Open Resources for Community College Algebra

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Portland Community College Faculty

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This book uses WeBWorK to provide most of its exercises, which may be used for online homework. WeBWorK was created by Mike Gage and Arnie Pizer, and has benefited from over 25 years of contributions from open source developers. In 2013, Chris Hughes, Alex Jordan, and Carl Yao programmed most of the WeBWorK questions in this book with a PCC curriculum development grant.

The javascript library MathJax, created and maintained by David Cervone, Volker Sorge, Christian Lawson-Perfect, and Peter Krautzberger allows math content to render nicely on screen in the HTML eBook. Additionally, MathJax makes web accessible mathematics possible.

The PDF versions are built using the typesetting software LaTeX, created by Donald Knuth and enhanced by Leslie Lamport.

Each of these open technologies, along with many that we use but have not listed here, has been enhanced by many additional contributors over the past 40 years. We are indebted to these contributors for their many contributions. By releasing this book with an open license, we honor their dedication to open software and open education.

To All

HTML, PDF, and print. This book is freely available as an HTML eBook, a PDF for reading on a screen, and a PDF intended for printing. Additionally, a printed and bound copy is available for purchase at low cost. All versions offer the same content and are synchronized such that cross-references match across versions. They can each be found at pcc.edu/orcca.

There are some differences between the HTML eBook, PDF screen version, and PDF-for-print version.

- The HTML eBook offers interactive elements, easier navigation than print, and its content is accessible in ways that a PDF cannot be. It has content (particularly in appendices) that is omitted from the PDF versions for the sake of economy. It requires no more software than a modern web browser with internet access.

- Two PDF versions can be downloaded and then accessed without the internet. One version is intended to be read on a screen with a PDF viewer. This version retains full color, has its text centered on the page, and includes hyperlinking. The other version is intended for printing on two-sided paper and then binding. Most of its color has been converted with care to gray scale. Text is positioned to the left or right of each page in a manner to support two-sided binding. Hyperlinks have been disabled.

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The graphs and other images that appear in this manual may be copied in various file formats using the HTML eBook version. Below each image are links to `.png`, `.eps`, `.svg`, `.pdf`, and `.tex` files for the image.

Mathematical content can be copied from the HTML eBook. To copy math content into *MS Word*, right-click or control-click over the math content, and click to `Show Math As MathML Code`. Copy the resulting code, and `Paste Special` into *Word*. In the `Paste Special` menu, paste it as `Unformatted Text`. To copy math content into LaTeX source, right-click or control-click over the math content, and click to `Show Math As TeX Commands`.

Tables can be copied from the HTML eBook version and pasted into applications like *MS Word*. However, mathematical content within tables will not always paste correctly without a little extra effort as described above.

Accessibility. The HTML eBook version is intended to meet or exceed web accessibility standards. If you encounter an accessibility issue, please report it.

- All graphs and images should have meaningful alt text that communicates what a sighted person would see, without necessarily giving away anything that is intended to be deduced from the image.
- All math content is rendered using MathJax. MathJax has a contextual menu that can be accessed in several ways, depending on what operating system and browser you are using. The most common way is to right-click or control-click on some piece of math content.
- In the MathJax contextual menu, you may set options for triggering a zoom effect on math content, and also by what factor the zoom will be. Also in the MathJax contextual menu, you can enable the Explorer, which allows for sophisticated navigation of the math content.
- A screen reader will generally have success verbalizing the math content from MathJax. With certain screen reader and browser combinations, you may need to set some configuration settings in the MathJax contextual menu.

Tablets and Smartphones. PreTeXt documents like this book are "mobile-friendly." When you view the HTML version, the display adapts to whatever screen size or window size you are using. A math teacher will usually recommend that you do not study from the small screen on a phone, but if it's necessary, the HTML eBook gives you that option.

WeBWorK for Online Homework. Most exercises are available in a ready-to-use collection of WeBWorK problem sets. Visit webwork.pcc.edu/webwork2/orcca-demonstration to see a demonstration WeBWorK course where guest login is enabled. Anyone interested in using these problem sets may contact the project leads. The WeBWorK set defintion files and supporting files should be available for download from pcc.edu/orcca.

Odd Answers. The answers to the odd homework exercises at the end of each section are not printed in the PDF versions for economy. Instead, a separate PDF with the odd answers is available through pcc.edu/orcca. Additionally, the odd answers are printed in an appendix in the HTML eBook.

Interactive and Static Examples. Traditionally, a math textbook has examples throughout each section. This textbook uses two types of "example":

Static These are labeled "Example." Static examples may or may not be subdivided into a "statement" followed by a walk-through solution. This is basically what traditional examples from math textbooks do.

Active These are labeled "Checkpoint." In the HTML version, active examples have WeBWorK answer blanks where a reader may try submitting an answer. In the PDF output, active examples are almost indistinguishable from static examples, but there is a WeBWorK icon indicating that a reader could interact more actively using the eBook. Generally, a walk-through solution is provided immediately following the answer blank.

Some readers using the HTML eBook will skip the opportunity to try an active example and go straight to its solution. That is OK. Some readers will try an active example once and then move on to just read the solution. That is also OK. Some readers will tough it out for a period of time and resist reading the solution until they answer the active example themselves.

For readers of the PDF, the expectation is to read the example and its solution just as they would read a static example.

A reader is *not* required to try submitting an answer to an active example before moving on. A reader *is* expected to read the solution to an active example, even if they succeed on their own at finding an answer.

Reading Questions. Each section has a few "reading questions" immediately before the exercises. These may be treated as regular homework questions, but they are intended to be something more. The intention is that reading questions could be used in certain classroom models as a tool to encourage students to do their assigned reading, and as a tool to measure what basic concepts might have been misunderstood by students following the reading.

At some point it will be possible for students to log in to the HTML eBook and record answers to reading questions for an instructor to review. The infrastructure for that feature is not yet in place at the time of printing this edition, but please check pcc.edu/orcca for updates.

Alternative Video Lessons. Most sections open with an alternative video lesson (that is only visible in the HTML eBook). These video play lists are managed through a YouTube account, and it is possible to swap videos out for better ones at any time, provided that does not disrupt courses at PCC. Please contact us if you would like to submit a different video into these video collections.

Pedagogical Decisions

The authors and the greater PCC faculty have taken various stances on certain pedagogical and notational questions that arise in basic algebra instruction. We attempt to catalog these decisions here, although this list is certainly incomplete. If you find something in the book that runs contrary to these decisions, please let us know.

- Basic math is addressed in an appendix. For the course sequence taught at PCC, this content is prerequisite and not within the scope of this book. However it is quite common for students in the basic algebra sequence to have skills deficiencies in these areas, so we include the basic math appendix. It should be understood that the content there does not attempt to teach basic math from first principles. It is itended to be more of a review.
- Interleaving is our preferred approach, compared to a proficiency-based approach. To us, this means that once the book covers a topic, that topic will be appear in subsequent sections and chapters in indirect ways.
- We round decimal results to four significant digits, or possibly fewer leaving out trailing zeros. We do this to maintain consistency with the most common level of precision that WeBWorK uses to assess decimal answers. We generally *round*, not *truncate*, and we use the $\approx$ symbol. For example, $\pi \approx 3.142$ and Portland's population is ≈ 609500. On rare occasions where it is the better option, we truncate and use an ellipsis. For example, $\pi = 3.141\ldots$.
- We offer *alternative* video lessons associated with each section, found at the top of most sections in the HTML eBook. We hope these videos provide readers with an alternative to whatever is in the reading, but there may be discrepancies here and there between the video content and reading content.
- We believe in opening a topic with some level of application rather than abstract examples, whenever that is possible. From applications and practical questions, we move to motivate more abstract definitions and notation. At first this may feel backwards to some instructors, with some easier examples *following* more difficult contextual examples.
- Linear inequalities are not strictly separated from linear equations. The section that teaches how to solve $2x + 3 = 8$ is immediately followed by the section teaching how to solve $2x + 3 < 8$. Our aim is to not treat inequalities as an add-on optional topic, but rather to show how intimately related they are to corresponding equations.
- When issues of "proper formatting" of student work arise, we value that the reader understand *why* such things help the reader to communicate outwardly. We believe that mathematics is about more

than understanding a topic, but also about understanding it well enough to communicate results to others. For example we promote progression of equations like

$$\begin{aligned} 1+1+1 &= 2+1 \\ &= 3 \end{aligned}$$

instead of

$$1+1+1 = 2+1 = 3.$$

We want students to understand that the former method makes their work easier for a reader to read. It is not simply a matter of "this is the standard and this is how it's done."

- When solving equations (or systems of linear equations), most examples should come with a check, intended to communicate to students that checking is part of the process. In Chapters 1–4, these checks will be complete simplifications using order of operations one step at a time. The later sections may have more summary checks where steps are skipped or carried out together, or we promote entering expressions into a calculator to check.

- Within a section, any first context-free example of solving some equation (or system) should summarize with some variant of both "the solution is…" and "the solution set is…." Later examples can mix it up, but always offer at least one of these.

- With applications of linear equations (not including linear systems), we limit applications to situations where the setup will be in the form $x+$expression-in-$x = C$ and also to certain rate problems where the setup will be in the form $at+bt = C$. There are other classes of application problem (mixing problems, interest problems, …) which can be handled with a system of two equations, and we reserve these until linear systems are covered.

- With simplifications of rational expressions in one variable, we always include domain restrictions that are lost in the simplification. For example, we would write $\frac{x(x+1)}{x+1} = x$, for $x \neq -1$. With *multivariable* rational expressions, we are content to ignore domain restrictions lost during simplification.

Entering WeBWorK Answers

This preface offers some guidance with syntax for WeBWorK answers. WeBWorK answer blanks appear in the active reading examples (called "checkpoints") in the HTML eBook version of the book. If you are using WeBWorK for online homework, then you will also enter answers into WeBWorK answer blanks there.

Basic Arithmetic. The five basic arithmetic operations are: addition, subtraction, multiplication, and raising to a power. The symbols for addition and subtraction are $+$ and $-$, and both of these are directly avialable on most keyboards as `+` and `-`.

On paper, multiplication is sometimes written using $\times$ and sometimes written using $\cdot$ (a centered dot). Since these symbols are not available on most keyboards, WeBWorK uses `*` instead, which is often `shift-8` on a full keyboard.

On paper, division is sometimes written using $\div$, sometimes written using a fraction layout like $\frac{4}{2}$, and sometimes written just using a slash, $/$. The slash is available on most full keyboards, near the question mark. WeBWorK uses `/` to indicate division.

On paper, raising to a power is written using a two-dimensional layout like 4^2. Since we don't have a way to directly type that with a simple keyboard, calculators and computers use the caret character, `^`, as in `4^2`. The character is usually `shift-6`.

Roots and Radicals. On paper, a square root is represented with a radical symbol like $\sqrt{}$. Since a keyboard does not usually have this symbol, WeBWorK and many computer applications use `sqrt( )` instead. For example, to enter $\sqrt{17}$, type `sqrt(17)`.

Higher-index radicals are written on paper like $\sqrt[4]{12}$. Again we have no direct way to write this using most keyboards. In *some* WeBWorK problems it is possible to type something like `root(4, 12)` for the fourth root of twelve. However this is not enabled for all WeBWorK problems.

As an alternative that you may learn about in a later chapter, $\sqrt[4]{12}$ is mathematically equal to $12^{1/4}$, so it can be typed as `12^(1/4)`. Take note of the parentheses, which very much matter.

Common Hiccups with Grouping Symbols. Suppose you wanted to enter $\frac{x+1}{2}$. You might type `x+1/2`, but this is not right. The computer will use the order of operations and do your division first, dividing 1 by 2. So the computer will see $x+\frac{1}{2}$. To address this, you would need to use grouping symbols like parentheses, and type something like `(x+1)/2`.

Suppose you wanted to enter $6^{1/4}$, and you typed `6^1/4`. This is not right. The order of operations places a higher priority on exponentiation than division, so it calculates 6^1 first and then divides the result by 4. That is simply not the same as raising 6 to the $\frac{1}{4}$ power. Again the way to address this is to use grouping symbols, like `6^(1/4)`.

Entering Decimal Answers. Often you will find a decimal answer with decimal places that go on and on. You are allowed to round, but not by too much. WeBWorK generally looks at how many *significant digits* you use, and generally expects you to use ***four or more*** correct significant digits.

"Significant digits" and "places past the decimal" are not the same thing. To count significant digits, read the number left to right and look for the first nonzero digit. Then count all the digits to the right including that first one.

The number 102.3 has four significant digits, but only one place past the decimal. This number could be a correct answer to a WeBWorK question. The number 0.0003 has one significant digit and four places past the decimal. This number might cause you trouble if you enter it, because maybe the "real" answer was 0.0003091, and rounding to 0.0003 was too much rounding.

Special Symbols. There are a handful of special symbols that are easy to write on paper, but it's not clear how to type them. Here are WeBWorK's expectations.

Symbol	Name	How to Type
∞	infinity	`infinity` or `inf`
π	pi	`pi`
$\cup$	union	`U`
$\mathbb{R}$	the real numbers	`R`
$\mid$	such that	`|` (`shift-\`, where `\` is above the `enter` key)
$\leq$	less than or equal to	`<=`
$\geq$	greater than or equal to	`>=`
$\neq$	not equal to	`!=`

Contents

Part I

Linear Equations and Lines

Chapter 1

Variables, Expressions, and Equations

1.1 Variables and Evaluating Expressions

Algebra helps people solve mathematical problems that are just a bit too complicated to solve in your head. This book is meant to cover basic principles and skills that people need to become successful with algebra. The first things to learn about are *variables*, *algebraic expressions*, *equations*, and *inequalities*. In this section, we'll focus on variables and expressions. In the remainder of this chapter, we'll focus on *equations* and *inequalities*.

1.1.1 Introduction to Variables

When we want to represent an unknown or changing numerical quantity, we use a **variable**. For example, if you'd like to discuss the gas mileage of various cars, you could use the symbol "g" as a variable to represent a car's gas mileage. The mileage might be 25 mpg, 30 mpg, or something else. ("mpg" stands for "miles per gallon".) If we agree to use mpg for the units of measure, g might be a place holder for 25, 30, or some other number. Since we are using a variable and not a specific number, we can discuss gas mileage for Honda Civics, Ford Explorers, and all other makes and models at the same time, even though these makes and models have different gas mileages.

When variables stand for actual physical quantities, it's good to use letters that clearly represent those quantities. For example, it's wise to use g to represent *g*as mileage. This helps the people who might read your work to understand it better.

It is also important to identify what unit of measurement goes along with each variable you use, and clearly tell your reader this. For example, suppose you are working with $g = 25$. A car whose gas mileage is 25 mpg is very different from a car whose gas mileage is 25 kpg (kilometers per gallon). So it would be important to tell readers that g represents gas mileage *in miles per gallon*.

Checkpoint 1.1.2 Identify a variable you might use to represent each quantity. Then identify what units would be most appropriate.

a. Let ____ be the age of a student, measured in ____________.

b. Let $\boxed{}$ be the amount of time passed since a driver left Portland, Oregon, bound for Boise, Idaho, measured in $\boxed{}$.

c. Let $\boxed{}$ be the area of a two-bedroom apartment, measured in $\boxed{}$.

Explanation.

a. The unknown quantity is age, which we generally measure in years. So we could say:

   "Let a be the age of a student, measured in years."

b. The amount of time passed is the unknown quantity. Since this is a drive from Portland to Boise, it would make sense to measure this in hours. So we could say:

   "Let t be the amount of time passed since a driver left Portland, Oregon, bound for Boise, Idaho, measured in hours."

c. The unknown quantity is area. Apartment area is usually measured in square feet. So we'll say:

   "Let A be the area of a two-bedroom apartment, measured in ft^2."

Unless an algebra problem specifies which letter(s) to use, we may *choose* which letter(s) to use for our variable(s). However *without* any context to a problem, x, y, and z are the most common letters used as variables, and you may see these variables (especially x) a lot.

Also note that the units we use are often determined indirectly by other information given in an algebra problem. For example, if we're told that a car has used so many *gallons* of gas after traveling so many *miles*, then this suggests we should measure gas mileage in mpg.

1.1.2 Algebraic Expressions

An **algebraic expression** is any combination of variables and numbers using arithmetic operations. The following are all examples of algebraic expressions:

$$x+1 \qquad 2\ell+2w \qquad \frac{\sqrt{x}}{y+1} \qquad nRT$$

Note that this definition of "algebraic expression" does *not* include anything with an equals sign in it.

Example 1.1.3 The expression:

$$\frac{5}{9}(F-32)$$

can be used to convert a temperature in degrees Fahrenheit to degrees Celsius. To do this, we need a Fahrenheit temperature, F. Then we can **evaluate** the expression. This means replacing its variable(s) (in this case, F) with specific numbers and finding the result as a single, simplified number.

Let's convert the temperature 89 °F to the Celsius scale by evaluating the expression.

$$\begin{aligned}
\frac{5}{9}(F-32) &= \frac{5}{9}(89-32) && \text{Review order of operations in A.5.}\\
&= \frac{5}{9}(57) && \text{Review fraction multiplication in A.2.}\\
&= \frac{285}{9} \approx 31.67
\end{aligned}$$

This shows us that 89 °F is equivalent to approximately 31.67 °C.

Warning 1.1.4 Correct Vocabulary. The steps in Example 1.1.3 are not considered "solving" anything. "Solving" is a word you might be tempted to use, because in everyday English you are "finding an answer." In algebra, there is a special meaning for "solving" something, and that will come soon in Section 1.5. When we substitute values in for variables and then compute the result, the technical thing to say is we are "evaluating an expression."

Checkpoint 1.1.5 Try evaluating the temperature expression for yourself.

a. If a temperature is 50°F, what is that temperature measured in Celsius?

b. If a temperature is −20°F, what is that temperature measured in Celsius?

Explanation.

a. $$\begin{aligned}\frac{5}{9}(F-32) &= \frac{5}{9}(50-32)\\ &= \frac{5}{9}(18)\\ &= \frac{5}{1}(2)\\ &= 10\end{aligned}$$

So 50°F is equavalent to 10°C.

b. $$\begin{aligned}\frac{5}{9}(F-32) &= \frac{5}{9}(-20-32)\\ &= \frac{5}{9}(-52)\\ &= -\frac{260}{9}\\ &\approx -28.89\end{aligned}$$

So −20°F is equavalent to about −28.89°C.

Example 1.1.6 Allowing Variables to Vary. With the help of technology, it is possible to quickly evaluate expressions as variables vary. In the GeoGebra applet in Figure 1.1.7, you may slide the value of F and see how a computer can quickly calculate the correspoding Celsius temperature.

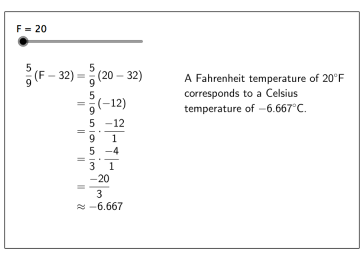


Figure 1.1.7: Allowing Variables to Vary

Example 1.1.8 Target heart rate. According to the American Heart Association, a person's maximum heart rate, in beats per minute (bpm), is given by $220 - a$, where a is their age in years.

a. Determine the maximum heart rate for someone who is 31 years old.

b. A person's *target* heart rate for moderate exercise is 50% to 70% of their maximum heart rate. If they want to reach 65% of their maximum heart rate during moderate exercise, we'd use the expression $0.65(220-a)$, where a is their age in years. Determine the target heart rate at this 65% level for someone who is 31 years old.

Explanation. Both parts ask us to evaluate an expression.

a. Since a is defined as age in years, we evaluate this expression by substituting a with 31:

$$\begin{aligned} 220 - a &= 220 - 31 \\ &= 189 \end{aligned}$$

This tells us that the maximum heart rate for someone who is 31 years old is 189 bpm.

b. We again substitute a with 31, but this time using the target heart rate expression:

$$\begin{aligned} 0.65(220 - a) &= 0.65(220 - 31) \\ &= 0.65(189) \\ &= 122.85 \end{aligned}$$

This tells us that the target heart rate for someone who is 31 years old undertaking moderate exercise is 122.85 bpm.

Checkpoint 1.1.9 We can use the expression $\frac{p}{100}(220-a)$ to represent a person's target heart rate when their target rate is $p\%$ of their maximum heart rate, and they are a years old.

Determine the target heart rate at the 53% level for moderate exercise for someone who is 56 years old.

At the 53% level, the target heart rate for moderate exercise for someone who is 56 years old is $\boxed{}$ beats per minute.

Explanation. The expression is $\frac{p}{100}(220 - a)$, and we must sustitute in 53 for p and 56 for a.

$$\begin{aligned} \frac{p}{100}(220 - a) &= \frac{53}{100}(220 - 56) \\ &= \frac{53}{100}(164) \\ &= \frac{53}{25}(41) \\ &= 86.92 \end{aligned}$$

So at the 53% level, the target heart rate for moderate exercise for someone who is 56 years old is 86.92 beats per minute.

Checkpoint 1.1.10 Rising Rents. From January, 2011, to October, 2016, an expression estimating the average rent of a one-bedroom apartment in Portland, Oregon, is given by $10.173x + 974.78$, where x is the number of months since January, 2011.

a. According to this model, what was the average rent of a one-bedroom apartment in Portland in January, 2011?

b. According to this model, what was the average rent of a one-bedroom apartment in Portland in January, 2016?

Explanation.

a. This model uses x as the number of months since January, 2011. So for January, 2011, x is 0:

$$\begin{aligned} 10.173x + 974.48 &= 10.173(0) + 974.48 \\ &\approx 974.48 \end{aligned}$$

According to this model, the average monthly rent for a one-bedroom apartment in Portland, Oregon, in January, 2011, was \$974.78.

b. The date we are given is January, 2016, which is 5 years after January, 2011. Recall that x is the number of *months* since January, 2011. So we need to use $x = 5 \cdot 12 = 60$:

$$\begin{aligned} 10.173x + 974.48 &= 10.173(60) + 974.48 \\ &\approx 1584.86 \end{aligned}$$

According to this model, the average monthly rent for a one-bedroom apartment in Portland, Oregon, in January, 2016, was \$1584.86.

1.1.3 Evaluating Expressions with Exponents, Absolute Value, and Radicals

Algebraic expressions will often have exponents, absolute value bars, and radicals. This does not change the basic approach to evaluating them.

Example 1.1.11 Tsunami Speed. The speed of a tsunami (in meters per second) can be modeled by $\sqrt{9.8d}$, where d is the depth of the tsunami (in meters). Determine the speed of a tsunami that has a depth of $30\,\mathrm{m}$ to four significant digits.

Explanation. Using $d = 30$, we find:

$$\begin{aligned} \sqrt{9.8d} &= \sqrt{9.8(30)} && \text{Review order of operations in A.5.} \\ &= \sqrt{294} && \text{Review square root in A.3.} \\ &\approx \overbrace{17.14}^{\text{four}}6428\ldots \end{aligned}$$

The speed of tsunami with a depth of $30\,\mathrm{m}$ is about $17.15\,\frac{\mathrm{m}}{\mathrm{s}}$.

Up to now, we have been evaluating expressions, but we can evaluate formulas in the same way. A **formula** usually has a single variable that represents the output of an expression. For example, the expression for a person's maximum heart rate in beats per minute, $220 - a$, can be written as the formula, $H = 220 - a$. When we substitute a value for a we are "evaluating" the formula.

Checkpoint 1.1.12 Tent Height. While camping, the height inside a tent when you are d feet from the north side of the tent is given by the formula $h = -2|d - 3| + 6$, where h is in feet.

a. When you are 5 ft from the north side, the height is ____.

b. When you are 2.5 ft from the north side, the height is ____.

Explanation.

a. When $d = 5$, we have:

$$\begin{aligned} h &= -2\left|d-3\right|+6 \\ &= -2\left|5-3\right|+6 && \text{Review order of operations in A.5.} \\ &= -2\left|2\right|+6 && \text{Review absolute value in A.3.} \\ &= -2(2)+6 \\ &= -4+6 \\ &= 2 \end{aligned}$$

So when you are 5 ft from the north side, the height of the tent is 2 ft.

b. When $d = 2.5$, we have:

$$\begin{aligned} h &= -2\left|d-3\right|+6 \\ &= -2\left|2.5-3\right|+6 \\ &= -2\left|-0.5\right|+6 \\ &= -2(0.5)+6 \\ &= -1+6 \\ &= 5 \end{aligned}$$

So when you are 2.5 ft from the north side, the height of the tent is 5 ft.

Checkpoint 1.1.13 Mortgage Payments. If we borrow L dollars for a home mortgage loan at an annual interest rate r, and intend to pay off the loan after n months, then the amount we should pay each month M, in dollars, is given by the formula

$$M = \frac{rL\left(1+\frac{r}{12}\right)^n}{12\left(\left(1+\frac{r}{12}\right)^n-1\right)}$$

If we borrow \$200,000 at an interest rate of 6% with the intent to pay off the loan in 30 years, what should our monthly payment be? (Using a calculator is appropriate here.)

Explanation. We must use $L = 200000$. Because the interest rate is a percentage, $r = 0.06$ (not 6). The variable n is supposed to be a number of *months*, but we will pay off the loan in 30 *years*. Therefore we take $n = 360$.

$$\begin{aligned} M = \frac{rL\left(1+\frac{r}{12}\right)^n}{12\left(\left(1+\frac{r}{12}\right)^n-1\right)} &= \frac{(0.06)(200000)\left(1+\frac{0.06}{12}\right)^{360}}{12\left(\left(1+\frac{0.06}{12}\right)^{360}-1\right)} \\ &= \frac{(0.06)(200000)(1+0.005)^{360}}{12\left((1+0.005)^{360}-1\right)} \\ &\approx \frac{(0.06)(200000)(6.022575\ldots)}{12\left(6.022575\ldots-1\right)} \\ &\approx \frac{(0.06)(200000)(6.022575\ldots)}{12(5.022575\ldots)} \\ &\approx \frac{72270.90\ldots}{60.2709\ldots} \\ &\approx 1199.10 \end{aligned}$$

Our monthly payment should be \$1,199.10.

Warning 1.1.14 Rounding Too Much. You might have noticed in the explanation to Exercise 1.1.13 that during the computations, many decimal places were recorded at each step. Recording lots of decimal places might be very important in some computations. If you round in the middle of your work, you have changed the numbers a little bit from what they *really* should be. As computations proceed, this little error can become larger and larger, leaving you with a final result that is too far off to be considered correct. So the best practice is to always keep lots of decimal places in all your computations, and then at the very end you may round more if that is appropriate.

1.1.4 Evaluating Expressions with Negative Numbers

When we substitute negative numbers into an expression, it's important to use parentheses around them or else it's easy to forget that a *negative* number is being raised to a power.

Example 1.1.15 Evaluate x^2 if $x = -2$.

We substitute:

$$\begin{aligned} x^2 &= (-2)^2 \\ &= 4 \end{aligned}$$

If we don't use parentheses, we would have:

$$\begin{aligned} x^2 &= -2^2 \qquad \text{incorrect!} \\ &= -4 \end{aligned}$$

The original expression, x^2, takes x and squares it, so we want to do the same thing to the number -2. But with -2^2, the number -2 is not being squared. Since the exponent has higher priority than the negation in the order of operations, it's just the number 2 that is being squared and then the result is negated. With $(-2)^2$ the number -2 *is* being squared, which is what we want.

So it is wise to always use some parentheses when substituting in any negative number.

Checkpoint 1.1.16 Evaluate and simplify the following expressions for $x = -5$ and $y = -2$:

a. x^3y^2 b. $(-2x)^3$ c. $-3x^2y$

Explanation. You may review multiplying negative numbers in Section A.1.

a. $$\begin{aligned} x^3y^2 &= (-5)^3(-2)^2 \\ &= (-125)(4) \\ &= -500 \end{aligned}$$

b. $$\begin{aligned} (-2x)^3 &= (-2(-5))^3 \\ &= (10)^3 \\ &= 1000 \end{aligned}$$

c. $$\begin{aligned} -3x^2y &= -3(-5)^2(-2) \\ &= -3(25)(-2) \\ &= 150 \end{aligned}$$

1.1.5 Reading Questions

1. Describe a situation where it might be better to use a letter other than x, y, or z as a variable.

2. What is the difference between an "algebraic expression" and a "formula," as described in this section? (Other math resources may define these terms differently.)

3. What should you watch out for when substituting a negative number in for a variable?

4. In Figure 1.1.7, when you change the value of F, why do some values of F cause there to be more steps in the calculation than other values of F?

1.1.6 Exercises

1. Identify a variable you might use to represent each quantity. Then identify what units would be most appropriate.

a. Let $\boxed{}$ be the depth of a swimming pool, measured in $\boxed{}$.

b. Let $\boxed{}$ be the weight of a dog, measured in $\boxed{}$.

2. Identify a variable you might use to represent each quantity. Then identify what units would be most appropriate.

a. Let $\boxed{}$ be the amount of time a person sleeps each night, measured in $\boxed{}$.

b. Let $\boxed{}$ be the surface area of a patio, measured in $\boxed{}$.

Evaluating Expressions

3. Evaluate $x - 4$ for $x = -6$.

4. Evaluate $x + 9$ for $x = -4$.

5. Evaluate $2 - x$ for $x = -1$.

6. Evaluate $-5 - x$ for $x = 1$.

7. Evaluate $8x - 10$ for $x = 3$.

8. Evaluate $x - 1$ for $x = 6$.

9. Evaluate $-9p$ for $p = -6$.

10. Evaluate $-4q$ for $q = 7$.

11. Evaluate the expression x^2:

a. For $x = 6$.

b. For $x = -7$.

12. Evaluate the expression x^2:

a. For $x = 3$.

b. For $x = -2$.

13. Evaluate the expression $-y^2$:

a. For $y = 5$.

b. For $y = -4$.

14. Evaluate the expression $-y^2$:

a. For $y = 3$.

b. For $y = -5$.

15. Evaluate the expression r^3:

a. For $r = 2$.

b. For $r = -3$.

16. Evaluate the expression r^3:

a. For $r = 5$.

b. For $r = -5$.

17. a. Evaluate $5x^2$ when $x = 2$.

b. Evaluate $(5x)^2$ when $x = 2$.

18. a. Evaluate $3x^2$ when $x = 2$.

b. Evaluate $(3x)^2$ when $x = 2$.

19. Evaluate $-10(t + 7)$ for $t = 5$.

20. Evaluate $-6(x + 5)$ for $x = -3$.

21. Evaluate $\dfrac{9x - 9}{6x}$ for $x = -10$.

22. Evaluate $\dfrac{4y - 2}{6y}$ for $y = 4$.

23. Evaluate $-5c - 2b$ for $c = 4$ and $b = 2$.

24. Evaluate $6A - 3C$ for $A = -8$ and $C = -4$.

25. Evaluate $\dfrac{-6}{C} - \dfrac{7}{A}$ for $C = 5$ and $A = -6$.

26. Evaluate $\dfrac{-5}{m} - \dfrac{7}{b}$ for $m = 2$ and $b = -6$.

27. Evaluate $\dfrac{-4p+4C-10}{4p-3C}$ for $p=-3$ and $C=-1$.

28. Evaluate $\dfrac{-q+6A+2}{-8q+7A}$ for $q=6$ and $A=-6$.

29. Evaluate $(y+6)^2+8$ for $y=-5$.

30. Evaluate $\frac{1}{6}(r-2)^2+3$ for $r=8$.

31. Evaluate $-\left(9a^2+2a+2\right)$ for $a=1$.

32. Evaluate $-\left(c^2+8c+2\right)$ for $c=-6$.

33. Evaluate $(7A)^3$ for $A=-2$.

34. Evaluate $(-5C)^3$ for $C=-5$.

35. Evaluate $(5m)^2$ for $m=2$.

36. Evaluate $(-8p)^3$ for $p=8$.

37. Evaluate $\sqrt{q+4}-3$ for $q=21$.

38. Evaluate $\sqrt{y+8}-1$ for $y=-4$.

39. Evaluate $-\left(4\sqrt{r-3}+7\right)$ for $r=52$.

40. Evaluate $7-4\sqrt{a+5}$ for $a=11$.

41. Evaluate $|b-6|+2$ for $b=-8$.

42. Evaluate $|A+2|-3$ for $A=4$.

43. Evaluate $-(5|C-9|+9)$ for $C=-2$.

44. Evaluate $5-5|m-1|$ for $m=-9$.

45. Evaluate $\frac{y_2-y_1}{x_2-x_1}$ for $x_1=8$, $x_2=3$, $y_1=8$, and $y_2=-1$.

46. Evaluate $\frac{y_2-y_1}{x_2-x_1}$ for $x_1=10$, $x_2=-4$, $y_1=-5$, and $y_2=-6$.

47. Evaluate $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ for $x_1=4$, $x_2=12$, $y_1=-7$, and $y_2=-13$.

48. Evaluate $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ for $x_1=-8$, $x_2=0$, $y_1=-7$, and $y_2=-1$.

49. Evaluate the algebraic expression $-5a+b$ for $a=\frac{3}{4}$ and $b=\frac{5}{3}$.

50. Evaluate the algebraic expression $-7a+b$ for $a=\frac{4}{9}$ and $b=\frac{9}{2}$.

51. Evaluate each algebraic expression for the given value(s):
$\dfrac{5+2|y-x|}{x+2y}$, for $x=12$ and $y=13$:

52. Evaluate each algebraic expression for the given value(s):
$\dfrac{4+2|y-x|}{x+4y}$, for $x=6$ and $y=-13$:

To convert a temperature measured in degrees Fahrenheit to degrees Celsius, there is a formula:

$$C=\frac{5}{9}(F-32)$$

where C represents the temperature in degrees Celsius and F represents the temperature in degrees Fahrenheit.

53. If a temperature is 95°F, what is that temperature measured in Celsius?

54. If a temperature is 113°F, what is that temperature measured in Celsius?

55. A formula for converting miles into kilometers is

$$K=1.61M$$

where M is a number of miles, and K is the corresponding number of kilometers.
Use the formula to find the number of kilometers that corresponds to six miles.
______ kilometers corresponds to six miles.

56. A formula for converting pounds into kilograms is

$$K=0.45P$$

where P is a number of pounds, and K is the corresponding number of kilograms.
Use the formula to find the number of kilograms that corresponds to eighteen pounds.
______ kilograms corresponds to eighteen pounds.

The formula

$$y = \frac{1}{2}\,a\,t^2 + v_0\,t + y_0$$

gives the vertical position of an object, at time t, thrown with an initial velocity v_0, from an initial position y_0 in a place where the acceleration of gravity is a. The acceleration of gravity on earth is $-9.8\ \frac{\text{m}}{\text{s}^2}$. It is negative, because we consider the upward direction as positive in this situation, and gravity pulls down.

57. What is the height of a baseball thrown with an initial velocity of $v_0 = 60\ \frac{\text{m}}{\text{s}}$, from an initial position of $y_0 = 76$ m, and at time $t = 12$ s?
Twelve seconds after the baseball was thrown, it was ______ high in the air.

58. What is the height of a baseball thrown with an initial velocity of $v_0 = 65\ \frac{\text{m}}{\text{s}}$, from an initial position of $y_0 = 58$ m, and at time $t = 4$ s?
Four seconds after the baseball was thrown, it was ______ high in the air.

The percentage of births in the U.S. delivered via C-section can be given by the following formula for the years since 1996:

$$p = 0.8(y - 1996) + 21$$

In this formula y is a year after 1996 and p is the percentage of births delivered via C-section for that year.

59. What percentage of births in the U.S. were delivered via C-section in the year 2004?
______ of births in the U.S. were delivered via C-section in the year 2004.

60. What percentage of births in the U.S. were delivered via C-section in the year 2006?
______ of births in the U.S. were delivered via C-section in the year 2006.

Target heart rate for moderate exercise is 50% to 70% of maximum heart rate. If we want to represent a certain percent of an individual's maximum heart rate, we'd use the formula

$$\text{rate} = p(220 - a)$$

where p is the percent, and a is age in years.

61. Determine the target heart rate at 63% level for someone who is 17 years old. Round your answer to an integer.
The target heart rate at 63% level for someone who is 17 years old is ______ beats per minute.

62. Determine the target heart rate at 65% level for someone who is 57 years old. Round your answer to an integer.
The target heart rate at 65% level for someone who is 57 years old is ______ beats per minute.

The diagonal length (D) of a rectangle with side lengths L and W is given by:

$$D = \sqrt{L^2 + W^2}$$

63. Determine the diagonal length of rectangles with $L = 24$ ft and $W = 7$ ft.
The diagonal length of rectangles with $L = 24$ ft and $W = 7$ ft is ______.

64. Determine the diagonal length of rectangles with $L = 21$ ft and $W = 20$ ft.
The diagonal length of rectangles with $L = 21$ ft and $W = 20$ ft is ______.

65. The height inside a camping tent when you are d feet from the edge of the tent is given by

$$h = -2|d - 4.2| + 7$$

where h stands for height in feet.
Determine the height when you are:

a. 6.7 ft from the edge.

The height inside a camping tent when you are 6.7 ft from the edge of the tent is ______.

b. 3.2 ft from the edge.

The height inside a camping tent when you are 3.2 ft from the edge of the tent is ______.

66. The height inside a camping tent when you are d feet from the edge of the tent is given by

$$h = -1.5|d - 4.6| + 7$$

where h stands for height in feet.
Determine the height when you are:

a. 6.2 ft from the edge.

The height inside a camping tent when you are 6.2 ft from the edge of the tent is ______.

b. 2.2 ft from the edge.

The height inside a camping tent when you are 2.2 ft from the edge of the tent is ______.

67. The height inside a camping tent when you are d ft from the edge of the tent is given by:

$$-1.5\,|d - 4| + 6$$

Determine the height when you are:

(a) 2 ft from the edge

(b) 6.5 ft from the edge

68. The diagonal length of a rectangle with side lengths L and W is given by:

$$\sqrt{L^2 + W^2}$$

Determine the diagonal length of rectangles with:

(a) length 5 cm and width 12 cm

(b) length 4 ft and width 10 ft

1.2 Combining Like Terms

In Section 1.1, we worked with algebraic expressions. Algebraic expressions can be large and complicated, and anything we can do to write the same expression in a simplified form is helpful. The most basic skill for simplfying an algebraic expression is finding parts of the expression that have a certain something in common that allows them to be combined into one. *Combining like terms* is the topic of this section.

1.2.1 Identifying Terms

Definition 1.2.2 In an algebraic expression, the **terms** are quantities being added together. ◊

Example 1.2.3 List the terms in the expression $2\ell + 2w$.

Explanation. The expression has two terms that are being added, 2ℓ and $2w$.

If there is any subtraction, we can rewrite the expression using addition to make it easier to see exactly what the terms are and what sign each term has.

Example 1.2.4 List the terms in the expression $-3x^2 + 5x - 4$.

Explanation. We can rewrite this expression as $-3x^2 + 5x + (-4)$ to see that the terms are $-3x^2$, $5x$, and -4.

Once you learn to recognize that subtraction represents a negative term, you don't need to rewrite subtraction as addition.

Example 1.2.5 List the terms in the expression $3\,\text{cm} + 2\,\text{cm} - 3\,\text{cm} + 2\,\text{cm}$.

Explanation. This expression has four terms: $3\,\text{cm}$, $2\,\text{cm}$, $-3\,\text{cm}$, and $2\,\text{cm}$.

Checkpoint 1.2.6 List the terms in the expression $5x - 4x + 10z$.

Explanation. The terms are $5x$, $-4x$, and $10z$.

1.2.2 Combining Like Terms

In the examples above, you may have wanted to combine terms in some cases. For example, if you have $3\,\text{cm} + 2\,\text{cm}$, it is natural to add those together to get $5\,\text{cm}$. That works because their units (cm) are the same. This idea applies to some other kinds of terms that don't have units. For example, with $2x + 3x$, we have 2 *somethings* and then we have 3 more of the same thing. All together, we have 5 of those things. So $2x + 3x$ is the same as $5x$.

Terms in an algebraic expression that can be combined like these last examples are called **like terms**.

- Sometimes terms are like terms because they have the same variable, like with $2x+3x$, which simplfies to $5x$.
- Sometimes terms are like terms because they have the same units, like with $3\,\text{cm}+2\,\text{cm}$, which simplfies to $5\,\text{cm}$.
- Sometimes terms are like terms because they have something else in common, like with $3\sqrt{7} + 2\sqrt{7}$, which simplfies to $5\sqrt{7}$.

Example 1.2.7 In the expressions below, look for like terms and then simplify where possible by adding or subtracting.

a. $5\,\mathrm{in} + 20\,\mathrm{in}$

b. $16\,\mathrm{ft}^2 + 4\,\mathrm{ft}$

c. 2🍎 + 5🍎

d. $5\,\mathrm{min} + 50\,\mathrm{ft}$

e. 5🐶 − 2🐱

f. $20\,\mathrm{m} - 6\,\mathrm{m}$

Explanation. We can combine terms with the same units, but we cannot combine units such as minutes and feet, or cats and dogs. We can combine the like terms by adding or subtracting their numerical parts.

a. $5\,\mathrm{in} + 20\,\mathrm{in} = 25\,\mathrm{in}$

b. $16\,\mathrm{ft}^2 + 4\,\mathrm{ft}$ cannot be simplified

c. 2🍎 + 5🍎 = 7🍎

d. $5\,\mathrm{min} + 50\,\mathrm{ft}$ cannot be simplified

e. 5🐶 − 2🐱 cannot be simplified

f. $20\,\mathrm{m} - 6\,\mathrm{m} = 14\,\mathrm{m}$

One of the examples from Example 1.2.7 was $16\,\mathrm{ft}^2 + 4\,\mathrm{ft}$. The units on these two terms may look similar, but they are very different. $16\,\mathrm{ft}^2$ is a measurement of how much area something has. $4\,\mathrm{ft}$ is a measurement of how long something is. Figure 1.2.8 illustrates this.

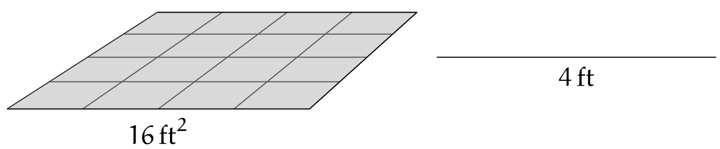


Figure 1.2.8: There is no way to add $16\,\mathrm{ft}^2$ to $4\,\mathrm{ft}$.

Checkpoint 1.2.9 Which expressions have like terms that you can combine?

a. $10x + 3y$ (☐ can ☐ cannot) be combined.

b. $4x - 8x$ (☐ can ☐ cannot) be combined.

c. $9y - 4y$ (☐ can ☐ cannot) be combined.

d. $-6x + 17z$ (☐ can ☐ cannot) be combined.

e. $-3x - 7x$ (☐ can ☐ cannot) be combined.

f. $5t + 8t^2$ (☐ can ☐ cannot) be combined.

Explanation. The terms that we can combine have the same variable part, including any exponents.

a. $10x + 3y$ cannot be combined.

b. $4x - 8x = -4x$

c. $9y - 4y = 5y$

d. $-6x + 17z$ cannot be combined.

e. $-3x - 7x = -10x$

f. $5t + 8t^2$ cannot be combined.

Example 1.2.10 Simplify the expression $20x - 16x + 4y$, if possible, by combining like terms.

Explanation. This expression has two like terms, $20x$ and $-16x$, which we can combine.

$$20x - 16x + 4y = 4x + 4y$$

Note that we cannot combine $4x$ and $4y$ because x and y are different.

Example 1.2.11 Simplify the expression $100x + 100x^2$, if possible, by combining like terms.

Explanation. This expression cannot be simplified because the variable parts are not the same. We cannot add x and x^2 just like we cannot add feet (a measure of length) and square feet (a measure of area).

Example 1.2.12 Simplify the expression $-10r + 2s - 5t$, if possible, by combining like terms.

Explanation. This expression cannot be simplified because there are not any like terms.

Example 1.2.13 Simplify the expression $y + 5y$, if possible, by combining like terms.

Explanation. This expression can be thought of as $1y + 5y$. When we have a single y, the numerical part 1 is not usually written. Now we have two like terms, $1y$ and $5y$. We will add those together:

$$\begin{aligned} y + 5y &= 1y + 5y \\ &= 6y \end{aligned}$$

So far we have combined terms with whole numbers and integers, but we can also combine like terms when the numerical parts are decimals or fractions.

Example 1.2.14 Simplify the expression $x - 0.15x$, if possible, by combining like terms.

Explanation. Note that this expression can be rewritten as $1.00x - 0.15x$, and combined like this:

$$\begin{aligned} x - 0.15x &= 1.00x - 0.15x \\ &= 0.85x \end{aligned}$$

Checkpoint 1.2.15 Simplify each expression, if possible, by combining like terms.

a. $x + 0.25x$

b. $\frac{4}{9}x - \frac{7}{10}y + \frac{2}{3}x$

c. $\frac{5}{6}y - \frac{8}{15}y + \frac{2}{3}x^2$

d. $4x + 1.5y - 9z$

Explanation.

a. Rewrite this expression as $1.00x + 0.25x$ and simplify to get $1.25x$.

b. This expression has two like terms that can be combined: $\frac{4}{9}x$ and $\frac{2}{3}x$. To combine them, we need to add the fractions $\frac{4}{9} + \frac{2}{3}$. (You may find a review of fraction addition in Section A.2.) Here, we have

$$\begin{aligned} \frac{4}{9} + \frac{2}{3} &= \frac{4}{9} + \frac{6}{9} \\ &= \frac{10}{9} \end{aligned}$$

so $\frac{4}{9}x + \frac{2}{3}x = \frac{10}{9}x$. And together with the third term, the answer is $\frac{10}{9}x - \frac{7}{10}y$.

c. In this expression we can combine the y terms, but we need to subtract the fractions $\frac{5}{6} - \frac{8}{15}$. (You may find a review of fraction subtraction in Section A.2.) Here, we have

$$\begin{aligned} \frac{5}{6} - \frac{8}{15} &= \frac{25}{30} - \frac{16}{30} \\ &= \frac{9}{30} \\ &= \frac{3}{10} \end{aligned}$$

so $\frac{5}{6}y - \frac{8}{15}y = \frac{3}{10}y$. And together with the third term, the answer is $\frac{3}{10}y + \frac{2}{3}x^2$.

d. This expression cannot be simplified further because there are not any like terms.

Remark 1.2.16 The Difference Between Terms and Factors. We have learned that terms are quantities that are added, such as $3x$ and $-2x$ in $3x-2x$. These are different from **factors**, which are parts that are multiplied together. For example, the term $2x$ has two factors: 2 and x (with the multiplication symbol implied between them). The term $2\pi r$ has three factors: 2, π, and r.

1.2.3 Reading Questions

1. What should you be careful with when there is subtraction in an algebraic expression and you are identifying its terms?

2. Describe at least two different ways in which a pair of terms are considered to be "like terms."

3. Describe the difference between "terms" and "factors" in an algebraic expression. Give examples.

1.2.4 Exercises

Review and Warmup

1. Add the following.

a. $4 + (-6)$

b. $6 + (-2)$

c. $9 + (-9)$

2. Add the following.

a. $4 + (-7)$

b. $9 + (-3)$

c. $9 + (-9)$

3. Add the following.

a. $-9 + 5$

b. $-4 + 5$

c. $-2 + 2$

4. Add the following.

a. $-6 + 5$

b. $-2 + 8$

c. $-2 + 2$

5. Subtract the following.

a. $1 - 9$

b. $10 - 2$

c. $6 - 17$

6. Subtract the following.

a. $1 - 7$

b. $7 - 1$

c. $6 - 12$

7. Subtract the following.

a. $-4 - 5$

b. $-7 - 5$

c. $-4 - 4$

8. Subtract the following.

a. $-3 - 1$

b. $-7 - 4$

c. $-9 - 9$

9. Subtract the following.

a. $-3 - (-6)$

b. $-8 - (-1)$

c. $-5 - (-5)$

10. Subtract the following.

a. $-4 - (-8)$

b. $-5 - (-1)$

c. $-5 - (-5)$

Counting, Identifying, and Combining Terms Count the number of terms in each expression.

11. a. $-5t - 9y - 5x - 5x^2$

b. $5x^2 + 8 - 8s^2$

c. $3y - 8x$

d. $4z + 2s^2 + 6y^2 - 7s$

12. a. $3t + 5s - 4y + 6t^2$

b. $-4z^2 - 9y^2$

c. $3t^2 + 5y^2 + 4y + 7$

d. $-2y - 6t^2 - y$

13. a. $-2t + 8.3x - 3.6y + 3y$

b. $-y^2 - 2s^2 + 4.5x$

c. $6.6y - 2.1y$

d. $-4.3x$

14. a. $-t$

b. $-8.9z - 1.8y + 8.1s$

c. $-1.8x - 5.3 + 6.2y + 8.4$

d. $-8.5z - 8.2 + 4.9s + 3.4$

List the terms in each expression.

15. a. $t + 7s^2$

b. $-8t$

c. $z + 9z$

d. $4s^2 - y - 8s + 3$

16. a. $-5t - 4x$

b. $-7x^2 + 5 - 4t - 8y$

c. $-5y^2 - 8x^2 - 9y^2$

d. $3t^2 - 5x^2$

17. a. $4.5t + 4.9t$

b. $0.7s^2 + 2.4t - 8.1x$

c. $7.4t + 5.1y + 5.1s - 5x^2$

d. $-6.5z$

18. a. $-7.8t^2 + 0.5t + 1.6x^2$

b. $-5.5y$

c. $1.8z + 2y$

d. $3.4t + 7y + 0.1s - 5.6s$

19. a. $3.2t + 7.7 - 6.3y^2$

b. $9t + 8.9s + 8.6s + 7.5$

c. $2.3z^2 + 7t - 1.6s$

d. $-7.5t^2 - 6.1t^2$

20. a. $5.2t^2 - 3.2x + 8.6x^2 + 0.1s$

b. $7.5z + 1.1x^2$

c. $-5.8t^2 - 3.9z^2$

d. $5.4z + 5.9y + 9z^2$

Simplify each expression, if possible, by combining like terms.

21. a. $-8t + 2t$

b. $4z + 7z$

c. $-5z + 8z$

d. $9y^2 + 3x^2$

22. a. $4t - 9s$

b. $5x - z$

c. $6s - 9s$

d. $2x + 5x$

23.
a. $-4z+3z$
b. $6x-9x^2-3x$
c. $-7t^2-6s+7t^2$
d. $7z^2-7s^2+4z$

24.
a. $z+9s$
b. $8x^2+9s+2t^2-2t$
c. $9t^2-6y-5z^2-9x$
d. $9s-8s$

25.
a. $-8z-21s^2$
b. $94s^2+70s+51s$
c. $-50z-51z$
d. $-90y-70y+38y+44$

26.
a. $-36z+14z^2+92s^2$
b. $40x^2+99x$
c. $-18x^2+91y^2-70s^2+59x^2$
d. $-5t^2+46y^2+99t^2$

27.
a. $2.5z-3.6z+4.6z^2$
b. $3.9x^2-3.5x+1.1x+8.9x$
c. $3.7z^2-0.4z^2$
d. $-4.3z+4.4z$

28.
a. $-6.7z^2+6t-8.1t$
b. $5.9y-7.3t+2t$
c. $-6.5y^2-2.8t^2-7.5s^2$
d. $-8.2y^2+4.5z^2$

29.
a. $6z-\frac{1}{7}z+4z$
b. $\frac{7}{3}y-\frac{1}{2}y^2$
c. $\frac{3}{7}t^2-\frac{1}{2}x^2+\frac{6}{5}x^2+\frac{2}{7}x$
d. $y+2y-\frac{5}{8}y+3y$

30.
a. $-\frac{8}{3}z^2-7z^2+3+\frac{3}{4}z^2$
b. $-t+2y+\frac{1}{3}t-\frac{1}{7}x$
c. $-y-2y$
d. $s+\frac{1}{3}-\frac{4}{9}t$

31.
a. $-3z+\frac{2}{3}z+7z-\frac{2}{5}z$
b. $\frac{8}{5}z-\frac{8}{3}x^2+\frac{3}{2}z^2$
c. $-x+\frac{7}{2}t-s$
d. $\frac{5}{9}z-6t$

32.
a. $9z^2+\frac{6}{5}z^2$
b. $\frac{4}{3}t-\frac{4}{3}t$
c. $\frac{3}{2}x+\frac{1}{2}z-\frac{1}{3}y$
d. $\frac{5}{8}s+\frac{3}{4}s+s-\frac{3}{2}s$

1.3 Comparison Symbols and Notation for Intervals

As you know, 8 is larger than 3; that's a specific comparison between two numbers. We can also make a comparison between two less specific numbers, like if we say that average rent in Portland in 2016 is larger than it was in 2009. That makes a comparison using unspecified amounts. In the first half of this section, we will go over the mathematical shorthand notation for making these kinds of comparisons.

In Oregon in 2019, only people who are 18 years old or older can vote in statewide elections.[1] Does that seem like a statement about the number 18? Maybe. But it's also a statement about numbers like 37 and 62: it says that people of these ages may vote as well. So the above is actually a statement about a large collection of numbers, not just 18. In the second half of this section, we will get into the mathematical notation for large collections of numbers like this.

1.3.1 Comparison Symbols

In everyday language you can say something like "8 is larger than 3." In mathematical writing, it's not convenient to write that out in English. Instead the symbol "$>$" has been adopted, and it's used like this:

$$8 > 3$$

and read out loud as "8 is greater than 3." The symbol "$>$" is called the **greater-than symbol.**

Checkpoint 1.3.2

a. Use mathematical notation to write "11.5 is greater than 4.2."

b. Use mathematical notation to write "age is greater than 20."

Explanation.

a. $11.5 > 4.2$

b. We can just write the word age to represent age, and write $\text{age} > 20$. Or we could use an abbreviation like a for age, and write $a > 20$. Or, it is common to use x as a generic abbreviation, and we could write $x > 20$.

Remark 1.3.3 At some point in history, someone felt that $>$ was a good symbol for "is greater than." In "$8 > 3$," the tall side of the symbol is with the larger of the two numbers, and the small pointed side is with the smaller of the two numbers.

Another way to remember how the greater-than symbol works is to imagine the symbol as the open mouth of an alligator, or whatever your favorite animal is. And then remind yourself that the alligator is hungry and it wants to eat the larger number.

We have to be careful when negative numbers are part of the comparison. Is -8 larger or smaller than -3? In some sense -8 is larger, because if you owe someone 8 dollars, that's *more* than owing them 3 dollars. But that is not how the $>$ symbol works. This symbol is meant to tell you which number is farther to the right on a number line. And if that's how it goes, then -3 is larger.

[1] Some states like Washington allow 17-year-olds to vote in primary elections provided they will be 18 by the general election.

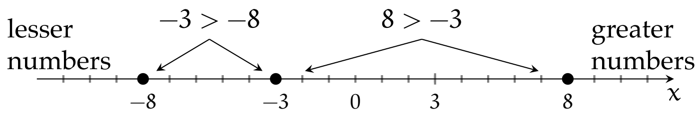
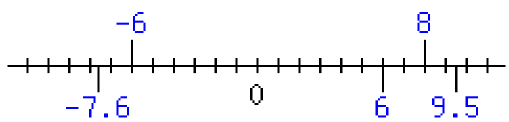


Figure 1.3.4: How the $>$ symbol works.

Checkpoint 1.3.5 Use the $>$ symbol to arrange the following numbers in order from greatest to least. For example, your answer might look like `4 > 3 > 2 > 1 > 0`.

$$-7.6 \quad 6 \quad -6 \quad 9.5 \quad 8$$

Explanation. We can order these numbers by placing these numbers on a number line.

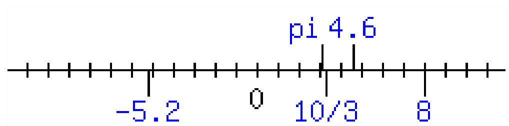


And so we see the answer is $9.5 > 8 > 6 > -6 > -7.6$.

Checkpoint 1.3.6 Use the $>$ symbol to arrange the following numbers in order from greatest to least. For example, your answer might look like `4 > 3 > 2 > 1 > 0`.

$$-5.2 \quad \pi \quad \frac{10}{3} \quad 4.6 \quad 8$$

Explanation. We can order these numbers by placing these numbers on a number line. Knowing or computing their decimals helps with this: $\pi \approx 3.141\ldots$ and $\frac{10}{3} \approx 3.333\ldots$.



And so we see the answer is $8 > 4.6 > 3.33333 > 3.14159 > -5.2$.

The greater-than symbol has a close relative, the **greater-than-or-equal-to symbol**, "$\geq$." It means just like it sounds: the first number is either greater than the second number or equal to it. These are all true statements:

$$8 \geq 3 \qquad 3 \geq -8 \qquad 3 \geq 3$$

but one of these three statements is false:

$$8 > 3 \qquad 3 > -8 \qquad 3 \overset{\text{no}}{>} 3$$

Remark 1.3.7 While it may not seem helpful to write $3 \geq 3$ when you could write $3 = 3$, the $\geq$ symbol is quite useful when specific numbers aren't used on at least one side, like in these examples:

$$(\text{hourly pay rate}) \geq (\text{minimum wage}) \qquad (\text{age of a voter}) \geq 18$$

Sometimes you want to emphasize that one number is *less than* another number instead of emphasizing which number is greater. To do this, we have symbols that are reversed from $>$ and $\geq$. The symbol "$<$" is the **less-than symbol** and it's used like this:

$$3 < 8$$

and read out loud as "3 is less than 8." To help remember which symbol is the "less than" sign and which is the "greater than" sign, notice that you can make a *L*ess than sign with your *L*eft hand.

Table 1.3.8 gives the complete list of all six comparison symbols. Note that we've only discussed three in this section so far, but you already know the equals symbol and have likely also seen the symbol "$\neq$," which means "not equal to."

Symbol	Means	True	True	False
$=$	equals	$13 = 13$	$\frac{5}{4} = 1.25$	$5 \stackrel{\text{no}}{=} 6$
$>$	is greater than	$13 > 11$	$\pi > 3$	$9 \stackrel{\text{no}}{>} 9$
$\geq$	is greater than or equal to	$13 \geq 11$	$3 \geq 3$	$10.2 \stackrel{\text{no}}{\geq} 11.2$
$<$	is less than	$-3 < 8$	$\frac{1}{2} < \frac{2}{3}$	$2 \stackrel{\text{no}}{<} -2$
$\leq$	is less than or equal to	$-3 \leq 8$	$3 \leq 3$	$\frac{4}{5} \stackrel{\text{no}}{\leq} \frac{3}{5}$
$\neq$	is not equal to	$10 \neq 20$	$\frac{1}{2} \neq 1.2$	$\frac{3}{8} \stackrel{\text{no}}{\neq} 0.375$

Table 1.3.8: Comparison Symbols

1.3.2 Set-Builder and Interval Notation

If you say

$$(\text{age of a voter}) \geq 18$$

and have a particular voter in mind, what is that person's age? *Maybe* they are 18, but maybe they are older. It's helpful to use a variable a to represent age (in years) and then to visualize the possibilities with a number line, as in Figure 1.3.9.

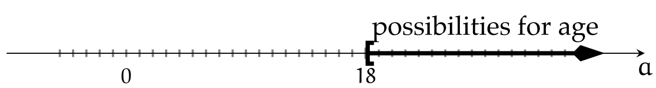


Figure 1.3.9: $(\text{age of a voter}) \geq 18$

The shaded portion of the number line in Figure 1.3.9 is a mathematical **interval**. For now, that just means a collection of certain numbers. In this case, it's all the numbers 18 and above.

The number line in Figure 1.3.9 is a *graphical* representation of a collection of certain numbers. We have two notations, set-builder notation and interval notation, that we also use to represent such collections of numbers.

Definition 1.3.10 Set-Builder Notation. Set-builder notation attempts to directly tell you the condition that numbers in the interval satisfy. In general, we write set-builder notation like:

$$\{x \mid \text{condition on } x\}$$

and read it out loud as "the set of all x such that …." For example,

$$\{x \mid x \geq 18\}$$

is read out loud as "the set of all x such that x is greater than or equal to 18." The breakdown is as follows.

$\{x \mid x \geq 18\}$	the set of
$\{x \mid x \geq 18\}$	all x
$\{x \mid x \geq 18\}$	such that
$\{x \mid x \geq 18\}$	x is greater than or equal to 18

◊

Definition 1.3.11 Interval Notation. Interval notation describes a collection of numbers by telling you where the collection "starts" and "stops". For example, in Figure 1.3.9, the interval starts at 18. To the right, the interval extends forever and has no end, so we use the ∞ symbol (meaning "infinity"). This particular interval is denoted:

$$[18, \infty)$$

Why use "[" on one side and ")" on the other? The square bracket tells us that 18 *is* part of the interval and the round parenthesis tells us that ∞ is *not* part of the interval. (And how could it be, since ∞ is not even a number?)

There are four types of infinite intervals. Take note of the different uses of round parentheses and square brackets.

Figure 1.3.12: An **open, infinite** interval denoted by (a, ∞) means all numbers a or larger, *not* including a.

Figure 1.3.13: A **closed, infinite** interval denoted by $[a, \infty)$ means all numbers a or larger, *including* a.

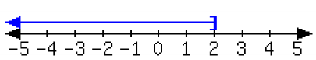
Figure 1.3.14: An **open, infinite** interval denoted by $(-\infty, a)$ means all numbers a or smaller, *not* including a.

Figure 1.3.15: A **closed, infinite** interval denoted by $(-\infty, a]$ means all numbers a or smaller, *including* a.

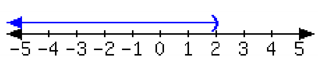
◊

Checkpoint 1.3.16 Interval and Set-Builder Notation from Number Lines. For each interval expressed in the number lines, give the interval notation and set-builder notation.

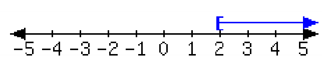
a.



b.



c.



Explanation.

a. Since all numbers less than or equal to 2 are shaded, the set-builder notation is $\{x \mid x \leq 2\}$. The shaded interval "starts" at $-\infty$ and ends at 2 (including 2) so the interval notation is $(-\infty, 2]$.

b. Since all numbers less than to 2 are shaded, the set-builder notation is $\{x \mid x < 2\}$. The shaded interval "starts" at $-\infty$ and ends at 2 (excluding 2) so the interval notation is $(-\infty, 2)$.

c. Since all numbers greater than or equal to 2 are shaded, the set-builder notation is $\{x \mid x \geq 2\}$. The shaded interval starts at 2 (including 2) and "ends" at ∞, so the interval notation is $[2, \infty)$.

Remark 1.3.17 Alternative Convention for Sketching Intervals. When graphing an interval, an alternative convention is to use open circles and filled-in circles. An open circle can be used in place of a round parenthesis, and a filled-in circle can be used in place of a square bracket, as in this example which corresponds to the interval $(a, b]$.

1.3.3 Reading Questions

1. How many inequality symbols are there?

2. What is the difference between the interval $[3, \infty)$ and the interval $(3, \infty)$? More generally, what do square brackets and round parentheses mean in the context of interval notation?

3. The set of numbers $\{x \mid x \leq 10\}$ is being expressed using $\boxed{}$ notation.

1.3.4 Exercises

Review and Warmup

1. Write the decimal number as a fraction.
0.65

2. Write the decimal number as a fraction.
0.75

3. Write the decimal number as a fraction.
8.85

4. Write the decimal number as a fraction.
9.45

5. Write the decimal number as a fraction.
0.108

6. Write the decimal number as a fraction.
0.274

7. Write the fraction as a decimal number.

a. $\frac{4}{5}$

b. $\frac{7}{8}$

8. Write the fraction as a decimal number.

a. $\frac{5}{8}$

b. $\frac{1}{4}$

9. Write the mixed number as a decimal number.

a. $3\frac{5}{16} = \boxed{}$

b. $1\frac{11}{20} = \boxed{}$

10. Write the mixed number as a decimal number.

a. $3\frac{1}{16} = \boxed{}$

b. $1\frac{1}{8} = \boxed{}$

Ordering Numbers Use the $>$ symbol to arrange the following numbers in order from greatest to least. For example, your answer might look like `4 > 3 > 2 > 1 > 0`.

11.
$$5 \quad -10 \quad -1 \quad 7 \quad -5$$

12.
$$7 \quad 3 \quad 8 \quad -1 \quad -6$$

13.
$$9.02 \quad -3.9 \quad -4.17 \quad -4.71 \quad -4.81$$

14.
$$-8.8 \quad 8.8 \quad 3.47 \quad 7.64 \quad -8.7$$

15.
$$-7 \quad 3 \quad -\frac{39}{5} \quad 5 \quad -\frac{33}{7}$$

16.
$$-\frac{31}{4} \quad \frac{1}{2} \quad \frac{26}{3} \quad -9 \quad -7$$

17.
$$\frac{1}{3} \quad 9 \quad \pi \quad -2 \quad \frac{2}{3} \quad \frac{\pi}{2}$$

18.
$$2 \quad \frac{1}{3} \quad \frac{\pi}{2} \quad 0 \quad \sqrt{2} \quad \frac{1}{2}$$

True/False

19. Decide if each comparison is true or false.

a. $-8 = -9$ ($\square$ True $\square$ False)

b. $-1 \neq -1$ ($\square$ True $\square$ False)

c. $-6 > -6$ ($\square$ True $\square$ False)

d. $-10 \neq 9$ ($\square$ True $\square$ False)

e. $7 \geq -5$ ($\square$ True $\square$ False)

f. $2 = 2$ ($\square$ True $\square$ False)

20. Decide if each comparison is true or false.

a. $-8 \leq -8$ ($\square$ True $\square$ False)

b. $-7 \geq -7$ ($\square$ True $\square$ False)

c. $-7 \leq 5$ ($\square$ True $\square$ False)

d. $0 < -4$ ($\square$ True $\square$ False)

e. $7 > 7$ ($\square$ True $\square$ False)

f. $-10 \neq -10$ ($\square$ True $\square$ False)

21. Decide if each comparison is true or false.

a. $-\frac{1}{9} > -\frac{2}{18}$ ($\square$ True $\square$ False)

b. $-\frac{20}{6} = \frac{20}{6}$ ($\square$ True $\square$ False)

c. $\frac{0}{7} \neq \frac{0}{7}$ ($\square$ True $\square$ False)

d. $\frac{6}{9} = \frac{6}{9}$ ($\square$ True $\square$ False)

e. $-\frac{10}{5} < -\frac{20}{10}$ ($\square$ True $\square$ False)

f. $-\frac{5}{2} > \frac{43}{8}$ ($\square$ True $\square$ False)

22. Decide if each comparison is true or false.

a. $-\frac{10}{3} \geq -\frac{30}{9}$ ($\square$ True $\square$ False)

b. $\frac{31}{4} = \frac{11}{6}$ ($\square$ True $\square$ False)

c. $\frac{5}{3} < \frac{19}{8}$ ($\square$ True $\square$ False)

d. $\frac{8}{4} = \frac{16}{8}$ ($\square$ True $\square$ False)

e. $-\frac{62}{7} \leq \frac{47}{8}$ ($\square$ True $\square$ False)

f. $\frac{43}{5} \neq -\frac{3}{2}$ ($\square$ True $\square$ False)

Comparisons Choose $<$, $>$, or $=$ to make a true statement.

23. $-\dfrac{9}{8}$ ($\square <$ $\square >$ $\square =$) $-\dfrac{5}{3}$

24. $-\dfrac{8}{5}$ ($\square <$ $\square >$ $\square =$) $-\dfrac{2}{3}$

25. $\frac{2}{3} + \frac{5}{2}$ ($\square <$ $\square >$ $\square =$) $\frac{3}{4} \div \frac{3}{5}$

26. $\frac{2}{5} + \frac{4}{3}$ ($\square <$ $\square >$ $\square =$) $\frac{1}{5} \div \frac{4}{5}$

27. $\frac{10}{9} \div \frac{10}{9}$ ($\square <$ $\square >$ $\square =$) $\frac{4}{10} - \frac{2}{5}$

28. $\frac{12}{11} \div \frac{12}{11}$ ($\square <$ $\square >$ $\square =$) $\frac{5}{10} - \frac{1}{2}$

29. $-5\frac{2}{3}$ ($\square <$ $\square >$ $\square =$) -5

30. $-9\frac{1}{2}$ ($\square <$ $\square >$ $\square =$) -9

31. $-9\frac{1}{2}$ ($\square <$ $\square >$ $\square =$) 3

32. $-2\frac{1}{3}$ ($\square <$ $\square >$ $\square =$) 1

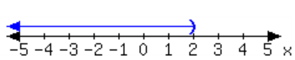
33. $\left|-\dfrac{3}{4}\right|$ ($\square <$ $\square >$ $\square =$) $|0.75|$

34. $\left|-\dfrac{2}{5}\right|$ ($\square <$ $\square >$ $\square =$) $|0.4|$

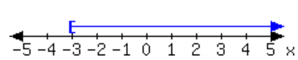
Set-builder and Interval Notation For each interval expressed in the number lines, give the interval notation and set-builder notation.

35.

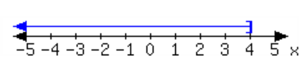
a.



b.

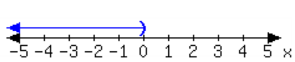


c.

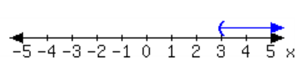


36.

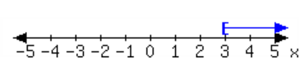
a.



b.

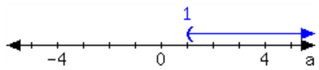


c.



For the interval expressed in the number line, write it using set-builder notation and interval notation.

37.



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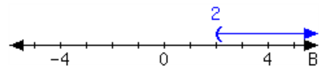

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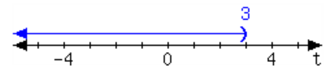

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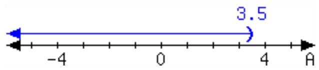

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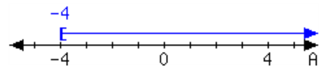

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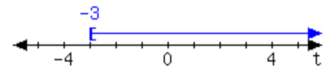

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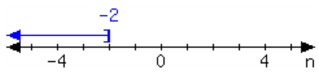

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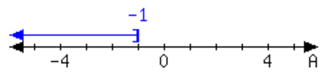


Convert to Interval Notation A set is written using set-builder notation. Write it using interval notation.

45. $\{x \mid x \leq -7\}$ **46.** $\{x \mid x \leq 2\}$ **47.** $\{x \mid x \geq 5\}$ **48.** $\{x \mid x \geq 7\}$

49. $\{x \mid x < 9\}$ **50.** $\{x \mid x < -9\}$ **51.** $\{x \mid x > -7\}$ **52.** $\{x \mid x > -5\}$

53. $\{x \mid -3 > x\}$ **54.** $\{x \mid -9 > x\}$ **55.** $\{x \mid 2 \geq x\}$ **56.** $\{x \mid 4 \geq x\}$

57. $\{x \mid 7 \leq x\}$ **58.** $\{x \mid 9 \leq x\}$ **59.** $\{x \mid -10 < x\}$ **60.** $\{x \mid -7 < x\}$

61. $\left\{x \mid \frac{3}{7} < x\right\}$ **62.** $\left\{x \mid \frac{4}{7} < x\right\}$ **63.** $\left\{x \mid x \leq -\frac{5}{3}\right\}$ **64.** $\left\{x \mid x \leq -\frac{6}{7}\right\}$

65. $\{x \mid x \leq 0\}$ **66.** $\{x \mid 0 < x\}$

1.4 Equations and Inequalities as True/False Statements

This section introduces the concepts of algebraic *equations* and *inequalities*, and what it means for a number to be a *solution* to an equation or inequality.

1.4.1 Equations, Inequalities, and Solutions

An **equation** is two algebraic expressions with an equals sign between them. The two expressions can be relatively simple or more complicated:

A simple equation:

$$x + 1 = 2$$

A more complicated equation:

$$\left(x^2 + y^2 - 1\right)^3 = x^2y^3$$

An **inequality** is similar to an equation, but the sign between the expressions is $<$, $\leq$, $>$, $\geq$, or $\neq$.

A simple inequality:

$$x \geq 15$$

A more complicated inequality:

$$x^2 + y^2 < 1$$

The simplest equations and inequalities have numbers and no variables. When this happens, the equation is either *true* or *false*. The following equations and inequalities are *true* statements:

$$2 = 2 \qquad -4 = -4 \qquad 2 > 1 \qquad -2 < -1 \qquad 3 \geq 3$$

The following equations and inequalities are *false* statements:

$$2 = 1 \qquad -4 = 4 \qquad 2 < 1 \qquad -2 \geq -1 \qquad 0 \neq 0$$

There will be times when doing algebra will lead us to an equation like $2 = 1$, which of course we know to be a false equation. To recognize that this is false, we will write $1 \stackrel{\text{no}}{=} 2$. This is different from writing $1 \neq 2$, because that is a *true* inequality. And when we want to explicitly recognize that an equation or inequality is true, we will use a checkmark, like with $2 \stackrel{\checkmark}{=} 2$.

A **linear expression** in one variable is an expression in the form $ax + b$, where a and b are numbers, $a \neq 0$, and x is a variable. For example, $2x + 1$ and $3y + \frac{1}{2}$ are linear expressions.

The following examples are a little harder to identify as linear expressions in one variable, but they are.

- $2x$ is linear, with $b = 0$.
- $y + 3$ is linear, with $a = 1$.
- $17 - q$ is linear, with $a = -1$, $b = 17$ and the two terms are written in reverse order.
- $2.1t + 3 + 8t - 1.4$ is linear (because it simplifies to $10.1t + 1.6$).

