}{3q} = \frac{1}{A}$$

55. Solve this equation for r:

$$\frac{1}{y} = \frac{3}{r+5}$$

56. Solve this equation for n:

$$\frac{1}{r} = \frac{9}{n+6}$$

Solving Rational Equations Using Technology Use technology to solve the equation

57.

$$\frac{10}{x^2+3} = \frac{x+1}{x+5}.$$

58.

$$\frac{x-9}{x^5+1} = -3x-7.$$

59.

$$\frac{1}{x} + \frac{1}{x^2} = \frac{1}{x^3}.$$

60.

$$\frac{12x}{x-5} + \frac{3}{x+1} = \frac{x-5}{x^2}.$$

61.

$$2x - \frac{1}{x+4} = \frac{3}{x+6}.$$

62.

$$\frac{1}{x^2-1} - \frac{2}{x-4} = \frac{3}{x-2}.$$

Application Problems

63. Jessica and Derick are working together to paint a room. If Jessica paints the room alone, it would take her 9 hours to complete the job. If Derick paints the room alone, it would take him 18 hours to complete the job. Answer the following question:

If they work together, it would take them $\boxed{}$ hours to complete the job. Use a decimal in your answer if needed.

64. There are three pipes at a tank. To fill the tank, it would take Pipe A 4 hours, Pipe B 12 hours, and Pipe C 3 hours. Answer the following question:

If all three pipes are turned on, it would take $\boxed{}$ hours to fill the tank.

65. Neil and Nathan are working together to paint a room. Neil works 5.5 times as fast as Nathan does. If they work together, it took them 11 hours to complete the job. Answer the following questions:

If Neil paints the room alone, it would take him ______ hours to complete the job.

If Nathan paints the room alone, it would take him ______ hours to complete the job.

66. Two pipes are being used to fill a tank. Pipe A can fill the tank 4.5 times as fast as Pipe B does. When both pipes are turned on, it takes 9 hours to fill the tank. Answer the following questions:

If only Pipe A is turned on, it would take ______ hours to fill the tank.

If only Pipe B is turned on, it would take ______ hours to fill the tank.

67. Aleric and Carmen worked together to paint a room, and it took them 4 hours to complete the job. If they work alone, it would take Carmen 6 more hours than Aleric to complete the job. Answer the following questions:

If Aleric paints the room alone, it would take him ______ hours to complete the job.

If Carmen paints the room alone, it would take her ______ hours to complete the job.

68. If both Pipe A and Pipe B are turned on, it would take 2 hours to fill a tank. If each pipe is turned on alone, it takes Pipe B 3 fewer hours than Pipe A to fill the tank. Answer the following questions:

If only Pipe A is turned on, it would take ______ hours to fill the tank.

If only Pipe B is turned on, it would take ______ hours to fill the tank.

69. Town A and Town B are 720 miles apart. A boat traveled from Town A to Town B, and then back to Town A. Since the river flows from Town B to Town A, the boat's speed was 30 miles per hour faster when it traveled from Town B to Town A. The whole trip took 20 hours. Answer the following questions:

The boat traveled from Town A to Town B at the speed of ______ miles per hour.

The boat traveled from Town B back to Town A at the speed of ______ miles per hour.

70. A river flows at 13 miles per hour. A boat traveled with the current from Town A to Town B, which are 360 miles apart. Then, the boat turned around, and traveled against the current to reach Town C, which is 100 miles away from Town B. The second leg of the trip (Town B to Town C) took the same time as the first leg (Town A to Town B). During this whole trip, the boat was driving at a constant still-water speed. Answer the following question:

During this trip, the boat's speed on still water was ______ miles.

71. A river flows at 8 miles per hour. A boat traveled with the current from Town A to Town B, which are 130 miles apart. The boat stayed overnight at Town B. The next day, the water's current stopped, and boat traveled on still water to reach Town C, which is 200 miles away from Town B. The second leg of the trip (Town B to Town C) took 5 hours longer than the first leg (Town A to Town B). During this whole trip, the boat was driving at a constant still-water speed. Find this speed.

Note that you should not consider the unreasonable answer.

During this trip, the boat's speed on still water was ☐ miles per hour.

72. Town A and Town B are 600 miles apart. With a constant still-water speed, a boat traveled from Town A to Town B, and then back to Town A. During this whole trip, the river flew from Town A to Town B at 11 miles per hour. The whole trip took 10 hours. Answer the following question:

During this trip, the boat's speed on still water was ☐ miles per hour.

73. Town A and Town B are 200 miles apart. With a constant still-water speed of 45 miles per hour, a boat traveled from Town A to Town B, and then back to Town A. During this whole trip, the river flew from Town B to Town A at a constant speed. The whole trip took 9 hours. Answer the following question:

During this trip, the river's speed was ☐ miles per hour.

74. Suppose that a large pump can empty a swimming pool in 33 hr and that a small pump can empty the same pool in 49 hr. If both pumps are used at the same time, how long will it take to empty the pool?

If both pumps are used at the same time, it will take ☐ to empty the pool.

75. The winner of a 6 mi race finishes 15.05 min ahead of the second-place runner. On average, the winner ran 0.6 $\frac{\text{mi}}{\text{hr}}$ faster than the second place runner. Find the average running speed for each runner.

The winner's average speed was ☐ and the second-place runner's average speed was ☐.

76. In still water a tugboat can travel 20 $\frac{\text{mi}}{\text{hr}}$. It travels 43 mi upstream and then 43 mi downstream in a total time of 4.35 hr. Find the speed of the current.

The current's speed is ☐.

77. Without any wind an airplane flies at 246 $\frac{\text{mi}}{\text{hr}}$. The plane travels 620 mi into the wind and then returns with the wind in a total time of 5.07 hr. Find the average speed of the wind.

The wind's speed is .

78. When there is a 22.4 $\frac{\text{mi}}{\text{hr}}$ wind, an airplane can fly 800 mi with the wind in the same time that it can fly 642 mi against the wind. Find the speed of the plane when there is no wind.

The plane's airspeed is .

79. It takes one employee 4.5 hr longer to mow a football field than it does a more experienced employee. Together they can mow the grass in 2.8 hr. How long does it take each person to mow the football field working alone?

The more experienced worker takes to mow the field alone, and the less experienced worker takes .

80. It takes one painter 19 hr longer to paint a house than it does a more experienced painter. Together they can paint the house in 25 hr. How long does it take for each painter to paint the house working alone?

The more experienced painter takes to paint the house alone, and the less experienced painter takes .

13.6 Rational Functions and Equations Chapter Review

13.6.1 Introduction to Rational Functions

In Section 13.1 we learned about rational functions and explored them with tables and graphs.

Example 13.6.1 Graphs of Rational Functions. In an apocalypse, a zombie infestation begins with 1 zombie and spreads rapidly. The population of zombies can be modeled by $Z(x) = \frac{200000x+100}{5x+100}$, where x is the number of days after the apocalypse began. Use technology to graph the function and answer these questions:

a. How many zombies are there 2 days after the apocalypse began?

b. After how many days will the zombie population be 20,000?

c. As time goes on, the population will level off at about how many zombies?

Explanation. We will graph the function with technology. After adjusting window settings, we have:

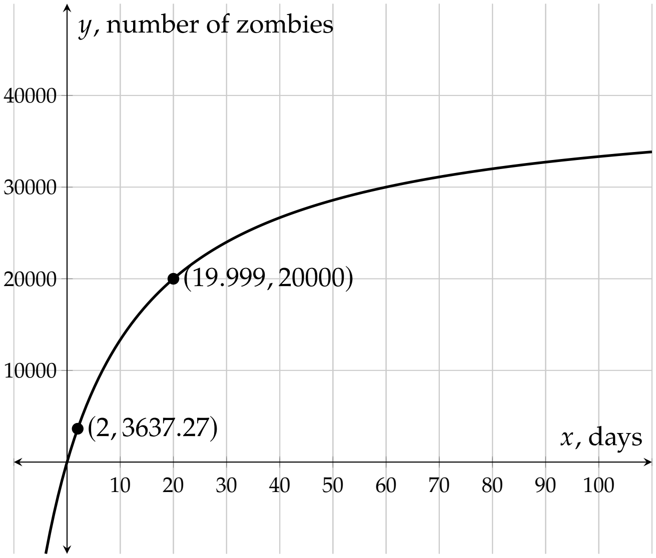


Figure 13.6.2: Graph of $y = Z(x) = \frac{200000x+100}{5x+100}$

a. To find the number of zombies after 2 days, we locate the point $(2, 3637.27)$. Since we can only have a whole number of zombies, we round to 3,637 zombies.

b. To find the number of days it will take for the zombie population reach 20,000, we locate the point $(19.999, 20000)$ so it will take about 20 days.

c. When we look far to the right on the graph using technology we can see that the population will level off at about 40,000 zombies.

13.6.2 Multiplication and Division of Rational Expressions

In Section 13.2 we covered how to simplify rational expressions. It is very important to list any domain restrictions from factors that are canceled. We also multiplied and divided rational expressions.

Example 13.6.3 Simplifying Rational Expressions. Simplify the expression $\frac{8t+4t^2-12t^3}{1-t}$.

Explanation. To begin simplifying this expression, we will rewrite each polynomial in descending order. Then we'll factor out the GCF, including the constant -1 from both the numerator and denominator because their leading terms are negative.

$$\begin{aligned}\frac{8t+4t^2-12t^3}{1-t} &= \frac{-12t^3+4t^2+8t}{-t+1}\\ &= \frac{-4t(3t^2-t-2)}{-(t-1)}\\ &= \frac{-4t(3t+2)(t-1)}{-(t-1)}\\ &= \frac{-4t(3t+2)\cancel{(t-1)}}{-\cancel{(t-1)}}\\ &= \frac{-4t(3t+2)}{-1}, \text{ for } t \neq 1\\ &= 4t(3t+2), \text{ for } t \neq 1\end{aligned}$$

Example 13.6.4 Multiplication of Rational Functions and Expressions. Multiply the rational expressions: $\frac{r^3s}{4t} \cdot \frac{2t^2}{r^2s^3}$.

Explanation. Note that we won't need to factor anything in this problem, and can simply multiply across and then simplify. With multivariable expressions, this textbook ignores domain restrictions.

$$\begin{aligned}\frac{r^3s}{4t} \cdot \frac{2t^2}{r^2s^3} &= \frac{r^3s \cdot 2t^2}{4t \cdot r^2s^3}\\ &= \frac{2r^3st^2}{4r^2s^3t}\\ &= \frac{rt}{2s^2}\end{aligned}$$

Example 13.6.5 Division of Rational Functions and Expressions. Divide the rational expressions: $\frac{2x^2+8xy}{x^2-4x+3} \div \frac{x^3+4x^2y}{x^2+4x-5}$.

Explanation. To divide rational expressions, we multiply by the reciprocal of the second fraction. Then we will factor and cancel any common factors. With multivariable expressions, this textbook ignores domain restrictions.

$$\begin{aligned}\frac{2x^2+8xy}{x^2-4x+3} \div \frac{x^3+4x^2y}{x^2+4x-5} &= \frac{2x^2+8xy}{x^2-4x+3} \cdot \frac{x^2+4x-5}{x^3+4x^2y}\\ &= \frac{2x\cancel{(x+4y)}}{\cancel{(x-1)}(x-3)} \cdot \frac{\cancel{(x-1)}(x+5)}{x^2\cancel{(x+4y)}}\end{aligned}$$

$$\begin{aligned}
&= \frac{2x}{x-3} \cdot \frac{x+5}{x^2} \\
&= \frac{2(x+5)}{x(x-3)}
\end{aligned}$$

13.6.3 Addition and Subtraction of Rational Expressions

In Section 13.3 we covered how to add and subtract rational expressions.

Example 13.6.6 Addition and Subtraction of Rational Expressions with the Same Denominator. Add the rational expressions: $\frac{5x}{x+5} + \frac{25}{x+5}$.

Explanation. These expressions already have a common denominator:

$$\begin{aligned}
\frac{5x}{x+5} + \frac{25}{x+5} &= \frac{5x+25}{x+5} \\
&= \frac{5\cancel{(x+5)}}{\cancel{x+5}} \\
&= \frac{5}{1}, \text{ for } x \neq -5 \\
&= 5, \text{ for } x \neq -5
\end{aligned}$$

Note that we didn't stop at $\frac{5x+25}{x+5}$. If possible, we must simplify the numerator and denominator.

Example 13.6.7 Addition and Subtraction of Rational Expressions with Different Denominators. Add and subtract the rational expressions: $\frac{6y}{y+2} + \frac{y}{y-2} - 7$

Explanation. The denominators can't be factored, so we'll find the least common denominator and build each expression to that denominator. Then we will be able to combine the numerators and simplify the expression.

$$\begin{aligned}
\frac{6y}{y+2} + \frac{y}{y-2} - 7 &= \frac{6y}{y+2} \cdot \frac{y-2}{y-2} + \frac{y}{y-2} \cdot \frac{y+2}{y+2} - 7 \cdot \frac{(y-2)(y+2)}{(y-2)(y+2)} \\
&= \frac{6y(y-2)}{(y-2)(y+2)} + \frac{y(y+2)}{(y-2)(y+2)} - \frac{7(y-2)(y+2)}{(y-2)(y+2)} \\
&= \frac{6y^2 - 12y + y^2 + 2y - \overset{\downarrow}{(}7(y^2-4)\overset{\downarrow}{)}}{(y-2)(y+2)} \\
&= \frac{6y^2 - 12y + y^2 + 2y - 7y^2 + 28}{(y-2)(y+2)} \\
&= \frac{-10y + 28}{(y-2)(y+2)} \\
&= \frac{-2(5y-14)}{(y-2)(y+2)}
\end{aligned}$$

13.6.4 Complex Fractions

In Section 13.4 we covered how to simplify a rational expression that has fractions in the numerator and/or denominator.

Example 13.6.8 Simplifying Complex Fractions. Simplify the complex fraction $\dfrac{\frac{2t}{t^2-9}+3}{\frac{6}{t+3}+\frac{1}{t-3}}$.

Explanation. First, we check all quadratic polynomials to see if they can be factored and factor them:

$$\frac{\frac{2t}{t^2-9}+3}{\frac{6}{t+3}+\frac{1}{t-3}}=\frac{\frac{2t}{(t-3)(t+3)}+3}{\frac{6}{t+3}+\frac{1}{t-3}}$$

Next, we identify the common denominator of the three fractions, which is $(t+3)(t-3)$. We then multiply the main numerator and denominator by that expression:

$$\begin{aligned}
\frac{\frac{2t}{(t-3)(t+3)}+3}{\frac{6}{t+3}+\frac{1}{t-3}}&=\frac{\frac{2t}{(t-3)(t+3)}+3}{\frac{6}{t+3}+\frac{1}{t-3}}\cdot\frac{(t-3)(t+3)}{(t-3)(t+3)}\\
&=\frac{\frac{2t}{\cancel{(t-3)(t+3)}}\cancel{(t-3)(t+3)}+3(t-3)(t+3)}{\frac{6}{\cancel{t+3}}(t-3)\cancel{(t+3)}+\frac{1}{\cancel{t-3}}\cancel{(t-3)}(t+3)}\\
&=\frac{2t+3(t-3)(t+3)}{6(t-3)+1(t+3)}\text{ for }t\neq-3,t\neq3\\
&=\frac{2t+3(t^2-9)}{6t-18+t+3}\text{ for }t\neq-3,t\neq3\\
&=\frac{2t+3t^2-27}{7t-15}\text{ for }t\neq-3,t\neq3\\
&=\frac{3t^2+2t-27}{7t-15}\text{ for }t\neq-3,t\neq3
\end{aligned}$$

Both the numerator and denominator are prime polynomials so this expression can neither factor nor simplify any further.

13.6.5 Solving Rational Equations

In Section 13.5 we covered how to solve rational equations. We looked at rate problems, solved for a specified variable and used technology to solve rational equations.

Example 13.6.9 Solving Rational Equations. Two pipes are being used to fill a large tank. Pipe B can fill the tank twice as fast as Pipe A can. When both pipes are turned on, it takes 12 hours to fill the tank. Write and solve a rational equation to answer the following questions:

a. If only Pipe A is turned on, how many hours would it take to fill the tank?

b. If only Pipe B is turned on, how many hours would it take to fill the tank?

Explanation. Since both pipes can fill the tank in 12 hours, they fill $\frac{1}{12}$ of the tank together each hour. We will let a represent the number of hours it takes pipe A to fill the tank alone, so pipe A will fill $\frac{1}{a}$ of the tank each hour. Pipe B can fill the tank twice as fast so it fills $2\cdot\frac{1}{a}$ of the tank each hour or $\frac{2}{a}$. When they are both turned on, they fill $\frac{1}{a}+\frac{2}{a}$ of the tank each hour.

Now we can write this equation:

$$\frac{1}{a}+\frac{2}{a}=\frac{1}{12}$$

To clear away denominators, we multiply both sides of the equation by the common denominator of 12 and a, which is $12a$:

$$\begin{aligned}
\frac{1}{a}+\frac{2}{a}&=\frac{1}{12}\\
12a\cdot\left(\frac{1}{a}+\frac{2}{a}\right)&=12a\cdot\frac{1}{12}\\
12a\cdot\frac{1}{a}+12a\cdot\frac{2}{a}&=12a\cdot\frac{1}{12}\\
12+24&=a\\
36&=a\\
a&=36
\end{aligned}$$

The possible solution $a=36$ should be checked

$$\begin{aligned}
\frac{1}{36}+\frac{2}{36}&\stackrel{?}{=}\frac{1}{12}\\
\frac{3}{36}&\stackrel{\checkmark}{=}\frac{1}{12}
\end{aligned}$$

So it is a solution.

a. If only Pipe A is turned on, it would take 36 hours to fill the tank.

b. Since Pipe B can fill the tank twice as fast, it would take half the time, or 18 hours to fill the tank.

Example 13.6.10 Solving Rational Equations for a Specific Variable. Solve the rational equation $y=\frac{2x+5}{3x-1}$ for x.

Explanation. To get the x out of the denominator, our first step will be to multiply each side by the LCD, which is $3x-1$. Then we'll isolate all terms containing x, factor out x, and then finish solving for that variable.

$$\begin{aligned}
y&=\frac{2x+5}{3x-1}\\
y\cdot(3x-1)&=\frac{2x+5}{\cancel{3x-1}}\cdot\cancel{(3x-1)}\\
3xy-y&=2x+5\\
3xy&=2x+5+y\\
3xy-2x&=y+5\\
x(3y-2)&=y+5\\
\frac{x(3y-2)}{3y-2}&=\frac{y+5}{3y-2}\\
x&=\frac{y+5}{3y-2}
\end{aligned}$$

Example 13.6.11 Solving Rational Equations Using Technology. Solve the equation $\frac{1}{x+2}+1=\frac{10x}{x^2+5}$ using graphing technology.

Explanation.

We will define $f(x)=\frac{1}{x+2}+1$ and $g(x)=\frac{10x}{x^2+5}$, and then find a window where we can see all of the points of intersection.

Since the two functions intersect at approximately $(-2.309,-2.235)$, $(0.76,1.362)$ and $(8.549,1.095)$, the solutions to $\frac{1}{x+2}+1=\frac{10x}{x^2+5}$ are approximately -2.309, 0.76 and 8.549. The solution set is approximately $\{-2.309\ldots,0.76\ldots,8.549\ldots\}$.

