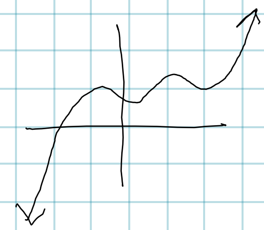
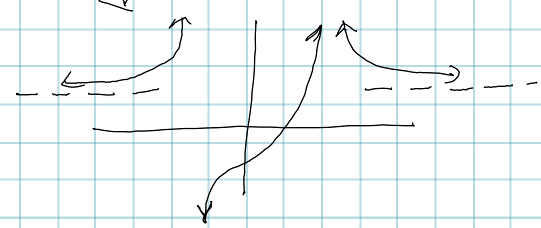
4.3 We have studied
- polynomial functions →

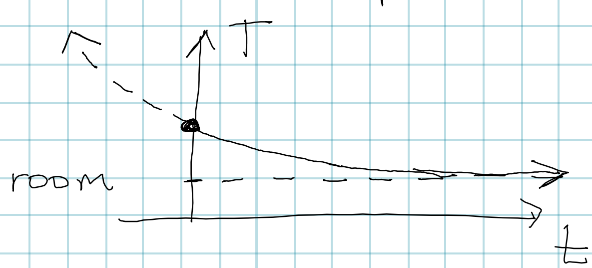


- rational functions



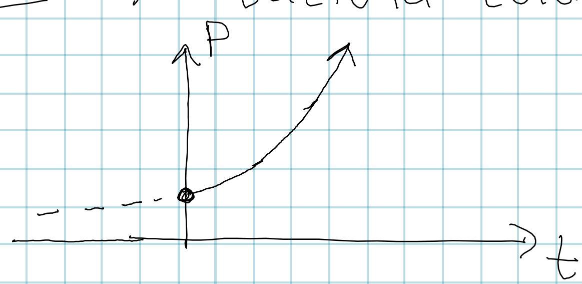
These are not enough for real-world applications

Ex Hot cup of coffee; it's cooling



this is a new shape!

Ex A bacteria colony growing unconstrained



this is a new shape!

Ex You are hired by an eccentric zillionaire

On day 0, paid 1¢
On day 1, paid 2¢
2, paid 4¢
3 paid 8¢

doubles every day.

⋮

on day 10 paid $1024¢ = \$10.24$

⋮

on day 30 paid $1,073,741,824¢$

$= \$10,737,418.24$

An exponential function

is a function whose formula

is written $f(x) = a \cdot b^x$

$(\text{Ex } f(x) = 1 \cdot 2^x)$

x	$f(x)$
0	1
1	2
2	4
3	8

$\times 2$

$\times 2$

$\times 2$

1

$2 = 1 \times 2$

$4 = 1 \times 2 \times 2$

$8 = 1 \times 2 \times 2 \times 2$

on day 3

$f(3) = 1 \times 2^3$

$(f(x) = a \cdot b^x)$

In $a \cdot b^x$

the initial value

It's the output when x is 0 (the y-intercept is at $(0, a)$)

every unit of time, we multiply by b...

b is the growth factor

Note $f(0) = a \cdot b^0$

$f(0) = a \cdot 1$

$f(0) = a$

at the beginning, output is a

$b^3 = 1 \cdot b \cdot b \cdot b$

$b^2 = 1 \cdot b \cdot b$

$b^1 = 1 \cdot b$

$b^0 = 1$

$b^{-1} = 1 \div b$

$b^{-2} = 1 \div b \div b$

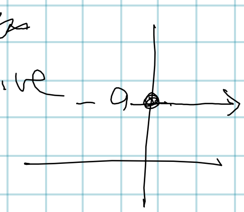
Constraints on a, b

$f(x) = a \cdot b^x$

$a \neq 0$, or you have a boring function

$b \neq 1$

$b = 1$ would give

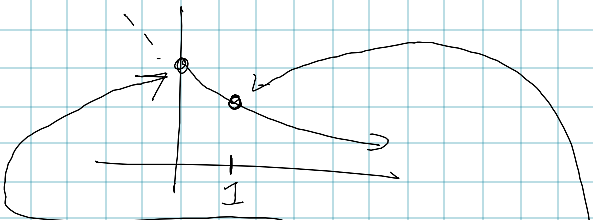


b cannot be negative! can't raise neg #'s to fractional powers

b has to be positive;
b cannot be 1

b is in $(0, 1)$ or b is in $(1, \infty)$

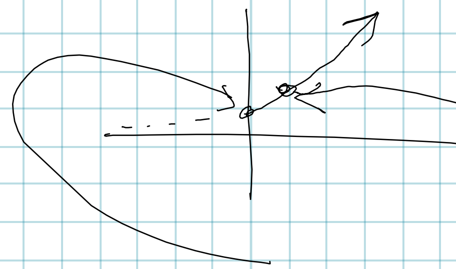
you multiply by a small number...