Definition 1.4.2 Linear Equation and Linear Inequality. A **linear equation** in one variable is any equation where one side is a linear expression in that variable, and the other side is either a constant number, or is another linear expression in that variable. A **linear inequality** in one variable is defined similarly, just with an inequality symbol instead of an equals sign. ◊

The following are some linear equations in one variable:

$$4 - y = 5 \qquad 4 - z = 5z \qquad 0 = \frac{1}{2}p$$

$$3 - 2(q + 2) = 10 \qquad \sqrt{2}r + 3 = 10 \qquad \frac{s}{2} + 3 = 5$$

(Note that r is outside the square root symbol.)

In a linear equation in one variable, the variable cannot appear with an exponent (other than 1 or 0), and the variable cannot be inside a root symbol (square root, cube root, etc.), absolute value bars, or in a denominator.

The following are not linear equations in one variable:

$$\begin{aligned} 1 + 2 &= 3 && \text{(There is no variable.)} \\ 4 - 2y^2 &= 5 && \text{(The exponent of } y \text{ is 2.)} \\ \sqrt{2r} + 3 &= 10 && (r \text{ is inside the square root.)} \\ \frac{2}{s} + 3 &= 5 && (s \text{ is in a denominator.)} \end{aligned}$$

Equations arise from real-world math problems, sometimes from simple problems, and sometimes from hard ones.

Example 1.4.3 A parking meter requires you pay \$2.50 for one hour. You have been inserting quarters, dimes, and nickels into the meter, and it says that you have inserted \$1.85. How much more do you need to pay?

You might have a simple way to answer that question, using subtraction. But there is an equation hidden in this story. Since we are asked "How much more do you need to pay?", let's use a variable to represent that: x. We've already paid \$1.85, and in total we need to pay \$2.50. So we need

$$1.85 + x = 2.50$$

This is an equation arising from this scenario.

With the equation in Example 1.4.3, if we substitute in 0.65 for x, the resulting equation is true.

$$1.85 + 0.65 \stackrel{\checkmark}{=} 2.50$$

If we substitute in any other number for x, the resulting equation is false. This motivates what it means to be a *solution* to an equation.

Definition 1.4.4 When an equation (or inequality) has one variable, a **solution** is any number that you could substitute in for the variable that would result in a true equation (or inequality). ◊

Example 1.4.5 A Solution. Consider the equation $y + 2 = 3$, which has only one variable, y. If we substitute in 1 for y and then simplify:

$$\begin{aligned} y + 2 &= 3 \\ 1 + 2 &\stackrel{?}{=} 3 \\ 3 &\stackrel{\checkmark}{=} 3 \end{aligned}$$

we get a true equation. So we say that 1 is a solution to $y + 2 = 3$. Notice that we used a question mark at first because we are unsure if the equation is true or false until the end.

If replacing a variable with a value makes a false equation or inequality, that number is not a solution.

Example 1.4.6 Not a Solution. Consider the inequality $x + 4 > 5$, which has only one variable, x. If we substitute in 0 for x and then simplify:

$$\begin{aligned} x + 4 &> 5 \\ 0 + 4 &\stackrel{?}{>} 5 \\ 4 &\stackrel{\text{no}}{>} 5 \end{aligned}$$

we get a false inequality. So we say that 0 is *not* a solution to $x + 4 > 5$.

Example 1.4.7 Allowing Variables to Vary. With the help of technology, it is possible to quickly evaluate expressions as variables vary. In the GeoGebra applet in Figure 1.4.8, you may slide the value of q and see how a computer can quickly calculate each side of the equation to determine if that value of q is a solution.

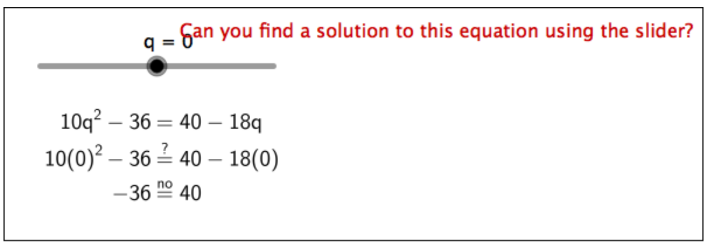


Figure 1.4.8: Allowing Variables to Vary

1.4.2 Checking Possible Solutions

Given an equation or an inequality (with one variable), checking if some particular number is a solution is just a matter of replacing the value of the variable with the specified number and determining if the resulting equation/inequality is true or false. This may involve some arithmetic and simplification.

Example 1.4.9 Is 8 a solution to $x^2 - 5x = \sqrt{2x} + 20$?

To find out, substitute in 8 for x and see what happens.

$$\begin{aligned} x^2 - 5x &= \sqrt{2x} + 20 \\ 8^2 - 5(8) &\stackrel{?}{=} \sqrt{2(8)} + 20 \\ 64 - 5(8) &\stackrel{?}{=} \sqrt{16} + 20 \\ 64 - 40 &\stackrel{?}{=} 4 + 20 \\ 24 &\stackrel{\checkmark}{=} 24 \end{aligned}$$

So yes, 8 is a solution to $x^2 - 5x = \sqrt{2x} + 20$.

Example 1.4.10 Is -5 a solution to $\sqrt{169-y^2} = y^2 - 2y$?

To find out, substitute in -5 for y and see what happens.

$$\begin{aligned}
\sqrt{169-y^2} &= y^2 - 2y \\
\sqrt{169-(-5)^2} &\stackrel{?}{=} (-5)^2 - 2(-5) \\
\sqrt{169-25} &\stackrel{?}{=} 25 - 2(-5) \\
\sqrt{144} &\stackrel{?}{=} 25 - (-10) \\
12 &\stackrel{\text{no}}{=} 35
\end{aligned}$$

So no, -5 is not a solution to $\sqrt{169-y^2} = y^2 - 2y$.

But is -5 a solution to the *inequality* $\sqrt{169-y^2} \le y^2 - 2y$? Yes, because substituting -5 in for y would give you

$$12 \le 35,$$

which is true.

Checkpoint 1.4.11 Is -3 a solution for x in the equation $2x - 3 = 5 - (4 + x)$? Evaluating the left and right sides gives:

$$\begin{array}{ccc}
2x-3 & = & 5-(4+x) \\
____ & \stackrel{?}{=} & ____
\end{array}$$

So -3 (☐ is ☐ is not) a solution to $2x - 3 = 5 - (4 + x)$.

Explanation. We will substitute x with -3 in the equation and simplify each side to determine if the statement is true or false:

$$\begin{aligned}
2x - 3 &= 5 - (4 + x) \\
2(-3) - 3 &\stackrel{?}{=} 5 - (4 + (-3)) \\
-6 - 3 &\stackrel{?}{=} 5 - (1) \\
-9 &\stackrel{\text{no}}{=} 4
\end{aligned}$$

Since $-9 = 4$ is not true, -3 is not a solution for x in the equation $2x - 3 = 5 - (4 + x)$.

Checkpoint 1.4.12 Is $\frac{1}{3}$ a solution for t in the equation $2t = 4\left(t - \frac{1}{2}\right)$? Evaluating the left and right sides gives:

$$\begin{array}{ccc}
2t & = & 4\left(t-\frac{1}{2}\right) \\
____ & \stackrel{?}{=} & ____
\end{array}$$

So $\frac{1}{3}$ (☐ is ☐ is not) a solution to $2t = 4\left(t - \frac{1}{2}\right)$.

Explanation. We will substitute t with $\frac{1}{3}$ in the equation and simplify each side to determine if the statement is true or false:

$$\begin{aligned}
2t &= 4\left(t - \frac{1}{2}\right) \\
2\left(\frac{1}{3}\right) &\stackrel{?}{=} 4\left(\frac{1}{3} - \frac{1}{2}\right)
\end{aligned}$$

This is not going to be true, since the left side is positive and the right side is negative. So $\frac{1}{3}$ is not a solution for t in the equation $2t = 4\left(t - \frac{1}{2}\right)$.

Checkpoint 1.4.13 Is -2 a solution to $y^2 + y - 5 \leq y - 1$? Evaluating the left and right sides gives:

$$\begin{array}{ccc} y^2 + y - 5 & \leq & y - 1 \\ \_\_\_\_ & \stackrel{?}{\leq} & \_\_\_\_ \end{array}$$

So -2 (☐ is ☐ is not) a solution to $y^2 + y - 5 \leq y - 1$.

Explanation. We will substitute y with -2 in the inequality and simplify each side to determine if the statement is true or false:

$$\begin{aligned} y^2 + y - 5 &\leq y - 1 \\ (-2)^2 + (-2) - 5 &\stackrel{?}{\leq} -2 - 1 \\ 4 - 2 - 5 &\stackrel{?}{\leq} -3 \\ 2 - 5 &\stackrel{?}{\leq} -3 \\ -3 &\stackrel{\checkmark}{\leq} -3 \end{aligned}$$

This is true. So -2 is a solution for y in the inequality $y^2 + y - 5 \leq y - 1$.

Checkpoint 1.4.14 Is 2 a solution to $\frac{z+3}{z-1} = \sqrt{18z}$? Evaluating the left and right sides gives:

$$\begin{array}{ccc} \dfrac{z+3}{z-1} & = & \sqrt{18z} \\ \_\_\_\_ & \stackrel{?}{=} & \_\_\_\_ \end{array}$$

So 2 (☐ is ☐ is not) a solution to $\frac{z+3}{z-1} = \sqrt{18z}$.

Explanation. We will substitute z with 2 in the equation and simplify each side to determine if the statement is true or false:

$$\begin{aligned} \frac{z+3}{z-1} &= \sqrt{18z} \\ \frac{2+3}{2-1} &\stackrel{?}{=} \sqrt{18(2)} \\ \frac{5}{1} &\stackrel{?}{=} \sqrt{36} \\ 5 &\stackrel{\text{no}}{=} 6 \end{aligned}$$

The equation is false. So 2 is not a solution for z in the equation $\frac{z+3}{z-1} = \sqrt{18z}$.

Checkpoint 1.4.15 Is -3 a solution to $x^2 + x + 1 = \frac{3x+2}{x+2}$? Evaluating the left and right sides gives:

$$\begin{array}{ccc} x^2 + x + 1 & = & \dfrac{3x+2}{x+2} \\ \_\_\_\_ & \stackrel{?}{=} & \_\_\_\_ \end{array}$$

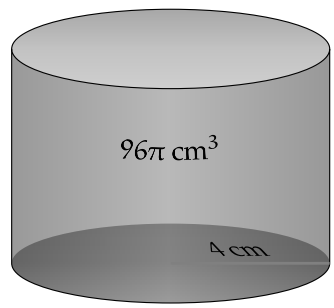
So -3 (☐ is ☐ is not) a solution to $x^2 + x + 1 = \frac{3x+2}{x+2}$.

. We will substitute x with -3 in the equation and simplify each side to determine if the statement is true or false:

$$\begin{aligned} x^2+x+1 &= \frac{3x+2}{x+2} \\ (-3)^2+(-3)+1 &\stackrel{?}{=} \frac{3(-3)+2}{-3+2} \\ 9-3+1 &\stackrel{?}{=} \frac{-9+2}{-1} \\ 6+1 &\stackrel{?}{=} \frac{-7}{-1} \\ 7 &\stackrel{\checkmark}{=} 7 \end{aligned}$$

This is true. So -3 is a solution for x in the inequality $x^2+x+1 \leq \frac{3x+2}{x+2}$.

Example 1.4.16 Cylinder Volume.



A cylinder's volume is related to its radius and its height by:

$$V = \pi r^2 h,$$

where V is the volume, r is the base's radius, and h is the height. If we know the volume is $96\pi\,\text{cm}^3$ and the radius is $4\,\text{cm}$, then we have:

$$96\pi = 16\pi h$$

Is $4\,\text{cm}$ the height of the cylinder? In other words, is 4 a solution to $96\pi = 16\pi h$? We will substitute h in the equation with 4 to check:

$$\begin{aligned} 96\pi &= 16\pi h \\ 96\pi &\stackrel{?}{=} 16\pi \cdot 4 \\ 96\pi &\stackrel{\text{no}}{=} 64\pi \end{aligned}$$

Since $96\pi = 64\pi$ is false, $h = 4$ does *not* satisfy the equation $96\pi = 16\pi h$.

Next, we will try $h = 6$:

$$\begin{aligned} 96\pi &= 16\pi h \\ 96\pi &\stackrel{?}{=} 16\pi \cdot 6 \\ 96\pi &\stackrel{\checkmark}{=} 96\pi \end{aligned}$$

When $h = 6$, the equation $96\pi = 16\pi h$ is true. This tells us that 6 *is* a solution to $96\pi = 16\pi h$.

Remark 1.4.17 Note that we did not approximate π with 3.14 or any other approximation. We often leave π as π throughout our calculations. If we need to round, we do so as a final step.

Example 1.4.18 Jaylen has budgeted a maximum of \$300 for an appliance repair. The total cost of the repair can be modeled by $89 + 110(h - 0.25)$, where \$89 is the initial cost and \$110 is the hourly labor charge after the first quarter hour. Is 2 hours a solution for h in the inequality $89 + 110(h - 0.25) \leq 300$?

To determine if $h = 2$ satisfies the inequality, we will replace h with 2 and check if the statement is true:

$$\begin{aligned} 89 + 110(h - 0.25) &\leq 300 \\ 89 + 110(2 - 0.25) &\stackrel{?}{\leq} 300 \\ 89 + 110(1.75) &\stackrel{?}{\leq} 300 \\ 89 + 192.5 &\stackrel{?}{\leq} 300 \\ 281.5 &\stackrel{\checkmark}{\leq} 300 \end{aligned}$$

So we find that 2 is a solution for h in the inequality $89 + 110(h - 0.25) \leq 300$. In context, this means that Jaylen would stay within their \$300 budget if there is only 2 hours of labor.

1.4.3 Reading Questions

1. Is the equation in Example 1.4.3, $1.85 + x = 2.50$, a linear equation?

2. Give your own example of an equation in one variable that is not a linear equation.

3. Do you believe it is possible for an inequality to have more than one solution? Do you believe it is possible for an equation to have more than one solution?

4. There are two solutions to the equation in Example 1.4.7. What are they?

1.4.4 Exercises

Review and Warmup

1. Evaluate $6 - x$ for $x = 0$.

2. Evaluate $-1 - x$ for $x = 2$.

3. Evaluate $-8x + 5$ for $x = 4$.

4. Evaluate $5x - 8$ for $x = 7$.

5. Evaluate $-5(t + 9)$ for $t = -2$.

6. Evaluate $-(x + 6)$ for $x = -9$.

7. Evaluate the expression $\frac{1}{7}(x+1)^2 - 7$ when $x = -8$.

8. Evaluate the expression $\frac{1}{3}(x+2)^2 - 4$ when $x = -5$.

9. Evaluate the expression $-16t^2 + 64t + 128$ when $t = -2$.

10. Evaluate the expression $-16t^2 + 64t + 128$ when $t = -4$.

Identifying Linear Equations and Inequalities

11. Are the equations below linear equations in one variable?

a. $-4.12z = 1$ (☐ is ☐ is not) a linear equation in one variable.

b. $7 + 4y^2 = 24$ (☐ is ☐ is not) a linear equation in one variable.

c. $\sqrt{1 - 0.5p} = 9$ (☐ is ☐ is not) a linear equation in one variable.

d. $x - 8z^2 = -11$ (☐ is ☐ is not) a linear equation in one variable.

e. $4q + 8 = 0$ (☐ is ☐ is not) a linear equation in one variable.

f. $2\pi r = 4\pi$ (☐ is ☐ is not) a linear equation in one variable.

12. Are the equations below linear equations in one variable?

a. $\sqrt{-3.3z - 8} = 1$ (☐ is ☐ is not) a linear equation in one variable.

b. $1.55z = 4$ (☐ is ☐ is not) a linear equation in one variable.

c. $9z - 2V^2 = -26$ (☐ is ☐ is not) a linear equation in one variable.

d. $7V^2 - 6 = 9$ (☐ is ☐ is not) a linear equation in one variable.

e. $6q - 16 = -1$ (☐ is ☐ is not) a linear equation in one variable.

f. $2\pi r = 10\pi$ (☐ is ☐ is not) a linear equation in one variable.

13. Are the equations below linear equations in one variable?

a. $V\sqrt{30} = 23$ (☐ is ☐ is not) a linear equation in one variable.

b. $-0.44r = -7$ (☐ is ☐ is not) a linear equation in one variable.

c. $q^2 + z^2 = 34$ (☐ is ☐ is not) a linear equation in one variable.

d. $\pi r^2 = 99\pi$ (☐ is ☐ is not) a linear equation in one variable.

e. $4prV = -27$ (☐ is ☐ is not) a linear equation in one variable.

f. $6 - 3p = -21$ (☐ is ☐ is not) a linear equation in one variable.

14. Are the equations below linear equations in one variable?

a. $z^2 + y^2 = -45$ (☐ is ☐ is not) a linear equation in one variable.

b. $V\sqrt{30} = -64$ (☐ is ☐ is not) a linear equation in one variable.

c. $9Vyz = -18$ (☐ is ☐ is not) a linear equation in one variable.

d. $-2.43V = -52$ (☐ is ☐ is not) a linear equation in one variable.

e. $15z - 1 = -34$ (☐ is ☐ is not) a linear equation in one variable.

f. $\pi r^2 = 33\pi$ (☐ is ☐ is not) a linear equation in one variable.

15. Are the inequalities below linear inequalities in one variable?

a. $-4x^2 - 3z^2 > 1$ (☐ is ☐ is not) a linear inequality in one variable.

b. $-2 \geq 5 - 10p$ (☐ is ☐ is not) a linear inequality in one variable.

c. $6x^2 - 8V > -81$ (☐ is ☐ is not) a linear inequality in one variable.

16. Are the inequalities below linear inequalities in one variable?

a. $-3y^2 - 6q \leq 44$ (☐ is ☐ is not) a linear inequality in one variable.

b. $2 > 5 - 14x$ (☐ is ☐ is not) a linear inequality in one variable.

c. $2p^2 + 6y^2 < 1$ (☐ is ☐ is not) a linear inequality in one variable.

17. Are the inequalities below linear inequalities in one variable?

a. $-3.9y < 80$ (☐ is ☐ is not) a linear inequality in one variable.

b. $\sqrt{4r} - 14 < 5$ (☐ is ☐ is not) a linear inequality in one variable.

c. $129 \leq -5144y - 2965q$ (☐ is ☐ is not) a linear inequality in one variable.

18. Are the inequalities below linear inequalities in one variable?

a. $-4.2z \geq -58$ (☐ is ☐ is not) a linear inequality in one variable.

b. $-73 \leq 3916t + 9643p$ (☐ is ☐ is not) a linear inequality in one variable.

c. $\sqrt{4y} + 2 \leq 4$ (☐ is ☐ is not) a linear inequality in one variable.

Checking a Solution for an Equation

19. Is -1 a solution for x in the equation $x - 2 = -1$? (☐ Yes ☐ No)

20. Is 1 a solution for x in the equation $x - 7 = -5$? (☐ Yes ☐ No)

21. Is 7 a solution for r in the equation $-6 - r = -13$? (☐ Yes ☐ No)

22. Is 3 a solution for t in the equation $-8 - t = -11$? (☐ Yes ☐ No)

23. Is -7 a solution for t in the equation $-9t + 7 = 70$? (☐ Yes ☐ No)

24. Is 6 a solution for x in the equation $x - 6 = 0$? (☐ Yes ☐ No)

25. Is -2 a solution for x in the equation $8x - 5 = -7x - 20$? (☐ Yes ☐ No)

26. Is -8 a solution for y in the equation $-4y + 10 = -7y - 14$? (☐ Yes ☐ No)

27. Is 7 a solution for y in the equation $8(y + 11) = 19y$? (☐ Yes ☐ No)

28. Is -3 a solution for y in the equation $3(y - 8) = 11y$? (☐ Yes ☐ No)

29. Is -10 a solution for r in the equation $4(r - 13) = 11(r + 1)$? (☐ Yes ☐ No)

30. Is -4 a solution for r in the equation $14(r + 1) = 5(r + 10)$? (☐ Yes ☐ No)

31. Is $\frac{1}{3}$ a solution for x in the equation $6x - 3 = -2$? (☐ Yes ☐ No)

32. Is $\frac{17}{9}$ a solution for x in the equation $9x - 10 = 7$? (☐ Yes ☐ No)

33. Is $\frac{5}{2}$ a solution for x in the equation $-\frac{2}{3}x + 1 = 0$? (☐ Yes ☐ No)

34. Is $-\frac{5}{6}$ a solution for x in the equation $-\frac{4}{3}x - \frac{2}{3} = \frac{4}{9}$? (☐ Yes ☐ No)

35. Is 3 a solution for x in the equation $-\frac{10}{3}x - 8 = \frac{9}{4}x - \frac{355}{36}$? (☐ Yes ☐ No)

36. Is $-\frac{2}{9}$ a solution for y in the equation $\frac{10}{3}y + \frac{9}{4} = -\frac{1}{4}y - \frac{111}{8}$? (☐ Yes ☐ No)

Checking a Solution for an Inequality Decide whether each value is a solution to the given inequality.

37. $-3x + 24 > 9$

a. $x = 5$ (☐ is ☐ is not) a solution.

b. $x = -5$ (☐ is ☐ is not) a solution.

c. $x = 0$ (☐ is ☐ is not) a solution.

d. $x = 13$ (☐ is ☐ is not) a solution.

38. $4x - 5 > 3$

a. $x = 0$ (☐ is ☐ is not) a solution.

b. $x = -8$ (☐ is ☐ is not) a solution.

c. $x = 5$ (☐ is ☐ is not) a solution.

d. $x = 2$ (☐ is ☐ is not) a solution.

39. $4x - 18 \geq -2$

a. $x = 3$ (☐ is ☐ is not) a solution.

b. $x = 0$ (☐ is ☐ is not) a solution.

c. $x = 4$ (☐ is ☐ is not) a solution.

d. $x = 12$ (☐ is ☐ is not) a solution.

40. $-5x - 3 \geq -8$

a. $x = 5$ (☐ is ☐ is not) a solution.

b. $x = -7$ (☐ is ☐ is not) a solution.

c. $x = 1$ (☐ is ☐ is not) a solution.

d. $x = 0$ (☐ is ☐ is not) a solution.

41. $5x - 8 \leq 7$

a. $x = 2$ (☐ is ☐ is not) a solution.

b. $x = 11$ (☐ is ☐ is not) a solution.

c. $x = 0$ (☐ is ☐ is not) a solution.

d. $x = 3$ (☐ is ☐ is not) a solution.

42. $2x - 9 \leq 1$

a. $x = 5$ (☐ is ☐ is not) a solution.

b. $x = 4$ (☐ is ☐ is not) a solution.

c. $x = 0$ (☐ is ☐ is not) a solution.

d. $x = 8$ (☐ is ☐ is not) a solution.

Checking Solutions for Application Problems

43. A triangle's area is 66 square meters. Its height is 12 meters. Suppose we wanted to find how long is the triangle's base. A triangle's area formula is

$$A = \frac{1}{2}bh$$

where A stands for area, b for base and h for height. If we let b be the triangle's base, in meters, we can solve this problem using the equation:

$$66 = \frac{1}{2}(b)(12)$$

Check whether 11 is a solution for b of this equation. (☐ Yes ☐ No)

44. A triangle's area is 114 square meters. Its height is 19 meters. Suppose we wanted to find how long is the triangle's base. A triangle's area formula is

$$A = \frac{1}{2}bh$$

where A stands for area, b for base and h for height. If we let b be the triangle's base, in meters, we can solve this problem using the equation:

$$114 = \frac{1}{2}(b)(19)$$

Check whether 24 is a solution for b of this equation. (☐ Yes ☐ No)

45. When a plant was purchased, it was 2 inches tall. It grows 0.5 inches per day. How many days later will the plant be 8 inches tall?
Assume the plant will be 8 inches tall d days later. We can solve this problem using the equation:

$$0.5d + 2 = 8$$

Check whether 15 is a solution for d of this equation. (☐ Yes ☐ No)

46. When a plant was purchased, it was 1.3 inches tall. It grows 0.6 inches per day. How many days later will the plant be 11.5 inches tall?
Assume the plant will be 11.5 inches tall d days later. We can solve this problem using the equation:

$$0.6d + 1.3 = 11.5$$

Check whether 19 is a solution for d of this equation. (☐ Yes ☐ No)

47. A water tank has 283 gallons of water in it, and it is being drained at the rate of 14 gallons per minute. After how many minutes will there be 31 gallons of water left?
Assume the tank will have 31 gallons of water after m minutes. We can solve this problem using the equation:

$$283 - 14m = 31$$

Check whether 19 is a solution for m of this equation. (☐ Yes ☐ No)

48. A water tank has 264 gallons of water in it, and it is being drained at the rate of 16 gallons per minute. After how many minutes will there be 40 gallons of water left?
Assume the tank will have 40 gallons of water after m minutes. We can solve this problem using the equation:

$$264 - 16m = 40$$

Check whether 17 is a solution for m of this equation. (☐ Yes ☐ No)

49. A cylinder's volume is 162π cubic centimeters. Its height is 18 centimeters. Suppose we wanted to find how long is the cylinder's radius. A cylinder's volume formula is

$$V = \pi r^2 h$$

where V stands for volume, r for radius and h for height. Let r represent the cylinder's radius, in centimeters. We can solve this problem using the equation:

$$162\pi = \pi r^2(18)$$

Check whether 9 is a solution for r of this equation. (☐ Yes ☐ No)

50. A cylinder's volume is 1280π cubic centimeters. Its height is 20 centimeters. Suppose we wanted to find how long is the cylinder's radius. A cylinder's volume formula is

$$V = \pi r^2 h$$

where V stands for volume, r for radius and h for height. Let r represent the cylinder's radius, in centimeters. We can solve this problem using the equation:

$$1280\pi = \pi r^2(20)$$

Check whether 8 is a solution for r of this equation. (☐ Yes ☐ No)

51. A country's national debt was 140 million dollars in 2010. The debt increased at 20 million dollars per year. If this trend continues, when will the country's national debt increase to 640 million dollars?
Assume the country's national debt will become 640 million dollars y years after 2010. We can solve this problem using the equation:

$$20y + 140 = 640$$

Check whether 26 is a solution for y of this equation. (☐ Yes ☐ No)

52. A country's national debt was 100 million dollars in 2010. The debt increased at 20 million dollars per year. If this trend continues, when will the country's national debt increase to 360 million dollars?
Assume the country's national debt will become 360 million dollars y years after 2010. We can solve this problem using the equation:

$$20y + 100 = 360$$

Check whether 13 is a solution for y of this equation. (☐ Yes ☐ No)

53. A school district has a reserve fund worth 32.8 million dollars. It plans to spend 2.2 million dollars per year. After how many years, will there be 13 million dollars left? Assume there will be 13 million dollars left after y years. We can solve this problem using the equation:

$$32.8 - 2.2y = 13$$

Check whether 11 is a solution for y of this equation. (☐ Yes ☐ No)

54. A school district has a reserve fund worth 31.1 million dollars. It plans to spend 2.3 million dollars per year. After how many years, will there be 15 million dollars left? Assume there will be 15 million dollars left after y years. We can solve this problem using the equation:

$$31.1 - 2.3y = 15$$

Check whether 8 is a solution for y of this equation. (☐ Yes ☐ No)

55. A rectangular frame's perimeter is 7 feet. If its length is 2.5 feet, suppose we want to find how long is its width. A rectangle's perimeter formula is

$$P = 2(l + w)$$

where P stands for perimeter, l for length and w for width. We can solve this problem using the equation:

$$7 = 2(2.5 + w)$$

Check whether 1 is a solution for w of this equation. (☐ Yes ☐ No)

56. A rectangular frame's perimeter is 8.8 feet. If its length is 2.6 feet, suppose we want to find how long is its width. A rectangle's perimeter formula is

$$P = 2(l + w)$$

where P stands for perimeter, l for length and w for width. We can solve this problem using the equation:

$$8.8 = 2(2.6 + w)$$

Check whether 6.2 is a solution for w of this equation. (☐ Yes ☐ No)

1.5 Solving One-Step Equations

We have learned how to check whether a specific number is a solution to an equation or inequality. In this section, we will begin learning how to *find* the solution(s) to basic equations ourselves.

1.5.1 Imagine Filling in the Blanks

Let's start with a very simple situation—so simple, that you might have success entirely in your head without writing much down. It's not exactly the algebra we hope you will learn, but the example may serve as a good warm up.

Example 1.5.2 A number plus 2 is 6. What is that number?

You may be so familiar with basic arithmetic that you know the answer already. The *algebra* approach is to translate "A number plus 2 is 6" into a math statement—in this case, an equation:

$$x + 2 = 6$$

where x is the number we are trying to find. And then ask what should be substituted in for x to make the equation true.

Now, how do you determine what x is? One valid option is to just *imagine* what number you could put in place of x that would result in a true equation.

- Would 0 work? No, that would mean $0 + 2 = 6$, which is false.
- Would 17 work? No, that would mean $17 + 2 = 6$, which is false.
- Would 4 work? Yes, because $4 + 2 = 6$ is a true equation.

So one solution to $x + 2 = 6$ is 4. No other numbers are going to be solutions, because when you add 2 to something smaller than 4, the result is going to be smaller than 6, and when you add 2 to something larger than 4, the result is going to be larger than 6.

This approach might work for you to solve *very basic* equations, but in general equations are going to be too complicated to solve in your head this way. So we move on to more systematic approaches.

1.5.2 The Basic Principle of Algebra

Let's revisit Example 1.5.2, thinking it through differently.

Example 1.5.3 If a number plus 2 is 6, what is the number?

If a number *plus* 2 is 6, the number is a little smaller than 6, and we should be able to *subtract* 2 from 6 and get that unknown number. Doing that: $6 - 2 = 4$. Thinking things through this way, we are using the *opposite* operation from addition: subtraction.

Let's try this strategy with another riddle.

Example 1.5.4 If a number minus 2 is 6, what is the number? Now we have a number a little larger than 6 in mind. This time, if we *add* 2 to 6 we will find the unknown number. So the unknown number is $6 + 2 = 8$.

Does this strategy work with multiplication and division?

Example 1.5.5 If a number times 2 is 6, what is the number? The mystery number is small, since it gets multipled by 2 to make 6. If we *divide* 6 by 2, we will find the unknown number. So the unknown number is $\frac{6}{2} = 3$.

Example 1.5.6 If a number divided by 2 equals 6, what is the number? Now we must have had a larger number to start with, since cutting it in half made 6. If we *multiply* 6 by 2, we find the unknown number is $6 \cdot 2 = 12$.

Abbreviation for "pound". Why is "lb" the abbreviation for "pound"? It has a connection to a balance scale, which is the symbol for the Zodiac sign ***Lib**ra*.

These examples explore an important principle for solving an equation—applying an opposite arithmetic operation. We can revisit Example 1.5.2 and more intentionally apply this strategy. If a number plus 2 is 6, what is the number? As is common in algebra, we use x to represent the unknown number. The question translates into the math equation

$$x + 2 = 6.$$

Try to envision the equals sign as the middle of a balanced scale. The left side has 2 one-pound objects and a block with unknown weight x lb. Together, the weight on the left is $x + 2$. The right side has 6 one-pound objects. Figure 1.5.7 shows the scale.

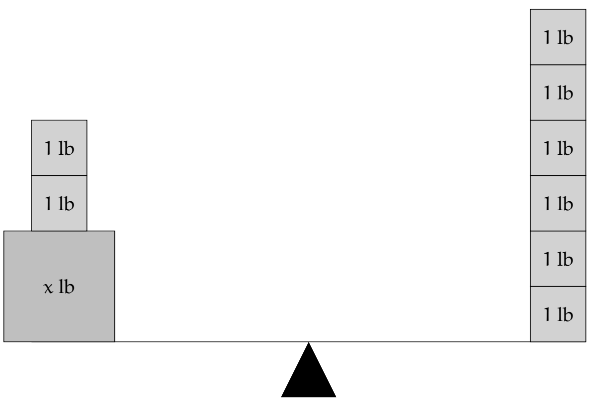


Figure 1.5.7: Balance scale representing $x + 2 = 6$.

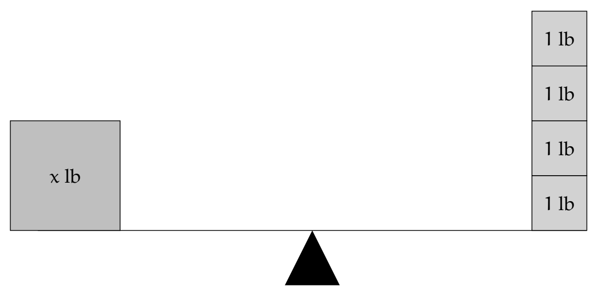


Figure 1.5.8: Balance scale representing the solution to $x + 2 = 6$, after taking away 2 from each side.

To find the weight of the unknown block, we can take away 2 one-pound blocks from *each* side of the scale (to keep the scale balanced). Figure 1.5.8 shows the solution.

An equation is like a balanced scale, as the two sides of the equation are equal. In the same way that we can take away 2 lb from *each* side of a balanced scale, we can subtract 2 from *each* side of the equation. So instead of two pictures of balance scales, we can use algebra symbols and solve the equation $x + 2 = 6$ in the following manner:

$$\begin{aligned} x + 2 &= 6 && \text{a balanced scale} \\ x + 2 - 2 &= 6 - 2 && \text{remove the same quantity from each side} \\ x &= 4 && \text{still balanced; now it tells you the solution} \end{aligned}$$

It's important to note that each line shows what is called an **equivalent equation**. In other words, each equation shown is algebraically equivalent to the one above it and will have exactly the same solution(s). The final equivalent equation $x = 4$ tells us that the **solution** to the equation is 4. The **solution set** to this equation is the set that lists every solution to the equation. For this example, the solution set is $\{4\}$.

In Figure 1.5.9, try adding or subtracting something to each side of the equation to find its solution.

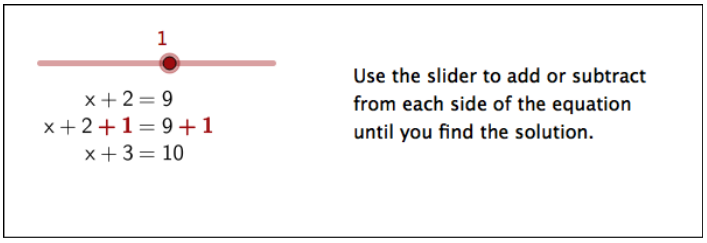


Figure 1.5.9: Allowing Variables to Vary

We have learned we can add or subtract the same number on both sides of the equals sign, just like we can add or remove the same amount of weight on a balanced scale. Can we multiply and divide the same number on both sides of the equals sign? Let's look at Example 1.5.5 again: If a number times 2 is 6, what is the number? Another balance scale can help visualize this.

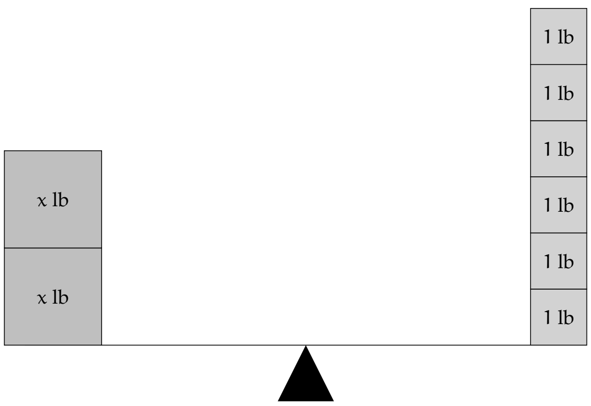


Figure 1.5.10: Balance scale representing the equation $2x = 6$.

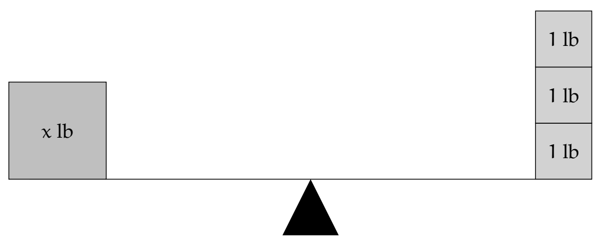


Figure 1.5.11: Balance scale representing the solution to $2x = 6$, after cutting each side in half.

Currently, the scale is balanced. If we cut the weight in half on both sides, the scale should still be balanced.

We can see from the scale that $x = 3$ is correct. Removing half of the weight from each side of the scale is like dividing both sides of an equation by 2:

$$\begin{aligned} 2x &= 6 \\ \frac{2x}{2} &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

The equivalent equation in this example is $x = 3$, which tells us that the solution to the equation is 3 and the solution set is $\{3\}$.

Remark 1.5.12 Note that when we divide each side of an equation by a number, we use the fraction line in place of the division symbol. The fact that $\frac{6}{2} = 6 \div 2$ is a reminder that the fraction line and division symbol have the same purpose. The division symbol is rarely used in later math courses.

Similarly, we could multiply each side of an equation by 2, just like we can keep a scale balanced if we double the weight on each side. We will summarize these properties.

Fact 1.5.13 Properties of Equivalent Equations. *If there is an equation Left = Right, we can do the following to obtain an equivalent equation.*

$\textit{Left} + c = \textit{Right} + c$	$\textit{Left} - c = \textit{Right} - c$	$(\textit{Left}) \cdot c = (\textit{Right}) \cdot c$	$\frac{\textit{Left}}{c} = \frac{\textit{Right}}{c}$
(add the same number to each side)	*(subtract the same number from each side)*	*(multiply each side of the equation by the same* non-zero *number)*	*(divide each side of the equation by the same* non-zero *number)*

1.5.3 Solving One-Step Equations and Stating Solution Sets

Notice that when we solved equations in Subsection 1.5.2, the final equation looked like $x =$ number, where the variable x is separated from other numbers and stands alone on one side of the equals sign. The goal of solving any equation is to *isolate the variable* in this same manner.

Putting together both strategies (applying the opposite operation and balancing equations like a scale) that we just explored, we summarize how to solve a one-step linear equation.

Process 1.5.14 Steps to Solving Simple (One-Step) Linear Equations.

Apply *Apply the opposite operation to both sides of the equation. If a number was added to the variable, subtract that number, and vice versa. If the variable was multiplied by a number, divide by that number, and vice versa.*

Check *Check the solution. This means, verify that what you think is the solution actually solves the equation. For one reason, it's human to have made a simple arithmetic mistake, and by checking you will protect yourself from this. For another reason, there are situations where solving an equation will tell you that certain numbers are* possible *solutions, but they do not actually solve the original equation. Checking solutions will catch these situations.*

Summarize *State the solution set, or in the case of application problems, summarize the result using a complete sentence and appropriate units.*

Let's look at a few examples.

Example 1.5.15 Solve for y in the equation $7 + y = 3$.

Explanation.

To isolate y, we need to remove 7 from the left side. Since 7 is being *added* to y, we need to *subtract* 7 from each side of the equation.

$$\begin{aligned} 7 + y &= 3 \\ 7 + y - 7 &= 3 - 7 \\ y &= -4 \end{aligned}$$

We should always check the solution when we solve equations. For this problem, we will substitute y in the original equation with -4:

$$\begin{aligned} 7 + y &= 3 \\ 7 + (-4) &\stackrel{?}{=} 3 \\ 3 &\stackrel{\checkmark}{=} 3 \end{aligned}$$

The solution -4 is checked, so the solution set is $\{-4\}$.

Checkpoint 1.5.16 Solve for z in the equation $-7.3 + z = 5.1$.

Explanation. There are two ways to think about this one, but the overall goal is to remove that -7.3 from the left side. Since that negative number is being added to z, we could *subtract* -7.3 from both sides. It is just as effective to *add* positive 7.3.

$$\begin{aligned} -7.3 + z &= 5.1 \\ -7.3 + z + 7.3 &= 5.1 + 7.3 \\ z &= 12.4 \end{aligned}$$

We will check the solution by substituting z in the original equation with 12.4:

$$\begin{aligned} -7.3 + z &= 5.1 \\ -7.3 + (12.4) &\stackrel{?}{=} 5.1 \\ 5.1 &\stackrel{\checkmark}{=} 5.1 \end{aligned}$$

The solution 12.4 is checked and the solution set is $\{12.4\}$.

Checkpoint 1.5.17 Solve for a in the equation $10 = -2a$.

Explanation. To isolate the variable a, we need to divide each side by -2 (because a is being *multiplied* by -2). One common mistake is to add 2 to each side. This would not isolate a, but would instead leave us with the expression $-2a + 2$ on the right-hand side.

$$\begin{aligned} 10 &= -2a \\ \frac{10}{-2} &= \frac{-2a}{-2} \\ -5 &= a \end{aligned}$$

We will check the solution by substituting a in the original equation with -5:

$$\begin{aligned} 10 &= -2a \\ 10 &\stackrel{?}{=} -2(-5) \\ 10 &\stackrel{\checkmark}{=} 10 \end{aligned}$$

The solution -5 is checked and the solution set is $\{-5\}$.

Note that in solving the equation in Checkpoint 1.5.17 we found that $-5 = a$, and did not bother to write $a = -5$. All that really matters is that we ended with a clear statement of what a must be equal to.

Example 1.5.18 The formula for a circle's circumference is $c = \pi d$, where c stands for circumference, d stands for diameter, and π is a constant with the value of $3.1415926\ldots$. If a circle's circumference is 12π ft, find this circle's diameter.

Explanation. The circumference is given as 12π feet. Approximating π with 3.14, this means the circumference is approximately 37.68 ft. It is nice to have a rough understanding of how long the circumference is, but if we use 3.14 instead of π, we are using a slightly smaller number than π, and the result of any calculations we do would not be as accurate. This is why we will use the symbol π throughout solving this equation and round only at the end in the conclusion summary (if necessary).

We will substitute c in the formula with 12π and solve for d:

$$\begin{aligned} c &= \pi d \\ 12\pi &= \pi d \\ \frac{12\pi}{\pi} &= \frac{\pi d}{\pi} \\ 12 &= d \end{aligned}$$

So the circle's diameter is 12 ft.

Example 1.5.19 Solve for b in $-b = 2$.

Explanation. Note that b is not yet isolated as there is a negative sign in front of it. One way to solve for b is to "negate" both sides:

$$\begin{aligned} -b &= 2 \\ -(-b) &= -(2) \\ b &= -2 \end{aligned}$$

We removed the negative sign from $-b$ by negating both sides. A second way to remove the negative sign -1 from $-b$ is to divide both sides by -1. If you view the original $-b$ as $-1 \cdot b$, then this approach resembles the solution from Checkpoint 1.5.17.

$$\begin{aligned} -b &= 2 \\ -1 \cdot b &= 2 \\ \frac{-1 \cdot b}{-1} &= \frac{2}{-1} \\ b &= -2 \end{aligned}$$

A third way to remove the original negative sign is to use the fact that $-1 \cdot (-b) = b$. So we could multiply on each side by -1.

$$\begin{aligned} -b &= 2 \\ -1 \cdot (-b) &= -1 \cdot (2) \\ b &= -2 \end{aligned}$$

We will check the solution by substituting b in the original equation with -2:

$$\begin{aligned} -b &= 2 \\ -(-2) &\stackrel{\checkmark}{=} 2 \end{aligned}$$

The solution -2 is checked and the solution set is $\{-2\}$.

1.5.4 Solving One-Step Equations Involving Fractions

When equations have fractions, solving them will make use of the same principles. You may need to use fraction arithmetic, and there may be special considerations that will make the calculations easier. So we have separated the following examples.

Example 1.5.20 Solve for g in $\frac{2}{3} + g = \frac{1}{2}$.

Explanation.

In Section 2.3, we will learn a skill to avoid fraction operations entirely in equations like this one. For now, let's solve the equation by using subtraction to isolate g:

$$\begin{aligned} \frac{2}{3} + g &= \frac{1}{2} \\ \frac{2}{3} + g - \frac{2}{3} &= \frac{1}{2} - \frac{2}{3} \\ g &= \frac{3}{6} - \frac{4}{6} \\ g &= -\frac{1}{6} \end{aligned}$$

We will check the solution by substituting g in the original equation with $-\frac{1}{6}$:

$$\begin{aligned} \frac{2}{3} + g &= \frac{1}{2} \\ \frac{2}{3} + \left(-\frac{1}{6}\right) &\stackrel{?}{=} \frac{1}{2} \\ \frac{4}{6} + \left(-\frac{1}{6}\right) &\stackrel{?}{=} \frac{1}{2} \\ \frac{3}{6} &\stackrel{\checkmark}{=} \frac{1}{2} \end{aligned}$$

The solution $-\frac{1}{6}$ is checked and the solution set is $\left\{-\frac{1}{6}\right\}$.

Checkpoint 1.5.21 Solve for q in the equation $q - \frac{3}{7} = \frac{3}{2}$.

Explanation. To remove the $\frac{3}{7}$ from the left side, we need to *add* $\frac{3}{7}$ to each side of the equation.

$$\begin{aligned} q - \frac{3}{7} &= \frac{3}{2} \\ q - \frac{3}{7} + \frac{3}{7} &= \frac{3}{2} + \frac{3}{7} \\ q &= \frac{21}{14} + \frac{6}{14} \\ q &= \frac{27}{14} \end{aligned}$$

We will check the solution by substituting q in the original equation with $\frac{27}{14}$:

$$\begin{aligned} q - \frac{3}{7} &= \frac{3}{2} \\ \frac{27}{14} - \frac{3}{7} &\stackrel{?}{=} \frac{3}{2} \\ \frac{27}{14} - \frac{6}{14} &\stackrel{?}{=} \frac{3}{2} \\ \frac{21}{14} &\stackrel{?}{=} \frac{3}{2} \\ \frac{3}{2} &\stackrel{\checkmark}{=} \frac{3}{2} \end{aligned}$$

The solution $\frac{27}{14}$ is checked and the solution set is $\left\{\frac{27}{14}\right\}$.

Example 1.5.22 Solve for c in $\frac{c}{5} = 4$.

Explanation.

Note that the fraction line here implies division, so our variable c is being divided by 5. The opposite operation is to *multiply* by 5:

$$\begin{aligned} \frac{c}{5} &= 4 \\ 5 \cdot \frac{c}{5} &= 5 \cdot 4 \\ c &= 20 \end{aligned}$$

We will check the solution by substituting c in the original equation with 20:

$$\begin{aligned} \frac{c}{5} &= 4 \\ \frac{20}{5} &\stackrel{\checkmark}{=} 4 \end{aligned}$$

The solution 20 is checked and the solution set is $\{20\}$.

Example 1.5.23 Solve for d in $-\frac{1}{3}d = 6$.

Explanation. It's true that in this example, the variable d is *multiplied* by $-\frac{1}{3}$. This means that *dividing* each side by $-\frac{1}{3}$ would be a valid strategy for solving this equation. However, dividing by a fraction could lead to human error, so consider this alternative strategy: multiply by -3.

$$\begin{aligned} -\frac{1}{3}d &= 6 \\ (-3) \cdot \left(-\frac{1}{3}d\right) &= (-3) \cdot 6 \\ d &= -18 \end{aligned}$$

If you choose to divide each side by $-\frac{1}{3}$, that will work out as well:

$$\begin{aligned} -\frac{1}{3}d &= 6 \\ \frac{-\frac{1}{3}d}{-\frac{1}{3}} &= \frac{6}{-\frac{1}{3}} \\ d &= \frac{6}{1} \cdot \frac{-3}{1} && \text{Review fraction division in A.2.} \\ d &= -18 \end{aligned}$$

This gives the same solution.

We will check the solution by substituting d in the original equation with -18:

$$\begin{aligned} -\frac{1}{3}d &= 6 \\ -\frac{1}{3} \cdot (-18) &\stackrel{?}{=} 6 \\ 6 &\stackrel{\checkmark}{=} 6 \end{aligned}$$

The solution -18 is checked and the solution set is $\{-18\}$.

Example 1.5.24 Solve for x in $\frac{3x}{4} = 10$.

Explanation. The variable x appears to have *two* operations that apply to it: first multiplication by 3, and then division by 4. But note that

$$\frac{3x}{4} = \frac{3}{4} \cdot \frac{x}{1} = \frac{3}{4}x.$$

If we view the left side this way, we can get away with solving the equation in one step, by multiplying on each side by the reciprocal of $\frac{3}{4}$.

$$\begin{aligned} \frac{3x}{4} &= 10 \\ \frac{3}{4}x &= 10 \\ \frac{4}{3} \cdot \frac{3}{4}x &= \frac{4}{3} \cdot 10 \\ x &= \frac{4}{3} \cdot \frac{10}{1} \\ x &= \frac{40}{3} \end{aligned}$$

We will check the solution by substituting x in the original equation with $\frac{40}{3}$:

$$\begin{aligned} \frac{3x}{4} &= 10 \\ \frac{3\left(\frac{40}{3}\right)}{4} &\stackrel{?}{=} 10 \\ \frac{40}{4} &\stackrel{?}{=} 10 \\ 10 &\stackrel{\checkmark}{=} 10 \end{aligned}$$

The solution $\frac{40}{3}$ is checked and the solution set is $\left\{\frac{40}{3}\right\}$.

Checkpoint 1.5.25 Solve for H in the equation $\frac{-7H}{12} = \frac{2}{3}$.

Explanation. The left side is effectively the same things as $-\frac{7}{12}H$, so multiplying by $-\frac{12}{7}$ will isolate H.

$$\begin{aligned} \frac{-7H}{12} &= \frac{2}{3} \\ -\frac{7}{12}H &= \frac{2}{3} \\ \left(-\frac{12}{7}\right) \cdot \left(-\frac{7}{12}H\right) &= \left(-\frac{12}{7}\right) \cdot \frac{2}{3} \\ H &= -\frac{4}{7} \cdot \frac{2}{1} \\ H &= -\frac{8}{7} \end{aligned}$$

We will check the solution by substituting H in the original equation with $-\frac{8}{7}$:

$$\begin{aligned} \frac{-7H}{12} &= \frac{2}{3} \\ \frac{-7\left(-\frac{8}{7}\right)}{12} &\stackrel{?}{=} \frac{2}{3} \\ \frac{8}{12} &\stackrel{?}{=} \frac{2}{3} \\ \frac{2}{3} &\stackrel{\checkmark}{=} \frac{2}{3} \end{aligned}$$

The solution $-\frac{8}{7}$ is checked and the solution set is $\left\{-\frac{8}{7}\right\}$.

1.5.5 Reading Questions

1. If you imagine the equation $2x + 3 = 11$ as a balance scale with boxes on each side, how many boxes do you imagine on the left side? How many *types* of boxes do you imagine on the left side?

2. What is the opposite operation of multiplying by a negative number?

3. Every time you solve an equation, there is something you should do to guarantee success. Describe what that thing is that you should do.

1.5.6 Exercises

Review and Warmup

1. Add the following.

a. $-10 + (-1)$

b. $-4 + (-6)$

c. $-1 + (-8)$

2. Add the following.

a. $-10 + (-2)$

b. $-6 + (-7)$

c. $-1 + (-10)$

3. Add the following.

a. $2 + (-9)$

b. $9 + (-2)$

c. $7 + (-7)$

4. Add the following.

a. $3 + (-6)$

b. $5 + (-3)$

c. $7 + (-7)$

5. Add the following.

a. $-8 + 3$

b. $-4 + 8$

c. $-4 + 4$

6. Add the following.

a. $-10 + 4$

b. $-1 + 10$

c. $-4 + 4$

7. Evaluate the following.

a. $\frac{-27}{-3}$

b. $\frac{42}{-6}$

c. $\frac{-64}{8}$

8. Evaluate the following.

a. $\frac{-12}{-2}$

b. $\frac{20}{-4}$

c. $\frac{-24}{8}$

9. Do the following multiplications.

a. $9 \cdot \frac{2}{3}$

b. $12 \cdot \frac{2}{3}$

c. $15 \cdot \frac{2}{3}$

10. Do the following multiplications.

a. $28 \cdot \frac{2}{7}$

b. $35 \cdot \frac{2}{7}$

c. $42 \cdot \frac{2}{7}$

11. Evaluate the following.

a. $\frac{-8}{-1}$

b. $\frac{7}{-1}$

c. $\frac{100}{-100}$

d. $\frac{-15}{-15}$

e. $\frac{12}{0}$

f. $\frac{0}{-4}$

12. Evaluate the following.

a. $\frac{-7}{-1}$

b. $\frac{4}{-1}$

c. $\frac{150}{-150}$

d. $\frac{-18}{-18}$

e. $\frac{12}{0}$

f. $\frac{0}{-9}$

Solving One-Step Equations with Addition/Subtraction Solve the equation.

13. $y + 7 = 17$

14. $r + 5 = 9$

15. $r + 1 = -3$

16. $t + 8 = -1$

17. $3 = t + 8$

18. $4 = x + 6$

19. $-10 = x - 7$

20. $-10 = x - 9$

21. $y + 79 = 0$

22. $y + 51 = 0$

23. $r - 3 = -1$

24. $r - 10 = -4$

25. $0 = t - 59$

26. $0 = t - 25$

27. $-13 = x - 10$

28. $-15 = x - 6$

29. $x - (-9) = 13$

30. $y - (-5) = 10$

31. $-5 = y - (-3)$

32. $-1 = r - (-5)$

33. $5 + r = -4$

34. $3 + t = -4$

35. $-8 = -9 + t$

36. $2 = -3 + x$

37. $x+\frac{7}{6}=\frac{5}{6}$

38. $x+\frac{3}{4}=\frac{3}{4}$

39. $\frac{10}{9}=y-\frac{8}{9}$

40. $\frac{4}{7}=y-\frac{2}{7}$

41. $-\frac{4}{3}+r=-\frac{5}{6}$

42. $-\frac{10}{9}+r=-\frac{17}{18}$

43. $\frac{6}{5}+m=-\frac{1}{8}$

44. $\frac{10}{9}+p=-\frac{1}{4}$

Solving One-Step Equations with Multiplication/Division Solve the equation.

45. $7t=42$

46. $4x=40$

47. $36=-3x$

48. $42=-6y$

49. $0=-26c$

50. $0=40A$

51. $\frac{1}{6}C=8$

52. $\frac{1}{3}m=3$

53. $-5=\frac{p}{9}$

54. $-1=\frac{q}{6}$

55. $\frac{3}{8}y=9$

56. $\frac{7}{13}t=14$

57. $\frac{2}{3}a=3$

58. $\frac{7}{3}c=6$

59. $\frac{8}{3}=-\frac{A}{10}$

60. $\frac{5}{6}=-\frac{C}{10}$

61. $2m=-7$

62. $8p=-5$

63. $-16=-10q$

64. $-15=-10y$

65. $\frac{7}{8}=\frac{r}{32}$

66. $\frac{3}{4}=\frac{a}{12}$

67. $-\frac{c}{54}=\frac{2}{9}$

68. $-\frac{A}{35}=\frac{10}{7}$

69. $-\frac{C}{16}=-\frac{5}{4}$

70. $-\frac{m}{20}=-\frac{7}{10}$

71. $-\frac{3}{7}=\frac{9p}{10}$

72. $-\frac{7}{4}=\frac{8q}{9}$

73. $\frac{x}{18}=\frac{5}{2}$

74. $\frac{x}{21}=\frac{5}{3}$

75. $\frac{9}{4}=\frac{x}{20}$

76. $\frac{3}{5}=\frac{x}{45}$

Comparisons

77. Solve the equation.
a. $6r=36$
b. $6+x=36$

78. Solve the equation.
a. $2r=20$
b. $2+x=20$

79. Solve the equation.
a. $20=-5t$
b. $20=-5+y$

80. Solve the equation.
a. $24=-3t$
b. $24=-3+y$

81. Solve the equation.
a. $-t=6$
b. $-x=-6$

82. Solve the equation.
a. $-x=14$
b. $-t=-14$

83. Solve the equation.
a. $-\frac{1}{2}r=7$
b. $-\frac{1}{2}b=-7$

84. Solve the equation.
a. $-\frac{1}{6}a=4$
b. $-\frac{1}{6}y=-4$

85. Solve the equation.
a. $30=-\frac{10}{7}b$
b. $-30=-\frac{10}{7}m$

86. Solve the equation.
a. $20=-\frac{5}{8}A$
b. $-20=-\frac{5}{8}b$

87. Solve the equation.
a. $8r=24$
b. $45x=80$

88. Solve the equation.
a. $3t=6$
b. $20r=64$

89. Solve the equation.
a. $35=-7t$
b. $80=-35y$

90. Solve the equation.
a. $28=-7t$
b. $60=-9x$

Challenge

91. Write a linear equation whose solution is $x = 5$. You may not write an equation whose left side is just "x" or whose right side is just "x."

There are infinitely many correct answers to this problem. *Be creative.* After finding an equation that works, see if you can come up with a different one that also works.

92. Fill in the blanks with the numbers 18 and 67 (using each number only once) to create an equation where x has the greatest possible value.

a. $\boxed{} + x = \boxed{}$

b. $\boxed{} = \boxed{} \cdot x$

1.6 Solving One-Step Inequalities

In this section, we learn that solving basic inequalities is not that different from solving basic equations.

With one small complication, we can use very similar properties to Fact 1.5.13 when we solve inequalities (as opposed to equations). Here are some numerical examples.

Add to both sides	If $2 < 4$, then $2 + 1 \overset{\checkmark}{<} 4 + 1$.
Subtract from both sides	If $2 < 4$, then $2 - 1 \overset{\checkmark}{<} 4 - 1$.
Multiply on both sides by a *positive* number	If $2 < 4$, then $3 \cdot 2 \overset{\checkmark}{<} 3 \cdot 4$.
Divide on both sides by a *positive* number	If $2 < 4$, then $\frac{2}{2} \overset{\checkmark}{<} \frac{4}{2}$.

However, something interesting happens when we multiply or divide by the same *negative* number on both sides of an inequality: the direction reverses! To understand why, consider Figure 1.6.2, where the numbers 2 and 4 are multiplied by the negative number -1.

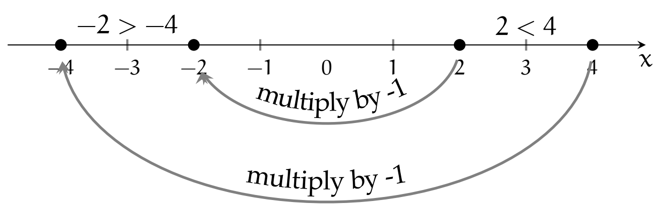


Figure 1.6.2: When two numbers are multiplied by a negative number, their relationship changes.

So even though $2 < 4$, if we multiply both sides by -1, we have the false inequality $-2 \overset{\text{no}}{<} -4$. The *true* inequality is $-2 > -4$.

Fact 1.6.3 Changing the Direction of the Inequality Sign. *When we multiply or divide each side of an inequality by the same* negative *number, the inequality sign must change direction. Do* not *change the inequality sign when multiplying/dividing by a positive number, or when adding/subtracting by any number.*

Example 1.6.4 Solve the inequality $-2x \geq 12$. State the solution set graphically, using interval notation, and using set-builder notation. (Interval notation and set-builder notation are discussed in Section 1.3.)

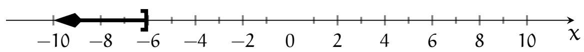
Explanation. To solve this inequality, we will divide each side by -2:

$$\begin{aligned} -2x &\geq 12 \\ \frac{-2x}{-2} &\leq \frac{12}{-2} && \text{Note the change in direction.} \\ x &\leq -6 \end{aligned}$$

Note that the inequality sign changed direction at the step where we divided each side of the inequality by a *negative* number.

When we solve a linear *inequality*, there are usually infinitely many solutions. (Unlike when we solve a linear equation and only have one solution.) For this example, any number less than or equal to -6 is a solution.

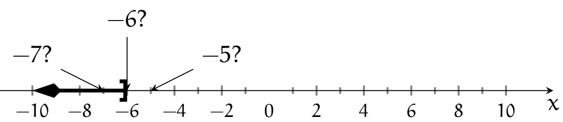
There are at least three ways to represent the solution set for the solution to an inequality: graphically, with set-builder notation, and with interval notation. Graphically, we represent the solution set as:



Using interval notation, we write the solution set as $(-\infty, -6]$. Using set-builder notation, we write the solution set as $\{x \mid x \leq -6\}$.

As with equations, we should check solutions to catch both human mistakes as well as for possible extraneous solutions (numbers which were *possible* solutions according to algebra, but which actually do not solve the inequality).

Since there are infinitely many solutions, it's impossible to literally check them all. We found that all values of x for which $x \leq -6$ are solutions. One approach is to check that one number less than -6 (any number, your choice) satisfies the inequality. *And* that -6 satisfies the inequality. *And* that one number greater than -6 (any number, your choice) does *not* satsify the inequality.



$$\begin{aligned} -2x &\geq 12 \\ -2(-7) &\overset{?}{\geq} 12 \\ 14 &\overset{\checkmark}{\geq} 12 \end{aligned} \qquad \begin{aligned} -2x &\geq 12 \\ -2(-6) &\overset{?}{\geq} 12 \\ 12 &\overset{\checkmark}{\geq} 12 \end{aligned} \qquad \begin{aligned} -2x &\geq 12 \\ -2(-5) &\overset{?}{\geq} 12 \\ 12 &\overset{\text{no}}{\geq} 12 \end{aligned}$$

Thus both -7 and -6 are solutions, while -5 is not. This is evidence that our solution set is correct, and it's valuable in that making these checks would likely help us catch an error if we had made one. While it certainly does take time and space to make three checks like this, it has its value.

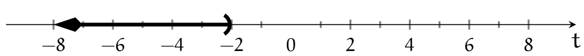
Example 1.6.5 Solve the inequality $t + 7 < 5$. State the solution set graphically, using interval notation, and using set-builder notation.

Explanation. To solve this inequality, we will subtract 7 from each side. There is not much difference between this process and solving the *equation* $t + 7 = 5$, because we are not going to multiply or divide by negative numbers.

$$\begin{aligned} t + 7 &< 5 \\ t + 7 - 7 &< 5 - 7 \\ t &< -2 \end{aligned}$$

Note again that the direction of the inequality did not change, since we did not multiply or divide each side of the inequality by a negative number at any point.

Graphically, we represent this solution set as:



Using interval notation, we write the solution set as $(-\infty, -2)$. Using set-builder notation, we write the solution set as $\{t \mid t < -2\}$.

We should check that some number less than -2 *is* a solution, but that -2 and some number greater than -2 are *not* solutions.

$$\begin{aligned} t + 7 &< 5 \\ -10 + 7 &\overset{?}{<} 5 \end{aligned} \qquad \begin{aligned} t + 7 &< 5 \\ -2 + 7 &\overset{?}{<} 5 \end{aligned} \qquad \begin{aligned} t + 7 &< 5 \\ 0 + 7 &\overset{?}{<} 5 \end{aligned}$$

$$-3 \overset{\checkmark}{<} 5 \qquad 5 \overset{\text{no}}{<} 5 \qquad 7 \overset{\text{no}}{<} 5$$

So our solution is reasonably checked.

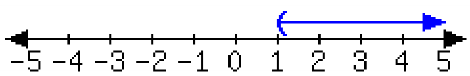
Checkpoint 1.6.6 Solve the inequality $x - 5 > -4$. State the solution set using interval notation and using set-builder notation.

Explanation. To solve this inequality, we will add 5 to each side.

$$\begin{aligned} x - 5 &> -4 \\ x - 5 + 5 &> -4 + 5 \\ x &> 1 \end{aligned}$$

Note again that the direction of the inequality did not change, since we did not multiply or divide each side of the inequality by a negative number at any point.

Graphically, we represent this solution set as:



Using interval notation, we write the solution set as $(1, \infty)$. Using set-builder notation, we write the solution set as $\{x \mid x > 1\}$.

We should check that some number less than 1 is *not* a solution, that 1 itself is *not* a solution, and that some number greater than 1 *is* a solution.

$$\begin{array}{ccc} x - 5 > -4 & x - 5 > -4 & x - 5 > -4 \\ 0 - 5 \overset{?}{>} -4 & 1 - 5 \overset{?}{>} -4 & 10 - 5 \overset{?}{>} -4 \\ -5 \overset{\text{no}}{>} -4 & -4 \overset{\text{no}}{>} -4 & 5 \overset{\checkmark}{>} -4 \end{array}$$

So our solution is reasonably checked.

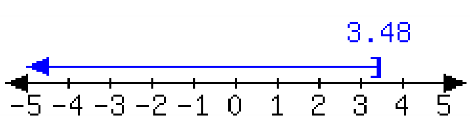
Checkpoint 1.6.7 Solve the inequality $-\frac{1}{2}z \geq -1.74$. State the solution set using interval notation and using set-builder notation.

Explanation. To solve this inequality, we will multiply by -2 to each side.

$$\begin{aligned} -\frac{1}{2}z &\geq -1.74 \\ (-2)\left(-\frac{1}{2}z\right) &\leq (-2)(-1.74) \\ z &\leq 3.48 \end{aligned}$$

In this exercise, we *did* multiply by a negative number and so the direction of the inequality sign changed.

Graphically, we represent this solution set as:



Using interval notation, we write the solution set as $(-\infty, 3.48]$. Using set-builder notation, we write the solution set as $\{z \mid z \leq 3.48\}$.

We should check that some number less than 3.48 *is* a solution, that 3.48 itself *is* a solution, and also that some number greater than 3.48 is *not* a solution.

$$\begin{array}{rrr}
-\frac{1}{2}z \geq -1.74 & -\frac{1}{2}z \geq -1.74 & -\frac{1}{2}z \geq -1.74 \\
-\frac{1}{2}(0) \stackrel{?}{\geq} -1.74 & -\frac{1}{2}(3.48) \stackrel{?}{\geq} -1.74 & -\frac{1}{2}(10) \stackrel{?}{\geq} -1.74 \\
0 \stackrel{\checkmark}{\geq} -1.74 & -1.74 \stackrel{\checkmark}{\geq} -1.74 & -5 \stackrel{\text{no}}{\geq} -1.74
\end{array}$$

So our solution is reasonably checked.

Reading Questions

1. What are three ways to express the solution set to a linear inequality?

2. When you go through the motions of solving a simple linear inequality, what step(s) might you need to take where something special happens that you don't have to worry about when solving a simple linear equation?

3. Why does checking the solution set to an inequality take more effort than checking the solution set to an equation?

Exercises

Review and Warmup

1. Add the following.

a. $-9+(-3)$

b. $-7+(-6)$

c. $-2+(-9)$

2. Add the following.

a. $-9+(-1)$

b. $-5+(-3)$

c. $-2+(-7)$

3. Add the following.

a. $4+(-8)$

b. $10+(-2)$

c. $5+(-5)$

4. Add the following.

a. $5+(-9)$

b. $7+(-3)$

c. $5+(-5)$

5. Add the following.

a. $-6+5$

b. $-4+9$

c. $-6+6$

6. Add the following.

a. $-8+1$

b. $-1+6$

c. $-6+6$

7. Evaluate the following.

a. $\frac{-27}{-9}$

b. $\frac{35}{-5}$

c. $\frac{-35}{7}$

8. Evaluate the following.

a. $\frac{-72}{-8}$

b. $\frac{28}{-7}$

c. $\frac{-60}{6}$

9. Do the following multiplications.

a. $12 \cdot \frac{3}{4}$

b. $16 \cdot \frac{3}{4}$

c. $20 \cdot \frac{3}{4}$

10. Do the following multiplications.

a. $20 \cdot \frac{4}{5}$

b. $25 \cdot \frac{4}{5}$

c. $30 \cdot \frac{4}{5}$

11. Evaluate the following.

a. $\frac{-5}{-1}$

b. $\frac{9}{-1}$

c. $\frac{120}{-120}$

d. $\frac{-15}{-15}$

e. $\frac{8}{0}$

f. $\frac{0}{-7}$

12. Evaluate the following.

a. $\frac{-4}{-1}$

b. $\frac{7}{-1}$

c. $\frac{170}{-170}$

d. $\frac{-18}{-18}$

e. $\frac{8}{0}$

f. $\frac{0}{-2}$

Solving One-Step Inequalities using Addition/Subtraction Solve this inequality.

13. $x+5>6$ **14.** $x+5>9$ **15.** $x-1\leq 7$ **16.** $x-1\leq 6$

17. $2\leq x+9$ **18.** $3\leq x+7$ **19.** $3>x-10$ **20.** $4>x-9$

Solving One-Step Inequalities using Multiplication/Division Solve this inequality.

21. $5x\leq 10$ **22.** $5x\leq 20$ **23.** $7x>10$ **24.** $4x>1$

25. $-2x\geq 8$ **26.** $-3x\geq 9$ **27.** $6\geq -3x$ **28.** $16\geq -4x$

29. $7<-x$ **30.** $8<-x$ **31.** $-x\leq 9$ **32.** $-x\leq 10$

33. $\dfrac{1}{3}x>2$ **34.** $\dfrac{2}{9}x>8$ **35.** $-\dfrac{4}{5}x\leq 8$ **36.** $-\dfrac{5}{2}x\leq 15$

37. $-12<\dfrac{6}{7}x$ **38.** $-21<\dfrac{7}{5}x$ **39.** $-16<-\dfrac{8}{7}x$ **40.** $-18<-\dfrac{9}{8}x$

41. $5x>-15$ **42.** $2x>-8$ **43.** $-6<-2x$ **44.** $-6<-3x$

45. $\dfrac{5}{6}\geq\dfrac{x}{12}$ **46.** $\dfrac{7}{6}\geq\dfrac{x}{36}$ **47.** $-\dfrac{z}{24}<-\dfrac{5}{8}$ **48.** $-\dfrac{z}{40}<-\dfrac{9}{8}$

Challenge

49. Choose the correct inequality or equal sign to make the relation true.

a. Let x and y be integers, such that $x<y$.
Then $x-y$ (□ $<$ □ $>$ □ $=$) $y-x$.

b. Let x and y be integers, such that $1<x<y$.
Then xy (□ $<$ □ $>$ □ $=$) $x+y$.

c. Let x and y be rational numbers, such that $0<x<y<1$.
Then xy (□ $<$ □ $>$ □ $=$) $x+y$.

d. Let x and y be integers, such that $x<y$.
Then $x+2y$ (□ $<$ □ $>$ □ $=$) $2x+y$.

1.7 Algebraic Properties and Simplifying Expressions

We know that if we have two apples and add three more, then our result is the same as if we'd had three apples and added two more. In this section, we'll examine this and other basic properties we know about numbers, and extend them to variable expressions.

1.7.1 Identities and Inverses

We start with some definitions. The number 0 is called the **additive identity**. It gets this name because adding 0 to a number does not change the "identity" of that number. If the sum of two numbers is the additive identity, 0, these two numbers are called **additive inverses**. For example, 2 is the additive inverse of -2, and the additive inverse of -2 is 2.

Similarly, the number 1 is called the **multiplicative identity**. It gets this name because multiplying a number by 1 does not change the "identity" of that number. If the product of two numbers is the multiplicative identity, 1, these two numbers are called **multiplicative inverses**. For example, 2 is the multiplicative inverse of $\frac{1}{2}$, and the multiplicative inverse of $-\frac{2}{3}$ is $-\frac{3}{2}$. The multiplicative inverse of a number is also called the **reciprocal** of that number.

1.7.2 Introduction to Algebraic Properties

Commutative Property. When we compute the area of a rectangle, we generally multiply the length by the width. Does the result change if we multiply the width by the length?

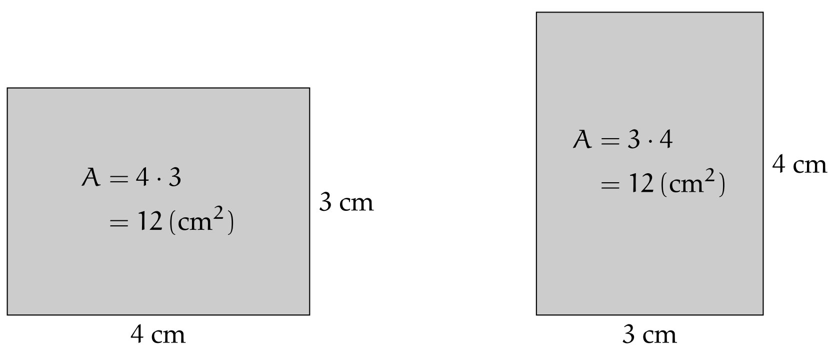


Figure 1.7.2: Horizontal and Vertical Rectangles

We can see $3 \cdot 2 = 2 \cdot 3$. If we denote the length of a rectangle with ℓ and the width with w, this implies $\ell w = w\ell$. The fact that we can reverse the order is known as the **commutative property of multiplication**. There is a similar property for addition, as with $1 + 2 = 2 + 1$, called the **commutative property of addition**. However, there is no commutative property of subtraction or division, because for example $2 - 1 \neq 1 - 2$ and $\frac{4}{2} \neq \frac{2}{4}$.

Associative Property. Let's extend the rectangle to a rectangular prism with width $w = 4\,\text{cm}$, depth $d = 3\,\text{cm}$, and height $h = 2\,\text{cm}$. To compute the volume of this solid, we multiply the width, depth and height, which we write as wdh.

In the following figure, on the left side, we multiply the width and depth first, and then multiply the height. On the right side, we multiply the width and height first, and then multiply the depth. Let's compare the products.