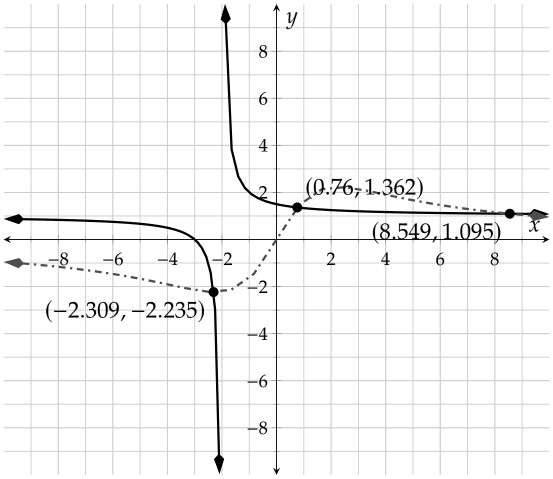
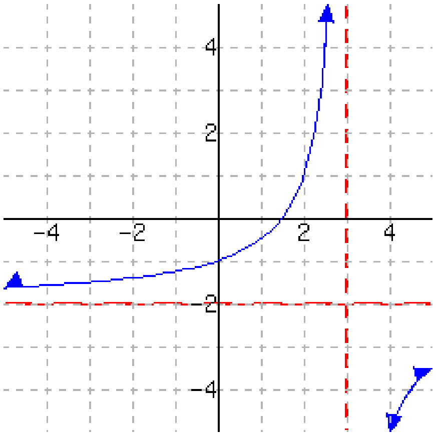


Figure 13.6.12: Graph of $f(x)=\frac{1}{x+2}+1$ and $g(x)=\frac{10x}{x^2+5}$

Exercises

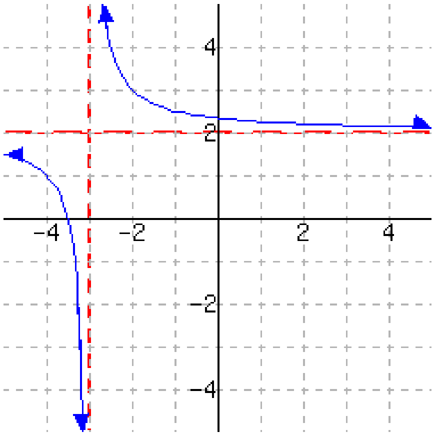
Introduction to Rational Functions

1. A function is graphed.



This function has domain ☐ and range ☐.

2. A function is graphed.



This function has domain ☐ and range ☐.

3. The population of deer in a forest can be modeled by

$$P(x) = \frac{780x + 2660}{3x + 7}$$

where x is the number of years in the future. Answer the following questions.

a. How many deer live in this forest this year?

b. How many deer will live in this forest 24 years later? Round your answer to an integer.

c. After how many years, the deer population will be 276? Round your answer to an integer.

d. Use a calculator to answer this question: As time goes on, the population levels off at about how many deer?

4. The population of deer in a forest can be modeled by

$$P(x) = \frac{400x + 1950}{4x + 5}$$

where x is the number of years in the future. Answer the following questions.

a. How many deer live in this forest this year?

b. How many deer will live in this forest 27 years later? Round your answer to an integer.

c. After how many years, the deer population will be 143? Round your answer to an integer.

d. Use a calculator to answer this question: As time goes on, the population levels off at about how many deer?

5. In a certain store, cashiers can serve 55 customers per hour on average. If x customers arrive at the store in a given hour, then the average number of customers C waiting in line can be modeled by the function

$$C(x) = \frac{x^2}{3025 - 55x}$$

where $x < 55$.

Answer the following questions with a graphing calculator. Round your answers to integers.

a. If 42 customers arrived in the store in the past hour, there are approximately ☐ customers waiting in line.

b. If there are 7 customers waiting in line, approximately ☐ customers arrived in the past hour.

6. In a certain store, cashiers can serve 60 customers per hour on average. If x customers arrive at the store in a given hour, then the average number of customers C waiting in line can be modeled by the function

$$C(x) = \frac{x^2}{3600 - 60x}$$

where $x < 60$.

Answer the following questions with a graphing calculator. Round your answers to integers.

a. If 51 customers arrived in the store in the past hour, there are approximately ☐ customers waiting in line.

b. If there are 2 customers waiting in line, approximately ☐ customers arrived in the past hour.

7. The concentration of a drug in a patient's blood stream, in milligrams per liter, can be modeled by the function $C(t) = \frac{6t}{t^2+4}$, where t is the number of hours since the drug is injected. Answer the following question with technology. Round your answer to two decimal places if needed.

$\boxed{}$ hours since injection, the drug's concentration is at the maximum value of $\boxed{}$ milligrams per liter.

8. The concentration of a drug in a patient's blood stream, in milligrams per liter, can be modeled by the function $C(t) = \frac{7t}{t^2+8}$, where t is the number of hours since the drug is injected. Answer the following question with technology. Round your answer to two decimal places if needed.

$\boxed{}$ hours since injection, the drug's concentration is at the maximum value of $\boxed{}$ milligrams per liter.

Multiplication and Division of Rational Expressions

9. Simplify this expression.

$$\frac{-t^2+6ty-5y^2}{t^2-25y^2} = \boxed{}$$

10. Simplify this expression.

$$\frac{-t^2-8tx-15x^2}{t^2-25x^2} = \boxed{}$$

11. Simplify the function formula, and if applicable, write the restricted domain.

$$G(x) = \frac{x^4+8x^3+16x^2}{3x^4+13x^3+4x^2}$$

Reduced $G(x) = \boxed{}$

12. Simplify the function formula, and if applicable, write the restricted domain.

$$h(x) = \frac{x^4-4x^3+4x^2}{2x^4-3x^3-2x^2}$$

Reduced $h(x) = \boxed{}$

13. Simplify this expression, and if applicable, write the restricted domain.

$$\frac{y^2-16y}{y^2-16} \cdot \frac{y^2-4y}{y^2-19y+48} = \boxed{}$$

14. Simplify this expression, and if applicable, write the restricted domain.

$$\frac{y^2-4y}{y^2-4} \cdot \frac{y^2-2y}{y^2-3y-4} = \boxed{}$$

15. Simplify this expression, and if applicable, write the restricted domain.

$$\frac{9r^2-25}{3r^2+8r+5} \div (5-3r) = \boxed{}$$

16. Simplify this expression, and if applicable, write the restricted domain.

$$\frac{25r^2-9}{5r^2+8r+3} \div (3-5r) = \boxed{}$$

17. Simplify this expression.

$$\frac{r^3}{r^2y+4r} \div \frac{1}{r^2y^2+5ry+4} = \boxed{}$$

18. Simplify this expression.

$$\frac{t^4}{t^2y+4t} \div \frac{1}{t^2y^2+5ty+4} = \boxed{}$$

Addition and Subtraction of Rational Expressions Add or subtract the rational expressions to a single rational expression and then simplify. If applicable, state the restricted domain.

19. $\dfrac{1}{t-4}-\dfrac{8}{t^2-16}=$

20. $\dfrac{1}{x+1}+\dfrac{2}{x^2-1}=$

21. $-\dfrac{5x}{x^2-3x+2}-\dfrac{5x}{x-1}=$

22. $\dfrac{8y}{y^2+2y-3}+\dfrac{2y}{y+3}=$

23. $\dfrac{y^2-20}{y^2+5y}-\dfrac{y-4}{y}=$

24. $\dfrac{y^2-10}{y^2-2y}-\dfrac{y+5}{y}=$

Add or subtract the rational expressions to a single rational expression and then simplify.

25. $-\dfrac{4r}{3t^5}-\dfrac{6}{5rt}=$

26. $-\dfrac{6r}{5y^3}+\dfrac{3}{2ry}=$

27. $\dfrac{6tx}{t^2-4tx-5x^2}-\dfrac{t}{t-5x}=$

28. $-\dfrac{6ty}{t^2+14ty+40y^2}-\dfrac{t}{t+10y}=$

Complex Fractions

29. Calculate the following. Use an improper fraction in your answer.

a. $\dfrac{\frac{3}{7}}{\frac{3}{8}}=$

b. $\dfrac{\frac{x}{y}}{\frac{r}{t}}=$

30. Calculate the following. Use an improper fraction in your answer.

a. $\dfrac{\frac{3}{8}}{\frac{3}{5}}=$

b. $\dfrac{\frac{x}{t}}{\frac{y}{r}}=$

31. Simplify this expression, and if applicable, write the restricted domain.

$$\frac{\frac{8}{b-1}-4}{\frac{1}{b-1}+\frac{1}{b-10}}=$$

32. Simplify this expression, and if applicable, write the restricted domain.

$$\frac{\frac{4}{b-1}-7}{\frac{1}{b-1}+\frac{1}{b-8}}=$$

33. Simplify this expression, and if applicable, write the restricted domain.

$$\frac{\frac{2y}{y^2-9}-1}{\frac{3}{y+3}+\frac{2}{y-3}}=$$

34. Simplify this expression, and if applicable, write the restricted domain.

$$\frac{\frac{2r}{r^2-9}+3}{\frac{3}{r+3}+\frac{1}{r-3}}=$$

35. Simplify this expression.

$$\frac{\frac{p}{q}}{\frac{4p}{3q^2}} =$$

36. Simplify this expression.

$$\frac{\frac{m}{n}}{\frac{6m}{5n^2}} =$$

37. Simplify this expression.

$$\frac{\frac{3}{t}}{3+\frac{3r}{2}} =$$

38. Simplify this expression.

$$\frac{\frac{2}{x}}{2-\frac{2y}{3}} =$$

Solving Rational Equations Solve the equation.

39. $\dfrac{2}{x-1} + \dfrac{4}{x+7} = -\dfrac{2}{x^2+6x-7}$

40. $\dfrac{3}{y+7} - \dfrac{9}{y-4} = -\dfrac{9}{y^2+3y-28}$

41. $\dfrac{1}{y-5} - \dfrac{5}{y^2-5y} = \dfrac{1}{4}$

42. $\dfrac{1}{y+8} + \dfrac{8}{y^2+8y} = -\dfrac{1}{4}$

43. $-\dfrac{6}{r+9} + \dfrac{2r}{r-3} = \dfrac{8}{r^2+6r-27}$

44. $\dfrac{3}{r-1} + \dfrac{6r}{r-7} = \dfrac{9}{r^2-8r+7}$

45. $\dfrac{t-3}{t+1} - \dfrac{6}{t-4} = 6$

46. $\dfrac{t+7}{t-9} + \dfrac{6}{t+6} = -1$

47. Solve this equation for r:

$$\frac{1}{x} = \frac{3}{r+4}$$

48. Solve this equation for n:

$$\frac{1}{r} = \frac{3}{n+8}$$

49. Use technology to solve the equation

$$2x - \frac{1}{x+4} = \frac{3}{x+6}.$$

50. Use technology to solve the equation

$$\frac{1}{x^2-1} - \frac{2}{x-4} = \frac{3}{x-2}.$$

51. Two pipes are being used to fill a tank. Pipe A can fill the tank 4.5 times as fast as Pipe B does. When both pipes are turned on, it takes 18 hours to fill the tank. Answer the following questions:

If only Pipe A is turned on, it would take ☐ hours to fill the tank.

If only Pipe B is turned on, it would take ☐ hours to fill the tank.

52. Two pipes are being used to fill a tank. Pipe A can fill the tank 5.5 times as fast as Pipe B does. When both pipes are turned on, it takes 11 hours to fill the tank. Answer the following questions:

If only Pipe A is turned on, it would take ☐ hours to fill the tank.

If only Pipe B is turned on, it would take ☐ hours to fill the tank.

53. Town A and Town B are 360 miles apart. A boat traveled from Town A to Town B, and then back to Town A. Since the river flows from Town B to Town A, the boat's speed was 15 miles per hour faster when it traveled from Town B to Town A. The whole trip took 20 hours. Answer the following questions:

The boat traveled from Town A to Town B at the speed of ☐ miles per hour.

The boat traveled from Town B back to Town A at the speed of ☐ miles per hour.

54. Town A and Town B are 560 miles apart. A boat traveled from Town A to Town B, and then back to Town A. Since the river flows from Town B to Town A, the boat's speed was 30 miles per hour faster when it traveled from Town B to Town A. The whole trip took 28 hours. Answer the following questions:

The boat traveled from Town A to Town B at the speed of ☐ miles per hour.

The boat traveled from Town B back to Town A at the speed of ☐ miles per hour.

CHAPTER 14

Radical Functions and Equations

14.1 Introduction to Radical Functions

We learned the basics of square roots in Section 8.2. The study of radicals is much broader than our first attempt covered and we need to expand our investigation. To do so, we will first look at an example that makes use of a topic covered in Section 8.3.

A #10 washer has a 5.6 mm inner diameter and is 1.2 mm thick. We will let d represent the outer diameter, measured in mm.

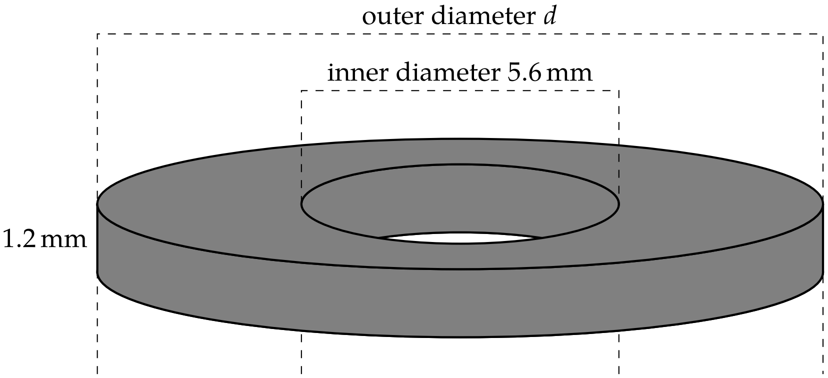


Figure 14.1.2: A Diagram of a #10 Washer

The amount of steel, M, in mg that it takes to make the washer with outer diameter d is approximated by the formula

$$M = 7.59d^2 - 238$$

Note that if you know the value of M ahead of time, this is a quadratic equation. We will now solve the equation for d using the square root method.

$$\begin{aligned} M &= 7.59d^2 - 238 \\ M + 238 &= 7.59d^2 \\ \frac{M + 238}{7.59} &= d^2 \end{aligned}$$

$$d = \sqrt{\frac{M + 238}{7.59}} \qquad \text{or} \qquad d = -\sqrt{\frac{M + 238}{7.59}}$$

Since we know that the diameter cannot really be negative, our formula for d must be

$$d = \sqrt{\frac{M+238}{7.59}}$$

This formula finds the diameter that the washer must be when you input the amount of steel used, M. Figure 14.1.3 shows a graph of this relationship.

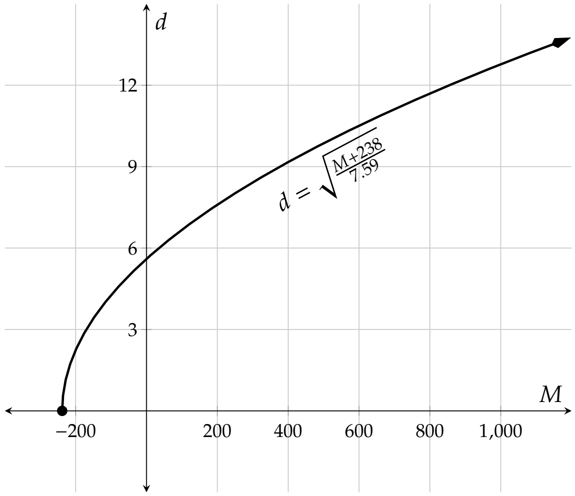


Figure 14.1.3: A Graph of $d = \sqrt{\frac{M+238}{7.59}}$

We also know that M cannot be negative, so we will cut the graph to begin at $M = 0$. This formula tells us that if we plan on using, for example, 1000 mg of steel (about as much as in a large paper clip) that we can find the the outer diameter for the washer that will be created:

$$\begin{aligned} d &= \sqrt{\frac{1000+238}{7.59}} \\ &\approx 12.8 \end{aligned}$$

So the washer's outer diameter must be about 12.8 mm for to have a mass of 1000 mg.

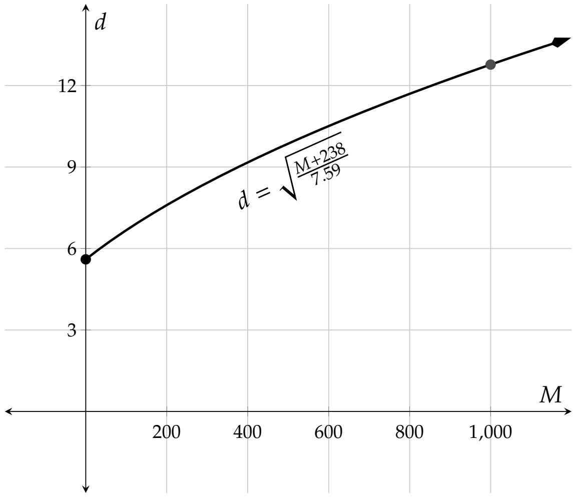


Figure 14.1.4: A Revised Graph of $d = \sqrt{\frac{M+238}{7.59}}$

Note that the vertical intercept of the graph is $(0, 5.6)$, which says that a washer that uses no steel at all (is that really a washer?) would have an outer diameter of 5.6 mm. This is the "smallest" possible washer with an inner diameter of 5.6 mm, even though it would technically be massless. Perhaps the implied domain of this function should be $(0, \infty)$ to exclude 0 mass.

Square roots often appear when we consider formulas from geometry like the washer problem, and they also show up in topics like in statistics (where $\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2}$ finds the standard deviation), in chemistry (where $v_{rms} = \sqrt{\frac{3RT}{M_m}}$ finds the velocity of a particle), and in physics (where $m(v) = \frac{m_0}{\sqrt{1-v^2/c^2}}$ finds the mass of an object as its velocity nears the speed of light). There are many more examples to give, but we need a firmer understanding of radicals to properly study these things, so it's time to venture into deeper waters.

14.1.1 The Square Root Function

Example 14.1.5 Gilberto is an artist who etches designs into square copper plates of different sizes. Customers can order the size they would like.

a. Build a function that calculates the length of a plate's side given its area. Explore the function with a table and graph.

b. One customer ordered a plate with an area of 6.25 square feet. Calculate the length of its side.

c. Find the domain and range of the function from Part a.

Explanation.

a. We know the formula to calculate a square's area is $A = \ell^2$, where A stands for a square's area, and ℓ is the length of the square's side. To build a function to calculate ℓ, we solve for ℓ in the formula:

$$A = \ell^2$$

$$\ell = \sqrt{A} \qquad \text{or} \qquad \ell = -\sqrt{A}$$

The formula is $\ell = \sqrt{A}$. We don't consider the negative solution in this context since negative length doesn't make sense. Since ℓ depends on A, we can use function notation and write $f(A) = \sqrt{A}$. We will make a table, plot points, and look at the graph of $\ell = \sqrt{A}$.