$f(t) = 300(0.9)^t$

90%

you multiply by a big number



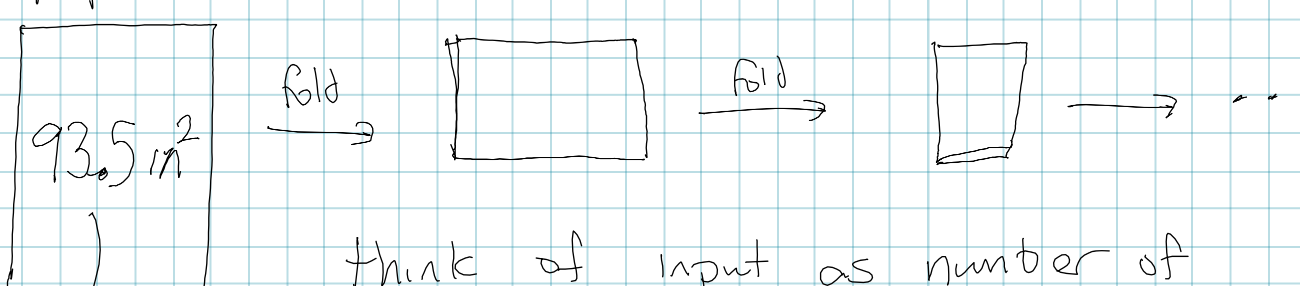
$f(t) = A\,(1.13)^t$

$b > 1$

mult by 113% every unit of time

Ex

A piece of $8.5'' \times 11''$ paper



think of input as number of folds...
output is the visible area...

Start with 93.5, every fold causes mult by $\frac{1}{2}$.

So we have $f(n) = 93.5 \cdot \left(\frac{1}{2}\right)^n$

how many folds

So after 15 folds, how much area is left?

$f(15) = 93.5 \cdot \left(\frac{1}{2}\right)^{15}$

$= 0.002853\ldots \text{ in}^2$

Third piece of vocab:

$f(t) = 187\,(1.08)^t$

side note
applications tend to have time as input.

growth factor 1.08
every unit of time multiply by 1.08

$1 + \underbrace{0.08}_{8\%}$

Same!