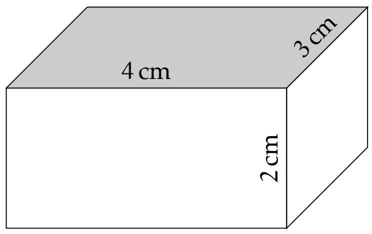


Figure 1.7.3: $(4 \cdot 3) \cdot 2 = 24$

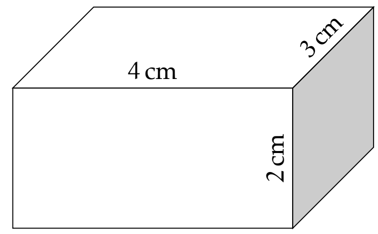


Figure 1.7.4: $4 \cdot (3 \cdot 2) = 24$

We can see $(wd)h = w(dh)$. We haven't changed the order that the three variables are written left to right, but we have moved parentheses to change what operation is highest priority in the order of operations. This is known as the **associative property of multiplication**. Theres is a similar property for addition, as with $(1+2)+3 = 1+(2+3)$, called the **associative property of addition**. However, there is no associative property of subtraction or division, because for example $(3-2)-1 \neq 3-(2-1)$ and $(2 \div 2) \div 2 \neq 2 \div (2 \div 2)$.

Distributive Property. The final property we'll explore is called the **distributive property**, which involves both multiplication and addition. To conceptualize this property, let's consider what happens if we buy 3 boxes that each contain one apple and one pear. This will have the same total cost as if we'd bought 3 apples and 3 pears. We write this algebraically:

$$3(a + p) = 3a + 3p.$$

Visually, we can see that it's just a means of re-grouping: (🍎 + 🍐) + (🍎 + 🍐) + (🍎 + 🍐) = (🍎 + 🍎 + 🍎) + (🍐 + 🍐 + 🍐).

1.7.3 Summary of Algebraic Properties

List 1.7.5: Algebraic Properties

Let a, b, and c represent real numbers, variables, or algebraic expressions. Then the following properties hold:

Commutative Property of Multiplication $a \cdot b = b \cdot a$

Associative Property of Multiplication $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

Commutative Property of Addition $a + b = b + a$

Associative Property of Addition $a + (b + c) = (a + b) + c$

Distributive Property $a(b + c) = ab + ac$

Let's practice these properties in the following exercises.

Checkpoint 1.7.6

a. Use the commutative property of multiplication to write an equivalent expression to $53m$.

b. Use the associative property of multiplication to write an equivalent expression to $3(5n)$.

c. Use the commutative property of addition to write an equivalent expression to $q + 84$.

d. Use the associative property of addition to write an equivalent expression to $x + (20 + c)$.

e. Use the distributive property to write an equivalent expression to $3(r + 7)$ that has no grouping symbols.

Explanation.

a. To use the commutative property of multiplication, we change the order in which two factors are multiplied:

$$\begin{aligned} &53m \\ &= m \cdot 53. \end{aligned}$$

b. To use the associative property of multiplication, we leave factors written in their original order, but change the grouping symbols so that a different multiplication has higher priority:

$$\begin{aligned} &3(5n) \\ &= (3 \cdot 5)n. \end{aligned}$$

You may further simplify by carrying out the multiplication between the two numbers:

$$\begin{aligned} &3(5n) \\ &= (3 \cdot 5)n \\ &= 15n. \end{aligned}$$

c. To use the commutative property of addition, we change the order in which two terms are added:

$$\begin{aligned} &q + 84 \\ &= 84 + q. \end{aligned}$$

d. To use the associative property of addition, we leave terms written in their original order, but change the grouping symbols so that a different addition has higher priority:

$$\begin{aligned} &x + (20 + c) \\ &= (x + 20) + c. \end{aligned}$$

e. To use the distributive property, we multiply the number outside the parentheses, 3, with each term inside the parentheses:

$$\begin{aligned} &3(r + 7) \\ &= 3 \cdot r + 3 \cdot 7 \\ &= 3r + 21. \end{aligned}$$

1.7.4 Applying the Commutative, Associative, and Distributive Properties

Like Terms. One of the main ways that we will use the commutative, associative, and distributive properties is to simplify expressions. In order to do this, we need to recognize **like terms**, as discussed in Section 1.2. We combine like terms when we take an expression like $2a + 3a$ and write the result as $5a$. The formal process actually involves using the distributive property:

$$\begin{aligned} 2a + 3a &= (2 + 3)a \\ &= 5a \end{aligned}$$

In practice, however, it's more helpful to think of this as having 2 of an object and then an additional 3 of that same object. In total, we then have 5 of that object.

Example 1.7.7 Where possible, simplify the following expressions by combining like terms.
a. $6c + 12c - 5c$ b. $-5q^2 - 3q^2$ c. $x - 5y + 4x$ d. $2x - 3y + 4z$

Explanation.

a. All three terms are like terms, so they may combined. We combine them two at a time:

$$\begin{aligned} 6c + 12c - 5c &= 18c - 5c \\ &= 13c \end{aligned}$$

b. The two terms $-5q^2$ and $-3q^2$ are like terms, so we may combine them:

$$-5q^2 - 3q^2 = -8q^2$$

c. The two terms x and $4x$ are like terms, while the other term is different. Using the associative and commutative properties of addition in the first step allows us to place the two like terms next to each other, and then combine them:

$$\begin{aligned} x - 5y + 4x &= x + 4x + (-5y) \\ &= 1x + 4x + (-5y) \\ &= 5x - 5y \end{aligned}$$

Note the expression x is the same as $1x$. Usually we don't write the "1" as it is implied. However, it's helpful when combining like terms to remember that $x = 1x$. (Similarly, $-x$ is equal to $-1x$, which can be helpful when combining $-x$ with like terms.)

d. The expression $2x - 3y + 4z$ cannot be simplified as there are no like terms.

Adding Expressions. When we add an expression like $4x - 5$ to an expression like $3x - 7$, we write them as follows:

$$(4x - 5) + (3x - 7)$$

In order to remove the given sets of parentheses and apply the commutative property of addition, we will rewrite the subtraction operation as "adding the opposite":

$$4x + (-5) + 3x + (-7)$$

At this point we can apply the commutative property of addition and then combine like terms. Here's how the entire problem will look:

$$\begin{aligned}(4x-5)+(3x-7) &= 4x+(-5)+3x+(-7)\\ &= 4x+3x+(-5)+(-7)\\ &= 7x+(-12)\\ &= 7x-12\end{aligned}$$

Remark 1.7.8 Once we become more comfortable simplifying such expressions, we will simply write this kind of simplification in one step:

$$(4x-5)+(3x-7) = 7x-12$$

Example 1.7.9 Use the associative, commutative, and distributive properties to simplify the following expressions as much as possible.

a. $(2x+3)+(4x+5)$

b. $(-5x+3)+(4x-7)$

Explanation.

a. We will remove parentheses, and then combine like terms:

$$\begin{aligned}(2x+3)+(4x+5) &= 2x+3+4x+5\\ &= 2x+4x+3+5\\ &= 6x+8\end{aligned}$$

b. We will remove parentheses, and then combine like terms:

$$\begin{aligned}(-5x+3)+(4x-7) &= -5x+3+4x+(-7)\\ &= -x+(-4)\\ &= -x-4\end{aligned}$$

Applying the Distributive Property with Negative Coefficients. Applying the distributive property in an expression such as $2(3x+4)$ is fairly straightforward, in that this becomes $2(3x)+2(4)$ which we then simplify to $6x+8$. Applying the distributive property is a little trickier when subtraction or a negative constant is involved, for example, with the expression $2(3x-4)$. Recalling that subtraction is defined as "adding the opposite," we can change the subtraction of positive 4 to the addition of negative 4:

$$2\big(3x+(-4)\big)$$

Now when we distribute, we obtain:

$$2(3x)+2(-4)$$

As a final step, we see that this simplifies to:

$$6x-8$$

Remark 1.7.10 We can also extend the distributive property to use subtraction, and state that $a(b-c) = ab-ac$. With this property, we would simplify $2(3x-4)$ more efficiently:

$$\begin{aligned}2(3x-4) &= 2(3x)-2(4)\\ &= 6x-8\end{aligned}$$

Example 1.7.11 Apply the distributive property to each expression and simplify it as much as possible.

a. $-3(5x + 7)$

b. $2(-4x - 1)$

Explanation.

a. We will distribute -3 to the $5x$ and 7:

$$\begin{aligned} -3(5x + 7) &= -3(5x) + (-3)(7) \\ &= -15x - 21 \end{aligned}$$

b. We will distribute 2 to the $-4x$ and -1:

$$\begin{aligned} 2(-4x - 1) &= 2(-4x) - 2(1) \\ &= -8x - 2 \end{aligned}$$

Checkpoint 1.7.12 Use the distributive property to write an equivalent expression to $-4(y - 7)$ that has no grouping symbols.

Explanation. To use the distributive property, we multiply the number outside the parentheses, -4, with each term inside the parentheses:

$$\begin{aligned} -4(y - 7) &= -4 \cdot y - 4(-7) \\ &= -4y + 28 \end{aligned}$$

Subtracting Expressions. To subtract one expression from another expression, such as $(5x + 9) - (3x + 2)$, we will again rely on the fact that subtraction is defined as "adding the opposite." To add the *opposite* of an expression, we will technically distribute a constant factor of -1 and simplify from there:

$$\begin{aligned} (5x + 9) - (3x + 2) &= (5x + 9) + (-1)(3x + 2) \\ &= 5x + 9 + (-1)(3x) + (-1)(2) \\ &= 5x + 9 + (-3x) + (-2) \\ &= 2x + 7 \end{aligned}$$

Remark 1.7.13 The above example demonstrates *how* we apply the distributive property in order to subtract two expressions. But in practice, it can be pretty cumbersome. A shorter (and often clearer) approach is to instead subtract every term in the expression we are subtracting, which is shown like this:

$$\begin{aligned} (5x + 9) - (3x + 2) &= 5x + 9 - 3x - 2 \\ &= 2x + 7 \end{aligned}$$

Example 1.7.14 Use the associative, commutative, and distributive properties to simplify the following expressions as much as possible.

a. $(-6x + 4) - (3x - 7)$

b. $(-2x - 5) - (-4x - 6)$

Explanation.

a. We will remove parentheses using the distributive property, and then combine like terms:

$$\begin{aligned} (-6x + 4) - (3x - 7) &= -6x + 4 - 3x - (-7) \\ &= -6x + 4 - 3x + 7 \\ &= -9x + 11 \end{aligned}$$

b. We will remove parentheses using the distributive property, and then combine like terms:

$$(-2x - 5) - (-4x - 6) = -2x - 5 - (-4x) - (-6)$$

$$
\begin{aligned}
&= -2x - 5 + 4x + 6 \\
&= 2x + 1
\end{aligned}
$$

1.7.5 The Role of the Order of Operations in Applying the Commutative, Associative, and Distributive Properties

When simplifying an expression such as $3 + 4(5x + 7)$, we need to respect the order of operations. Since the terms inside the parentheses are not like terms, there is nothing to simplify there. The next highest priority operation is multiplying the 4 by $(5x + 7)$. This must be done *before* anything happens with the adding of that 3. We cannot say $3 + 4(5x + 7) = 7(5x + 7)$, because that would mean we treated the addition as having higher priority than the multiplication.

So to simplify $3 + 4(5x + 7)$, we will first examine the multiplication of 4 with $(5x + 7)$, and here we may apply the distributive property. After that, we will use the commutative and associative properties:

$$
\begin{aligned}
3 + 4(5x + 7) &= 3 + 4(5x) + 4(7) \\
&= 3 + 20x + 28 \\
&= 20x + 3 + 28 \\
&= 20x + 31
\end{aligned}
$$

Example 1.7.15 Simplify the following expressions using the commutative, associative, and distributive properties.

a. $4 - (3x - 9)$ b. $5x + 9(-2x + 3)$ c. $5(x - 9) + 4(x + 4)$

Explanation.

a. We will remove parentheses using the distributive property, and then combine like terms:

$$
\begin{aligned}
4 - (3x - 9) &= 4 - 3x - (-9) \\
&= 4 - 3x + 9 \\
&= -3x + 13
\end{aligned}
$$

b. We will remove parentheses using the distributive property, and then combine like terms:

$$
\begin{aligned}
5x + 9(-2x + 3) &= 5x + 9(-2x) + 9(3) \\
&= 5x - 18x + 27 \\
&= -13x + 27
\end{aligned}
$$

c. We will remove parentheses using the distributive property, and then combine like terms:

$$
\begin{aligned}
5(x - 9) + 4(x + 4) &= 5x - 45 + 4x + 16 \\
&= 9x - 29
\end{aligned}
$$

Checkpoint 1.7.16 Use the distributive property to simplify $6 - 7(-8 + 7r)$ completely.

Explanation. We first use distributive property to get rid of parentheses, and then combine like terms:

$$\begin{aligned} 6 - 7(-8 + 7r) &= 6 + (-7)(-8 + 7r) \\ &= 6 + (-7)(-8) + (-7)(7r) \\ &= 6 + 56 - 49r \\ &= 62 - 49r \\ &= -49r + 62 \end{aligned}$$

Note that either of the last two expressions are acceptable final answers.

1.7.6 Reading Questions

1. Why is the number 1 known as the "multiplicative identity"?

2. Consider the expression $138 + 25 + 5$. By the order of operations, you would add this as $(138 + 25) + 5$. (Meaning you would start by adding $138 + 25$.)

Which property of algebra allows you to view that as $138 + (25 + 5)$? (Meaning you could start with an easy $25 + 5$, and follow with an easy $138 + 30$.)

3. Describe a certin thing that you should watch out for when it comes to using the distributive law. (It is something that has to do with negative numbers and/or subtraction.)

1.7.7 Exercises

Review and Warmup

1. Count the number of terms in each expression.

a. $-6s - 3y$

b. $-9x - 6z$

c. $9s^2 + y - 9y^2 + 7t$

d. $4z + 6x^2 - 3t + 7y$

2. Count the number of terms in each expression.

a. $-5t^2 + 5t^2$

b. $8t - 7z^2 + 5 - 5s^2$

c. $7y$

d. $2y + 6z + x$

3. List the terms in each expression.

a. $-2.9t - 5x + 5.7y$

b. $-7.7z + 0.6s$

c. $6.9y + 4.5z - 8.5 + 1.6t$

d. $5.1z - 6y$

4. List the terms in each expression.

a. $-1.3t^2 + 3.2z + 3.7x$

b. $3.7z$

c. $8.4y^2 - 3z$

d. $8.2t + 3.6t^2$

5. List the terms in each expression.

a. $0.3t - 6.7s^2 + 1.7t + 7.9t^2$

b. $-0.9x - 5.7x^2 + 6.5$

c. $5.5y^2 + 6.7z^2$

d. $0.6t^2 - 7.2s - 5s^2$

6. List the terms in each expression.

a. $1.9t^2 + 1.5 - 0.3y + 6.7t$

b. $6.6t + 6.2x^2$

c. $2.1t - 8.8t^2 - 2s + 3.6x$

d. $-1.9z + 4.4y^2$

7. Simplify each expression, if possible, by combining like terms.

a. $4t - 9 + 4s - 6s$

b. $t + 7x$

c. $-6s - 7$

d. $-9x + 3x$

8. Simplify each expression, if possible, by combining like terms.

a. $5t^2 + t$

b. $5y + 5y - 7y + 9t$

c. $2t^2 - 7s^2 + 4s$

d. $4x + 2 + 9s^2 + s$

These exercises involve the concepts of like terms and the commutative, associative, and distributive properties.

9. What is the additive inverse of -7?

10. What is the additive inverse of -4?

11. What is the multiplicative inverse of -2?

12. What is the multiplicative inverse of -3?

13. Use the associative property of addition to write an equivalent expression to $x + (51 + m)$.

14. Use the associative property of addition to write an equivalent expression to $m + (24 + r)$.

15. Use the associative property of addition to write an equivalent expression to $7 + (13 + n)$.

16. Use the associative property of addition to write an equivalent expression to $20 + (1 + q)$.

17. Use the associative property of multiplication to write an equivalent expression to $7(5x)$.

18. Use the associative property of multiplication to write an equivalent expression to $4(9r)$.

19. Use the commutative property of addition to write an equivalent expression to $a + 92$.

20. Use the commutative property of addition to write an equivalent expression to $b + 57$.

21. Use the commutative property of addition to write an equivalent expression to $4t + 11$.

22. Use the commutative property of addition to write an equivalent expression to $2p + 53$.

23. Use the commutative property of addition to write an equivalent expression to $10(m + 53)$.

24. Use the commutative property of addition to write an equivalent expression to $5(n + 18)$.

25. Use the commutative property of multiplication to write an equivalent expression to $84q$.

26. Use the commutative property of multiplication to write an equivalent expression to $49x$.

27. Use the commutative property of multiplication to write an equivalent expression to $14 + 7r$.

28. Use the commutative property of multiplication to write an equivalent expression to $79 + 2t$.

29. Use the commutative property of multiplication to write an equivalent expression to $5(b + 45)$.

30. Use the commutative property of multiplication to write an equivalent expression to $9(y + 87)$.

31. Use the distributive property to write an equivalent expression to $4(m + 6)$ that has no grouping symbols.

32. Use the distributive property to write an equivalent expression to $5(m + 8)$ that has no grouping symbols.

33. Use the distributive property to write an equivalent expression to $-10(n-8)$ that has no grouping symbols.

34. Use the distributive property to write an equivalent expression to $-4(q+1)$ that has no grouping symbols.

35. Use the distributive property to write an equivalent expression to $-(x-3)$ that has no grouping symbols.

36. Use the distributive property to write an equivalent expression to $-(r-10)$ that has no grouping symbols.

37. Use the distributive property to simplify $7+9(9+5t)$ completely.

38. Use the distributive property to simplify $4+3(7+4b)$ completely.

39. Use the distributive property to simplify $7-8(-4-7q)$ completely.

40. Use the distributive property to simplify $2-8(4+3x)$ completely.

41. Use the distributive property to simplify $4-(-1+5m)$ completely.

42. Use the distributive property to simplify $10-(8-n)$ completely.

43. Use the distributive property to simplify $7-(-6q-5)$ completely.

44. Use the distributive property to simplify $4-(9x+4)$ completely.

45. Use the distributive property to simplify $\frac{9}{7}(-6-7r)$ completely.

46. Use the distributive property to simplify $\frac{6}{5}(-6+3t)$ completely.

47. Use the distributive property to simplify $\frac{3}{10}\left(-9+\frac{3}{4}b\right)$ completely.

48. Use the distributive property to simplify $\frac{9}{5}\left(7+\frac{3}{7}c\right)$ completely.

49. The expression $c+m+r$ would be ambiguous if we did not have a left-to-right reading convention. Use grouping symbols to emphasize the order that these additions should be carried out.
Use the associative property of addition to write an equivalent (but different) algebraic expression.

50. The expression $m+y+r$ would be ambiguous if we did not have a left-to-right reading convention. Use grouping symbols to emphasize the order that these additions should be carried out.
Use the associative property of addition to write an equivalent (but different) algebraic expression.

51. A student has (correctly) simplified an algebraic expression in the following steps. Between each pair of steps, identify the algebraic property that justifies moving from one step to the next.

$$8(n+6)+9n$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$=(8n+48)+9n$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$=(48+8n)+9n$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$= 48 + (8n + 9n)$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$= 48 + (8 + 9)\,n$$

$$= 48 + 17n$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$= 17n + 48$$

52. A student has (correctly) simplified an algebraic expression in the following steps. Between each pair of steps, identify the algebraic property that justifies moving from one step to the next.

$$5(q + 2) + 7q$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$= (5q + 10) + 7q$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$= (10 + 5q) + 7q$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$= 10 + (5q + 7q)$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$= 10 + (5 + 7)\,q$$

$$= 10 + 12q$$

(☐ commutative property of addition ☐ commutative property of multiplication ☐ associative property of addition ☐ associative property of multiplication ☐ distributive property)

$$= 12q + 10$$

53. The number of students enrolled in math courses at Portland Community College has grown over the years. The formulas

$$M = 0.31x + 3.3 \quad W = 0.43x + 4.3 \quad N = 0.01x + 0.1$$

describe the numbers (of thousands) of men, women, and gender-non-binary students enrolled in math courses at PCC x years after 2005. (Note this is an exercise using randomized numbers, not actual data.) Give a simplified formula for the total number T of thousands of students at PCC taking math classes x years after 2005. Be sure to give the entire formula, starting with `T=`.

54. The number of students enrolled in math courses at Portland Community College has grown over the years. The formulas

$$M = 0.35x + 5.3 \quad W = 0.56x + 3.4 \quad N = 0.01x + 0.2$$

describe the numbers (of thousands) of men, women, and gender-non-binary students enrolled in math courses at PCC x years after 2005. (Note this is an exercise using randomized numbers, not actual data.) Give a simplified formula for the total number T of thousands of students at PCC taking math classes x years after 2005. Be sure to give the entire formula, starting with `T=`.

55. Fully simplify $(-3x + 5) + (x - 3)$.

56. Fully simplify $(4x + 1) + (-x + 7)$.

57. Fully simplify $(-5x - 7) + (-x - 3)$.

58. Fully simplify $(6x - 3) + (-x - 6)$.

59. Fully simplify $(7x + 1) + (-x + 2)$.

60. Fully simplify $(-8x - 6) + (-x - 6)$.

61. Fully simplify $-9(-3x - 7) - 5(7x - 8)$.

62. Fully simplify $-9x + 3 + 6(x - 2)$.

63. Fully simplify $2(6x - 7) - 7(-5x - 3)$.

64. Fully simplify $-3(4x + 1) + 7(9x - 5)$.

65. Fully simplify $-4(9x - 5) - 8(-3x - 1)$.

66. Fully simplify $5(6x + 9) + 9(-6x + 3)$.

67. Fully simplify $6(2x - 3) - (3x + 7)$.

68. Fully simplify $-7(9x + 5) + 2(-6x + 9)$.

1.8 Modeling with Equations and Inequalities

One purpose of learning math is to be able to model real-life situations and then use the model to ask and answer questions about the situation. In this lesson, we will examine the basics of modeling to set up an equation (or inequality).

1.8.1 Setting Up Equations for Rate Models

To set up an equation modeling a real world scenario, the first thing we need to do is identify what variable we will use. The variable we use will be determined by whatever is unknown in our problem statement. Once we've identified and defined our variable, we'll use the numerical information provided to set up our equation.

Example 1.8.2 A savings account starts with \$500. Each month, an automatic deposit of \$150 is made. Write an equation that represents the number of months it will take for the balance to reach \$1,700.

Explanation.

To set up an equation, we might start by making a table in order to identify a general pattern for the total amount in the account after m months. In Figure 1.8.3, we find the pattern is that after m months, the total amount saved is $500 + 150m$.
Using this pattern, we determine that an equation representing when the total savings equals \$1700 is:

$$500 + 150m = 1700$$

Months Since Saving Started	Total Amount Saved (in Dollars)
0	500
1	$500 + 150 = 650$
2	$500 + 150(2) = 800$
3	$500 + 150(3) = 950$
4	$500 + 150(4) = 1100$
$\vdots$	$\vdots$
m	$500 + 150m$

Figure 1.8.3: Amount in Savings Account

Right now we are not interested in actually solving this equation and finding the number of months m until the savings has reached \$1700. The skill of *setting up* that equation is challenging enough, and this section only focuses on that setup.

Example 1.8.4 A bathtub contains $2.5\,\text{ft}^3$ of water. More water is being poured in at a rate of $1.75\,\text{ft}^3$ per minute. Write an equation representing when the amount of water in the bathtub will reach $6.25\,\text{ft}^3$.

Explanation.

Since this problem refers to *when* the amount of water will reach a certain amount, we immediately know that the unknown quantity is time. As the volume of water in the tub is measured in ft^3 per minute, we know that time needs to be measured in minutes. We'll define t to be the number of minutes that water is poured into the tub. To determine this equation, we'll start by making a table of values:

Minutes Water Has Been Poured	Total Amount of Water (in ft^3)
0	2.5
1	$2.5 + 1.75 = 4.25$
2	$2.5 + 1.75(2) = 6$
3	$2.5 + 1.75(3) = 7.75$
$\vdots$	$\vdots$
t	$2.5 + 1.75t$

Figure 1.8.5: Amount of Water in the Bathtub

Using this pattern, we determine that the equation representing when the amount will be 6.25 ft^3 is:

$$2.5 + 1.75t = 6.25$$

1.8.2 Setting Up Equations for Percent Problems

Section A.4 reviews some basics of working with percentages, and even solves some one-step equations that are set up using percentages. Here we look at some scenarios where there is an equation to set up based on percentages, but it things are a little more complicated than with a one-step equation. One important fact is that when doing math with percentages, we always start by rewriting the percentage as a decimal. For example, 18% should be written as 0.18 if you are going to us it to do algebra or arithmetic.

Example 1.8.6 Jakobi's annual salary as a nurse in Portland, Oregon, is \$73,290. His salary increased by 4% from last year. Write a linear equation modeling this scenario, where the unknown value is Jakobi's salary last year.

Explanation. We want to work with Jakobi's salary last year. So we'll introduce s, defined to be Jakobi's salary last year (in dollars). To set up the equation, we need to think about how he arrived at this year's salary. To get to this year's salary, his employer took last year's salary and added 4% to it. Conceptually, this means we have:

$$(\text{last year's salary}) + (4\% \text{ of last year's salary}) = (\text{this year's salary})$$

We'll represent 4% of last year's salary with $0.04s$ since 0.04 is the decimal representation of 4%. This means that the equation we set up is:

$$s + 0.04s = 73290$$

Checkpoint 1.8.7 Kirima offered to pay the bill and tip at a restaurant where she and her friends had dinner. In total she paid \$150, which made the tip come out to a little more than 19%. We'd like to know what was the bill before tip. Set up an equation for this situation.

Explanation. A common mistake is to translate a question like this into "what is 19% of \$150?" as a way to calculate the tip amount, and then subtract that from \$150. But that is not how tipping works. The 19%is applied to the *original bill*, not the *final total*. If we let x represent the original bill, then:

$$\overset{\text{bill}}{x} \; \overset{\text{plus}}{+} \; \overset{\text{19\%}}{0.19} \; \overset{\text{of}}{\cdot} \; \overset{\text{bill}}{x} \; \overset{\text{is}}{=} \; \overset{\$150}{150}$$

Example 1.8.8 The price of a refrigerator after a 15% discount is \$612. Write a linear equation modeling this scenario, where the original price of the refrigerator (before the discount was applied) is the unknown quantity.

Explanation. We'll let c be the original price of the refrigerator. To obtain the discounted price, we take the original price and subtract 15% of that amount. Conceptually, this looks like:

$$(\text{original price}) - (15\% \text{ of the original price}) = (\text{discounted price})$$

Since the amount of the discount is 15% of the original price, we'll represent this with $0.15c$. The equation we set up is then:

$$c - 0.15c = 612$$

Checkpoint 1.8.9 A shirt is on sale at 20% off. The current price is \$51.00. Write an equation based on this scenario where the variable represents the shirt's original price.

Explanation. Let x represent the original price of the shirt. Since 20% is removed to bring the cost to \$51, we can set up the equation:

$$\overset{\text{original}}{x} \overset{\text{minus}}{-} \overset{20\%}{0.20} \overset{\text{of}}{\cdot} \overset{\text{original}}{x} \overset{\text{is}}{=} \overset{\$51}{51}$$

1.8.3 Setting Up Inequalities for Models

In general, we'll model using inequalities when we want to determine a maximum or minimum value. To identify that an inequality is needed instead of an equality, we'll look for phrases like *at least, at most, at a minimum* or *at a maximum*.

Example 1.8.10 The car share company car2go has a one-time registration fee of \$5 and charges \$14.99 per hour for use of their vehicles. Hana wants to use car2go and has a maximum budget of \$300. Write a linear inequality representing this scenario, where the unknown quantity is the number of hours she uses their vehicles.

Explanation. We'll let h be the number of hours that Hana uses car2go. We need the initial cost and the cost from the hourly charge to be less than or equal to \$300, which we set up as:

$$5 + 14.99h \leq 300$$

Example 1.8.11 When an oil tank is decommissioned, it is drained of its remaining oil and then re-filled with an inert material, such as sand. A cylindrical oil tank has a volume of 275 gal and is being filled with sand at a rate of 700 gal per hour. Write a linear inequality representing this scenario, where the time it takes for the tank to overflow with sand is the unknown quantity.

Explanation. The unknown in this scenario is time, so we'll define t to be the number of hours that sand is poured into the tank. (Note that we chose hours based on the rate at which the sand is being poured.) We'll represent the amount of sand poured in as $700t$ as each hour an additional 700 gal are added. Given that we want to know when this amount exceeds 275 gal, we set this equation up as:

$$700t > 275$$

1.8.4 Translating Phrases into Algebraic Expressions and Equations/Inequalities

Void of context, there are certain short phrases and expressions in English that have mathematical meaning, and might show up in a modeling scenario. The following table shows how to translate some of these common phrases into algebraic expressions.

English Phrases	Math Expressions
the sum of 2 and a number	$x+2$ or $2+x$
2 more than a number	$x+2$ or $2+x$
a number increased by 2	$x+2$ or $2+x$
a number and 2 together	$x+2$ or $2+x$
the difference between a number and 2	$x-2$
the difference of 2 and a number	$2-x$
2 less than a number	$x-2$ (*not* $2-x$)
a number decreased by 2	$x-2$
2 decreased by a number	$2-x$
2 subtracted from a number	$x-2$
a number subtracted from 2	$2-x$
the product of 2 and a number	$2x$
twice a number	$2x$
a number times 2	$x \cdot 2$ or $2x$
two thirds of a number	$\frac{2}{3}x$
25% of a number	$0.25x$
the quotient of a number and 2	$x/2$
the quotient of 2 and a number	$2/x$
the ratio of a number and 2	$x/2$
the ratio of 2 and a number	$2/x$

Table 1.8.12: Translating English Phrases into Math Expressions

We can extend this to setting up equations and inequalities. Let's look at some examples. The key is to break a complicated phrase or sentence into smaller parts, identifying key vocabulary such as "is," "of," "greater than," "at most," etc.

English Sentences	Math Equations and Inequalities
The sum of 2 and a number is 6.	$x+2=6$
2 less than a number is at least 6.	$x-2\geq 6$
Twice a number is at most 6.	$2x\leq 6$
6 is the quotient of a number and 2.	$6=\frac{x}{2}$
4 less than twice a number is greater than 10.	$2x-4>10$
Twice the difference between 4 and a number is 10.	$2(4-x)=10$
The product of 2 and the sum of 3 and a number is less than 10.	$2(x+3)<10$
The product of 2 and a number, subtracted from 5, yields 8.	$5-2x=8$
Two thirds of a number subtracted from 10 is 2.	$10-\frac{2}{3}x=2$
25% of the sum of 7 and a number is 2.	$0.25(x+7)=2$

Table 1.8.13: Translating English Sentences into Math Equations

A certain amount of practice with translating these English phrases and sentences into math expressions, equations, and inequalities can be helpful for word problems that do have context. In the exercises for this section, you will find such practice exercises.

1.8.5 Reading Questions

1. It is common to come across a word problem where there is some kind of rate. In a problem like that, it can help you to understand the pattern if you make a ____.

2. It is common to come across a word problem where some percent is either added or subtracted from an unknown original value. With the approach described in theis section for setting up an equation, how many times will you use the variable in such an equation?

3. Is there any difference between these three phrases, or do they all mean the same thing?

- ten subtracted from a number
- ten less than a number
- ten minus a number

1.8.6 Exercises

Review and Warmup

1. Identify a variable you might use to represent each quantity. And identify what units would be most appropriate.

a. Let ____ be the area of a house, measured in ____.

b. Let ____ be the age of a dog, measured in ____.

c. Let ____ be the amount of time passed since a driver left Seattle, Washington, bound for Portland, Oregon, measured in ____.

2. Identify a variable you might use to represent each quantity. And identify what units would be most appropriate.

a. Let ____ be the age of a person, measured in ____.

b. Let ____ be the distance traveled by a driver that left Portland, Oregon, bound for Boise, Idaho, measured in ____.

c. Let ____ be the surface area of the walls of a room, measured in ____.

Modeling with Linear Equations

3. Sherial's annual salary as a radiography technician is \$38,494.00. Her salary increased by 1.3% from last year. What was her salary last year?
Assume her salary last year was s dollars. Write an equation to model this scenario. There is no need to solve it.

4. Ken's annual salary as a radiography technician is \$42,066.00. His salary increased by 2.6% from last year. What was his salary last year?
Assume his salary last year was s dollars. Write an equation to model this scenario. There is no need to solve it.

5. A bicycle for sale costs \$180.71, which includes 6.3% sales tax. What was the cost before sales tax?
Assume the bicycle's price before sales tax is p dollars. Write an equation to model this scenario. There is no need to solve it.

6. A bicycle for sale costs \$210.60, which includes 5.3% sales tax. What was the cost before sales tax?
Assume the bicycle's price before sales tax is p dollars. Write an equation to model this scenario. There is no need to solve it.

7. The price of a washing machine after 25% discount is \$172.50. What was the original price of the washing machine (before the discount was applied)?
Assume the washing machine's price before the discount is p dollars. Write an equation to model this scenario. There is no need to solve it.

8. The price of a washing machine after 15% discount is \$221.00. What was the original price of the washing machine (before the discount was applied)?
Assume the washing machine's price before the discount is p dollars. Write an equation to model this scenario. There is no need to solve it.

9. The price of a restaurant bill, including an 10% gratuity charge, was \$110.00. What was the price of the bill before gratuity was added?
Assume the bill without gratuity is b dollars. Write an equation to model this scenario. There is no need to solve it.

10. The price of a restaurant bill, including an 17% gratuity charge, was \$11.70. What was the price of the bill before gratuity was added?
Assume the bill without gratuity is b dollars. Write an equation to model this scenario. There is no need to solve it.

11. In May 2016, the median rent price for a one-bedroom apartment in a city was reported to be \$904.50 per month. This was reported to be an increase of 0.5% over the previous month. Based on this reporting, what was the median price of a one-bedroom apartment in April 2016?
Assume the median price of a one-bedroom apartment in April 2016 was p dollars. Write an equation to model this scenario. There is no need to solve it.

12. In May 2016, the median rent price for a one-bedroom apartment in a city was reported to be \$1,002.00 per month. This was reported to be an increase of 0.2% over the previous month. Based on this reporting, what was the median price of a one-bedroom apartment in April 2016?
Assume the median price of a one-bedroom apartment in April 2016 was p dollars. Write an equation to model this scenario. There is no need to solve it.

13. Briana is driving an average of 41 miles per hour, and she is 123 miles away from home. After how many hours will she reach his home?
Assume Briana will reach home after h hours. Write an equation to model this scenario. There is no need to solve it.

14. Ryan is driving an average of 44 miles per hour, and he is 83.6 miles away from home. After how many hours will he reach his home?
Assume Ryan will reach home after h hours. Write an equation to model this scenario. There is no need to solve it.

15. Uhaul charges an initial fee of \$31.90 and then \$0.89 per mile to rent a 15-foot truck for a day. If the total bill is \$182.31, how many miles were driven?
Assume m miles were driven. Write an equation to model this scenario. There is no need to solve it.

16. Uhaul charges an initial fee of \$34.10 and then \$0.75 per mile to rent a 15-foot truck for a day. If the total bill is \$94.85, how many miles were driven?
Assume m miles were driven. Write an equation to model this scenario. There is no need to solve it.

17. A cat litter box has a rectangular base that is 24 inches by 12 inches. What will the height of the cat litter be if 3 cubic feet of cat litter is poured? (Hint: $1\text{ ft}^3 = 1728\text{ in}^3$)
Assume h inches will be the height of the cat litter if 3 cubic feet of cat litter is poured. Write an equation to model this scenario. There is no need to solve it.

18. A cat litter box has a rectangular base that is 24 inches by 24 inches. What will the height of the cat litter be if 4 cubic feet of cat litter is poured? (Hint: $1\text{ ft}^3 = 1728\text{ in}^3$)
Assume h inches will be the height of the cat litter if 4 cubic feet of cat litter is poured. Write an equation to model this scenario. There is no need to solve it.

19. Ibuprofen for infants comes in a liquid form and contains 30 milligrams of ibuprofen for each 0.75 milliliters of liquid. If a child is to receive a dose of 50 milligrams of ibuprofen, how many milliliters of liquid should they be given?
Assume l milliliters of liquid should be given. Write an equation to model this scenario. There is no need to solve it.

20. Ibuprofen for infants comes in a liquid form and contains 35 milligrams of ibuprofen for each 0.875 milliliters of liquid. If a child is to receive a dose of 60 milligrams of ibuprofen, how many milliliters of liquid should they be given?
Assume l milliliters of liquid should be given. Write an equation to model this scenario. There is no need to solve it.

21. The property taxes on a 2400-square-foot house are \$2,832.00 per year. Assuming these taxes are proportional, what are the property taxes on a 1600-square-foot house?
Assume property taxes on a 1600-square-foot house is t dollars. Write an equation to model this scenario. There is no need to solve it.

22. The property taxes on a 1900-square-foot house are \$2,489.00 per year. Assuming these taxes are proportional, what are the property taxes on a 1500-square-foot house?
Assume property taxes on a 1500-square-foot house is t dollars. Write an equation to model this scenario. There is no need to solve it.

Modeling with Linear Inequalities

23. A truck that hauls water is capable of carrying a maximum of 1800 lb. Water weighs $8.3454\,\frac{\text{lb}}{\text{gal}}$, and the plastic tank on the truck that holds water weighs 74 lb.
Assume the truck can carry a maximum of g gallons of water. Write an *inequality* to model this scenario. There is no need to solve it.

24. A truck that hauls water is capable of carrying a maximum of 2800 lb. Water weighs $8.3454\,\frac{\text{lb}}{\text{gal}}$, and the plastic tank on the truck that holds water weighs 80 lb.
Assume the truck can carry a maximum of g gallons of water. Write an *inequality* to model this scenario. There is no need to solve it.

25. Lindsay's maximum lung capacity is 6.8 liters. If her lungs are full and she exhales at a rate of 0.8 liters per second, write an *inequality* that models when she will still have at least 0.56 liters of air left in his lungs. There is no need to solve it.

26. Lily's maximum lung capacity is 7.3 liters. If her lungs are full and she exhales at a rate of 0.8 liters per second, write an *inequality* that models when she will still have at least 3.94 liters of air left in his lungs. There is no need to solve it.

27. A swimming pool is being filled with water from a garden hose at a rate of 9 gallons per minute. If the pool already contains 100 gallons of water and can hold up to 334 gallons, set up an *inequality* modeling how much time can pass without the pool overflowing. There is no need to solve it.

28. A swimming pool is being filled with water from a garden hose at a rate of 7 gallons per minute. If the pool already contains 10 gallons of water and can hold up to 108 gallons, set up an *inequality* modeling how much time can pass without the pool overflowing. There is no need to solve it.

29. An engineer is designing a cylindrical springform pan (the kind of pan a cheesecake is baked in). The pan needs to be able to hold a volume at least 33 cubic inches and have a diameter of 6 inches. Write an *inequality* modeling possible height of the pan. There is no need to solve it.

30. An engineer is designing a cylindrical springform pan (the kind of pan a cheesecake is baked in). The pan needs to be able to hold a volume at least 96 cubic inches and have a diameter of 7 inches. Write an *inequality* modeling possible height of the pan. There is no need to solve it.

Translating English Phrases into Math Expressions and Equations Translate the following phrase or sentence into a math expression or equation (whichever is appropriate).

31. five more than a number

32. one less than a number

33. the sum of a number and eight

34. the difference between a number and four

35. the difference between one and a number

36. the difference between seven and a number

37. four subtracted from a number

38. ten added to a number

39. seven decreased by a number

40. three increased by a number

41. a number decreased by ten

42. a number increased by six

43. four times a number, decreased by five

44. ten times a number, decreased by nine

45. six less than four times a number

46. two less than eight times a number

47. nine less than the quotient of three and a number

48. five more than the quotient of six and a number

49. Three times a number is twenty-four.

50. Nine times a number is thirty-six.

51. The sum of forty and a number is fifty-six.

52. The sum of eighteen and a number is twenty-three.

53. The quotient of a number and three is thirteen thirds.

54. The quotient of a number and thirty-one is one thirty-first.

55. The quotient of three and a number is one eighth.

56. The quotient of twenty and a number is five thirds.

57. The sum of four times a number and ten is 194.

58. The sum of twice a number and twenty-two is eighty-eight.

59. Two less than six times a number is sixty-four.

60. Two less than four times a number is 118.

61. The product of eight and a number, added to eight, is 344.

62. The product of six and a number, increased by four, is 178.

63. The product of three and a number added to seven, is sixty-nine.

64. The product of eight and a number increased by three, is 424.

65. one seventh of a number

66. one fourth of a number

67. twenty-seven forty-seconds of a number

68. seventeen forty-fourths of a number

69. a number decreased by two twelfths of itself

70. a number increased by eight thirtieths of itself

71. A number decreased by two thirds is three elevenths of that number.

72. A number increased by one sixth is three elevenths of that number.

73. One less than the product of three elevenths and a number gives two ninths of that number.

74. Three more than the product of two thirds and a number gives one third of that number.

Challenge

75. Last year, Joan received a 1% raise. This year, she received a 2% raise. Her current wage is $11.07 an hour. What was her wage before the two raises?

1.9 Variables, Expressions, and Equations Chapter Review

1.9.1 Variables and Evaluating Expressions

In Section 1.1 we covered the definitions of variables and expressions, and how to evaluate an expression with a particular number.

Evaluating Expressions. When we evaluate an expression's value, we substitute each variable with its given value.

Example 1.9.1 Evaluate the value of $\frac{5}{9}(F-32)$ if $F=212$.

$$\begin{aligned}\frac{5}{9}(F-32) &= \frac{5}{9}(212-32)\\ &= \frac{5}{9}(180)\\ &= 100\end{aligned}$$

Substituting a Negative Number. When we substitute a variable with a negative number, it's important to use parentheses around the number.

Example 1.9.2 Evaluate the following expressions if $x=-3$.

a. $\begin{aligned}x^2 &= (-3)^2\\ &= 9\end{aligned}$

b. $\begin{aligned}x^3 &= (-3)^3\\ &= (-3)(-3)(-3)\\ &= -27\end{aligned}$

c. $\begin{aligned}-x^2 &= -(-3)^2\\ &= -9\end{aligned}$

d. $\begin{aligned}-x^3 &= -(-3)^3\\ &= -(-27)\\ &= 27\end{aligned}$

1.9.2 Combining Like Terms

In Section 1.2 we covered the definitions of a term and how to combine like terms.

Example 1.9.3 List the terms in the expression $5x-3y+\frac{2w}{3}$.

Explanation. The expression has three terms that are being added, $5x$, $-3y$ and $\frac{2w}{3}$.

Example 1.9.4 Simplify the expression $5x-3x^2+2x+5x^2$, if possible, by combining like terms.

Explanation. This expression has four terms: $5x$, $-3x^2$, $2x$, and $5x^2$. Both $5x$ and $2x$ are like terms; also $-3x^2$ and $5x^2$ are like terms. When we combine like terms, we get:

$$5x-3x^2+2x+5x^2 = 7x+2x^2$$

Note that we cannot combine $7x$ and $2x^2$ because x and x^2 represent different quantities.

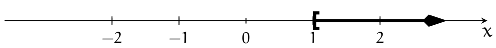
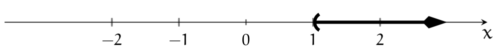
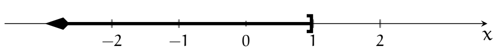
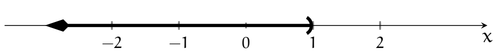
1.9.3 Comparison Symbols and Notation for Intervals

The following are symbols used to compare numbers.

Symbol	Means	True	True	False
$=$	equals	$13 = 13$	$\frac{5}{4} = 1.25$	$5 \overset{\text{no}}{=} 6$
$>$	is greater than	$13 > 11$	$\pi > 3$	$9 \overset{\text{no}}{>} 9$
$\geq$	is greater than or equal to	$13 \geq 11$	$3 \geq 3$	$11.2 \overset{\text{no}}{\geq} 10.2$
$<$	is less than	$-3 < 8$	$\frac{1}{2} < \frac{2}{3}$	$2 \overset{\text{no}}{<} -2$
$\leq$	is less than or equal to	$-3 \leq 8$	$3 \leq 3$	$\frac{4}{5} \overset{\text{no}}{\leq} \frac{3}{5}$
$\neq$	is not equal to	$10 \neq 20$	$\frac{1}{2} \neq 1.2$	$\frac{3}{8} \overset{\text{no}}{\neq} 0.375$

Table 1.9.5: Comparison Symbols

The following are some examples of set-builder notation and interval notation.

Graph	Set-builder Notation	Interval Notation
	$\{x \mid x \geq 1\}$	$[1, \infty)$
	$\{x \mid x > 1\}$	$(1, \infty)$
	$\{x \mid x \leq 1\}$	$(-\infty, 1]$
	$\{x \mid x < 1\}$	$(-\infty, 1)$

1.9.4 Equations and Inequalities as True/False Statements

In Section 1.4 we covered the definitions of an equation and an inequality, as well as how to verify if a particular number is a solution to them.

Checking Possible Solutions. Given an equation or an inequality (with one variable), checking if some particular number is a solution is just a matter of replacing the value of the variable with the specified number and determining if the resulting equation/inequality is true or false. This may involve some amount of arithmetic simplification.

Example 1.9.6 Is -5 a solution to $2(x + 3) - 2 = 4 - x$?

Explanation. To find out, substitute in -5 for x and see what happens.

$$\begin{aligned} 2(x+3) - 2 &= 4 - x \\ 2((-5)+3) - 2 &\overset{?}{=} 4 - (-5) \\ 2(-2) - 2 &\overset{?}{=} 9 \\ -4 - 2 &\overset{?}{=} 9 \\ -6 &\overset{\text{no}}{=} 9 \end{aligned}$$

So no, -5 is not a solution to $2(x + 3) - 2 = 4 - x$.

1.9.5 Solving One-Step Equations

In Section 1.5 we covered to to add, subtract, multiply, or divide on both sides of an equation to isolate the variable, summarized in Fact 1.5.13. We also learned how to state our answer, either as a solution or a solution set. Last, we discussed how to solve equations with fractions.

Solving One-Step Equations. When we solve linear equations, we use Properties of Equivalent Equations and follow an algorithm to solve a linear equation.

Example 1.9.7 Solve for g in $\frac{1}{2} = \frac{2}{3} + g$.

Explanation.

We will subtract $\frac{2}{3}$ on both sides of the equation:

$$\begin{aligned} \frac{1}{2} &= \frac{2}{3} + g \\ \frac{1}{2} - \frac{2}{3} &= \frac{2}{3} + g - \frac{2}{3} \\ \frac{3}{6} - \frac{4}{6} &= g \\ -\frac{1}{6} &= g \end{aligned}$$

We will check the solution by substituting g in the original equation with $-\frac{1}{6}$:

$$\begin{aligned} \frac{1}{2} &= \frac{2}{3} + g \\ \frac{1}{2} &\stackrel{?}{=} \frac{2}{3} + \left(-\frac{1}{6}\right) \\ \frac{1}{2} &\stackrel{?}{=} \frac{4}{6} + \left(-\frac{1}{6}\right) \\ \frac{1}{2} &\stackrel{\checkmark}{=} \frac{3}{6} \end{aligned}$$

The solution $-\frac{1}{6}$ is checked and the solution set is $\left\{-\frac{1}{6}\right\}$.

1.9.6 Solving One-Step Inequalities

In Section 1.6 we covered how solving inequalities is very much like how we solve equations, except that if we multiply or divide by a negative we switch the inequality sign.

Solving One-Step Inequalities. When we solve linear inequalities, we also use Properties of Equivalent Equations with one small complication: When we multiply or divide by the same *negative* number on both sides of an inequality, the direction reverses!

Example 1.9.8 Solve the inequality $-2x \geq 12$. State the solution set with both interval notation and set-builder notation.

Explanation. To solve this inequality, we will divide each side by -2:

$$\begin{aligned} -2x &\geq 12 \\ \frac{-2x}{-2} &\leq \frac{12}{-2} && \text{Note the change in direction.} \\ x &\leq -6 \end{aligned}$$

- The inequality's solution set in interval notation is $(-\infty, -6]$.
- The inequality's solution set in set-builder notation is $\{x \mid x \leq -6\}$.

1.9.7 Algebraic Properties and Simplifying Expressions

In Section 1.7 we covered the definitions of the identities and inverses, and the various algebraic properties. We then learned about the order of operations.

Example 1.9.9 Use the associative, commutative, and distributive properties to simplify the expression $5x + 9(-2x+3)$ as much as possible.

Explanation. We will remove parentheses by the distributive property, and then combine like terms:

$$\begin{aligned} 5x + 9(-2x+3) &= 5x + 9(-2x+3) \\ &= 5x + 9(-2x) + 9(3) \\ &= 5x - 18x + 27 \\ &= -13x + 27 \end{aligned}$$

1.9.8 Modeling with Equations and Inequalities

In Section 1.8 we covered how to translate phrases into mathematics, and how to set up equations and inequalities for application models.

Modeling with Equations and Inequalities. To set up an equation modeling a real world scenario, the first thing we need to do is to identify what variable we will use. The variable we use will be determined by whatever is unknown in our problem statement. Once we've identified and defined our variable, we'll use the numerical information provided in the equation to set up our equation.

Example 1.9.10 A bathtub contains $2.5\,\text{ft}^3$ of water. More water is being poured in at a rate of $1.75\,\text{ft}^3$ per minute. When will the amount of water in the bathtub reach $6.25\,\text{ft}^3$?

Explanation. Since the question being asked in this problem starts with "when," we immediately know that the unknown is time. As the volume of water in the tub is measured in ft^3 per minute, we know that time needs to be measured in minutes. We'll defined t to be the number of minutes that water is poured into the tub. Since each minute there are $1.75\,\text{ft}^3$ of water added, we will add the expression $1.75t$ to 2.5 to obtain the total amount of water. Thus the equation we set up is:

$$2.5 + 1.75t = 6.25$$

1.9.9 Exercises

Variables and Evaluating Expressions

1. Evaluate the expression y^2:

a. For $y = 8$.

b. For $y = -3$.

2. Evaluate the expression y^2:

a. For $y = 5$.

b. For $y = -7$.

3. Evaluate $\dfrac{2r-9}{9r}$ for $r = -9$.

4. Evaluate $\dfrac{6r-2}{9r}$ for $r = 5$.

5. Evaluate the expression $\dfrac{1}{3}(x+4)^2 - 4$ when $x = -7$.

6. Evaluate the expression $\dfrac{1}{4}(x+4)^2 - 2$ when $x = -8$.

To convert a temperature measured in degrees Fahrenheit to degrees Celsius, there is a formula:

$$C = \frac{5}{9}(F - 32)$$

where C represents the temperature in degrees Celsius and F represents the temperature in degrees Fahrenheit.

7. If a temperature is 5°F, what is that temperature measured in Celsius?

8. If a temperature is 14°F, what is that temperature measured in Celsius?

The percentage of births in the U.S. delivered via C-section can be given by the following formula for the years since 1996:

$$p = 0.8(y - 1996) + 21$$

In this formula y is a year after 1996 and p is the percentage of births delivered via C-section for that year.

9. What percentage of births in the U.S. were delivered via C-section in the year 2001?

[ ] of births in the U.S. were delivered via C-section in the year 2001.

10. What percentage of births in the U.S. were delivered via C-section in the year 2003?

[ ] of births in the U.S. were delivered via C-section in the year 2003.

Combining Like Terms

11. Count the number of terms in each expression.

a. $8t + y + 8s^2$

b. $3t - 5x$

c. $-5z^2 + 7z$

d. $-3y^2 - 3x - 5x + 2z^2$

12. Count the number of terms in each expression.

a. $-9t^2 + 9t^2 + 6z^2$

b. $2t - 9t + 5s - 1$

c. $6x^2 - 7s + x + 9y$

d. $9t^2 + 8z^2$

13. List the terms in each expression.

a. $-6.9t - 1.1x + 3.4x^2$

b. $0.7s^2 + 5.3y^2 - 1.5$

c. $-3.4x^2 - 2.1t$

d. $-7.9s - 8.6t^2$

14. List the terms in each expression.

a. $-5.2t^2 + 7.1z + 1.4t^2 - 0.5s$

b. $1.3t - 7.7x$

c. $-5.6x^2 + 8.5y - 8.9x^2$

d. $-0.1y$

Simplify each expression, if possible, by combining like terms.

15. a. $-\frac{1}{2}t - \frac{3}{7} - \frac{9}{2}t^2 - t^2$

b. $-\frac{5}{3}s^2 - \frac{1}{6}s^2 + 9z^2 + \frac{1}{6}s$

c. $-2t + \frac{5}{8}z$

d. $-\frac{3}{7}y^2 - 2y^2 + \frac{6}{7}$

16. a. $\frac{2}{3}t^2 + \frac{9}{7}t^2$

b. $-\frac{2}{3}y^2 + \frac{7}{6}s^2 - \frac{3}{8}s^2$

c. $\frac{7}{4}z + s^2 + \frac{1}{3}z^2$

d. $-9z^2 + 2y^2 + \frac{3}{4}z$

Comparison Symbols and Notation for Intervals

17. Decide if each comparison is true or false.

a. $-\frac{3}{3} \neq -\frac{6}{6}$ (☐ True ☐ False)

b. $\frac{29}{4} < \frac{7}{4}$ (☐ True ☐ False)

c. $-\frac{5}{9} < -\frac{5}{9}$ (☐ True ☐ False)

d. $\frac{7}{4} \geq -\frac{58}{7}$ (☐ True ☐ False)

e. $\frac{19}{4} > -\frac{15}{8}$ (☐ True ☐ False)

f. $\frac{82}{9} = -\frac{41}{5}$ (☐ True ☐ False)

18. Decide if each comparison is true or false.

a. $-\frac{6}{3} = -\frac{12}{6}$ (☐ True ☐ False)

b. $\frac{9}{2} < \frac{9}{2}$ (☐ True ☐ False)

c. $-\frac{16}{3} > \frac{27}{8}$ (☐ True ☐ False)

d. $\frac{11}{6} \leq -\frac{6}{4}$ (☐ True ☐ False)

e. $\frac{9}{2} \geq -\frac{76}{8}$ (☐ True ☐ False)

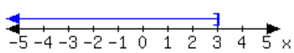
f. $\frac{77}{9} \neq \frac{5}{2}$ (☐ True ☐ False)

19. Choose $<$, $>$, or $=$ to make a true statement.
$\frac{2}{3} + \frac{4}{5}$ (☐ $<$ ☐ $>$ ☐ $=$) $\frac{4}{3} \div \frac{1}{4}$

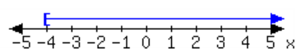
20. Choose $<$, $>$, or $=$ to make a true statement.
$\frac{3}{2} + \frac{1}{3}$ (☐ $<$ ☐ $>$ ☐ $=$) $\frac{4}{5} \div \frac{4}{5}$

21. For each interval expressed in the number lines, give the interval notation and set-builder notation.

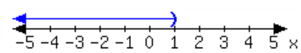
a.



b.

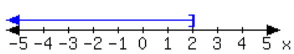


c.

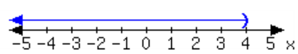


22. For each interval expressed in the number lines, give the interval notation and set-builder notation.

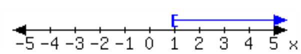
a.



b.



c.



23. A set is written using set-builder notation. Write it using interval notation.
$\{x \mid x \leq 6\}$

24. A set is written using set-builder notation. Write it using interval notation.
$\{x \mid x \leq 8\}$

Equations and Inequalities as True/False Statements

25. Is -2 a solution for x in the equation $-6x - 10 = 2x + 6$? (☐ Yes ☐ No)

26. Is -10 a solution for x in the equation $3x + 5 = 2x - 4$? (☐ Yes ☐ No)

27. Is $\frac{3}{4}$ a solution for x in the equation $-\frac{5}{9}x - \frac{1}{2} = -\frac{67}{54}$? (☐ Yes ☐ No)

28. Is 6 a solution for y in the equation $\frac{3}{7}y + \frac{1}{6} = \frac{5}{21}$? (☐ Yes ☐ No)

29. When a plant was purchased, it was 1 inches tall. It grows 0.6 inches per day. How many days later will the plant be 12.4 inches tall?

Assume the plant will be 12.4 inches tall d days later. We can solve this problem using the equation:

$$0.6d + 1 = 12.4$$

Check whether 22 is a solution for d of this equation. (☐ Yes ☐ No)

30. When a plant was purchased, it was 2.4 inches tall. It grows 0.7 inches per day. How many days later will the plant be 11.5 inches tall?

Assume the plant will be 11.5 inches tall d days later. We can solve this problem using the equation:

$$0.7d + 2.4 = 11.5$$

Check whether 15 is a solution for d of this equation. (☐ Yes ☐ No)

Solving One-Step Equations

31. Solve the equation. $r - 4 = 4$

32. Solve the equation. $t - 10 = -7$

33. Solve the equation. $\frac{6}{7} + p = -\frac{5}{8}$

34. Solve the equation. $\frac{6}{5} + x = -\frac{7}{6}$

35. Solve the equation. $\frac{5}{3} = \frac{x}{27}$

36. Solve the equation. $\frac{9}{4} = \frac{x}{28}$

37. Solve the equation.

a. $2y = 16$

b. $2 + x = 16$

38. Solve the equation.

a. $6y = 36$

b. $6 + t = 36$

39. Solve the equation.

a. $54 = -9r$

b. $54 = -9 + x$

40. Solve the equation.

a. $14 = -7r$

b. $14 = -7 + x$

41. Solve the equation.

a. $30 = -\frac{10}{7}m$

b. $-30 = -\frac{10}{7}p$

42. Solve the equation.

a. $20 = -\frac{4}{5}p$

b. $-20 = -\frac{4}{5}A$

Solving One-Step Inequalities Solve this inequality.

43. $5 > x - 6$

44. $1 > x - 10$

45. $2x \leq 6$

46. $3x \leq 6$

47. $-3x \geq 12$

48. $-4x \geq 12$

49. $\frac{7}{2}x > 7$

50. $\frac{8}{7}x > 8$

51. $-\frac{9}{5}x \leq 9$

52. $-\frac{10}{7}x \leq 40$

Algebraic Properties and Simplifying Expressions

53. What is the additive inverse of -8?

54. What is the additive inverse of -6?

55. What is the multiplicative inverse of -4?

56. What is the multiplicative inverse of -1?

57. Use the associative property of addition to write an equivalent expression to $a + (19 + y)$.

58. Use the associative property of addition to write an equivalent expression to $q + (18 + p)$.

59. Use the associative property of multiplication to write an equivalent expression to $7(5m)$.

60. Use the associative property of multiplication to write an equivalent expression to $4(9p)$.

61. Use the commutative property of addition to write an equivalent expression to $4q + 91$.

62. Use the commutative property of addition to write an equivalent expression to $7y + 56$.

63. Use the commutative property of multiplication to write an equivalent expression to $21r$.

64. Use the commutative property of multiplication to write an equivalent expression to $87a$.

65. Use the distributive property to write an equivalent expression to $6(c + 10)$ that has no grouping symbols.

66. Use the distributive property to write an equivalent expression to $4(r + 7)$ that has no grouping symbols.

67. Use the distributive property to simplify $8 - 7(-9 - 7n)$ completely.

68. Use the distributive property to simplify $6 - 9(-9 - 10m)$ completely.

69. Use the distributive property to simplify $3p - 5p\left(7 - 10p^2\right)$ completely.

70. Use the distributive property to simplify $8q - 2q\left(1 - 10q^4\right)$ completely.

71. Fully simplify $-(-5x - 4) + 3(7x + 8)$.

72. Fully simplify $2(x + 8) - 4(-x + 3)$.

Modeling with Equations and Inequalities

73. A bicycle for sale costs \$139.36, which includes 7.2% sales tax. What was the cost before sales tax?
Assume the bicycle's price before sales tax is p dollars. Write an equation to model this scenario. There is no need to solve it.

74. A bicycle for sale costs \$169.76, which includes 6.1% sales tax. What was the cost before sales tax?
Assume the bicycle's price before sales tax is p dollars. Write an equation to model this scenario. There is no need to solve it.

75. The price of a washing machine after 5% discount is \$180.50. What was the original price of the washing machine (before the discount was applied)?
Assume the washing machine's price before the discount is p dollars. Write an equation to model this scenario. There is no need to solve it.

76. The price of a washing machine after 25% discount is \$157.50. What was the original price of the washing machine (before the discount was applied)?
Assume the washing machine's price before the discount is p dollars. Write an equation to model this scenario. There is no need to solve it.

77. Lisa is driving an average of 53 miles per hour, and she is 106 miles away from home. After how many hours will she reach his home?
Assume Lisa will reach home after h hours. Write an equation to model this scenario. There is no need to solve it.

78. Emiliano is driving an average of 57 miles per hour, and he is 228 miles away from home. After how many hours will he reach his home?
Assume Emiliano will reach home after h hours. Write an equation to model this scenario. There is no need to solve it.

Translate the following phrase or sentence into a math expression or equation (whichever is appropriate).

79. seven less than eight times a number

80. three less than three times a number

81. ten more than the quotient of seven and a number

82. seven more than the quotient of ten and a number

83. One less than three times a number yields sixty-eight.

84. One less than eight times a number yields 335.

85. The product of five and a number increased by three, yields 125.

86. The product of three and a number added to six, yields forty-five.

87. A number increased by three fourths is three tenths of that number.

88. A number decreased by two sevenths is three tenths of that number.

Chapter 2

Linear Equations and Inequalities

2.1 Solving Multistep Linear Equations

We learned how to solve one-step equations in Section 1.5. In this section, we will solve equations that need more than one step.

2.1.1 Solving Two-Step Equations

Example 2.1.2 A water tank can hold up to 140 gallons of water, but it starts with only 5 gallons. A tap is turned on, pouring 15 gallons of water into the tank every minute. After how many minutes will the tank be full? Let's find a pattern first.

Minutes Since Tap Was Turned on	Amount of Water in the Tank (in Gallons)
0	5
1	$15 \cdot 1 + 5 = 20$
2	$15 \cdot 2 + 5 = 35$
3	$15 \cdot 3 + 5 = 50$
4	$15 \cdot 4 + 5 = 65$
$\vdots$	$\vdots$
m	$15m + 5$

Figure 2.1.3: Amount of Water in the Tank

We can see that after m minutes, the tank has $15m + 5$ gallons of water. This makes sense since the tap pours 15 gallons into the tank for each of m minutes, and it has 5 gallons to start with. To find when the tank will be full with 140 gallons, we can write the equation

$$15m + 5 = 140$$

First, we need to isolate the variable term, $15m$. In other words, we need to "remove" the 5 from the left side of the equals sign. We can do this by subtracting 5 from both sides of the equation. Once the variable term is isolated, we can eliminate its coefficient and solve for m.

The full process is:

$$\begin{aligned}
15m + 5 &= 140 \\
15m + 5 - 5 &= 140 - 5 \\
15m &= 135 \\
\frac{15m}{15} &= \frac{135}{15} \\
m &= 9
\end{aligned}$$

We should check the solution by substituting m with 9 in the original equation:

$$\begin{aligned}
15m + 5 &= 140 \\
15(9) + 5 &\stackrel{?}{=} 140 \\
135 + 5 &\stackrel{\checkmark}{=} 140
\end{aligned}$$

The solution 9 is checked.

This problem had *context*. It was not simply solving an equation, rather it comes with the story of the tank filling with water. So we report a conclusion that uses that context: In summary, the tank will be full after 9 minutes.

In solving the two-step equation in Example 2.1.2, we first isolated the variable expression $15m$ and then eliminated the coefficient of 15 by dividing each side of the equation by 15. These two steps are at the heart of our approach to solving linear equations. For more complicated equations, we may need to simplify one or both sides first. Below is a general approach that summarizes all of this.

Process 2.1.4 Steps to Solve Linear Equations.

Simplify *Simplify the expressions on each side of the equation by distributing and combining like terms.*

Separate *Use addition or subtraction to separate the variable terms and constant terms so that they are on different sides of the equation.*

Clear Coefficient *Use multiplication or division to eliminate the variable term's coefficient.*

Check *Check the solution in the original equation. Substitute values into the original equation and use the order of operations to simplify both sides. It's important to use the order of operations alone rather than properties like the distributive law. Otherwise you might repeat the same arithmetic errors made while solving and fail to catch an incorrect solution.*

Summarize *State the solution set. Or in the case of an application problem, summarize the result in a complete sentence using appropriate units.*

Example 2.1.5 Solve for y in the equation $7 - 3y = -8$.

Explanation. To solve, we first separate the variable terms and constant terms to different sides of the equation. Then we eliminate the variable term's coefficient.

$$\begin{aligned}
7 - 3y &= -8 \\
7 - 3y - 7 &= -8 - 7 \\
-3y &= -15 \\
\frac{-3y}{-3} &= \frac{-15}{-3} \\
y &= 5
\end{aligned}$$

Checking the solution $y = 5$:

$$\begin{aligned}
7 - 3y &= -8 \\
7 - 3(5) &\stackrel{?}{=} -8 \\
7 - 15 &\stackrel{\checkmark}{=} -8
\end{aligned}$$

So the solution to the equation $7 - 3y = -8$ is 5 and the solution set is $\{5\}$.

2.1.2 Solving Multistep Linear Equations

Example 2.1.6 Ahmed has saved \$2,500.00 in his savings account and is going to start saving \$550.00 per month. Julia has saved \$4,600.00 in her savings account and is going to start saving \$250.00 per month. If this situation continues, how many months will it take Ahmed catch up with Julia in savings?

Ahmed saves \$550.00 per month, so he can save $550m$ dollars in m months. With the \$2,500.00 he started with, after m months he has $550m + 2500$ dollars. Similarly, after m months, Julia has $250m + 4600$ dollars. To find when those two accounts will have the same amount of money, we can write the equation

$$550m + 2500 = 250m + 4600.$$

$$\begin{aligned}
550m + 2500 &= 250m + 4600 \\
550m + 2500 - 2500 &= 250m + 4600 - 2500 \\
550m &= 250m + 2100 \\
550m - 250m &= 250m + 2100 - 250m \\
300m &= 2100 \\
\frac{300m}{300} &= \frac{2100}{300} \\
m &= 7
\end{aligned}$$

Checking the solution 7:

$$\begin{aligned}
550m + 2500 &= 250m + 4600 \\
550(7) + 2500 &\stackrel{?}{=} 250(7) + 4600 \\
3850 + 2500 &\stackrel{?}{=} 1750 + 4600 \\
6350 &\stackrel{\checkmark}{=} 6350
\end{aligned}$$

Ahmed will catch up with Julia's savings after 7 months.

Example 2.1.7 Solve for x in $5 - 2x = 5x - 9$.

Explanation.

$$\begin{aligned}
5 - 2x &= 5x - 9 \\
5 - 2x - 5 &= 5m - 9 - 5 \\
-2x &= 5x - 14 \\
-2x - 5x &= 5x - 14 - 5x \\
-7x &= -14 \\
\frac{-7x}{-7} &= \frac{-14}{-7} \\
x &= 2
\end{aligned}$$

Checking the solution 2:

$$\begin{aligned}
5 - 2x &= 5x - 9 \\
5 - 2(2) &\stackrel{?}{=} 5(2) - 9 \\
5 - 4 &\stackrel{?}{=} 10 - 9 \\
1 &\stackrel{\checkmark}{=} 1
\end{aligned}$$

Therefore the solution is 2 and the solution set is $\{2\}$.

Example 2.1.8 In Example 2.1.7, we could have moved variable terms to the right side of the equals sign, and number terms to the left side. We chose not to. There's no reason we *couldn't* have moved variable terms to the right side though. Let's compare:

$$\begin{aligned}
5 - 2x &= 5x - 9 \\
5 - 2x + 9 &= 5x - 9 + 9 \\
14 - 2x &= 5x \\
14 - 2x + 2x &= 5x + 2x \\
14 &= 7x \\
\frac{14}{7} &= \frac{7x}{7}
\end{aligned}$$

$$2 = x$$

Lastly, we could save a step by moving variable terms and number terms in one step:

$$\begin{aligned}
5 - 2x &= 5x - 9\\
5 - 2x + 2x + 9 &= 5x - 9 + 2x + 9\\
14 &= 7x\\
\frac{14}{7} &= \frac{7x}{7}\\
2 &= x
\end{aligned}$$

For the sake of a slow and careful explanation, the examples in this chapter will move variable terms and number terms separately.

Checkpoint 2.1.9 Solve the equation.

$9y + 3 = y + 83$

Explanation. The first step is to subtract terms in order to separate the variable and non-variable terms.

$$\begin{aligned}
9y + 3 &= y + 83\\
9y + 3 - \mathbf{y} - \mathbf{3} &= y + 83 - \mathbf{y} - \mathbf{3}\\
8y &= 80\\
\frac{8y}{8} &= \frac{80}{8}\\
y &= 10
\end{aligned}$$

The solution to this equation is 10. To stress that this is a value assigned to y, some report $y = 10$. We can also say that the solution set is $\{10\}$, or that $y \in \{10\}$. If we substitute 10 in for y in the original equation $9y + 3 = y + 83$, the equation will be true. Please check this on your own; it is an important habit.

The next example requires combining like terms.

Example 2.1.10 Solve for n in $n - 9 + 3n = n - 3n$.

Explanation. To start solving this equation, we'll need to combine like terms. After this, we can put all terms containing n on one side of the equation and finish solving for n.

$$\begin{aligned}
n - 9 + 3n &= n - 3n\\
4n - 9 &= -2n\\
4n - 9 - 4n &= -2n - 4n\\
-9 &= -6n\\
\frac{-9}{-6} &= \frac{-6n}{-6}\\
\frac{3}{2} &= n
\end{aligned}$$

Checking the solution $\frac{3}{2}$:

$$\begin{aligned}
n - 9 + 3n &= n - 3n\\
\frac{3}{2} - 9 + 3\left(\frac{3}{2}\right) &\stackrel{?}{=} \frac{3}{2} - 3\left(\frac{3}{2}\right)\\
\frac{3}{2} - 9 + \frac{9}{2} &\stackrel{?}{=} \frac{3}{2} - \frac{9}{2}\\
\frac{12}{2} - 9 &\stackrel{?}{=} -\frac{6}{2}\\
6 - 9 &\stackrel{\checkmark}{=} -3
\end{aligned}$$

The solution to the equation $n - 9 + 3n = n - 3n$ is $\frac{3}{2}$ and the solution set is $\left\{\frac{3}{2}\right\}$.

Checkpoint 2.1.11 Solve the equation.
$-6+7=-5t-t-11$

Explanation. The first step is simply to combine like terms.

$$\begin{aligned}
-6+7&=-5t-t-11\\
1&=-6t-11\\
1+\mathbf{11}&=-6t-11+\mathbf{11}\\
12&=-6t\\
\frac{12}{-6}&=\frac{-6t}{-6}\\
-2&=t\\
t&=-2
\end{aligned}$$

The solution to this equation is -2. To stress that this is a value assigned to t, some report $t=-2$. We can also say that the solution set is $\{-2\}$, or that $t\in\{-2\}$. If we substitute -2 in for t in the original equation $-6+7=-5t-t-11$, the equation will be true. Please check this on your own; it is an important habit.

We should be careful when we distribute a negative sign into the parentheses, like in the next example.

Example 2.1.12 Solve for a in $4-(3-a)=-2-2(2a+1)$.

Explanation. To solve this equation, we will simplify each side of the equation, manipulate it so that all variable terms are on one side and all constant terms are on the other, and then solve for a:

$$\begin{aligned}
4-(3-a)&=-2-2(2a+1)\\
4-3+a&=-2-4a-2\\
1+a&=-4-4a\\
1+a+4a&=-4-4a+4a\\
1+5a&=-4\\
1+5a-1&=-4-1\\
5a&=-5\\
\frac{5a}{5}&=\frac{-5}{5}\\
a&=-1
\end{aligned}$$

Checking the solution -1:

$$\begin{aligned}
4-(3-a)&=-2-2(2a+1)\\
4-(3-(-1))&\stackrel{?}{=}-2-2(2(-1)+1)\\
4-(4)&\stackrel{?}{=}-2-2(-1)\\
0&\stackrel{?}{=}-2+2\\
0&\stackrel{\checkmark}{=}0
\end{aligned}$$

Therefore the solution to the equation is -1 and the solution set is $\{-1\}$.

2.1.3 Revisiting Applications from Section 1.8

In Section 1.8, we set up equations given some background information, but we didn't try to solve the equations yet. Now we can do that.

Example 2.1.13 Here we revisit Example 1.8.2.

A savings account starts with \$500. Each month, an automatic deposit of \$150 is made. Find the number of months it will take for the balance to reach \$1,700.

Explanation.

To set up an equation, we might start by making a table in order to identify a general pattern for the total amount in the account after m months. In Figure 2.1.14, we find the pattern is that after m months, the total amount saved is $500 + 150m$. Using this pattern, we determine that an equation representing when the total savings equals $1700 is:

$$500 + 150m = 1700$$

Months Since Saving Started	Total Amount Saved (in Dollars)
0	500
1	$500 + 150 = 650$
2	$500 + 150(2) = 800$
3	$500 + 150(3) = 950$
4	$500 + 150(4) = 1100$
⋮	⋮
m	$500 + 150m$

Figure 2.1.14: Amount in Savings Account

To solve this equation, we can start by subtracting 500 from each side. Then we cna divide each side by 150.

$$\begin{aligned}
500 + 150m &= 1700\\
500 + 150m - 500 &= 1700 - 500\\
150m &= 1200\\
\frac{150m}{150} &= \frac{1200}{150}\\
m &= 8
\end{aligned}$$

Checking the solution 8:

$$\begin{aligned}
500 + 150m &= 1700\\
500 + 150(8) &\stackrel{?}{=} 1700\\
500 + 1200 &\stackrel{\checkmark}{=} 1700
\end{aligned}$$

So 8 is the solution, and it checks out. This means it will take 8 months for the account balance to reach $1700.