A	$f(A)$	Points on the Curve
0	$\sqrt{0} = 0$	$(0, 0)$
1	$\sqrt{1} = 1$	$(1, 1)$
4	$\sqrt{4} = 2$	$(4, 2)$
6.25	$\sqrt{6.25} = 2.5$	$(6.25, 2.5)$
9	$\sqrt{9} = 3$	$(9, 3)$

Table 14.1.6: Values of $f(A) = \sqrt{A}$

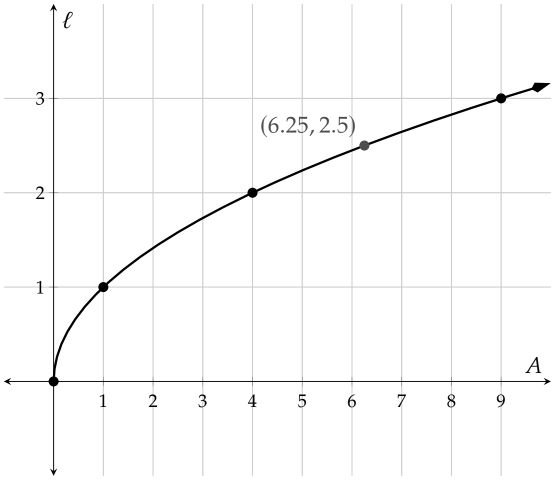


Figure 14.1.7: Graph of $y = \sqrt{A}$

b. The point $(6.25, 2.5)$ implies that a square plate with an area of 6.25 square feet would have a length of 2.5 feet on each side.

c. According to the graph, the function's domain is $[0, \infty)$ because the graph goes forever to the right from $A = 0$. This should make some sense because you cannot take the square root of a negative number. The function's range is $[0, \infty)$ because the graph seems to go up forever starting at $\ell = 0$. This should also make sense because a square root never gives you a negative number as an answer.

Fact 14.1.8 Domain of a Square Root Functions. *To algebraically find the domain of a square root function, set the radicand (the expression under the radical) greater than or equal to* 0 *and solve for the variable. The solution set to* *that inequality is the domain of the function.*

Example 14.1.9 Algebraically find the domain of the function g where $g(x) = \sqrt{2x-4}+1$ and then find the range by making a graph.

Explanation. Using Fact 14.1.8 to find the function's domain, we set the radicand greater than or equal to zero and solve:

$$\begin{aligned} 2x-4 &\geq 0\\ 2x &\geq 4\\ x &\geq 2 \end{aligned}$$

The function's domain is $[2,\infty)$ in interval notation.

To find the function's range, we use technology to look at a graph of the function. The graph shows that the function's range is $[1,\infty)$. The graph also verifies the function's domain is indeed $[2,\infty)$.

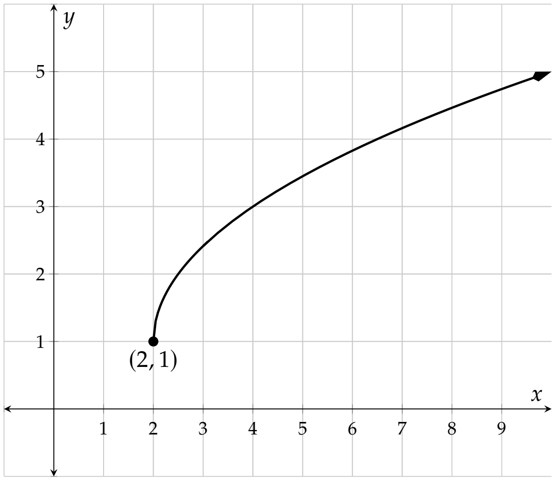


Figure 14.1.10: Graph of $g(x) = \sqrt{2x-4}+1$

Example 14.1.11 Algebraically find the domain and graphically find the range of the function h where $h(x) = 2-3\sqrt{8-5x}$.

Explanation. To find the function's domain, we set the radicand to be greater than or equal to zero:

$$\begin{aligned} 8-5x &\geq 0\\ -5x &\geq -8\\ \frac{-5x}{-5} &\leq \frac{-8}{-5}\\ x &\leq \frac{8}{5} \end{aligned}$$

So, the function's domain is $\left(-\infty,\frac{8}{5}\right]$. The 2 and 3 in the function do not play a role in the domain, although they do alter the range which we will find now by making a graph.

From the graph we can see that the range is all numbers below (or equal to) the y-value 2. In interval notation, this would be written $(-\infty, 2]$.

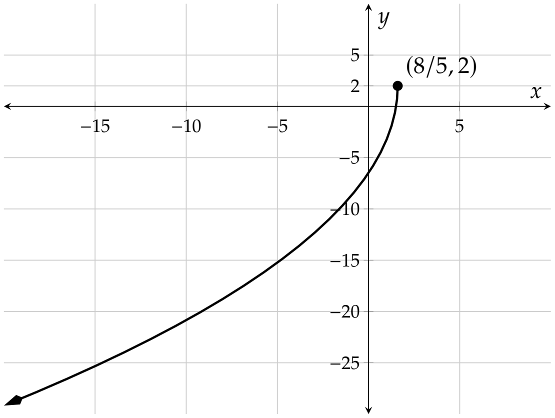


Figure 14.1.12: Graph of $h(x) = 2 - 3\sqrt{8 - 5x}$

Example 14.1.13 When an object is dropped, the time it takes to hit the ground can be modeled by

$$t = \sqrt{\frac{d}{16}}$$

where t is in seconds, and d is the initial height of the object in feet. Use graphing technology to create a graph and answer the following questions.

a. In a science experiment, Amaka's class drops a beanbag from the top of a 100-foot-tall building. How long will it take for the beanbag to hit the ground?

b. Her class then goes to a second building, drops the beanbag from the top, and uses a stopwatch to measures the time it takes to hit the ground. If it takes 3 seconds for the beanbag to hit the ground, how tall is the building?

Explanation.

With graphing technology, and after adjusting the window settings, we can see the graph of $t = \sqrt{\frac{d}{16}}$ and some important points.

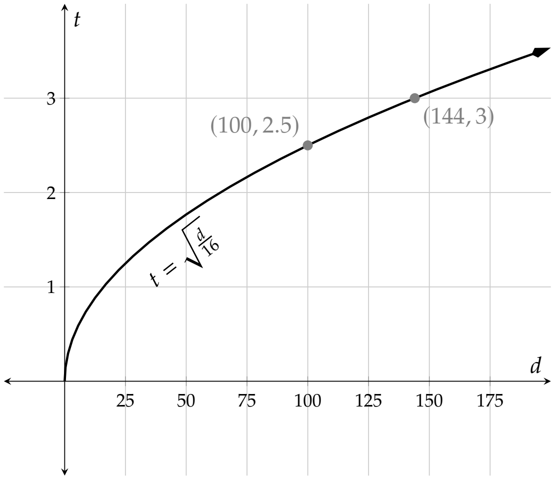


Figure 14.1.14: Graph of $t = \sqrt{\frac{d}{16}}$

a. We look for a d-value of 100 feet and find the point $(100, 2.5)$ on the graph. This means it will take the beanbag 2.5 seconds to hit the ground if it's released from the top of a 100-foot-tall building.

b. This time we look for a t-value of 3 seconds and find the point $(144, 3)$. This means the beanbag will fall approximately 144 feet in 3 seconds, so the second building is approximately 144 feet tall.

14.1.2 The Distance Formula

A square root is used in calculating the distance between two points on a coordinate plane. We learned the Pythagorean Theorem in Section 8.3.2. In a coordinate plane, we can use the Pythagorean Theorem to calculate the distance between any two points.

Example 14.1.15 Calculate the distance between $(2, 3)$ and $(5, 7)$.

Explanation. First, we will sketch a graph of those two points.

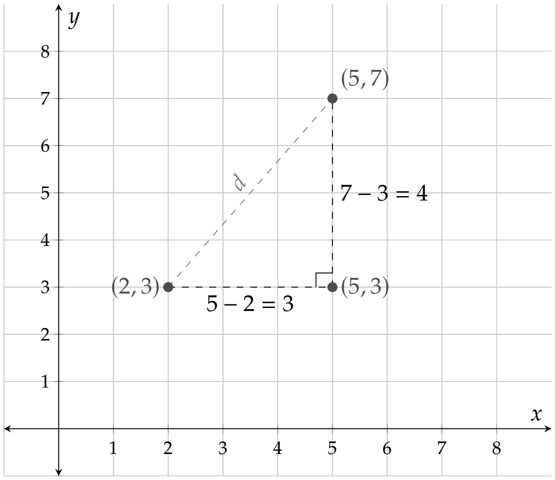


Figure 14.1.16: Calculating the Distance Between $(2, 3)$ and $(5, 7)$

To calculate the distance between $(2, 3)$ and $(5, 7)$, we sketch a right triangle as in the figure and then use the Pythagorean Theorem:

$$\begin{aligned}
d^2 &= (5-2)^2 + (7-3)^2 \\
d^2 &= 3^2 + 4^2 \\
d^2 &= 9 + 16 \\
d^2 &= 25 \\
d &= \sqrt{25} \\
d &= 5
\end{aligned}$$

In conclusion, the distance between $(2, 3)$ and $(5, 7)$ is 5. Note that in our calculations, we didn't need to show $d = \pm\sqrt{25}$ because distance must have positive values.

With the same method, we can derive a formula to calculate the distance between any two points.

Example 14.1.17 With a generic first and second point, we will use subscripts to identify the first pair (x_1, y_1) and the second pair (x_2, y_2). Calculate the distance between the generic points (x_1, y_1) and (x_2, y_2).

Explanation. First, we will sketch a graph of those two points. We will put the image from last example side by side with the new image, so it's clear that we are using the same method.

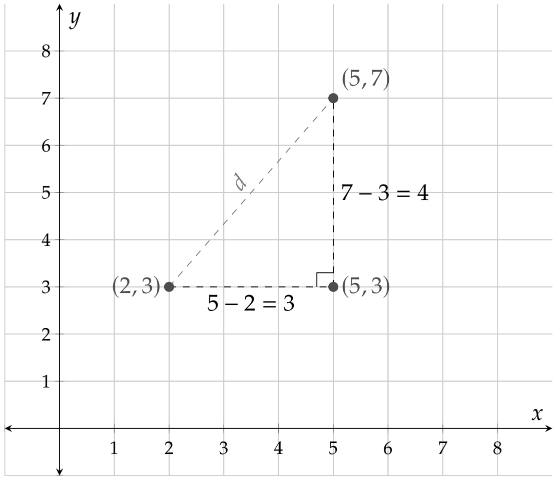


Figure 14.1.18: Calculating the Distance Between $(2, 3)$ and $(5, 7)$

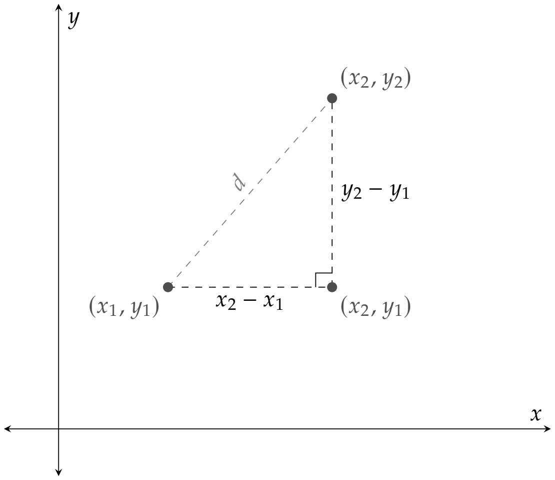


Figure 14.1.19: Calculating the Distance Between (x_1, y_1) and (x_2, y_2)

To calculate the distance between (x_1, y_1) and (x_2, y_2), we sketch a right triangle as in the figure and then use Pythagorean Theorem:

$$\begin{aligned}
d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\end{aligned}$$

Fact 14.1.20 The Distance Formula. *The distance between two points (x_1, y_1) and (x_2, y_2), is given by the formula:*

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

With this formula, we can calculate the distance between two points without sketching a graph.

Example 14.1.21 Find the distance between $(-2, 4)$ and $(5, -20)$.

Explanation. To calculate the distance between $(-2, 4)$ and $(5, -20)$, we use the distance formula. It's good practice to mark each value with the corresponding variables in the formula. Again, (x_1, y_1) stands for the first point's coordinates, and (x_2, y_2) stands for the second point's coordinates:

$$(\overset{x_1}{-2}, \overset{y_1}{4}), (\overset{x_2}{5}, \overset{y_2}{-20})$$

We have:

$$\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
d &= \sqrt{(5 - (-2))^2 + ((-20) - 4)^2} \\
d &= \sqrt{(7)^2 + (-24)^2} \\
d &= \sqrt{49 + 576} \\
d &= \sqrt{625}
\end{aligned}$$

> $$d = 25$$
>
> The distance between $(-2, 4)$ and $(5, -20)$ is 25 units.

Warning 14.1.22. Note that it's good practice to add parentheses around negative values when we do substitutions. For example, when we substitute x with -7 in x^2, we should write

$$x^2 = (-7)^2 = 49 \text{ correct } \checkmark$$

We should *not* write

$$x^2 = -7^2 = -49 \text{ incorrect}$$

14.1.3 Cube Root Function

The square of 2 is 4, so the square root of 4 is 2.

Similarly, the cube of 2 is 8, so the cube root of 8 is 2. We write

$$\sqrt[3]{8} = 2$$

It's helpful to memorize the first few perfect cube numbers and their cube roots:

$$\begin{aligned} 0^3 &= 0 & \sqrt[3]{0} &= 0 \\ 1^3 &= 1 & \sqrt[3]{1} &= 1 \\ 2^3 &= 8 & \sqrt[3]{8} &= 2 \\ 3^3 &= 27 & \sqrt[3]{27} &= 3 \\ 4^3 &= 64 & \sqrt[3]{64} &= 4 \\ 5^3 &= 125 & \sqrt[3]{125} &= 5 \end{aligned}$$

One major difference between a cube root and a square root is that we can find the cube root of negative numbers. For example:

$$(-4)^3 = -64, \text{ so } \sqrt[3]{-64} = -4$$

However, $\sqrt{-64}$ is non-real and in general we cannot take the square root of a negative number.

Remark 14.1.23. Many calculators don't have a cube root button. If yours does, it might look like $\sqrt[n]{}$ and you will tell the calculator both to enter a number for the "n" as well as the radicand. Many calculators also allow you to type something like `root(3,8)` for $\sqrt[3]{8}$, for example.

Another way to calculate the cube root on a calculator is to use the exponent button (which is usually marked with the caret symbol, `^`) with a *reciprocal* power. For example, to calculate $\sqrt[3]{8}$, you may type `8^(1/3)`. We will explain why $\sqrt[3]{8} = 8^{\frac{1}{3}}$ in Section 14.2. For now, just learn how to use a calculator to calculate the cube root of a given number.

We can also estimate the value of a cube root, like $\sqrt[3]{10}$, by knowing the perfect cubes nearby:

$$\sqrt[3]{8} = 2 \qquad \sqrt[3]{10} = ? \qquad \sqrt[3]{27} = 3$$

Since 10 is between the perfect cubes 8 and 27, $\sqrt[3]{10}$ must be between 2 and 3, and closer to 2. We can use a calculator to verify $\sqrt[3]{10} \approx 2.154$

Let's build a table and graph the cube root function.

x	$g(x) = \sqrt[3]{x}$	Points on the Curve
-8	$\sqrt[3]{-8} = -2$	$(-8, -2)$
-1	$\sqrt[3]{-1} = -1$	$(-1, -1)$
0	$\sqrt[3]{0} = 0$	$(0, 0)$
1	$\sqrt[3]{1} = 1$	$(1, 1)$
8	$\sqrt[3]{8} = 2$	$(8, 2)$

Table 14.1.24: Values of $g(x) = \sqrt[3]{x}$

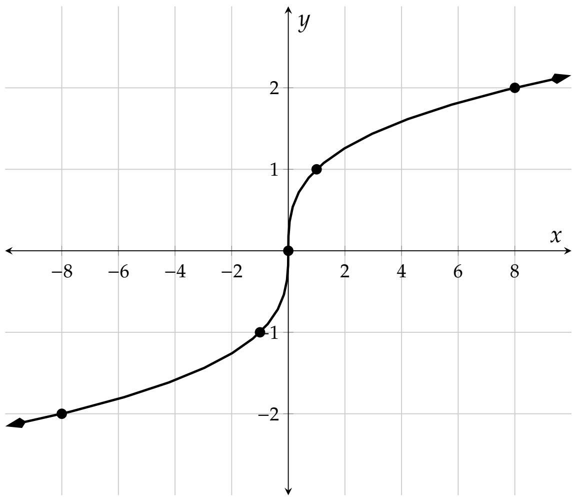


Figure 14.1.25: Graph of $g(x) = \sqrt[3]{x}$

Both the domain and range of the *cube* root function are $(-\infty, \infty)$. Compare this with the domain and range of the *square* root function, which are each $[0, \infty)$. The reason for the difference is that we cannot take the *square* root of negative numbers, but we can take the *cube* root of negative numbers (and when we do, we get negative numbers as the output).

Remark 14.1.26. It is helpful to be able to quickly sketch the graphs of the following types of basic functions:

$$f(x) = c \qquad f(x) = mx + b \qquad f(x) = a(x-h)^2 + k \qquad f(x) = |x|$$

$$f(x) = \frac{1}{x} \qquad f(x) = \frac{1}{x^2} \qquad f(x) = \sqrt{x} \qquad f(x) = \sqrt[3]{x}$$

Now with the graph of the cube root function, you have seen all of these shapes in this book.

Example 14.1.27 Nasim makes solid copper spheres for their grounding and healing properties. A sphere's radius can be calculated by the formula $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$, where $r(V)$ stands for the sphere's radius for a given volume V. If Nasim uses 2 cubic inches of copper per sphere, what diameter should he list on his website? Round your answer to two decimal places.

Explanation. First, to find the radius we will substitute 2 in for V, and we have:

$$\begin{aligned} r(V) &= \sqrt[3]{\frac{3V}{4\pi}} \\ r(2) &= \sqrt[3]{\frac{3(2)}{4\pi}} \\ &\approx 0.78 \end{aligned}$$

The spheres will have a radius of approximately 0.78 in.

When we calculate $\sqrt[3]{\frac{3(2)}{4\pi}}$ with a calculator, we enter `(3*2/(4□))^(1/3)`. To find the diameter, we multiply the radius by 2 to get 1.56 in. Nasim can advertise the spheres to be 1.56 inches in diameter.

14.1.4 Other Roots

Similar to the cube root, there is the fourth root, and the fifth root, and so on, as in the following examples:

$$\sqrt[4]{16} = 2 \text{ because } 2^4 = 16 \qquad \sqrt[5]{-32} = -2 \text{ because } (-2)^5 = -32 \qquad \sqrt[6]{64} = 2 \text{ because } 2^6 = 64,$$

To calculate the fifth root of -32 with a calculator, try typing `(-32)^(1/5)` or `root(5,-32)`.

Definition 14.1.28. The **index** of a radical is the number "n" in $\sqrt[n]{}$. The symbol $\sqrt[n]{}$ is read "the nth root." The plural of index is **indices**, as in "we can evaluate radicals of multiple indices in a single expression."

Fact 14.1.29 Domain of Radical Functions. *To find the domain of any* even *indexed radical function, set the radicand greater than or equal to zero. The solution set is the domain of the function.*

The domain of any odd *indexed radical of a polynomial is* $(-\infty, \infty)$.

