Rules for Manipulating Logarithms

In these exercises, practice the rules for combining and expanding logarithm expressions. Then solve some application problems, using the logarithm rules to make the solving process go more smoothly.

1. Simplify/expand each of the following expressions, using the rules of logarithms and the rules of exponents. Do not use a calculator. The goal should always be to make it so that any number or expression *inside* a logarithm is as simple as possible. This may mean that you have more than one logarithm in your answer.

a) $\log_3 3^{-15}$

$= -15$

b) $\log_{12}(2) + \log_{12}(72)$

$= \log_{12}(2 \cdot 72)$

$= \log_{12}(144) = 2$

c) $7^{\log_7(5)+\log_7 3}$

$= 7^{\log_7(5 \cdot 3)} = 7^{\log_7(15)}$

$= 15$

d) $\log_4\left(\frac{x}{4}\right)$

$= \log_4(x) - \log_4(4)$

$= \log_4(x) - 1$

e) $\log_8(x8^x)$

$= \log_8(x) + \log_8(8^x)$

$= \log_8(x) + x$

f) $\log_2\left(\frac{x}{y^2}\right)$

$= \log_2(x) - \log_2(y^2)$

$= \log_2(x) - 2\log_2(y)$

g) $\log_6\left(\frac{\sqrt{x^2+1}}{x^2-2}\right)$

$= \log_6(\sqrt{x^2+1}) - \log_6(x^2-2)$

$= \frac{1}{2}\log_6(x^2+1) - \log_6(x^2-2)$

h) $\log\left(\left(\frac{x^2+4x+3}{(x+7)^2}\right)^{1/3}\right)$

$= \frac{1}{3}\log\left(\frac{x^2+4x+3}{(x+7)^2}\right) = \frac{1}{3}\left[\log(x^2+4x+3) - \log((x+7)^2)\right]$

$= \frac{1}{3}\left[\log(x+3) + \log(x+1) - 2\log|x+7|\right]$

2. Rewrite each expression as a *single* logarithm. That is, your answer should be like:

$$\log_{\text{some base}}(\text{some expression})$$

a) $4\log a + 3\log b$

$= \log a^4 + \log b^3$

$= \log(a^4 b^3)$

b) $\log_2(x^2-4) - \log_2(x+2)$

$= \log_2\left(\frac{x^2-4}{x+2}\right) = \log_2(x-2)$

c) $\log_2(x^2-4) - 2\log_2(x+2)$

$= \log_2(x^2-4) - \log_2((x+2)^2)$

$= \log_2\left(\frac{x^2-4}{(x+2)^2}\right) = \log_2\left(\frac{x-2}{x+2}\right)$

d) $\log_3\left(\frac{\overbrace{x^2+5x+4}^{(x+4)(x+1)}}{x^2-9}\right) - \log_3\left(\frac{\overbrace{x^2+7x+6}^{(x+1)(x+6)}}{x-3}\right)$

$= \log_3\left(\frac{(x+4)(x+1)}{(x+3)(x-3)} \cdot \frac{(x-3)}{(x+1)(x+6)}\right) = \log_3\left(\frac{(x+4)}{(x+3)(x+6)}\right)$

e) $5\log_b(2x^2) + \frac{1}{2}\log_b(2x+1)$

$= \log_b(32x^{10}) + \log_b(\sqrt{2x+1})$

$= \log_b(32x^{10}\sqrt{2x+1})$

f) $4\log_6(2x-7) - 2\log_6(2x+7) - \log_6 x$

$= \log_6((2x-7)^4) - \log_6((2x+7)^2) - \log_6(x)$

$= \log_6\left(\frac{(2x-7)^4}{(2x+7)^2 \cdot x}\right)$

3. In the application problems that follow, set up an equation and apply log (base 10) to both sides as soon as doing so would help solve the equation. Don't use other problem-solving techniques we have learned, such as graphing, using the calculator's solve tool, or using a logarithm with a special base. Give the answer in exact form *and* as a decimal approximation.

a) If the black bear population in florida was 11 000 in 1953 and decreased by 6% per year, when did the population reach half of its 1953 value?

$$11000 \cdot (0.94)^t = 5500$$
$$(0.94)^t = \tfrac{1}{2}$$
$$\log(0.94^t) = \log(\tfrac{1}{2})$$
$$t \cdot \log(0.94) = \log(\tfrac{1}{2})$$
$$t = \frac{\log(1/2)}{\log(0.94)} \approx 11.2 \text{ yrs}$$

b) If the city of Portland is growing exponentially by 1.08% per year, and is currently 584 000, how long will it take until the population reaches 700 000?

$$584000\,(1.0108)^t = 700000$$
$$1.0108^t = \frac{700}{584}$$
$$\log(1.0108^t) = \log\left(\frac{700}{584}\right)$$
$$t \cdot \log(1.0108) = \log\left(\frac{700}{584}\right)$$
$$t = \frac{\log\left(\frac{700}{584}\right)}{\log(1.0108)} = 16.86\ldots$$

c) A piece of paper is 0.1 mm thick, and every time you fold it in half you have something that is twice as tall as it used to be. How many times would you need to fold a piece of paper in half in order to have something that is 1 m thick?

change to 1cm, = 10 mm

$$0.1 \cdot 2^n = 10$$
$$2^n = 100$$
$$\log(2^n) = \log(100)$$
$$n \log(2) = 2$$
$$n = \frac{2}{\log 2} \approx 6.64$$

You'd need 7 folds.

d) Your great-grandparents left a trust fund for you, initially worth \$1.1 million, earning 6% annual interest. How many years until the trust fund is valued at \$3 million?

$$1.1\,(1.06)^t = 3$$
$$1.06^t = \frac{3}{1.1}$$
$$\log(1.06^t) = \log\left(\frac{3}{1.1}\right)$$
$$t \cdot \log(1.06) = \log(3/1.1)$$
$$t = \frac{\log(3/1.1)}{\log(1.06)} \approx 17.218\ldots$$

e) Recall that Carbon-14 decays with a decay rate of 0.0120968% per year. Suppose that a piece of cloth made from fresh cotton should have about 25 ng of C-14 per square inch. You find an ancient piece of clothing made from the same cotton at an architectural site, and it has 18 ng of C-14 per square inch. How old is this piece of clothing?

$$25\,(0.999879032)^t = 18$$
$$(0.999879032)^t = 0.72$$
$$\log(0.999879032)^t = \log(0.72)$$
$$t \cdot \log(0.999879032) = \log(0.72)$$
$$t = \frac{\log(0.72)}{\log(0.999879032)} \approx 2715$$

It's about 2715 yrs old.