Example 2.1.15 Here we revisit Example 1.8.4.

A bathtub contains $2.5\,\text{ft}^3$ of water. More water is being poured in at a rate of $1.75\,\text{ft}^3$ per minute. How long will it be until the amount of water in the bathtub reaches $6.25\,\text{ft}^3$?

Explanation.

Since this problem refers to *when* the amount of water will reach a certain amount, we immediately know that the unknown quantity is time. As the volume of water in the tub is measured in ft^3 per minute, we know that time needs to be measured in minutes. We'll define t to be the number of minutes that water is poured into the tub. To determine this equation, we'll start by making a table of values:

Minutes Water Has Been Poured	Total Amount of Water (in ft^3)
0	2.5
1	$2.5 + 1.75 = 4.25$
2	$2.5 + 1.75(2) = 6$
3	$2.5 + 1.75(3) = 7.75$
⋮	⋮
t	$2.5 + 1.75t$

Figure 2.1.16: Amount of Water in the Bathtub

Using this pattern, we determine that the equation representing when the amount will be $6.25\,\text{ft}^3$ is:

$$2.5 + 1.75t = 6.25$$

To solve this equation, we can start by subtracting 2.5 from each side. Then we cna divide each side by 1.75.

$$\begin{aligned} 2.5+1.75t &= 6.25 \\ 2.5+1.75t-2.5 &= 6.25-2.5 \\ 1.75t &= 3.75 \\ \frac{1.75t}{1.75} &= \frac{3.75}{1.75} \\ t &= \frac{375}{175} = \frac{15}{7} \approx 2.143 \end{aligned}$$

Checking the solution $\frac{15}{7}$:

$$\begin{aligned} 2.5+1.75t &= 6.25 \\ 2.5+1.75\left(\frac{15}{7}\right) &\stackrel{?}{=} 6.25 \\ 2.5+\frac{7}{4}\cdot\frac{15}{7} &\stackrel{?}{=} 6.25 \\ 2.5+\frac{15}{4} &\stackrel{?}{=} 6.25 \\ 2.5+3.75 &\stackrel{\checkmark}{=} 6.25 \end{aligned}$$

So $\frac{15}{7}$ (about 2.143) is the solution, and it checks out. This means it will take about 2.143 minutes for the tub to fill to $6.25\,\text{ft}^3$.

Example 2.1.17 Here we revisit Example 1.8.6.

Jakobi's annual salary as a nurse in Portland, Oregon, is \$73,290. His salary increased by 4% from last year. What was Jakobi's salary last year?

Explanation. We need to find Jakobi's salary last year. So we'll introduce s, defined to be Jakobi's salary last year (in dollars). To set up the equation, we need to think about how he arrived at this year's salary. To get to this year's salary, his employer took last year's salary and added 4% to it. Conceptually, this means we have:

$$(\text{last year's salary}) + (4\% \text{ of last year's salary}) = (\text{this year's salary})$$

We'll represent 4% of last year's salary with $0.04s$ since 0.04 is the decimal representation of 4%. This means that the equation we set up is:

$$s+0.04s = 73290$$

To solve this equation, we can start by simplifing the left side, and proceed from there.

$$\begin{aligned} s+0.04s &= 73290 \\ 1s+0.04s &= 73290 \\ 1.04s &= 73290 \\ \frac{1.04s}{1.04} &= \frac{73290}{1.04} \\ s &\approx 70471.15 \end{aligned}$$

Checking the solution 70471.15:

$$\begin{aligned} s+0.04s &= 73290 \\ 70471.15+0.04(70471.15) &\stackrel{?}{=} 73290 \\ 70471.15+2818.846 &\stackrel{?}{=} 73290 \\ 73289.996 &\stackrel{?}{=} 73290 \end{aligned}$$

In the check, those values are not equal, but they are very close. And it is reasonable to believe the only reason they are at all different comes from when we rounded the real solution to 70471.15. So Jakobi's salary was \$70471.15 last year.

Checkpoint 2.1.18 Here we revisit Exercise 1.8.7.

Kirima offered to pay the bill and tip at a restaurant where she and her friends had dinner. In total she paid \$150, which made the tip come out to a little more than 19%. What was the bill before tip?

Explanation. A common mistake is to translate a question like this into "what is 19% of \$150?" as a way to calculate the tip amount, and then subtract that from \$150. But that is not how tipping works. If we let x

represent the original bill, then:

$$\begin{aligned}
\overset{\text{bill}}{x} \overset{\text{plus}}{+} \overset{19\%}{0.19} \overset{\text{of}}{\cdot} \overset{\text{bill}}{x} &\overset{\text{is}}{=} \overset{\$150}{150}\\
1.00x + 0.19x &= 150\\
1.19x &= 150\\
\frac{1.19x}{1.19} &= \frac{150}{1.19}\\
x &\approx 126.05
\end{aligned}$$

So the original bill was about \$126.05.

Example 2.1.19 Here we revisit Example 1.8.8.

The price of a refrigerator after a 15% discount is \$612. What was the price before the discount?

Explanation. We'll let c be the original price of the refrigerator. To obtain the discounted price, we take the original price and subtract 15% of that amount. Conceptually, this looks like:

$$(\text{original price}) - (15\% \text{ of the original price}) = (\text{discounted price})$$

Since the amount of the discount is 15% of the original price, we'll represent this with $0.15c$. The equation we set up is then:

$$c - 0.15c = 612$$

To solve this equation, we can start by simplifing the left side, and proceed from there.

$$\begin{aligned}
c - 0.15c &= 612\\
1.00c - 0.15c &= 612\\
0.85c &= 612\\
\frac{0.85c}{0.85} &= \frac{612}{0.85}\\
c &= 720
\end{aligned}$$

Checking the solution 720:

$$\begin{aligned}
c - 0.15c &= 612\\
720 - 0.15(720) &\stackrel{?}{=} 612\\
720 - 108 &\stackrel{\checkmark}{=} 612
\end{aligned}$$

The solution 720 checks out. So the refrigerator cost \$720 before the discount was applied.

Checkpoint 2.1.20 Here we revisit Exercise 1.8.9.

A shirt is on sale at 20% off. The current price is \$51.00. What was the shirt's original price?

Explanation. Let x represent the original price of the shirt. Since 20% is removed to bring the cost to \$51, we can set up and solve the equation:

$$\begin{aligned}
\overset{\text{original}}{x} \overset{\text{minus}}{-} \overset{20\%}{0.20} \overset{\text{of}}{\cdot} \overset{\text{original}}{x} &\overset{\text{is}}{=} \overset{\$51}{51}\\
1.00x - 0.20x &= 51\\
0.8x &= 51\\
\frac{0.8x}{0.8} &= \frac{51}{0.8}\\
x &= 63.75
\end{aligned}$$

The shirt's original price was \$63.75.

2.1.4 Differentiating between Simplifying Expressions, Evaluating Expressions and Solving Equations

Let's look at the following similar, yet different examples.

Example 2.1.21 Simplify the expression $10 - 3(x + 2)$.
Explanation.

$$\begin{aligned} 10 - 3(x+2) &= 10 - 3x - 6 \\ &= -3x + 4 \end{aligned}$$

An equivalent result is $4 - 3x$. Note that our final result is an *expression*.

Example 2.1.22 Evaluate the expression $10 - 3(x + 2)$ when $x = 2$.
Explanation. We will substitute $x = 2$ into the expression:

$$\begin{aligned} 10 - 3(x+2) &= 10 - 3(2+2) \\ &= 10 - 3(4) \\ &= 10 - 12 \\ &= -2 \end{aligned}$$

So when $x = 2$, $10 - 3(x + 2) = -2$.
Note that our final result here is a *numerical value*.

Example 2.1.23 Solve the equation $10 - 3(x + 2) = x - 16$.
Explanation.

$$\begin{aligned} 10 - 3(x+2) &= x - 16 \\ 10 - 3x - 6 &= x - 16 \\ -3x + 4 &= x - 16 \\ -3x + 4 - 4 &= x - 16 - 4 \\ -3x &= x - 20 \\ -3x - x &= x - 20 - x \\ -4x &= -20 \\ \frac{-4x}{-4} &= \frac{-20}{-4} \\ x &= 5 \end{aligned}$$

So the solution set is $\{5\}$. (We should probably check that.)
Note that our final result here is a *solution set*.

Here is a summary collection of the distinctions that you should understand between simplifying expressions, evaluating expressions and solving equations.

List 2.1.24: A summary the differences among simplifying expressions, evaluating expressions and solving equations.

- An expression like $10-3(x+2)$ can be simplified to $-3x+4$ (as in Example 2.1.21). However we cannot "solve" an expression like this.
- As variables take different values, an expression can be evaluated to different values. In Example 2.1.22, when $x=2$, $10-3(x+2)=-2$; but when $x=3$, $10-3(x+2)=-5$.
- An equation connects two expressions with an equals sign. In Example 2.1.23, $10-3(x+2)=x-16$ has the expression $10-3(x+2)$ on the left side of equals sign, and the expression $x-16$ on the right side.
- When we solve the equation $10-3(x+2)=x-16$, we are looking for a number which makes those two expressions have the same value. In Example 2.1.23, we found the solution to be 5. It makes both $10-3(x+2)$ and $x-16=$ equal to -11.

2.1.5 Reading Questions

1. Describe the five steps that you might need to go through when solving a general linear equation in one variable.

2. In percent questions like tipping at a restaurant when you know the total bill, or purchasing something where sales tax is applied and you know the total charge, describe a common misunderstanding of what number to apply the percentage to.

3. Explain what is wrong with saying "I need to solve $3x+x-8$."

2.1.6 Exercises

Warmup and Review Solve the equation.

1. $r+4=1$ **2.** $r+1=-7$ **3.** $t-7=1$ **4.** $t-4=-2$
5. $21=-3t$ **6.** $10=-5x$ **7.** $\frac{5}{4}t=3$ **8.** $\frac{4}{5}a=9$

Solving Two-Step Equations Solve the equation.

9. $7c+4=25$ **10.** $4A+2=30$ **11.** $10C-1=-101$ **12.** $6m-5=-11$
13. $-6=3p+3$ **14.** $-70=9q+2$ **15.** $18=6y-6$ **16.** $-28=3t-4$
17. $-3a+4=22$ **18.** $-6c+1=13$ **19.** $-9A-9=18$ **20.** $-4C-6=-30$
21. $9=-m+2$ **22.** $17=-p+7$ **23.** $8q+80=0$ **24.** $5y+35=0$

Application Problems for Solving Two-Step Equations

25. A gym charges members \$25 for a registration fee, and then \$20 per month. You became a member some time ago, and now you have paid a total of \$405 to the gym. How many months have passed since you joined the gym?

$\boxed{}$ months have passed since you joined the gym.

26. Your cell phone company charges a \$16 monthly fee, plus \$0.17 per minute of talk time. One month your cell phone bill was \$75.50. How many minutes did you spend talking on the phone that month?

You spent $\boxed{}$ talking on the phone that month.

27. A school purchased a batch of T-shirts from a company. The company charged \$6 per T-shirt, and gave the school a \$65 rebate. If the school had a net expense of \$2,575 from the purchase, how many T-shirts did the school buy?

The school purchased $\boxed{}$ T-shirts.

28. James hired a face-painter for a birthday party. The painter charged a flat fee of \$75, and then charged \$5.50 per person. In the end, James paid a total of \$141. How many people used the face-painter's service?

$\boxed{}$ people used the face-painter's service.

29. A certain country has 465.5 million acres of forest. Every year, the country loses 6.65 million acres of forest mainly due to deforestation for farming purposes. If this situation continues at this pace, how many years later will the country have only 206.15 million acres of forest left? (Use an equation to solve this problem.)

After $\boxed{}$ years, this country would have 206.15 million acres of forest left.

30. Randi has \$86 in his piggy bank. He plans to purchase some Pokemon cards, which costs \$1.95 each. He plans to save \$52.85 to purchase another toy. At most how many Pokemon cards can he purchase?

Write an equation to solve this problem.

Randi can purchase at most $\boxed{}$ Pokemon cards.

Solving Equations with Variable Terms on Both Sides Solve the equation.

31. $10p + 5 = p + 59$

32. $8q + 8 = q + 29$

33. $-4y + 3 = -y - 27$

34. $-10r + 7 = -r - 56$

35. $6 - 2a = 5a + 55$

36. $3 - 5c = 5c + 103$

37. $8A + 6 = 5A + 8$

38. $2C + 6 = 10C + 5$

39. a. $9m + 4 = 2m + 32$

b. $2r + 4 = 9r - 17$

40. a. $6p + 10 = 2p + 42$

b. $2n + 10 = 6n - 6$

Application Problems for Solving Equations with Variable Terms on Both Sides Use a linear equation to solve the word problem.

41. Two trees are 8 feet and 13.5 feet tall. The shorter tree grows 3 feet per year; the taller tree grows 2.5 feet per year. How many years later would the shorter tree catch up with the taller tree?

It would take the shorter tree $\boxed{}$ years to catch up with the taller tree.

42. Massage Heaven and Massage You are competitors. Massage Heaven has 2700 registered customers, and it gets approximately 550 newly registered customers every month. Massage You has 6700 registered customers, and it gets approximately 350 newly registered customers every month.

How many months would it take Massage Heaven to catch up with Massage You in the number of registered customers?

These two companies would have approximately the same number of registered customers $\boxed{}$ months later.

43. Two truck rental companies have different rates. V-Haul has a base charge of $55.00, plus $0.65 per mile. W-Haul has a base charge of $50.40, plus $0.70 per mile. For how many miles would these two companies charge the same amount?

If a driver drives $\boxed{}$ miles, those two companies would charge the same amount of money.

44. Massage Heaven and Massage You are competitors. Massage Heaven has 9800 registered customers, but it is losing approximately 300 registered customers every month. Massage You has 2650 registered customers, and it gets approximately 350 newly registered customers every month. How many months would it take Massage Heaven to catch up with Massage You in the number of registered customers?

These two companies would have approximately the same number of registered customers $\boxed{}$ months later.

45. Anthony has $75.00 in his piggy bank, and he spends $2.00 every day. Ken has $10.00 in his piggy bank, and he saves $4.50 every day.

If they continue to spend and save money this way, how many days later would they have the same amount of money in their piggy banks?

$\boxed{}$ days later, Anthony and Ken will have the same amount of money in their piggy banks.

46. Penelope has $100.00 in her piggy bank, and she spends $1.50 every day. Haley has $8.00 in her piggy bank, and she saves $2.50 every day.

If they continue to spend and save money this way, how many days later would they have the same amount of money in their piggy banks?

$\boxed{}$ days later, Penelope and Haley will have the same amount of money in their piggy banks.

Solving Linear Equations with Like Terms Solve the equation.

47. $5C + 2C + 2 = 16$

48. $2m + 5m + 2 = 58$

49. $8p + 9 + 3 = 60$

50. $5q + 4 + 2 = 21$

51. $-1 + 8 = -2y - y - 20$

52. $-4 + 2 = -5r - r - 56$

53. $3y + 6 - 6y = 27$

54. $5y + 10 - 6y = 19$

55. $-7r + 5 + r = -37$

56. $-5r + 8 + r = 4$

57. $52 = -10m - 3 - m$

58. $-87 = -7p - 7 - p$

59. $3 - q - q = -2 + (-5)$

60. $9 - y - y = -6 + 17$

61. $4 - 3r - 10 = -6$

62. $2 - 10a - 7 = -5$

63. $b - 9 - 8b = -7 - 2b + 28$

64. $A - 6 - 3A = -5 - 8A + 5$

65. $-6B + 4B = 2 - 3B - 3$

66. $-9m + 5m = 9 - 10m - 57$

67. $5p + 5 = -6p + 5 - 10p$

68. $2q + 9 = -4q + 9 - 10q$

69. $-9 + 11 = 8y - 10 - 3y + 5 - 6y$

70. $-6 + (-10) = 5r - 10 - 8r + 4 + 2r$

Application Problems for Solving Linear Equations with Like Terms

71. A 130-meter rope is cut into two segments. The longer segment is 10 meters longer than the shorter segment. Write and solve a linear equation to find the length of each segment. Include units.

The segments are $\boxed{}$ and $\boxed{}$ long.

72. In a doctor's office, the receptionist's annual salary is \$135,000 less than that of the doctor. Together, the doctor and the receptionist make \$207,000 per year. Find each person's annual income.

The receptionist's annual income is $\boxed{}$. The doctor's annual income is $\boxed{}$.

73. Selena and Charity went picking strawberries. Selena picked 100 fewer strawberries than Charity did. Together, they picked 214 strawberries. How many strawberries did Charity pick?

Charity picked $\boxed{}$ strawberries.

74. Stephanie and Haley collect stamps. Haley collected 30 fewer than three times the number of Stephanie's stamps. Altogether, they collected 626 stamps. How many stamps did Stephanie and Haley collect?

Stephanie collected $\boxed{}$ stamps. Haley collected $\boxed{}$ stamps.

75. Emily and Sherial sold girl scout cookies. Emily's sales were \$35 more than five times of Sherial's. Altogether, their sales were \$725. How much did each girl sell?

Emily's sales were $\boxed{}$. Sherial's sales were $\boxed{}$.

76. A hockey team played a total of 129 games last season. The number of games they won was 13 more than three times of the number of games they lost.

Write and solve an equation to answer the following questions.

The team lost $\boxed{}$ games. The team won $\boxed{}$ games.

77. After a 55% increase, a town has 1395 people. What was the population before the increase?

Before the increase, the town's population was $\boxed{}$.

78. After a 45% increase, a town has 145 people. What was the population before the increase?

Before the increase, the town's population was $\boxed{}$.

Solving Linear Equations Involving Distribution Solve the equation.

79. $4(r+2)=28$
80. $10(a+10)=190$
81. $7(b-7)=-91$
82. $3(A-4)=-6$
83. $72=-9(B+1)$
84. $-66=-6(m+5)$
85. $12=-3(n-9)$
86. $72=-9(q-3)$
87. $-(y-8)=9$
88. $-(r-2)=10$
89. $-7=-(8-a)$
90. $4=-(5-b)$
91. $2(5A-7)=56$
92. $8(9B-7)=88$
93. $18=-3(3-3m)$
94. $-2=-2(7-2n)$
95. $4+6(q+7)=76$
96. $2+8(y+6)=42$
97. $5-10(r+6)=5$
98. $4-7(a+6)=-101$
99. $28=4-8(b-6)$
100. $37=10-3(A-6)$
101. $4-8(B-6)=124$
102. $2-10(m-6)=2$
103. $9=10-(2-n)$
104. $-1=8-(4-q)$
105. $2-(x+7)=-15$
106. $5-(r+9)=-8$
107. a. $6+(a+5)=1$
b. $6-(a+5)=1$
108. a. $3+(b+2)=4$
b. $3-(b+2)=4$

Solve the equation.

109. $5(A+9)-10(A-1)=60$

110. $3(B+3)-8(B-6)=67$

111. $1+9(m-5)=-53-(1-4m)$

112. $4+7(n-3)=-17-(6-4n)$

113. $8(q-6)-q=68-3(2+5q)$

114. $6(x-10)-x=-39-3(7+5x)$

115. $7(-8r+4)=14(-3-5r)$

116. $5(-8a+8)=10(-10-5a)$

117. $5+2(3-5b)=-3(b-3)+2$

118. $14+5(5-3A)=-3(A-12)+3$

Application Problems for Solving Linear Equations Involving Distribution

119. A rectangle's perimeter is 78 cm. Its base is 26 cm.
Its height is ______.

120. A rectangle's perimeter is 50 m. Its width is 8 m. Use an equation to solve for the rectangle's length.
Its length is ______.

121. A rectangle's perimeter is 112 in. Its length is 8 in longer than its width. Use an equation to find the rectangle's length and width.
Its width is ______.
Its length is ______.

122. A rectangle's perimeter is 180 cm. Its length is 2 times as long as its width. Use an equation to find the rectangle's length and width.
It's width is ______.
Its length is ______.

123. A rectangle's perimeter is 92 ft. Its length is 4 ft shorter than four times its width. Use an equation to find the rectangle's length and width.
Its width is ______.
Its length is ______.

124. A rectangle's perimeter is 170 ft. Its length is 1 ft longer than three times its width. Use an equation to find the rectangle's length and width.
Its width is ______.
Its length is ______.

Comparisons

125. Solve the equation.

a. $-t+9=9$

b. $-A+9=-9$

c. $-c-9=9$

d. $-x-9=-9$

126. Solve the equation.

a. $-b+3=3$

b. $-p+3=-3$

c. $-q-3=3$

d. $-r-3=-3$

127. a. Solve $r+1=6$.

b. Evaluate $r+1$ when $r=5$.

128. a. Solve $r-5=4$.

b. Evaluate $r-5$ when $r=9$.

129. a. Solve $5(r-6)-3=-18$.

b. Evaluate $5(r-6)-3$ when $r=3$.

c. Simplify $5(r-6)-3$.

130. a. Solve $4(t+2)-9=11$.

b. Evaluate $4(t+2)-9$ when $t=3$.

c. Simplify $4(t+2)-9$.

131. Choose True or False for the following questions about the difference between expressions and equations.

a. $9x - 7$ is an expression. (☐ True ☐ False)

b. We can check whether $x = 1$ is a solution of $9x - 7 = -7x + 9$. (☐ True ☐ False)

c. $9x - 7 = -7x + 9$ is an equation. (☐ True ☐ False)

d. $-7x + 9$ is an equation. (☐ True ☐ False)

e. $9x - 7 = -7x + 9$ is an expression. (☐ True ☐ False)

f. We can evaluate $9x - 7$ when $x = 1$ (☐ True ☐ False)

g. We can evaluate $9x - 7 = -7x + 9$ when $x = 1$ (☐ True ☐ False)

h. We can check whether $x = 1$ is a solution of $9x - 7$. (☐ True ☐ False)

132. Choose True or False for the following questions about the difference between expressions and equations.

a. $7x - 9$ is an equation. (☐ True ☐ False)

b. $-9x + 7 = 7x - 9$ is an equation. (☐ True ☐ False)

c. We can evaluate $-9x + 7 = 7x - 9$ when $x = 1$ (☐ True ☐ False)

d. We can evaluate $-9x + 7$ when $x = 1$ (☐ True ☐ False)

e. We can check whether $x = 1$ is a solution of $-9x + 7 = 7x - 9$. (☐ True ☐ False)

f. $-9x + 7 = 7x - 9$ is an expression. (☐ True ☐ False)

g. We can check whether $x = 1$ is a solution of $-9x + 7$. (☐ True ☐ False)

h. $-9x + 7$ is an expression. (☐ True ☐ False)

Challenge

133. Think of a number. Add four to your number. Now double that. Then add six. Then halve it. Finally, subtract 7. What is the result? Do you always get the same result, regardless of what number you start with? How does this work? Explain using algebra.

134. Write a linear equation whose solution is $x = -9$. You may not write an equation whose left side is just "x" or whose right side is just "x."

There are infinitely many correct answers to this problem. Be creative. After finding an equation that works, see if you can come up with a different one that also works.

2.2 Solving Multistep Linear Inequalities

We learned how to solve one-step inequalities in Section 1.6. In this section, we will solve linear inequalities that need more than one step.

2.2.1 Solving Multistep Inequalities

When solving a linear inequality, we follow the same steps in Process 2.1.4. The only difference is that when we multiply or divide by a negative number on both sides of an inequality, the direction of the inequality symbol must switch.

Process 2.2.2 Steps to Solve Linear Inequalities.

Simplify *Simplify the expressions on each side of the inequality by distributing and combining like terms.*

Separate *Use addition or subtraction to isolate the variable terms and constant terms (numbers) so that they are on different sides of the inequality symbol.*

Clear Coefficient *Use multiplication or division to eliminate the variable term's coefficient.* If you multiply or divide each side by a negative number, switch the direction of the inequality symbol.

Check *A solution to a linear inequality has a "boundary number". In the original inequality, check a number less than the boundary number, the boundary number itself, and a number greater than the boundary number to confirm what should be a solution is indeed a solution, and what should not is not. (This can be time-consuming, so use your judgment about when you might get away with skipping this.)*

Summarize *State the solution set or (in the case of an application problem) summarize the result in a complete sentence using appropriate units.*

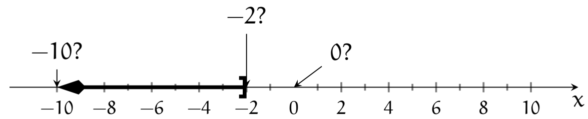
Example 2.2.3 Solve for t in the inequality $-3t + 5 \geq 11$. Write the solution set in both set-builder notation and interval notation.

Explanation.

$$\begin{aligned}
-3t + 5 &\geq 11\\
-3t + 5 - 5 &\geq 11 - 5\\
-3t &\geq 6\\
\frac{-3t}{-3} &\leq \frac{6}{-3}\\
t &\leq -2
\end{aligned}$$

Note that when we divided both sides of the inequality by -3, we had to switch the direction of the inequality symbol. At this point we think that the solution set in set-builder notation is $\{t \mid t \leq -2\}$, and the solution set in interval notation is $(-\infty, -2]$.

Since there are infinitely many solutions, it's impossible to literally check them all. We believe that all values of t for which $t \leq -2$ are solutions. We check that one number less than -2 (any number, your choice) satisfies the inequality. *And* that -2 satisfies the inequality. *And* that one number greater than -2 (any number, your choice) does *not* satisfy the inequality. We choose to check the values -10, -2, and 0.



$$\begin{array}{rrr}
-3t+5 \geq 12 & -3t+5 \geq 12 & -3t+5 \geq 11 \\
-3(-10)+5 \stackrel{?}{\geq} 12 & -3(-2)+5 \stackrel{?}{\geq} 12 & -3(0)+5 \stackrel{?}{\geq} 11 \\
30+5 \stackrel{?}{\geq} 11 & 6+5 \stackrel{?}{\geq} 11 & 0+5 \stackrel{?}{\geq} 11 \\
35 \stackrel{\checkmark}{\geq} 11 & 11 \stackrel{\checkmark}{\geq} 11 & 5 \stackrel{\text{no}}{\geq} 11
\end{array}$$

So both −10 and −2 are solutions as expected, while 0 is not. This is evidence that our solution set is correct, and it's valuable in that making these checks would likely help us catch an error if we had made one. While it certainly does take time and space to make three checks like this, it has its value.

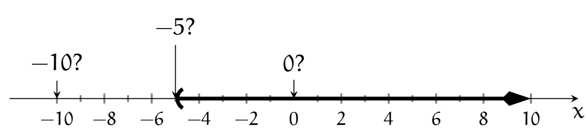
Example 2.2.4 Solve for z in the inequality $(6z+5)-(2z-3) > -12$. Write the solution set in both set-builder notation and interval notation.

Explanation.

$$\begin{aligned}
(6z+5)-(2z-3) &> -12 \\
6z+5-2z+3 &> -12 \\
4z+8 &> -12 \\
4z+8-8 &> -12-8 \\
4z &> -20 \\
\frac{4z}{4} &> \frac{-20}{4} \\
z &> -5
\end{aligned}$$

Note that we divided both sides of the inequality by 4 and since this is a positive number we *did not* need to switch the direction of the inequality symbol. At this point we think that the solution set in set-builder notation is $\{z \mid z > -5\}$, and the solution set in interval notation is $(-5, \infty)$.

Since there are infinitely many solutions, it's impossible to literally check them all. We believe that all values of z for which $z > -5$ are solutions. We check that one number less than −5 (any number, your choice) does *not* satisfy the inequality. *And* that −5 does *not* satisfy the inequality. *And* that one number greater than −5 (any number, your choice) *does* satisfy the inequality. We choose to check the values −10, −5, and 0.



$$\begin{array}{rrr}
(6z+5)-(2z-3) > -12 & (6z+5)-(2z-3) > -12 & (6z+5)-(2z-3) > -12 \\
(6(-10)+5)-(2(-10)-3) \stackrel{?}{>} -12 & (6(-5)+5)-(2(-5)-3) \stackrel{?}{>} -12 & (6(0)+5)-(2(0)-3) \stackrel{?}{>} -12 \\
(-60+5)-(-20-3) \stackrel{?}{>} -12 & (-30+5)-(-10-3) \stackrel{?}{>} -12 & (0+5)-(0-3) \stackrel{?}{>} -12 \\
-55-(-23) \stackrel{?}{>} -12 & -25-(-13) \stackrel{?}{>} -12 & 5-(-3) \stackrel{?}{>} -12
\end{array}$$

$$-32 \stackrel{\text{no}}{>} -12 \qquad -12 \stackrel{\text{no}}{>} -12 \qquad 8 \stackrel{\checkmark}{>} -12$$

So both -10 and -5 are not solutions as expected, while 0 is a solution. This is evidence that our solution set is correct. The solution set in set-builder notation is $\{z \mid z > -5\}$. The solution set in interval notation is $(-5, \infty)$.

Checkpoint 2.2.5 Solve for x in $-2-2(2x+1) > 4-(3-x)$. Write the solution set in both set-builder notation and interval notation.

Explanation.

$$\begin{aligned}
-2-2(2x+1) &> 4-(3-x)\\
-2-4x-2 &> 4-3+x\\
-4x-4 &> x+1\\
-4x-4-x &> x+1-x\\
-5x-4 &> 1\\
-5x-4+4 &> 1+4\\
-5x &> 5\\
\frac{-5x}{-5} &< \frac{5}{-5}\\
x &< -1
\end{aligned}$$

Note that when we divided both sides of the inequality by -5, we had to switch the direction of the inequality symbol. We should check the solution set, but we omit that here. The solution set in set-builder notation is $\{x \mid x < -1\}$. The solution set in interval notation is $(-\infty, -1)$.

Example 2.2.6 When a stopwatch started, the pressure inside a gas container was 4.2 atm (one atm is standard atmospheric pressure). As the container was heated, the pressure increased by 0.7 atm per minute. The maximum pressure the container can handle was 21.7 atm. Heating must be stopped once the pressure reaches 21.7 atm. Over what time interval was the container in a safe state?

Explanation. The pressure increases by 0.7 atm per minute, so it increases by $0.7m$ after m minutes. Counting in the original pressure of 4.2 atm, pressure in the container can be modeled by $0.7m + 4.2$, where m is the number of minutes since the stop watch started.

The container is safe when the pressure is 21.7 atm or lower. We can write and solve this inequality:

$$\begin{aligned}
0.7m + 4.2 &\le 21.7\\
0.7m + 4.2 - 4.2 &\le 21.7 - 4.2\\
0.7m &\le 17.5\\
\frac{0.7m}{0.7} &\le \frac{17.5}{0.7}\\
m &\le 25
\end{aligned}$$

In summary, the container was safe as long as $m \le 25$. Assuming that m also must be greater than or equal to zero, this means $0 \le m \le 25$. We can write this as the time interval as $[0, 25]$. Thus the container was safe between 0 minutes and 25 minutes.

2.2.2 Reading Questions

1. When solving an inequality, what are the conditions when you have to reverse the direction of the inequality symbol?

2. How is the solution set to a linear inequality different from the solution set to a linear equation?

2.2.3 Exercises

Review and Warmup Solve this inequality.

1. $x+3>8$ **2.** $x+4>6$ **3.** $4>x-9$ **4.** $5>x-7$ **5.** $5x\leq 10$

6. $2x\leq 8$ **7.** $6\geq -2x$ **8.** $6\geq -3x$ **9.** $\frac{4}{7}x>8$ **10.** $\frac{5}{4}x>20$

11. A swimming pool is being filled with water from a garden hose at a rate of 10 gallons per minute. If the pool already contains 70 gallons of water and can hold 230 gallons, after how long will the pool overflow?

Assume m minutes later, the pool would overflow. Write an equation to model this scenario. There is no need to solve it.

12. An engineer is designing a cylindrical springform pan. The pan needs to be able to hold a volume of 248 cubic inches and have a diameter of 12 inches. What's the minimum height it can have? (Hint: The formula for the volume of a cylinder is $V=\pi r^2 h$).

Assume the pan's minimum height is h inches. Write an equation to model this scenario. There is no need to solve it.

Solving Multistep Linear Inequalities Solve this inequality.

13. $9x+4>31$ **14.** $10x+10>70$ **15.** $26\geq 3x-4$

16. $18\geq 4x-2$ **17.** $45\leq 9-4x$ **18.** $21\leq 6-5x$

19. $-6x-2<-44$ **20.** $-7x-9<-23$ **21.** $3\geq -8x+3$

22. $2\geq -9x+2$ **23.** $-4>5-x$ **24.** $-7>1-x$

25. $3(x+2)\geq 12$ **26.** $4(x+6)\geq 56$ **27.** $7t+5<5t+11$

28. $8t+2<2t+62$ **29.** $-7z+8\leq -z-46$ **30.** $-8z+5\leq -z-16$

Solve this inequality.

31. $a-8-5a>-6-6a+10$ **32.** $a-10-7a>-2-9a+4$

33. $-6p+4-6p\geq 2p+4$ **34.** $-9p+10-4p\geq 3p+10$

35. $44<-4(p-7)$ **36.** $-10<-5(p-3)$

37. $-(x-10)\geq 16$ **38.** $-(x-6)\geq 11$

39. $4\leq 8-4(z-9)$ **40.** $81\leq 9-9(z-6)$

41. $5-(y+8)<-9$ **42.** $1-(y+6)<1$

43. $1+10(x-8)<-38-(6-3x)$ **44.** $2+8(x-3)<-19-(6-5x)$

Applications

45. You are riding in a taxi and can only pay with cash. You have to pay a flat fee of \$25, and then pay \$2.60 per mile. You have a total of \$155 in your pocket. Let x be the number of miles the taxi will drive you. You want to know how many miles you can afford.

a. Write an inequality to represent this situation in terms of how many miles you can afford.

b. Solve this inequality. At most how many miles can you afford?

c. Use interval notation to express the number of miles you can afford.

46. You are riding in a taxi and can only pay with cash. You have to pay a flat fee of \$30, and then pay \$3.30 per mile. You have a total of \$261 in your pocket. Let x be the number of miles the taxi will drive you. You want to know how many miles you can afford.

a. Write an inequality to represent this situation in terms of how many miles you can afford.

b. Solve this inequality. At most how many miles can you afford?

c. Use interval notation to express the number of miles you can afford.

47. A car rental company offers the following two plans for renting a car.
Plan A: \$30 per day and 16 cents per mile
Plan B: \$47 per day with free unlimited mileage
How many miles must one drive in order to justify choosing Plan B?

One must drive more than ______ miles to justify choosing Plan B. In other words, it's more economical to use plan B if your number of miles driven will be in the interval ______.

48. A car rental company offers the following two plans for renting a car.
Plan A: \$27 per day and 18 cents per mile
Plan B: \$50 per day with free unlimited mileage
How many miles must one drive in order to justify choosing Plan B?

One must drive more than ______ miles to justify choosing Plan B. In other words, it's more economical to use plan B if your number of miles driven will be in the interval ______.

49. You are offered two different sales jobs. The first company offers a straight commission of 8% of the sales. The second company offers a salary of \$240 per week *plus* 4% of the sales. How much would you have to sell in a week in order for the straight commission offer to be at least as good?

You'd have to sell more than ______ worth of goods for the straight commission to be better for you. In other words, the dollar amount of goods sold would have to be in the interval ______.

50. You are offered two different sales jobs. The first company offers a straight commission of 7% of the sales. The second company offers a salary of \$370 per week *plus* 4% of the sales. How much would you have to sell in a week in order for the straight commission offer to be at least as good?

You'd have to sell more than ______ worth of goods for the straight commission to be better for you. In other words, the dollar amount of goods sold would have to be in the interval ______.

2.3 Linear Equations and Inequalities with Fractions

This section discusses a technique that might make it easier for you to solve linear equations and inequalities when there are fractions present.

2.3.1 Introduction

So far, in our last step of solving for a variable we have divided each side of the equation by a constant, as in:

$$\begin{aligned} 2x &= 10 \\ \frac{2x}{2} &= \frac{10}{2} \\ x &= 5 \end{aligned}$$

If we have a coefficient that is a fraction, we *could* proceed in exactly the same manner:

$$\begin{aligned} \frac{1}{2}x &= 10 \\ \frac{\frac{1}{2}x}{\frac{1}{2}} &= \frac{10}{\frac{1}{2}} \\ x &= 10 \cdot \frac{2}{1} = 20 \end{aligned}$$

What if our equation or inequality was more complicated though, for example $\frac{1}{4}x + \frac{2}{3} = \frac{1}{6}$? We would have to first do a lot of fraction arithmetic in order to then divide each side by the coefficient of x. An alternate approach is to instead *multiply* each side of the equation by just the right number that eliminates the denominator(s). In the equation $\frac{1}{2}x = 10$, we could simply multiply each side of the equation by 2, which would eliminate the denominator of 2:

$$\begin{aligned} \frac{1}{2}x &= 10 \\ 2 \cdot \left(\frac{1}{2}x\right) &= 2 \cdot 10 \\ x &= 20 \end{aligned}$$

For more complicated equations, multiply each side of the equation by the least common denominator (LCD) of all fractions appearing in the equation. In this way we can take one step to turn an equation full of fractions into an equation with no fractions.

2.3.2 Eliminating Denominators

Example 2.3.2
Deshawn planted a sapling in his yard that was 4 ft tall. The tree will grow $\frac{2}{3}$ of a foot every year. How many years will it take for his tree to be 10 ft tall?
Since the tree grows $\frac{2}{3}$ of a foot every year, we can use a table to help write a formula modeling the tree's growth:

Years Passed	Tree's Height (ft)
0	4
1	$4 + \frac{2}{3}$
2	$4 + \frac{2}{3} \cdot 2$
$\vdots$	$\vdots$
y	$4 + \frac{2}{3}y$

From this, we've determined that y years since the tree was planted, the tree's height will be $4 + \frac{2}{3}y$ feet. To find when Deshawn's tree will be 10 feet tall, we set up the equation

$$4 + \frac{2}{3}y = 10$$

To solve the equation, we take note of the fraction and its denominator 3. As the very first step, multiplying by 3 on each side will leave us with no fractions.

$$\begin{aligned}
4+\frac{2}{3}y&=10\\
3\cdot\left(4+\frac{2}{3}y\right)&=3\cdot 10\\
3\cdot 4+3\cdot\frac{2}{3}y&=30\\
12+2y&=30\\
2y&=18\\
y&=9
\end{aligned}$$

Now we will check the solution 9 in the equation $4+\frac{2}{3}y=10$:

$$\begin{aligned}
4+\frac{2}{3}y&=10\\
4+\frac{2}{3}(9)&\stackrel{?}{=}10\\
4+6&\stackrel{\checkmark}{=}10
\end{aligned}$$

In summary, it will take 9 years for Deshawn's tree to reach 10 feet tall. The point of this example was to demonstrate that clearing denominators can make an equation fairly easy to solve.

Example 2.3.3 Solve for x in $\frac{1}{4}x+\frac{2}{3}=\frac{1}{6}$.

Explanation. To solve this equation, we first need to identify the LCD of all fractions in the equation. On the left side we have $\frac{1}{4}$ and $\frac{2}{3}$. On the right side we have $\frac{1}{6}$. The LCD of $3, 4$, and 6 is 12, so we will multiply each side of the equation by 12 in order to eliminate *all* of the denominators:

$$\begin{aligned}
\frac{1}{4}x+\frac{2}{3}&=\frac{1}{6}\\
12\cdot\left(\frac{1}{4}x+\frac{2}{3}\right)&=12\cdot\frac{1}{6}\\
12\cdot\left(\frac{1}{4}x\right)+12\cdot\left(\frac{2}{3}\right)&=12\cdot\frac{1}{6}\\
3x+8&=2\\
3x&=-6\\
\frac{3x}{3}&=\frac{-6}{3}\\
x&=-2
\end{aligned}$$

Checking the solution -2:

$$\begin{aligned}
\frac{1}{4}x+\frac{2}{3}&=\frac{1}{6}\\
\frac{1}{4}(-2)+\frac{2}{3}&\stackrel{?}{=}\frac{1}{6}\\
-\frac{2}{4}+\frac{2}{3}&\stackrel{?}{=}\frac{1}{6}\\
-\frac{6}{12}+\frac{8}{12}&\stackrel{?}{=}\frac{1}{6}\\
\frac{2}{12}&\stackrel{\checkmark}{=}\frac{1}{6}
\end{aligned}$$

The solution is therefore -2 and the solution set is $\{-2\}$.

Checkpoint 2.3.4 Solve for z in $-\frac{2}{5}z-\frac{3}{2}=-\frac{1}{2}z+\frac{4}{5}$.

Explanation. The first thing we need to do is identify the LCD of all denominators in this equation. Since the denominators are 2 and 5, the LCD is 10. So as our first step, we will multiply each side of the equation by 10 in order to eliminate all denominators:

$$\begin{aligned}
-\frac{2}{5}z-\frac{3}{2} &= -\frac{1}{2}z+\frac{4}{5}\\
10\cdot\left(-\frac{2}{5}z-\frac{3}{2}\right) &= 10\cdot\left(-\frac{1}{2}z+\frac{4}{5}\right)\\
10\left(-\frac{2}{5}z\right)-10\left(\frac{3}{2}\right) &= 10\left(-\frac{1}{2}z\right)+10\left(\frac{4}{5}\right)\\
-4z-15 &= -5z+8\\
z-15 &= 8\\
z &= 23
\end{aligned}$$

Checking the solution 23:

$$\begin{aligned}
-\frac{2}{5}z-\frac{3}{2} &= -\frac{1}{2}z+\frac{4}{5}\\
-\frac{2}{5}(23)-\frac{3}{2} &\stackrel{?}{=} -\frac{1}{2}(23)+\frac{4}{5}\\
-\frac{46}{5}-\frac{3}{2} &\stackrel{?}{=} -\frac{23}{2}+\frac{4}{5}\\
-\frac{46}{5}\cdot\frac{2}{2}-\frac{3}{2}\cdot\frac{5}{5} &\stackrel{?}{=} -\frac{23}{2}\cdot\frac{5}{5}+\frac{4}{5}\cdot\frac{2}{2}\\
-\frac{92}{10}-\frac{15}{10} &\stackrel{?}{=} -\frac{115}{10}+\frac{8}{10}\\
-\frac{107}{10} &\stackrel{\checkmark}{=} -\frac{107}{10}
\end{aligned}$$

Thus the solution is 23 and so the solution set is $\{23\}$.

Example 2.3.5 Solve for a in the equation $\frac{2}{3}(a+1)+5=\frac{1}{3}$.

Explanation.

There are two fractions appearing here, but both have the same denominator 3. So we can multiply by 3 on each side to clear denominators.

$$\begin{aligned}
\frac{2}{3}(a+1)+5 &= \frac{1}{3}\\
3\cdot\left(\frac{2}{3}(a+1)+5\right) &= 3\cdot\frac{1}{3}\\
3\cdot\frac{2}{3}(a+1)+3\cdot 5 &= 1\\
2(a+1)+15 &= 1\\
2a+2+15 &= 1\\
2a+17 &= 1\\
2a &= -16\\
a &= -8
\end{aligned}$$

Check the solution -8 in the equation $\frac{2}{3}(a+1)+5=\frac{1}{3}$, we find that:

$$\begin{aligned}
\frac{2}{3}(a+1)+5 &= \frac{1}{3}\\
\frac{2}{3}(-8+1)+5 &\stackrel{?}{=} \frac{1}{3}\\
\frac{2}{3}(-7)+5 &\stackrel{?}{=} \frac{1}{3}\\
-\frac{14}{3}+\frac{15}{3} &\stackrel{\checkmark}{=} \frac{1}{3}
\end{aligned}$$

The solution is therefore -8 and the solution set is $\{-8\}$.

Example 2.3.6 Solve for b in the equation $\frac{2b+1}{3} = \frac{2}{5}$.

Explanation.

The structure of the equation is a little different from previous examples, but we can still see two denominators 3 and 5, and find their LCM is 15.

$$\begin{aligned}\frac{2b+1}{3} &= \frac{2}{5}\\ 15 \cdot \frac{2b+1}{3} &= 15 \cdot \frac{2}{5}\\ 5(2b+1) &= 6\\ 10b+5 &= 6\\ 10b &= 1\\ b &= \frac{1}{10}\end{aligned}$$

Checking the solution $\frac{1}{10}$:

$$\begin{aligned}\frac{2b+1}{3} &= \frac{2}{5}\\ \frac{2\left(\frac{1}{10}\right)+1}{3} &\stackrel{?}{=} \frac{2}{5}\\ \frac{\frac{1}{5}+1}{3} &\stackrel{?}{=} \frac{2}{5}\\ \frac{\frac{1}{5}+\frac{5}{5}}{3} &\stackrel{?}{=} \frac{2}{5}\\ \frac{\frac{6}{5}}{3} &\stackrel{?}{=} \frac{2}{5}\\ \frac{6}{5}\cdot\frac{1}{3} &\stackrel{\checkmark}{=} \frac{2}{5}\end{aligned}$$

The solution is $\frac{1}{10}$ and the solution set is $\left\{\frac{1}{10}\right\}$.

Example 2.3.7 Suppose we want to know the total cost for a box of cereal that weighs 18 ounces, assuming it costs the same per ounce as the 21-ounce box. Letting C be this unknown cost (in dollars), we could set up the following proportion:

$$\begin{aligned}\frac{\text{cost in dollars}}{\text{weight in oz}} &= \frac{\text{cost in dollars}}{\text{weight in oz}}\\ \frac{\$3.99}{21\,\text{oz}} &= \frac{\$C}{18\,\text{oz}}\end{aligned}$$

We take a moment to rewrite the equation without units:

$$\frac{3.99}{21} = \frac{C}{18}$$

Next we want to recognize that each side contains a fraction. Our usual approach for solving this type of equation is to multiply each side by the least common denominator (LCD). In this case, the LCD of 21 and 18 is 126. As with many other proportions we solve, it is often easier to just multiply each side by the common denominator of $18 \cdot 21$, which we know will make each denominator cancel:

$$\begin{aligned}\frac{3.99}{21} &= \frac{C}{18}\\ 18\cdot 21\cdot\frac{3.99}{21} &= \frac{C}{18}\cdot 18\cdot 21\\ 18\cdot\cancel{21}\frac{3.99}{\cancel{21}} &= \frac{C}{\cancel{18}}\cdot\cancel{18}\cdot 21\\ 71.82 &= 21C\\ \frac{71.82}{21} &= \frac{21C}{21}\end{aligned}$$

$$C = 3.42$$

So assuming the cost is proportional to the cost of the 21-ounce box, the cost for an 18-ounce box of cereal would be \$3.42.

Example 2.3.8 In a science lab, a container had 21 ounces of water at 9:00 AM. Water has been evaporating at the rate of 3 ounces every 5 minutes. When will there be 8 ounces of water left?

Explanation. Since the container has been losing 3 oz of water every 5 minutes, it loses $\frac{3}{5}$ oz every minute. In m minutes since 9:00 AM, the container would lose $\frac{3}{5}m$ oz of water. Since the container had 21 oz of water at the beginning, the amount of water in the container can be modeled by $21 - \frac{3}{5}m$ (in oz).

To find when there would be 8 oz of water left, we write and solve this equation:

$$\begin{aligned}
21 - \frac{3}{5}m &= 8 \\
5 \cdot \left(21 - \frac{3}{5}x\right) &= 5 \cdot 8 \\
5 \cdot 21 - 5 \cdot \frac{3}{5}x &= 40 \\
105 - 3m &= 40 \\
105 - 3m - 105 &= 40 - 105 \\
-3m &= -65 \\
\frac{-3m}{-3} &= \frac{-65}{-3} \\
m &= \frac{65}{3}
\end{aligned}$$

Checking the solution $\frac{65}{3}$:

$$\begin{aligned}
21 - \frac{3}{5}m &= 8 \\
21 - \frac{3}{5}\left(\frac{65}{3}\right) &\stackrel{?}{=} 8 \\
21 - 13 &\stackrel{\checkmark}{=} 8
\end{aligned}$$

Therefore the solution is $\frac{65}{3}$. As a mixed number, this is $21\frac{2}{3}$. In context, this means that 21 minutes and 40 seconds after 9:00 AM, at 9:21:40 AM, the container will have 8 ounces of water left.

2.3.3 Creating and Solving Proportions

Proportions can be used to solve many real-life applications where two quantities vary together. For example, if your home is worth more, your property tax will be more. If you have a larger amount of liquid Tylenol, you have more milligrams of the drug dissolved in that liquid. The key to using proportions is to first set up a ratio where all values are known. We then set up a second ratio that will be proportional to the first, but has an unknown value.

Example 2.3.9 Property taxes for a residential property are proportional to the assessed value of the property. A certain home is assessed at \$234,100 and its annual property taxes are \$2,518.92. What are the annual property taxes for the house next door that is assessed at \$287,500?

Explanation. Let T be the annual property taxes (in dollars) for a property assessed at \$287,500. We can write and solve this proportion:

$$\begin{aligned}
\frac{\text{tax}}{\text{property value}} &= \frac{\text{tax}}{\text{property value}} \\
\frac{2518.92}{234100} &= \frac{T}{287500}
\end{aligned}$$

The least common denominator of this proportion is rather large, so we will instead multiply each side by 234100 and 287500 and simplify from there:

$$\begin{aligned}\frac{2518.92}{234100} &= \frac{T}{287500}\\ 234100 \cdot 287500 \cdot \frac{2518.92}{234100} &= \frac{T}{287500} \cdot 234100 \cdot 287500\\ 287500 \cdot 2518.92 &= T \cdot 234100\\ \frac{287500 \cdot 2518.92}{234100} &= \frac{234100T}{234100}\\ T &\approx 3093.50\end{aligned}$$

The property taxes for a property assessed at \$287,500 are \$3,093.50.

Example 2.3.10 Tagging fish is a means of estimating the size of the population of fish in a lake. A sample of fish is taken, tagged, and then redistributed into the lake. Later when another sample is taken, some of those fish will have tags. The number of tagged fish are assumed to be proportional to the total number of fish. We can look at that relationship from the perspective of the entire lake, or just the second sample, and we get two ratios that should be proportional.

$$\frac{\text{number of tagged fish in sample}}{\text{number of fish in sample}} = \frac{\text{number of tagged fish total}}{\text{number of fish total}}$$

Assume that 90 fish are caught and tagged. Once they are redistributed, a sample of 200 fish is taken. Of these, 7 are tagged. Estimate how many fish total are in the lake.

Explanation. Let n be the number of fish in the lake. We can set up a proportion for this scenario:

$$\frac{7}{200} = \frac{90}{n}$$

To solve for n, which is in a denominator, we'll need to multiply each side by both 200 and n:

$$\begin{aligned}\frac{7}{200} &= \frac{90}{n}\\ 200 \cdot n \cdot \frac{7}{200} &= \frac{90}{n} \cdot 200 \cdot n\\ \cancel{200} \cdot n \cdot \frac{7}{\cancel{200}} &= \frac{90}{\cancel{n}} \cdot 200 \cdot \cancel{n}\\ 7n &= 1800\\ \frac{7n}{7} &= \frac{1800}{7}\\ n &\approx 2471.4286\end{aligned}$$

According to this sample, we can estimate that there are about 2471 fish in the lake.

WeBWorK **Checkpoint 2.3.11** Infant Tylenol contains 160 mg of acetaminophen in each 5 mL of liquid. If Bao's baby is prescribed 60 mg of acetaminophen, how many milliliters of liquid should he give the baby?

Explanation. Assume Bao should give q milliliters of liquid medicine, so we can set up the following proportion:

$$\begin{aligned}
\frac{\text{amount of liquid medicine in mL}}{\text{amount of acetaminophen in mg}} &= \frac{\text{amount of liquid medicine in mL}}{\text{amount of acetaminophen in mg}} \\
\frac{5\,\text{mL}}{160\,\text{mg}} &= \frac{q\,\text{mL}}{60\,\text{mg}} \\
\frac{5}{160} &= \frac{q}{60} \\
160 \cdot 60 \cdot \frac{5}{160} &= \frac{q}{60} \cdot 160 \cdot 60 \\
60 \cdot 5 &= q \cdot 160 \\
300 &= 160q \\
\frac{300}{160} &= \frac{160q}{160} \\
q &= 1.875
\end{aligned}$$

So to give 60 mg of acetaminophen to his baby, Bao should give 1.875 mL of liquid medication.

Example 2.3.12 Sarah is an architect and she's making a scale model of a building. The actual building will be 30 ft tall. In the model, the height of the building will be 2 in. How tall should she make the model of a person who is 5 ft 6 in tall so that the model is to scale?

Explanation. Let h be the height of the person in Sarah's model, which we'll measure in inches. We'll create a proportion that compares the building and person's heights in the model to their heights in real life:

$$\begin{aligned}
\frac{\text{height of model building in inches}}{\text{height of actual building in feet}} &= \frac{\text{height of model person in inches}}{\text{height of actual person in feet}} \\
\frac{2\,\text{in}}{30\,\text{ft}} &= \frac{h\,\text{in}}{5\,\text{ft}\,6\text{in}}
\end{aligned}$$

Before we can just eliminate the units, we'll need to convert 5 ft 6 in to feet:

$$\frac{2\,\text{in}}{30\,\text{ft}} = \frac{h\,\text{in}}{5.5\,\text{ft}}$$

Now we can remove the units and continue solving:

$$\begin{aligned}
\frac{2}{30} &= \frac{h}{5.5} \\
30 \cdot 5.5 \cdot \frac{2}{30} &= \frac{h}{5.5} \cdot 30 \cdot 5.5 \\
5.5 \cdot 2 &= h \cdot 30 \\
11 &= 30h \\
\frac{11}{30} &= \frac{30h}{30} \\
\frac{11}{30} &= h \\
h &\approx 0.3667
\end{aligned}$$

Sarah should make the model of a person who is 5 ft 6 in tall be $\frac{11}{30}$ inches (about 0.3667 inches) tall.

2.3.4 Solving Inequalities with Fractions

The notion of clearing denominators can also apply when solving a linear inequality. Remember that with inequalities, the only difference in the process is that the inequality sign reverses direction whenever we multiply or divide each side by a negative number.

Example 2.3.13 Solve for x in the inequality $\frac{3}{4}x - 2 > \frac{4}{5}x$. Write the solution set in both set-builder notation and interval notation.

Explanation. The LCM of the denominators is 20, so we start out multiplying each side of the inequality by 20.

$$\begin{aligned}
\frac{3}{4}x - 2 &> \frac{4}{5}x \\
20 \cdot \left(\frac{3}{4}x - 2\right) &> 20 \cdot \frac{4}{5}x \\
20 \cdot \frac{3}{4}x - 20 \cdot 2 &> 16x \\
15x - 40 &> 16x \\
15x - 40 - 15x &> 16x - 15x \\
-40 &> x \\
x &< -40
\end{aligned}$$

The solution set in set-builder notation is $\{x \mid x < -40\}$. Note that it's equivalent to write $\{x \mid -40 > x\}$, but it's easier to understand if we write x first in an inequality. The solution set in interval notation is $(-\infty, -40)$.

Checkpoint 2.3.14 Solve for y in the inequality $\frac{4}{7} - \frac{4}{3}y \leq \frac{2}{3}$. Write the solution set in both set-builder notation and interval notation.

Explanation. The LCM of the denominators is 21, so we start out multiplying each side of the inequality by 21.

$$\begin{aligned}
\frac{4}{7} - \frac{4}{3}y &\leq \frac{2}{3} \\
21 \cdot \left(\frac{4}{7} - \frac{4}{3}y\right) &\leq 21 \cdot \left(\frac{2}{3}\right) \\
21\left(\frac{4}{7}\right) - 21\left(\frac{4}{3}y\right) &\leq 21\left(\frac{2}{3}\right) \\
12 - 28y &\leq 14 \\
-28y &\leq 2 \\
\frac{-28y}{-28} &\geq \frac{2}{-28} \\
y &\geq -\frac{1}{14}
\end{aligned}$$

Note that when we divided each side of the inequality by -28, the inequality symbol reversed direction. The solution set in set-builder notation is $\left\{y \mid y \geq -\frac{1}{14}\right\}$. The solution set in interval notation is $\left[-\frac{1}{14}, \infty\right)$.

Example 2.3.15 In a certain class, a student's grade is calculated by the average of their scores on three tests. Aidan scored 78% and 54% on the first two tests. If he wants to earn at least a grade of C (70%), what's the lowest score he needs to earn on the third exam?

Explanation. Assume Aidan will score $x\%$ on the third test. To make his average test score greater than or equal to 70%, we write and solve this inequality:

$$\begin{aligned} \frac{78+54+x}{3} &\geq 70 \\ \frac{132+x}{3} &\geq 70 \\ 3\cdot\frac{132+x}{3} &\geq 3\cdot 70 \\ 132+x &\geq 210 \\ x &\geq 78 \end{aligned}$$

To earn at least a C grade, Aidan needs to score at least 78% on the third test.

2.3.5 Reading Questions

1. What does LCD stand for? And what does it really *mean*?

2. When you clear denominators from an equation like $\frac{2}{3}x+5=\frac{2}{7}$, you will multiply by 21. At first, what are the *two* things that you multiply by 21, possibly requiring you to use some parentheses?

3. What is a proportional equation?

2.3.6 Exercises

Review and Warmup

1. Multiply: $4\cdot\dfrac{2}{3}$

2. Multiply: $7\cdot\dfrac{3}{8}$

3. Multiply: $25\cdot\left(-\dfrac{6}{5}\right)$

4. Multiply: $30\cdot\left(-\dfrac{7}{10}\right)$

5. Do the following multiplications.

a. $32\cdot\frac{5}{8}$

b. $40\cdot\frac{5}{8}$

c. $48\cdot\frac{5}{8}$

6. Do the following multiplications.

a. $14\cdot\frac{5}{7}$

b. $21\cdot\frac{5}{7}$

c. $28\cdot\frac{5}{7}$

Solving Linear Equations with Fractions Solve the equation.

7. $\dfrac{x}{10}+95=2x$

8. $\dfrac{r}{7}+68=5r$

9. $\dfrac{t}{4}+5=9$

10. $\dfrac{b}{9}+3=9$

11. $6-\dfrac{c}{5}=0$

12. $2-\dfrac{B}{9}=-3$

13. $-3=9-\dfrac{4C}{3}$

14. $-1=5-\dfrac{2n}{7}$

15. $2p=\dfrac{3p}{2}+6$

16. $2x = \frac{5x}{8} + 44$

17. $80 = \frac{4}{9}y + 4y$

18. $36 = \frac{2}{5}t + 2t$

19. $78 - \frac{7}{8}b = 4b$

20. $51 - \frac{5}{6}c = 2c$

21. $3B = \frac{2}{7}B + 1$

22. $3C = \frac{10}{9}C + 8$

23. $\frac{5}{4} - 7n = 2$

24. $\frac{9}{2} - 4p = 2$

25. $\frac{3}{8} - \frac{1}{8}x = 1$

26. $\frac{7}{4} - \frac{1}{4}y = 8$

27. $\frac{2t}{9} - 6 = -\frac{74}{9}$

28. $\frac{4b}{7} - 4 = -\frac{68}{7}$

29. $\frac{2}{5} + \frac{8}{5}c = 3c$

30. $\frac{4}{9} + \frac{8}{9}B = 3B$

31. $\frac{4C}{7} - \frac{30}{7} = -\frac{2}{7}C$

32. $\frac{2n}{5} - \frac{6}{5} = -\frac{1}{5}n$

33. $\frac{10p}{9} + \frac{7}{10} = p$

34. $\frac{6x}{7} + \frac{5}{6} = x$

35. $\frac{4y}{3} - 69 = -\frac{5}{2}y$

36. $\frac{6t}{7} - 47 = -\frac{5}{2}t$

37. $-\frac{9}{10}a + 42 = \frac{3a}{20}$

38. $-\frac{5}{2}c + 19 = \frac{9c}{4}$

39. $\frac{3B}{10} - 10B = \frac{3}{20}$

40. $\frac{5C}{6} - 6C = \frac{7}{12}$

41. $\frac{9n}{2} + \frac{8}{7} = \frac{1}{8}n$

42. $\frac{7p}{6} + \frac{8}{5} = \frac{1}{4}p$

43. $\frac{5}{4}x = \frac{2}{7} + \frac{4x}{3}$

44. $\frac{1}{2}y = \frac{5}{3} + \frac{2y}{5}$

45. $\frac{7}{8} = \frac{t}{32}$

46. $\frac{9}{4} = \frac{a}{12}$

47. $-\frac{c}{15} = \frac{10}{3}$

48. $-\frac{B}{35} = \frac{10}{7}$

49. $-\frac{C}{12} = -\frac{3}{4}$

50. $-\frac{n}{20} = -\frac{7}{10}$

51. $-\frac{3}{7} = \frac{8p}{9}$

52. $-\frac{3}{4} = \frac{9x}{7}$

53. $\frac{9}{10} = \frac{y+3}{50}$

54. $\frac{5}{6} = \frac{t+1}{30}$

55. $\frac{7}{4} = \frac{a-7}{7}$

56. $\frac{3}{10} = \frac{c-7}{5}$

57. $\frac{A-6}{4} = \frac{A+2}{6}$

58. $\frac{C-10}{2} = \frac{C+10}{4}$

59. $\frac{n+5}{6} - \frac{n-7}{12} = \frac{5}{3}$

60. $\frac{p+9}{4} - \frac{p-4}{8} = \frac{15}{8}$

61. $\frac{x}{3} - 1 = \frac{x}{6}$

62. $\frac{y}{7} - 9 = \frac{y}{10}$

63. $\frac{t}{4} - 1 = \frac{t}{5} + 2$

64. $\frac{a}{2} - 1 = \frac{a}{6} + 1$

65. $\frac{4}{9}c + \frac{2}{9} = \frac{7}{9}c + \frac{1}{3}$

66. $\frac{5}{3}A + \frac{4}{3} = \frac{8}{3}A + 2$

Solve the equation.

67. $\frac{5C+8}{2} - \frac{4-C}{4} = \frac{3}{7}$

68. $\frac{3n+8}{4} - \frac{1-n}{8} = \frac{2}{3}$

69. $33 = \frac{p}{5} + \frac{p}{6}$

70. $26 = \frac{x}{3} + \frac{x}{10}$

71. $-2y - \frac{4}{5} = \frac{7}{5}y + \frac{4}{3}$

72. $\frac{4}{9}t - 1 = \frac{3}{4}t - 4$

Solve the equation.

73. $-\frac{5}{2}a - 1 = \frac{7}{4}a - \frac{3}{10}$

74. $\frac{7}{8}c - \frac{3}{2} = -\frac{3}{4}c + \frac{2}{3}$

75. a. $-\frac{A}{5}+2=-3$

b. $\frac{-t}{5}+2=-3$

c. $\frac{y}{-5}+2=-3$

d. $\frac{-r}{-5}+2=-3$

76. a. $-\frac{C}{5}+3=1$

b. $\frac{-q}{5}+3=1$

c. $\frac{c}{-5}+3=1$

d. $\frac{-p}{-5}+3=1$

Applications

77. Kimball is jogging in a straight line. He got a head start of 10 meters from the starting line, and he ran 5 meters every 8 seconds. After how many seconds will Kimball be 30 meters away from the starting line?

Kimball will be 30 meters away from the starting line $\boxed{}$ seconds since he started running.

78. Brent is jogging in a straight line. He started at a place 36 meters from the starting line, and ran toward the starting line at the speed of 5 meters every 7 seconds. After how many seconds will Brent be 21 meters away from the starting line?

Brent will be 21 meters away from the starting line $\boxed{}$ seconds since he started running.

79. Joseph had only \$7.00 in his piggy bank, and he decided to start saving more. He saves \$4.00 every 5 days. After how many days will he have \$23.00 in the piggy bank?

Joseph will save \$23.00 in his piggy bank after $\boxed{}$ days.

80. Brandon has saved \$41.00 in his piggy bank, and he decided to start spending them. He spends \$2.00 every 7 days. After how many days will he have \$31.00 left in the piggy bank?

Brandon will have \$31.00 left in his piggy bank after $\boxed{}$ days.

81. According to a salad recipe, each serving requires 2 teaspoons of vegetable oil and 8 teaspoons of vinegar. If 17 teaspoons of vegetable oil were used, how many teaspoons of vinegar should be used?

If 17 teaspoons of vegetable oil were used, $\boxed{}$ teaspoons of vinegar should be used.

82. According to a salad recipe, each serving requires 5 teaspoons of vegetable oil and 20 teaspoons of vinegar. If 84 teaspoons of vinegar were used, how many teaspoons of vegetable oil should be used?

If 84 teaspoons of vinegar were used, $\boxed{}$ teaspoons of vegetable oil should be used.

83. Jay makes \$108 every eight hours he works. How much will he make if he works twenty-four hours this week?

If Jay works twenty-four hours this week, he will make $\boxed{}$.

84. Casandra makes \$198 every twelve hours she works. How much will she make if she works thirty hours this week?

If Casandra works thirty hours this week, she will make $\boxed{}$.

85. A mutual fund consists of 63% stock and 37% bond. In other words, for each 63 dollars of stock, there are 37 dollars of bond. For a mutual fund with \$2,810.00 of stock, how many dollars of bond are there?

For a mutual fund with \$2,810.00 of stock, there are approximately ______ of bond.

86. A mutual fund consists of 72% stock and 28% bond. In other words, for each 72 dollars of stock, there are 28 dollars of bond. For a mutual fund with \$2,460.00 of bond, how many dollars of stock are there?

For a mutual fund with \$2,460.00 of bond, there are approximately ______ of stock.

87. Eileen jogs every day. Last month, she jogged 6.5 hours for a total of 37.05 miles. At this speed, if Eileen runs 31.5 hours, how far can she run?

At this speed, Eileen can run ______ in 31.5 hours.

88. Donna jogs every day. Last month, she jogged 16.5 hours for a total of 57.75 miles. At this speed, how long would it take Donna to run 140 miles?

At this speed, Donna can run 140 mi in ______.

89. Kimball purchased 5.3 pounds of apples at the total cost of \$7.42. If he purchases 6.6 pounds of apples at this store, how much would it cost?

It would cost ______ to purchase 6.6 pounds of apples.

90. Alejandro purchased 3.1 pounds of apples at the total cost of \$6.20. If the price doesn't change, how many pounds of apples can Alejandro purchase with \$12.20?

With \$12.20, Alejandro can purchase ______ of apples.

91. Dawn collected a total of 1411 stamps over the past 17 years. At this rate, how many stamps would she collect in 27 years?

At this rate, Dawn would collect ______ stamps in 27 years.

92. Corey collected a total of 1330 stamps over the past 14 years. At this rate, how many years would it take he to collect 2090 stamps?

At this rate, Corey can collect 2090 stamps in ______ years.

93. In a city, the owner of a house valued at 260 thousand dollars needs to pay \$821.60 in property tax. At this tax rate, how much property tax should the owner pay if a house is valued at 650 thousand dollars?

The owner of a 650-thousand-dollar house should pay ______ in property tax.

94. In a city, the owner of a house valued at 480 thousand dollars needs to pay \$2,299.20 in property tax. At this tax rate, if the owner of a house paid \$3,113.50 of property tax, how much is the house worth?

If the owner of a house paid \$3,113.50 of property tax, the house is worth ______ thousand dollars.

95. To try to determine the health of the Rocky Mountain elk population in the Wenaha Wildlife Area, the Oregon Department of Fish and Wildlife caught, tagged, and released 39 Rocky Mountain elk. A week later, they returned and observed 42 Rocky Mountain elk, 9 of which had tags. Approximately how many Rocky Mountain elk are in the Wenaha Wildlife Area?

There are approximately ______ elk in the wildlife area.

96. To try to determine the health of the black-tailed deer population in the Jewell Meadow Wildlife Area, the Oregon Department of Fish and Wildlife caught, tagged, and released 28 black-tailed deer. A week later, they returned and observed 63 black-tailed deer, 18 of which had tags. Approximately how many black-tailed deer are in the Jewell Meadow Wildlife Area?

There are approximately $\boxed{}$ deer in the wildlife area.

97. A restaurant used 1015 lb of vegetable oil in 25 days. At this rate, how many pounds of vegetable oil will be used in 42 days?

The restaurant will use $\boxed{}$ of vegetable oil in 42 days.

98. A restaurant used 735.9 lb of vegetable oil in 33 days. At this rate, 1271.1 lb of oil will last how many days?

The restaurant will use 1271.1 lb of vegetable oil in $\boxed{}$ days.

Solving Inequalities with Fractions Solve this inequality.

99. $\frac{x}{3} + 30 \geq 2x$

100. $\frac{x}{4} + 44 \geq 3x$

101. $\frac{3}{4} - 6y < 6$

102. $\frac{5}{4} - 3y < 5$

103. $-\frac{1}{6}t > \frac{4}{5}t - 58$

104. $-\frac{1}{2}t > \frac{2}{7}t - 11$

105. $\frac{9}{10} \geq \frac{x}{40}$

106. $\frac{7}{10} \geq \frac{x}{60}$

107. $-\frac{z}{8} < -\frac{3}{2}$

108. $-\frac{z}{24} < -\frac{9}{4}$

109. $\frac{x}{7} - 4 \leq \frac{x}{3}$

110. $\frac{x}{7} - 4 \leq \frac{x}{5}$

111. $\frac{y-8}{6} \geq \frac{y+1}{4}$

112. $\frac{y-5}{6} \geq \frac{y+6}{4}$

Solve this inequality.

113. $\frac{5}{4} < \frac{x+1}{6} - \frac{x-10}{12}$

114. $\frac{3}{2} < \frac{x+8}{6} - \frac{x-4}{12}$

Applications

115. Your grade in a class is determined by the average of three test scores. You scored 75 and 87 on the first two tests. To earn at least 82 for this course, how much do you have to score on the third test? Let x be the score you will earn on the third test.

a. Write an inequality to represent this situation.

b. Solve this inequality. What is the minimum that you have to earn on the third test in order to earn a 82 for the course?

c. You cannot score over 100 on the third test. Use interval notation to represent the range of scores you can earn on the third test in order to earn at least 82 for this course.

116. Your grade in a class is determined by the average of three test scores. You scored 70 and 85 on the first two tests. To earn at least 77 for this course, how much do you have to score on the third test? Let x be the score you will earn on the third test.

a. Write an inequality to represent this situation.

b. Solve this inequality. What is the minimum that you have to earn on the third test in order to earn a 77 for the course?

c. You cannot score over 100 on the third test. Use interval notation to represent the range of scores you can earn on the third test in order to earn at least 77 for this course.

Challenge

117. The ratio of girls to boys in a preschool is 2 to 7. If there are 63 kids in the school, how many girls are there in the preschool?

2.4 Special Solution Sets

Most of the time after solving a linear equation in one variable, you have an equivalent equation similar to, say $x = 3$, that explicitly tells you what value the variable needs to be. And there is only one solution, 3 in this case. Similarly, after solving a linear inequality, you typically have a statement like $x < 5$, and the solution set is represented with either $(-\infty, 5)$ in interval notation or $\{x \mid x < 5\}$ in set-builder notation.

Occasionally you run into a linear equation or inequality that doesn't work this way. A linear equation might have may more solutions than just one. Or it might have none at all. An inequality might have no solutions, or maybe every real number is a solution. In this section we explore these equations and inequalities.

2.4.1 Special Solution Sets

Recall that for the equation $x + 2 = 5$, there is only one number which will make the equation true: 3. This means that the solution is 3, and we write the **solution set** as $\{3\}$. We say the equation's solution set has one **element**, 3.

We'll now explore equations that have *all* real numbers as solutions or *no* real numbers at all as solutions.

Example 2.4.2 Solve for x in $3x = 3x + 4$.

To solve this equation, we need to move all terms containing x to one side of the equals sign:

$$\begin{aligned} 3x &= 3x + 4 \\ 3x - 3x &= 3x + 4 - 3x \\ 0 &= 4 \end{aligned}$$

Notice that x is no longer present in the equation. What value can we substitute into x everywhere that you see x in the equation to make $0 = 4$ true? Nothing! There is no x to substitute a value into and change the equation from being $0 = 4$. We say this equation has no solution. Or, the equation has an empty solution set. We can write this as $\emptyset$, or $\{\,\}$, which are the symbols for the **empty set** (not to be confused with the number 0.)

The equation $0 = 4$ is known a **false statement**. It is false no matter what x is. It indicates there is no solution to the original equation.

Example 2.4.3 Solve for x in $2x + 1 = 2x + 1$.

We will move all terms containing x to one side of the equals sign:

$$\begin{aligned} 2x + 1 &= 2x + 1 \\ 2x + 1 - 2x &= 2x + 1 - 2x \\ 1 &= 1 \end{aligned}$$

At this point, x is no longer present in the equation. What value can we substitute into x to make $1 = 1$ true? Any number! This means that all real numbers are solutions to the equation $2x + 1 = 2x + 1$. We say this equation's solution set contains *all real numbers*. We can write this set using set-builder notation as $\{x \mid x \text{ is a real number}\}$ or using interval notation as $(-\infty, \infty)$.

The equation $1 = 1$ is unambiguously true, since it is true no matter what x is. It indicates that all real numbers are solutions to the original linear equation.

Remark 2.4.4 What would have happened if we had continued solving after we obtained $1 = 1$ in Example 2.4.3?

$$\begin{aligned} 1 &= 1 \\ 1 - 1 &= 1 - 1 \\ 0 &= 0 \end{aligned}$$

As we can see, all we found was another unambiguously true equation.