Example 14.1.30

a. Let $g(x) = 7 - 3\sqrt[4]{10 - 5x}$. Algebraically find g's domain and graphically find g's range.

b. Let $h(x) = 4\sqrt[5]{2x - 5} + 1$. Algebraically find h's domain and graphically find h's range.

Explanation.

a. First note that the index of this function is 4. By Fact 14.1.29, to find the domain of this function we must set the radicand greater or equal to zero and solve.

$$\begin{aligned} 10 - 5x &\geq 0 \\ -5x &\geq -10 \\ x &\leq \frac{-10}{-5} \\ x &\leq 2 \end{aligned}$$

So, the domain of g must be $(-\infty, 2]$.

To find the function's range, we use technology to graph the function. By the graph, the function's range is $(-\infty, 7]$.

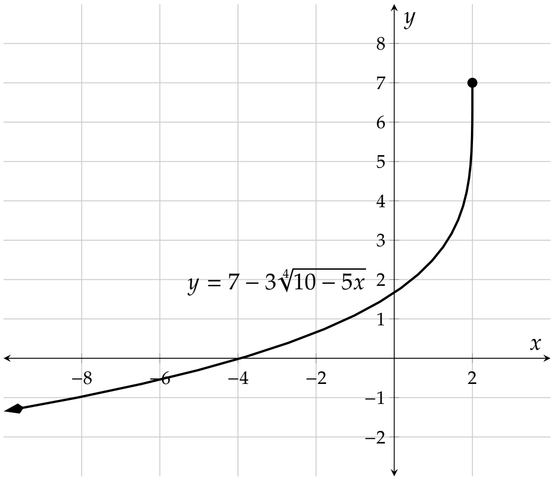


Figure 14.1.31: Graph of $y = 7 - 3\sqrt[4]{10 - 5x}$

b. First note that the index of the function h for $h(x) = 4\sqrt[5]{2x-5}+1$ is 5. By Fact 14.1.29, the domain is $(-\infty, \infty)$.

To find the function's range, we use technology to graph the function. According to the graph, the function's range is also the set of all real numbers.

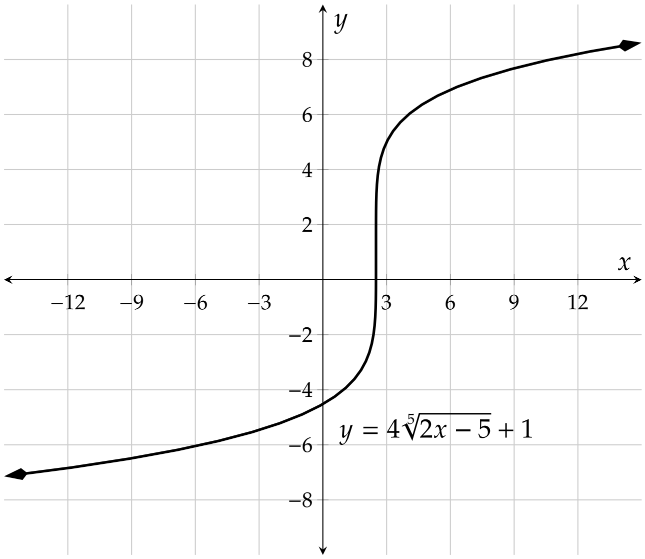


Figure 14.1.32: Graph of $y = 4\sqrt[5]{2x-5}+1$

Exercises

Review and Warmup

1. Evaluate the following.

$-\sqrt{81} =$ ☐.

2. Evaluate the following.

$-\sqrt{121} =$ ☐.

3. Evaluate the following.

$\sqrt{-144} =$ ☐.

4. Evaluate the following.

$\sqrt{-4} =$ ☐.

5. Evaluate the following.

$\sqrt{\frac{4}{81}} =$ ☐.

6. Evaluate the following.

$\sqrt{\frac{9}{25}} =$ ☐.

7. Without using a calculator, evaluate the expression.

a. $\sqrt[3]{8} =$ ☐

b. $\sqrt[3]{-8} =$ ☐

c. $-\sqrt[3]{8} =$ ☐

8. Without using a calculator, evaluate the expression.

a. $\sqrt[4]{16} =$ ☐

b. $\sqrt[4]{-16} =$ ☐

c. $-\sqrt[4]{16} =$ ☐

9. Without using a calculator, estimate the value of $\sqrt{65}$:

(☐ 8.94 ☐ 7.94 ☐ 7.06 ☐ 8.06)

10. Without using a calculator, estimate the value of $\sqrt{61}$:
(☐ 7.19 ☐ 7.81 ☐ 8.81 ☐ 8.19)

11. Without using a calculator, estimate the value of $\sqrt[3]{68}$:
(☐ 4.08 ☐ 3.08 ☐ 3.92 ☐ 4.92)

12. Without using a calculator, estimate the value of $\sqrt[3]{62}$:
(☐ 3.04 ☐ 4.96 ☐ 3.96 ☐ 4.04)

Domain and Range Find the domain of the function.

13. $f(x) = \sqrt{10 - x}$

14. $g(x) = \sqrt{7 - x}$

15. $g(x) = \sqrt{3 + 20x}$

16. $h(x) = \sqrt{9 + 14x}$

17. $F(x) = \sqrt[3]{-9x - 1}$

18. $G(x) = \sqrt[3]{-7x - 7}$

19. $G(x) = \sqrt[4]{45 - 5x}$

20. $H(x) = \sqrt[4]{-9 - 3x}$

21. $K(x) = -\frac{8}{\sqrt{x+5}}$

22. $f(x) = \frac{6}{\sqrt{x-7}}$

23. $f(x) = \sqrt{x} + 7$

24. $f(x) = \sqrt{x} + 5$

25. $f(x) = \sqrt{-3 - 7x}$

26. $f(x) = \sqrt{-2 - 7x}$

27. $f(x) = \frac{1}{\sqrt{18-5x}}$

28. $f(x) = \frac{1}{\sqrt{40-56x}}$

29. $f(x) = \frac{1}{\sqrt{29x+62}}$

30. $f(x) = \frac{1}{\sqrt{2x+85}}$

31. $f(x) = \frac{x+75}{\sqrt{53x^2+4}}$

32. $f(x) = \frac{x+28}{\sqrt{26x^2+15}}$

33. $f(x) = \sqrt[11]{x} + 26$

34. $f(x) = \sqrt[9]{x} + 37$

35. $f(x) = \frac{1}{\sqrt[7]{27x+48}}$

36. $f(x) = \frac{1}{\sqrt[3]{92x+59}}$

37. $f(x) = \frac{1}{\sqrt[6]{7x+70}}$

38. $f(x) = \frac{1}{\sqrt[10]{3x+81}}$

39. Use technology to find the range of the function K defined by $K(x) = \sqrt{4 - x} - 3$.

40. Use technology to find the range of the function f defined by $f(x) = \sqrt{2 - x} - 5$.

41. Use technology to find the range of the function f defined by $f(x) = \frac{0.1}{\sqrt{x-4}} + 3$.

42. Use technology to find the range of the function g defined by $g(x) = \frac{0.5}{\sqrt{x+1}} + 1$.

Applications If an object is dropped with no initial velocity, the time since the drop, in seconds, can be calculated by the function

$$T(h) = \sqrt{\frac{2h}{g}}$$

where h is the distance the object traveled in feet. The variable g is the gravitational acceleration on earth, and we can round it to $32\frac{ft}{s^2}$ for this problem.

43. a. After ______ seconds since the release, the object would have traveled 24 feet.

b. After 2.9 seconds since the release, the object would have traveled ______ feet.

44. a. After ______ seconds since the release, the object would have traveled 29 feet.

b. After 2.4 seconds since the release, the object would have traveled ______ feet.

A factory manufactures toy plastic balls. For a ball with a certain volume, V in cubic centimeters, the ball's radius can be calculated by the formula

$$r(V) = \sqrt[3]{\frac{3V}{4\pi}}$$

45. a. If a ball's volume is 350 cubic centimeters, its radius must be ______ centimeters.

b. If a ball's radius is 3.2 centimeters, its volume would be ______ cubic centimeters.

46. a. If a ball's volume is 400 cubic centimeters, its radius must be ______ centimeters.

b. If a ball's radius is 2.7 centimeters, its volume would be ______ cubic centimeters.

The speed of a tsunami, in meters per second, can be modeled by the function $S(d) = \sqrt{9.8d}$, where d is the depth of water in meters. Answer the following question with technology.

47. A tsunami's speed at 900 meters below the sea level is ______ meters per second.

48. A tsunami's speed at ______ meters below the sea level is 80 meters per second.

Distance Formula

49. Find the distance between the points $(-3, 3)$ and $(-24, 23)$.

50. Find the distance between the points $(14, -9)$ and $(19, 3)$.

51. Find the distance between the points $(8, -10)$ and $(11, -4)$.

52. Find the distance between the points $(-6, 7)$ and $(-1, 17)$.

14.2 Radical Expressions and Rational Exponents

Recall that in Remark 14.1.23, we learned to calculate the cube root of a number, say $\sqrt[3]{8}$, we can type `8^(1/3)` into a calculator. This suggests that $\sqrt[3]{8} = 8^{1/3}$. In this section, we will learn why this is true, and how to simplify expressions with rational exponents.

Many learners will find a review of exponent rules to be helpful before continuing with the current section. Section 2.9 covers an introduction to exponent rules, and there is more in Section 6.1. The basic rules are summarized in List 6.1.15. These rules are still true and we can use them throughout this section whenever they might help.

14.2.1 Radical Expressions and Rational Exponents

Compare the following calculations:

$$\begin{aligned}\sqrt{9}\cdot\sqrt{9} &= 3\cdot 3\\ &= 9\end{aligned} \qquad\qquad \begin{aligned}9^{1/2}\cdot 9^{1/2} &= 9^{1/2+1/2}\\ &= 9^{1}\\ &= 9\end{aligned}$$

If we rewrite the above calculations with exponents, we have:

$$\left(\sqrt{9}\right)^2 = 9 \qquad\qquad \left(9^{1/2}\right)^2 = 9$$

Since $\sqrt{9}$ and $9^{1/2}$ are both positive, and squaring either of them generates the same number, we conclude that

$$\sqrt{9} = 9^{1/2}$$

We can verify this result by entering `9^(1/2)` into a calculator, and we get `3`. In general for any non-negative real number a, we have:

$$\sqrt{a} = a^{1/2}$$

Similarly, when a is non-negative we can prove:

$$\sqrt[3]{a} = a^{1/3} \qquad \sqrt[4]{a} = a^{1/4} \qquad \sqrt[5]{a} = a^{1/5} \qquad \cdots$$

Let's summarize this information with a new exponent rule.

Fact 14.2.2 Radicals and Rational Exponents Rule. *If n is any natural number, and a is any non-negative real number or function with non-negative outputs, then*

$$a^{1/n} = \sqrt[n]{a}.$$

Additionally, if n is an odd *natural number, then even when a is negative, we still have $a^{1/n} = \sqrt[n]{a}$.*

Warning 14.2.3 Exponents on Negative Bases. Some computers and calculators follow different conventions when there is an exponent on a negative base. To see an example of this, visit *WolframAlpha* and try entering `cuberoot(-8)`, and then try `(-8)^(1/3)`, and you will get different results. `cuberoot(-8)` will come out as -2, but `(-8)^(1/3)` will come out as a certain non-real complex number. Most likely, the graphing technology you are using *does* behave as in Fact 14.2.2, but you should confirm this.

With this relationship, we can re-write radical expressions as expressions with rational exponents.

Example 14.2.4 Evaluate $\sqrt[4]{9}$ with a calculator. Round your answer to two decimal places.

Since $\sqrt[4]{9} = 9^{1/4}$, we press the following buttons on a calculator to get the value: `9^(1/4)`. So, we see that $\sqrt[4]{9} \approx 1.73$.

For many examples that follow, we will not need a calculator. We will, however, need to recognize the roots in Table 14.2.5.

Square Roots	Cube Roots	4th-Roots	5th-Roots	Roots of Powers of 2
$\sqrt{1} = 1$	$\sqrt[3]{1} = 1$	$\sqrt[4]{1} = 1$	$\sqrt[5]{1} = 1$	
$\sqrt{4} = 2$	$\sqrt[3]{8} = 2$	$\sqrt[4]{16} = 2$	$\sqrt[5]{32} = 2$	$\sqrt{4} = 2$
$\sqrt{9} = 3$	$\sqrt[3]{27} = 3$	$\sqrt[4]{81} = 3$		$\sqrt[3]{8} = 2$
$\sqrt{16} = 4$	$\sqrt[3]{64} = 4$			$\sqrt[4]{16} = 2$
$\sqrt{25} = 5$	$\sqrt[3]{125} = 5$			$\sqrt[5]{32} = 2$
$\sqrt{36} = 6$				$\sqrt[6]{64} = 2$
$\sqrt{49} = 7$				$\sqrt[7]{128} = 2$
$\sqrt{64} = 8$				$\sqrt[8]{256} = 2$
$\sqrt{81} = 9$				$\sqrt[9]{512} = 2$
$\sqrt{100} = 10$				$\sqrt[10]{1024} = 2$
$\sqrt{121} = 11$				
$\sqrt{144} = 12$				

Table 14.2.5: Small Roots of Appropriate Natural Numbers

Example 14.2.6 Convert the radical expression $\sqrt[3]{5}$ into an expression with a rational exponent and simplify it if possible.

$\sqrt[3]{5} = 5^{1/3}$. No simplification is possible since the cube root of 5 is not a perfect integer appearing in Table 14.2.5.

Example 14.2.7 Write the expressions in radical form using Fact 14.2.2 and simplify the results.

a. $4^{1/2}$

b. $(-9)^{1/2}$

c. $-16^{1/4}$

d. $64^{-1/3}$

e. $(-27)^{1/3}$

f. $3^{1/2} \cdot 3^{1/2}$

g. 12^0

Explanation.

a. $$\begin{aligned} 4^{1/2} &= \sqrt{4} \\ &= 2 \end{aligned}$$

b. $(-9)^{1/2} = \sqrt{-9}$ This value is non-real.

c. Without parentheses around -16, the negative sign in this problem should be left out of the radical.

$$\begin{aligned} -16^{1/4} &= -\sqrt[4]{16} \\ &= -2 \end{aligned}$$

d. $$\begin{aligned}64^{-1/3} &= \frac{1}{64^{1/3}}\\ &= \frac{1}{\sqrt[3]{64}}\\ &= \frac{1}{4}\end{aligned}$$

e. $$\begin{aligned}(-27)^{1/3} &= \sqrt[3]{-27}\\ &= -3\end{aligned}$$

f. $$\begin{aligned}3^{1/2} \cdot 3^{1/2} &= \sqrt{3} \cdot \sqrt{3}\\ &= \sqrt{3 \cdot 3}\\ &= \sqrt{9}\\ &= 3\end{aligned}$$

g. $12^0 = 1$

Fact 14.2.2 applies to variables in expressions just as much as it does to numbers.

Remark 14.2.8. In general, it is easier to do algebra with rational exponents on variables than with radicals of variables. You should use Fact 14.2.2 to convert from rational exponents to radicals on variables *only as a last step* in simplifying.

Example 14.2.9 Write the expressions as simplified as they can be using radicals.

a. $2x^{-1/2}$

b. $(5x)^{1/3}$

c. $\left(-27x^{12}\right)^{1/3}$

d. $\left(\frac{16x}{81y^8}\right)^{1/4}$

Explanation.

a. Note that in this example the exponent is only applied to the x. Making this type of observation should be our first step for each of these exercises.

$$\begin{aligned}2x^{-1/2} &= \frac{2}{x^{1/2}} && \text{by the Negative Exponent Rule}\\ &= \frac{2}{\sqrt{x}} && \text{by the Radicals and Rational Exponents Rule}\end{aligned}$$

b. In this exercise, the exponent applies to both the 5 and x.

$$(5x)^{1/3} = \sqrt[3]{5x} \qquad \text{by the Radicals and Rational Exponents Rule}$$

We could choose to simplify our answer in a different way. Note that neither one is technically preferred over the other except that perhaps the first way is simpler.

$$\begin{aligned}(5x)^{1/3} &= 5^{1/3}x^{1/3} && \text{by the Product to a Power Rule}\\ &= \sqrt[3]{5}\sqrt[3]{x} && \text{by the Radicals and Rational Exponents Rule}\end{aligned}$$

c. As in the previous exercise, we have a choice as to how to simplify this expression. Here we should note that we *do* know what the cube root of −27 is, so we will take the path to splitting up the expression, using the Product to a Power Rule, before applying the root.

$$\begin{aligned}\left(-27x^{12}\right)^{1/3} &= (-27)^{1/3}\cdot\left(x^{12}\right)^{1/3} && \text{by the Product to a Power Rule}\\ &= (-27)^{1/3}\cdot\left(x^{12\cdot 1/3}\right) && \text{by the Power to a Power Rule}\\ &= \sqrt[3]{-27}\cdot x^4 && \text{by the Radicals and Rational Exponents Rule}\\ &= -3x^4\end{aligned}$$

d. We'll use the exponent rule for a fraction raised to a power.

$$\begin{aligned}\left(\frac{16x}{81y^8}\right)^{1/4} &= \frac{(16x)^{1/4}}{\left(81y^8\right)^{1/4}} && \text{by the Quotient to a Power Rule}\\ &= \frac{16^{1/4}\cdot x^{1/4}}{81^{1/4}\cdot\left(y^8\right)^{1/4}} && \text{by the Product to a Power Rule}\\ &= \frac{16^{1/4}\cdot x^{1/4}}{81^{1/4}\cdot y^2}\\ &= \frac{\sqrt[4]{16}\cdot\sqrt[4]{x}}{\sqrt[4]{81}\cdot y^2} && \text{by the Radicals and Rational Exponents Rule}\\ &= \frac{2\sqrt[4]{x}}{3y^2}\end{aligned}$$

Fact 14.2.2 describes what can be done when there is a fractional exponent and the numerator is a 1. The numerator doesn't have to be a 1 though and we need guidance for that situation.

Fact 14.2.10 Radicals and Rational Exponents Rule. *If m and n are natural numbers such that $\frac{m}{n}$ is a reduced fraction, and a is any non-negative real number or function that takes non-negative values, then*

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m .$$

Additionally, if n is an odd *natural number, then even when a is negative, we still have $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$.*

Remark 14.2.11. By Fact 14.2.10, there are two ways to express $a^{m/n}$ as a radical, both

$$a^{m/n} = \sqrt[n]{a^m} \qquad \text{and} \qquad a^{m/n} = \left(\sqrt[n]{a}\right)^m$$

There are different times to use each formula. In general, use $a^{m/n} = \sqrt[n]{a^m}$ for variables and $a^{m/n} = \left(\sqrt[n]{a}\right)^m$ for numbers.

Example 14.2.12

a. Consider the expression $27^{4/3}$. Use both versions of Fact 14.2.10 to explain part of Remark 14.2.11.

b. Consider the expression $x^{4/3}$. Use both versions of Fact 14.2.10 to explain the other part of Remark 14.2.11.