every of unit of time, add 8% to the output

relative growth rate

$r = 0.08 = 8\%$

$$\boxed{\begin{aligned} b &= 1 + r \\ r &= b - 1 \end{aligned}}$$

Ex $f(t) = 100(1.3)^t$

Find rel. growth rate $\rightarrow$ 30%

$r = b - 1$

$r = 1.3 - 1$

$r = 0.3 = 30\%$

Ex $f(t) = 250(0.75)^t$

Find rel. growth rate

$b < 1$ getting smaller each unit time

growth factor $= 0.75$

$r = b - 1$

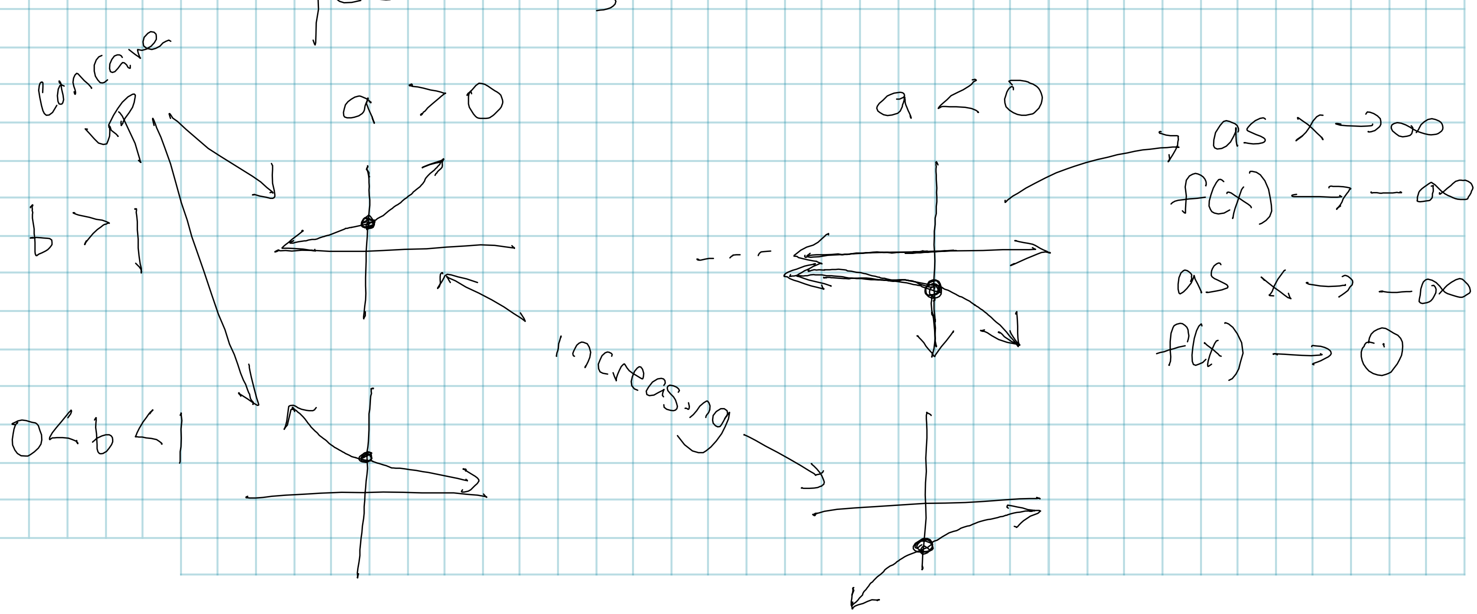
$r = 0.75 - 1$

$r = -0.25 = -25\%$

relative growth rate is -25%.

relative decay rate is 25%

Catalog of possible shapes $\longrightarrow$



Ex An island had no rats...
A ship landed and 30 rats colonized the island. This population grew by 15%/month.

initial value $a = 30$

$r = 0.15$
$b = 1.15$ per month

- After 2 months, how many rats?
- After 2 years, how many rats?
- What is the relative growth rate per year?

We have $P(t) = 30 \cdot (1.15)^t$

- $P(2) = 30\,(1.15)^2 \approx 40$ rats
- $P(\underset{\text{24 months is 2 years}}{24}) = 30\,(\underbrace{1.15}_{\text{per month growth factor}})^{24} \approx 859$ rats

- Look 1 year forward!

$$P(12) = 30\,(1.15)^{12} \approx 160$$

$$30 \xrightarrow{\text{1 year}} 160$$

$\times\,?$

1-year growth factor

$$30 \cdot ? = 160$$

$$? = 5.33\ldots$$

533%

1-year growth rate $r = b - 1$

$r = 5.33 - 1$
$= 4.33 \approx 433\%$

So we have a relative growth rate of 433%/year. We had 15%/month...

With relative growth rates (%/unit of time) you cannot simply multiply to change units like

$$\cancel{15\%/\text{month} \cdot 12\ \text{months}/\text{year} = 180\%/\text{year}}$$

Ex Solve $1000000 = 30(1.15)^t$

initial rat pop

growth factor per month

"When will the rat pop reach 1 million?"

$$33{,}333.\overline{3} = \frac{1000000}{30} = 1.15^t$$

use logarithms...

use technology

demo with TI-89

Basic Exponent Rules

$7^2 \cdot 7^3 = 7^5$ (same base → add exponents)

$5^8 \cdot 3^8 = 15^8$ (same exponent → multiply bases)

$(7^2)^3 = 7^6$ (power to a power → mult by powers)

Ex $f(t) = 2 \cdot 3^t \cdot 3^{3t}$ rewrite this in $a \cdot b^t$ form

$= 2 \cdot 3^{4t}$

$= 2(3^4)^t$

$= 2(81)^t$ → has $a = 2$, $b = 81$

Ex $g(t) = 7(1.08)^t(3.2)^t$ rewrite in standard form

$= 7(1.08 \cdot 3.2)^t$

$= 7(3.456)^t$ ✓

Initial value $= 7$

growth factor $= 3.456$

rel. growth rate

$r = b - 1$

$= 2.456$

$= 245.6\%$

Ex $g(t) = 2^{t+1} \cdot 2^{3t}$ rewrite in standard form

$= 2^{4t+1}$

$= 2^{4t} \cdot 2^{1}$

$= 2 \cdot (2^{4})^{t}$

$= 2 \cdot (16)^{t}$

$a = 2$ $b = 16$

Ex Same task: $h(t) = 8^{t} \cdot 2^{\frac{t+1}{2}}$

$= (2^{3})^{t} \cdot 2^{\frac{t+1}{2}}$

$= 2^{3t} \cdot 2^{\frac{t+1}{2}}$

$= 2^{3t + \frac{t+1}{2}}$

$= 2^{\frac{6t}{2} + \frac{t+1}{2}} = 2^{\frac{7t+1}{2}} = \left(2^{1/2}\right)^{7t+1}$

$= \left(\sqrt{2}\right)^{7t+1} = \left(\sqrt{2}\right)^{7t} \cdot \left(\sqrt{2}\right)^{1}$

$= \sqrt{2} \cdot \left(\sqrt{2}^{\,7}\right)^{t}$

$= \sqrt{2} \cdot \left(\sqrt{128}\right)^{t}$

a $b = \sqrt{128}$