Warning 2.4.5 Note that there is a very important difference between when our solving process ends with $0 = 0$ and when it ends with $x = 0$. The first equation is true for all real numbers, and the solution set is $(-\infty, \infty)$. The second has only one solution, 0, and the solution set can be written as $\{0\}$.

Example 2.4.6 Solve for t in the inequality $4t + 5 > 4t + 2$.

To solve for t, we will first subtract $4t$ from each side to get all terms containing t on one side:

$$\begin{aligned} 4t + 5 &> 4t + 2 \\ 4t + 5 - 4t &> 4t + 2 - 4t \\ 5 &> 2 \end{aligned}$$

Notice that again, the variable t is no longer contained in the inequality. We then need to consider which values of t make the inequality true. The answer is that *all values* of t make $5 > 2$, which we know is a very strange sentence. So our solution set is all real numbers, which we can write as $\{t \mid t \text{ is a real number}\}$, or as $(-\infty, \infty)$.

Example 2.4.7 Solve for x in the inequality $-5x + 1 \leq -5x$.

To solve for x, we will first add $5x$ to each side to get all terms containing x on one side:

$$\begin{aligned} -5x + 1 &\leq -5x \\ -5x + 1 + 5x &\leq -5x + 5x \\ 1 &\leq 0 \end{aligned}$$

Once more, the variable x is absent. So we can ask ourselves, "For which values of x is $1 \leq 0$ true?" The answer is *none*, and so there is no solution to this inequality. We can write the solution set using $\emptyset$.

Remark 2.4.8 Again consider what would have happened if we had continued solving after we obtained $1 \leq 0$ in Example 2.4.7.

$$\begin{aligned} 1 &\leq 0 \\ 1 - 1 &\leq 0 - 1 \\ 0 &\leq -1 \end{aligned}$$

As we can see, all we found was another false statement—a different inequality that is not true for any real number.

Let's summarize the two special cases when solving linear equations and inequalities.

List 2.4.9: Special Solution Sets for Equations and Inequalities

All Real Numbers When solving an equation or inequality boils down to an unambiguously true equation or inequality such as $2 = 2$ or $0 < 2$, all real numbers are solutions. We write this solution set as either $(-\infty, \infty)$ or $\{x \mid x \text{ is a real number}\}$.

No Solution When solving an equation or inequality boils down to a *false statement* such as $0 = 2$ or $0 > 2$, no real number is a solution. We write this solution set as either $\{\,\}$ or $\emptyset$ or write the words "no solution exists."

2.4.2 Further Examples

Example 2.4.10 Solve for a in $\frac{2}{3}(a+1) - \frac{5}{6} = \frac{2}{3}a$.

To solve this equation for a, we recall the technique of multiplying each side of the equation by the LCD of all fractions. Here, this means that we will multiply each side by 6 as our first step. After that, we'll be able to simplify each side of the equation and continue solving for a:

$$\begin{aligned}
\frac{2}{3}(a+1) - \frac{5}{6} &= \frac{2}{3}a \\
6 \cdot \left(\frac{2}{3}(a+1) - \frac{5}{6}\right) &= 6 \cdot \frac{2}{3}a \\
6 \cdot \frac{2}{3}(a+1) - 6 \cdot \frac{5}{6} &= 6 \cdot \frac{2}{3}a \\
4(a+1) - 5 &= 4a \\
4a + 4 - 5 &= 4a \\
4a - 1 &= 4a \\
4a - 1 - 4a &= 4a - 4a \\
-1 &= 0
\end{aligned}$$

The statement $-1 = 0$ is false, so the equation has no solution. We can write the solution set as the empty set, $\emptyset$.

Example 2.4.11 Solve for x in the equation $3(x+2) - 8 = (5x+4) - 2(x+1)$.

To solve for x, we will first need to simplify the left side and right side of the equation as much as possible by distributing and combining like terms:

$$\begin{aligned}
3(x+2) - 8 &= (5x+4) - 2(x+1) \\
3x + 6 - 8 &= 5x + 4 - 2x - 2 \\
3x - 2 &= 3x + 2
\end{aligned}$$

From here, we'll want to subtract $3x$ from each side:

$$\begin{aligned}
3x - 2 - 3x &= 3x + 2 - 3x \\
-2 &= 2
\end{aligned}$$

As the equation $-2 = 2$ is not true for any value of x, there is no solution to this equation. We can write the solution set as the empty set, $\emptyset$.

Example 2.4.12 Solve for z in the inequality $\frac{3z}{5} + \frac{1}{2} \leq \left(\frac{z}{10} + \frac{3}{4}\right) + \left(\frac{z}{2} - \frac{1}{4}\right)$.

To solve for z, we will first need to multiply each side of the inequality by the LCD, which is 20. After that, we'll finish solving by putting all terms containing a variable on one side of the inequality:

$$\begin{aligned}
\frac{3z}{5} + \frac{1}{2} &\leq \left(\frac{z}{10} + \frac{3}{4}\right) + \left(\frac{z}{2} - \frac{1}{4}\right) \\
20 \cdot \left(\frac{3z}{5} + \frac{1}{2}\right) &\leq 20 \cdot \left(\left(\frac{z}{10} + \frac{3}{4}\right) + \left(\frac{z}{2} - \frac{1}{4}\right)\right) \\
20 \cdot \left(\frac{3z}{5}\right) + 20 \cdot \left(\frac{1}{2}\right) &\leq 20 \cdot \left(\frac{z}{10} + \frac{3}{4}\right) + 20 \cdot \left(\frac{z}{2} - \frac{1}{4}\right) \\
20 \cdot \left(\frac{3z}{5}\right) + 20 \cdot \left(\frac{1}{2}\right) &\leq 20 \cdot \left(\frac{z}{10}\right) + 20 \cdot \left(\frac{3}{4}\right) + 20 \cdot \left(\frac{z}{2}\right) - 20 \cdot \left(\frac{1}{4}\right) \\
12z + 10 &\leq 2z + 15 + 10z - 5 \\
12z + 10 &\leq 12z + 10 \\
12z + 10 - 12z &\leq 12z + 10 - 12z \\
10 &\leq 10
\end{aligned}$$

As the equation $10 \leq 10$ is true for all values of z, all real numbers are solutions to this inequality. Thus the solution set is $\{z \mid z \text{ is a real number}\}$, or $(-\infty, \infty)$ in interval notation.

2.4.3 Reading Questions

1. Given a linear equation in one variable, what are the possibilities for how many solutions it could have? One solution? Two solutions? Other possibilities?

2. How will you know when a linear equation or inequality has no solutions?

3. How will you know when all numbers are solutions to a linear equation or inequality?

2.4.4 Exercises

Review and Warmup Solve the equation.

1. $5a + 2 = 12$

2. $10c + 1 = 51$

3. $-5A - 8 = -43$

4. $-8C - 5 = 35$

5. $-10m + 8 = -m - 10$

6. $-7p + 2 = -p - 52$

7. $16 = -4(q - 6)$

8. $130 = -10(y - 10)$

Solving Equations with Special Solution Sets Solve the equation.

9. $6r = 6r + 4$

10. $4a = 4a + 7$

11. $10c + 2 = 10c + 2$

12. $6A + 6 = 6A + 6$

13. $3C - 2 - 4C = -7 - C + 5$

14. $9m - 5 - 10m = -9 - m + 4$

15. $-7 - 10p + 3 = -p + 13 - 9p$

16. $-5 - 6q + 1 = -q + 12 - 5q$

17. $8(y - 9) = 8(y - 4)$

18. $6(r - 5) = 6(r - 3)$

19. $2(5-4a)-(8a-10)=22-2(6+8a)$

20. $4(9-2b)-(8b-5)=16-2(7+8b)$

21. $15-4(5+5A)=-21A-(5-A)$

22. $19-3(8+4C)=-13C-(5-C)$

23. Solve the equation.

a. $9m+3=3m+3$

b. $9m+3=9m+3$

c. $9m+3=9m+6$

24. Solve the equation.

a. $7p+7=5p+7$

b. $7p+7=7p+7$

c. $7p+7=7p+10$

Solving Inequalities with Special Solution Sets Solve this inequality.

25. $10x>10x+9$

26. $2x>2x+6$

27. $-4x\leq-4x-8$

28. $-4x\leq-4x-1$

29. $-7+6x+14\geq 6x+7$

30. $-1+6x+4\geq 6x+3$

31. $-6+8x+16<8x+10$

32. $-10+8x+16<8x+6$

33. $-10-6z+4>-z+5-5z$

34. $-10-10z+8>-z+1-9z$

35. $2(k-8)\leq 2(k-1)$

36. $4(k-6)\leq 4(k-3)$

37. $4x\leq 4x+8$

38. $6x\leq 6x+5$

39. $4(2-10m)-(8m-4)>7-2(10+24m)$

40. $4(7-4m)-(6m-4)>9-2(5+11m)$

Challenge

41. Fill in the right side of the equation to create a linear equation that meets the description.

a. Create a linear equation with solution set $\{2\}$.

$16(x+4)$

b. Create a linear equation with *infinitely many solutions*.

$16(x+4)$

2.5 Isolating a Linear Variable

In this section, we solve for a variable in a linear equation, even when there is more than one variable.

2.5.1 Solving for a Variable

The formula of calculating a rectangle's area is $A = \ell w$, where ℓ stands for the rectangle's length, and w stands for width. So when a rectangle's length and width are given, we can easily calculate its area.

But what if we know a rectangle's *area* and *length*, and we need to calculate its width?

If a rectangle's area is $12\,\mathrm{m}^2$, and its length is $4\,\mathrm{m}$, we could find its width this way:

$$\begin{aligned} A &= \ell w \\ 12 &= 4w \\ \frac{12}{4} &= \frac{4w}{4} \\ 3 &= w \\ w &= 3 \end{aligned}$$

We can go through the same motions without using the specific numbers for A and ℓ:

$$\begin{aligned} A &= \ell w \\ \frac{A}{\ell} &= \frac{\ell w}{\ell} \\ \frac{A}{\ell} &= w \\ w &= \frac{A}{\ell} \end{aligned}$$

Now if we want to find the width when $A = 12$ and $\ell = 4$, we have a formula: $w = \frac{A}{\ell}$. But the formula could also quickly tell us the width when $A = 100$ and $\ell = 20$, or when when $A = 23.47$ and $\ell = 2.71$, or any other variation. This formula, $w = \frac{A}{\ell}$, is a handy version of the original equation in situations where w is the unknown.

Remark 2.5.2 Note that in solving for A, we divided each side of the equation by ℓ. The operations that we apply, and the order in which we do them, are determined by the operations in the original equation. In the original equation $A = \ell w$, we saw that w was *multiplied* by ℓ, and so we knew that in order to undo that operation, we would need to *divide* each side by ℓ. We continue to use this process of "un-doing" operations throughout this section.

Example 2.5.3 Solve for R in $P = R - C$. (This is the relationship between profit, revenue, and cost.)

Explanation. To solve for R, we first want to note that C is *subtracted* from R. To undo this, we *add* C to each side of the equation:

$$\begin{aligned} P &= \overset{\downarrow}{R} - C \\ P + C &= \overset{\downarrow}{R} - C + C \\ P + C &= \overset{\downarrow}{R} \\ R &= P + C \end{aligned}$$

Example 2.5.4 Solve for x in $y = mx + b$. (This is a line's equation in slope-intercept form, studied more in Section 3.5.)

Explanation. In the equation $y = mx + b$, we see that x is multiplied by m and then b is added to that. Our first step will be to isolate mx, which we'll do by subtracting b from each side of the equation:

$$\begin{aligned} y &= m\overset{\downarrow}{x} + b \\ y - b &= m\overset{\downarrow}{x} + b - b \\ y - b &= m\overset{\downarrow}{x} \end{aligned}$$

Now that we have mx by itself, we note that x is multiplied by m. To undo this, we divide each side of the equation by m:

$$\begin{aligned} \frac{y-b}{m} &= \frac{m\overset{\downarrow}{x}}{m} \\ \frac{y-b}{m} &= \overset{\downarrow}{x} \\ x &= \frac{y-b}{m} \end{aligned}$$

Warning 2.5.5 It's important to note in Example 2.5.4 that each *side* was divided by m. We can't simply divide y by m, as the equation would no longer be equivalent.

Example 2.5.6 Solve for b in $A = \frac{1}{2}bh$. (This is the area formula for a triangle.)

Explanation. To solve for b, we need to determine what operations need to be undone. The expression $\frac{1}{2}bh$ has multiplication between $\frac{1}{2}$ and b and h. As a first step, we will multiply each side of the equation by 2 in order to eliminate the $\frac{1}{2}$:

$$\begin{aligned} A &= \frac{1}{2}\overset{\downarrow}{b}h \\ 2 \cdot A &= 2 \cdot \frac{1}{2}\overset{\downarrow}{b}h \\ 2A &= \overset{\downarrow}{b}h \end{aligned}$$

Next we undo the multiplication between b and h by dividing each side by h:

$$\begin{aligned} \frac{2A}{h} &= \frac{\overset{\downarrow}{b}h}{h} \\ \frac{2A}{h} &= \overset{\downarrow}{b} \\ b &= \frac{2A}{h} \end{aligned}$$

Checkpoint 2.5.7 Solve for y in $2x + 5y = 10$. (This is a linear equation in standard form, studied more in Section 3.7.)

Explanation. To solve for y, we first isolate $5y$ by subtracting $2x$ from each side of the equation. After that, we can divide each side by 5 to finish solving for y:

$$\begin{aligned}
2x + 5\overset{\downarrow}{y} &= 10\\
2x + 5\overset{\downarrow}{y} - 2x &= 10 - 2x\\
5\overset{\downarrow}{y} &= 10 - 2x\\
\frac{5\overset{\downarrow}{y}}{5} &= \frac{10 - 2x}{5}\\
y &= \frac{10 - 2x}{5}
\end{aligned}$$

Example 2.5.8 Solve for F in $C = \frac{5}{9}(F-32)$. (This represents the relationship between temperature in degrees Celsius and degrees Fahrenheit.)

Explanation. To solve for F, note that it is contained inside parentheses. To isolate the expression $F - 32$, we want to eliminate the $\frac{5}{9}$ outside those parentheses. One way we can undo this multiplication is to divide each side by $\frac{5}{9}$. A better technique is to multiply each side by the reciprocal of $\frac{5}{9}$, which is $\frac{9}{5}$:

$$\begin{aligned}
C &= \frac{5}{9}(\overset{\downarrow}{F} - 32)\\
\frac{9}{5} \cdot C &= \frac{9}{5} \cdot \frac{5}{9}(\overset{\downarrow}{F} - 32)\\
\frac{9}{5}C &= \overset{\downarrow}{F} - 32
\end{aligned}$$

Now that we have $F - 32$, we simply need to add 32 to each side to finish solving for F:

$$\begin{aligned}
\frac{9}{5}C + 32 &= \overset{\downarrow}{F} - 32 + 32\\
\frac{9}{5}C + 32 &= \overset{\downarrow}{F}\\
F &= \frac{9}{5}C + 32
\end{aligned}$$

2.5.2 Reading Questions

1. Suppose you want to solve the equation $mq + b = T$ for q. What would be wrong with dividing on each side by m to get $\frac{mq}{m} + b = \frac{T}{m}$?

2. How do you undo dividing by R?

2.5.3 Exercises

Review and Warmup Solve the equation.

1. $8q + 4 = 52$

2. $5y + 3 = 53$

3. $-10r - 1 = 29$

4. $-4a - 9 = -29$

5. $-5b + 3 = -b - 21$

6. $-10A + 7 = -A - 20$

7. $-63 = -7(B - 1)$

8. $4 = -4(m - 5)$

Solving for a Variable

9. a. Solve $t + 5 = 13$ for t.
b. Solve $y + r = p$ for y.

10. a. Solve $t + 1 = 7$ for t.
b. Solve $r + b = c$ for r.

11. a. Solve $x - 9 = -7$ for x.
b. Solve $y - C = -7$ for y.

12. a. Solve $x - 5 = -3$ for x.
b. Solve $t - y = -3$ for t.

13. a. Solve $-y + 1 = -9$ for y.
b. Solve $-r + A = a$ for r.

14. a. Solve $-y + 3 = -7$ for y.
b. Solve $-x + q = r$ for x.

15. a. Solve $3r = 30$ for r.
b. Solve $at = m$ for t.

16. a. Solve $5r = 10$ for r.
b. Solve $ct = y$ for t.

17. a. Solve $\frac{r}{3} = 10$ for r.
b. Solve $\frac{x}{y} = n$ for x.

18. a. Solve $\frac{t}{5} = 4$ for t.
b. Solve $\frac{r}{q} = y$ for r.

19. a. Solve $4t + 3 = 35$ for t.
b. Solve $xy + C = n$ for y.

20. a. Solve $6x + 6 = 24$ for x.
b. Solve $by + a = r$ for y.

21. a. Solve $xt = n$ for x.
b. Solve $xt = n$ for t.

22. a. Solve $yt = x$ for y.
b. Solve $yt = x$ for t.

23. a. Solve $y + x = C$ for y.
b. Solve $y + x = C$ for x.

24. a. Solve $r + x = t$ for r.
b. Solve $r + x = t$ for x.

25. a. Solve $br + B = q$ for B.
b. Solve $br + B = q$ for b.

26. a. Solve $xt + m = C$ for m.
b. Solve $xt + m = C$ for x.

27. a. Solve $y = Cn + c$ for n.
b. Solve $y = Cn + c$ for C.

28. a. Solve $t = aq + r$ for q.
b. Solve $t = aq + r$ for a.

29. a. Solve $10 = \frac{1}{2}b \cdot 2$ for b.
b. Solve $A = \frac{1}{2}b \cdot h$ for b.

30. a. Solve $8 = \frac{1}{2}b \cdot 2$ for b.
b. Solve $A = \frac{1}{2}b \cdot h$ for b.

31. Solve these linear equations for y.
a. $\frac{y}{5} + 10 = 12$
b. $\frac{y}{x} + 10 = a$

32. Solve these linear equations for y.
a. $\frac{y}{5} + 1 = 3$
b. $\frac{y}{t} + 1 = c$

33. Solve this linear equation for x.

$$y = mx - b$$

34. Solve this linear equation for x.

$$y = -mx + b$$

35. Solve this linear equation for r.

$$C = 2\pi r$$

36. Solve this linear equation for h.

$$V = \pi r^2 h$$

37. Solve this linear equation for t.

$$\frac{t}{x} + A = C$$

38. Solve this linear equation for x.

$$\frac{x}{t} + q = b$$

39. Solve this linear equation for x.

$$\frac{x}{2} + r = a$$

40. Solve this linear equation for y.

$$\frac{y}{9} + x = m$$

41. Solve this linear equation for b.

$$A = r - \frac{9b}{n}$$

42. Solve this linear equation for A.

$$t = B - \frac{9A}{c}$$

43. Solve this linear equation for x.

$$Ax + By = C$$

44. Solve this linear equation for y.

$$Ax + By = C$$

Solve the linear equation for y.

45. $-35x - 5y = 10$

46. $20x - 5y = 65$

47. $8x + 2y = 30$

48. $14x - 2y = -10$

49. $4x - y = 18$

50. $2x - y = -8$

51. $4x - 6y = -36$

52. $8x - 6y = -36$

53. $7y - 3x = 5$

54. $6x - 8y = 5$

55. $21x + 87y = 110$

56. $24y - 51x = 97$

2.6 Linear Equations and Inequalities Chapter Review

2.6.1 Solving Multistep Linear Equations

In Section 2.1 we covered the steps to solve a linear equation and the differences between simplifying expressions, evaluating expressions and solving equations.

Example 2.6.1 Solve for a in $4-(3-a)=-2-2(2a+1)$.

Explanation. To solve this equation, we will simplify each side of the equation, manipulate it so that all variable terms are on one side and all constant terms are on the other, and then solve for a:

$$\begin{aligned}
4-(3-a)&=-2-2(2a+1)\\
4-3+a&=-2-4a-2\\
1+a&=-4-4a\\
1+a+4a&=-4-4a+4a\\
1+5a&=-4\\
1+5a-1&=-4-1\\
5a&=-5\\
\frac{5a}{5}&=\frac{-5}{5}\\
a&=-1
\end{aligned}$$

Checking the solution -1 in the original equation, we get:

$$\begin{aligned}
4-(3-a)&=-2-2(2a+1)\\
4-(3-(-1))&\stackrel{?}{=}-2-2(2(-1)+1)\\
4-(4)&\stackrel{?}{=}-2-2(-1)\\
0&\stackrel{\checkmark}{=}0
\end{aligned}$$

Therefore the solution to the equation is -1 and the solution set is $\{-1\}$.

2.6.2 Solving Multistep Linear Inequalities

In Section 2.2 we covered how solving inequalities is very much like how we solve equations, except that if we multiply or divide by a negative we switch the inequality sign.

Example 2.6.2 Solve for x in $-2-2(2x+1)>4-(3-x)$. Write the solution set in both set-builder notation and interval notation.

Explanation.

$$\begin{aligned}
-2-2(2x+1)&>4-(3-x)\\
-2-4x-2&>4-3+x\\
-4x-4&>x+1\\
-4x-4-x&>x+1-x
\end{aligned}$$

$$\begin{aligned}
-5x - 4 &> 1 \\
-5x - 4 + 4 &> 1 + 4 \\
-5x &> 5 \\
\frac{-5x}{-5} &< \frac{5}{-5} \\
x &< -1
\end{aligned}$$

Note that when we divided both sides of the inequality by -5, we had to switch the direction of the inequality symbol.

The solution set in set-builder notation is $\{x \mid x < -1\}$. The solution set in interval notation is $(-\infty, -1)$.

2.6.3 Linear Equations and Inequalities with Fractions

In Section 2.3 we covered how to eliminate denominators in an equation with the LCD to help solve the equation.

Example 2.6.3 Solve for x in $\frac{1}{4}x + \frac{2}{3} = \frac{1}{6}$.

Explanation.

We'll solve by multiplying each side of the equation by 12:

$$\begin{aligned}
\frac{1}{4}x + \frac{2}{3} &= \frac{1}{6} \\
12 \cdot \left(\frac{1}{4}x + \frac{2}{3}\right) &= 12 \cdot \frac{1}{6} \\
12 \cdot \left(\frac{1}{4}x\right) + 12 \cdot \left(\frac{2}{3}\right) &= 12 \cdot \frac{1}{6} \\
3x + 8 &= 2 \\
3x &= -6 \\
\frac{3x}{3} &= \frac{-6}{3} \\
x &= -2
\end{aligned}$$

Checking the solution:

$$\begin{aligned}
\frac{1}{4}x + \frac{2}{3} &= \frac{1}{6} \\
\frac{1}{4}(-2) + \frac{2}{3} &\stackrel{?}{=} \frac{1}{6} \\
-\frac{2}{4} + \frac{2}{3} &\stackrel{?}{=} \frac{1}{6} \\
-\frac{6}{12} + \frac{8}{12} &\stackrel{?}{=} \frac{1}{6} \\
\frac{2}{12} &\stackrel{?}{=} \frac{1}{6} \\
\frac{1}{6} &\stackrel{\checkmark}{=} \frac{1}{6}
\end{aligned}$$

The solution is therefore -2. We write the solution set s $\{-2\}$.

2.6.4 Special Solution Sets

In Section 2.4 we covered linear equations that have no solutions and also linear equations that have infinitely many solutions. When solving linear inequalities, it's also possible that no solution exists or that all real numbers are solutions.

Example 2.6.4

a. Solve for x in the equation $3x = 3x + 4$.

b. Solve for t in the inequality $4t + 5 > 4t + 2$.

Explanation.

a. To solve this equation, we need to move all terms containing x to one side of the equals sign:

$$\begin{aligned} 3x &= 3x + 4 \\ 3x - 3x &= 3x + 4 - 3x \\ 0 &= 4 \end{aligned}$$

This equation has no solution. We write the solution set as $\emptyset$, which is the symbol for the empty set.

b. To solve for t, we will first subtract $4t$ from each side to get all terms containing t on one side:

$$\begin{aligned} 4t + 5 &> 4t + 2 \\ 4t + 5 - 4t &> 4t + 2 - 4t \\ 5 &> 2 \end{aligned}$$

All values of the variable t make the inequality true. The solution set is all real numbers, which we can write as $\{t \mid t \text{ is a real number}\}$ in set notation, or $(-\infty, \infty)$ in interval notation.

2.6.5 Isolating a Linear Variable

In Section 2.5 we covered how to solve an equation when there are multiple variables in the equation.

Example 2.6.5 Solve for x in $y = mx + b$.
Explanation.

$$\begin{aligned} y &= mx + b \\ y - b &= mx + b - b \\ y - b &= mx \\ \frac{y-b}{m} &= \frac{mx}{m} \\ \frac{y-b}{m} &= x \end{aligned}$$

2.6.6 Exercises

1. a. Solve $3(x + 7) - 7 = 35$.

b. Evaluate $3(x + 7) - 7$ when $x = 7$.

c. Simplify $3(x + 7) - 7$.

2. a. Solve $5(y - 4) + 7 = 22$.

b. Evaluate $5(y - 4) + 7$ when $y = 7$.

c. Simplify $5(y - 4) + 7$.

3. Solve the equation.
$-16 = -7b - 8 - b$

4. Solve the equation.
$14 = -3A - 2 - A$

5. Solve the equation.
$5 + 9(B - 3) = -7 - (8 - 2B)$

6. Solve the equation.
$3 + 10(m - 8) = -74 - (3 - 2m)$

7. Solve the equation.
$-6 - 7n + 3 = -n + 9 - 6n$

8. Solve the equation.
$-9 - 9q + 2 = -q + 7 - 8q$

9. Solve the equation.
$21 = \frac{x}{5} + \frac{x}{2}$

10. Solve the equation.
$22 = \frac{r}{3} + \frac{r}{8}$

11. Solve the equation.
$\frac{t-1}{6} = \frac{t+4}{8}$

12. Solve the equation.
$\frac{b-5}{4} = \frac{b+1}{6}$

13. Solve this inequality.
$3 - (y+6) < -11$

14. Solve this inequality.
$4 - (y+9) < 0$

15. Solve this inequality.
$8(k-7) \leq 8(k-3)$

16. Solve this inequality.
$10(k-5) \leq 10(k-1)$

17. Solve this inequality.
$5 + 9(x-7) < -34 - (4-4x)$

18. Solve this inequality.
$1 + 8(x-10) < -87 - (4-2x)$

19. Solve this inequality.
$-\frac{1}{4}t > \frac{2}{3}t - 22$

20. Solve this inequality.
$-\frac{5}{6}t > \frac{4}{3}t - 26$

21. Solve this linear equation for x.

$$Ax + By = C$$

22. Solve this linear equation for y.

$$Ax + By = C$$

23. Solve this linear equation for B.

$$C = a - \frac{7B}{x}$$

24. Solve this linear equation for m.

$$r = c - \frac{8m}{t}$$

25. Matthew has \$87 in his piggy bank. He plans to purchase some Pokemon cards, which costs \$2.85 each. He plans to save \$52.80 to purchase another toy. At most how many Pokemon cards can he purchase? Write an equation to solve this problem.

Matthew can purchase at most [ ] Pokemon cards.

26. Chris has \$89 in his piggy bank. He plans to purchase some Pokemon cards, which costs \$2.25 each. He plans to save \$71 to purchase another toy. At most how many Pokemon cards can he purchase? Write an equation to solve this problem.

Chris can purchase at most [ ] Pokemon cards.

27. Use a linear equation to solve the word problem.

Daniel has \$95.00 in his piggy bank, and he spends \$3.50 every day. Sydney has \$40.00 in her piggy bank, and she saves \$1.50 every day.

If they continue to spend and save money this way, how many days later would they have the same amount of money in their piggy banks?

[ ] days later, Daniel and Sydney will have the same amount of money in their piggy banks.

28. Use a linear equation to solve the word problem.

Sean has \$100.00 in his piggy bank, and he spends \$2.50 every day. Kurt has \$8.00 in his piggy bank, and he saves \$1.50 every day.

If they continue to spend and save money this way, how many days later would they have the same amount of money in their piggy banks?

[ ] days later, Sean and Kurt will have the same amount of money in their piggy banks.

29. Use a linear equation to solve the word problem.

Massage Heaven and Massage You are competitors. Massage Heaven has 4200 registered customers, and it gets approximately 700 newly registered customers every month. Massage You has 9450 registered customers, and it gets approximately 350 newly registered customers every month. How many months would it take Massage Heaven to catch up with Massage You in the number of registered customers?

These two companies would have approximately the same number of registered customers ______ months later.

30. Use a linear equation to solve the word problem.

Two truck rental companies have different rates. V-Haul has a base charge of \$60.00, plus \$0.35 per mile. W-Haul has a base charge of \$45.40, plus \$0.45 per mile. For how many miles would these two companies charge the same amount?

If a driver drives ______ miles, those two companies would charge the same amount of money.

31. A rectangle's perimeter is 146 ft. Its length is 2 ft shorter than four times its width. Use an equation to find the rectangle's length and width.

Its width is ______.

Its length is ______.

32. A rectangle's perimeter is 216 ft. Its length is 4 ft longer than three times its width. Use an equation to find the rectangle's length and width.

Its width is ______.

Its length is ______.

Chapter 3

Graphing Lines

3.1 Cartesian Coordinates

When we model a relationship between two variables visually, we use the *Cartesian coordinate system*. This section covers the basic vocabulary and ideas that come with the Cartesian coordinate system.

The Cartesian coordinate system identifies the location of every point in a plane. Basically, the system gives every point in a plane its own "address" in relation to a starting point. We'll use a street grid as an analogy. Here is a map with Carl's home at the center. The map also shows some nearby businesses. Assume each unit in the grid represents one city block.

René Descartes. Several ideas and conventions used in mathematics are attributed to (or at least named after) René Descartes[1]. The Cartesian coordinate system is one of these.

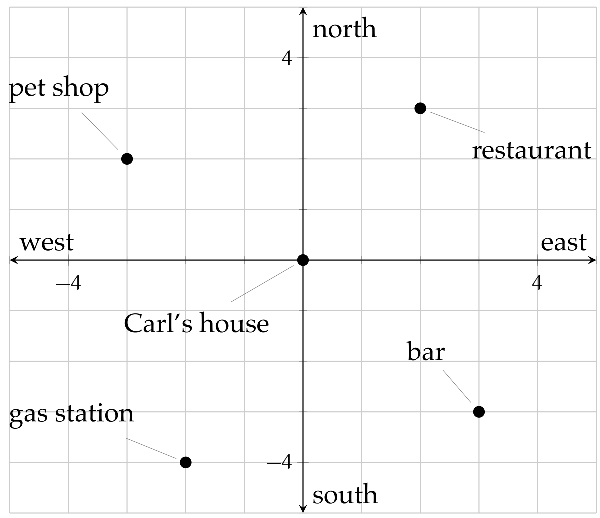


Figure 3.1.2: Carl's neighborhood

[1] en.wikipedia.org/wiki/René_Descartes

If Carl has an out-of-town guest who asks him how to get to the restaurant, Carl could say:

> "First go 2 blocks east (to the right on the map), then go 3 blocks north (up on the map)."

Two numbers are used to locate the restaurant. In the Cartesian coordinate system, these numbers are called **coordinates** and they are written as the **ordered pair** $(2, 3)$. The first coordinate, 2, represents distance traveled from Carl's house to the east (or to the right horizontally on the graph). The second coordinate, 3, represents distance to the north (up vertically on the graph).

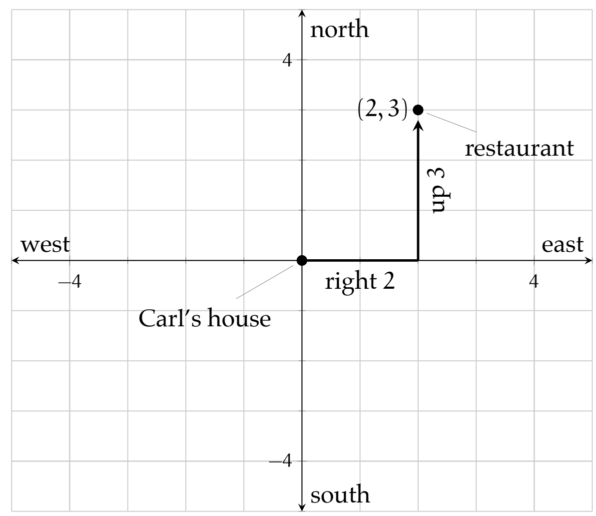


Figure 3.1.3: Carl's path to the restaurant

To travel from Carl's home to the pet shop, he would go 3 blocks west, and then 2 blocks north. In the Cartesian coordinate system, the *positive* directions are to the *right* horizontally and *up* vertically. The *negative* directions are to the *left* horizontally and *down* vertically. So the pet shop's Cartesian coordinates are $(-3, 2)$.

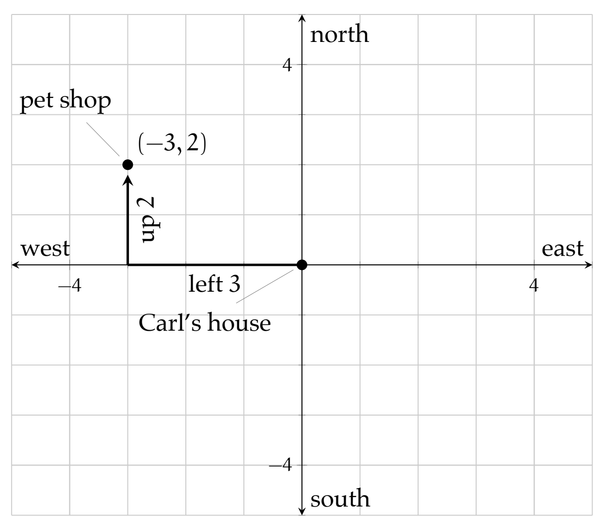


Figure 3.1.4: Carl's path to the pet shop

Remark 3.1.5 It's important to know that the order of Cartesian coordinates is (horizontal, vertical). This idea of communicating horizontal information *before* vertical information is consistent throughout most of mathematics.

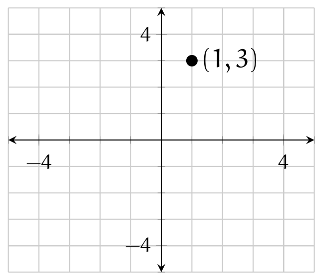
Checkpoint 3.1.6 Use Figure 3.1.2 to answer the following questions.

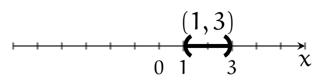
a. What are the coordinates of the bar?

b. What are the coordinates of the gas station?

c. What are the coordinates of Carl's house?

Warning 3.1.7 Notation Issue: Coordinates or Interval? Unfortunately, the notation for an ordered pair looks exactly like interval notation for an open interval. *Context* will help you understand if $(1,3)$ indicates the point 1 unit right of the origin and 3 units up, or if $(1,3)$ indicates the interval of all real numbers between 1 and 3.





Traditionally, the variable x represents numbers on the horizontal axis, so it is called the x**-axis**. The variable y represents numbers on the vertical axis, so it is called the y**-axis**. The axes meet at the point $(0,0)$, which is called the **origin**. Every point in the plane is represented by an **ordered pair**, (x,y).

In a Cartesian coordinate system, the map of Carl's neighborhood would look like this:

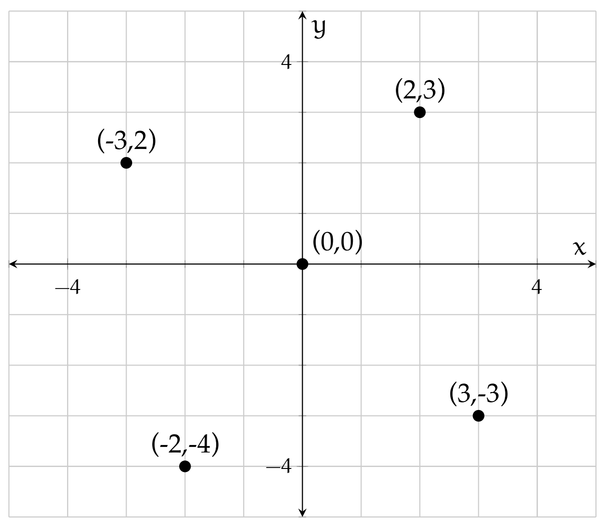


Figure 3.1.8: Carl's Neighborhood in a Cartesian Coordinate System

Definition 3.1.9 Cartesian Coordinate System. The Cartesian coordinate system[2] is a coordinate system that specifies each point uniquely in a plane by a pair of numerical coordinates, which are the signed (positive/negative) distances to the point from two fixed perpendicular directed lines, measured in the same unit of length. Those two reference lines are called the **horizontal axis** and **vertical axis**, and the point where they meet is the **origin**. The horizontal and vertical axes are often called the x**-axis** and y**-axis**.

The plane based on the x-axis and y-axis is called a **coordinate plane**. The ordered pair used to locate a point is called the point's **coordinates**, which consists of an x**-coordinate** and a y**-coordinate**. For example, the point $(1,2)$, has x-coordinate 1, and y-coordinate 2. The origin has coordinates $(0,0)$.

A Cartesian coordinate system is divided into four **quadrants**, as shown in Figure 3.1.10. The quadrants are traditionally labeled with Roman numerals.

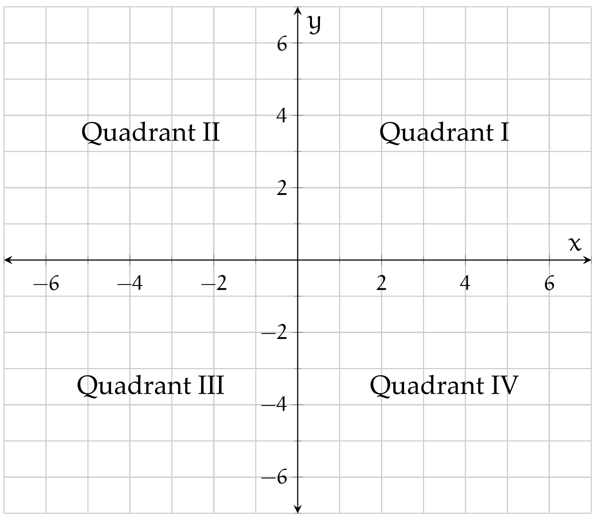
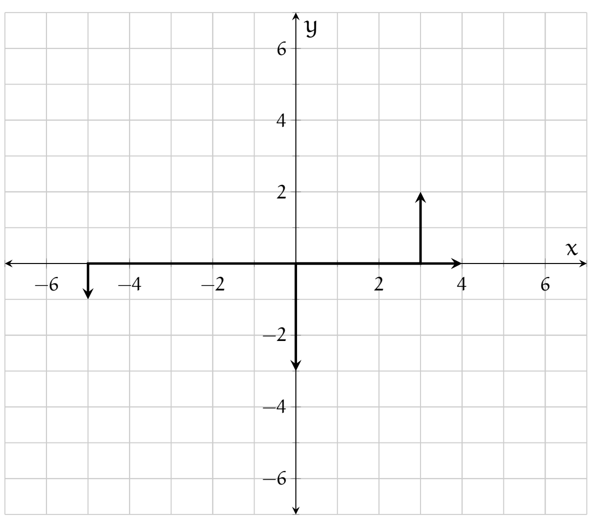


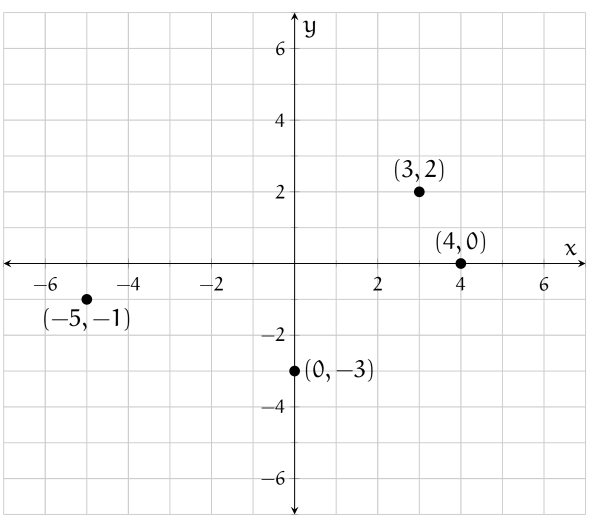
Figure 3.1.10: A Cartesian grid with four quadrants marked

◊

Example 3.1.11 On paper, sketch a Cartesian coordinate system with units, and then plot the following points: $(3, 2)$, $(-5, -1)$, $(0, -3)$, $(4, 0)$.

Explanation.





2 en.wikipedia.org/wiki/Cartesian_coordinate_system

Reading Questions

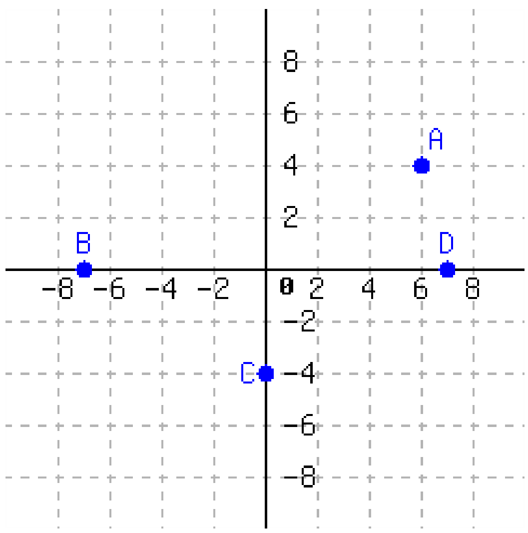
1. What are the coordinates of the gas station in the map of Carl's neighborhood?

2. A Cartesian coordinate system has seven "places" within that are worth noting. What are they? (For example, one of them is Quadrant I.)

Exercises

Identifying Coordinates Locate each point in the graph:

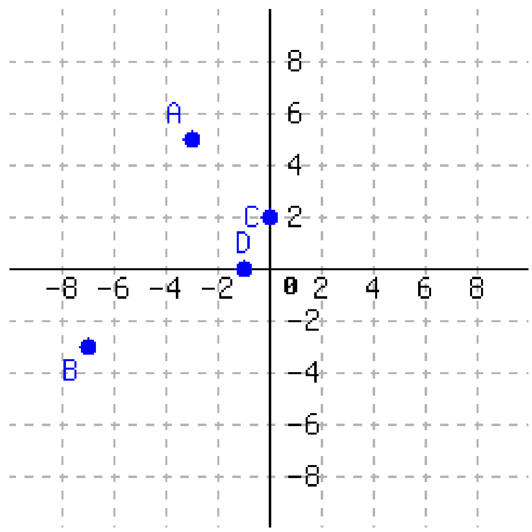
1.



Write each point's position as an ordered pair, like $(1, 2)$.

A = ______ B = ______
C = ______ D = ______

2.



Write each point's position as an ordered pair, like $(1, 2)$.

A = ______ B = ______
C = ______ D = ______

Creating Sketches of Graphs

3. Sketch the points $(8, 2)$, $(5, 5)$, $(-3, 0)$, and $(2, -6)$ on a Cartesian plane.

4. Sketch the points $(1, -4)$, $(-3, 5)$, $(0, 4)$, and $(-2, -6)$ on a Cartesian plane.

5. Sketch the points $(208, -50)$, $(97, 112)$, $(-29, 103)$, and $(-80, -172)$ on a Cartesian plane.

6. Sketch the points $(110, 38)$, $(-205, 52)$, $(-52, 125)$, and $(-172, -80)$ on a Cartesian plane.

7. Sketch the points $(5.5, 2.7)$, $(-7.3, 2.75)$, $(-\frac{10}{3}, \frac{1}{2})$, and $(-\frac{28}{5}, -\frac{29}{4})$ on a Cartesian plane.

8. Sketch the points $(1.9, -3.3)$, $(-5.2, -8.11)$, $(\frac{7}{11}, \frac{15}{2})$, and $(-\frac{16}{3}, \frac{19}{5})$ on a Cartesian plane.

9. Sketch a Cartesian plane and shade the quadrants where the x-coordinate is negative.

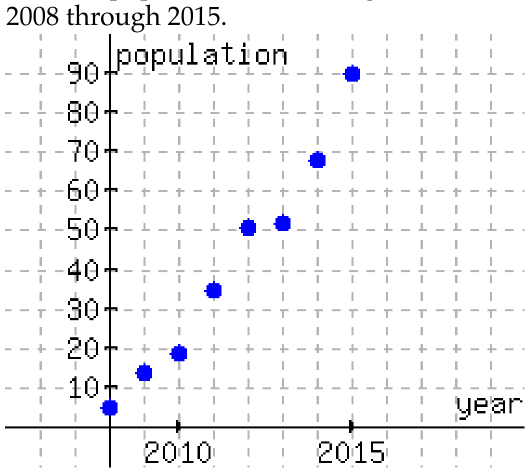
10. Sketch a Cartesian plane and shade the quadrants where the y-coordinate is positive.

11. Sketch a Cartesian plane and shade the quadrants where the x-coordinate has the same sign as the y-coordinate.

12. Sketch a Cartesian plane and shade the quadrants where the x-coordinate and the y-coordinate have opposite signs.

Cartesian Plots in Context

13. This graph gives the minimum estimates of the wolf population in Washington from 2008 through 2015.

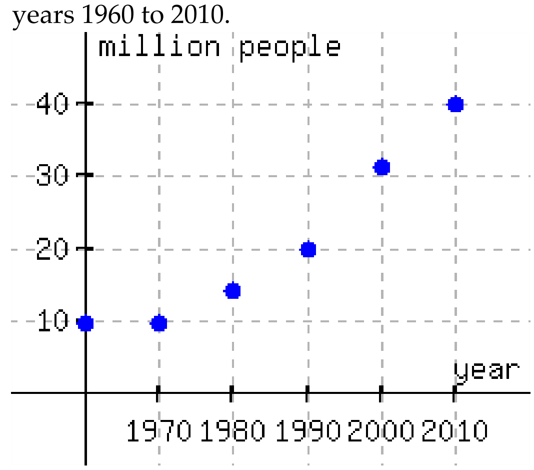


What are the Cartesian coordinates for the point representing the year 2011?

Between 2011 and 2012, the wolf population grew by ☐ wolves.

List at least three ordered pairs in the graph.

14. Here is a graph of the foreign-born US population (in millions) during Census years 1960 to 2010.



What are the Cartesian coordinates for the point representing the year 1980?

Between 1980 and 2000, the US population that is foreign-born increased by ☐ million people.

List at least three ordered pairs in the graph.

Regions in the Cartesian Plane

15. The point $(3, -10)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .
The point $(-5, 7)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .
The point $(8, 9)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .
The point $(-3, -9)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

16. The point $(6, 4)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .
The point $(4, -5)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .
The point $(-10, -4)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .
The point $(-10, 4)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

17. Assume the point (x, y) is in Quadrant II, locate the following points:
The point $(-x, y)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .
The point $(x, -y)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .
The point $(-x, -y)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

18. Assume the point (x, y) is in Quadrant IV, locate the following points:
The point $(-x, y)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

The point $(x, -y)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .
The point $(-x, -y)$ is in Quadrant (☐ I ☐ II ☐ III ☐ IV) .

19. Answer the following questions on the coordinate system:
For the point (x, y), if $x > 0$ and $y > 0$, then the point is in/on (☐ Quadrant I ☐ Quadrant II ☐ Quadrant III ☐ Quadrant IV ☐ the x-axis ☐ the y-axis) .
For the point (x, y), if $x > 0$ and $y < 0$, then the point is in/on (☐ Quadrant I ☐ Quadrant II ☐ Quadrant III ☐ Quadrant IV ☐ the x-axis ☐ the y-axis) .
For the point (x, y), if $y = 0$, then the point is in/on (☐ Quadrant I ☐ Quadrant II ☐ Quadrant III ☐ Quadrant IV ☐ the x-axis ☐ the y-axis) .
For the point (x, y), if $x < 0$ and $y < 0$, then the point is in/on (☐ Quadrant I ☐ Quadrant II ☐ Quadrant III ☐ Quadrant IV ☐ the x-axis ☐ the y-axis) .
For the point (x, y), if $x = 0$, then the point is in/on (☐ Quadrant I ☐ Quadrant II ☐ Quadrant III ☐ Quadrant IV ☐ the x-axis ☐ the y-axis) .
For the point (x, y), if $x < 0$ and $y > 0$, then the point is in/on (☐ Quadrant I ☐ Quadrant II ☐ Quadrant III ☐ Quadrant IV ☐ the x-axis ☐ the y-axis) .

Plotting Points and Choosing a Scale

20. What would be the difficulty with trying to plot $(12, 4)$, $(13, 5)$, and $(310, 208)$ all on the same graph?

21. The points $(3, 5)$, $(5, 6)$, $(7, 7)$, and $(9, 8)$ all lie on a straight line. What can go wrong if you make a plot of a Cartesian plane with these points marked, and you don't have tick marks that are evenly spaced apart?

3.2 Graphing Equations

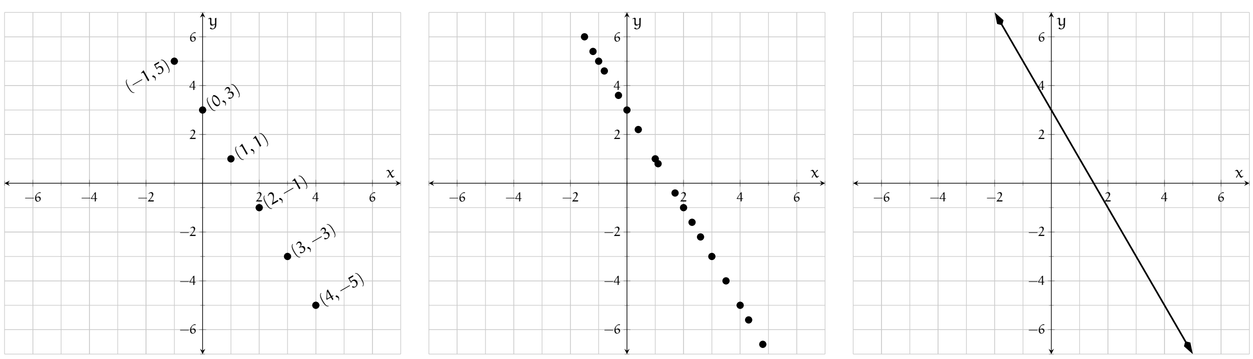
We have graphed *points* in a coordinate system, and now we will graph *lines* and *curves*.

A **graph** of an equation is a picture of that equation's solution set. For example, the graph of $y = -2x+3$ is shown in Figure 3.2.3(c). The graph plots the ordered pairs whose coordinates make $y = -2x + 3$ true. Figure 3.2.2 shows a few points that make the equation true.

$y = -2x + 3$	(x, y)
$5 \overset{\checkmark}{=} -2(-1) + 3$	$(-1, 5)$
$3 \overset{\checkmark}{=} -2(0) + 3$	$(0, 3)$
$1 \overset{\checkmark}{=} -2(1) + 3$	$(1, 1)$
$-1 \overset{\checkmark}{=} -2(2) + 3$	$(2, -1)$
$-3 \overset{\checkmark}{=} -2(3) + 3$	$(3, -3)$
$-5 \overset{\checkmark}{=} -2(4) + 3$	$(4, -5)$

Figure 3.2.2

Figure 3.2.2 tells us that the points $(-1, 5)$, $(0, 3)$, $(1, 1)$, $(2, -1)$, $(3, -3)$, and $(4, -5)$ are all solutions to the equation $y = -2x + 3$, and so they should all be shaded as part of that equation's graph. You can see them in Figure 3.2.3(a). But there are many more points that make the equation true. More points are plotted in Figure 3.2.3(b). Even more points are plotted in Figure 3.2.3(c)—so many, that together the points look like a straight line.



(a) A few points...

(b) More points...

(c) So many points it looks like a straight line

Figure 3.2.3: Graphs of the Equation $y = -2x + 3$

The graph of an equation shades all the points (x, y) that make the equation true once the x- and y-values are substituted in. Typically, there are *so many* points shaded, that the final graph appears to be a continuous line or curve that you could draw with one stroke of a pen.

Checkpoint 3.2.4 The point $(4, -5)$ is on the graph in Figure 3.2.3.(c). What happens when you substitute these values into the equation $y = -2x + 3$?

$$\begin{array}{ccc} y & = & -2x + 3 \\ \_\_\_\_ & \overset{?}{=} & \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \end{array}$$

This equation is (☐ true ☐ false) .

Checkpoint 3.2.5 Decide whether $(5,-2)$ and $(-10,-7)$ are on the graph of the equation $y=-\frac{3}{5}x+1$.

At $(5,-2)$:

$$\begin{aligned} y &= -\tfrac{3}{5}x+1 \\ \_\_ &\stackrel{?}{=} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \end{aligned}$$

This equation is (☐ true ☐ false) and $(5,-2)$ is (☐ part of ☐ not part of) the graph of $y=-\frac{3}{5}x+1$.

At $(-10,-7)$:

$$\begin{aligned} y &= -\tfrac{3}{5}x+1 \\ \_\_ &\stackrel{?}{=} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \end{aligned}$$

This equation is (☐ true ☐ false) and $(-10,-7)$ is (☐ part of ☐ not part of) the graph of $y=-\frac{3}{5}x+1$.

Explanation. If the point $(5,-2)$ is on $y=-\frac{3}{5}x+1$, once we substitute $x=5$ and $y=-2$ into the line's equation, the equation should be true. Let's try:

$$\begin{aligned} y &= -\frac{3}{5}x+1 \\ -2 &\stackrel{?}{=} -\frac{3}{5}(5)+1 \\ -2 &\stackrel{\checkmark}{=} -3+1 \end{aligned}$$

Because this last equation is true, we can say that $(5,-2)$ is on the graph of $y=-\frac{3}{5}x+1$.

However if we substitute $x=-10$ and $y=-7$ into the equation, it leads to $-7=7$, which is false. This tells us that $(-10,-7)$ is *not* on the graph.

To make our own graph of an equation with two variables x and y, we can choose some reasonable x-values, then calculate the corresponding y-values, and then plot the (x,y)-pairs as points. For many algebraic equations, connecting those points with a smooth curve will produce an excellent graph.

Example 3.2.6 Let's plot a graph for the equation $y=-2x+5$. We use a table to organize our work:

x	$y=-2x+5$	Point
-2		
-1		
0		
1		
2		

(a) Set up the table

x	$y=-2x+5$	Point
-2	$-2(-2)+5=9$	$(-2,9)$
-1	$-2(-1)+5=7$	$(-1,7)$
0	$-2(0)+5=5$	$(0,5)$
1	$-2(1)+5=3$	$(1,3)$
2	$-2(2)+5=1$	$(2,1)$

(b) Complete the table

Figure 3.2.7: Making a table for $y=-2x+5$

We use points from the table to graph the equation in Figure 3.2.8. First, we need a coordinate system to draw on. We will eventually need to see the x-values -2, -1, 0, 1, and 2. So drawing an x-axis that runs from -5 to 5 will be good enough. We will eventually need to see the y-values 9, 7, 5, 3, and 1. So drawing a y-axis that runs from -1 to 10 will be good enough.

Axes should always be labeled using the variable names they represent. In this case, with "x" and "y".

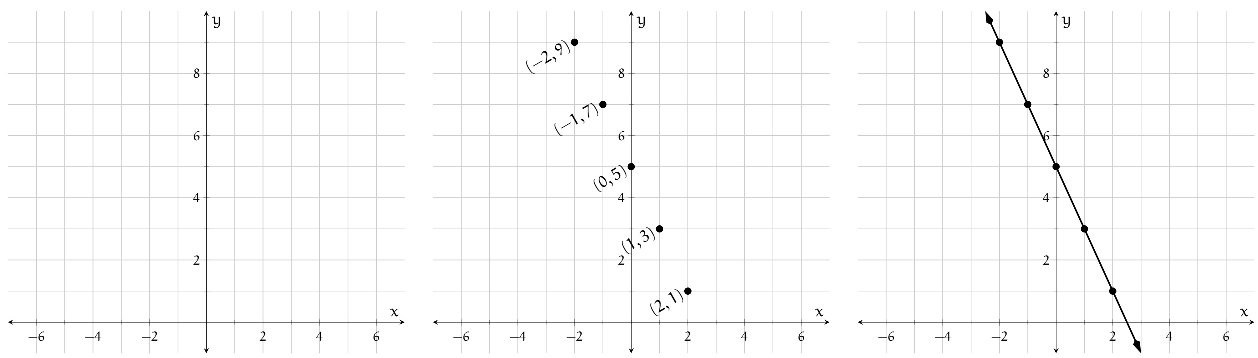
The axes should have tick marks that are spaced evenly. In this case it is fine to place the tick marks one unit apart (on both axes). Labeling at least some of the tick marks is necessary for a reader to understand the scale. Here we label the even-numbered ticks.

Then, connect the points with a smooth curve. Here, the curve is a straight line. Lastly, we can communicate that the graph extends further by sketching arrows on both ends of the line.

All that we have decided so far is drawn in Figure 3.2.8(a).

Next we carefully plot each point from the table in Figure 3.2.8(b). It's not necessary to label each point's coordinates, but we do so here.

Last we connect the points with a smooth curve. Here, the "curve" is a straight line. We communicate that the line extends further by putting arrowheads on both ends.



(a) Staging the coordinate system

(b) Use points from the table

(c) Connect the points in whatever pattern is apparent

Figure 3.2.8: Graphing the Equation $y = -2x + 5$

Remark 3.2.9 Note that our choice of x-values in the table was arbitrary. As long as we determine the coordinates of enough points to indicate the behavior of the graph, we may choose whichever x-values we like. Having a few negative x-values will be good. For simpler calculations, it's fine to choose -2, -1, 0, 1, and 2. However sometimes the equation has context that suggests using other x-values, as in the next examples.

Example 3.2.10 The gas tank in Sofia's car holds 14 gal of fuel. Over the course of a long road trip, her car uses fuel at an average rate of $0.032\,\frac{\text{gal}}{\text{mi}}$. If Sofia fills the tank at the beginning of a long trip, then the amount of fuel remaining in the tank, y, after driving x miles is given by the equation $y = 14 - 0.032x$. Make a suitable table of values and graph this equation.

Explanation. Choosing x-values from -2 to 2, as in our previous example, wouldn't make sense here. Sofia cannot drive a negative number of miles, and any long road trip is longer than 2 miles. So in this context, choose x-values that reflect the number of miles Sofia might drive in a day.

x	$y = 14 - 0.032x$	Point
20	13.36	$(20, 13.36)$
50	12.4	$(50, 12.4)$
80	11.44	$(80, 11.44)$
100	10.8	$(100, 10.8)$
200	7.6	$(200, 7.6)$

Figure 3.2.11: Make the table

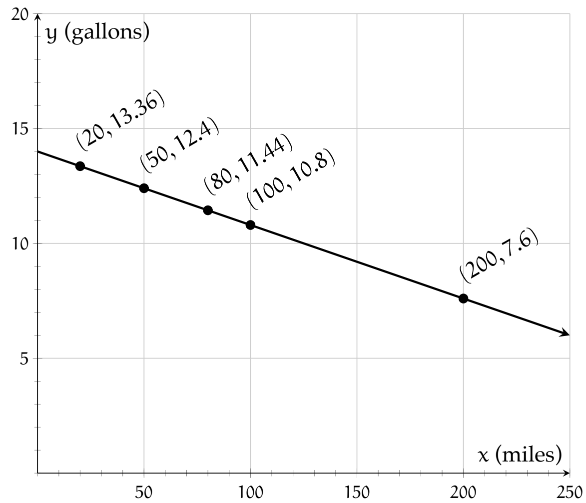


Figure 3.2.12: Make the graph

In Figure 3.2.12, notice how both axes are also labeled with *units* ("gallons" and "miles"). When the equation you plot has context like this example, including the units in the axes labels is very important to help anyone who reads your graph to understand it better.

In the table, x-values ran from 20 to 200, while y-values ran from 13.36 down to 7.6. So it was appropriate to make the x-axis cover something like from 0 to 250, and make the y-axis cover something like from 0 to 20. The scales on the two axes ended up being different.

Example 3.2.13 Plot a graph for the equation $y = \frac{4}{3}x - 4$.

Explanation. This equation doesn't have any context to help us choose x-values for a table. We could use x-values like -2, -1, and so on. But note the fraction in the equation. If we use an x-value like -2, we will have to multiply by the fraction $\frac{4}{3}$ which will leave us still holding a fraction. And then we will have to subtract 4 from that fraction. Since we know that everyone can make mistakes with that kind of arithmetic, maybe we can avoid it with a more wise selection of x-values.

If we use only multiples of 3 for the x-values, then multiplying by $\frac{4}{3}$ will leave us with an integer, which will be easy to subtract 4 from. So we decide to use -6, -3, 0, 3, and 6 for x.

x	$y = \frac{4}{3}x - 4$	Point
-6		
-3		
0		
3		
6		

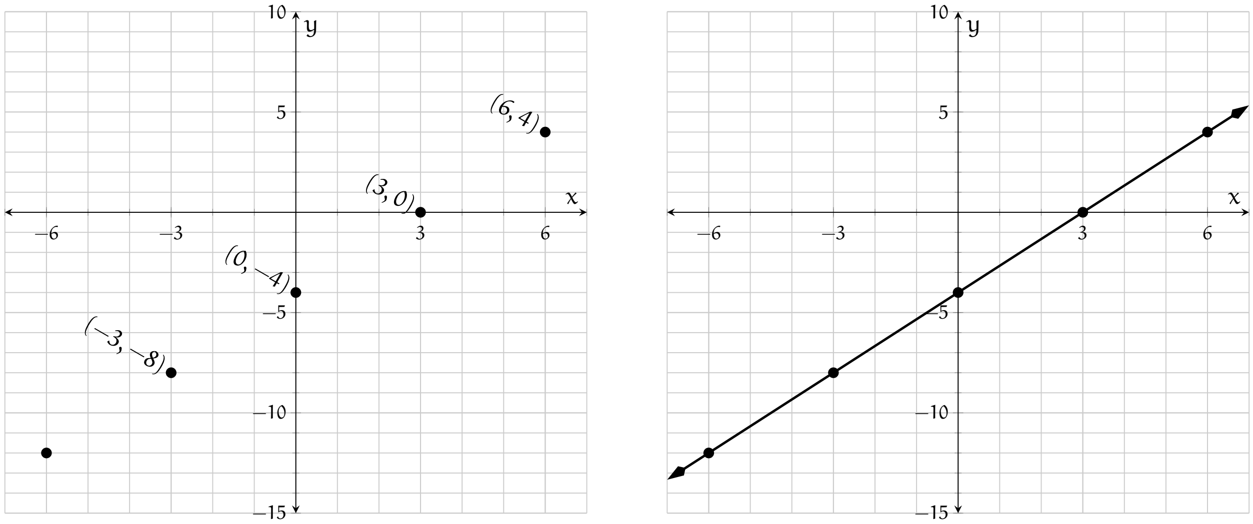
(a) Set up the table

x	$y = \frac{4}{3}x - 4$	Point
-6	$\frac{4}{3}(-6) - 2 = -12$	$(-6, -12)$
-3	$\frac{4}{3}(-3) - 2 = -8$	$(-3, -8)$
0	$\frac{4}{3}(0) - 2 = -4$	$(0, -4)$
3	$\frac{4}{3}(3) - 2 = 0$	$(3, 0)$
6	$\frac{4}{3}(6) - 2 = 4$	$(6, 4)$

(b) Complete the table

Figure 3.2.14: Making a table for $y = \frac{4}{3}x - 4$

We use points from the table to graph the equation. First, plot each point carefully. Then, connect the points with a smooth curve. Here, the curve is a straight line. Lastly, we can communicate that the graph extends further by sketching arrows on both ends of the line.



(a) Use points from the table **(b)** Connect the points in whatever pattern is apparent

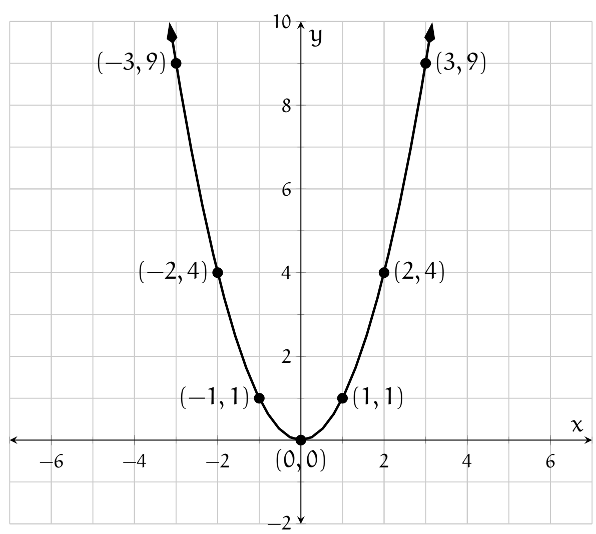
Figure 3.2.15: Graphing the Equation $y = \frac{4}{3}x - 4$

Not all equations make a straight line once they are plotted.

Example 3.2.16 Build a table and graph the equation $y = x^2$. Use x-values from -3 to 3.

Explanation.

x	$y = x^2$	Point
-3	$(-3)^2 = 9$	$(-3, 9)$
-2	$(-2)^2 = 4$	$(-2, 4)$
-1	$(-1)^2 = 1$	$(-1, 1)$
0	$(0)^2 = 0$	$(0, 0)$
1	$(1)^2 = 1$	$(0, 1)$
2	$(2)^2 = 4$	$(2, 4)$
3	$(3)^2 = 9$	$(3, 9)$



In this example, the points do not fall on a straight line. Many algebraic equations have graphs that are non-linear, where the points do not fall on a straight line. We connected the points with a smooth curve, sketching from left to right.

Reading Questions

1. When a point like $(5, 8)$ is on the graph of an equation, where the equation has variables x and y, what happens when you substitute in 5 for x and 8 for y?
2. What are all the things to label when you set up a Cartesian coordinate system?
3. When you start making a table for some equation, you have to choose some x-values. Explain three different ways to choose those x-values that were demonstrated in this section.
4. What is an example of an equation that does not make a straight line once you make a graph of it?

Exercises

Testing Points as Solutions Consider the equation

1. $y = 7x + 10$
   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.
   ☐ $(-5, -25)$ ☐ $(-4, -18)$
   ☐ $(2, 25)$ ☐ $(0, 12)$
2. $y = 8x + 6$
   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.
   ☐ $(-4, -26)$ ☐ $(0, 11)$
   ☐ $(5, 50)$ ☐ $(-3, -18)$
3. $y = -3x - 8$
   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.
   ☐ $(9, -35)$ ☐ $(0, -8)$ ☐ $(-4, 6)$
   ☐ $(-6, 10)$
4. $y = -2x - 2$
   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.
   ☐ $(4, -10)$ ☐ $(-9, 16)$
   ☐ $(0, -2)$ ☐ $(-4, 11)$
5. $y = \frac{2}{3}x - 4$
   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.
   ☐ $(0, 0)$ ☐ $(-6, -6)$
   ☐ $(-15, -14)$ ☐ $(15, 6)$
6. $y = \frac{2}{3}x - 1$
   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.
   ☐ $(12, 7)$ ☐ $(0, 0)$ ☐ $(-6, 0)$
   ☐ $(-15, -11)$
7. $y = -\frac{3}{4}x - 3$
   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.
   ☐ $(0, -3)$ ☐ $(8, -5)$
   ☐ $(-20, 14)$ ☐ $(-16, 9)$
8. $y = -\frac{3}{4}x - 5$
   Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.
   ☐ $(-16, 10)$ ☐ $(-4, -2)$
   ☐ $(4, -4)$ ☐ $(0, -5)$

Tables for Equations Make a table for the equation.

9. The first row is an example.

x	$y=-x+6$	Points
-3	9	$(-3,9)$
-2		
-1		
0		
1		
2		

10. The first row is an example.

x	$y=-x+7$	Points
-3	10	$(-3,10)$
-2		
-1		
0		
1		
2		

11. The first row is an example.

x	$y=5x+5$	Points
-3	-10	$(-3,-10)$
-2		
-1		
0		
1		
2		

12. The first row is an example.

x	$y=6x+1$	Points
-3	-17	$(-3,-17)$
-2		
-1		
0		
1		
2		

13. The first row is an example.

x	$y=-2x+8$	Points
-3	14	$(-3,14)$
-2		
-1		
0		
1		
2		

14. The first row is an example.

x	$y=-5x+4$	Points
-3	19	$(-3,19)$
-2		
-1		
0		
1		
2		

15. The first row is an example.

x	$y=\frac{3}{2}x+7$	Points
-6	-2	$(-6,-2)$
-4		
-2		
0		
2		
4		

16. The first row is an example.

x	$y=\frac{3}{8}x-5$	Points
-24	-14	$(-24,-14)$
-16		
-8		
0		
8		
16		

17. The first row is an example.

x	$y=-\frac{5}{4}x+4$	Points
-12	19	$(-12,19)$
-8		
-4		
0		
4		
8		

18. The first row is an example.

x	$y=-\frac{5}{6}x-4$	Points
-18	11	$(-18,11)$
-12		
-6		
0		
6		
12		

19.

x	$y = 7x$

20.

x	$y = 12x$

21.

x	$y = 8x + 8$

22.

x	$y = 10x + 2$

23.

x	$y = \frac{5}{3}x - 8$

24.

x	$y = \frac{18}{7}x - 6$

25.

x	$y = -\frac{5}{9}x - 4$

26.

x	$y = -\frac{6}{5}x - 1$

Cartesian Plots in Context

27. A certain water heater will cost you \$900 to buy and have installed. This water heater claims that its operating expense (money spent on electricity or gas) will be about \$31 per month. According to this information, the equation $y = 900 + 31x$ models the total cost of the water heater after x months, where y is in dollars. Make a table of at least five values and plot a graph of this equation.

28. You bought a new Toyota Corolla for \$18,600 with a zero interest loan over a five-year period. That means you'll have to pay \$310 each month for the next five years (sixty months) to pay it off. According to this information, the equation $y = 18600 - 310x$ models the loan balance after x months, where y is in dollars. Make a table of at least five values and plot a graph of this equation. Make sure to include a data point representing when you will have paid off the loan.

29. The pressure inside a full propane tank will rise and fall if the ambient temperature rises and falls. The equation $P = 0.1963(T+459.67)$ models this relationship, where the temperature T is measured in °F and the pressure and the pressure P is measured in $\frac{\text{lb}}{\text{in}^2}$. Make a table of at least five values and plot a graph of this equation. Make sure to use T-values that make sense in context.

30. A beloved coworker is retiring and you want to give her a gift of week-long vacation rental at the coast that costs \$1400 for the week. You might end up paying for it yourself, but you ask around to see if the other 29 office coworkers want to split the cost evenly. The equation $y = \frac{1400}{x}$ models this situation, where x people contribute to the gift, and y is the dollar amount everyone contributes. Make a table of at least five values and plot a graph of this equation. Make sure to use x-values that make sense in context.

Graphs of Equations

31. Create a table of ordered pairs and then make a plot of the equation $y = 2x + 3$.

32. Create a table of ordered pairs and then make a plot of the equation $y = 3x + 5$.

33. Create a table of ordered pairs and then make a plot of the equation $y = -4x + 1$.

34. Create a table of ordered pairs and then make a plot of the equation $y = -x - 4$.

35. Create a table of ordered pairs and then make a plot of the equation $y = \frac{5}{2}x$.

36. Create a table of ordered pairs and then make a plot of the equation $y = \frac{4}{3}x$.

37. Create a table of ordered pairs and then make a plot of the equation $y = -\frac{2}{5}x - 3$.

38. Create a table of ordered pairs and then make a plot of the equation $y = -\frac{3}{4}x + 2$.

39. Create a table of ordered pairs and then make a plot of the equation $y = x^2 + 1$.

40. Create a table of ordered pairs and then make a plot of the equation $y = (x - 2)^2$. Use x-values from 0 to 4.

41. Create a table of ordered pairs and then make a plot of the equation $y = -3x^2$.

42. Create a table of ordered pairs and then make a plot of the equation $y = -x^2 - 2x - 3$.

3.3 Exploring Two-Variable Data and Rate of Change

This section is about examining data that has been plotted on a Cartesian coordinate system, and then making observations. In some cases, we'll be able to turn those observations into useful mathematical calculations.

3.3.1 Modeling data with two variables

Using mathematics, we can analyze data from the world around us. We can use what we discover to understand the world better and make predictions. Here's an example with economic data from the US, plotted in a Cartesian plane.

For the years from 1990 to 2013, consider what percent of all income was held by the top 1% of wage earners. The table in Figure 3.3.2 gives the numbers, but any pattern there might not be apparent when looking at the data organized this way. Plotting the data in a Cartesian coordinates sytem can make an overall pattern or trend become visible.

year	%	year	%
1990	14	2002	17
1991	13	2003	18
1992	15	2004	20
1993	14	2005	22
1994	14	2006	23
1995	15	2007	24
1996	17	2008	21
1997	18	2009	18
1998	19	2010	20
1999	20	2011	20
2000	22	2012	22
2001	18	2013	20

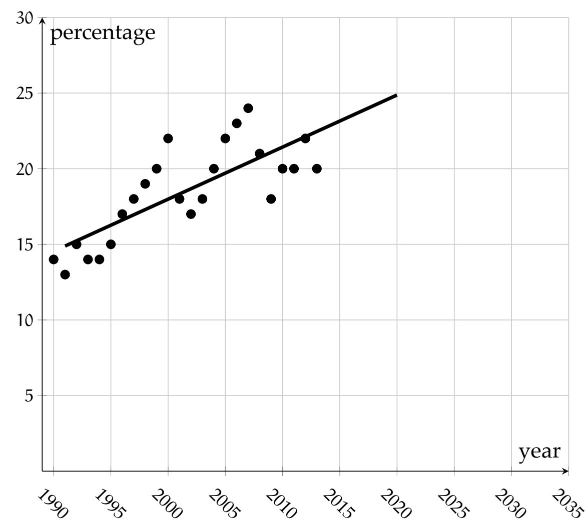


Figure 3.3.2: Share of all income held by the top 1% of wage earners

If this trend continues, what percentage of all income will the top 1 % have in the year 2030? If we model data in the chart with the trend line, we can estimate the value to be 28.6 %. This is one way math is used in real life.