Explanation.

a. The expression $27^{4/3}$ can be evaluated in the following two ways by Fact 14.2.10.

$$\begin{aligned} 27^{4/3} &= \sqrt[3]{27^4} && \text{by the first part of the Radicals and Rational Exponents Rule} \\ &= \sqrt[3]{531441} \\ &= 81 \end{aligned}$$

or

$$\begin{aligned} 27^{4/3} &= \left(\sqrt[3]{27}\right)^4 && \text{by the second part of the Radicals and Rational Exponents Rule} \\ &= 3^4 \\ &= 81 \end{aligned}$$

The calculations using $a^{m/n} = \left(\sqrt[n]{a}\right)^m$ worked with smaller numbers and can be done without a calculator. This is why we made the general recommendation in Remark 14.2.11.

b. The expression $x^{4/3}$ can be evaluated in the following two ways by Fact 14.2.10.

$$x^{4/3} = \sqrt[3]{x^4} \qquad \text{by the first part of Radicals and Rational Exponents Rule}$$

or

$$x^{4/3} = \left(\sqrt[3]{x}\right)^4 \qquad \text{by the second part of the Radicals and Rational Exponents Rule}$$

In this case, the simplification using $a^{m/n} = \sqrt[n]{a^m}$ is just shorter looking and easier to write. This is why we made the general recommendation in Remark 14.2.11.

Example 14.2.13 Simplify the expressions using Fact 14.2.10.

a. $8^{2/3}$

b. $16^{-3/2}$

c. $-16^{3/4}$

d. $\left(-\frac{27}{64}\right)^{2/3}$

Explanation.

a.

$$\begin{aligned} 8^{2/3} &= \left(\sqrt[3]{8}\right)^2 && \text{by the second part of the Radicals and Rational Exponents Rule} \\ &= 2^2 \\ &= 4 \end{aligned}$$

b.

$$\begin{aligned} 16^{-3/2} &= \frac{1}{16^{3/2}} && \text{by the Negative Exponent Rule} \\ &= \frac{1}{\left(\sqrt{16}\right)^3} && \text{by the second part of the Radicals and Rational Exponents Rule} \\ &= \frac{1}{4^3} \\ &= \frac{1}{64} \end{aligned}$$

c.

$$\begin{aligned} -16^{3/4} &= -\left(\sqrt[4]{16}\right)^3 && \text{by the second part of the Radicals and Rational Exponents Rule} \\ &= -2^3 \\ &= -8 \end{aligned}$$

d. In this problem the negative can be associated with either the numerator or the denominator, but not both. We choose the numerator.

$$\begin{aligned} \left(-\frac{27}{64}\right)^{2/3} &= \left(\sqrt[3]{-\frac{27}{64}}\right)^2 && \text{by the second part of the Radicals and Rational Exponents Rule} \\ &= \left(\frac{\sqrt[3]{-27}}{\sqrt[3]{64}}\right)^2 \\ &= \left(\frac{-3}{4}\right)^2 \\ &= \frac{(-3)^2}{(4)^2} \\ &= \frac{9}{16} \end{aligned}$$

While we are looking at the algebra of $x^{m/n}$, we should briefly examine a graph to see what this type of function can look like. Fractional powers can make some fairly interesting graphs. We invite you to play with these graphs on your favorite graphing program.

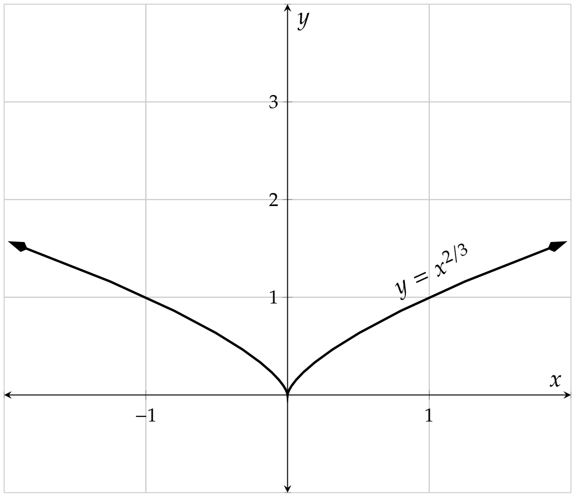


Figure 14.2.14: A Graph of $y = x^{2/3}$

14.2.2 More Expressions with Rational Exponents

To recap, here is a "complete" list of exponent rules.

Product Rule $a^n \cdot a^m = a^{n+m}$

Power to a Power Rule $(a^n)^m = a^{n \cdot m}$

Product to a Power Rule $(ab)^n = a^n \cdot b^n$

Quotient Rule $\dfrac{a^n}{a^m} = a^{n-m}$, as long as $a \neq 0$

Quotient to a Power Rule $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$, as long as $b \neq 0$

Zero Exponent Rule $a^0 = 1$ for $a \neq 0$

Negative Exponent Rule $a^{-n} = \frac{1}{a^n}$

Negative Exponent Reciprocal Rule $\frac{1}{a^{-n}} = a^n$

Negative Exponent on Fraction Rule $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$

Radical and Rational Exponent Rule $x^{1/n} = \sqrt[n]{x}$

Radical and Rational Exponent Rule $x^{m/n} = \left(\sqrt[n]{x}\right)^m$, usually for numbers

Radical and Rational Exponent Rule $x^{m/n} = \sqrt[n]{x^m}$, usually for variables

List 14.2.15: Complete List of Exponent Rules

Example 14.2.16 Convert the following radical expressions into expressions with rational exponents, and simplify them if possible.

a. $\dfrac{1}{\sqrt{x}}$

b. $\dfrac{1}{\sqrt[3]{25}}$

Explanation.

a.

$$\begin{aligned} \frac{1}{\sqrt{x}} &= \frac{1}{x^{1/2}} && \text{by the Radicals and Rational Exponents Rule} \\ &= x^{-1/2} && \text{by the Negative Exponent Rule} \end{aligned}$$

b.

$$\begin{aligned} \frac{1}{\sqrt[3]{25}} &= \frac{1}{25^{1/3}} && \text{by the Radicals and Rational Exponents Rule} \\ &= \frac{1}{\left(5^2\right)^{1/3}} \\ &= \frac{1}{5^{2 \cdot 1/3}} && \text{by the Power to a Power Rule} \\ &= \frac{1}{5^{2/3}} \\ &= 5^{-2/3} && \text{by the Negative Exponent Rule} \end{aligned}$$

Learners of these simplifications often find it challenging, so we now include a plethora of examples of varying difficulty.

Example 14.2.17 Use exponent properties in List 14.2.15 to simplify the expressions, and write all final versions using radicals.

a. $2w^{7/8}$

b. $\frac{1}{2}y^{-1/2}$

c. $(27b)^{2/3}$

d. $\left(-8p^6\right)^{5/3}$

e. $\sqrt{x^3} \cdot \sqrt[4]{x}$

f. $h^{1/3} + h^{1/3} + h^{1/3}$

g. $\frac{\sqrt{z}}{\sqrt[3]{z}}$

h. $\sqrt{\sqrt[4]{q}}$

i. $3\left(c^{1/2} + d^{1/2}\right)^2$

j. $3\left(4k^{2/3}\right)^{-1/2}$

Explanation.

a.

$$2w^{7/8} = 2\sqrt[8]{w^7} \qquad \text{by the Radicals and Rational Exponents Rule}$$

b.

$$\begin{aligned} \frac{1}{2}y^{-1/2} &= \frac{1}{2}\frac{1}{y^{1/2}} && \text{by the Negative Exponent Rule} \\ &= \frac{1}{2}\frac{1}{\sqrt{y}} && \text{by the Radicals and Rational Exponents Rule} \\ &= \frac{1}{2\sqrt{y}} \end{aligned}$$

c.

$$\begin{aligned}
(27b)^{2/3} &= (27)^{2/3} \cdot (b)^{2/3} && \text{by the Product to a Power Rule}\\
&= \left(\sqrt[3]{27}\right)^2 \cdot \sqrt[3]{b^2} && \text{by the Radicals and Rational Exponents Rule}\\
&= 3^2 \cdot \sqrt[3]{b^2}\\
&= 9\sqrt[3]{b^2}
\end{aligned}$$

d.

$$\begin{aligned}
\left(-8p^6\right)^{5/3} &= (-8)^{5/3} \cdot \left(p^6\right)^{5/3} && \text{by the Product to a Power Rule}\\
&= (-8)^{5/3} \cdot p^{6 \cdot 5/3} && \text{by the Power to a Power Rule}\\
&= \left(\sqrt[3]{-8}\right)^5 \cdot p^{10} && \text{by the Radicals and Rational Exponents Rule}\\
&= (-2)^5 \cdot p^{10}\\
&= -32p^{10}
\end{aligned}$$

e.

$$\begin{aligned}
\sqrt{x^3} \cdot \sqrt[4]{x} &= x^{3/2} \cdot x^{1/4} && \text{by the Radicals and Rational Exponents Rule}\\
&= x^{3/2+1/4} && \text{by the Product Rule}\\
&= x^{6/4+1/4}\\
&= x^{7/4}\\
&= \sqrt[4]{x^7} && \text{by the Radicals and Rational Exponents Rule}
\end{aligned}$$

f.

$$\begin{aligned}
h^{1/3} + h^{1/3} + h^{1/3} &= 3h^{1/3}\\
&= 3\sqrt[3]{h} && \text{by the Radicals and Rational Exponents Rule}
\end{aligned}$$

g.

$$\begin{aligned}
\frac{\sqrt{z}}{\sqrt[3]{z}} &= \frac{z^{1/2}}{z^{1/3}} && \text{by the Radicals and Rational Exponents Rule}\\
&= z^{1/2-1/3} && \text{by the Quotient Rule}\\
&= z^{3/6-2/6}\\
&= z^{1/6}\\
&= \sqrt[6]{z} && \text{by the Radicals and Rational Exponents Rule}
\end{aligned}$$

h.

$$\begin{aligned}
\sqrt{\sqrt[4]{q}} &= \sqrt{q^{1/4}} && \text{by the Radicals and Rational Exponents Rule}\\
&= \left(q^{1/4}\right)^{1/2} && \text{by the Radicals and Rational Exponents Rule}
\end{aligned}$$

$$\begin{aligned} &= q^{1/4 \cdot 1/2} && \text{by the Power to a Power Rule} \\ &= q^{1/8} \\ &= \sqrt[8]{q} && \text{by the Radicals and Rational Exponents Rule} \end{aligned}$$

i.

$$\begin{aligned} 3\left(c^{1/2}+d^{1/2}\right)^2 &= 3\left(c^{1/2}+d^{1/2}\right)\left(c^{1/2}+d^{1/2}\right) \\ &= 3\left(\left(c^{1/2}\right)^2 + 2c^{1/2}\cdot d^{1/2} + \left(d^{1/2}\right)^2\right) \\ &= 3\left(c^{1/2\cdot 2} + 2c^{1/2}\cdot d^{1/2} + d^{1/2\cdot 2}\right) \\ &= 3\left(c + 2c^{1/2}\cdot d^{1/2} + d\right) \\ &= 3\left(c + 2(cd)^{1/2} + d\right) && \text{by the Product to a Power Rule} \\ &= 3\left(c + 2\sqrt{cd} + d\right) && \text{by the Radicals and Rational Exponents Rule} \\ &= 3c + 6\sqrt{cd} + 3d \end{aligned}$$

j.

$$\begin{aligned} 3\left(4k^{2/3}\right)^{-1/2} &= \frac{3}{\left(4k^{2/3}\right)^{1/2}} && \text{by the Negative Exponent Rule} \\ &= \frac{3}{4^{1/2}\left(k^{2/3}\right)^{1/2}} && \text{by the Product to a Power Rule} \\ &= \frac{3}{4^{1/2}k^{2/3\cdot 1/2}} && \text{by the Power to a Power Rule} \\ &= \frac{3}{4^{1/2}k^{1/3}} \\ &= \frac{3}{\sqrt{4}\sqrt[3]{k}} && \text{by the Radicals and Rational Exponents Rule} \\ &= \frac{3}{2\sqrt[3]{k}} \end{aligned}$$

We will end a with a short application on rational exponents. Kepler's Laws of Orbital Motion[1] describe how planets orbit stars and how satellites orbit planets. In particular, his third law has a rational exponent, which we will now explore.

Example 14.2.18 Kepler and the Satellite. Kepler's third law of motion says that for objects with a roughly circular orbit that the time (in hours) that it takes to make one full revolution around the planet, T, is proportional to three-halves power of the distance (in kilometers) from the center of the planet to the satellite, r. For the Earth, it looks like this:

$$T = \frac{2\pi}{\sqrt{G\cdot M_E}}r^{3/2}$$

---

[1] en.wikipedia.org/wiki/Kepler%27s_laws_of_planetary_motion

In this case, both G and M_E are constants. G stands for the universal gravitational constant[a] where G is about $8.65 \times 10^{-13}\ \frac{\text{km}^3}{\text{kg}\,\text{h}^2}$ and M_E stands for the mass of the Earth[b] where M_E is about 5.972×10^{24} kg. Inputting these values into this formula yields a simplified version that looks like this:

$$T \approx 2.76 \times 10^{-6} r^{3/2}$$

Most satellites orbit in what is called low Earth orbit[c], including the international space station which orbits at about 340 km above from Earth's surface. The Earth's average radius is about 6380 km. Find the period of the international space station.

Explanation. The formula has already been identified, but the input takes just a little thought. The formula uses r as the distance from the center of the Earth to the satellite, so to find r we need to combine the radius of the Earth and the distance to the satellite above the surface of the Earth.

$$\begin{aligned} r &= 340 + 6380 \\ &= 6720 \end{aligned}$$

Now we can input this value into the formula and evaluate.

$$\begin{aligned} T &\approx 2.76 \cdot 10^{-6} r^{3/2} \\ &\approx 2.76 \cdot 10^{-6} (6720)^{3/2} \\ &\approx 2.76 \cdot 10^{-6} \left(\sqrt{6720}\right)^3 \\ &\approx 1.52 \end{aligned}$$

The formula tells us that it takes a little more than an hour and a half for the ISS to orbit the Earth! That works out to 15 or 16 sunrises per day.

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[a] en.wikipedia.org/wiki/Gravitational_constant
[b] en.wikipedia.org/wiki/Earth_mass
[c] en.wikipedia.org/wiki/Low_Earth_orbit

Exercises

Review and Warmup

1. Evaluate the following.

a. $\sqrt{36} =$

b. $\sqrt{1} =$

c. $\sqrt{4} =$

2. Evaluate the following.

a. $\sqrt{49} =$

b. $\sqrt{81} =$

c. $\sqrt{36} =$

3. Evaluate the following.

a. $\sqrt{\frac{81}{16}} =$

b. $\sqrt{-\frac{144}{121}} =$

4. Evaluate the following.

a. $\sqrt{\dfrac{100}{49}} = \boxed{}$

b. $\sqrt{-\dfrac{16}{9}} = \boxed{}$

5. Evaluate the following.

Do not use a calculator.

a. $\sqrt{144} = \boxed{}$

b. $\sqrt{1.44} = \boxed{}$

c. $\sqrt{14400} = \boxed{}$

6. Evaluate the following.

Do not use a calculator.

a. $\sqrt{4} = \boxed{}$

b. $\sqrt{0.04} = \boxed{}$

c. $\sqrt{400} = \boxed{}$

7. Use the properties of exponents to simplify the expression.

$r^4 \cdot r^{19}$

8. Use the properties of exponents to simplify the expression.

$x^6 \cdot x^{12}$

9. Use the properties of exponents to simplify the expression.

$\left(y^5\right)^4$

10. Use the properties of exponents to simplify the expression.

$\left(t^7\right)^{11}$

11. Use the properties of exponents to simplify the expression.

$\left(\dfrac{5x^7}{8}\right)^3 = \boxed{}$

12. Use the properties of exponents to simplify the expression.

$\left(\dfrac{7x^8}{6}\right)^2 = \boxed{}$

13. Use the properties of exponents to simplify the expression.

$\left(-10x^{10}\right)^3$

14. Use the properties of exponents to simplify the expression.

$\left(-6t^{11}\right)^2$

15. Use the properties of exponents to simplify the expression.

$\dfrac{t^3}{t} = \boxed{}$

16. Use the properties of exponents to simplify the expression.

$\dfrac{y^5}{y^4} = \boxed{}$

17. Rewrite the expression simplified and using only positive exponents.

$x^{-10} \cdot x^8 = \boxed{}$

18. Rewrite the expression simplified and using only positive exponents.

$y^{-4} \cdot y^2 = \boxed{}$

19. Rewrite the expression simplified and using only positive exponents.

$(-6y^{-16}) \cdot (2y^{10}) = \boxed{}$

20. Rewrite the expression simplified and using only positive exponents.

$(-3r^{-10}) \cdot (6r^2) = \boxed{}$

Calculations Without using a calculator, evaluate the expression.

21. a. $64^{\frac{1}{2}} = \boxed{}$

b. $(-64)^{\frac{1}{2}} = \boxed{}$

c. $-64^{\frac{1}{2}} = \boxed{}$

22. a. $81^{\frac{1}{2}} = \boxed{}$

b. $(-81)^{\frac{1}{2}} = \boxed{}$

c. $-81^{\frac{1}{2}} = \boxed{}$

23. a. $125^{\frac{1}{3}} = \boxed{}$

b. $(-125)^{\frac{1}{3}} = \boxed{}$

c. $-125^{\frac{1}{3}} = \boxed{}$

24. a. $8^{\frac{1}{3}} = \boxed{}$

b. $(-8)^{\frac{1}{3}} = \boxed{}$

c. $-8^{\frac{1}{3}} = \boxed{}$

25. $8^{-\frac{2}{3}} = \boxed{}$

26. $32^{-\frac{2}{5}} = \boxed{}$

27. $\left(\frac{1}{27}\right)^{-\frac{2}{3}} = \boxed{}$

28. $\left(\frac{1}{27}\right)^{-\frac{2}{3}} = \boxed{}$

29. $\sqrt[4]{16^3} = \boxed{}$

30. $\sqrt[4]{81^3} = \boxed{}$

31. $\sqrt[5]{1024} = \boxed{}$

32. $\sqrt[3]{64} = \boxed{}$

33. a. $\sqrt[3]{1} = \boxed{}$

b. $\sqrt[3]{-1} = \boxed{}$

c. $-\sqrt[3]{1} = \boxed{}$

34. a. $\sqrt[3]{8} = \boxed{}$

b. $\sqrt[3]{-8} = \boxed{}$

c. $-\sqrt[3]{8} = \boxed{}$

35. a. $\sqrt[4]{1} = \boxed{}$

b. $\sqrt[4]{-1} = \boxed{}$

c. $-\sqrt[4]{1} = \boxed{}$

36. a. $\sqrt[4]{16} = \boxed{}$

b. $\sqrt[4]{-16} = \boxed{}$

c. $-\sqrt[4]{16} = \boxed{}$

37. $\sqrt[4]{10^3} = \boxed{}$

38. $\sqrt[3]{8^2} = \boxed{}$

39. $\sqrt[3]{-\frac{27}{64}} = \boxed{}.$

40. $\sqrt[3]{-\frac{27}{125}} = \boxed{}.$

Convert Radicals to Fractional Exponents Use rational exponents to write the expression.

41. $\sqrt{b} = \boxed{}$

42. $\sqrt[8]{x} = \boxed{}$

43. $\sqrt[5]{5y + 7} = \boxed{}$

44. $\sqrt[4]{2z + 1} = \boxed{}$

45. $\sqrt[7]{t} = \boxed{}$

46. $\sqrt[4]{r} = \boxed{}$

47. $\frac{1}{\sqrt{m^3}} =$ ☐

48. $\frac{1}{\sqrt[7]{n^4}} =$ ☐

Convert Fractional Exponents to Radicals Convert the expression to radical notation.