Does that trend line have an equation like those we looked at in Section 3.2? Is it even correct to look at this data set and decide that a straight line is a good model?

3.3.2 Patterns in Tables

Example 3.3.3 Find a pattern in each table. What is the missing entry in each table? Can you describe each pattern in words and/or mathematics?

black	white
big	small
short	tall
few	

USA	Washington
UK	London
France	Paris
Mexico	

1	2
2	4
3	6
5	

Figure 3.3.4: Patterns in 3 tables

Explanation.

black	white
big	small
short	tall
few	*many*

USA	Washington
UK	London
France	Paris
Mexico	*Mexico City*

1	2
2	4
3	6
5	*10*

Figure 3.3.5: Patterns in 3 tables

First table Each word on the right has the opposite meaning of the word to its left.

Second table Each city on the right is the capital of the country to its left.

Third table Each number on the right is double the number to its left.

We can view each table as assigning each input in the left column a corresponding output in the right column. In the first table, for example, when the input "big" is on the left, the output "small" is on the right. The first table's function is to output a word with the opposite meaning of each input word. (This is not a numerical example.)

The third table *is* numerical. And its function is to take a number as input, and give twice that number as its output. Mathematically, we can describe the pattern as "$y = 2x$," where x represents the input, and y represents the output. Labeling the table mathematically, we have Figure 3.3.6.

x (input)	y (output)
1	2
2	4
3	6
5	10
10	20
Pattern: $y = 2x$	

Figure 3.3.6: Table with a mathematical pattern

The equation $y = 2x$ summarizes the pattern in the table. For each of the following tables, find an equation that describes the pattern you see. Numerical pattern recognition may or may not come naturally for you. Either way, pattern recognition is an important mathematical skill that anyone can develop. Solutions for these exercises provide some ideas for recognizing patterns.

Checkpoint 3.3.7 Write an equation in the form $y = \ldots$ suggested by the pattern in the table.

x	y
0	10
1	11
2	12
3	13

Explanation. One approach to pattern recognition is to look for a relationship in each row. Here, the y-value in each row is always 10 more than the x-value. So the pattern is described by the equation $y = x + 10$.

Checkpoint 3.3.8 Write an equation in the form $y = \ldots$ suggested by the pattern in the table.

x	y
0	-1
1	2
2	5
3	8

Explanation. The relationship between x and y in each row is not as clear here. Another popular approach for finding patterns: in each column, consider how the values change from one row to the next. From row to row, the x-value increases by 1. Also, the y-value increases by 3 from row to row.

	x	y	
	0	-1	
$+1 \rightarrow$	1	2	$\leftarrow +3$
$+1 \rightarrow$	2	5	$\leftarrow +3$
$+1 \rightarrow$	3	8	$\leftarrow +3$

Since row-to-row change is always 1 for x and is always 3 for y, the *rate of change* from one row to another row is always the same: 3 units of y for every 1 unit of x. This suggests that $y = 3x$ *might* be a good equation for the table pattern. But if we try to make a table with that pattern:

x	y using $y = 3x$	Actual y
0	0	-1
1	3	2
2	6	5
3	9	8

We find that the values from $y = 3x$ are 1 too large. So now we make an adjustment. The equation $y = 3x - 1$ describes the pattern in the table.

Checkpoint 3.3.9 Write an equation in the form $y = \ldots$ suggested by the pattern in the table.

x	y
0	0
1	1
2	4
3	9

Explanation. Looking for a relationship in each row here, we see that each y-value is the square of the corresponding x-value. That may not be obvious to you. It comes down to recognizing what square numbers are. So the equation is $y = x^2$.

What if we had tried the approach we used in the previous exercise, comparing change from row to row in each column?

$$\begin{array}{rccl} & x & y & \\ & 0 & 0 & \\ +1 \rightarrow & 1 & 1 & \leftarrow +1 \\ +1 \rightarrow & 2 & 4 & \leftarrow +3 \\ +1 \rightarrow & 3 & 9 & \leftarrow +5 \end{array}$$

Here, the rate of change is *not* constant from one row to the next. While the x-values are increasing by 1 from row to row, the y-values increase more and more from row to row. Do you notice that there is a pattern there as well? Mathematicians are interested in relationships with patterns.

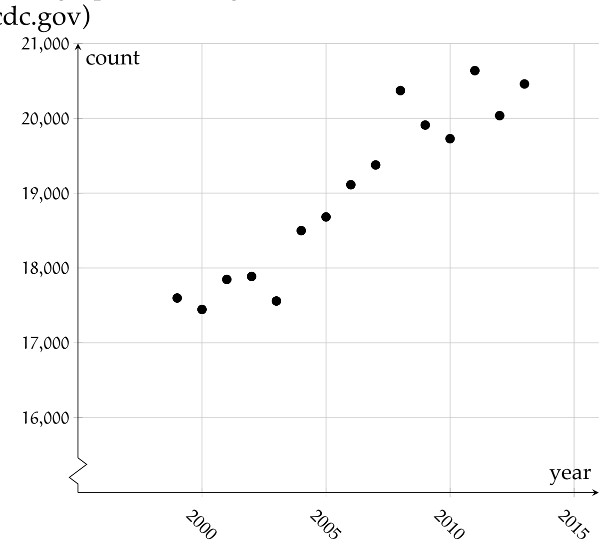
3.3.3 Rate of Change

For an hourly wage-earner, the amount of money they earn depends on how many hours they work. If a worker earns \$15 per hour, then 10 hours of work corresponds to \$150 of pay. Working *one* additional hour will change 10 hours to 11 hours; and this will cause the \$150 in pay to rise by *fifteen* dollars to \$165 in pay. Any time we compare how one amount changes (dollars earned) as a consequence of another amount changing (hours worked), we are talking about a **rate of change**.

Given a table of two-variable data, between any two rows we can compute a **rate of change**.

Example 3.3.10 The following data, given in both table and graphed form, gives the counts of invasive cancer diagnoses in Oregon over a period of time. (wonder.cdc.gov)

Year	Invasive Cancer Incidents
1999	17,599
2000	17,446
2001	17,847
2002	17,887
2003	17,559
2004	18,499
2005	18,682
2006	19,112
2007	19,376
2008	20,370
2009	19,909
2010	19,727
2011	20,636
2012	20,035
2013	20,458



What is the **rate of change** in Oregon invasive cancer diagnoses between 2000 and 2010? The total (net) change in diagnoses over that timespan is

$$19727 - 17446 = 2281$$

meaning that there were 2281 more invasive cancer incidents in 2010 than in 2000. Since 10 years passed (which you can calculate as $2010 - 2000$), the rate of change is 2281 diagnoses per 10 years, or

$$\frac{2281 \text{ diagnoses}}{10 \text{ year}} = 228.1 \, \frac{\text{diagnoses}}{\text{year}}.$$

We read that last quantity as "228.1 diagnoses per year." This rate of change means that between the years 2000 and 2010, there were 228.1 more diagnoses *each* year, on average. This is just an average over those ten years—it does not mean that the diagnoses grew by exactly this much each year. % We dare not interpret *why* that increase existed, % just that it did. % If you are interested in examining causal relationships that exist in real life, % we strongly recommend a statistics course or two in your future!

Checkpoint 3.3.11 Use the data in Example 3.3.10 to find the rate of change in Oregon invasive cancer diagnoses between 1999 and 2002.

And what was the rate of change between 2003 and 2011?

Explanation. To find the rate of change between 1999 and 2002, calculate

$$\frac{17887 - 17599}{2002 - 1999} = 96.$$

So the rate of change was 96.

To find the rate of change between 2003 and 2011, calculate

$$\frac{20636 - 17559}{2011 - 2003} = 384.625.$$

So the rate of change was 384.625.

We are ready to give a formal definition for "rate of change". Considering our work from Example 3.3.10 and Checkpoint 3.3.11, we settle on:

Definition 3.3.12 Rate of Change. If (x_1, y_1) and (x_2, y_2) are two data points from a set of two-variable data, then the **rate of change** between them is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

(The Greek letter delta, Δ, is used to represent "change in" since it is the first letter of the Greek word for "difference.") ◊

In Example 3.3.10 and Checkpoint 3.3.11 we found three rates of change. Figure 3.3.13 highlights the three pairs of points that were used to make these calculations.

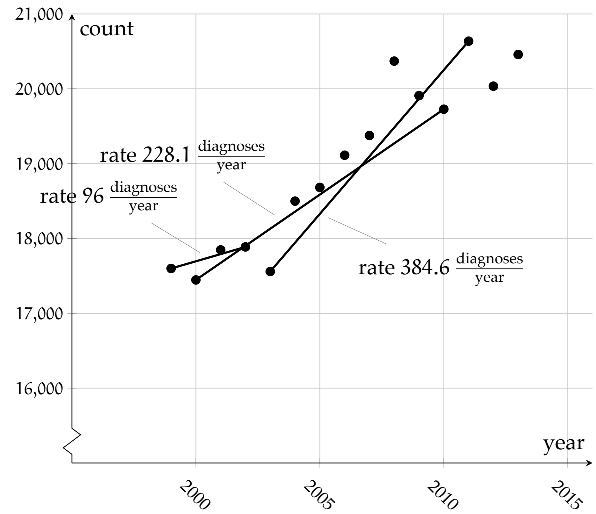


Figure 3.3.13

Note how the larger the numerical rate of change between two points, the steeper the line is that connects them. This is such an important observation, we'll put it in an official remark.

Remark 3.3.14 The rate of change between two data points is intimately related to the steepness of the line segment that connects those points.

1. The steeper the line, the larger the rate of change, and vice versa.
2. If one rate of change between two data points equals another rate of change between two different data points, then the corresponding line segments will have the same steepness.
3. We always measure rate of change from left to right. When a line segment between two data points slants up from left to right, the rate of change between those points will be positive. When a line segment between two data points slants down from left to right, the rate of change between those points will be negative.

In the solution to Checkpoint 3.3.8, the key observation was that the rate of change from one row to the next was constant: 3 units of increase in y for every 1 unit of increase in x. Graphing this pattern in Figure 3.3.15, we see that every line segment here has the same steepness, so the whole picture is a straight line.

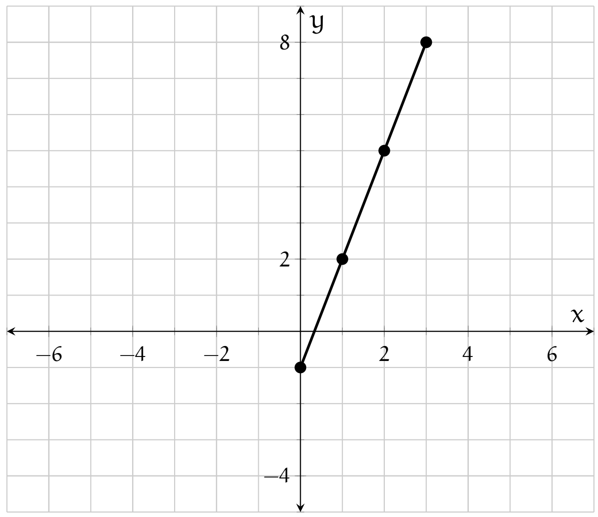


Figure 3.3.15

Whenever the rate of change is constant no matter which two (x, y)-pairs (or data pairs) are chosen from a data set, then you can conclude the graph will be a straight line *even without making the graph*. We call this kind of relationship a **linear** relationship. We'll study linear relationships in more detail throughout this chapter. Right now in this section, we feel it is important to simply identify if data has a linear relationship or not.

Checkpoint 3.3.16 Is there a linear relationship in the table?

x	y
-8	3.1
-5	2.1
-2	1.1
1	0.1

(☐ The relationship is linear ☐ The relationship is not linear)

Explanation. From one x-value to the next, the change is always 3. From one y-value to the next, the change is always -1. So the rate of change is always $\frac{-1}{3} = -\frac{1}{3}$. Since the rate of change is constant, the data have a linear relationship.

Checkpoint 3.3.17 Is there a linear relationship in the table?

x	y
11	208
13	210
15	214
17	220

(☐ The relationship is linear ☐ The relationship is not linear)

Explanation. The rate of change between the first two points is $\frac{210-208}{13-11} = 1$. The rate of change between the last two points is $\frac{220-214}{17-15} = 3$. This is one way to demonstrate that the rate of change differs for different pairs of points, so this pattern is not linear.

Checkpoint 3.3.18 Is there a linear relationship in the table?

x	y
3	-2
6	-8
8	-12
12	-20

(☐ The relationship is linear ☐ The relationship is not linear)

Explanation. The changes in x from one row to the next are $+3$,$+2$, and $+8$. That's not a consistent pattern, but we need to consider rates of change between points. The rate of change between the first two points is $\frac{-8-(-2)}{6-3} = -2$. The rate of change between the next two points is $\frac{-12-(-8)}{8-6} = -2$. And the rate of change between the last two points is $\frac{-20-(-12)}{12-8} = -2$. So the rate of change, -2, is constant regardless of which pairs we choose. That means these pairs describe a linear relationship.

Let's return to the data that we opened the section with, in Figure 3.3.2. Is that data linear? Well, yes and no. To be completely honest, it's not linear. It's easy to pick out pairs of points where the steepness changes from one pair to the next. In other words, the points do not all fall into a single line.

However if we step back, there does seem to be an overall upward trend that is captured by the line someone has drawn over the data. Points *on this line* do have a linear pattern. Let's estimate the rate of change between some points on this line. We are free to use any points to do this, so let's make this calculation easier by choosing points we can clearly identify on the graph: $(1991, 15)$ and $(2020, 25)$.

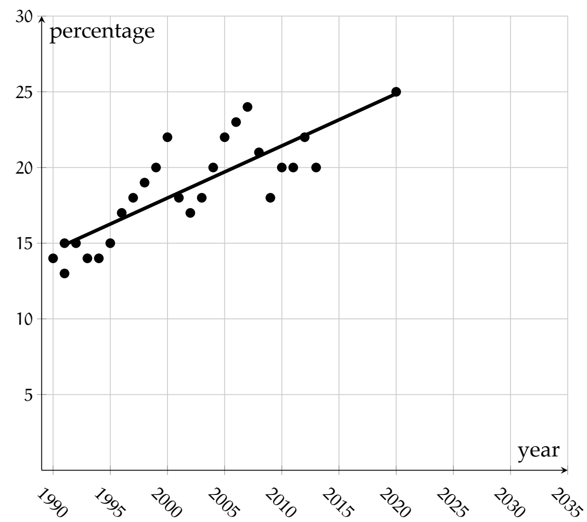


Figure 3.3.19: Share of all income held by the top 1 %, United States, 1990–2013 (www.epi.org)

The rate of change between those two points is

$$\frac{(25-15)\text{ percentage points}}{(2020-1991)\text{years}} = \frac{10\text{ percentage points}}{29\text{years}} \approx 0.3448\frac{\text{percentage points}}{\text{year}}.$$

So we might say that *on average* the rate of change expressed by this data is 0.3448 percentage points per year.

3.3.4 Reading Questions

1. Given a table of data with x- and y-values, explain how to calculate the rate of change from one row to the next.

2. If there is a table of data with x- and y-values, and the plot of all that data makes a straight line, what is true about the rates of change as you move from row to row in the table?

3. What does it mean for a rate of change to be positive (or negative) with regard to a graph with two points plotted?

3.3.5 Exercises

Finding Patterns Write an equation in the form $y = \ldots$ suggested by the pattern in the table.

1.

x	y
-2	-6
-1	-3
0	0
1	3
2	6

2.

x	y
3	12
4	16
5	20
6	24
7	28

3.

x	y
5	13
6	14
7	15
8	16
9	17

4.

x	y
6	12
7	13
8	14
9	15
10	16

5.

x	y
15	20
0	5
8	13
16	21
4	9

6.

x	y
17	12
13	8
16	11
8	3
3	-2

7.

x	y
25	5
4	2
1	1
9	3
16	4

8.

x	y
-5	5
-1	1
-2	2
5	5
1	1

9.

x	y
5	25
6	36
7	49
8	64
9	81

10.

x	y
2	4
4	16
6	36
8	64
10	100

11.

x	y
42	$\frac{1}{42}$
94	$\frac{1}{94}$
93	$\frac{1}{93}$
11	$\frac{1}{11}$
15	$\frac{1}{15}$

12.

x	y
53	$\frac{1}{53}$
60	$\frac{1}{60}$
35	$\frac{1}{35}$
83	$\frac{1}{83}$
13	$\frac{1}{13}$

Linear Relationships Does the following table show that x and y have a linear relationship? (☐ yes ☐ no)

13.

x	y
0	32
1	39
2	46
3	53
4	60
5	67

14.

x	y
0	92
1	100
2	108
3	116
4	124
5	132

15.

x	y
6	60
7	57
8	54
9	51
10	48
11	45

16.

x	y
0	60
1	58
2	56
3	54
4	52
5	50

17.

x	y
4	34
5	50
6	82
7	146
8	274
9	530

18.

x	y
9	523
10	1035
11	2059
12	4107
13	8203
14	16395

19.

x	y
0	4
1	5
2	12
3	31
4	68
5	129

20.

x	y
2	25
3	44
4	81
5	142
6	233
7	360

21.

x	y
-8	60.57
-7	60.86
-6	61.15
-5	61.44
-4	61.73
-3	62.02

22.

x	y
1	39.95
2	41.35
3	42.75
4	44.15
5	45.55
6	46.95

23.

x	y
9	150
11	166
14	190
16	206
18	222
19	230

24.

x	y
3	67
5	83
7	99
10	123
14	155
19	195

Calculating Rate of Change

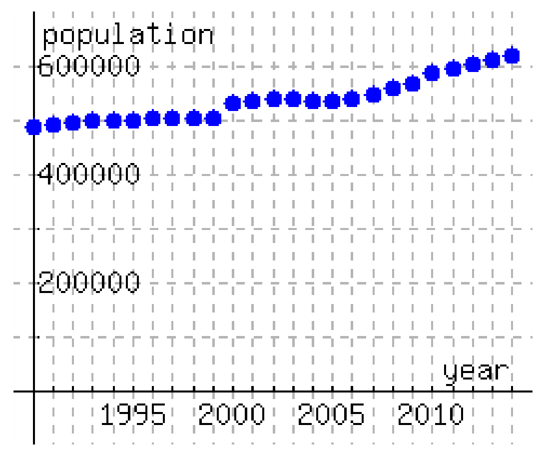
25. This table gives population estimates for Portland, Oregon from 1990 through 2014.

Year	Population	Year	Population
1990	487849	2003	539546
1991	491064	2004	533120
1992	493754	2005	534112
1993	497432	2006	538091
1994	497659	2007	546747
1995	498396	2008	556442
1996	501646	2009	566143
1997	503205	2010	585261
1998	502945	2011	593859
1999	503637	2012	602954
2000	529922	2013	609520
2001	535185	2014	619360
2002	538803		

Find the rate of change in Portland population between 1991 and 1996.
And what was the rate of change between 2008 and 2014?
List all the years where there is a negative rate of change between that year and the next year.

26. This table and graph gives population estimates for Portland, Oregon from 1990 through 2014.

Year	Population	Year	Population
1990	487849	2003	539546
1991	491064	2004	533120
1992	493754	2005	534112
1993	497432	2006	538091
1994	497659	2007	546747
1995	498396	2008	556442
1996	501646	2009	566143
1997	503205	2010	585261
1998	502945	2011	593859
1999	503637	2012	602954
2000	529922	2013	609520
2001	535185	2014	619360
2002	538803		



Between what two years that are two years apart was the rate of change highest?
What was that rate of change?

3.4 Slope

In Section 3.3, we observed that a constant rate of change between points produces a linear relationship, whose graph is a straight line. Such a constant rate of change has a special name, *slope*, and we'll explore slope in more depth here.

3.4.1 What is slope?

Given a graph with points, when the rate of change from one point to the next never changes, those points must fall on a straight line, as in Figure 3.4.2. Rather than say "constant rate of change" in every such situation, there is a specific word for this.

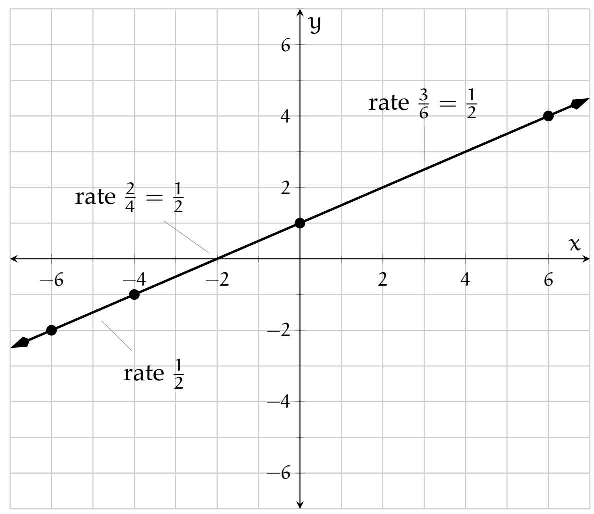


Figure 3.4.2: Between successive points, the rate of change is always 1/2.

Definition 3.4.3 Slope. When x and y are two variables where the rate of change between any two points is always the same, we call this common rate of change the **slope**. Since having a constant rate of change means the graph will be a straight line, its also called the **slope of the line**. ◊

Considering the definition for rate of change, this means that when x and y are two variables where the rate of change between any two points is always the same, then you can calculate slope, m, by finding two distinct data points (x_1, y_1) and (x_2, y_2), and calculating

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{3.4.1}$$

A slope is a rate of change. So if there are units for the horizontal and vertical variables, then there will be units for the slope. The slope will be measured in $\frac{\text{vertical units}}{\text{horizontal units}}$. If the slope is nonzero, we say that there is a **linear relationship** between x and y. When the slope is 0, we say that y is **constant** with respect to x.

Slope m. Why is the letter m commonly used as the symbol for "slope?" Some believe that it comes from the French word "monter" which means "to climb."

Here are some scenarios with different slopes. As you read each scenario, note how a slope is more meaningful with units.

- If a tree grows 2.5 feet every year, its rate of change in height is the same from year to year. So the height and time have a linear relationship where the slope is 2.5 $\frac{\text{ft}}{\text{yr}}$.

- If a company loses 2 million dollars every year, its rate of change in reserve funds is the same from year to year. So the company's reserve funds and time have a linear relationship where the slope is -2 million dollars per year.

- If Sakura is an adult who has stopped growing, her rate of change in height is the same from year to year—it's zero. So the slope is $0\,\frac{\text{in}}{\text{yr}}$. Sakura's height is constant with respect to time.

Remark 3.4.4 A useful phrase for remembering the definition of slope is "rise over run." Here, "rise" refers to "change in y," Δy, and "run" refers to "change in x," Δx. Be careful though. As we have learned, the horizontal direction comes *first* in mathematics, followed by the vertical direction. The phrase "rise over run" reverses this. (It's a bit awkward to say, but the phrase "run under rise" puts the horizontal change first.)

Example 3.4.5 Yara's Savings. On Dec. 31, Yara had only \$50 in her savings account. For the the new year, she resolved to deposit \$20 into her savings account each week, without withdrawing any money from the account.

Yara keeps her resolution, and her account balance increases steadily by \$20 each week. That's a constant rate of change, so her account balance and time have a linear relationship with slope $20\,\frac{\text{dollars}}{\text{wk}}$.

We can model the balance, y, in dollars, in Yara's savings account x weeks after she started making deposits with an equation. Since Yara started with \$50 and adds \$20 each week, then x weeks after she started making deposits,

$$y = 50 + 20x \tag{3.4.2}$$

where y is a dollar amount. Notice that the slope, $20\,\frac{\text{dollars}}{\text{wk}}$, serves as the multiplier for x weeks.

We can also consider Yara's savings using a table as in Figure 3.4.6.

	x, weeks since DEC. 31	y, savings account balance (dollars)	
	0	50	
x increases by $1 \longrightarrow$	1	70	$\longleftarrow y$ increases by 20
x increases by $1 \longrightarrow$	2	90	$\longleftarrow y$ increases by 20
x increases by $2 \longrightarrow$	4	130	$\longleftarrow y$ increases by 40
x increases by $3 \longrightarrow$	7	190	$\longleftarrow y$ increases by 60
x increases by $5 \longrightarrow$	12	290	$\longleftarrow y$ increases by 100

Figure 3.4.6: Yara's savings

In first few rows of the table, we see that when the number of weeks x increases by 1, the balance y increases by 20. The row-to-row rate of change is $\frac{20\text{ dollars}}{1\text{ wk}} = 20\,\frac{\text{dollars}}{\text{wk}}$, the slope. In any table for a linear relationship, whenever x increases by 1 unit, y will increase by the slope.

In further rows, notice that as row-to-row change in x increases, row-to-row change in y increases proportionally to preserve the constant rate of change. Looking at the change in the last two rows of the table, we see x increases by 5 and y increases by 100, which gives a rate of change of $\frac{100\text{ dollars}}{5\text{ wk}} = 20\,\frac{\text{dollars}}{\text{wk}}$, the value of the slope again.

On a graph of Yara's savings, we can "see" the rates of change between consecutive rows of the table by including **slope triangles**.

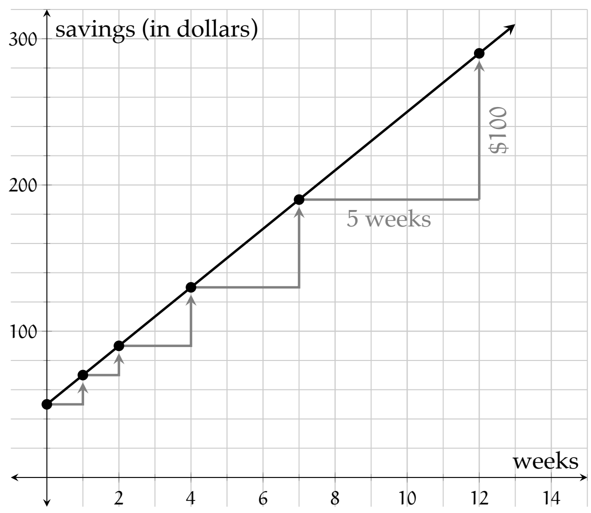


Figure 3.4.7: Yara's savings

The large slope triangle indicates that when 5 weeks pass, Yara saves \$100. This is the rate of change between the last two rows of the table, $\frac{100}{5} = 20\,\frac{\text{dollars}}{\text{wk}}$. The smaller slope triangles indicate, from left to right, the rates of change $\frac{20\text{ dollars}}{1\text{ wk}}$, $\frac{20\text{ dollars}}{1\text{ wk}}$, $\frac{40\text{ dollars}}{2\text{ wk}}$, and $\frac{60\text{ dollars}}{3\text{ wk}}$ respectively. All of these rates simplify to the slope, $20\,\frac{\text{dollars}}{\text{wk}}$.

Every slope triangle on the graph of Yara's savings has the same shape (geometrically, they are called similar triangles) since the ratio of vertical change to horizontal change is always $20\,\frac{\text{dollars}}{\text{wk}}$. On any graph of any line, we can draw a slope triangle and compute slope as "rise over run."

Of course, we could draw a slope triangle on the other side of the line:

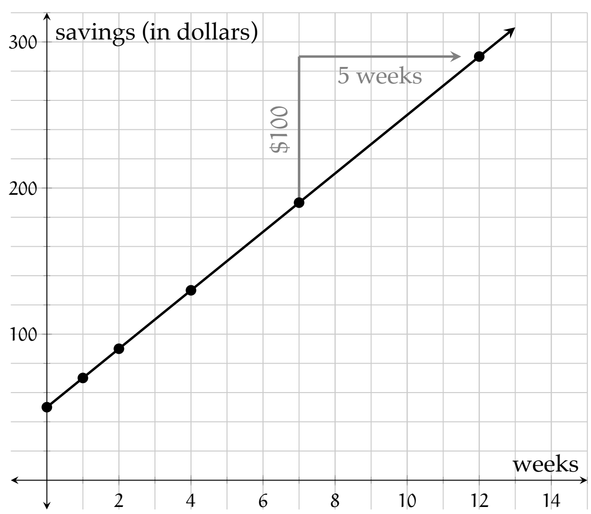


Figure 3.4.8: Yara's savings

This slope triangle works just as well for identifying "rise" and "run," but it focuses on vertical change before horizontal change. For consistency with mathematical conventions, we will generally draw slope triangles showing horizontal change followed by vertical change, as in Figure 3.4.7.

Example 3.4.9 The following graph of a line models the amount of gas, in gallons, in Kiran's gas tank as they drive their car. Find the line's slope, and interpret its meaning in this context.

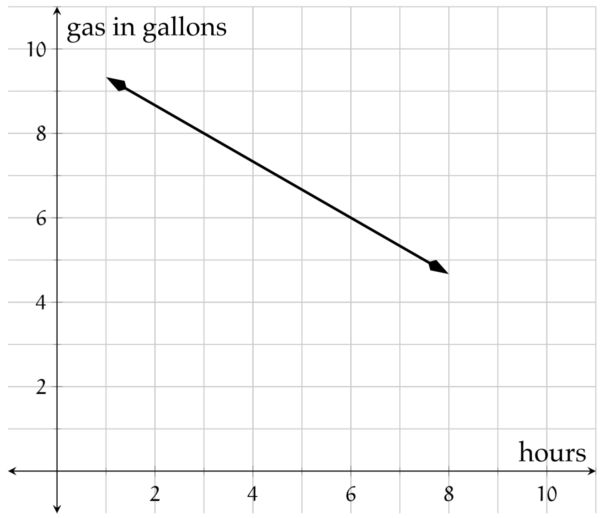


Figure 3.4.10: Amount of gas in Kiran's gas tank

Explanation. To find a line's slope using its graph, we first identify two points on it, and then draw a slope triangle. Naturally, we would want to choose two points whose x- and y-coordinates are easy to identify exactly based on the graph. We will pick the two points where $x = 3$ and $x = 6$, because they are right on grid line crossings:

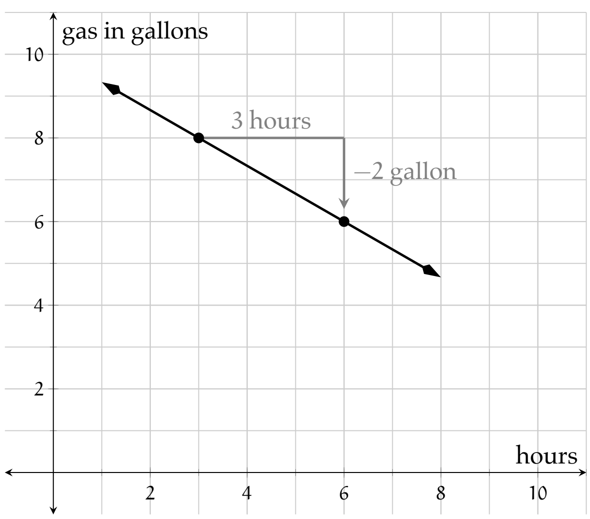


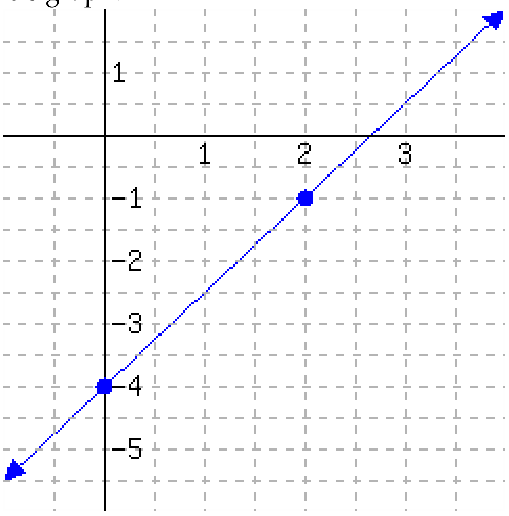
Figure 3.4.11: A Good Slope Triangle

Notice that the *change* in y is negative, because the amount of gas is decreasing. Since we chose points with integer coordinates, we can easily calculate the slope:

$$\text{slope} = \frac{-2\,\text{gallons}}{3\,\text{miles}} = -\frac{2}{3}\,\frac{\text{gal}}{\text{mi}}.$$

In the given context, this slope implies gas in the tank is *decreasing* at the rate of $\frac{2}{3}\,\frac{\text{gal}}{\text{h}}$. Since this slope is written as a fraction, another way to understand it is that Kiran is using up 2 gallons of gas every 3 hours.

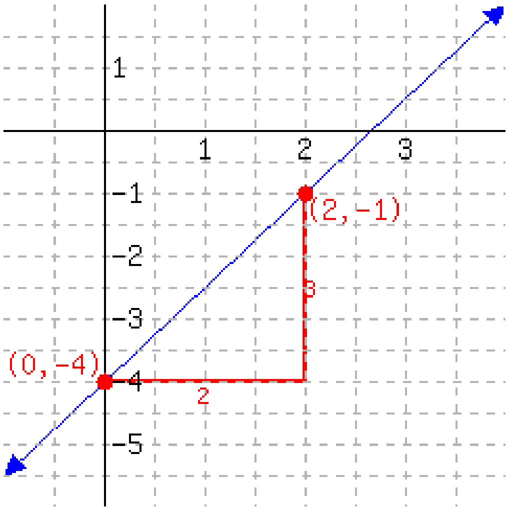
Checkpoint 3.4.12 Below is a line's graph.



The slope of this line is ______.

Explanation. To find the slope of a line from its graph, we first need to identify two points that the line passes through. It is wise to choose points with integer coordinates. For this problem, we choose $(0, -4)$ and $(2, -1)$.

Next, we sketch a slope triangle and find the *rise* and *run*. In the sketch below, the rise is 3 and the run is 2.



$$\begin{aligned} \text{slope} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{3}{2} \end{aligned}$$

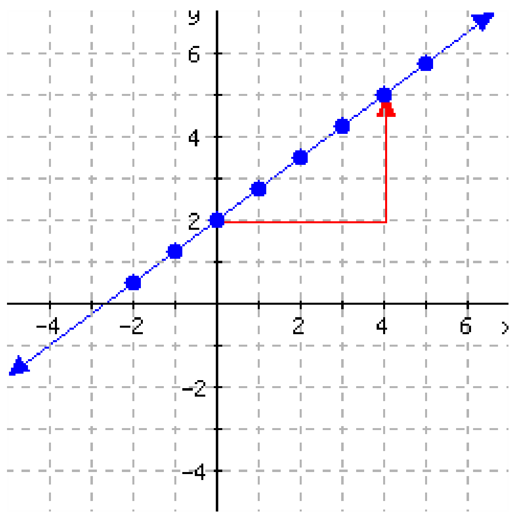
This line's slope is $\frac{3}{2}$.

Checkpoint 3.4.13 Make a table and plot the equation $y = \frac{3}{4}x + 2$, which makes a straight line. Use the plot to determine the slope of this line.

Explanation. First, we choose some x-values to make a table, and compute the corresponding y-values.

x	$y = \frac{3}{4}x + 2$	Point
-2	$\frac{3}{4}(-2) + 2 = 0.5$	$(-2, 0.5)$
-1	$\frac{3}{4}(-1) + 2 = 1.25$	$(-1, 1.25)$
0	$\frac{3}{4}(0) + 2 = 2$	$(0, 2)$
1	$\frac{3}{4}(1) + 2 = 2.75$	$(1, 2.75)$
2	$\frac{3}{4}(2) + 2 = 3.5$	$(2, 3.5)$
3	$\frac{3}{4}(3) + 2 = 4.25$	$(3, 4.25)$
4	$\frac{3}{4}(4) + 2 = 5$	$(4, 5)$
5	$\frac{3}{4}(5) + 2 = 5.75$	$(5, 5.75)$

This table lets us plot the graph and identify a slope triangle that is easy to work with.



Since the slope triangle runs 4 units and then rises 3 units, the slope is $\frac{3}{4}$.

3.4.2 Comparing Slopes

It's useful to understand what it means for different slopes to appear on the same coordinate system.

Example 3.4.14 Effie, Ivan and Cleo are in a foot race. Figure 3.4.15 models the distance each has traveled in the first few seconds. Each runner takes a second to accelerate up to their running speed, but then runs at a constant speed. So they are then traveling with a constant rate of change, and the straight line portions of their graphs have a slope. Find each line's slope, and interpret its meaning in this context. What comparisons can you make with these runners?

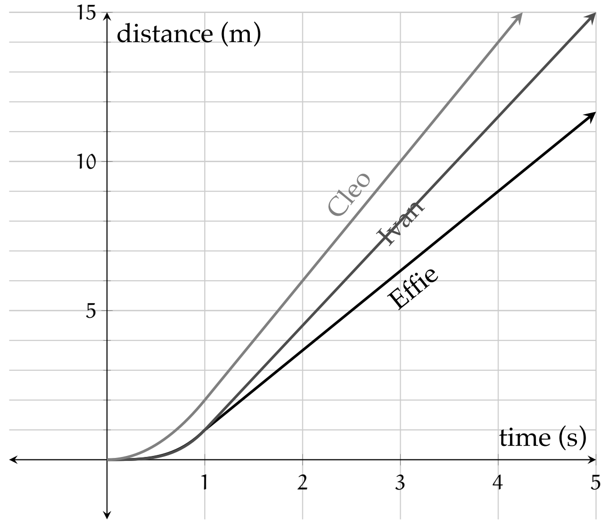


Figure 3.4.15: A three-way foot race

We will draw slope triangles to find each line's slope.

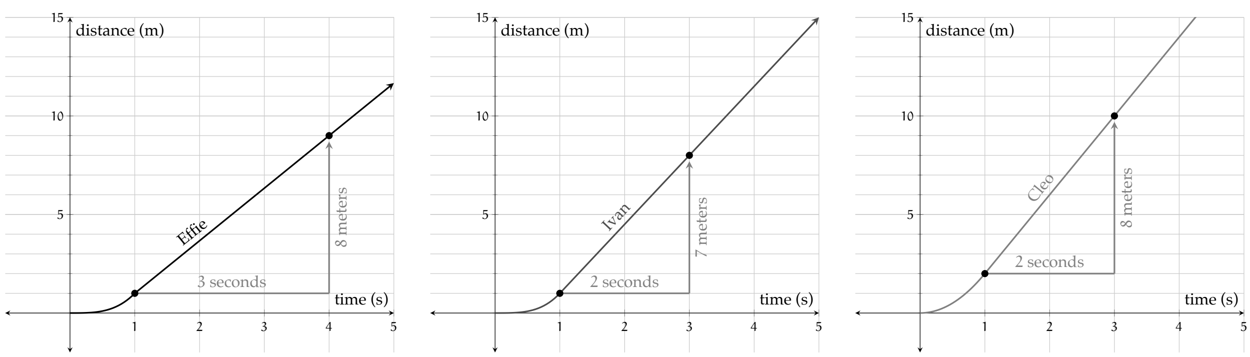


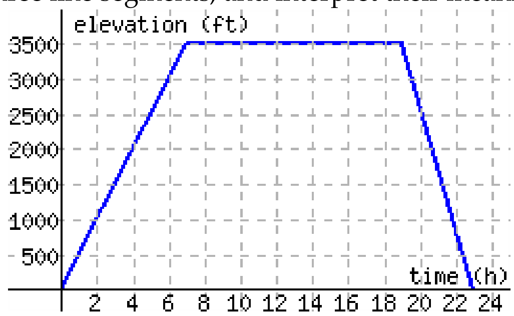
Figure 3.4.16: Find the Slope of Each Line

Using the slope equation (3.4.1), we have:

- Effie's slope is $\frac{8\,\mathrm{m}}{3\,\mathrm{s}} \approx 2.666\,\frac{\mathrm{m}}{\mathrm{s}}$.
- Ivan's slope is $\frac{7\,\mathrm{m}}{2\,\mathrm{s}} = 3.5\,\frac{\mathrm{m}}{\mathrm{s}}$.
- Cleo's slope is $\frac{8\,\mathrm{m}}{2\,\mathrm{s}} = 4\,\frac{\mathrm{m}}{\mathrm{s}}$.

In a time-distance graph, the slope of a line represents speed. The slopes in these examples and the running speeds of these runners are measured in $\frac{\mathrm{m}}{\mathrm{s}}$. A relationship we can see is that the more sharply a line is slanted, the bigger the slope is. This should make sense because for each passing second, the faster person travels longer, making a slope triangle's height taller. This means that, numerically, we can tell that Cleo is the fastest runner (and Effie is the slowest) just by comparing the slopes $4 > 3.5 > 2.666$.

Checkpoint 3.4.17 Jogging on Mt. Hood. Kato is training for a race up the slope of Mt. Hood, from Sandy to Government Camp, and then back. The graph below models his elevation from his starting point as time passes. Find the slopes of the three line segments, and interpret their meanings in this context.



a. What is the slope of the first segment?

b. What is the slope of the second segment?

c. What is the slope of the third segment?

Explanation. The first segment started at $(0, 0)$ and stopped at $(7, 3500)$. This implies, Kato started at the starting point, traveled 7 hours and reached a point 3500 feet higher in elevation from the starting point. The slope of the line is

$$\frac{\Delta y}{\Delta x} = \frac{3500\,\text{ft}}{7\,\text{h}} = 500\,\frac{\text{ft}}{\text{h}}.$$

In context, Kato was running, gaining 500 feet in elevation per hour.

What happened in the second segment, which started at $(7, 3500)$ and ended at $(19, 3500)$? This implies he started this portion 3500 feet higher in elevation from the starting point, and didn't change elevation for 19 hours. The slope of the line is

$$\frac{\Delta y}{\Delta x} = \frac{0\,\text{ft}}{12\,\text{h}} = 0\,\frac{\text{ft}}{\text{h}}$$

In context, Kato was running but neither gaining nor losing elevation.

The third segment started from $(19, 3500)$ and stopped at $(23, 0)$. This implies, Kato started this part of his trip from a spot 3500 feet higher in elevation from the starting point, traveled for 4 hours and returned to the starting elevation. The slope of the line is

$$\frac{\Delta y}{\Delta x} = \frac{-3500\,\text{ft}}{4\,\text{h}} = -875\,\frac{\text{ft}}{\text{h}}$$

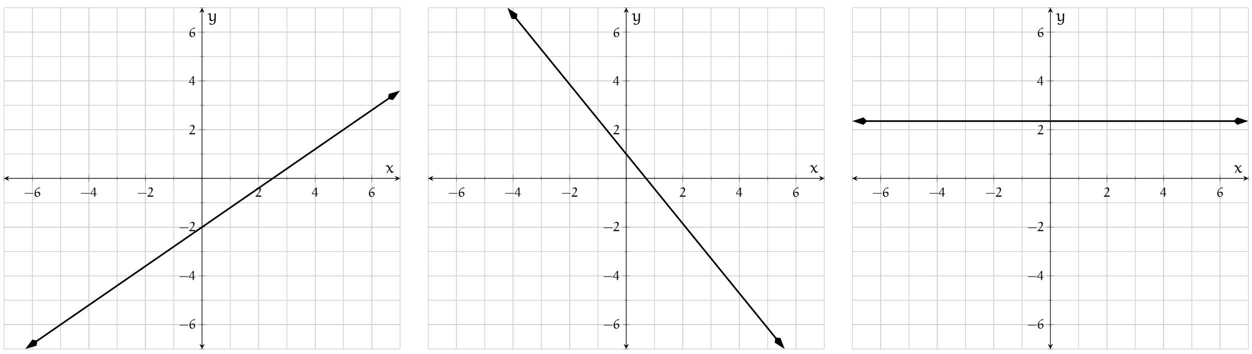
In context, Kato was running, dropping in elevation by 875 feet per hour.

Some important properties are demonstrated in Exercise 3.4.17.

Fact 3.4.18 The Relationship Between Slope and Increase/Decrease. *In a linear relationship, as the* x*-value increases (in other words as you read its graph from left to right):*

- *if the* y*-values increase (in other words, the line goes upward), its slope is positive.*
- *if the* y*-values decrease (in other words, the line goes downward), its slope is negative.*
- *if the* y*-values don't change (in other words, the line is flat, or horizontal), its slope is* 0*.*

These properties are summarized graphically in Figure 3.4.19.



(a) A line with positive slope **(b)** A line with negative slope **(c)** A line with a slope of zero

Figure 3.4.19

3.4.3 Finding Slope by Two Given Points

Several times in this section we computed a slope by drawing a slope triangle. That's not really necessary if you have coordinates for two points that a line passes through. In fact, sometimes it's impractical to draw a slope triangle.[1] Here we will stress how to find a line's slope without drawing a slope triangle.

Example 3.4.20 Your neighbor planted a sapling from Portland Nursery in his front yard. Ever since, for several years now, it has been growing at a constant rate. By the end of the third year, the tree was 15 ft tall; by the end of the sixth year, the tree was 27 ft tall. What's the tree's rate of growth (i.e. the slope)?

We *could* sketch a graph for this scenario, and include a slope triangle. If we did that, it would look like:

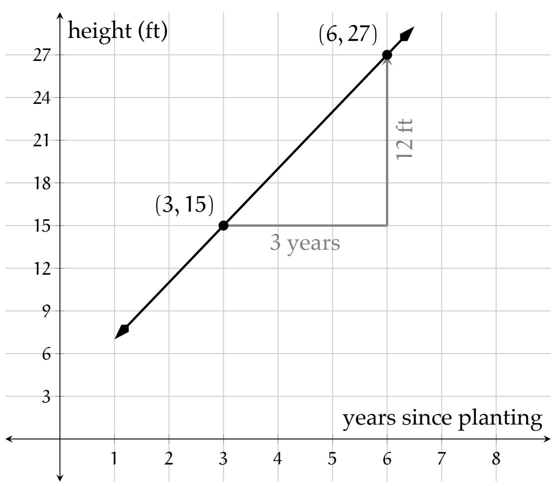


Figure 3.4.21: Height of a Tree

By the slope triangle and Equation (3.4.1) we have:

$$\begin{aligned}\text{slope} = m &= \frac{\Delta y}{\Delta x}\\ &= \frac{12\,\text{ft}}{3\,\text{yr}}\\ &= 4\frac{\text{ft}}{\text{yr}}\end{aligned}$$

So the tree is growing at a rate of $4\,\frac{\text{ft}}{\text{yr}}$.

But hold on. Did we really *need* this picture? The "rise" of 12 came from a subtraction of two y-values: $27 - 15$. And the "run" of 3 came from a subtraction of two x-values: $6 - 3$.

Here is a picture-free approach. We know that after 3 yr, the height is 15 ft. As an ordered pair, that

[1]For instance if you only have specific information about two points that are too close together to draw a triangle, or if you cannot clearly see precise coordinates where you might start and stop your slope triangle.

information gives us the point $(3, 15)$ which we can label as $(\overset{x_1}{3}, \overset{y_1}{15})$. Similarly, the background information tells us to consider $(6, 27)$, which we label as $(\overset{x_2}{6}, \overset{y_2}{27})$. Here, x_1 and y_1 represent the first point's x-value and y-value, and x_2 and y_2 represent the second point's x-value and y-value.

Now we can write an alternative to Equation (3.4.1):

$$\text{slope} = m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{3.4.3}$$

This is known as the **slope formula**. The following graphs help to understand why this formula works. Basically, we are still using a slope triangle to calculate the slope.

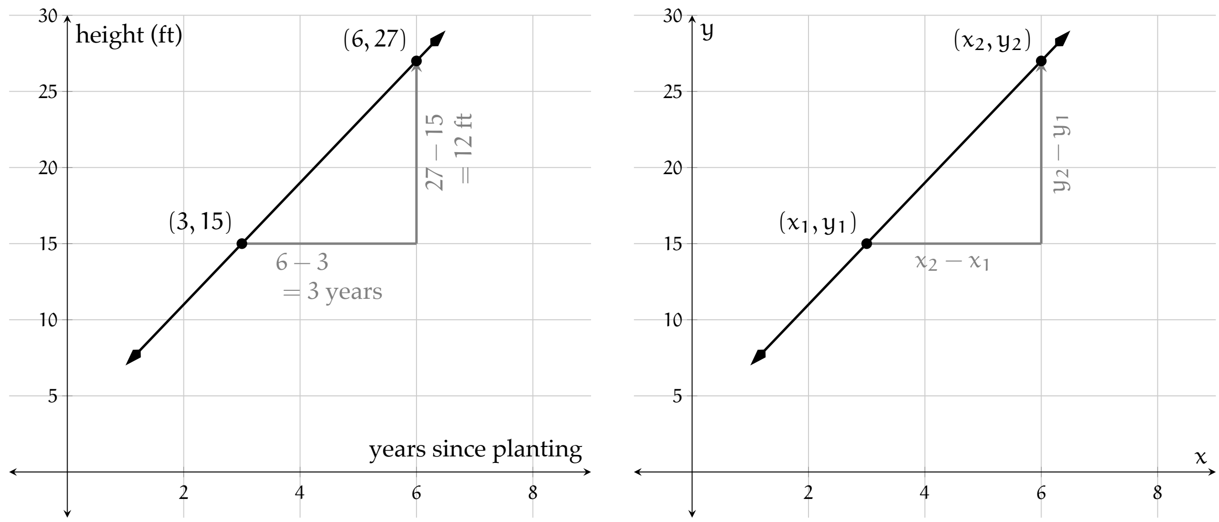


Figure 3.4.22: Understanding the slope formula

Remark 3.4.23 It's important to use subscript instead of superscript in the slope equation, because y^2 means to take the number y and square it. Whereas y_2 tells you that there are at least two y-values in the conversation, and y_2 is the second of them.

The beauty of the slope formula (3.4.3) is that to find a line's slope, we don't need to draw a slope triangle any more. Let's look at an example.

Example 3.4.24 A line passes the points $(-5, 25)$ and $(4, -2)$. Find this line's slope.

Explanation. If you are new to this formula, it may help to label each number before using the formula. The two given points are:

$$(\overset{x_1}{-5}, \overset{y_1}{25}) \text{ and } (\overset{x_2}{4}, \overset{y_2}{-2})$$

Now apply the slope formula (3.4.3):

$$\begin{aligned} \text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2 - 25}{4 - (-5)} \end{aligned}$$

$$= \frac{-27}{9}$$
$$= -3$$

Note that we used parentheses when substituting negative numbers in x_1 and y_1. This is a good habit to protect yourself from making errors with subtraction and double negatives.

Checkpoint 3.4.25 A line passes through the points $(-4, 15)$ and $(12, -13)$. Find this line's slope.

Explanation. To find a line's slope, we can use the slope formula:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

First, we mark which number corresponds to which variable in the formula:

$$(-4, 15) \longrightarrow (x_1, y_1)$$

$$(12, -13) \longrightarrow (x_2, y_2)$$

Now we substitute these numbers into the corresponding variables in the slope formula:

$$\begin{aligned} \text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-13 - 15}{12 - (-4)} \\ &= \frac{-28}{16} \\ &= -\frac{7}{4} \end{aligned}$$

So the line's slope is $-\frac{7}{4}$.

3.4.4 Reading Questions

1. Have you memorized a formula for finding the slope between two points using their coordinates?
2. What is an important thing to do with slope to make it more meaningful in an application problem?
3. Drawing a slope triangle can be helpful to think about slope. But what might happen that could make it impractical to draw a slope triangle?

3.4.5 Exercises

Review and Warmup

1. Reduce the fraction $\frac{7}{56}$.
2. Reduce the fraction $\frac{5}{45}$.
3. Reduce the fraction $\frac{10}{14}$.
4. Reduce the fraction $\frac{25}{40}$.
5. Reduce the fraction $\frac{21}{168}$.
6. Reduce the fraction $\frac{6}{36}$.

7. Reduce the fraction $\frac{315}{245}$.

8. Reduce the fraction $\frac{350}{105}$.

9. Reduce the fraction $\frac{70}{10}$.

10. Reduce the fraction $\frac{112}{14}$.

Slope and Points

11. A line passes through the points $(3, 12)$ and $(9, 36)$. Find this line's slope.

12. A line passes through the points $(5, 18)$ and $(8, 33)$. Find this line's slope.

13. A line passes through the points $(2, 5)$ and $(7, 0)$. Find this line's slope.

14. A line passes through the points $(4, -21)$ and $(10, -51)$. Find this line's slope.

15. A line passes through the points $(-4, -12)$ and $(-6, -14)$. Find this line's slope.

16. A line passes through the points $(-2, 2)$ and $(-7, -8)$. Find this line's slope.

17. A line passes through the points $(-1, 0)$ and $(1, -2)$. Find this line's slope.

18. A line passes through the points $(-1, 2)$ and $(3, -18)$. Find this line's slope.

19. A line passes through the points $(-1, 7)$ and $(-6, 17)$. Find this line's slope.

20. A line passes through the points $(-4, 6)$ and $(-7, 12)$. Find this line's slope.

21. A line passes through the points $(3, 11)$ and $(-6, -13)$. Find this line's slope.

22. A line passes through the points $(21, 14)$ and $(-14, -26)$. Find this line's slope.

23. A line passes through the points $(-8, 5)$ and $(8, -7)$. Find this line's slope.

24. A line passes through the points $(-24, 17)$ and $(16, 2)$. Find this line's slope.

25. A line passes through the points $(4, -4)$ and $(-4, -4)$. Find this line's slope.

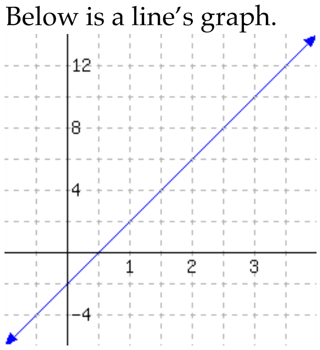
26. A line passes through the points $(2, -2)$ and $(-2, -2)$. Find this line's slope.

27. A line passes through the points $(0, -1)$ and $(0, 1)$. Find this line's slope.

28. A line passes through the points $(3, -2)$ and $(3, 3)$. Find this line's slope.

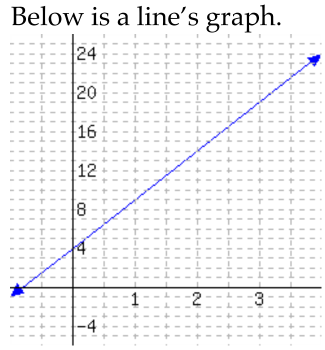
Slope and Graphs

29. Below is a line's graph.



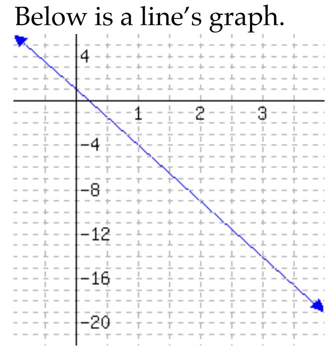
The slope of this line is ________.

30. Below is a line's graph.



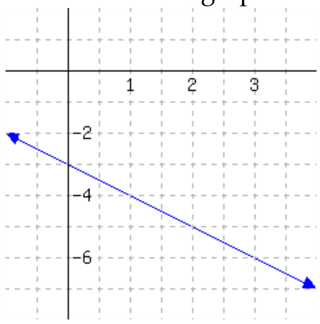
The slope of this line is ________.

31. Below is a line's graph.



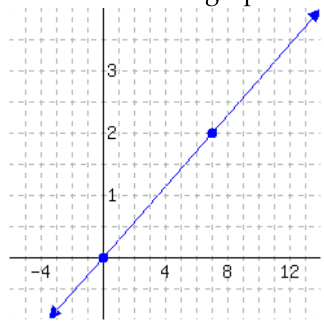
The slope of this line is ________.

32. Below is a line's graph.



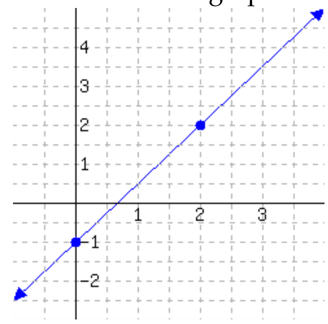
The slope of this line is .

33. Below is a line's graph.



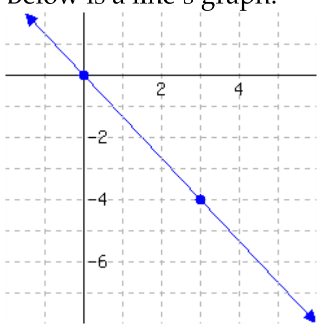
The slope of this line is .

34. Below is a line's graph.



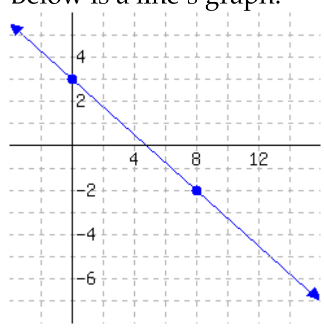
The slope of this line is .

35. Below is a line's graph.



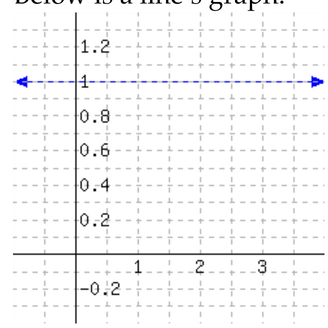
The slope of this line is .

36. Below is a line's graph.



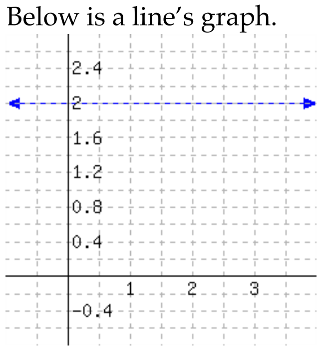
The slope of this line is .

37. Below is a line's graph.



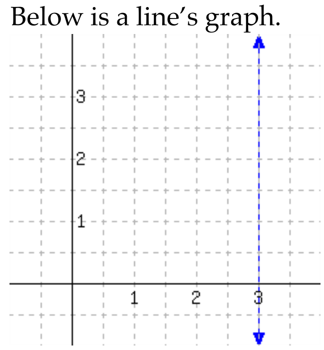
The slope of this line is .

38. Below is a line's graph.



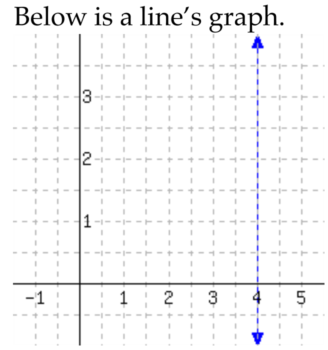
The slope of this line is .

39. Below is a line's graph.



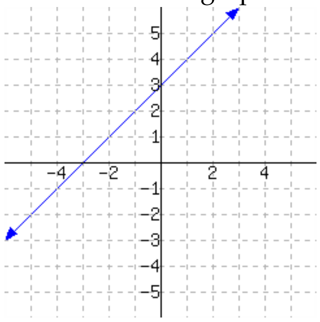
The slope of this line is .

40. Below is a line's graph.



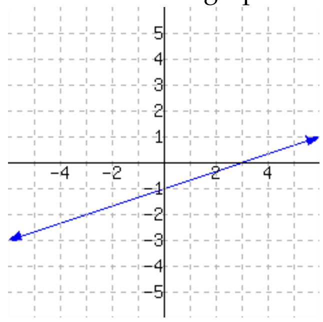
The slope of this line is .

41. Below is a line’s graph.



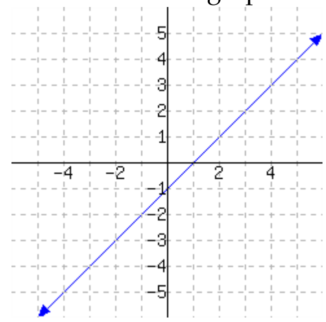
The slope of this line is ______.

42. Below is a line’s graph.



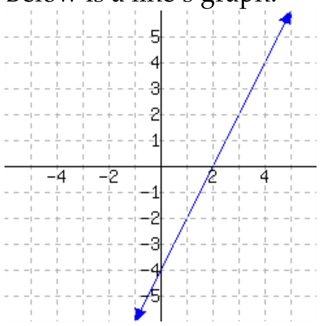
The slope of this line is ______.

43. Below is a line’s graph.



The slope of this line is ______.

44. Below is a line’s graph.



The slope of this line is ______.

Slope in Context

45. By your cell phone contract, you pay a monthly fee plus some money for each minute you use the phone during the month. In one month, you spent 270 minutes on the phone, and paid \$29.15. In another month, you spent 340 minutes on the phone, and paid \$32.30. What is the rate (in dollars per minute) that the phone company is charging you? That is, what is the slope of the line if you plotted the bill versus the number of minutes spent on the phone?

The rate is ______ per minute.

46. By your cell phone contract, you pay a monthly fee plus some money for each minute you use the phone during the month. In one month, you spent 210 minutes on the phone, and paid \$23.45. In another month, you spent 310 minutes on the phone, and paid \$27.95. What is the rate (in dollars per minute) that the phone company is charging you? That is, what is the slope of the line if you plotted the bill versus the number of minutes spent on the phone?

The rate is ______ per minute.

47. A company set aside a certain amount of money in the year 2000. The company spent exactly the same amount from that fund each year on perks for its employees. In 2003, there was still \$474,000 left in the fund. In 2007, there was \$306,000 left. What is the rate (in dollars per year) at which this company is spending from this fund?

The company is spending [ ] per year on perks for its employees.

48. A company set aside a certain amount of money in the year 2000. The company spent exactly the same amount from that fund each year on perks for its employees. In 2004, there was still \$677,000 left in the fund. In 2006, there was \$585,000 left. What is the rate (in dollars per year) at which this company is spending from this fund?

The company is spending [ ] per year on perks for its employees.

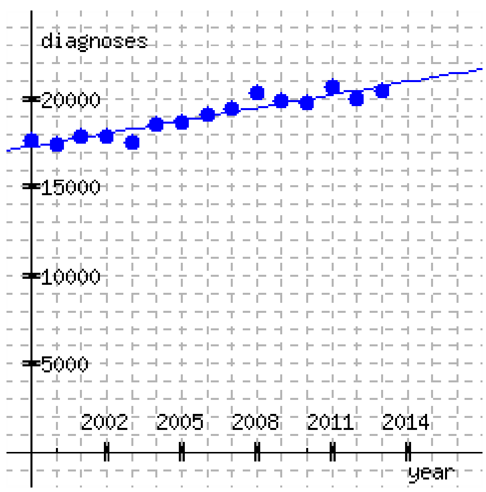
49. A biologist has been observing a tree's height. Eleven months into the observation, the tree was 16.59 feet tall. Seventeen months into the observation, the tree was 18.33 feet tall. What is the rate at which the tree is growing? In other words, what is the slope if you plotted height versus time?

50. A biologist has been observing a tree's height. Fourteen months into the observation, the tree was 20.24 feet tall. Twenty months into the observation, the tree was 20.9 feet tall. What is the rate at which the tree is growing? In other words, what is the slope if you plotted height versus time?

51. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way. Five minutes since the experiment started, the gas had a mass of 234 grams. Seventeen minutes since the experiment started, the gas had a mass of 156 grams. At what rate is the gas leaking?

52. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way. Eight minutes since the experiment started, the gas had a mass of 119 grams. Fourteen minutes since the experiment started, the gas had a mass of 98.6 grams. At what rate is the gas leaking?

53. The graph plots the number of invasive cancer diagnoses in Oregon over time, and a trend-line has been drawn.



Estimate the slope of the trend-line.

Challenge

54. True or False: A slope of $\frac{1}{7}$ is steeper than a slope of 0.1. (☐ true ☐ false)

55. True or False: A slope of $\frac{5}{7}$ is steeper than a slope of 0.7. (☐ true ☐ false)

3.5 Slope-Intercept Form

In this section, we will explore what is perhaps the most common way to write the equation of a line. It's known as slope-intercept form.

3.5.1 Slope-Intercept Definition

Recall Example 3.4.5, where Yara had \$50 in her savings account when the year began, and decided to deposit \$20 each week without withdrawing any money. In that example, we model using x to represent how many weeks have passed. After x weeks, Yara has added $20x$ dollars. And since she started with \$50, she has

$$y = 20x + 50$$

in her account after x weeks. In this example, there is a constant rate of change of 20 dollars per week, so we call that the slope as discussed in Section 3.4. We also saw in Figure 3.4.7 that plotting Yara's balance over time gives us a straight-line graph.

The graph of Yara's savings has some things in common with almost every straight-line graph. There is a slope, and there is a place where the line crosses the y-axis. Figure 3.5.3 illustrates this in the abstract.

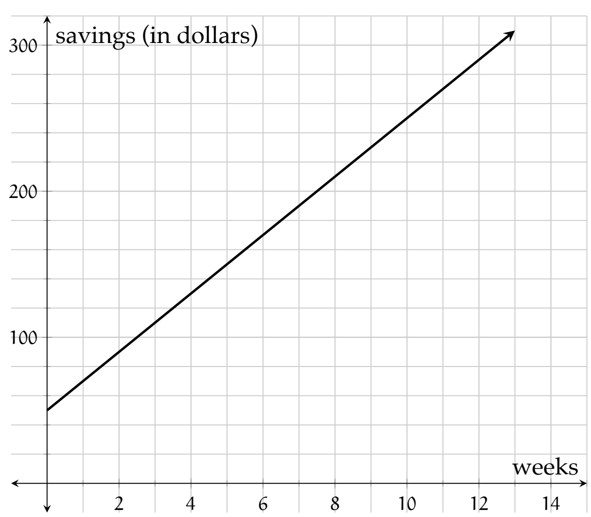


Figure 3.5.2: Yara's savings

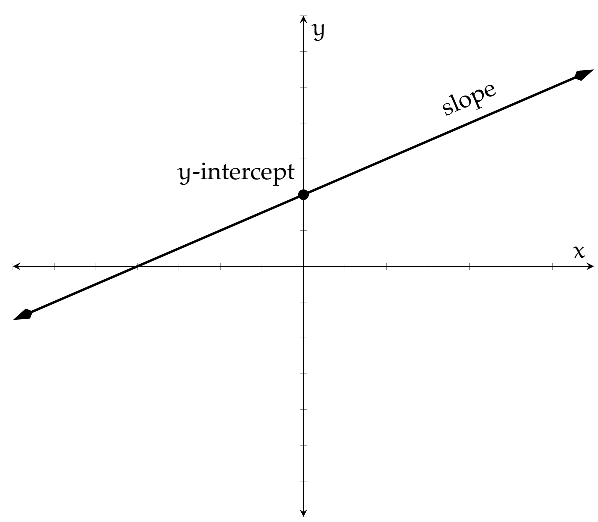


Figure 3.5.3: Generic line with slope and y-intercept

We already have an accepted symbol, m, for the slope of a line. The **y-intercept** is a *point* on the y-axis where the line crosses. Since it's on the y-axis, the x-coordinate of this point is 0. It is standard to call the point $(0, b)$ the y-intercept, and call the number b the y-coordinate of the y-intercept.

Checkpoint 3.5.4 Use Figure 3.4.7 to answer this question.

What was the value of b in the plot of Yara's savings?
What is the y-intercept?

Explanation. The line crosses the y-axis at $(0, 50)$, so the value of b is 50. And the y-intercept is $(0, 50)$.

One way to write the equation for Yara's savings was

$$y = 20x + 50$$

where both $m = 20$ and $b = 50$ are immediately visible in the equation. Now we are ready to generalize this.

Definition 3.5.5 Slope-Intercept Form. When x and y have a linear relationship where m is the slope and $(0, b)$ is the y-intercept, one equation for this relationship is

$$y = mx + b \tag{3.5.1}$$

and this equation is called the **slope-intercept form** of the line. It is called this because the slope and y-intercept are immediately discernible from the numbers in the equation. ♢

Checkpoint 3.5.6 What are the slope and y-intercept for each of the following line equations?

Equation	Slope	y-intercept
$y = 3.1x + 1.78$		
$y = -17x + 112$		
$y = \frac{3}{7}x - \frac{2}{3}$		
$y = 13 - 8x$		
$y = 1 - \frac{2x}{3}$		
$y = 2x$		
$y = 3$		

Explanation. In the first three equations, simply read the slope m according to slope-intercept form. The slopes are 3.1, -17, and $\frac{3}{7}$.

The fourth equation was written with the terms not in the slope-intercept form order. It could be written $y = -8x + 13$, and then it is clear that its slope is -8. In any case, the slope is the coefficient of x.

The fifth equation is also written with the terms not in the slope-intercept form order. Changing the order of the terms, it could be written $y = -\frac{2x}{3} + 1$, but this still does not match the pattern of slope-intercept form. Considering how fraction multiplication works, $\frac{2x}{3} = \frac{2}{3} \cdot \frac{x}{1} = \frac{2}{3}x$. So we can write this equation as $y = -\frac{2}{3}x + 1$, and we see the slope is $-\frac{2}{3}$.

The last two equations could be written $y = 2x + 0$ and $y = 0x + 3$, allowing us to read their slopes as 2 and 0.

For the y-intercepts, remember that we are expected to answer using an ordered pair $(0, b)$, not just a single number b. We can simply read that the first two y-intercepts are $(0, 1.78)$ and $(0, 112)$.

The third equation does not exactly match the slope-intercept form, until you view it as $y = \frac{3}{7}x + \left(-\frac{2}{3}\right)$, and then you can see that its y-intercept is $\left(0, -\frac{2}{3}\right)$.

With the fourth equation, after rewriting it as $y = -8x + 13$, we can see that its y-intercept is $(0, 13)$.

We already explored rewriting the fifth equation as $y = -\frac{2}{3}x + 1$, where we can see that its y-intercept is $(0, 1)$.

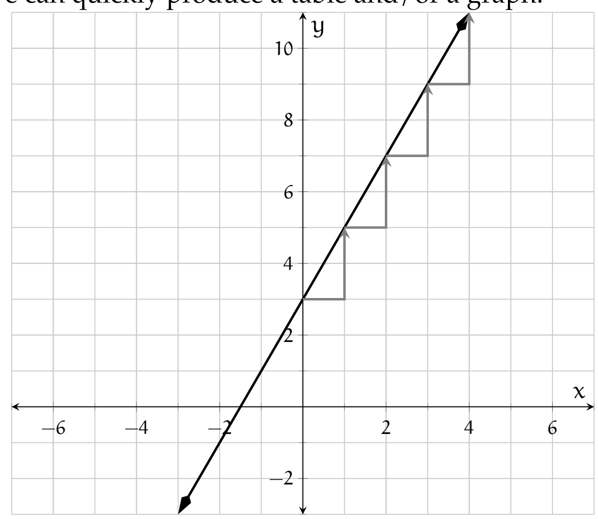
The last two equations could be written $y = 2x + 0$ and $y = 0x + 3$, allowing us to read their y-intercepts as $(0, 0)$ and $(0, 3)$.

Alternatively, we know that y-intercepts happen where $x = 0$, and substituting $x = 0$ into each equation gives you the y-value of the y-intercept.

Remark 3.5.7 The number b is the y-value when $x = 0$. Therefore it is common to refer to b as the **initial value** or **starting value** of a linear relationship.

Example 3.5.8 With a simple equation like $y = 2x + 3$, we can see that this is a line whose slope is 2 and which has initial value 3. So starting at $y = 3$ on the y-axis, each time we increase the x-value by 1, the y-value increases by 2. With these basic observations, we can quickly produce a table and/or a graph.

	x	y	
start on y-axis $\longrightarrow$	0	3	$\longleftarrow$ initial value
increase by 1 $\longrightarrow$	1	5	$\longleftarrow$ increase by 2
increase by 1 $\longrightarrow$	2	7	$\longleftarrow$ increase by 2
increase by 1 $\longrightarrow$	3	9	$\longleftarrow$ increase by 2
increase by 1 $\longrightarrow$	4	11	$\longleftarrow$ increase by 2



Example 3.5.9 Decide whether data in the table has a linear relationship. If so, write the linear equation in slope-intercept form (3.5.1).

x-values	y-values
0	−4
2	2
5	11
9	23

Explanation. To assess whether the relationship is linear, we have to recall from Section 3.3 that we should examine rates of change between data points. Note that the changes in y-values are not consistent. However, the rates of change are calculated as follows:

- When x increases by 2, y increases by 6. The first rate of change is $\frac{6}{2} = 3$.
- When x increases by 3, y increases by 9. The second rate of change is $\frac{9}{3} = 3$.
- When x increases by 4, y increases by 12. The third rate of change is $\frac{12}{4} = 3$.

Since the rates of change are all the same, 3, the relationship is linear and the slope m is 3. According to the table, when $x = 0$, $y = -4$. So the starting value, b, is -4. So in slope-intercept form, the line's equation is $y = 3x - 4$.

Checkpoint 3.5.10 Decide whether data in the table has a linear relationship. If so, write the linear equation in slope-intercept form. This may not be as easy as the previous example. Read the solution for a full explanation.

x-values	y-values
3	-2
6	-8
8	-12
11	-18

The data (☐ does ☐ does not) have a linear relationship, because: (☐ changes in x are not constant ☐ rates of change between data points are constant ☐ rates of change between data points are not constant)

The slope-intercept form of the equation for this line is $\boxed{}$.

Explanation. To assess whether the relationship is linear, we examine rates of change between data points.

- The first rate of change is $\frac{-6}{3} = -2$.
- The second rate of change is $\frac{-4}{2} = -2$.
- The third rate of change is $\frac{-6}{3} = -2$.

Since the rates of change are all the same, -2, the relationship is linear and the slope m is -2.

So we know that the slope-intercept equation is $y = -2x + b$, but what number is b? The table does not directly tell us what the initial y-value is. One approach is to use any point that we know the line passes through, and use algebra to solve for b. We know the line passes through $(3, -2)$, so

$$\begin{aligned} y &= -2x + b \\ -2 &= -2(3) + b \\ -2 &= -6 + b \\ 4 &= b \end{aligned}$$

So the equation is $y = -2x + 4$.

3.5.2 Graphing Slope-Intercept Equations

Example 3.5.11 The conversion formula for a Celsius temperature into Fahrenheit is $F = \frac{9}{5}C + 32$. This appears to be in slope-intercept form, except that x and y are replaced with C and F. Suppose you are asked to graph this equation. How will you proceed? You *could* make a table of values as we do in Section 3.2 but that takes time and effort. Since the equation here is in slope-intercept form, there is a nicer way.

Since this equation is for starting with a Celsius temperature and obtaining a Fahrenheit temperature, it makes sense to let C be the horizontal axis variable and F be the vertical axis variable. Note the slope is $\frac{9}{5}$ and the vertical intercept (here, the F-intercept) is $(0, 32)$.

1. Set up the axes using an appropriate window and labels. Considering the freezing temperature of water (0° Celsius or 32° Fahrenheit), and the boiling temperature of water (100° Celsius or 212° Fahrenheit), it's reasonable to let C run through at least 0 to 100 and F run through at least 32 to 212.

2. Plot the F-intercept, which is at $(0, 32)$.

3. Starting at the F-intercept, use slope triangles to reach the next point. Since our slope is $\frac{9}{5}$, that suggests a "run" of 5 and a "rise" of 9 might work. But as Figure 3.5.12 indicates, such slope triangles are too tiny. You can actually use any fraction equivalent to $\frac{9}{5}$ to plot using the slope, as in $\frac{18}{10}$, $\frac{90}{50}$, $\frac{900}{50}$, or $\frac{45}{25}$

which all reduce to $\frac{9}{5}$. Given the size of our graph, we will use $\frac{90}{50}$ to plot points, where we will try a "run" of 50 and a "rise" of 90.

4. Connect your points with a straight line, use arrowheads, and label the equation.

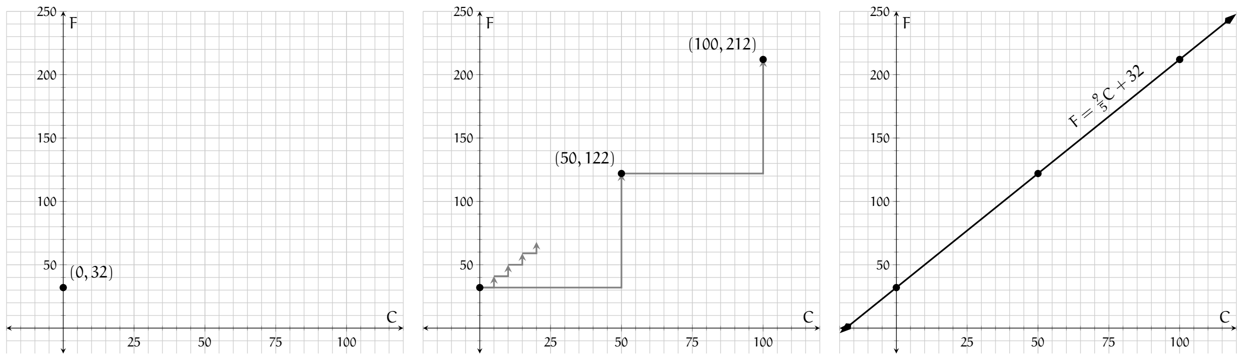
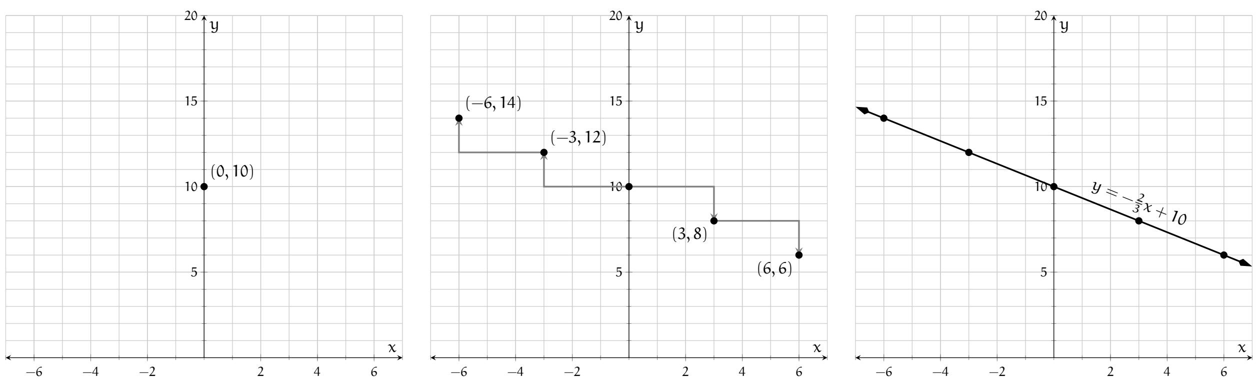


Figure 3.5.12: Graphing $F = \frac{9}{5}C + 32$

Example 3.5.13 Graph $y = -\frac{2}{3}x + 10$.



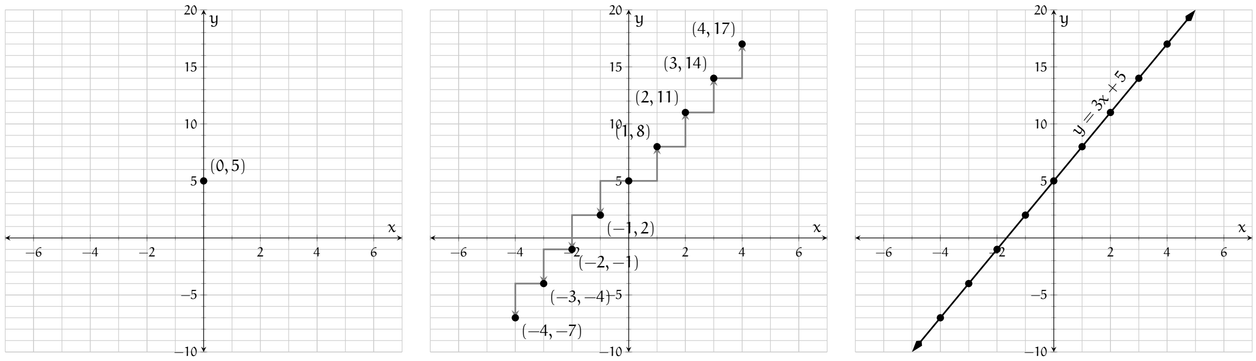
(a) Setting up the axes in an appropriate window and making sure that the y-intercept will be visible, and that any "run" and "rise" amounts we wish to use will not make triangles that are too big or too small.

(b) The slope is $-\frac{2}{3} = \frac{-2}{3} = \frac{2}{-3}$. So we can try using a "run" of 3 and a "rise" of -2 or a "run" of -3 and a "rise" of 2.

(c) Connecting the points with a straight line and adding labels.

Figure 3.5.14: Graphing $y = -\frac{2}{3}x + 10$

Example 3.5.15 Graph $y = 3x + 5$.



(a) Setting up the axes to make sure that the y-intercept will be visible, and that any "run" and "rise" amounts we wish to use will not make triangles that are too big or too small.

(b) The slope is a whole number 3. Every 1 unit forward causes a change of positive 3 in the y-values.

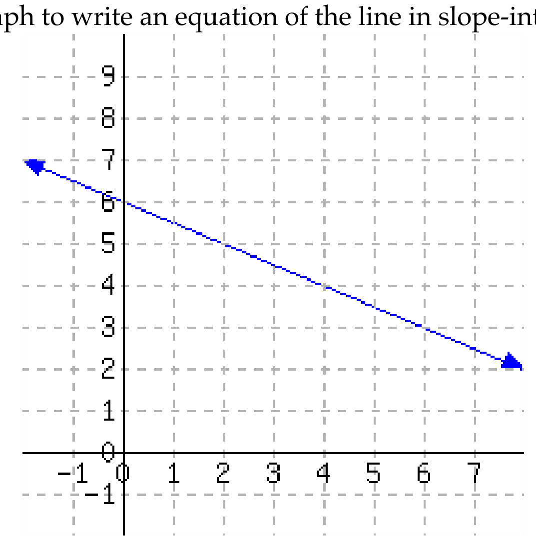
(c) Connecting the points with a straight line and adding labels.

Figure 3.5.16: Graphing $y = 3x + 5$

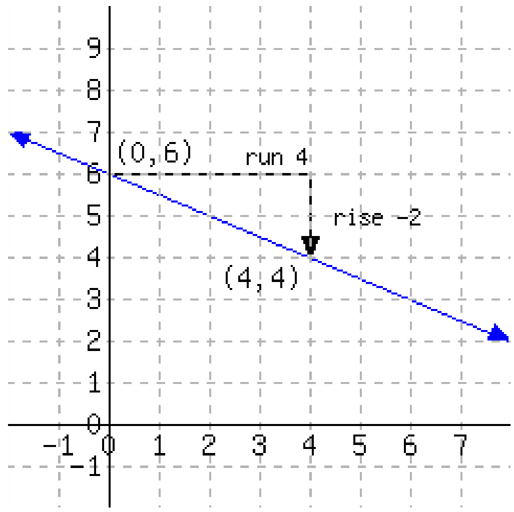
3.5.3 Writing a Slope-Intercept Equation Given a Graph

We can write a linear equation in slope-intercept form based on its graph. We need to be able to calculate the line's slope and see its y-intercept.

Checkpoint 3.5.17 Use the graph to write an equation of the line in slope-intercept form.



Explanation. On the line, pick two points with easy-to-read integer coordinates so that we can calculate slope. It doesn't matter which two points we use; the slope will be the same.



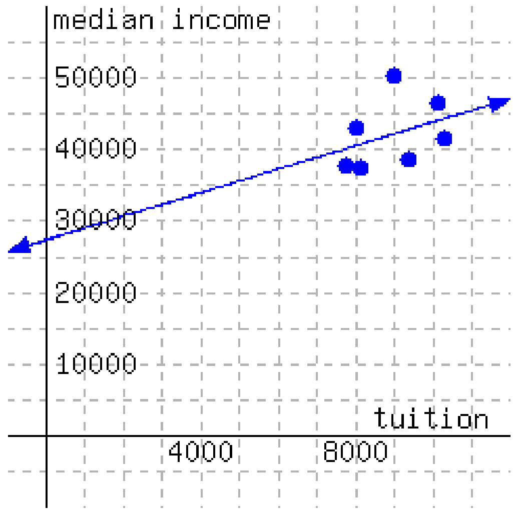
Using the slope triangle, we can calculate the line's slope:

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{-2}{4} = -\frac{1}{2}.$$

From the graph, we can see the y-intercept is $(0, 6)$.
With the slope and y-intercept found, we can write the line's equation:

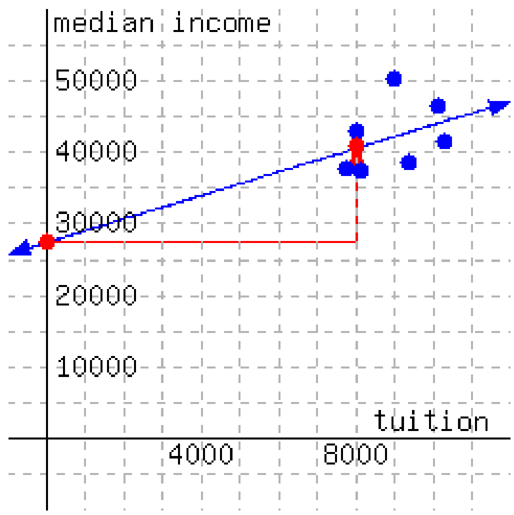
$$y = -\frac{1}{2}x + 6.$$

Checkpoint 3.5.18 There are seven public four-year colleges in Oregon. The graph plots the annual in-state tuition for each school on the x-axis, and the median income of former students ten years after first enrolling on the y-axis.



Write an equation for this line in slope-intercept form.

Explanation. Do your best to identify two points on the line. We go with $(0, 27500)$ and $(11000, 45000)$.



$$\frac{\Delta y}{\Delta x} = \frac{45000 - 27500}{11000 - 0} = \frac{17500}{11000} \approx 1.591$$

So the slope is about 1.591 dollars of median income per dollar of tuition. This is only an estimate since we are not certain the two points we chose are actually on the line.

Estimating the y-intercept to be at $(0, 27500)$, we have $y = 1.591x + 27500$.

3.5.4 Writing a Slope-Intercept Equation Given Two Points

The idea that any two points uniquely determine a line has been understood for thousands of years in many cultures around the world. Once you have two specific points, there is a straightforward process to find the slope-intercept form of the equation of the line that connects them.

Example 3.5.19 Find the slope-intercept form of the equation of the line that passes through the points $(0, 5)$ and $(8, -5)$.

Explanation. We are trying to write down $y = mx + b$, but with specific numbers for m and b. So the first step is to find the slope, m. To do this, recall the slope formula (3.4.3) from Section 3.4. It says that if a line passes through the points (x_1, y_1) and (x_2, y_2), then the slope is found by the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Applying this to our two points $(\overset{x_1}{0}, \overset{y_1}{5})$ and $(\overset{x_2}{8}, \overset{y_2}{-5})$, we see that the slope is:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-5 - 5}{8 - 0} \\ &= \frac{-10}{8} = -\frac{5}{4} \end{aligned}$$

We are trying to write $y = mx + b$. Since we already found the slope, we know that we want to write $y = -\frac{5}{4}x + b$ but we need a specific number for b. We *happen* to know that one point on this line is $(0, 5)$, which is on the y-axis because its x-value is 0. So $(0, 5)$ is this line's y-intercept, and therefore $b = 5$. So, our equation is

$$y = -\frac{5}{4}x + 5.$$

Example 3.5.20 Find the slope-intercept form of the equation of the line that passes through the points $(3, -8)$ and $(-6, 1)$.

Explanation. The first step is always to find the slope between our two points: $(\overset{x_1}{3}, \overset{y_1}{-8})$ and $(\overset{x_2}{-6}, \overset{y_2}{1})$. Using the slope formula (3.4.3) again, we have:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1-(-8)}{-6-3} \\ &= \frac{9}{-9} \\ &= -1 \end{aligned}$$

Now that we have the slope, we can write $y = -1x + b$, which simplifies to $y = -x + b$. Unlike in Example 3.5.19, we are not given the value of b because neither of our two given points have an x-value of 0. The trick to finding b is to remember that we have two points that we know make the equation true! This means all we have to do is substitute *either* point into the equation for x and y and solve for b. Let's arbitrarily choose $(3, -8)$ to plug in.

$$\begin{aligned} y &= -x + b \\ -8 &= -(3) + b && \text{(Now solve for } b\text{.)} \\ -8 &= -3 + b \\ -8 + 3 &= -3 + b + 3 \\ -5 &= b \end{aligned}$$

In conclusion, the equation for which we were searching is $y = -x - 5$.

Don't be tempted to plug in values for x and y at this point. The general equation of a line in any form should have (at least one, and in this case two) variables in the final answer.

Checkpoint 3.5.21 Find the slope-intercept form of the equation of the line that passes through the points $(-3, 150)$ and $(0, 30)$.

Explanation. The first step is always to find the slope between our points: $(\overset{x_1}{-3}, \overset{y_1}{150})$ and $(\overset{x_2}{0}, \overset{y_2}{30})$. Using the slope formula, we have:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{30-150}{0-(-3)} \\ &= \frac{-120}{3} \\ &= -40 \end{aligned}$$

Now we can write $y = -40x + b$ and to find b we need look no further than one of the given points: $(0, 30)$. Since the x-value is 0, the value of b must be 30. So, the slope-intercept form of the line is

$$y = -40x + 30$$

Checkpoint 3.5.22 Find the slope-intercept form of the equation of the line that passes through the points $\left(-3, \frac{3}{4}\right)$ and $\left(-6, -\frac{17}{4}\right)$.

Explanation. First find the slope through our points: $\left(-3, \frac{3}{4}\right)$ and $\left(-6, -\frac{17}{4}\right)$.

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-\frac{17}{4} - \frac{3}{4}}{-6 - (-3)} \\
&= \frac{\frac{-20}{4}}{-3} \\
&= \frac{-5}{-3} \\
&= \frac{5}{3}
\end{aligned}$$

So far we have $y = \frac{5}{3}x + b$. Now we need to solve for b since neither of the points given were the vertical intercept. Recall that to do this, we will choose one of the two points and plug it into our equation. We choose $\left(-3, \frac{3}{4}\right)$.

$$\begin{aligned}
y &= \frac{5}{3}x + b \\
\frac{3}{4} &= \frac{5}{3}(-3) + b \\
\frac{3}{4} &= -5 + b \\
\frac{3}{4} + 5 &= -5 + b + 5 \\
\frac{3}{4} + \frac{20}{4} &= b \\
\frac{23}{4} &= b
\end{aligned}$$

Lastly, we write our equation.

$$y = \frac{5}{3}x + \frac{23}{4}$$

3.5.5 Modeling with Slope-Intercept Form

We can model many relatively simple relationships using slope-intercept form, and then solve related questions using algebra. Here are a few examples.

Example 3.5.23 Uber is a ride-sharing company. Its pricing in Portland factors in how much time and how many miles a trip takes. But if you assume that rides average out at a speed of 30 mph, then their pricing scheme boils down to a base of \$7.35 for the trip, plus \$3.85 per mile. Use a slope-intercept equation and algebra to answer these questions.

a. How much is the fare if a trip is 5.3 miles long?

b. With \$100 available to you, how long of a trip can you afford?

Explanation. The rate of change (slope) is \$3.85 per mile, and the starting value is \$7.35. So the slope-intercept equation is

$$y = 3.85x + 7.35.$$

In this equation, x stands for the number of miles in a trip, and y stands for the amount of money to be charged.

If a trip is 5.3 miles long, we substitute $x = 5.3$ into the equation and we have:

$$\begin{aligned} y &= 3.85x + 7.35 \\ &= 3.85(5.3) + 7.35 \\ &= 20.405 + 7.35 \\ &= 27.755 \end{aligned}$$

And the 5.3-mile ride will cost you about \$27.76. (We say "about," because this was all assuming you average 30 mph.)

Next, to find how long of a trip would cost \$100, we substitute $y = 100$ into the equation and solve for x:

$$\begin{aligned} y &= 3.85x + 7.35 \\ 100 &= 3.85x + 7.35 \\ 100 - 7.35 &= 3.85x \\ 92.65 &= 3.85x \\ \frac{92.65}{3.85} &= x \\ 24.06 &\approx x \end{aligned}$$

So with \$100 you could afford a little more than a 24-mile trip.

Checkpoint 3.5.24 In a certain wildlife reservation in Africa, there are approximately 2400 elephants. Sadly, the population has been decreasing by 30 elephants per year. Use a slope-intercept equation and algebra to answer these questions.

a. If the trend continues, what would the elephant population be 15 years from now?

b. If the trend continues, how many years will it be until the elephant population dwindles to 1200?

Explanation. The rate of change (slope) is -30 elephants per year. Notice that since we are losing elephants, the slope is a negative number. The starting value is 2400 elephants. So the slope-intercept equation is

$$y = -30x + 2400.$$

In this equation, x stands for a number of years into the future, and y stands for the elephant population. To estimate the elephant population 15 years later, we substitute x in the equation with 15, and we have:

$$\begin{aligned} y &= -30x + 2400 \\ &= -30(15) + 2400 \\ &= -450 + 2400 \\ &= 1950 \end{aligned}$$

So if the trend continues, there would be 1950 elephants on this reservation 15 years later.

Next, to find when the elephant population would decrease to 1200, we substitute y in the equation with

1200, and solve for x:

$$\begin{aligned} y &= -30x + 2400 \\ 1200 &= -30x + 2400 \\ 1200 - 2400 &= -30x \\ -1200 &= -30x \\ \frac{-1200}{-30} &= x \\ 40 &= x \end{aligned}$$

So if the trend continues, 40 years later, the elephant population would dwindle to 1200.

3.5.6 Reading Questions

1. How does "slope-intercept form" get its name?
2. What are two phrases you can use for "b" in a slope-intercept form line equation?
3. Explain the two basic steps to graphing a line when you have the equation in slope-intercept form. (Not counting the step where you draw and label the axes and ticks.)

3.5.7 Exercises

Review and Warmup

1. Evaluate $2A + 10a$ for $A = 1$ and $a = -8$.
2. Evaluate $-5B + 3C$ for $B = 7$ and $C = -9$.
3. Evaluate
$$\frac{y_2 - y_1}{x_2 - x_1}$$
for $x_1 = 14$, $x_2 = -4$, $y_1 = -5$, and $y_2 = 2$:
4. Evaluate
$$\frac{y_2 - y_1}{x_2 - x_1}$$
for $x_1 = 18$, $x_2 = -19$, $y_1 = 12$, and $y_2 = -9$:

Identifying Slope and y-Intercept Find the line's slope and y-intercept.

5. A line has equation $y = 2x + 7$.
This line's slope is ______.
This line's y-intercept is ______.
6. A line has equation $y = 3x + 4$.
This line's slope is ______.
This line's y-intercept is ______.
7. A line has equation $y = -8x - 10$.
This line's slope is ______.
This line's y-intercept is ______.
8. A line has equation $y = -7x - 4$.
This line's slope is ______.
This line's y-intercept is ______.
9. A line has equation $y = x - 4$.
This line's slope is ______.
This line's y-intercept is ______.
10. A line has equation $y = x + 2$.
This line's slope is ______.
This line's y-intercept is ______.
11. A line has equation $y = -x + 5$.
This line's slope is ______.
This line's y-intercept is ______.
12. A line has equation $y = -x + 7$.
This line's slope is ______.
This line's y-intercept is ______.

13. A line has equation $y = -\frac{8}{9}x + 1$.

This line's slope is ______.

This line's y-intercept is ______.

14. A line has equation $y = -\frac{2}{7}x + 10$.

This line's slope is ______.

This line's y-intercept is ______.

15. A line has equation $y = \frac{1}{2}x + 8$.

This line's slope is ______.

This line's y-intercept is ______.

16. A line has equation $y = \frac{1}{4}x - 7$.

This line's slope is ______.

This line's y-intercept is ______.

17. A line has equation $y = 1 + 5x$.

This line's slope is ______.

This line's y-intercept is ______.

18. A line has equation $y = -7 + 6x$.

This line's slope is ______.

This line's y-intercept is ______.

19. A line has equation $y = 7 - x$.

This line's slope is ______.

This line's y-intercept is ______.

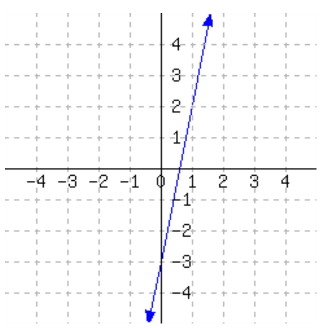
20. A line has equation $y = 8 - x$.

This line's slope is ______.

This line's y-intercept is ______.

Graphs and Slope-Intercept Form A line's graph is given. What is this line's slope-intercept equation?

21.



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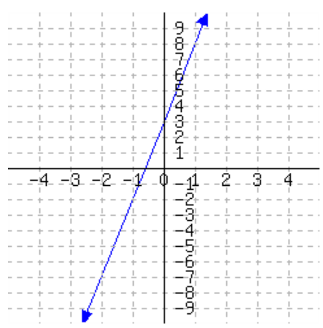

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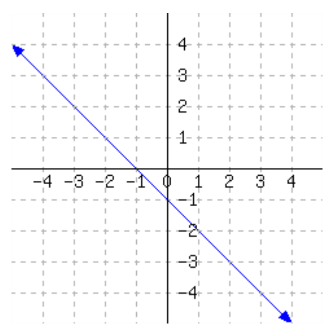

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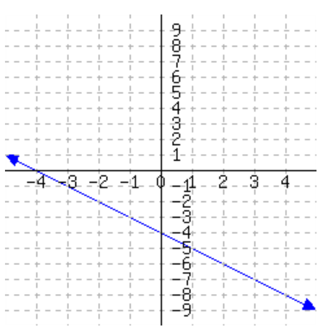

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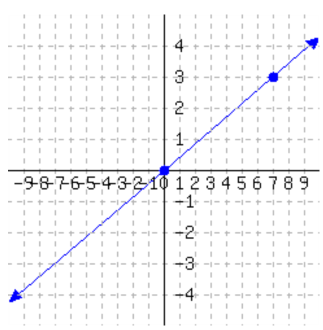

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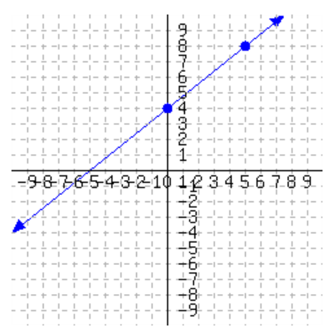

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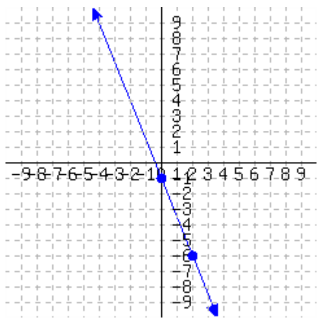

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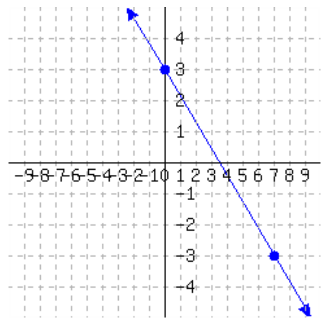


29. Graph the equation $y = 4x$.

30. Graph the equation $y = 5x$.

31. Graph the equation $y = -3x$.

32. Graph the equation $y = -2x$.

33. Graph the equation $y = \frac{5}{2}x$.

34. Graph the equation $y = \frac{1}{4}x$.

35. Graph the equation $y = -\frac{1}{3}x$.

36. Graph the equation $y = -\frac{5}{4}x$.

37. Graph the equation $y = 5x + 2$.

38. Graph the equation $y = 3x + 6$.

39. Graph the equation $y = -4x + 3$.

40. Graph the equation $y = -2x + 5$.

41. Graph the equation $y = x - 4$.

42. Graph the equation $y = x + 2$.

43. Graph the equation $y = -x + 3$.

44. Graph the equation $y = -x - 5$.

45. Graph the equation $y = \frac{2}{3}x + 4$.

46. Graph the equation $y = \frac{3}{2}x - 5$.

47. Graph the equation $y = -\frac{3}{5}x - 1$.

48. Graph the equation $y = -\frac{1}{5}x + 1$.

Writing a Slope-Intercept Equation Given Two Points Find the following line's equation in slope-intercept form.

49. The line passes through the points $(2, 12)$ and $(4, 20)$.

50. The line passes through the points $(4, 21)$ and $(1, 6)$.

51. The line passes through the points $(-5, 13)$ and $(3, -3)$.

52. The line passes through the points $(2, -14)$ and $(-1, 1)$.

53. The line passes through the points $(5, -12)$ and $(4, -11)$.

54. The line passes through the points $(1, -6)$ and $(-2, -3)$.

55. The line passes through the points $(7, 11)$ and $(21, 19)$.

56. The line passes through the points $(-9, -3)$ and $(9, 7)$.

57. The line passes through the points $(-5, 13)$ and $(-10, 19)$.

58. The line passes through the points $(6, -4)$ and $(3, 3)$.

Applications

59. A gym charges members \$40 for a registration fee, and then \$38 per month. You became a member some time ago, and now you have paid a total of \$534 to the gym. How many months have passed since you joined the gym?

[ ] months have passed since you joined the gym.

60. Your cell phone company charges a \$29 monthly fee, plus \$0.15 per minute of talk time. One month your cell phone bill was \$98. How many minutes did you spend talking on the phone that month?

You spent ______ talking on the phone that month.

61. A school purchased a batch of T-shirts from a company. The company charged \$3 per T-shirt, and gave the school a \$60 rebate. If the school had a net expense of \$960 from the purchase, how many T-shirts did the school buy?

The school purchased ______ T-shirts.

62. Laney hired a face-painter for a birthday party. The painter charged a flat fee of \$55, and then charged \$5.50 per person. In the end, Laney paid a total of \$181.50. How many people used the face-painter's service?

______ people used the face-painter's service.

63. A certain country has 277.29 million acres of forest. Every year, the country loses 3.51 million acres of forest mainly due to deforestation for farming purposes. If this situation continues at this pace, how many years later will the country have only 136.89 million acres of forest left? (Use an equation to solve this problem.)

After ______ years, this country would have 136.89 million acres of forest left.

64. Irene has \$77 in her piggy bank. She plans to purchase some Pokemon cards, which costs \$1.45 each. She plans to save \$65.40 to purchase another toy. At most how many Pokemon cards can he purchase?

Write an equation to solve this problem.

Irene can purchase at most ______ Pokemon cards.

65. By your cell phone contract, you pay a monthly fee plus \$0.05 for each minute you spend on the phone. In one month, you spent 290 minutes over the phone, and had a bill totaling \$32.50.

Let x be the number of minutes you spend on the phone in a month, and let y be your total cell phone bill for that month, in dollars. Use a linear equation to model your monthly bill based on the number of minutes you spend on the phone.

a. This line's slope-intercept equation is ______.

b. If you spend 150 minutes on the phone in a month, you would be billed ______.

c. If your bill was \$42.50 one month, you must have spent ______ minutes on the phone in that month.

66. A company set aside a certain amount of money in the year 2000. The company spent exactly \$38,000 from that fund each year on perks for its employees. In 2002, there was still \$702,000 left in the fund.

Let x be the number of years since 2000, and let y be the amount of money, in dollars, left in the fund that year. Use a linear equation to model the amount of money left in the fund after so many years.

a. The linear model's slope-intercept equation is ______.

b. In the year 2010, there was ______ left in the fund.

c. In the year ______, the fund will be empty.

67. A biologist has been observing a tree's height. This type of tree typically grows by 0.24 feet each month. Fourteen months into the observation, the tree was 15.06 feet tall.

Let x be the number of months passed since the observations started, and let y be the tree's height at that time, in feet. Use a linear equation to model the tree's height as the number of months pass.

a. This line's slope-intercept equation is ______.

b. 27 months after the observations started, the tree would be ______ feet in height.

c. ______ months after the observation started, the tree would be 25.86 feet tall.

68. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way. Each minute, they lose 7.7 grams. Five minutes since the experiment started, the remaining gas had a mass of 331.1 grams.

Let x be the number of minutes that have passed since the experiment started, and let y be the mass of the gas in grams at that moment. Use a linear equation to model the weight of the gas over time.

a. This line's slope-intercept equation is ______.

b. 34 minutes after the experiment started, there would be ______ grams of gas left.

c. If a linear model continues to be accurate, ______ minutes since the experiment started, all gas in the container will be gone.

69. A company set aside a certain amount of money in the year 2000. The company spent exactly the same amount from that fund each year on perks for its employees. In 2003, there was still $617,000 left in the fund. In 2007, there was $425,000 left.

Let x be the number of years since 2000, and let y be the amount of money, in dollars, left in the fund that year. Use a linear equation to model the amount of money left in the fund after so many years.

a. The linear model's slope-intercept equation is ______.

b. In the year 2010, there was ______ left in the fund.

c. In the year ______, the fund will be empty.

70. By your cell phone contract, you pay a monthly fee plus some money for each minute you use the phone during the month. In one month, you spent 300 minutes on the phone, and paid $15.50. In another month, you spent 360 minutes on the phone, and paid $16.40.

Let x be the number of minutes you talk over the phone in a month, and let y be your cell phone bill, in dollars, for that month. Use a linear equation to model your monthly bill based on the number of minutes you talk over the phone.

a. This linear model's slope-intercept equation is [ ].

b. If you spent 150 minutes over the phone in a month, you would pay [ ].

c. If in a month, you paid \$17.60 of cell phone bill, you must have spent [ ] minutes on the phone in that month.

71. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way.

Seven minutes since the experiment started, the gas had a mass of 248.2 grams.

Thirteen minutes since the experiment started, the gas had a mass of 204.4 grams.

Let x be the number of minutes that have passed since the experiment started, and let y be the mass of the gas in grams at that moment. Use a linear equation to model the weight of the gas over time.

a. This line's slope-intercept equation is [ ].

b. 34 minutes after the experiment started, there would be [ ] grams of gas left.

c. If a linear model continues to be accurate, [ ] minutes since the experiment started, all gas in the container will be gone.

72. A biologist has been observing a tree's height. 14 months into the observation, the tree was 16 feet tall. 20 months into the observation, the tree was 16.9 feet tall.

Let x be the number of months passed since the observations started, and let y be the tree's height at that time, in feet. Use a linear equation to model the tree's height as the number of months pass.

a. This line's slope-intercept equation is [ ].

b. 27 months after the observations started, the tree would be [ ] feet in height.

c. [ ] months after the observation started, the tree would be 21.85 feet tall.

Challenge

73. Line S has the equation $y = ax + b$ and Line T has the equation $y = cx + d$. Suppose $a > b > c > d > 0$.

a. What can you say about Line S and Line T, given that $a > c$? Give as much information about Line S and Line T as possible.

b. What can you say about Line S and Line T, given that $b > d$? Give as much information about Line S and Line T as possible.

3.6 Point-Slope Form

In Section 3.5, we learned that a linear equation can be written in slope-intercept form, $y = mx + b$. This section covers an alternative that is often more useful depending on the application: point-slope form.

3.6.1 Point-Slope Motivation and Definition

Starting in 1990, the population of the United States has been growing by about 2.865 million people per year. Also, back in 1990, the population was 253 million. Since the rate of growth has been roughly constant, a linear model is appropriate. Let's try to write an equation to model this.

We consider using slope-intercept form (3.5.1), but we would need to know the y-intercept, and nothing in the background tells us that. We'd need to know the population of the United States in the year 0, before there even was a United States.

We could do some side work to calculate the y-intercept, but let's try something else. Here are some things we know:

1. The slope equation is $m = \frac{y_2 - y_1}{x_2 - x_1}$.
2. The slope is $m = 2.865\,\frac{\text{million people}}{\text{year}}$, or $m = \frac{2.865\text{ million people}}{1\text{ year}}$.
3. One point on the line is $(1990, 253)$, because in *1990*, the population was *253* million.

If we use the generic (x, y) to represent a point *somewhere* on this line, then the rate of change between $(1990, 253)$ and (x, y) has to be 2.865. So

$$\frac{y - 253}{x - 1990} = 2.865.$$

There is good reason[1] to want to isolate y in this equation:

$$\begin{aligned} \frac{y - 253}{x - 1990} &= 2.865 \\ \frac{y - 253}{x - 1990} \cdot (x - 1990) &= 2.865 \cdot (x - 1990) \\ y - 253 &= 2.865(x - 1990) && \text{(could distribute, but not going to)} \\ y &= 2.865(x - 1990) + 253 \end{aligned}$$

This is a good place to stop. We have isolated y, and three *meaningful* numbers appear in the equation: the rate of growth, a certain year, and the population in that year. This is a specific example of *point-slope form*. Before we look deeper at point-slope form, let's continue reducing the line equation into slope-intercept form by distributing and combining like terms.

$$\begin{aligned} y &= 2.865(x - 1990) + 253 \\ y &= 2.865x - 5701.35 + 253 && \text{(distributed the 2.865)} \\ y &= 2.865x - 5448.35 && \text{combined like terms} \end{aligned}$$

One concern with slope-intercept form (3.5.1) is that it uses the y-intercept, which might be somewhat meaningless in the context of an application. For example, here we have found that the y-intercept is at

[1] It will help us to see that y (population) *depends* on x (whatever year it is).

$(0, -5448.35)$, but what practical use is that? It's nonsense to say that in the year 0, the population of the United States was -5448.35 million. It doesn't make sense to have a negative population. It doesn't make sense to talk about the United States population before there even was a United States. And it doesn't make sense to use this model for years earlier than 1990 because the background information says clearly that the rate of change we have applies to years 1990 and later.

For all these reasons, we prefer the equation when it was in the form

$$y = 2.865(x - 1990) + 253$$

Definition 3.6.2 Point-Slope Form. When x and y have a linear relationship where m is the slope and (x_0, y_0) is some specific point that the line passes through, one equation for this relationship is

$$y = m(x - x_0) + y_0 \tag{3.6.1}$$

and this equation is called the **point-slope form** of the line. It is called this because the slope and one point on the line are immediately discernible from the numbers in the equation.

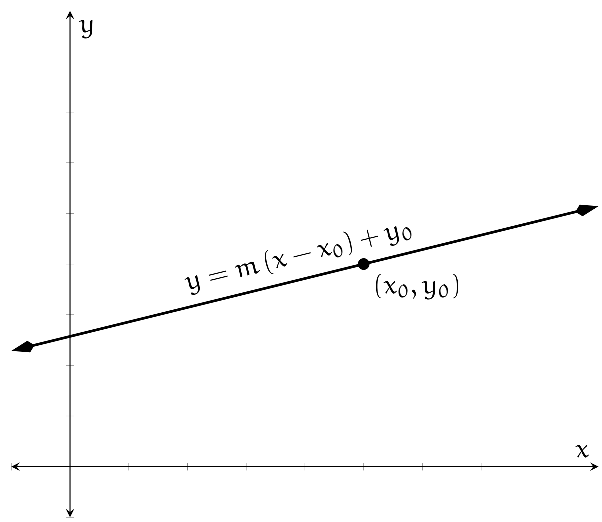


Figure 3.6.3

◊

There is a subtraction sign and an addition sign in point-slope form (3.6.1), and you may have trouble remembering which is which. But remember that that the line is supposed to pass through (x_0, y_0). So substituting x_0 in for x should leave y equal to y_0. And it does. For example, consider our example equation $y = 2.865(x - 1990) + 253$. Here, x_0 is 1990 and y_0 is 253. And:

$$\begin{aligned}
y &= 2.865(x - 1990) + 253 && \text{substitute 1990 for } x\ldots\\
y &= 2.865(1990 - 1990) + 253 &&\\
y &= 2.865(0) + 253 &&\\
y &= 253 && \ldots\text{and } y \text{ works out to be 253}
\end{aligned}$$

The subtraction is exactly where it needs to be to wipe out $m(x-x_0)$ and leave you with y_0. More generally:

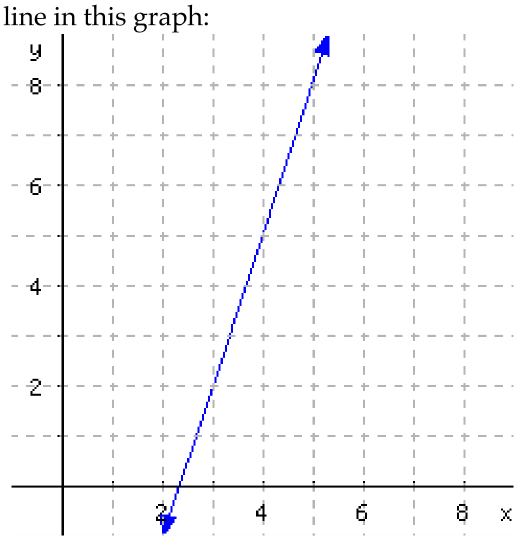
$$\underset{\overset{\downarrow}{y_0}}{y} = m(\overset{\overset{x_0}{\downarrow}}{x} - x_0) + y_0$$

Remark 3.6.4 Alternative Point-Slope Form. It is also common to define point-slope form as

$$y - y_0 = m\,(x - x_0) \tag{3.6.2}$$

by subtracting y_0 from each side. If you learn about point-slope form from some other resource, you may come across this. We feel that the $y = m\,(x - x_0) + y_0$ form will be more helpful with college algebra, statistics, and calculus.

Checkpoint 3.6.5 Consider the line in this graph:



a. Identify a point visible on this line that has integer coordinates, and write it as an ordered pair.

b. What is the slope of the line?

c. Use point-slope form to write an equation for this line, making use of a point with integer coordinates.

Explanation.

a. The visible points with integer coordinates are $(2, -1)$, $(3, 2)$, $(4, 5)$, and $(5, 8)$.

b. Several slope triangles are visible where the "run" is 1 and the "rise" is 3. So the slope is $\frac{3}{1} = 3$.

c. Using $(3, 2)$, the point-slope equation is $y = 3(x - 3) + 2$. (You could use other points, like $(2, -1)$, and get a different-looking equation like $y = 3(x - 2) + (-1)$ which simplifies to $y = 3(x - 2) - 1$.)

In Checkpoint 3.6.5, the solution explains that each of the following are acceptable equations for the same line:

$$y = 3(x - 3) + 2 \qquad\qquad y = 3(x - 2) - 1$$

The first uses $(3, 2)$ as a point on the line, and the second uses $(2, -1)$. Are those two equations really equivalent? Let's distribute and simplify each of them to get slope-intercept form (3.5.1).

$$\begin{aligned} y &= 3(x-3)+2 \\ y &= 3x-9+2 \\ y &= 3x-7 \end{aligned} \qquad\qquad \begin{aligned} y &= 3(x-2)-1 \\ y &= 3x-6-1 \\ y &= 3x-7 \end{aligned}$$

So, yes. It didn't matter which point we used to write a point-slope equation. We get different-looking equations that still represent the same line.

Point-slope form is preferable when we know a line's slope and a point on it, but we don't know the y-intercept. We recognie that distributing the slope and combining like terms can always be used to find the line's slope-intercept form.

Example 3.6.6 A spa chain has been losing customers at a roughly constant rate since the year 2010. In 2013, it had 2975 customers; in 2016, it had 2585 customers. Management estimated that the company will go out of business once its customer base decreases to 1800. If this trend continues, when will the company close?

The given information tells us two points on the line: $(2013, 2975)$ and $(2016, 2585)$. The slope formula (3.4.3) will give us the slope. After labeling those two points as $(\overset{x_1}{2013}, \overset{y_1}{2975})$ and $(\overset{x_2}{2016}, \overset{y_2}{2585})$, we have:

$$\begin{aligned} \text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2585 - 2975}{2016 - 2013} \\ &= \frac{-390}{3} \\ &= -130 \end{aligned}$$

And considering units, this means they are losing 130 customers per year.

Let's note that we could try to make an equation for this line in slope-intercept form, but then we would need to calculate the y-intercept, which in context would correspond to the number of customers in year 0. We could do it, but we'd be working with numbers that have no real-world meaning in this context.

For point-slope form, since we calculated the slope, we know at least this much:

$$y = -130(x - x_0) + y_0.$$

Now we can pick one of those two given points, say $(2013, 2975)$, and get the equation

$$y = -130(x - 2013) + 2975.$$

Note that all three numbers in this equation have meaning in the context of the spa chain. The -130 tells us how many customers are leaving per year, the 2013 represents a year, and the 2975 tells us the number of customers in that year.

We're ready to answer the question about when the chain might go out of business. We need to substitute the given value of 1800 into the appropriate place in our equation. In order to substitute correctly, we must clearly understand what the variable y represents including its units, and what the variable x represents including its units. Write down those definitions first in any question you attempt to answer. It will save you time in the long run, and avoid frustration. Knowing the variable y represents the number of customers in a given year allows one to understand that y must be replaced with 1800 customers. Knowing the variable x represents a year allows one to understand that x will *not* be substituted, because we are trying to *solve* for what year something happens.

After substituting y in the equation with 1800, we will solve for x, and find the answer we seek.

$$\begin{aligned} y &= -130(x-2013)+2975 \\ 1800 &= -130(x-2013)+2975 \\ 1800-2975 &= -130(x-2013) \\ -1175 &= -130(x-2013) \\ \frac{-1175}{-130} &= \frac{-130(x-2013)}{-130} \\ 9.038 &\approx x-2013 \\ 9.038+2013 &\approx x \\ 2022 &\approx x \end{aligned}$$

We find that at this rate the company is headed toward a collapse in 2022.

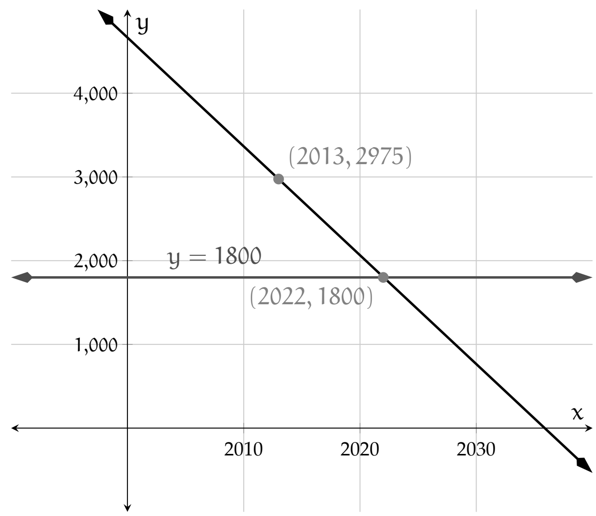
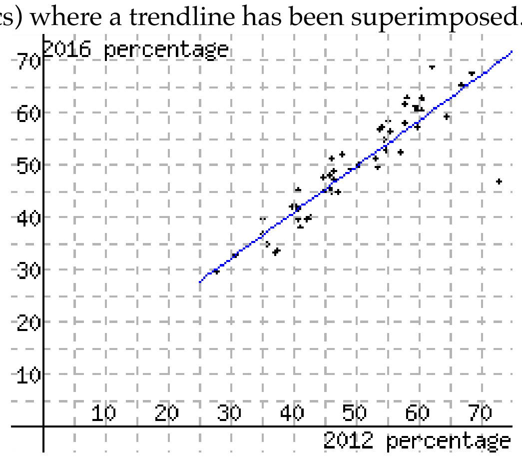


Figure 3.6.7: $y = -130(x-2013)+2975$

Figure 3.6.7 illustrates one line representing the spa's customer base, and another line representing the customer level that would cause the business to close. To make a graph of $y = -130(x-2013)+2975$, we first marked the point $(2013, 2975)$ and from there used the slope of -130.

Checkpoint 3.6.8 If we go state by state and compare the Republican presidential candidate's 2012 vote share (x) to the Republican presidential candidate's 2016 vote share (y), we get the following graph (called a "scatterplot", used in statistics) where a trendline has been superimposed.



Find a point-slope equation for this line. (Note that a slope-intercept equation would use the y-intercept coordinate b, and that would not be meaningful in context, since no state had anywhere near zero percent Republican vote.)

Explanation. We need to calculate slope first. And for that, we need to identify two points on the line. conveniently, the line appears to pass right through $(50, 50)$. We have to take a second point somewhere, and $(75, 72)$ seems like a reasonable roughly accurate choice. The slope equation gives us that

$$m = \frac{72-50}{75-50} = \frac{22}{25} = 0.88.$$

Using $(50, 50)$ as the point, the point-slope equation would then be

$$y = 0.88(x - 50) + 50.$$

3.6.2 Using Two Points to Build a Linear Equation

Since two points can determine a line's location, we can calculate a line's equation using just the coordinates from any two points it passes through.

Example 3.6.9 A line passes through $(-6, 0)$ and $(9, -10)$. Find this line's equation in both point-slope and slope-intercept form.

Explanation. We will use the slope formula (3.4.3) to find the slope first. After labeling those two points as $(\overset{x_1}{-6}, \overset{y_1}{0})$ and $(\overset{x_2}{9}, \overset{y_2}{-10})$, we have:

$$\begin{aligned}
\text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{ - }{ - } && \text{(reserving place holders)} \\
&= \frac{ - 0}{ - (-6)} && \text{(fill in first point, vertically)} \\
&= \frac{-10 - 0}{9 - (-6)} && \text{(fill in second point, vertically)} \\
&= \frac{-10}{15} \\
&= -\frac{2}{3}
\end{aligned}$$

The point-slope equation is $y = -\frac{2}{3}(x - x_0) + y_0$. Next, we will use $(9, -10)$ and substitute x_0 with 9 and y_0 with -10, and we have:

$$\begin{aligned}
y &= -\frac{2}{3}(x - x_0) + y_0 \\
y &= -\frac{2}{3}(x - 9) + (-10) \\
y &= -\frac{2}{3}(x - 9) - 10
\end{aligned}$$

Next, we will change the point-slope equation into slope-intercept form:

$$\begin{aligned}
y &= -\frac{2}{3}(x - 9) - 10 \\
y &= -\frac{2}{3}x + 6 - 10 \\
y &= -\frac{2}{3}x - 4
\end{aligned}$$

Checkpoint 3.6.10 A line passes through $(37, 40)$ and $(-11, -60)$. Find equations for this line using both point-slope and slope-intercept form.

An equation for this line in slope-intercept form is

Explanation. First, use the slope formula to find the slope of this line:

$$\begin{aligned} m = \frac{y_2 - y_1}{x_2 - x_1} &= \frac{-60 - 40}{-11 - 37} \\ &= \frac{-100}{-48} \\ &= \frac{25}{12}. \end{aligned}$$

The generic point-slope equation is $y = m(x - x_0) + y_0$. We have found the slope, m, and we may use $(37, 40)$ for (x_0, y_0). So an equation in point-slope form is $y = \frac{25}{12}(x - 37) + 40$.

To find a slope-intercept form equation, we can take the generic $y = mx + b$ and substitute in the value of m we found. Also, we know that $(x, y) = (-11, -60)$ should make the equation true. So we have

$$\begin{aligned} y &= mx + b \\ -60 &= \frac{25}{12}(-11) + b && \text{(now we may solve for } b\text{)} \\ -60 \cdot 12 &= \left(\frac{25}{12}(-11) + b\right) \cdot 12 && \text{(clear the denominator)} \\ -720 &= 25(-11) + 12b && \text{(distribute and multiply)} \\ -720 &= -275 + 12b \\ -720 + 275 &= 264 + 12b + 275 \\ -445 &= 12b \\ \frac{-445}{12} &= \frac{12b}{12} \\ b &= -\frac{445}{12}. \end{aligned}$$

So the slope-intercept equation is $y = \frac{25}{12}x - \frac{445}{12}$. Note that the slope-intercept equation has an unfriendly fraction, and the fact that this can happen is another reason to use the point-slope form whenever it is reasonable to do so.

3.6.3 More on Point-Slope Form

We can tell a lot about a linear equation now that we have learned both slope-intercept form (3.5.1) and point-slope form (3.6.1). For example, we can know that $y = 4x + 2$ is in slope-intercept form because it looks like $y = mx + b$. It will graph as a line with slope 4 and vertical intercept $(0, 2)$. Likewise, we know that the equation $y = -5(x - 3) + 2$ is in point-slope form because it looks like $y = m(x - x_0) + y_0$. It will graph as a line that has slope -5 and will pass through the point $(3, 2)$.

Example 3.6.11 For the equations below, state whether they are in slope-intercept form or point-slope form. Then identify the slope of the line and at least one point that the line will pass through.

a. $y = -3x + 2$

b. $y = 9(x + 1) - 6$

c. $y = 5 - x$

d. $y = -\frac{12}{5}(x - 9) + 1$

Explanation.

a. The equation $y = -3x + 2$ is in slope-intercept form. The slope is -3 and the vertical intercept is $(0, 2)$.

b. The equation $y = 9(x + 1) - 6$ is in point-slope form. The slope is 9 and the line passes through the point $(-1, -6)$.

c. The equation $y = 5 - x$ is almost in slope-intercept form. If we rearrange the right hand side to be $y = -x + 5$, we can see that the slope is -1 and the vertical intercept is $(0, 5)$.

d. The equation $y = -\frac{12}{5}(x - 9) + 1$ is in point-slope form. The slope is $-\frac{12}{5}$ and the line passes through the point $(9, 1)$.

Example 3.6.12
Consider the graph in Figure 3.6.13.

a. Using the point-slope form of a line, find three different linear equations for the line shown in the graph. Three integer-valued points are shown for convenience.

b. Determine the slope-intercept form of the equation of this line.

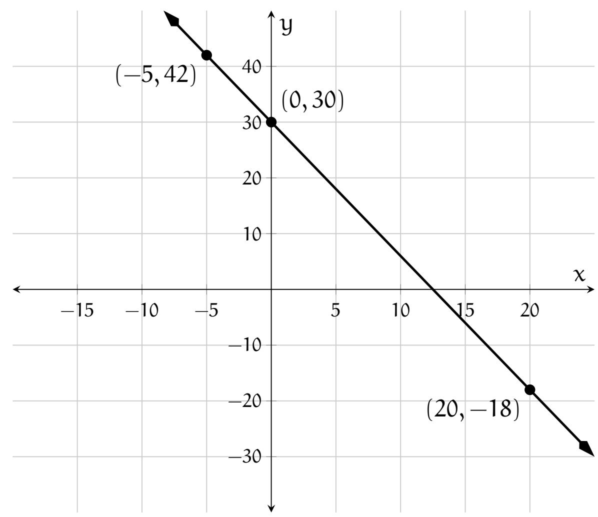


Figure 3.6.13

Explanation.

a. To write *any* of the equations representing this line in point-slope form, we must first find the slope of the line and we can use the slope formula (3.4.3) to do so. We will arbitrarily choose $(0, 30)$ and $(-5, 42)$ as the two points. Inputting these points into the slope formula yields:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{42 - 30}{-5 - 0} \\ &= \frac{12}{-5} \\ &= -\frac{12}{5} \end{aligned}$$

Thus the slope of the line is $-\frac{12}{5}$.

Next, we need to write an equation in point-slope form based on each point shown. Using the point

$(0, 30)$, we have:

$$y = -\frac{12}{5}(x - 0) + 30$$

(This simplifies to $y = -\frac{12}{5}x + 30$.)

The next point is $(20, -18)$. Using this point, we can write an equation for this line as:

$$y = -\frac{12}{5}(x - 20) - 18$$

Finally, we can also use the point $(-5, 42)$ to write an equation for this line:

$$y = -\frac{12}{5}(x - (-5)) + 42$$

which can also be written as:

$$y = -\frac{12}{5}(x + 5) + 42$$

b. As $(0, 30)$ is the vertical intercept, we can write the equation of this line in slope-intercept form as $y = -\frac{12}{5}x + 30$. It's important to note that each of the equations that were written in point-slope form simplify to this, making all four equations equivalent.

3.6.4 Reading Questions

1. Explain why there are some situations where point-slope form is preferable to slope-intercept form.

2. There are basically two steps to convert a point-slope form line equation into a slope-intercept form line equation. What are they?

3. If a line has equation $y = 2(x + 5) + 6$, we can see that the line passes through a certain point. To find the x-coordinate of that point, you might look at the 5 and have memorized that you should negate that. Instead, you could train yourself to look at the x and realize the important number is -5 because that is what it takes to __________.

3.6.5 Exercises

Review and Warmup

1. Evaluate $5B + 10A$ for $B = 8$ and $A = 2$.

2. Evaluate $10C - 8b$ for $C = -4$ and $b = 10$.

3. Evaluate

$$\frac{y_2 - y_1}{x_2 - x_1}$$

for $x_1 = -19$, $x_2 = 17$, $y_1 = 9$, and $y_2 = -19$:

4. Evaluate

$$\frac{y_2 - y_1}{x_2 - x_1}$$

for $x_1 = -15$, $x_2 = 3$, $y_1 = -15$, and $y_2 = 11$:

Point-Slope Form

5. A line's equation is given in point-slope form:
$y = 5(x-5)+28$
This line's slope is ______.
A point on this line that is apparent from the given equation is ______.

6. A line's equation is given in point-slope form:
$y = 2(x-1)+5$
This line's slope is ______.
A point on this line that is apparent from the given equation is ______.

7. A line's equation is given in point-slope form:
$y = -2(x+2)+5$
This line's slope is ______.
A point on this line that is apparent from the given equation is ______.

8. A line's equation is given in point-slope form:
$y = -3(x+4)+7$
This line's slope is ______.
A point on this line that is apparent from the given equation is ______.

9. A line's equation is given in point-slope form:
$y = \left(-\frac{7}{8}\right)(x+24)+20$
This line's slope is ______.
A point on this line that is apparent from the given equation is ______.

10. A line's equation is given in point-slope form:
$y = \frac{8}{5}(x+15)-28$
This line's slope is ______.
A point on this line that is apparent from the given equation is ______.

11. A line's equation is given in point-slope form:
$y = \frac{9}{2}(x-6)+24$
This line's slope is ______.
A point on this line that is apparent from the given equation is ______.

12. A line's equation is given in point-slope form:
$y = \left(-\frac{1}{8}\right)(x-8)-4$
This line's slope is ______.
A point on this line that is apparent from the given equation is ______.

13. A line passes through the points $(5, 15)$ and $(2, 9)$. Find this line's equation in point-slope form.
Using the point $(5, 15)$, this line's point-slope form equation is ______.
Using the point $(2, 9)$, this line's point-slope form equation is ______.

14. A line passes through the points $(2, 8)$ and $(5, 17)$. Find this line's equation in point-slope form.
Using the point $(2, 8)$, this line's point-slope form equation is ______.
Using the point $(5, 17)$, this line's point-slope form equation is ______.

15. A line passes through the points $(3, -1)$ and $(2, 2)$. Find this line's equation in point-slope form.
Using the point $(3, -1)$, this line's point-slope form equation is
__________.
Using the point $(2, 2)$, this line's point-slope form equation is
__________.

16. A line passes through the points $(-4, 17)$ and $(-1, 8)$. Find this line's equation in point-slope form.
Using the point $(-4, 17)$, this line's point-slope form equation is
__________.
Using the point $(-1, 8)$, this line's point-slope form equation is
__________.

17. A line passes through the points $(0, 9)$ and $(7, 15)$. Find this line's equation in point-slope form.
Using the point $(0, 9)$, this line's point-slope form equation is
__________.
Using the point $(7, 15)$, this line's point-slope form equation is
__________.

18. A line passes through the points $(-27, -13)$ and $(-18, -6)$. Find this line's equation in point-slope form.
Using the point $(-27, -13)$, this line's point-slope form equation is
__________.
Using the point $(-18, -6)$, this line's point-slope form equation is
__________.

19. A line's slope is 5. The line passes through the point $(2, 12)$. Find an equation for this line in both point-slope and slope-intercept form.

20. A line's slope is 5. The line passes through the point $(5, 26)$. Find an equation for this line in both point-slope and slope-intercept form.

21. A line's slope is -5. The line passes through the point $(-3, 17)$. Find an equation for this line in both point-slope and slope-intercept form.

22. A line's slope is -5. The line passes through the point $(2, -12)$. Find an equation for this line in both point-slope and slope-intercept form.

23. A line's slope is 1. The line passes through the point $(5, 2)$. Find an equation for this line in both point-slope and slope-intercept form.

24. A line's slope is 1. The line passes through the point $(1, -1)$. Find an equation for this line in both point-slope and slope-intercept form.

25. A line's slope is -1. The line passes through the point $(4, -6)$. Find an equation for this line in both point-slope and slope-intercept form.

26. A line's slope is -1. The line passes through the point $(5, -4)$. Find an equation for this line in both point-slope and slope-intercept form.

27. A line's slope is $\frac{7}{6}$. The line passes through the point $(-6, -4)$. Find an equation for this line in both point-slope and slope-intercept form.

28. A line's slope is $\frac{8}{3}$. The line passes through the point $(-9, -28)$. Find an equation for this line in both point-slope and slope-intercept form.

29. A line's slope is $-\frac{9}{8}$. The line passes through the point $(16, -13)$. Find an equation for this line in both point-slope and slope-intercept form.

30. A line's slope is $-\frac{1}{6}$. The line passes through the point $(18, 2)$. Find an equation for this line in both point-slope and slope-intercept form.

Point-Slope and Slope-Intercept Change this equation from point-slope form to slope-intercept form.

31. $y = 2(x+1) - 5$

32. $y = 2(x-3) + 10$

33. $y = -4(x-2) - 11$

34. $y = -4(x-2) - 12$

35. $y = \dfrac{6}{7}(x+7) - 1$

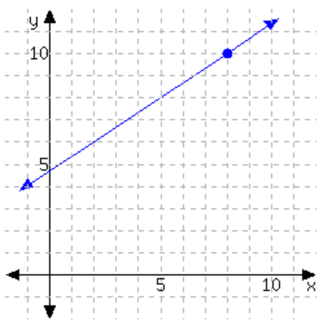
36. $y = \dfrac{7}{5}(x-15) + 25$

37. $y = -\dfrac{8}{9}(x-27) - 23$

38. $y = -\dfrac{9}{8}(x+8) + 7$

Point-Slope Form and Graphs Determine the point-slope form of the linear equation from its graph.

39.



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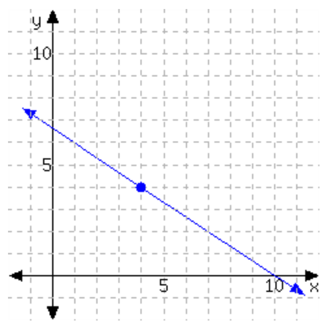

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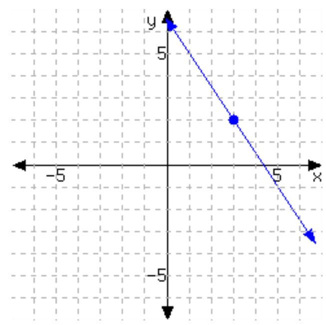

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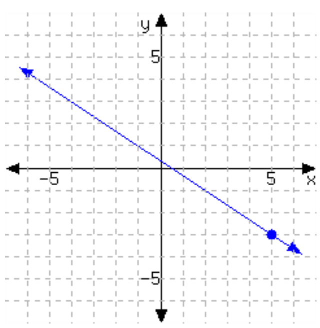

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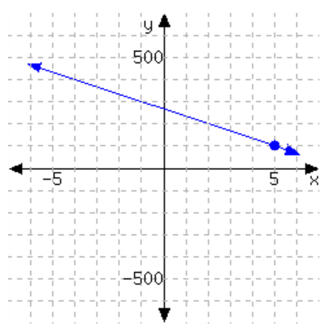

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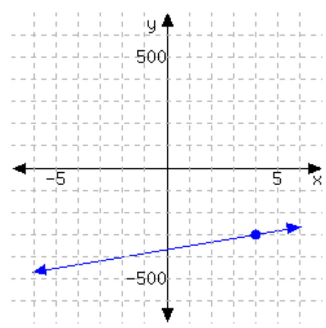

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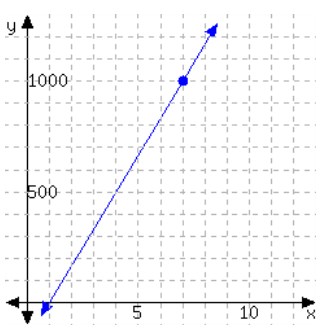

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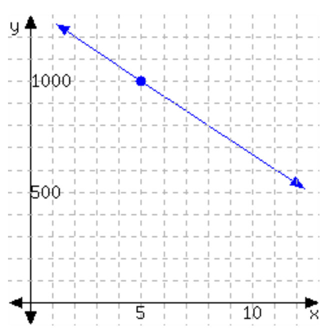

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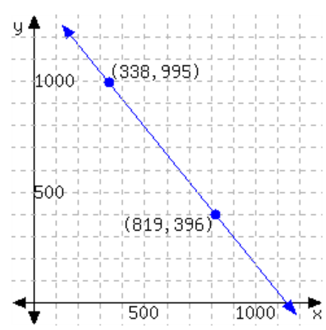

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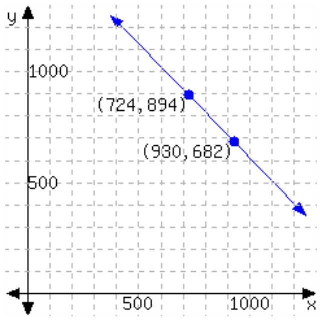

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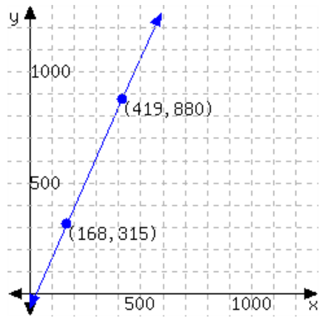

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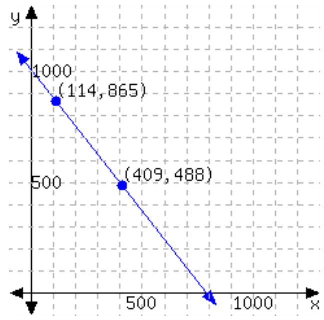


51. Graph the linear equation $y = -\frac{8}{3}(x - 4) - 5$ by identifying the slope and one point on this line.

52. Graph the linear equation $y = \frac{5}{7}(x + 3) + 2$ by identifying the slope and one point on this line.

53. Graph the linear equation $y = \frac{3}{4}(x + 2) + 1$ by identifying the slope and one point on this line.

54. Graph the linear equation $y = -\frac{5}{2}(x - 1) - 5$ by identifying the slope and one point on this line.

55. Graph the linear equation $y = -3(x - 9) + 4$ by identifying the slope and one point on this line.

56. Graph the linear equation $y = 7(x + 3) - 10$ by identifying the slope and one point on this line.

57. Graph the linear equation $y = 8(x + 12) - 20$ by identifying the slope and one point on this line.

58. Graph the linear equation $y = -5(x - 20) - 70$ by identifying the slope and one point on this line.

Applications

59. By your cell phone contract, you pay a monthly fee plus \$0.04 for each minute you spend on the phone. In one month, you spent 280 minutes over the phone, and had a bill totaling \$24.20.

Let x be the number of minutes you spend on the phone in a month, and let y be your total cell phone bill for that month, in dollars. Use a linear equation to model your monthly bill based on the number of minutes you spend on the phone.

a. A point-slope equation to model this is ______.

b. If you spend 170 minutes on the phone in a month, you would be billed ______.

c. If your bill was \$31.80 one month, you must have spent ______ minutes on the phone in that month.

60. A company set aside a certain amount of money in the year 2000. The company spent exactly \$34,000 from that fund each year on perks for its employees. In 2002, there was still \$889,000 left in the fund.

Let x be the number of years since 2000, and let y be the amount of money, in dollars, left in the fund that year. Use a linear equation to model the amount of money left in the fund after so many years.

a. A point-slope equation to model this is ______.

b. In the year 2011, there was [ ] left in the fund.

c. In the year [ ], the fund will be empty.

61. A biologist has been observing a tree's height. This type of tree typically grows by 0.21 feet each month. Thirteen months into the observation, the tree was 18.03 feet tall.

Let x be the number of months passed since the observations started, and let y be the tree's height at that time, in feet. Use a linear equation to model the tree's height as the number of months pass.

a. A point-slope equation to model this is [ ].

b. 28 months after the observations started, the tree would be [ ] feet in height.

c. [ ] months after the observation started, the tree would be 27.27 feet tall.

62. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way. Each minute, they lose 2.8 grams. Ten minutes since the experiment started, the remaining gas had a mass of 103.6 grams.

Let x be the number of minutes that have passed since the experiment started, and let y be the mass of the gas in grams at that moment. Use a linear equation to model the weight of the gas over time.

a. A point-slope equation to model this is [ ].

b. 37 minutes after the experiment started, there would be [ ] grams of gas left.

c. If a linear model continues to be accurate, [ ] minutes since the experiment started, all gas in the container will be gone.

63. A company set aside a certain amount of money in the year 2000. The company spent exactly the same amount from that fund each year on perks for its employees. In 2003, there was still $808,000 left in the fund. In 2006, there was $676,000 left.

Let x be the number of years since 2000, and let y be the amount of money, in dollars, left in the fund that year. Use a linear equation to model the amount of money left in the fund after so many years.

a. A point-slope equation to model this is [ ].

b. In the year 2010, there was [ ] left in the fund.

c. In the year [ ], the fund will be empty.

64. By your cell phone contract, you pay a monthly fee plus some money for each minute you use the phone during the month. In one month, you spent 290 minutes on the phone, and paid $33.85. In another month, you spent 340 minutes on the phone, and paid $37.10.

Let x be the number of minutes you talk over the phone in a month, and let y be your cell phone bill, in dollars, for that month. Use a linear equation to model your monthly bill based on the number of minutes you talk over the phone.

a. A point-slope equation to model this is $\boxed{}$.

b. If you spent 170 minutes over the phone in a month, you would pay $\boxed{}$.

c. If in a month, you paid $42.30 of cell phone bill, you must have spent $\boxed{}$ minutes on the phone in that month.

65. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way.

Six minutes since the experiment started, the gas had a mass of 81.6 grams.

Eleven minutes since the experiment started, the gas had a mass of 69.6 grams.

Let x be the number of minutes that have passed since the experiment started, and let y be the mass of the gas in grams at that moment. Use a linear equation to model the weight of the gas over time.

a. A point-slope equation to model this is $\boxed{}$.

b. 36 minutes after the experiment started, there would be $\boxed{}$ grams of gas left.

c. If a linear model continues to be accurate, $\boxed{}$ minutes since the experiment started, all gas in the container will be gone.

66. A biologist has been observing a tree's height. 13 months into the observation, the tree was 19.06 feet tall. 19 months into the observation, the tree was 19.78 feet tall.

Let x be the number of months passed since the observations started, and let y be the tree's height at that time, in feet. Use a linear equation to model the tree's height as the number of months pass.

a. A point-slope equation to model this is $\boxed{}$.

b. 28 months after the observations started, the tree would be $\boxed{}$ feet in height.

c. $\boxed{}$ months after the observation started, the tree would be 23.62 feet tall.

3.7 Standard Form

We've seen that a linear relationship can be expressed with an equation in slope-intercept form (3.5.1) or with an equation in point-slope form (3.6.1). There is a third form that you can use to write line equations. It's known as *standard form*.

3.7.1 Standard Form Definition

Imagine trying to gather donations to pay for a $10,000 medical procedure you cannot afford. Oversimplifying the mathematics a bit, suppose that there were only two types of donors in the world: those who will donate $20 and those who will donate $100. How many of each, or what combination, do you need to reach the funding goal? As in, if x people donate $20 and y people donate $100, what numbers could x and y be? The donors of the first type have collectively donated $20x$ dollars, and the donors of the second type have collectively donated $100y$. Reflect on the meaning of $20x$ and $100y$. Make sure you understand their meaning before reading on.

So altogether you'd need

$$20x + 100y = 10000$$

This is an example of a line equation in **standard form**.

Definition 3.7.2 Standard Form. It is always possible to write an equation for a line in the form

$$Ax + By = C \tag{3.7.1}$$

where A, B, and C are three numbers (each of which might be 0, although at least one of A and B must be nonzero). This form of a line equation is called **standard form**. In the context of an application, the meaning of A, B, and C depends on that context. This equation is called **standard form** perhaps because *any* line can be written this way, even vertical lines (which cannot be written using slope-intercept or point-slope form equations). ◊

Checkpoint 3.7.3 For each of the following equations, identify what form they are in.

$2.7x + 3.4y = -82$	(☐ slope-intercept ☐ point-slope ☐ standard ☐ other linear ☐ not linear)
$y = \frac{2}{7}(x-3) + \frac{1}{10}$	(☐ slope-intercept ☐ point-slope ☐ standard ☐ other linear ☐ not linear)
$12x - 3 = y + 2$	(☐ slope-intercept ☐ point-slope ☐ standard ☐ other linear ☐ not linear)
$y = x^2 + 5$	(☐ slope-intercept ☐ point-slope ☐ standard ☐ other linear ☐ not linear)
$x - y = 10$	(☐ slope-intercept ☐ point-slope ☐ standard ☐ other linear ☐ not linear)
$y = 4x + 1$	(☐ slope-intercept ☐ point-slope ☐ standard ☐ other linear ☐ not linear)

Explanation. $2.7x + 3.4y = -82$ is in standard form, with $A = 2.7$, $B = 3.4$, and $C = -82$.

$y = \frac{2}{7}(x-3) + \frac{1}{10}$ is in point-slope form, with slope $\frac{2}{7}$, and passing through $\left(3, \frac{1}{10}\right)$.

$12x - 3 = y + 2$ is linear, but not in any of the forms we have studied. Using algebra, you can rearrange it to read $y = 12x - 5$.

$y = x^2 + 5$ is not linear. The exponent on x is a dead giveaway.

$x - y = 10$ is in standard form, with $A = 1$, $B = -1$, and $C = 10$.

$y = 4x + 1$ is in slope-intercept form, with slope 4 and y-intercept at $(0, 1)$.

Returning to the example with donations for the medical procedure, let's examine the equation

$$20x + 100y = 10000.$$

What units are attached to all of the parts of this equation? Both x and y are numbers of people. The 10000 is in dollars. Both the 20 and the 100 are in dollars per person. Note how both sides of the equation are in dollars. On the right, that fact is clear. On the left, $20x$ is in dollars since 20 is in dollars per person, and x is in people. The same is true for $100y$, and the two dollar amounts $20x$ and $100y$ add to a dollar amount.

What is the slope of the linear relationship? It's not immediately visible since m is not part of the standard form equation. But we can use algebra to isolate y:

$$\begin{aligned} 20x + 100y &= 10000 \\ 100y &= -20x + 10000 \\ y &= \frac{-20x + 10000}{100} \\ y &= \frac{-20x}{100} + \frac{10000}{100} \\ y &= -\frac{1}{5}x + 100. \end{aligned}$$

And we see that the slope is $-\frac{1}{5}$. OK, what units are on that slope? As always, the units on slope are $\frac{\text{y-unit}}{\text{x-unit}}$. In this case that's $\frac{\text{person}}{\text{person}}$, which sounds a little weird and seems like it should be simplified away to unitless. But this slope of $-\frac{1}{5}\frac{\text{person}}{\text{person}}$ is saying that for every 5 extra people who donate \$20 each, you need 1 fewer person donating \$100 to still reach your goal.