49. $a^{\frac{3}{4}} =$ ☐

50. $b^{\frac{5}{6}} =$ ☐

51. $x^{\frac{5}{6}} =$ ☐

52. $r^{\frac{2}{3}} =$ ☐

53. $16^{\frac{1}{6}}z^{\frac{5}{6}} =$ ☐

54. $5^{\frac{1}{5}}t^{\frac{4}{5}} =$ ☐

55. Convert $r^{\frac{3}{7}}$ to a radical.

⊙ $\sqrt[3]{r^7}$
⊙ $\sqrt[7]{r^3}$

56. Convert $m^{\frac{5}{6}}$ to a radical.

⊙ $\sqrt[6]{m^5}$
⊙ $\sqrt[5]{m^6}$

57. Convert $n^{-\frac{5}{8}}$ to a radical.

⊙ $\frac{1}{\sqrt[5]{n^8}}$
⊙ $-\sqrt[8]{n^5}$
⊙ $-\sqrt[5]{n^8}$
⊙ $\frac{1}{\sqrt[8]{n^5}}$

58. Convert $a^{-\frac{4}{7}}$ to a radical.

⊙ $-\sqrt[4]{a^7}$
⊙ $-\sqrt[7]{a^4}$
⊙ $\frac{1}{\sqrt[4]{a^7}}$
⊙ $\frac{1}{\sqrt[7]{a^4}}$

59. Convert $5^{\frac{3}{7}}b^{\frac{5}{7}}$ to a radical.

⊙ $5^3 \cdot \sqrt[7]{b^5}$
⊙ $\sqrt[3]{5^7} \cdot \sqrt[5]{b^7}$
⊙ $\sqrt[7]{5^3b^5}$
⊙ $5^7 \cdot \sqrt[5]{b^7}$

60. Convert $2^{\frac{4}{5}}x^{\frac{2}{5}}$ to a radical.

⊙ $\sqrt[5]{2^4x^2}$
⊙ $\sqrt[4]{2^5} \cdot \sqrt{x^5}$
⊙ $2^4 \cdot \sqrt[5]{x^2}$
⊙ $2^5 \cdot \sqrt{x^5}$

Simplifying Expressions with Rational Exponents Simplify the expression, answering with rational exponents and not radicals.

61. $\sqrt[3]{y}\,\sqrt[3]{y} =$ ☐

62. $\sqrt[9]{z}\,\sqrt[9]{z} =$ ☐

63. $\sqrt[5]{32t^3} =$ ☐

64. $\sqrt[3]{125r} =$ ☐

65. $\frac{\sqrt[3]{8m}}{\sqrt[6]{m^5}} =$ ☐

66. $\frac{\sqrt{16n}}{\sqrt[6]{n^5}} =$ ☐

67. $\frac{\sqrt[5]{32a^4}}{\sqrt[10]{a^3}} =$ ☐

68. $\frac{\sqrt[3]{8b^2}}{\sqrt[6]{b}} =$ ☐

69. $\sqrt{c} \cdot \sqrt[6]{c^5} =$ ☐

70. $\sqrt[5]{y}\cdot\sqrt[10]{y^3} = \boxed{}$

71. $\sqrt[3]{\sqrt[4]{z}} = \boxed{}$

72. $\sqrt[4]{\sqrt[3]{t}} = \boxed{}$

73. $\sqrt{c}\,\sqrt[6]{c} = \boxed{}$

74. $\sqrt{n}\,\sqrt[7]{n} = \boxed{}$

14.3 More on Rationalizing the Denominator

In Section 8.2, we learned how to rationalize the denominator in simple expressions like $\frac{1}{\sqrt{2}}$. We will briefly review this topic and then extend the concept to the next level.

14.3.1 A Review of Rationalizing the Denominator

To remove radicals from the denominator of $\frac{1}{\sqrt{2}}$, we multiply the numerator and denominator by $\sqrt{2}$:

$$\begin{aligned}\frac{1}{\sqrt{2}} &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

We used the property:

$$\sqrt{x} \cdot \sqrt{x} = x, \text{ where } x \text{ is positive}$$

Example 14.3.2 Rationalize the denominator of the expressions.

a. $\frac{3}{\sqrt{6}}$

b. $\frac{\sqrt{5}}{\sqrt{72}}$

Explanation.

a. To rationalize the denominator of $\frac{3}{\sqrt{6}}$, we take the expression and multiply by a special version of 1 to make the radical in the denominator cancel.

$$\begin{aligned}\frac{3}{\sqrt{6}} &= \frac{3}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}}\\ &= \frac{3\sqrt{6}}{6}\\ &= \frac{\sqrt{6}}{2}\end{aligned}$$

b. Rationalizing the denominator of $\frac{\sqrt{5}}{\sqrt{72}}$ is slightly trickier. We could go the brute force method and multiply both the numerator and denominator by $\sqrt{72}$, and it would be effective; however, we should note that the $\sqrt{72}$ in the denominator can be *reduced* first. This will simplify future algebra.

$$\begin{aligned}\frac{\sqrt{5}}{\sqrt{72}} &= \frac{\sqrt{5}}{\sqrt{36 \cdot 2}}\\ &= \frac{\sqrt{5}}{\sqrt{36} \cdot \sqrt{2}}\\ &= \frac{\sqrt{5}}{6 \cdot \sqrt{2}}\end{aligned}$$

Now all that remains is to multiply the numerator and denominator by $\sqrt{2}$.

$$\begin{aligned}
&= \frac{\sqrt{5}}{6\cdot\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\\
&= \frac{\sqrt{10}}{6\cdot 2}\\
&= \frac{\sqrt{10}}{12}
\end{aligned}$$

14.3.2 Rationalize Denominator with Difference of Squares Formula

How can be remove the radical from the denominator of $\frac{1}{\sqrt{2}+1}$? Let's try multiplying the numerator and denominator by $\sqrt{2}$:

$$\begin{aligned}
\frac{1}{\sqrt{2}+1} &= \frac{1}{\left(\sqrt{2}+1\right)}\cdot\frac{\sqrt{2}}{\sqrt{2}}\\
&= \frac{\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}+1\cdot\sqrt{2}}\\
&= \frac{\sqrt{2}}{2+\sqrt{2}}
\end{aligned}$$

We removed one radical from the denominator, but created another. We need to find another method. The difference of squares formula will help:

$$(a+b)(a-b) = a^2 - b^2$$

Those two squares in $a^2 - b^2$ can remove square roots. To remove the radical from the denominator of $\frac{1}{\sqrt{2}+1}$, we multiply the numerator and denominator by $\sqrt{2}-1$:

$$\begin{aligned}
\frac{1}{\sqrt{2}+1} &= \frac{1}{\left(\sqrt{2}+1\right)}\cdot\frac{\left(\sqrt{2}-1\right)}{\left(\sqrt{2}-1\right)}\\
&= \frac{\sqrt{2}-1}{\left(\sqrt{2}\right)^2-(1)^2}\\
&= \frac{\sqrt{2}-1}{2-1}\\
&= \frac{\sqrt{2}-1}{1}\\
&= \sqrt{2}-1
\end{aligned}$$

Let's look at a few more examples.

Example 14.3.3 Rationalize the denominator in $\frac{\sqrt{7}-\sqrt{2}}{\sqrt{5}+\sqrt{3}}$.

Explanation. To remove radicals in $\sqrt{5}+\sqrt{3}$ with the difference of squares formula, we multiply it with

$\sqrt{5}-\sqrt{3}$.

$$\begin{aligned}\frac{\sqrt{7}-\sqrt{2}}{\sqrt{5}+\sqrt{3}} &= \frac{\sqrt{7}-\sqrt{2}}{\sqrt{5}+\sqrt{3}} \cdot \frac{\left(\sqrt{5}-\sqrt{3}\right)}{\left(\sqrt{5}-\sqrt{3}\right)} \\ &= \frac{\sqrt{7}\cdot\sqrt{5}-\sqrt{7}\cdot\sqrt{3}-\sqrt{2}\cdot\sqrt{5}-\sqrt{2}\cdot -\sqrt{3}}{\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2} \\ &= \frac{\sqrt{35}-\sqrt{21}-\sqrt{10}+\sqrt{6}}{5-3} \\ &= \frac{\sqrt{35}-\sqrt{21}-\sqrt{10}+\sqrt{6}}{2}\end{aligned}$$

Example 14.3.4 Rationalize the denominator in $\frac{\sqrt{3}}{3-2\sqrt{3}}$.

Explanation. To remove the radical in $3-2\sqrt{3}$ with the difference of squares formula, we multiply it with $3+2\sqrt{3}$.

$$\begin{aligned}\frac{\sqrt{3}}{3-2\sqrt{3}} &= \frac{\sqrt{3}}{(3-2\sqrt{3})} \cdot \frac{(3+2\sqrt{3})}{(3+2\sqrt{3})} \\ &= \frac{3\cdot\sqrt{3}+2\sqrt{3}\cdot\sqrt{3}}{(3)^2-\left(2\sqrt{3}\right)^2} \\ &= \frac{3\sqrt{3}+2\cdot 3}{9-2^2\left(\sqrt{3}\right)^2} \\ &= \frac{3\sqrt{3}+6}{9-4(3)} \\ &= \frac{3\left(\sqrt{3}+2\right)}{9-12} \\ &= \frac{3\left(\sqrt{3}+2\right)}{-3} \\ &= \frac{\sqrt{3}+2}{-1} \\ &= -\sqrt{3}-2\end{aligned}$$

Exercises

Review and Warmup Rationalize the denominator and simplify the expression.

1. $\frac{1}{\sqrt{6}} =$ ☐

2. $\frac{1}{\sqrt{7}} =$ ☐

3. $\frac{30}{\sqrt{10}} =$ ☐

4. $\frac{20}{\sqrt{10}} =$ ☐

5. $\frac{1}{\sqrt{28}} = \boxed{\quad\quad}$

6. $\frac{1}{\sqrt{45}} = \boxed{\quad\quad}$

7. $\frac{8}{\sqrt{180}} = \boxed{\quad\quad}$

8. $\frac{9}{\sqrt{72}} = \boxed{\quad\quad}$

Further Rationalizing a Denominator

9. Rationalize the denominator and simplify the expression.

$\frac{3}{\sqrt{m}} = \boxed{\quad\quad}$

10. Rationalize the denominator and simplify the expression.

$\frac{1}{\sqrt{n}} = \boxed{\quad\quad}$

11. Rationalize the denominator and simplify the expression.

$\sqrt{\frac{13}{14}} = \boxed{\quad\quad}$

12. Rationalize the denominator and simplify the expression.

$\sqrt{\frac{14}{15}} = \boxed{\quad\quad}$

13. Simplify this radical.

$\sqrt{\frac{11}{72}} = \boxed{\quad\quad}$

14. Simplify this radical.

$\sqrt{\frac{5}{24}} = \boxed{\quad\quad}$

Rationalizing the Denominator Using the Difference of Squares Formula Rationalize the denominator and simplify the expression.

15. $\frac{3}{\sqrt{11}+8} = \boxed{\quad\quad}$

16. $\frac{4}{\sqrt{2}+3} = \boxed{\quad\quad}$

17. $\frac{5}{\sqrt{17}+6} = \boxed{\quad\quad}$

18. $\frac{3}{\sqrt{6}+5} = \boxed{\quad\quad}$

19. $\frac{\sqrt{2}-12}{\sqrt{11}+10} = \boxed{\quad\quad}$

20. $\frac{\sqrt{5}-13}{\sqrt{13}+8} = \boxed{\quad\quad}$

21. $\frac{\sqrt{2}-14}{\sqrt{7}+5} = \boxed{\quad\quad}$

22. $\frac{\sqrt{5}-15}{\sqrt{13}+3} = \boxed{\quad\quad}$

14.4 Solving Radical Equations

In this section, we will learn how to solve equations involving radicals.

14.4.1 Solving Radical Equations

One common application of radicals is the Pythagorean Theorem. We already saw some examples in earlier sections. We will look at some other applications of radicals in this section.

The formula $T = 2\pi\sqrt{\frac{L}{g}}$ is used to calculate the period of a pendulum and is attributed to the scientist Christiaan Huygens[1]. In the formula, T stands for the pendulum's period (how long one back-and-forth oscillation takes) in seconds, L stands for the pendulum's length in meters, and g is approximately $9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$ which is the gravitational acceleration constant on Earth.

An engineer is designing a pendulum. Its period must be 10 seconds. How long should the pendulum's length be?

We will substitute 10 into the formula for T and also the value of g, and then solve for L:

$$\begin{aligned}
T &= 2\pi\sqrt{\frac{L}{g}} \\
10 &= 2\pi\sqrt{\frac{L}{9.8}} \\
\frac{1}{2\pi}\cdot 10 &= \frac{1}{2\pi}\cdot 2\pi\sqrt{\frac{L}{9.8}} \\
\frac{5}{\pi} &= \sqrt{\frac{L}{9.8}} \\
\left(\frac{5}{\pi}\right)^2 &= \left(\sqrt{\frac{L}{9.8}}\right)^2 && \text{canceling square root by squaring both sides} \\
\frac{25}{\pi^2} &= \frac{L}{9.8} \\
9.8\cdot\frac{25}{\pi^2} &= 9.8\cdot\frac{L}{9.8} \\
24.82 &\approx L
\end{aligned}$$

To build a pendulum with a period of 10 seconds, the pendulum's length should be approximately 24.82 meters.

Remark 14.4.2. The basic strategy to solve radical equations is to isolate the radical on one side of the equation and then square both sides to cancel the radical.

Remark 14.4.3. Squaring both sides of an equation is "dangerous," as it could create **extraneous solutions**, which will not make the equation true. For example, if we square both sides of $1 = -1$, we have:

$$\begin{aligned}
1 &= -1 && \text{false} \\
(1)^2 &= (-1)^2 && \text{square both sides}\ldots \\
1 &= 1 && \text{true}
\end{aligned}$$

---

[1] en.wikipedia.org/wiki/Christiaan_Huygens#Pendulums

By squaring both sides of an equation, we turned a false equation into a true one. This is why we *must check solutions* when we square both sides of an equation.

Example 14.4.4 Solve the equation $1+\sqrt{y-1}=4$ for y.

Explanation. We will isolate the radical first, and then square both sides.

$$\begin{aligned} 1+\sqrt{y-1}&=4\\ \sqrt{y-1}&=3\\ \left(\sqrt{y-1}\right)^2&=(3)^2\\ y-1&=9\\ y&=10 \end{aligned}$$

Because we squared both sides of an equation, we must check the solution. Substitute 10 into $1+\sqrt{y-1}=4$, and we have:

$$\begin{aligned} 1+\sqrt{10-1}&\overset{?}{=}4\\ 1+\sqrt{9}&\overset{?}{=}4\\ 1+3&\overset{\checkmark}{=}4 \end{aligned}$$

So, 10 is the solution to the equation $1+\sqrt{y-1}=4$.

Example 14.4.5 Solve the equation $5+\sqrt{q}=3$ for q.

Explanation. First, isolate the radical and square both sides.

$$\begin{aligned} 5+\sqrt{q}&=3\\ \sqrt{q}&=-2\\ \left(\sqrt{q}\right)^2&=(-2)^2\\ q&=4 \end{aligned}$$

Because we squared both sides of an equation, we must check the solution. Substitute 4 into $5+\sqrt{q}=3$, and we have:

$$\begin{aligned} 5+\sqrt{4}&\overset{?}{=}3\\ 5+2&\overset{?}{=}3\\ 7&\overset{\text{no}}{=}3 \end{aligned}$$

Thus, the potential solution -2 is actually extraneous and we have no real solutions to the equation $5+\sqrt{q}=3$. The solution set is the empty set, $\emptyset$.

Remark 14.4.6. In the previous example, it would be legitimate to observe earlier stages that there are no solutions. From the very beginning, how could 5 plus a positive quantity result in 3? Or at the second step, since square roots are non-negative, how could a square root equal -2?

You do not have to be able to make these observations. If you follow the general steps for solving radical equations *and* you remember to check the possible solutions you find, then that will be enough.

Example 14.4.7 Solve for z in $\sqrt{z}+2=z$.

Explanation. We will isolate the radical first, and then square both sides.

$$\begin{aligned}\sqrt{z}+2&=z\\ \sqrt{z}&=z-2\\ \left(\sqrt{z}\right)^2&=(z-2)^2\\ z&=z^2-4z+4\\ 0&=z^2-5z+4\\ 0&=(z-1)(z-4)\end{aligned}$$

$$\begin{aligned}z-1&=0 &&\text{or} & z-4&=0\\ z&=1 &&\text{or} & z&=4\end{aligned}$$

Because we squared both sides of an equation, we must check both solutions.

$$\begin{aligned}\sqrt{1}+2&\stackrel{?}{=}1 & \sqrt{4}+2&\stackrel{?}{=}4\\ 1+2&\stackrel{\text{no}}{=}1 & 2+2&\stackrel{?}{\checkmark}4\end{aligned}$$

It turned out that 1 is an extraneous solution, but 4 is a valid solution. So the equation has one solution: 4. The solution set is $\{4\}$.

Sometimes, we need to square both sides of an equation *twice* before finding the solutions, like in the next example.

Example 14.4.8 Solve the equation $\sqrt{p-5}=5-\sqrt{p}$ for p.

Explanation. We cannot isolate two radicals, so we will simply square both sides, and later try to isolate the remaining radical.

$$\begin{aligned}\sqrt{p-5}&=5-\sqrt{p}\\ \left(\sqrt{p-5}\right)^2&=\left(5-\sqrt{p}\right)^2\\ p-5&=25-10\sqrt{p}+p && \text{after expanding the binomial squared}\\ -5&=25-10\sqrt{p}\\ -30&=-10\sqrt{p}\\ \frac{-30}{-10}&=\frac{-10\sqrt{p}}{-10}\\ 3&=\sqrt{p}\\ (3)^2&=\left(\sqrt{p}\right)^2\\ 9&=p\end{aligned}$$

Because we squared both sides of an equation, we must check the solution by substituting 9 into $\sqrt{p-5}=$

$5 - \sqrt{p}$, and we have:

$$\begin{aligned} \sqrt{9-5} &\stackrel{?}{=} 5-\sqrt{9} \\ \sqrt{4} &\stackrel{?}{=} 5-3 \\ 2 &\stackrel{\checkmark}{=} 2 \end{aligned}$$

So 9 is the solution. The solution set is $\{9\}$.

Example 14.4.9 Solve the equation $\sqrt{2n-6} = 1 + \sqrt{n-2}$ for n.

Explanation. We cannot isolate two radicals, so we will simply square both sides, and later try to isolate the remaining radical.

$$\begin{aligned} \sqrt{2n-6} &= 1+\sqrt{n-2} \\ \left(\sqrt{2n-6}\right)^2 &= \left(1+\sqrt{n-2}\right)^2 \\ 2n-6 &= \left(1+\sqrt{n-2}\right)\left(1+\sqrt{n-2}\right) \\ 2n-6 &= 1+\sqrt{n-2}+\sqrt{n-2}+n-2 \\ 2n-6 &= 2\sqrt{n-2}+n-1 \text{ by combining like terms} \\ n-5 &= 2\sqrt{n-2} \end{aligned}$$

Note here that we can leave the coefficient 2 next to the radical in the squaring process. We will square the 2 also.

$$\begin{aligned} (n-5)^2 &= \left(2\sqrt{n-2}\right)^2 \\ n^2-10n+25 &= 4(n-2) \\ n^2-10n+25 &= 4n-8 \\ n^2-14n+33 &= 0 \\ (n-11)(n-3) &= 0 \end{aligned}$$

$$\begin{aligned} n-11 &= 0 & \text{or } n-3 &= 0 \\ n &= 11 & \text{or } n &= 3 \end{aligned}$$

So our two potential solutions are 11 and 3. We should now verify that they truly are solutions.

$$\begin{aligned} \sqrt{2(11)-6} &\stackrel{?}{=} 1+\sqrt{11-2} & \sqrt{2(3)-6} &\stackrel{?}{=} 1+\sqrt{3-2} \\ \sqrt{22-6} &\stackrel{?}{=} 1+\sqrt{9} & \sqrt{6-6} &\stackrel{?}{=} 1+\sqrt{1} \\ \sqrt{16} &\stackrel{?}{=} 1+3 & \sqrt{0} &\stackrel{?}{=} 1+1 \\ 4 &\stackrel{\checkmark}{=} 4 & 0 &\stackrel{\text{no}}{=} 2 \end{aligned}$$

So, 11 is the only solution. The solution set is $\{11\}$.

Let's look at an example of solving an equation with a cube root. There is very little difference between solving a cube-root equation and solving a square-root equation. Instead of *squaring* both sides, you *cube* both sides.

Example 14.4.10 Solve for q in $\sqrt[3]{2-q}+2=5$.

Explanation.

$$\begin{aligned}\sqrt[3]{2-q}+2&=5\\ \sqrt[3]{2-q}&=3\\ \left(\sqrt[3]{2-q}\right)^3&=(3)^3\\ 2-q&=27\\ -q&=25\\ q&=-25\end{aligned}$$

Unlike squaring both sides of an equation, raising both sides of an equation to the 3rd power will not create extraneous solutions. It's still good practice to check solution, though. This part is left as exercise.

14.4.2 Solving a Radical Equation with a Variable

We also need the ability to solve radical equations with variables, like in the next example that comes from physics. The strategy is the same: isolating the radical, and then raise both sides to a certain power to cancel the radical.

Example 14.4.11 The study of black holes has resulted in some interesting science. One fundamental concept about black holes is that there is a distance close enough to the black hole that not even light can escape, called the Schwarzschild radius[a] or the event horizon radius. To find the Schwarzschild radius, R_s, we set the formula for the escape velocity equal to the speed of light, c, and we get $c=\sqrt{\frac{2GM}{R_s}}$ which we need to solve for R_s. Note that G is a constant, and M is the mass of the black hole.

Explanation. We will start by taking the equation $c=\sqrt{\frac{2GM}{R_s}}$ and applying our standard radical-equation-solving techniques. Isolate the radical and square both sides:

$$\begin{aligned}c&=\sqrt{\frac{2GM}{R_s}}\\ c^2&=\left(\sqrt{\frac{2GM}{R_s}}\right)^2\\ c^2&=\frac{2GM}{R_s}\\ R_s\cdot c^2&=\frac{2GM}{R_s}\cdot R_s\\ R_sc^2&=2GM\\ \frac{R_sc^2}{c^2}&=\frac{2GM}{c^2}\\ R_s&=\frac{2GM}{c^2}\end{aligned}$$

So, the Schwarzschild radius can be found using the formula $R_s = \frac{2GM}{c^2}$.

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[a] en.wikipedia.org/wiki/Schwarzschild_radius

Example 14.4.12 The term redshift[a] refers to the Doppler effect[b] for light. When an object (like a star) is moving away from Earth at very fast speeds, the wavelength of the light emitted by the star is increased due to the distance between the planets increasing (and the constant speed of light). Increased wavelength makes light "redder." The opposite phenomenon is called blueshift[c]. It turns out that the formula to calculate the redshift for a star moving away from the Earth uses square roots:

$$f_r = f_s \cdot \sqrt{\frac{c-v}{c+v}}$$

where c stands for the constant speed of light in a vacuum, f_r represents the frequency of the light that the receiver on Earth sees, f_s represents the frequency of light that the source star emits, and v is the velocity that the star moving away from Earth. Solve this equation for v.

Explanation. We will take the original equation $f_r = f_s \cdot \sqrt{\frac{c-v}{c+v}}$ and follow the steps to solving a radical equation. We could isolate the radical and then square both sides, but in this case isolating the radical is not necessary. If we begin by squaring both sides, that too will eliminate the radical.

$$\begin{aligned}
f_r &= f_s \cdot \sqrt{\frac{c-v}{c+v}} \\
\left(f_r\right)^2 &= \left(f_s \cdot \sqrt{\frac{c-v}{c+v}}\right)^2 \\
f_r^2 &= f_s^2 \cdot \frac{c-v}{c+v} \\
f_r^2 \cdot (c+v) &= f_s^2 \cdot \frac{c-v}{c+v} \cdot (c+v) \\
f_r^2(c+v) &= f_s^2(c-v) \\
f_r^2 c + f_r^2 v &= f_s^2 c - f_s^2 v \\
f_r^2 v + f_s^2 v &= f_s^2 c - f_r^2 c \\
\left(f_s^2 + f_r^2\right) v &= \left(f_s^2 - f_r^2\right) c \\
v &= \frac{f_s^2 - f_r^2}{f_s^2 + f_r^2} c
\end{aligned}$$

This formula will tell us the velocity of the star away from Earth if we can know the respective frequencies of the starlight. This formula is used to demonstrate that the universe is expanding[d].

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[a] en.wikipedia.org/wiki/Redshift
[b] en.wikipedia.org/wiki/Doppler_effect
[c] https://en.wikipedia.org/wiki/Blueshift
[d] en.wikipedia.org/wiki/Metric_expansion_of_space

14.4.3 Graphing Technology

We can use technology to solve equations by finding where two graphs intersect.

Example 14.4.13 Solve the equation $1 - x = \sqrt{x+5}$ with technology.

Explanation.

We define $f(x) = 1 - x$ and $g(x) = \sqrt{x+5}$, and then look for the intersection(s) of the graphs. Since the two functions intersect at $(-1, 2)$, the solution to $1 - x = \sqrt{x+5}$ is -1. The solution set is $\{-1\}$.

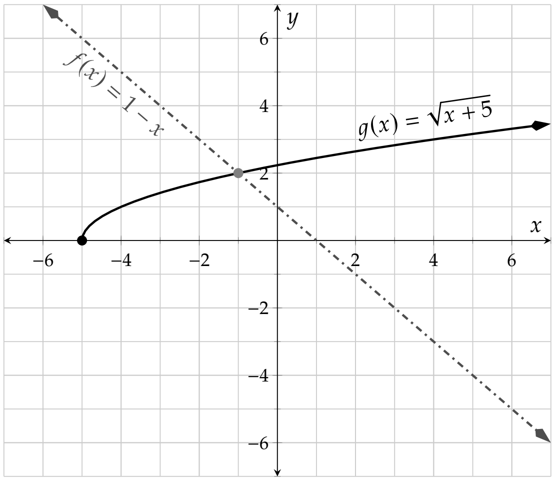

Figure 14.4.14: Graph of $f(x) = 1 - x$ and $g(x) = \sqrt{x+5}$

Before we finish with this example, we would like to illustrate why there are sometimes extraneous solutions to radical equations. It has to do with the squaring-both-sides step of the solving process.

A graph of a radical, for example $y = \sqrt{x+5}$, actually graphs as *half* of a sideways parabola, as you can see in Figure 14.4.14. When we square both sides of that equation, we get $y^2 = x + 5$ which actually graphs as a *complete* sideways parabola.

The curve and the line now intersect *twice*! This second solution (which is 4, by the way) is the extraneous solution that we would have found had we solved $1 - x = \sqrt{x+5}$ algebraically.

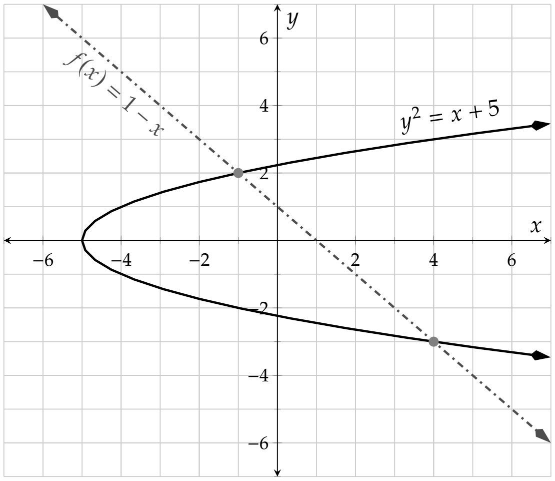

Figure 14.4.15: Graph of $f(x) = 1 - x$ and $y^2 = x + 5$

Example 14.4.16 Solve the equation $3 + \sqrt{x+4} = x - \sqrt{x-4}$ graphically using technology.

Explanation.

To solve the equation graphically, first we will assign the left side of the equation the label $m(x) = 3 + \sqrt{x+4}$ and the right side $n(x) = x - \sqrt{x-4}$. Next we will make graphs of both m and n on the same grid and look for their intersection point(s). Since the two functions intersect at about $(8.75, 6.571)$, the solution to $3 + \sqrt{x+4} = x - \sqrt{x-4}$ is 8.75. The solution set is $\{8.75\}$.

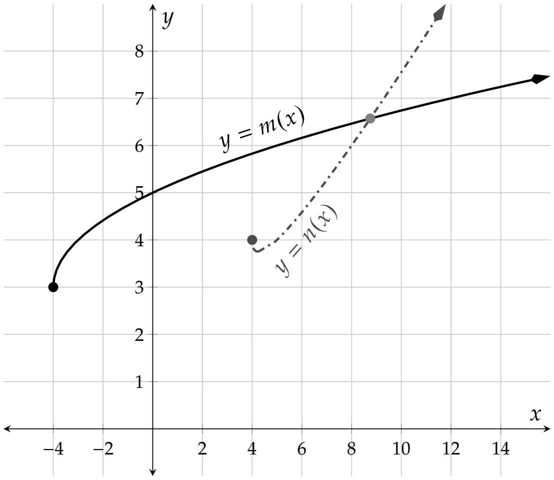

Figure 14.4.17: Graph of $m(x) = 3 + \sqrt{x+4}$ and $n(x) = x - \sqrt{x-4}$

Exercises

Review and Warmup Solve the equation.

1. $-7y + 3 = -y - 51$

2. $-4r + 6 = -r - 12$

3. $-155 = -5(10 - 3a)$

4. $-72 = -4(4 - 2c)$

5. $-24 = 3 - 9(A - 2)$

6. $18 = 9 - 3(C - 2)$

7. $(x + 5)^2 = 49$

8. $(x + 8)^2 = 16$

9. $x^2 + 21x + 108 = 0$

10. $x^2 - x - 90 = 0$

11. $x^2 - 16x + 54 = -9$

12. $x^2 + 3x - 32 = 8$

Solving Radical Equations Solve the equation.

13. $\sqrt{y} = 7$

14. $\sqrt{r} = 3$

15. $\sqrt{5r} = 25$

16. $\sqrt{3t} = 9$

17. $2\sqrt{t} = 10$

18. $4\sqrt{t} = 8$

19. $-3\sqrt{x} = 12$

20. $-2\sqrt{x} = 4$

21. $-2\sqrt{-1 - y} + 9 = -5$

22. $-5\sqrt{8 - y} + 9 = -11$

23. $\sqrt{2r + 80} = r$

24. $\sqrt{4r + 21} = r$

25. $\sqrt{t} + 6 = t$

26. $\sqrt{t} + 72 = t$

27. $t = \sqrt{t+1} + 5$

28. $x = \sqrt{x-3} + 9$

29. $\sqrt{x+8} = \sqrt{x} + 2$

30. $\sqrt{y+3} = \sqrt{y} + 1$

31. $\sqrt{y+3} = -1 - \sqrt{y}$

32. $\sqrt{r-7} = 1 - \sqrt{r}$

33. $\sqrt{6r} = 6$

34. $\sqrt{3t} = 9$

35. $\sqrt[3]{t-9} = 4$

36. $\sqrt[3]{t-6} = 8$

37. $\sqrt{3x+3} + 9 = 17$

38. $\sqrt{8x+6} + 7 = 15$

39. $\sqrt{y} + 42 = y$

40. $\sqrt{y} + 12 = y$

41. $\sqrt[3]{r-9} = 6$

42. $\sqrt[3]{r-5} = -6$

43. $r = \sqrt{r+3} + 9$

44. $t = \sqrt{t+1} + 89$

45. $\sqrt{38-t} = t + 4$

46. $\sqrt{148-x} = x + 8$

Solving Radical Equations Using Technology Use technology to solve the equation

47.

$$\sqrt{x-2.1} = \sqrt{x} - 4.$$

48.

$$\sqrt{x-2} = \sqrt{x} - 2.1.$$

Solving Radical Equations with Variables

49. Solve the equation for R. Assume that R is positive.

$$Z = \sqrt{L^2 + R^2}$$

$R =$ ☐.

50. According to the Pythagorean Theorem, the length c of the hypothenuse of a rectangular triangle can be found through the following equation:

$$c = \sqrt{a^2 + b^2}$$

Solve the equation for the length a of one of the triangle's legs.

$a =$ ☐.

51. In an electric circuit, resonance occurs when the frequency f, inductance L, and capacitance C fulfill the following equation:

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Solve the equation for the inductance L.

The frequency is measured in Hertz, the inductance in Henry, and the capacitance in Farad.

$L =$ ☐.

52. A pendulum has the length L. The time period T that it takes to once swing back and forth can be found with the following formula:

$$T = 2\pi\sqrt{\frac{L}{32}}$$

Solve the equation for the length L.

The length is measured in feet and the time period in seconds.

$L =$ ☐.

Radical Equation Applications According to the Pythagorean Theorem, the length c of the hypothenuse of a rectangular triangle can be found through the following equation.

$$c = \sqrt{a^2 + b^2}$$

53. If a rectangular triangle has a hypothenuse of 41 ft and one leg is 40 ft long, how long is the third side of the triangle?

The third side of the triangle is ☐ long.

54. If a rectangular triangle has a hypothenuse of 41 ft and one leg is 40 ft long, how long is the third side of the triangle?

The third side of the triangle is ☐ long.

In a coordinate system, the distance r of a point (x, y) from the origin $(0, 0)$ is given by the following equation.

$$r = \sqrt{x^2 + y^2}$$

55. If a point in a coordinate system is 5 cm away from the origin and its x coordinate is 4 cm, what is its y coordinate? Assume that y is positive.

$y =$ ☐.

56. If a point in a coordinate system is 5 cm away from the origin and its x coordinate is 4 cm, what is its y coordinate? Assume that y is positive.

$y =$ ☐.

57. A pendulum has the length L ft. The time period T that it takes to once swing back and forth is 4 s. Use the following formula to find its length.

$$T = 2\pi\sqrt{\frac{L}{32}}$$

The pendulum is ☐ long.

58. A pendulum has the length L ft. The time period T that it takes to once swing back and forth is 6 s. Use the following formula to find its length.

$$T = 2\pi\sqrt{\frac{L}{32}}$$

The pendulum is ☐ long.

Challenge Solve for x.

59.

$$\sqrt{1+\sqrt{6}} = \sqrt{2+\sqrt{\frac{1}{\sqrt{x}}}-1}$$

60.

$$\sqrt{1+\sqrt{7}} = \sqrt{2+\sqrt{\frac{1}{\sqrt{x}}}-1}$$

14.5 Radical Functions and Equations Chapter Review

14.5.1 Introduction to Radical Functions

In Section 14.1 we covered the square root and other root functions. We learned how to find the domain of radical functions algebraically and the range graphically. We also saw the distance formula which is an application of square roots.

Example 14.5.1 The Square Root Function. Algebraically find the domain of the function f where $f(x) = \sqrt{3x-1}+2$ and then find the range by making a graph.

Explanation. Using Fact 14.1.29 to find the function's domain, we set the radicand greater than or equal to zero and solve:

$$\begin{aligned} 3x-1 &\geq 0 \\ 3x &\geq 1 \\ x &\geq \frac{1}{3} \end{aligned}$$

The function's domain is $\left[\frac{1}{3}, \infty\right)$ in interval notation.

To find the function's range, we use technology to look at a graph of the function. The graph shows that the function's range is $[2, \infty)$. The graph also verifies the function's domain is indeed $\left[\frac{1}{3}, \infty\right)$.

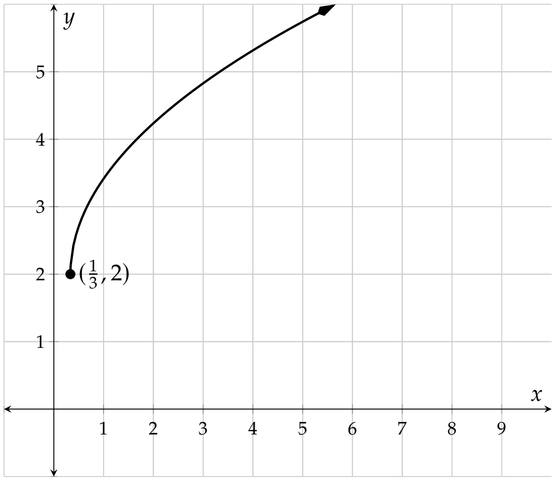

Figure 14.5.2: Graph of $f(x) = \sqrt{3x-1}+2$

Example 14.5.3 The Distance Formula. Find the distance between $(6, -13)$ and $(-4, 17)$.

Explanation. To calculate the distance between $(6, -13)$ and $(-4, 17)$, we use the distance formula. It's good practice to mark each value with the corresponding variables in the formula:

$$(\overset{x_1}{6}, \overset{y_1}{-13}), (\overset{x_2}{-4}, \overset{y_2}{17})$$

We have:

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$\begin{aligned}
d &= \sqrt{(-4-6)^2+(17-(-13))^2}\\
d &= \sqrt{(-10)^2+(30)^2}\\
d &= \sqrt{100+900}\\
d &= \sqrt{1000}\\
d &= \sqrt{100}\cdot\sqrt{10}\\
d &= 10\sqrt{10}
\end{aligned}$$

The distance between $(6,-13)$ and $(-4,17)$ is $10\sqrt{10}$ or approximately 31.62 units.

Example 14.5.4 The Cube Root Function. Algebraically find the domain and graphically find the range of the function g where $g(x)=-2\sqrt[3]{x+6}-1$.

Explanation. First note that the index of the function g for $g(x)=-2\sqrt[3]{x+6}-1$ is odd. By Fact 14.1.29, the domain is $(-\infty,\infty)$.

To find the function's range, we use technology to graph the function. According to the graph, the function's range is also $(-\infty,\infty)$.

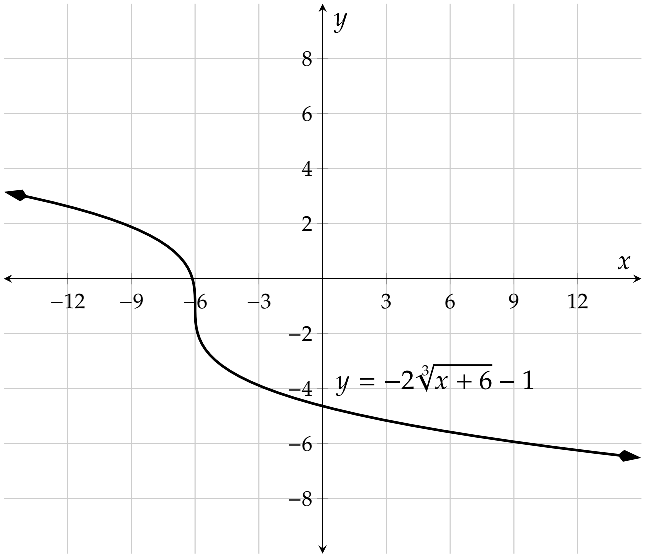

Figure 14.5.5: Graph of $y=g(x)=-2\sqrt[3]{x+6}-1$

Example 14.5.6 Other Roots. Algebraically find the domain and graphically find the range of the function h where $h(x)=8-\frac{5}{2}\sqrt[4]{6-2x}$.

Explanation. First note that the index of this function is 4, which is even. By Fact 14.1.29, to find the domain of this function we must set the radicand greater or equal to zero and solve.

$$\begin{aligned}
6-2x &\geq 0\\
-2x &\geq -6
\end{aligned}$$

$$x \le \frac{-6}{-2}$$
$$x \le 3$$

So, the domain of h is $(-\infty, 3]$.

To find the function's range, we use technology to graph the function. By the graph, the function's range is $(-\infty, 8]$.

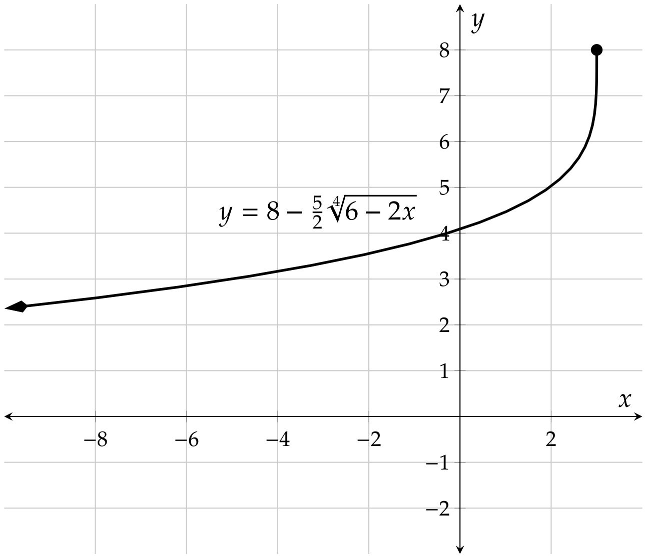

Figure 14.5.7: Graph of $y = h(x) = 8 - \frac{5}{2}\sqrt[4]{6-2x}$

14.5.2 Radical Expressions and Rational Exponents

In Section 14.2 we learned the rational exponent rule and added it to our list of exponent rules.

Example 14.5.8 Radical Expressions and Rational Exponents. Simplify the expressions using Fact 14.2.2 or Fact 14.2.10.

a. $100^{1/2}$

b. $(-64)^{-1/3}$

c. $-81^{3/4}$

d. $\left(-\frac{1}{27}\right)^{2/3}$

Explanation.

a. $$\begin{aligned} 100^{1/2} &= \left(\sqrt{100}\right) \\ &= 10 \end{aligned}$$

b. $$(-64)^{-1/3} = \frac{1}{(-64)^{1/3}} = \frac{1}{\left(\sqrt[3]{(-64)}\right)} = \frac{1}{-4}$$

c. $$-81^{3/4} = -\left(\sqrt[4]{81}\right)^3 = -3^3 = -27$$

d. In this problem the negative can be associated with either the numerator or the denominator, but not both. We choose the numerator.

$$\begin{aligned}\left(-\frac{1}{27}\right)^{2/3} &= \left(\sqrt[3]{-\frac{1}{27}}\right)^2\\ &= \left(\frac{\sqrt[3]{-1}}{\sqrt[3]{27}}\right)^2\\ &= \left(\frac{-1}{3}\right)^2\\ &= \frac{(-1)^2}{(3)^2}\\ &= \frac{1}{9}\end{aligned}$$

Example 14.5.9 More Expressions with Rational Exponents. Use exponent properties in List 14.2.15 to simplify the expressions, and write all final versions using radicals.

a. $7z^{5/9}$

b. $\frac{5}{4}x^{-2/3}$

c. $\left(-9q^5\right)^{4/5}$

d. $\sqrt{y^5} \cdot \sqrt[4]{y^2}$

e. $\frac{\sqrt{t^3}}{\sqrt[3]{t^2}}$

f. $\sqrt{\sqrt[3]{x}}$

g. $5\left(4 + a^{1/2}\right)^2$

h. $-6\left(2p^{-5/2}\right)^{3/5}$

Explanation.

a. $7z^{5/9} = 7\sqrt[9]{z^5}$

b.
$$\begin{aligned}\frac{5}{4}x^{-2/3} &= \frac{5}{4}\cdot\frac{1}{x^{2/3}}\\ &= \frac{5}{4}\cdot\frac{1}{\sqrt[3]{x^2}}\\ &= \frac{5}{4\sqrt[3]{x^2}}\end{aligned}$$

c.
$$\begin{aligned}\left(-9q^5\right)^{4/5} &= (-9)^{4/5}\cdot\left(q^5\right)^{4/5}\\ &= (-9)^{4/5}\cdot q^{5\cdot 4/5}\\ &= \left(\sqrt[5]{-9}\right)^4\cdot q^4\\ &= \left(q\sqrt[5]{-9}\right)^4\end{aligned}$$

d.
$$\begin{aligned}\sqrt{y^5}\cdot\sqrt[4]{y^2} &= y^{5/2}\cdot y^{2/4}\\ &= y^{5/2+2/4}\\ &= y^{10/4+1/4}\\ &= x^{11/4}\\ &= \sqrt[4]{x^{11}}\end{aligned}$$

e.
$$\begin{aligned}\frac{\sqrt{t^3}}{\sqrt[3]{t^2}} &= \frac{t^{3/2}}{t^{2/3}}\\ &= t^{3/2-2/3}\\ &= t^{9/6-4/6}\\ &= t^{5/6}\\ &= \sqrt[6]{t^5}\end{aligned}$$

f.
$$\begin{aligned}\sqrt{\sqrt[3]{x}} &= \sqrt{x^{1/3}}\\ &= \left(x^{1/3}\right)^{1/2}\\ &= x^{1/3\cdot 1/2}\\ &= x^{1/6}\\ &= \sqrt[6]{x}\end{aligned}$$

g.
$$\begin{aligned}5\left(4+a^{1/2}\right)^2 &= 5\left(4+a^{1/2}\right)\left(4+a^{1/2}\right)\\ &= 5\left(4^2+2\cdot 4\cdot a^{1/2}+\left(a^{1/2}\right)^2\right)\\ &= 5\left(16+8a^{1/2}+a^{1/2\cdot 2}\right)\\ &= 5\left(16+8a^{1/2}+a\right)\\ &= 5\left(16+8\sqrt{a}+a\right)\\ &= 80+40\sqrt{a}+5a\end{aligned}$$

h.
$$\begin{aligned}-6\left(2p^{-5/2}\right)^{3/5} &= -6\cdot 2^{3/5}\cdot p^{-5/2\cdot 3/5}\\ &= -6\cdot 2^{3/5}\cdot p^{-3/2}\\ &= -\frac{6\cdot 2^{3/5}}{p^{3/2}}\\ &= -\frac{6\sqrt[5]{2^3}}{\sqrt{p^3}}\\ &= -\frac{6\sqrt[5]{8}}{\sqrt{p^3}}\end{aligned}$$

14.5.3 More on Rationalizing the Denominator

In Section 14.3 we covered how to rationalize the denominator when it contains a single square root or a binomial.

Example 14.5.10 A Review of Rationalizing the Denominator. Rationalize the denominator of the expressions.

a. $\frac{12}{\sqrt{3}}$

b. $\frac{\sqrt{5}}{\sqrt{75}}$

Explanation.

a.

$$\frac{12}{\sqrt{3}} = \frac{12}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$

$$
\begin{aligned}
&= \frac{12\sqrt{3}}{3} \\
&= 4\sqrt{3}
\end{aligned}
$$

b. First we will simplify $\sqrt{75}$.

$$
\begin{aligned}
\frac{\sqrt{5}}{\sqrt{75}} &= \frac{\sqrt{5}}{\sqrt{25 \cdot 3}} \\
&= \frac{\sqrt{5}}{\sqrt{25} \cdot \sqrt{3}} \\
&= \frac{\sqrt{5}}{5\sqrt{3}}
\end{aligned}
$$

Now we can rationalize the denominator by multiplying the numerator and denominator by $\sqrt{3}$.

$$
\begin{aligned}
&= \frac{\sqrt{5}}{5\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\
&= \frac{\sqrt{15}}{5 \cdot 3} \\
&= \frac{\sqrt{15}}{15}
\end{aligned}
$$

Example 14.5.11 Rationalize Denominator with Difference of Squares Formula. Rationalize the denominator in $\frac{\sqrt{6}-\sqrt{5}}{\sqrt{3}+\sqrt{2}}$.

Explanation. To remove radicals in $\sqrt{3}+\sqrt{2}$ with the difference of squares formula, we multiply it with $\sqrt{3}-\sqrt{2}$.

$$
\begin{aligned}
\frac{\sqrt{6}-\sqrt{5}}{\sqrt{3}+\sqrt{2}} &= \frac{\sqrt{6}-\sqrt{5}}{\sqrt{3}+\sqrt{2}} \cdot \frac{\left(\sqrt{3}-\sqrt{2}\right)}{\left(\sqrt{3}-\sqrt{2}\right)} \\
&= \frac{\sqrt{6}\cdot\sqrt{3}-\sqrt{6}\cdot\sqrt{2}-\sqrt{5}\cdot\sqrt{3}-\sqrt{5}\cdot-\sqrt{2}}{\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2} \\
&= \frac{\sqrt{18}-\sqrt{12}-\sqrt{15}+\sqrt{10}}{9-4} \\
&= \frac{3\sqrt{2}-2\sqrt{3}-\sqrt{15}+\sqrt{10}}{5}
\end{aligned}
$$

14.5.4 Solving Radical Equations

In Section 14.4 we covered solving equations that contain a radical. We learned about extraneous solutions and the need to check our solutions.

Example 14.5.12 Solving Radical Equations. Solve for r in $r = 9 + \sqrt{r+3}$.

Explanation. We will isolate the radical first, and then square both sides.

$$\begin{aligned} r &= 9 + \sqrt{r+3} \\ r - 9 &= \sqrt{r+3} \\ (r-9)^2 &= \left(\sqrt{r+3}\right)^2 \\ r^2 - 18r + 81 &= r + 3 \\ r^2 - 19r + 78 &= 0 \\ (r-6)(r-13) &= 0 \end{aligned}$$

$$\begin{aligned} r - 6 &= 0 && \text{or } r - 13 && = 0 \\ r &= 6 && \text{or } r && = 13 \end{aligned}$$

Because we squared both sides of an equation, we must check both solutions.

$$\begin{aligned} 6 &\overset{?}{=} 9 + \sqrt{6+3} & 13 &\overset{?}{=} 9 + \sqrt{13+3} \\ 6 &\overset{?}{=} 9 + \sqrt{9} & 13 &\overset{?}{=} 9 + \sqrt{16} \\ 6 &\overset{\text{no}}{=} 9 + 3 & 13 &\overset{\checkmark}{=} 9 + 4 \end{aligned}$$

It turns out 6 is an extraneous solution and 13 is a valid solution. So the equation has one solution: 13. The solution set is $\{13\}$.

Example 14.5.13 Solving Radical Equations that Require Squaring Twice. Solve the equation $\sqrt{t+9} = -1 - \sqrt{t}$ for t.

Explanation. We cannot isolate two radicals, so we will simply square both sides, and later try to isolate the remaining radical.

$$\begin{aligned} \sqrt{t+9} &= -1 - \sqrt{t} \\ \left(\sqrt{t+9}\right)^2 &= \left(-1 - \sqrt{t}\right)^2 \\ t + 9 &= 1 + 2\sqrt{t} + t && \text{after expanding the binomial squared} \\ 9 &= 1 + 2\sqrt{t} \\ 8 &= 2\sqrt{t} \\ 4 &= \sqrt{t} \\ (4)^2 &= \left(\sqrt{t}\right)^2 \\ 16 &= t \end{aligned}$$

Because we squared both sides of an equation, we must check the solution by substituting 16 into

$\sqrt{t+9} = -1 - \sqrt{t}$, and we have:

$$\begin{aligned}\sqrt{t+9} &= -1 - \sqrt{t}\\ \sqrt{16+9} &\overset{?}{=} -1 - \sqrt{16}\\ \sqrt{25} &\overset{?}{=} -1 - 4\\ 5 &\overset{\text{no}}{=} -5\end{aligned}$$

Our solution did not check so there is no solution to this equation. The solution set is the empty set, which can be denotes { } or $\emptyset$.

Exercises

Introduction to Radical Functions Find the domain of the function.

1. $H(x) = \sqrt{8-x}$

2. $H(x) = \sqrt{5-x}$

3. $K(x) = \sqrt[3]{-9x+10}$

4. $f(x) = \sqrt[3]{5x-2}$

5. $g(x) = \sqrt[4]{18-3x}$

6. $h(x) = \sqrt[4]{-12-2x}$

7. Use technology to find the range of the function h defined by $h(x) = \sqrt{2-x} + 1$.

8. Use technology to find the range of the function F defined by $F(x) = \sqrt{-2-x} - 5$.

If an object is dropped with no initial velocity, the time since the drop, in seconds, can be calculated by the function

$$T(h) = \sqrt{\frac{2h}{g}}$$

where h is the distance the object traveled in feet. The variable g is the gravitational acceleration on earth, and we can round it to $32\frac{ft}{s^2}$ for this problem.

9. a. After ☐ seconds since the release, the object would have traveled 35 feet.

b. After 5 seconds since the release, the object would have traveled ☐ feet.

10. a. After ☐ seconds since the release, the object would have traveled 40 feet.

b. After 3.6 seconds since the release, the object would have traveled ☐ feet.

11. Find the distance between the points $(-8,-2)$ and $(57,70)$.

12. Find the distance between the points $(-10,-15)$ and $(45,33)$.

Radical Expressions and Rational Exponents Without using a calculator, evaluate the expression.

13. $\sqrt[4]{3^3} = \boxed{}$

14. $\sqrt[3]{13^2} = \boxed{}$

15. $\sqrt[3]{-\frac{8}{125}} = \boxed{}$.

16. $\sqrt[3]{-\frac{1}{8}} = \boxed{}$.

17. Use rational exponents to write the expression.

$\sqrt[5]{3m+9} = \boxed{}$

18. Use rational exponents to write the expression.

$\sqrt[4]{9n+3} = \boxed{}$

19. Convert the expression to radical notation.

$14^{\frac{1}{5}}a^{\frac{4}{5}} = \boxed{}$

20. Convert the expression to radical notation.

$3^{\frac{1}{3}}b^{\frac{2}{3}} = \boxed{}$

21. Convert $c^{-\frac{5}{7}}$ to a radical.

⊙ $-\sqrt[7]{c^5}$
⊙ $\frac{1}{\sqrt[7]{c^5}}$
⊙ $\frac{1}{\sqrt[5]{c^7}}$
⊙ $-\sqrt[5]{c^7}$

22. Convert $x^{-\frac{3}{8}}$ to a radical.

⊙ $\frac{1}{\sqrt[3]{x^8}}$
⊙ $\frac{1}{\sqrt[8]{x^3}}$
⊙ $-\sqrt[8]{x^3}$
⊙ $-\sqrt[3]{x^8}$

Simplify the expression, answering with rational exponents and not radicals.

23. $\dfrac{\sqrt{25y}}{\sqrt[10]{y^3}} = \boxed{}$

24. $\dfrac{\sqrt{9z}}{\sqrt[6]{z^5}} = \boxed{}$

25. $\sqrt[5]{t} \cdot \sqrt[10]{t^3} = \boxed{}$

26. $\sqrt[5]{m} \cdot \sqrt[10]{m^3} = \boxed{}$

27. $\sqrt{a}\sqrt[7]{a} = \boxed{}$

28. $\sqrt{t}\sqrt[8]{t} = \boxed{}$

More on Rationalizing the Denominator Rationalize the denominator and simplify the expression.

29. $\dfrac{9}{\sqrt{72}} = \boxed{}$

30. $\dfrac{4}{\sqrt{216}} = \boxed{}$

31. $\sqrt{\dfrac{7}{12}} = \boxed{}$

32. $\sqrt{\dfrac{11}{28}} = \boxed{}$

33. $\dfrac{4}{\sqrt{23}+6} = \boxed{}$

34. $\dfrac{5}{\sqrt{7}+9} = \boxed{}$

35. $\dfrac{\sqrt{5}-10}{\sqrt{7}+9} = \boxed{}$

36. $\dfrac{\sqrt{3}-11}{\sqrt{13}+6} = \boxed{}$

Solving Radical Equations Solve the equation.

37. $r = \sqrt{r+2} + 4$

38. $t = \sqrt{t+3} + 3$

39. $\sqrt{t+8} = \sqrt{t} - 4$

40. $\sqrt{x+8} = \sqrt{x} + 2$

41. $\sqrt{x} + 110 = x$

42. $\sqrt{y} + 56 = y$

43. $y = \sqrt{y+3} + 17$

44. $r = \sqrt{r+1} + 109$

45. $\sqrt{66-r}=r+6$

46. $\sqrt{29-r}=r+1$

47. Use technology to solve the equation $\sqrt{x-1.9}=\sqrt{x}-0.2$.

48. Use technology to solve the equation $\sqrt{x+2.3}=\sqrt{x}-3$.

49. According to the Pythagorean Theorem, the length c of the hypothenuse of a rectangular triangle can be found through the following equation:

$$c=\sqrt{a^2+b^2}$$

Solve the equation for the length a of one of the triangle's legs.

$a=$ ☐.

50. In a coordinate system, the distance r of a point (x, y) from the origin $(0, 0)$ is given by the following equation:

$$r=\sqrt{x^2+y^2}$$

Solve the equation for the coordinate y. Assume that y is positive.

If you need to use a square root, type *sqrt(23)* for $\sqrt{23}$ (for example).

$y=$ ☐.

According to the Pythagorean Theorem, the length c of the hypothenuse of a rectangular triangle can be found through the following equation.

$$c=\sqrt{a^2+b^2}$$

51. If a rectangular triangle has a hypothenuse of 13 ft and one leg is 12 ft long, how long is the third side of the triangle?

The third side of the triangle is ☐ long.

52. If a rectangular triangle has a hypothenuse of 17 ft and one leg is 15 ft long, how long is the third side of the triangle?

The third side of the triangle is ☐ long.

53. A pendulum has the length L ft. The time period T that it takes to once swing back and forth is 6 s. Use the following formula to find its length.

$$T=2\pi\sqrt{\frac{L}{32}}$$

The pendulum is ☐ long.

54. A pendulum has the length L ft. The time period T that it takes to once swing back and forth is 8 s. Use the following formula to find its length.

$$T=2\pi\sqrt{\frac{L}{32}}$$

The pendulum is ☐ long.

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