What is the y-intercept? Since we've already converted the equation into slope-intercept form, we can see that it is at $(0, 100)$. This tells us that if 0 people donate \$20, then you will need 100 people to each donate \$100.

What does a graph for this line look like? We've already converted into slope-intercept form, and we could use that to make the graph. But when given a line in standard form, there is another approach that is often used. Returning to

$$20x + 100y = 10000,$$

let's calculate the y-intercept and the x-intercept. Recall that these are *points* where the line crosses the y-axis and x-axis. To be on the y-axis means that $x = 0$, and to be on the x-axis means that $y = 0$. All these zeros make the resulting algebra easy to finish:

$$\begin{aligned} 20x + 100y &= 10000 \\ 20(0) + 100y &= 10000 \\ 100y &= 10000 \\ y &= \frac{10000}{100} \\ y &= 100 \end{aligned} \qquad\qquad \begin{aligned} 20x + 100y &= 10000 \\ 20x + 100(0) &= 10000 \\ 20x &= 10000 \\ x &= \frac{10000}{20} \\ x &= 500 \end{aligned}$$

So we have a y-intercept at $(0, 100)$ and an x-intercept at $(500, 0)$. If we plot these, we get to mark especially relevant points given the context, and then drawing a straight line between them gives us Figure 3.7.4.

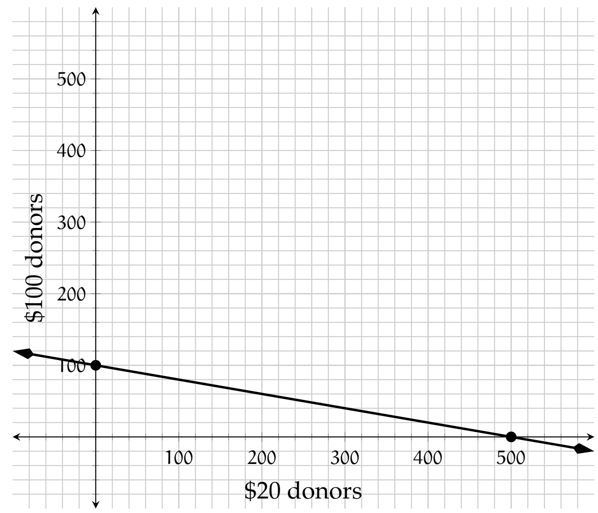


Figure 3.7.4

3.7.2 The x- and y-Intercepts

With a linear relationship (and other types of equations too), we are often interested in the x-intercept and y-intercept because they have special meaning in context. For example, in Figure 3.7.4, the x-intercept implies that if *no one* donates \$100, you need 500 people to donate \$20 to get us to \$10,000. And the y-intercept implies if *no one* donates \$20, you need 100 people to donate \$100. Let's look at another example.

Example 3.7.5 James owns a restaurant that uses about 32 lb of flour every day. He just purchased 1200 lb of flour. Model the amount of flour that remains x days later with a linear equation, and interpret the meaning of its x-intercept and y-intercept.

Since the rate of change is constant (−32 lb every day), and we know the initial value, we can model the amount of flour at the restaurant with a slope-intercept form (3.5.1) equation:

$$y = -32x + 1200$$

where x represents the number of days passed since the initial purchase, and y represents the amount of flour left (in lb.)

A line's x-intercept is in the form of $(x, 0)$, since to be on the x-axis, the y-coordinate must be 0. To find this line's x-intercept, we substitute y in the equation with 0, and solve for x:

$$\begin{aligned} y &= -32x + 1200 \\ 0 &= -32x + 1200 \\ 0 - 1200 &= -32x \\ -1200 &= -32x \\ \frac{-1200}{-32} &= x \\ 37.5 &= x \end{aligned}$$

So the line's x-intercept is at $(37.5, 0)$. In context this means the flour would last for 37.5 days.

A line's y-intercept is in the form of $(0, y)$. This line equation is already in slope-intercept form, so we can just see that its y-intercept is at $(0, 1200)$. In general though, we would substitute x in the equation with 0, and we have:

$$\begin{aligned} y &= -32x + 1200 \\ y &= -32(0) + 1200 \\ y &= 1200 \end{aligned}$$

So yes, the line's y-intercept is at $(0, 1200)$. This means that when the flour was purchased, there was 1200 lb of it. In other words, the y-intercept tells us one of the original pieces of information: in the beginning, James purchased 1200 lb of flour.

If a line is in standard form, it's often easiest to graph it using its two intercepts.

Example 3.7.6 Graph $2x - 3y = -6$ using its intercepts. And then use the intercepts to calculate the line's slope.

Explanation. To graph a line by its x-intercept and y-intercept, it might help to first set up a table like in Figure 3.7.7:

	x-value	y-value	Intercepts
x-intercept		0	
y-intercept	0		

Figure 3.7.7: Intercepts of $2x - 3y = -6$

A table like this might help you stay focused on the fact that we are searching for *two* points. As we've noted earlier, an x-intercept is on the x-axis, and so its y-coordinate must be 0. This is worth taking special note of: to find an x-intercept, y must be 0. This is why we put 0 in the y-value cell of the x-intercept. Similarly, a line's y-intercept has $x = 0$, and we put 0 into the x-value cell of the y-intercept.

Next, we calculate the line's x-intercept by substituting $y = 0$ into the equation

$$\begin{aligned} 2x - 3y &= -6 \\ 2x - 3(0) &= -6 \\ 2x &= -6 \\ x &= -3 \end{aligned}$$

So the line's x-intercept is $(-3, 0)$.

Similarly, we substitute $x = 0$ into the equation to calculate the y-intercept:

$$\begin{aligned} 2x - 3y &= -6 \\ 2(0) - 3y &= -6 \\ -3y &= -6 \\ y &= 2 \end{aligned}$$

So the line's y-intercept is $(0, 2)$.

Now we can complete the table:

	x-value	y-value	Intercepts
x-intercept	-3	0	$(-3, 0)$
y-intercept	0	2	$(0, 2)$

Figure 3.7.8: Intercepts of $2x - 3y = -6$

With both intercepts' coordinates, we can graph the line:

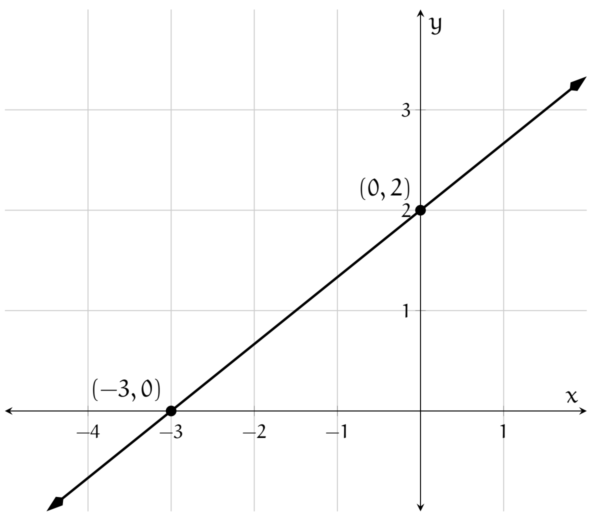


Figure 3.7.9: Graph of $2x - 3y = -6$

There is a slope triangle from the x-intercept to the origin up to the y-intercept. It tells us that the slope is

$$m = \frac{\Delta y}{\Delta x} = \frac{2}{3}.$$

This last example generalizes to a fact worth noting.

Fact 3.7.10 *If a line's x-intercept is at $(r, 0)$ and its y-intercept is at $(0, b)$, then the slope of the line is $-\frac{b}{r}$. (Unless the line passes through the origin, in which case both r and b equal 0, and then this fraction is undefined. And the slope of the line could be anything.)*

Checkpoint 3.7.11 Consider the line with equation $2x + 4.3y = \frac{1000}{99}$.

a. What is its x-intercept?

b. What is its y-intercept?

c. What is its slope?

Explanation.

a. To find the x-intercept:

$$\begin{aligned} 2x + 4.3y &= \frac{1000}{99} \\ 2x + 4.3(0) &= \frac{1000}{99} \\ 2x &= \frac{1000}{99} \\ x &= \frac{500}{99} \end{aligned}$$

So the x-intercept is at $\left(\frac{500}{99}, 0\right)$.

b. To find the y-intercept:

$$\begin{aligned}2x + 4.3y &= \frac{1000}{99}\\ 2(0) + 4.3y &= \frac{1000}{99}\\ 4.3y &= \frac{1000}{99}\\ y &= \frac{1}{4.3}\cdot\frac{1000}{99}\\ y &\approx 2.349\ldots\end{aligned}$$

So the y-intercept is at about $(0, 2.349)$.

c. Since we have the x- and y-intercepts, we can calculate the slope:

$$m \approx -\frac{2.349}{\frac{500}{99}} = -\frac{2.349\cdot 99}{500} \approx -0.4561.$$

3.7.3 Transforming between Standard Form and Slope-Intercept Form

Sometimes a linear equation arises in standard form, but it would be useful to see that equation in slope-intercept form (3.5.1). Or perhaps, vice versa.

A linear equation in slope-intercept form (3.5.1) tells us important information about the line: its slope m and y-intercept $(0, b)$. However, a line's standard form does not show those two important values. As a result, we often need to change a line's equation from standard form to slope-intercept form. Let's look at some examples.

Example 3.7.12 Change $2x - 3y = -6$ to slope-intercept form, and then graph it.

Explanation. Since a line in slope-intercept form looks like $y = \ldots$, we will solve for y in $2x - 3y = -6$:

$$\begin{aligned}2x - 3y &= -6\\ -3y &= -6 - 2x\\ -3y &= -2x - 6\\ y &= \frac{-2x-6}{-3}\\ y &= \frac{-2x}{-3} - \frac{6}{-3}\\ y &= \frac{2}{3}x + 2\end{aligned}$$

In the third line, we wrote $-2x-6$ on the right side, instead of $-6 - 2x$. The only reason we did this is because we are headed to slope-intercept form, where the x-term is traditionally written first.

Now we can see that the slope is $\frac{2}{3}$ and the y-intercept is at $(0, 2)$. With these things found, we can graph the line using slope triangles.

Compare this graphing method with the Graphing by Intercepts method in Example 3.7.6. We have more points in this graph, thus we can graph the line more accurately.

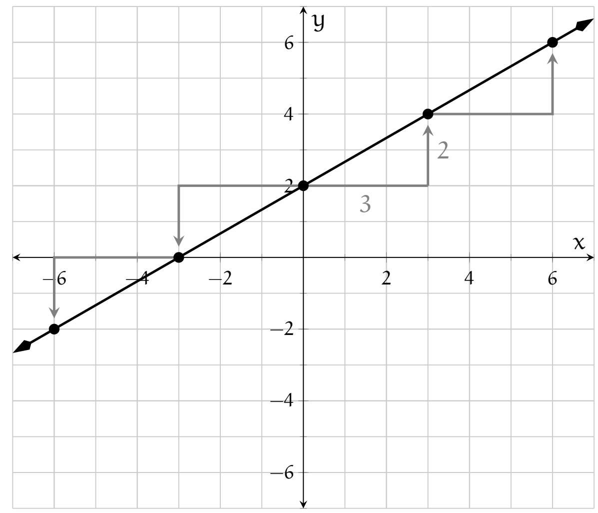


Figure 3.7.13: Graphing $2x - 3y = -6$ with Slope Triangles

Example 3.7.14 Graph $2x - 3y = 0$.

Explanation. First, we will try (and fail) to graph this line using its x- and y-intercepts.

Trying to find the x-intercept:

$$\begin{aligned} 2x - 3y &= 0 \\ 2x - 3(0) &= 0 \\ 2x &= 0 \\ x &= 0 \end{aligned}$$

So the line's x-intercept is at $(0, 0)$, at the origin.

Huh, that is *also* on the y-axis…

Trying to find the y-intercept:

$$\begin{aligned} 2x - 3y &= 0 \\ 2(0) - 3y &= 0 \\ -3y &= 0 \\ y &= 0 \end{aligned}$$

So the line's y-intercept is also at $(0, 0)$.

Since both intercepts are the same point, there is no way to use the intercepts alone to graph this line. So what can be done?

Several approaches are out there, but one is to convert the line equation into slope-intercept form:

$$\begin{aligned} 2x - 3y &= 0 \\ -3y &= 0 - 2x \\ -3y &= -2x \\ y &= \frac{-2x}{-3} \\ y &= \frac{2}{3}x \end{aligned}$$

So the line's slope is $\frac{2}{3}$, and we can graph the line using slope triangles and the intercept at $(0, 0)$, as in Figure 3.7.15.

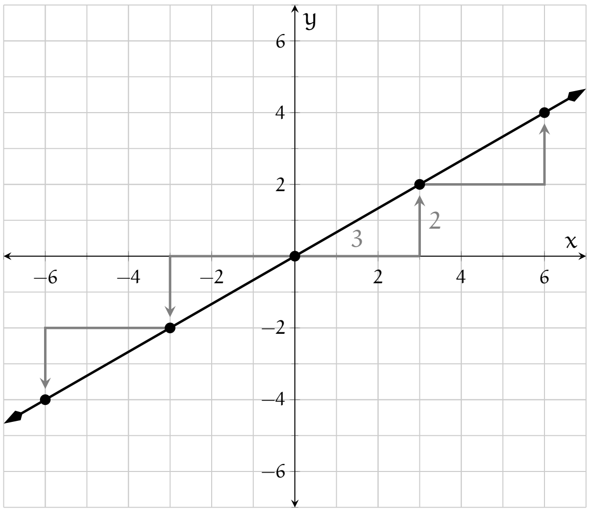


Figure 3.7.15: Graphing $2x - 3y = 0$ with Slope Triangles

In summary, if $C = 0$ in a standard form equation, it's convenient to graph it by first converting the equation to slope-intercept form (3.5.1).

Example 3.7.16 Write the equation $y = \frac{2}{3}x + 2$ in standard form.

Explanation. Once we subtract $\frac{2}{3}x$ on both sides of the equation, we have

$$-\frac{2}{3}x + y = 2$$

Technically, this equation is already in standard form $Ax + By = C$. However, you might like to end up with an equation that has no fractions, so you could multiply each side by 3:

$$\begin{aligned} 3 \cdot \left(-\frac{2}{3}x + y\right) &= 3 \cdot 2 \\ -x + 3y &= 6 \end{aligned}$$

3.7.4 Reading Questions

1. What kind of line can be written in standard form, but cannot be written in slope-intercept or point-slope form?

2. What are some reasons why you might care to find the x- and y-intercepts of a line?

3. What is not immediately apparent from standard form, that *is* immediately apparent from slope-intercept form and point-slope form?

3.7.5 Exercises

Review and Warmup Solve the linear equation for y.

1. $24x + 3y = 54$
2. $12x - 4y = -28$
3. $-3x - y = 17$
4. $-5x - y = -9$
5. $6x - 9y = 6$
6. $9x - 2y = -1$

Slope and y-intercept Find the line's slope and y-intercept.

7. A line has equation $-3x + y = 4$.
This line's slope is ______.
This line's y-intercept is ______.

8. A line has equation $-x - y = -6$.
This line's slope is ______.
This line's y-intercept is ______.

9. A line has equation $15x + 3y = -9$.
This line's slope is ______.
This line's y-intercept is ______.

10. A line has equation $12x - 3y = 6$.
This line's slope is ______.
This line's y-intercept is ______.

11. A line has equation $x + 4y = -8$.
This line's slope is ______.
This line's y-intercept is ______.

12. A line has equation $5x + 6y = 30$.
This line's slope is ______.
This line's y-intercept is ______.

13. A line has equation $6x - 7y = 28$.
This line's slope is ______.
This line's y-intercept is ______.

14. A line has equation $18x + 15y = 15$.
This line's slope is ______.
This line's y-intercept is ______.

15. A line has equation $4x - 24y = 0$.
This line's slope is ______.
This line's y-intercept is ______.

16. A line has equation $2x - 8y = 0$.
This line's slope is ______.
This line's y-intercept is ______.

17. A line has equation $6x + 9y = 4$.
This line's slope is ______.
This line's y-intercept is ______.

18. A line has equation $12x + 8y = 5$.
This line's slope is ______.
This line's y-intercept is ______.

Converting to Standard Form

19. Rewrite $y = 5x - 2$ in standard form.
20. Rewrite $y = 6x - 7$ in standard form.
21. Rewrite $y = \frac{7}{8}x - 5$ in standard form.
22. Rewrite $y = -\frac{8}{5}x + 3$ in standard form.

Graphs and Standard Form

23. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$7x + 2y = 14$$

	x-value	y-value	Location (as an ordered pair)
y-intercept	______	______	______
x-intercept	______	______	______

24. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$2x + 7y = -28$$

	x-value	y-value	Location (as an ordered pair)
y-intercept	______	______	____________
x-intercept	______	______	____________

25. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$2x - 5y = -30$$

	x-value	y-value	Location (as an ordered pair)
y-intercept	______	______	____________
x-intercept	______	______	____________

26. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$x - 3y = -3$$

	x-value	y-value	Location (as an ordered pair)
y-intercept	______	______	____________
x-intercept	______	______	____________

27. Find the x- and y-intercepts of the line with equation $4x + 6y = 24$. Then find one other point on the line. Use your results to graph the line.

28. Find the x- and y-intercepts of the line with equation $4x + 5y = -40$. Then find one other point on the line. Use your results to graph the line.

29. Find the x- and y-intercepts of the line with equation $5x - 2y = 10$. Then find one other point on the line. Use your results to graph the line.

30. Find the x- and y-intercepts of the line with equation $5x - 6y = -90$. Then find one other point on the line. Use your results to graph the line.

31. Find the x- and y-intercepts of the line with equation $x + 5y = -15$. Then find one other point on the line. Use your results to graph the line.

32. Find the x- and y-intercepts of the line with equation $6x + y = -18$. Then find one other point on the line. Use your results to graph the line.

33. Make a graph of the line $x + y = 2$.

34. Make a graph of the line $-5x - y = -3$.

35. Make a graph of the line $x + 5y = 5$.

36. Make a graph of the line $x - 2y = 2$.

37. Make a graph of the line $20x - 4y = 8$.

38. Make a graph of the line $3x + 5y = 10$.

39. Make a graph of the line $-3x + 2y = 6$.

40. Make a graph of the line $-4x - 5y = 10$.

41. Make a graph of the line $4x - 5y = 0$.

42. Make a graph of the line $5x + 7y = 0$.

Interpreting Intercepts in Context

43. Kara is buying some tea bags and some sugar bags. Each tea bag costs 8 cents, and each sugar bag costs 3 cents. She can spend a total of \$3.60.

Assume Kara will purchase x tea bags and y sugar bags. Use a linear equation to model the number of tea bags and sugar bags she can purchase.

Find this line's x-intercept, and interpret its meaning in this context.

- ⊙ ***A***. The x-intercept is (0, 120). It implies Kara can purchase 120 sugar bags with no tea bags.
- ⊙ ***B***. The x-intercept is (0,45). It implies Kara can purchase 45 sugar bags with no tea bags.
- ⊙ ***C***. The x-intercept is (45,0). It implies Kara can purchase 45 tea bags with no sugar bags.
- ⊙ ***D***. The x-intercept is (120,0). It implies Kara can purchase 120 tea bags with no sugar bags.

44. Corey is buying some tea bags and some sugar bags. Each tea bag costs 6 cents, and each sugar bag costs 5 cents. He can spend a total of \$1.50.

Assume Corey will purchase x tea bags and y sugar bags. Use a linear equation to model the number of tea bags and sugar bags he can purchase.

Find this line's y-intercept, and interpret its meaning in this context.

- ⊙ ***A***. The y-intercept is (0, 30). It implies Corey can purchase 30 sugar bags with no tea bags.
- ⊙ ***B***. The y-intercept is (0,25). It implies Corey can purchase 25 sugar bags with no tea bags.
- ⊙ ***C***. The y-intercept is (30,0). It implies Corey can purchase 30 tea bags with no sugar bags.
- ⊙ ***D***. The y-intercept is (25,0). It implies Corey can purchase 25 tea bags with no sugar bags.

45. An engine's tank can hold 60 gallons of gasoline. It was refilled with a full tank, and has been running without breaks, consuming 3 gallons of gas per hour.

Assume the engine has been running for x hours since its tank was refilled, and assume there are y gallons of gas left in the tank. Use a linear equation to model the amount of gas in the tank as time passes.

Find this line's x-intercept, and interpret its meaning in this context.

- ⊙ ***A***. The x-intercept is (60,0). It implies the engine will run out of gas 60 hours after its tank was refilled.
- ⊙ ***B***. The x-intercept is (20,0). It implies the engine will run out of gas 20 hours after its tank was refilled.
- ⊙ ***C***. The x-intercept is (0,20). It implies the engine started with 20 gallons of gas in its tank.
- ⊙ ***D***. The x-intercept is (0,60). It implies the engine started with 60 gallons of gas in its tank.

46. An engine's tank can hold 100 gallons of gasoline. It was refilled with a full tank, and has been running without breaks, consuming 2.5 gallons of gas per hour.

Assume the engine has been running for x hours since its tank was refilled, and assume there are y gallons of gas left in the tank. Use a linear equation to model the amount of gas in the tank as time passes.

Find this line's y-intercept, and interpret its meaning in this context.

⊙ ***A***. The y-intercept is (100,0). It implies the engine will run out of gas 100 hours after its tank was refilled.

⊙ ***B***. The y-intercept is (0,40). It implies the engine started with 40 gallons of gas in its tank.

⊙ ***C***. The y-intercept is (0,100). It implies the engine started with 100 gallons of gas in its tank.

⊙ ***D***. The y-intercept is (40,0). It implies the engine will run out of gas 40 hours after its tank was refilled.

47. A new car of a certain model costs \$46,800.00. According to Blue Book, its value decreases by \$2,600.00 every year.

Assume x years since its purchase, the car's value is y dollars. Use a linear equation to model the car's value.

Find this line's x-intercept, and interpret its meaning in this context.

⊙ ***A***. The x-intercept is (0,18). It implies the car would have no more value 18 years since its purchase.

⊙ ***B***. The x-intercept is (46800,0). It implies the car's initial value was 46800.

⊙ ***C***. The x-intercept is (18,0). It implies the car would have no more value 18 years since its purchase.

⊙ ***D***. The x-intercept is (0,46800). It implies the car's initial value was 46800.

48. A new car of a certain model costs \$65,000.00. According to Blue Book, its value decreases by \$2,600.00 every year.

Assume x years since its purchase, the car's value is y dollars. Use a linear equation to model the car's value.

Find this line's y-intercept, and interpret its meaning in this context.

⊙ ***A***. The y-intercept is (65000,0). It implies the car's initial value was 65000.

⊙ ***B***. The y-intercept is (25,0). It implies the car would have no more value 25 years since its purchase.

⊙ ***C***. The y-intercept is (0,65000). It implies the car's initial value was 65000.

⊙ ***D***. The y-intercept is (0,25). It implies the car would have no more value 25 years since its purchase.

Challenge

49. Fill in the variables A, B, and C in $Ax + By = C$ with the numbers $10, 11$ and 14. You may only use each number once.

a. For the steepest possible slope, A must be $\boxed{}$, B must be $\boxed{}$, and C must be $\boxed{}$.

b. For the shallowest possible slope, A must be $\boxed{}$, B must be $\boxed{}$, and C must be $\boxed{}$.

3.8 Horizontal, Vertical, Parallel, and Perpendicular Lines

The equations of horizontal and vertical lines distinguish them from other line equations enough to merit a special investigation. In addition, pairs of lines that are parallel or perpendicular to each other have interesting features and properties. This section examines the geometric features of these types of lines.

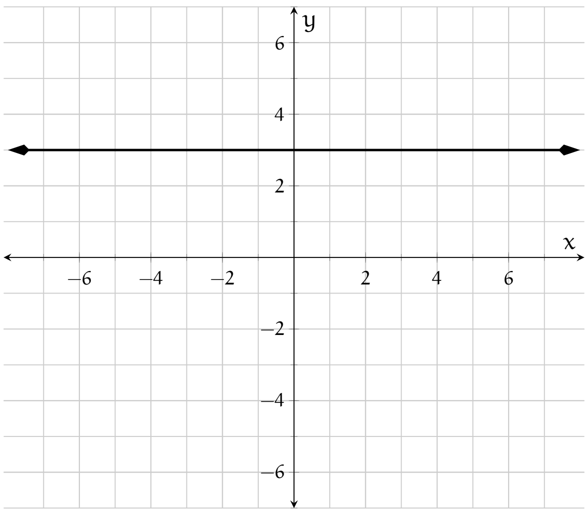


Figure 3.8.2: Horizontal Line

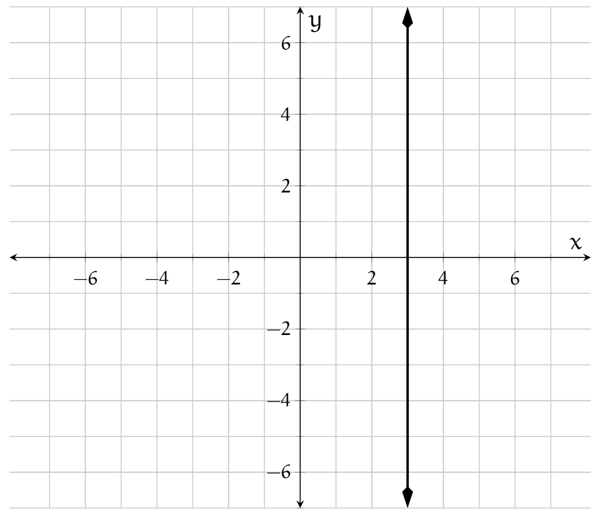


Figure 3.8.3: Vertical Line

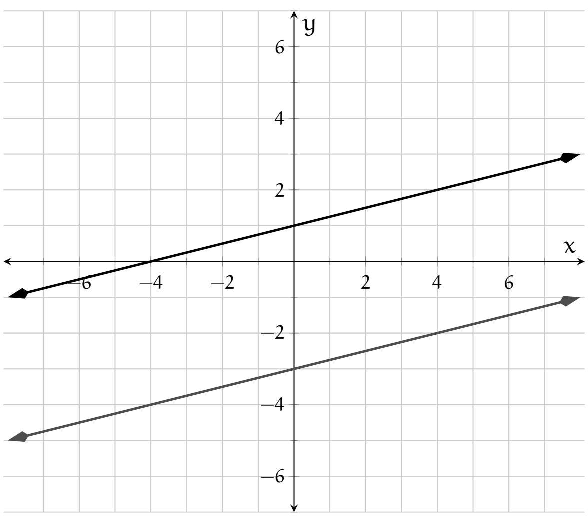


Figure 3.8.4: Two Parallel Lines

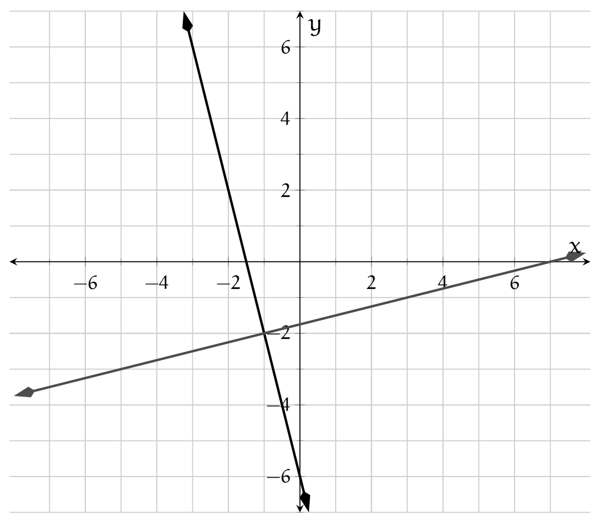


Figure 3.8.5: Two Perpendicular Lines

3.8.1 Horizontal Lines and Vertical Lines

We learned in Section 3.7 that all lines can be written in standard form (3.7.1). When either A or B equal 0, we end up with a horizontal or vertical line, as we will soon see. Let's take the standard form line equation,

$Ax + Bx = C$, and one at a time let $A = 0$ and $B = 0$ and simplify each equation.

$$\begin{aligned} Ax + By &= C \\ 0x + By &= C \\ By &= C \\ y &= \frac{C}{B} \\ y &= k \end{aligned} \qquad\qquad \begin{aligned} Ax + By &= C \\ Ax + 0y &= C \\ Ax &= C \\ x &= \frac{C}{A} \\ x &= h \end{aligned}$$

At the end we just renamed the constant numbers $\frac{C}{B}$ and $\frac{C}{A}$ to k and h because of tradition. What is important, is that you view h and k (as well as A, B, and C) as constants: numbers that have some specific value and don't change in the context of one problem.

Think about one of these equations: $y = k$. It says that the y-value is the same no matter where you are on the line. If you wanted to plot points on this line, you are free to move far to the left or far to the right on the x-axis, but then you always move up (or down) to make the y-value equal k. What does such a line look like?

Example 3.8.6
Let's plot the line with equation $y = 3$. (Note that this is the same as $0x+1y = 3$.) To plot some points, it doesn't matter what x-values we use. All that matters is that y is *always* 3.

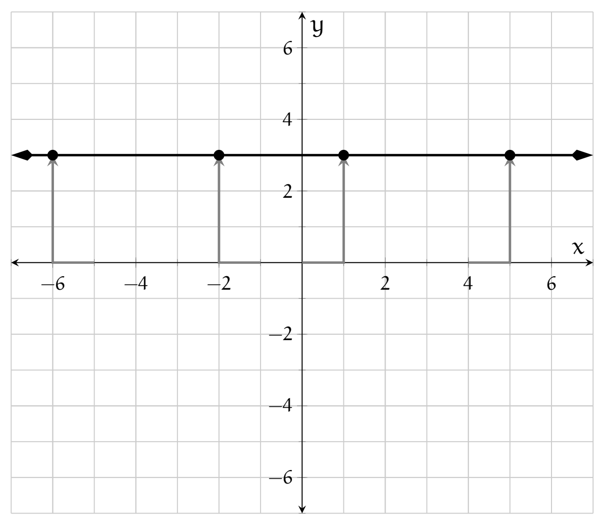


Figure 3.8.7: $y = 3$

A line like this is **horizontal**, parallel to the horizontal axis. All lines with an equation in the form

$$y = k$$

(or, in standard form, $0x + By = C$) are **horizontal**.

Example 3.8.8
Let's plot the line with equation $x = 5$. Points on the line always have $x = 5$, so if we wanted to make a table for plotting points, we are *required* to make all of the x-values be 5. From there, we have complete freedom to let y take any value. Here we take some random y-values.

x	y
5	-6
5	-2
5	1
5	5

These points are plotted in Figure 3.8.9.

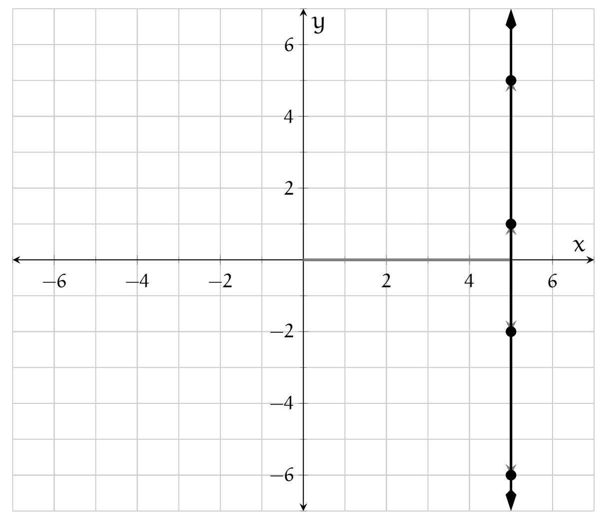


Figure 3.8.9: $x = 5$

Note that the equation for this line is the same as $x + 0y = 5$. An alternative for making a table is to choose our y-values first and substitute them into the equation.

y	$x + 0y = 5 \implies x = 5$	Ordered Pair
-6	$x + 0(-6) = 5 \implies x = 5$	$(5, -6)$
-2	$x + 0(-2) = 5 \implies x = 5$	$(5, -2)$
1	$x + 0(1) = 5 \implies x = 5$	$(5, 1)$
5	$x + 0(5) = 5 \implies x = 5$	$(5, 5)$

In each case no matter what value y is, we find that the equation tells us that $x = 5$.

A line like this is **vertical**, parallel to the vertical axis. All lines with an equation in the form

$$x = h$$

(or, in standard form, $Ax + 0y = C$) are vertical.

Example 3.8.10 Zero Slope. In Checkpoint 3.4.17, we learned that a horizontal line's slope is 0, because the distance doesn't change as time moves on. So the numerator in the slope formula (3.4.3) is 0. Now, if we know a line's slope and its y-intercept, we can use slope-intercept form (3.5.1) to write its equation:

$$\begin{aligned} y &= mx + b \\ y &= 0x + b \\ y &= b \end{aligned}$$

This provides us with an alternative way to think about equations of horizontal lines. They have a certain y-intercept b, and they have slope 0.

We use horizontal lines to model scenarios where there is no change in y-values, like when Kato stopped for 12 hours (he deserved a rest)!

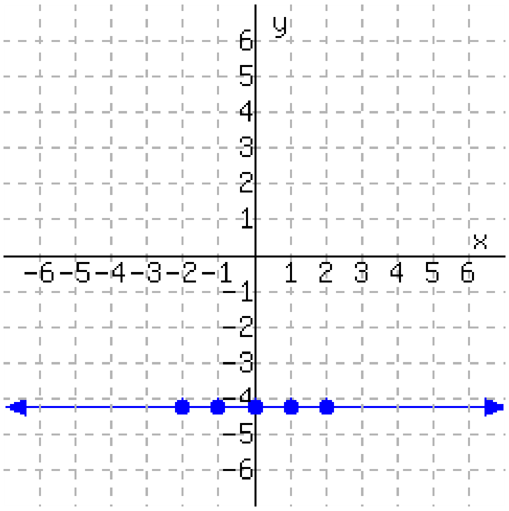
Checkpoint 3.8.11 Plotting Points. Suppose you need to plot the equation $y = -4.25$. Since the equation is in "$y =$" form, you decide to make a table of points. Fill out some points for this table.

x	y
__________	__________
__________	__________
__________	__________
__________	__________
__________	__________

Explanation. We can use whatever values for x that we like, as long as they are all different. The equation tells us the y-value has to be -4.25 each time.

x	y
-2	-4.25
-1	-4.25
0	-4.25
1	-4.25
2	-4.25

Now that we have a table, we could use its values to assist with plotting the line.



Example 3.8.12 Slope of a Vertical Line. What is the slope of a vertical line? Figure 3.8.13 shows three lines passing through the origin, each steeper than the last. In each graph, you can see a slope triangle that uses a "run" of 1 unit.

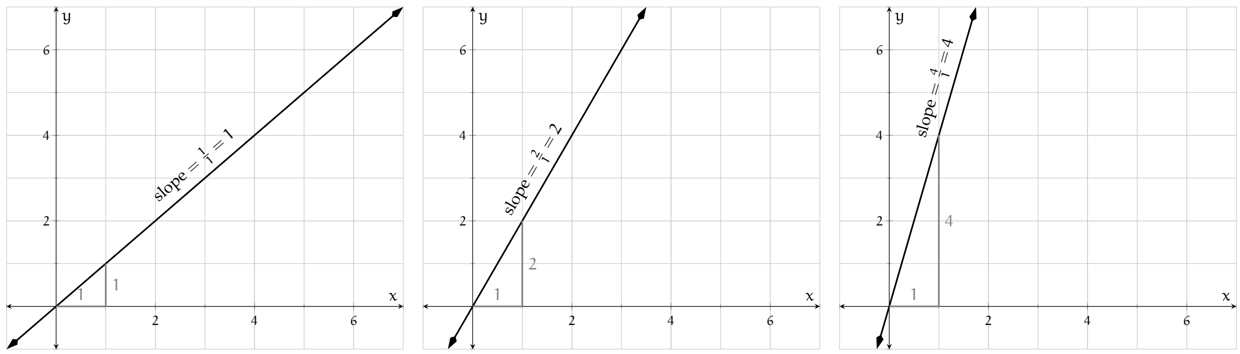


Figure 3.8.13

If we continued making the line steeper and steeper until it was vertical, the slope triangle would still have a "run" of 1, but the "rise" would become larger and larger with no upper limit. The slope would be $m = \frac{\text{very large}}{1}$. Actually if the line is vertical, the "rise" segment we've drawn, will never intercept the line. So the slope of a vertical line can be thought of as "infinitely large." We usually say that the slope of a vertical line is *undefined*. Some people say that a vertical line *has no slope*.

Fact 3.8.14 *The slope of a vertical line is undefined.*

Remark 3.8.15 Be careful not to mix up "no slope" (which means "its slope is undefined") with "has slope 0." If a line has slope 0, it *does* have a slope.

If you are familiar with NBA basketball, some players wear number 0. That's not the same thing as "not having a number". This is similar to the situation with having slope 0 versus not having slope.

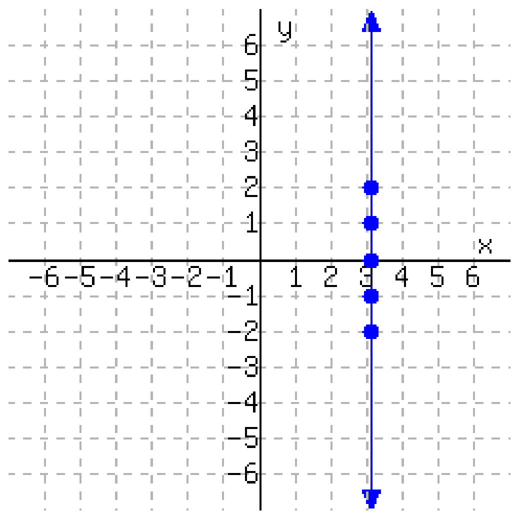
Checkpoint 3.8.16 Plotting Points. Suppose you need to plot the equation $x = 3.14$. You decide to try making a table of points. Fill out some points for this table.

x	y
__________	__________
__________	__________
__________	__________
__________	__________
__________	__________

Explanation. Since the equation says x is always the number 3.14, we have to use this for the x value in all the points. This is different from how we would plot a "$y =$" equation, where we would use several different x-values. We can use whatever values for y that we like, as long as they are all different.

x	y
3.14	-2
3.14	-1
3.14	0
3.14	1
3.14	2

The reason we made a table was to help with plotting the line.



Example 3.8.17 Let x represent the price of a new 60-inch television at Target on Black Friday (which was \$650), and let y be the number of hours you will watch something on this TV over its lifetime. What is the relationship between x and y?

Well, there is no getting around the fact that $x = 650$. As for y, without any extra information about your viewing habits, it could theoretically be as low as 0 or it could be anything larger than that. If we graph this scenario, we have to graph the equation $x = 650$ which we now know to give a vertical line, and we get Figure 3.8.18.

Figure 3.8.18: New TV: hours watched versus purchase price; negative y-values omitted since they make no sense in context

Horizontal Lines	Vertical Lines
A line is **horizontal** if and only if its equation can be written $$y = k$$ for some constant k.	A line is **vertical** if and only if its equation can be written $$x = h$$ for some constant h.
In standard form (3.7.1), any line with equation $$0x + By = C$$ is horizontal.	In standard form (3.7.1), any line with equation $$Ax + 0y = C$$ is vertical.
If the line with equation $y = k$ is horizontal, it has a y-intercept at $(0, k)$ and has slope 0.	If the line with equation $x = h$ is vertical, it has an x-intercept at $(h, 0)$ and its slope is *undefined*. Some say it has *no* slope, and some say the slope is *infinitely large*.
In the slope-intercept form (3.5.1), any line with equation $$y = 0x + b$$ is horizontal.	It's impossible to write the equation of a vertical line in slope-intercept form (3.5.1), because vertical lines do not have a defined slope.

Figure 3.8.19: Summary of Horizontal and Vertical Line Equations

3.8.2 Parallel Lines

Example 3.8.20
Two trees were planted in the same year, and their growth over time is modeled by the two lines in Figure 3.8.21. Use linear equations to model each tree's growth, and interpret their meanings in this context.
We can see Tree 1's equation is $y = \frac{2}{3}x + 2$, and Tree 2's equation is $y = \frac{2}{3}x + 5$. Tree 1 was 2 feet tall when it was planted, and Tree 2 was 5 feet tall when it was planted. Both trees have been growing at the same rate, $\frac{2}{3}$ feet per year, or 2 feet every 3 years.

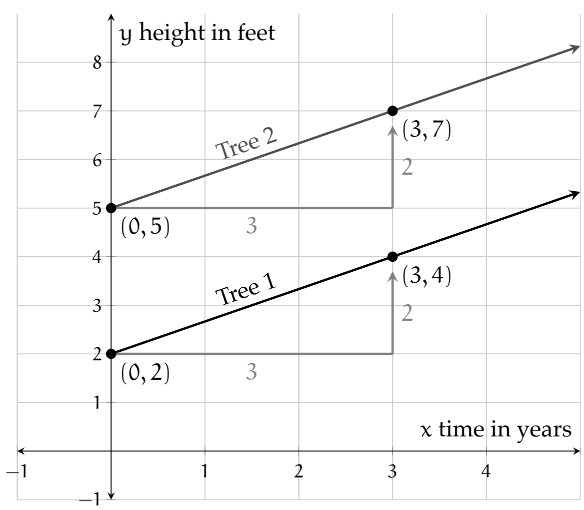


Figure 3.8.21: Two Trees' Growth Chart

An important observation right now is that those two lines are parallel. Why? For lines with positive slopes, the bigger a line's slope, the steeper the line is slanted. As a result, if two lines have the same slope, they are slanted at the same angle, thus they are parallel.

Fact 3.8.22 *Any two vertical lines are parallel to each other. For two non-vertical lines, they are parallel if and only if they have the same slope.*

Checkpoint 3.8.23 A line ℓ is parallel to the line with equation $y = 17.2x - 340.9$, but ℓ has y-intercept at $(0, 128.2)$. What is an equation for ℓ?

Explanation. Parallel lines have the same slope, and the slope of $y = 17.2x - 340.9$ is 17.2. So ℓ has slope 17.2. And we have been given that ℓ's y-intercept is at $(0, 128.2)$. So we can use slope-intercept form to write its equation as

$$y = 17.2x + 128.2.$$

Checkpoint 3.8.24 A line κ is parallel to the line with equation $y = -3.5x + 17$, but κ passes through the point $(-12, 23)$. What is an equation for κ?

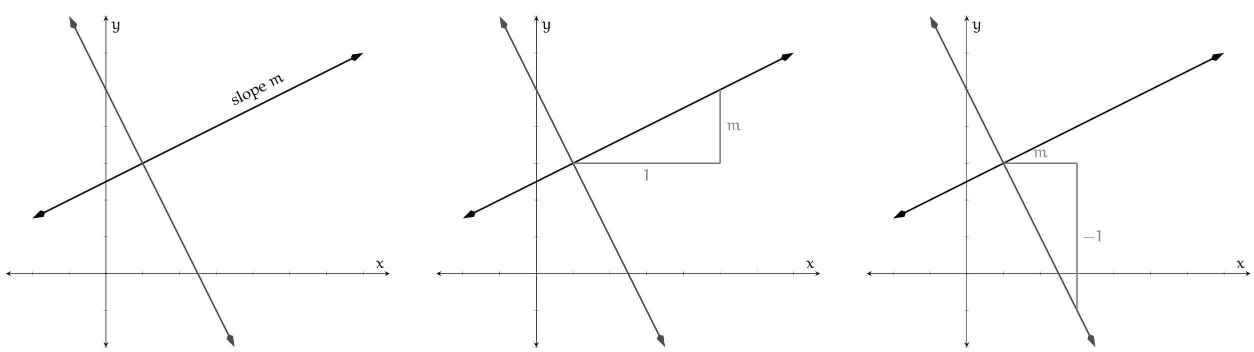
Explanation. Parallel lines have the same slope, and the slope of $y = -3.5x + 17$ is -3.5. So κ has slope -3.5. And we know a point that κ passes through, so we can use point-slope form to write its equation as

$$y = -3.5(x + 12) + 23.$$

3.8.3 Perpendicular Lines

The slopes of two perpendicular lines have a special relationship too.

Figure 3.8.25 walks you through an explanation of this relationship.



(a) Two generic perpendicular lines, where one has slope m.

(b) Since the one slope is m, we can draw a slope triangle with "run" 1 and "rise" m.

(c) A *congruent* slope triangle can be drawn for the perpendicular line. It's legs have the same lengths, but in different positions, and one is negative.

Figure 3.8.25: The relationship between slopes of perpendicular lines

The second line in Figure 3.8.25 has slope

$$\frac{\Delta y}{\Delta x} = \frac{-1}{m} = -\frac{1}{m}.$$

Fact 3.8.26 *A vertical line and a horizontal line are perpendicular. For two lines that are neither vertical nor horizontal, they are perpendicular if and only if the slope of one is the negative reciprocal of the slope of the other. That is, if one has slope m, the other has slope $-\frac{1}{m}$.*

Another way to say this is that the product of the slopes of two perpendicular lines is -1 (assuming both of the lines have a slope in the first place). That is, if there are two perpendicular lines and we let m_1 and m_2 represent their slopes, then $m_1 \cdot m_2 = -1$.

Not convinced? Here are three pairs of perpendicular lines where we can see if the pattern holds.

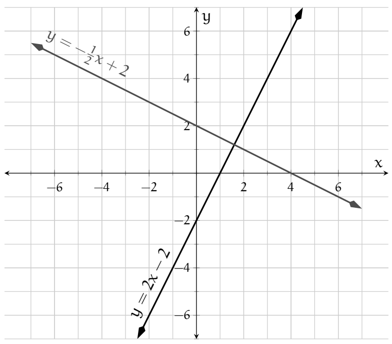


Figure 3.8.27: Graphing $y = 2x - 2$ and $y = -\frac{1}{2}x + 2$. Note the relationship between their slopes: $2 = -\frac{1}{-1/2}$

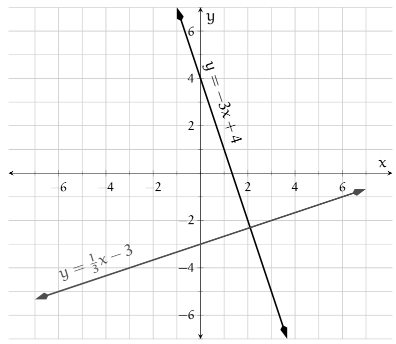


Figure 3.8.28: Graphing $y = -3x + 4$ and $y = \frac{1}{3}x - 3$. Note the relationship between their slopes: $-3 = -\frac{1}{1/3}$

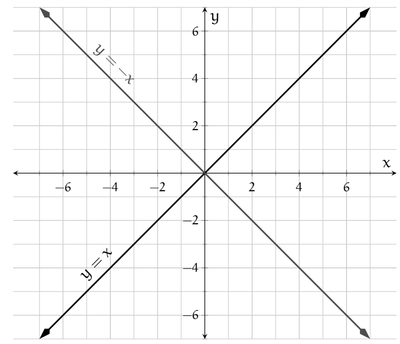


Figure 3.8.29: Graphing $y = x$ and $y = -x$. Note the relationship between their slopes: $1 = -\frac{1}{-1}$

Example 3.8.30 Line A passes through $(-2, 10)$ and $(3, -10)$. Line B passes through $(-4, -4)$ and $(8, -1)$. Determine whether these two lines are parallel, perpendicular or neither.

Explanation. We will use the slope formula to find both lines' slopes:

$$\begin{aligned}
\text{Line A's slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-10 - 10}{3 - (-2)} \\
&= \frac{-20}{5} \\
&= -4
\end{aligned}
\qquad
\begin{aligned}
\text{Line B's slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-1 - (-4)}{8 - (-4)} \\
&= \frac{3}{12} \\
&= \frac{1}{4}
\end{aligned}$$

Their slopes are not the same, so those two lines are not parallel.
The product of their slopes is $(-4) \cdot \frac{1}{4} = -1$, which means the two lines are perpendicular.

Checkpoint 3.8.31 Line A and Line B are perpendicular. Line A's equation is $2x + 3y = 12$. Line B passes through the point $(4, -3)$. Find an equation for Line B.

Explanation. First, we will find Line A's slope by rewriting its equation from standard form to slope-intercept form:

$$\begin{aligned}2x + 3y &= 12\\ 3y &= 12 - 2x\\ 3y &= -2x + 12\\ y &= \frac{-2x + 12}{3}\\ y &= -\frac{2}{3}x + 4\end{aligned}$$

So Line A's slope is $-\frac{2}{3}$. Since Line B is perpendicular to Line A, its slope is $-\frac{1}{-\frac{2}{3}} = \frac{3}{2}$. It's also given that Line B passes through $(4, -3)$, so we can write Line B's point-slope form equation:

$$\begin{aligned}y &= m(x - x_0) + y_0\\ y &= \frac{3}{2}(x - 4) - 3\end{aligned}$$

3.8.4 Reading Questions

1. Explain the difference between a line that has no slope and a line that has slope 0.

2. If you make a table of x- and y-values for either a horizontal line or a vertical line, what is going to happen in one of the two columns?

3. If you know two points on one line, and you know two points on a second line, what could you do to determine whether or not the two lines are perpendicular?

3.8.5 Exercises

Review and Warmup

1. Evaluate the following expressions. If the answer is undefined, you may answer with DNE (meaning "does not exist").

a. $\frac{7}{0}$

b. $\frac{0}{7}$

2. Evaluate the following expressions. If the answer is undefined, you may answer with DNE (meaning "does not exist").

a. $\frac{0}{8}$

b. $\frac{8}{0}$

3. A line passes through the points $(5, 8)$ and $(-3, 8)$. Find this line's slope.

4. A line passes through the points $(3, 10)$ and $(-1, 10)$. Find this line's slope.

5. A line passes through the points $(-8, -5)$ and $(-8, 3)$. Find this line's slope.

6. A line passes through the points $(-6, -1)$ and $(-6, 5)$. Find this line's slope.

7. Consider the equation:

$$y = 1$$

Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

☐ $(4, 1)$ ☐ $(-4, 1)$
☐ $(0, 7)$ ☐ $(1, 4)$

8. Consider the equation:

$$y = 1$$

Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

☐ $(-6, 1)$ ☐ $(5, 1)$
☐ $(0, 9)$ ☐ $(1, 2)$

9. Consider the equation:

$$x + 1 = 0$$

Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

☐ $(-1, 0)$ ☐ $(-1, 3)$
☐ $(0, -6)$ ☐ $(1, -1)$

10. Consider the equation:

$$x + 1 = 0$$

Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.

☐ $(1, -1)$ ☐ $(-1, 0)$
☐ $(0, -8)$ ☐ $(-1, 4)$

Tables for Horizontal and Vertical Lines

11. Fill out this table for the equation $y = 8$. The first row is an example.

x	y	Points
-3	8	$(-3, 8)$
-2	______	____________
-1	______	____________
0	______	____________
1	______	____________
2	______	____________

12. Fill out this table for the equation $y = 9$. The first row is an example.

x	y	Points
-3	9	$(-3, 9)$
-2	______	____________
-1	______	____________
0	______	____________
1	______	____________
2	______	____________

13. Fill out this table for the equation $x = -1$. The first row is an example.

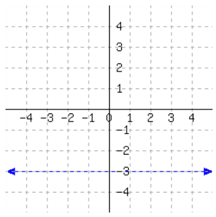
x	y	Points
-1	-3	$(-1, -3)$
______	-2	____________
______	-1	____________
______	0	____________
______	1	____________
______	2	____________

14. Fill out this table for the equation $x = -9$. The first row is an example.

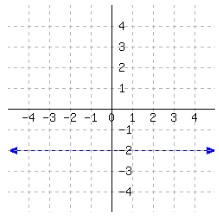
x	y	Points
-9	-3	$(-9, -3)$
______	-2	____________
______	-1	____________
______	0	____________
______	1	____________
______	2	____________

Line Equations A line's graph is shown. Write an equation for the line.

15.



16.



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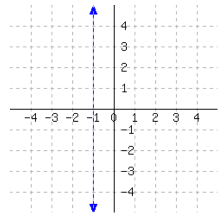
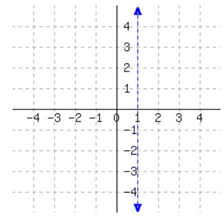


18.



19. A line passes through the points $(-2, 3)$ and $(-3, 3)$. Find an equation for this line.
An equation for this line is
[ ].

20. A line passes through the points $(5, 6)$ and $(2, 6)$. Find an equation for this line.
An equation for this line is
[ ].

21. A line passes through the points $(8, 1)$ and $(8, -5)$. Find an equation for this line.
An equation for this line is
[ ].

22. A line passes through the points $(10, -3)$ and $(10, 0)$. Find an equation for this line.
An equation for this line is
[ ].

Intercepts

23. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$x = -8$$

	x-value	y-value	Location
y-intercept	______	______	______________
x-intercept	______	______	______________

24. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$x = -6$$

	x-value	y-value	Location
y-intercept	______	______	______________
x-intercept	______	______	______________

25. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$y = -4$$

	x-value	y-value	Location
y-intercept	______	______	______________
x-intercept	______	______	______________

26. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$y = -1$$

	x-value	y-value	Location
y-intercept	______	______	____________
x-intercept	______	______	____________

Graphs of Horizontal and Vertical Lines

27. Graph the line $y = 1$.

28. Graph the line $y + 5 = 0$.

29. Graph the line $x = 2$.

30. Graph the line $x - 3 = 0$.

Parallel or Perpendicular?

31. Line m passes points $(1, 3)$ and $(-3, -9)$. Line n passes points $(5, 20)$ and $(4, 17)$. These two lines are (☐ parallel ☐ perpendicular ☐ neither parallel nor perpendicular) .

32. Line m passes points $(-21, 28)$ and $(-7, 16)$. Line n passes points $(-35, 33)$ and $(7, -3)$. These two linea are (☐ parallel ☐ perpendicular ☐ neither parallel nor perpendicular) .

33. Line m passes points $(-5, 8)$ and $(5, 6)$. Line n passes points $(-4, -16)$ and $(2, 14)$. These two lines are (☐ parallel ☐ perpendicular ☐ neither parallel nor perpendicular) .

34. Line m passes points $(-5, 15)$ and $(5, -1)$. Line n passes points $(-16, 0)$ and $(-24, -5)$. These two lines are (☐ parallel ☐ perpendicular ☐ neither parallel nor perpendicular) .

35. Line m passes points $(-3, -5)$ and $(-4, -4)$. Line n passes points $(-2, -9)$ and $(1, 3)$. These two lines are (☐ parallel ☐ perpendicular ☐ neither parallel nor perpendicular) .

36. Line m passes points $(6, -8)$ and $(4, -8)$. Line n passes points $(3, 7)$ and $(10, 7)$. These two lines are (☐ parallel ☐ perpendicular ☐ neither parallel nor perpendicular) .

37. Line m passes points $(-6, -2)$ and $(-6, -9)$. Line n passes points $(1, 3)$ and $(1, -1)$. These two lines are (☐ parallel ☐ perpendicular ☐ neither parallel nor perpendicular) .

38. Line m passes points $(-4, -9)$ and $(-4, 0)$. Line n passes points $(3, 10)$ and $(3, 1)$. These two lines are (☐ parallel ☐ perpendicular ☐ neither parallel nor perpendicular) .

Parallel and Perpendicular Line Equations

39. A line passes through the point $(9, 5)$, and it's parallel to the line $y = -2$. Find an equation for this line.

40. A line passes through the point $(-3, -2)$, and it's parallel to the line $y = 1$. Find an equation for this line.

41. A line passes through the point $(-10, 5)$, and it's parallel to the line $x = 3$. Find an equation for this line.

42. A line passes through the point $(4, -7)$, and it's parallel to the line $x = 5$. Find an equation for this line.

43. Line k has the equation $y = 4x - 2$.
Line ℓ is parallel to line k, but passes through the point $(4, 18)$.
Find an equation for line ℓ in both point-slope form and slope-intercept form.
An equation for ℓ in point-slope form is: ______.
An equation for ℓ in slope-intercept form is: ______.

44. Line k has the equation $y = 5x + 5$.
Line ℓ is parallel to line k, but passes through the point $(1, -5)$.
Find an equation for line ℓ in both point-slope form and slope-intercept form.
An equation for ℓ in point-slope form is: ______.
An equation for ℓ in slope-intercept form is: ______.

45. Line k has the equation $y = -\frac{1}{7}x - 3$.
Line ℓ is parallel to line k, but passes through the point $(-2, -\frac{5}{7})$.
Find an equation for line ℓ in both point-slope form and slope-intercept form.
An equation for ℓ in point-slope form is: ______.
An equation for ℓ in slope-intercept form is: ______.

46. Line k has the equation $y = -\frac{2}{9}x + 10$.
Line ℓ is parallel to line k, but passes through the point $(3, -\frac{11}{3})$.
Find an equation for line ℓ in both point-slope form and slope-intercept form.
An equation for ℓ in point-slope form is: ______.
An equation for ℓ in slope-intercept form is: ______.

47. Line k has the equation $y = -x + 9$.
Line ℓ is perpendicular to line k, and passes through the point $(-1, -4)$.
Find an equation for line ℓ.

48. Line k has the equation $y = 4x + 2$.
Line ℓ is perpendicular to line k and passes through the point $(2, \frac{9}{2})$.
Find an equation for line ℓ in both point-slope form and slope-intercept form.
An equation for ℓ in point-slope form is: ______.
An equation for ℓ in slope-intercept form is: ______.

49. Line k's equation is $y = -\frac{6}{5}x - 3$.
Line ℓ is perpendicular to line k and passes through the point $(2, \frac{14}{3})$.
Find an equation for line ℓ in both point-slope form and slope-intercept form.
An equation for ℓ in point-slope form is: ______.
An equation for ℓ in slope-intercept form is: ______.

50. Line k has the equation $x - 9y = -45$.
Line ℓ is perpendicular to line k and passes through the point $(1, -8)$.
Find an equation for line ℓ in both point-slope form and slope-intercept form.
An equation for ℓ in point-slope form is: ______.
An equation for ℓ in slope-intercept form is: ______.

Challenge

51. Prove that a triangle with vertices at the points $(1, 1)$, $(-4, 4)$, and $(-3, 0)$ is a right triangle.

3.9 Summary of Graphing Lines

The previous several sections have demonstrated several methods for plotting a graph of a linear equation. In this section, we review these methods.

We have learned three forms to write a linear equation:

- slope-intercept form

$$y = mx + b$$

- point-slope form

$$y = m(x - x_0) + y_0$$

- standard form

$$Ax + By = C$$

We have studied two special types of line:

- horizontal line

$$y = k$$

- vertical line

$$x = h$$

We have practiced three ways to graph a line:

- building a table of x- and y-values
- plotting *one* point (often the y-intercept) and drawing slope triangles
- plotting its x-intercept and y-intercept

3.9.1 Graphing Lines in Slope-Intercept Form

In the following examples we will graph $y = -2x + 1$, which is in slope-intercept form (3.5.1), with different methods and compare them.

Example 3.9.2 Building a Table of x- and y-values. First, we will graph $y = -2x + 1$ by building a table of values. In theory this method can be used for any type of equation, linear or not.

x-value	y-value	Point
-2	$y = -2(-2) + 1 = 5$	$(-2, 5)$
-1	$y = -2(-1) + 1 = 3$	$(-1, 3)$
0	$y = -2(0) + 1 = 1$	$(0, 1)$
1	$y = -2(1) + 1 = -1$	$(1, -1)$
2	$y = -2(2) + 1 = -3$	$(2, -3)$

Figure 3.9.3: Table for $y = -2x + 1$

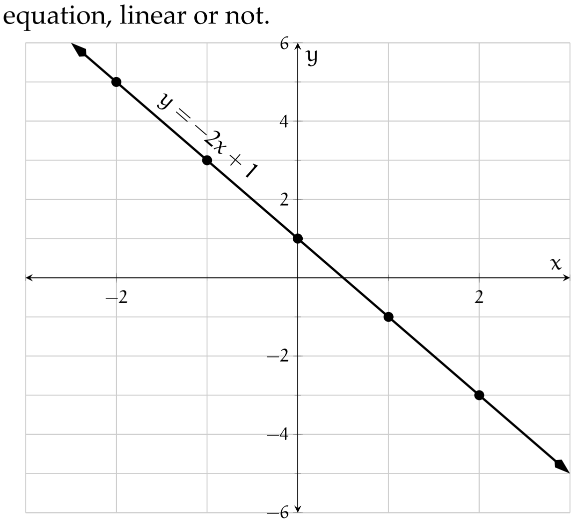


Figure 3.9.4: Graphing $y = -2x + 1$ using a table of values

Example 3.9.5 Using Slope Triangles. Although making a table is straightforward, the slope triangle method is faster and reinforces the true meaning of slope. With the slope triangle method, we first identify some point on the line. Given a line in slope-intercept form (3.5.1), we know the y-intercept. For the line $y = 2x + 1$, the y-intercept is $(0, 1)$. Plot this first, and then we can draw slope triangles in both directions to find more points.

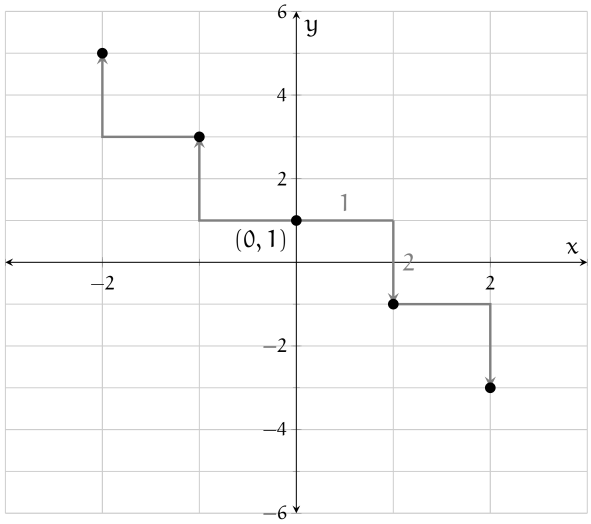


Figure 3.9.6: Marking a point and some slope triangles

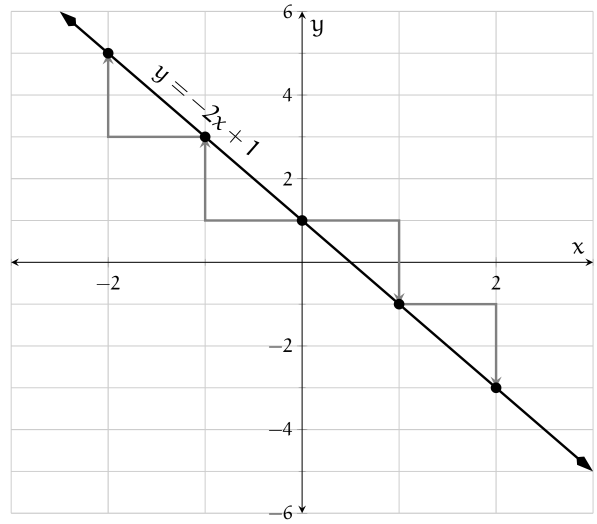


Figure 3.9.7: Graphing $y = -2x + 1$ by slope triangles

Compared to the table method, the slope triangle method:

- is less straightforward;
- doesn't take the time and space to make a table;
- doesn't involve lots of calculations where you might make a human error;
- shows slope triangles, which reinforces the meaning of slope.

Example 3.9.8 Using intercepts. If we use the x- and y-intercepts to plot $y = -2x + 1$, we have some calculation to do. While it is apparent that the y-intercept is at $(0, 1)$, where is the x-intercept?

Set $y = 0$.

$$\begin{aligned} y &= -2x + 1 \\ 0 &= -2x + 1 \\ 0 - 1 &= -2x \\ -1 &= -2x \\ \frac{-1}{-2} &= x \\ \frac{1}{2} &= x \end{aligned}$$

So the x-intercept is at $\left(\frac{1}{2}, 0\right)$. Plotting both intercepts:

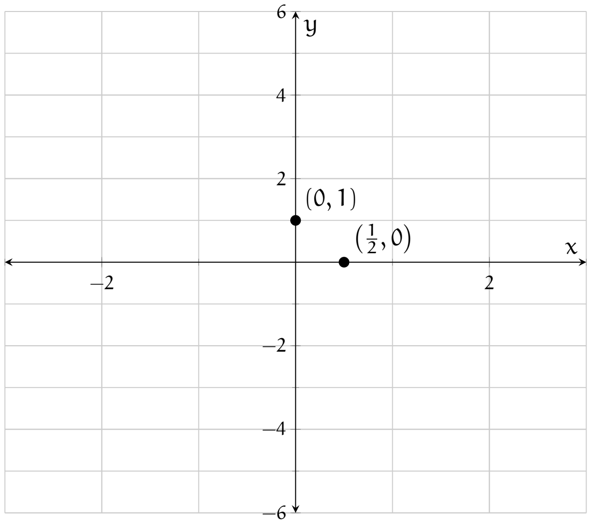


Figure 3.9.9: Marking intercepts

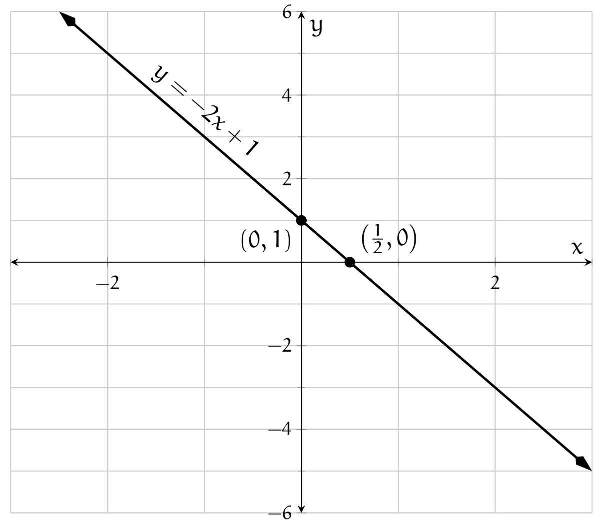


Figure 3.9.10: Using slope triangles

This worked, but here are some observations about why this method is not the greatest.

- We had to plot a point with a fraction in its coordinates.
- We only plotted two points and they turned out very close to each other, so even the slightest inaccuracy in our drawing skills could result in a line that is way off.

When a line is presented in slope-intercept form (3.5.1) and b is an integer, our opinion is that the slope triangle method is the best choice for making its graph.

3.9.2 Graphing Lines in Point-Slope Form

When we graph a line in point-slope form (3.6.1) like $y = \frac{2}{3}(x + 1) + 3$, the slope triangle method is the obvious choice. We can see a point on the line, $(-1, 3)$, and the slope is apparent: $\frac{2}{3}$. Here is the graph:

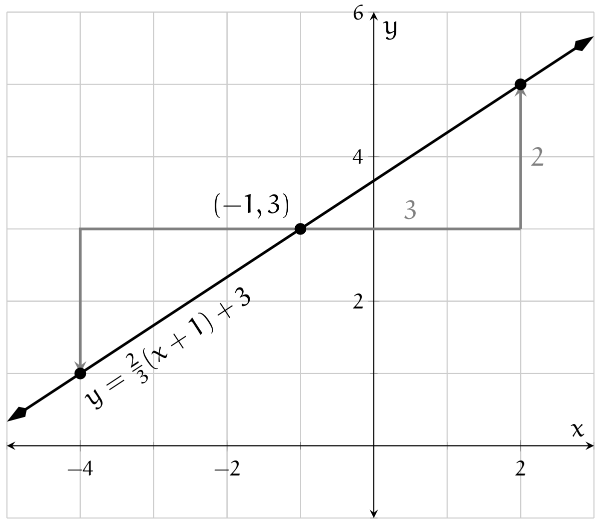


Figure 3.9.11: Graphing $y - 3 = \frac{2}{3}(x+1)$ using slope triangles

Other graphing methods would take more work and miss the purpose of point-slope form (3.6.1). To graph a line in point-slope form (3.6.1), we recommend always using slope triangles.

3.9.3 Graphing Lines in Standard Form

In the following examples we will graph $3x + 4y = 12$, which is in standard form (3.7.1), with different methods and compare them.

Example 3.9.12 Building a Table of x- and y-values. To make a table, we could substitute x for various numbers and use algebra to find the corresponding y-values. Let's start with $x = -2$, planning to move on to $x = -1, 0, 1, 2$.

$$\begin{aligned} 3x + 4y &= 12 \\ 3(-2) + 4y &= 12 \\ -6 + 4y &= 12 \\ 4y&12 + 6 \\ 4y &= 18 \\ y &= \frac{18}{4} = \frac{9}{2} \end{aligned}$$

The first point we found is $\left(-2, \frac{9}{2}\right)$. This has been a lot of calculation, and we ended up with a fraction we will have to plot. *And* we have to repeat this process a few more times to get more points for the table. The table method is generally not a preferred way to graph a line in standard form (3.7.1). Let's look at other options.

Example 3.9.13 Using intercepts. Next, we will try graphing $3x + 4y = 12$ using intercepts. We set up a small table to record the two intercepts:

	x-value	y-value	Intercept
x-intercept		0	
y-intercept	0		

We have to calculate the line's x-intercept by substituting $y = 0$ into the equation:

$$\begin{aligned} 3x + 4y &= 12 \\ 3x + 4(0) &= 12 \\ 3x &= 12 \\ x &= \frac{12}{3} \\ x &= 4 \end{aligned}$$

And similarly for the y-intercept:

$$\begin{aligned} 3x + 4y &= 12 \\ 3(0) + 4y &= 12 \\ 4y &= 12 \\ y &= \frac{12}{4} \\ y &= 3 \end{aligned}$$

So the line's x-intercept is at $(4, 0)$ and its y-intercept is at $(0, 3)$. Now we can complete the table and then graph the line:

	x-value	y-value	Intercepts
x-intercept	4	0	$(4, 0)$
y-intercept	0	3	$(0, 3)$

Figure 3.9.14: Intercepts of $3x + 4y = 12$

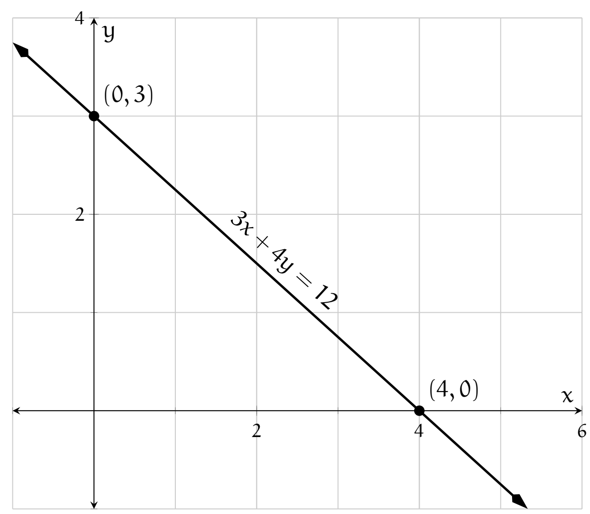


Figure 3.9.15: Graph of $3x + 4y = 12$

Example 3.9.16 With Slope Triangles.
We can always rearrange $3x + 4y = 12$ into slope-intercept form (3.5.1), and then graph it with the slope triangle method:

$$\begin{aligned} 3x + 4y &= 12 \\ 4y &= 12 - 3x \\ 4y &= -3x + 12 \\ y &= \frac{-3x + 12}{4} \\ y &= -\frac{3}{4}x + 3 \end{aligned}$$

With the y-intercept at $(0, 3)$ and slope $-\frac{3}{4}$, we can graph the line using slope triangles:

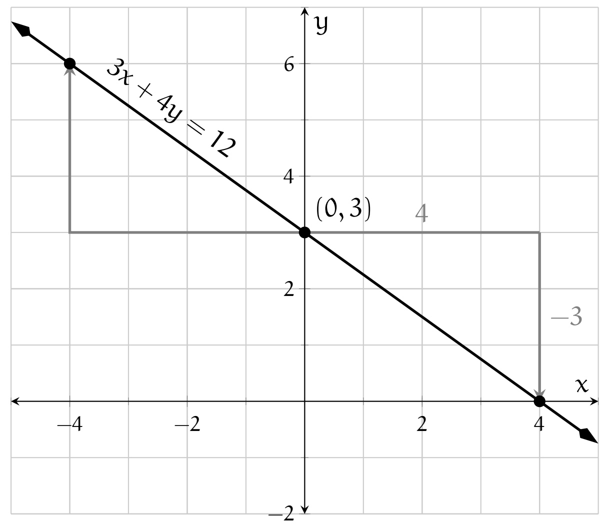


Figure 3.9.17: Graphing $3x + 4y = 12$ using slope triangles

Compared with the intercepts method, the slope triangle method takes more time, but produces points and so makes a more accurate graph. Also sometimes (as with Example 3.7.14) when we graph a standard form equation like $2x - 3y = 0$, the intercepts method doesn't work because both intercepts are actually at the same point (at the origin), and we have to resort to something else like slope triangles anyway.

Here are some observations about graphing a line equation that is in standard form (3.7.1):

- The intercepts method might be the quickest approach.
- The intercepts method only tells us two intercepts of the line. When we need to know more information, like the line's slope, and get a more accurate graph, we should take the time to convert the equation into slope-intercept form.
- When $C = 0$ in a standard form equation (3.7.1) we cannot use the intercepts method to plot the line anyway.

3.9.4 Graphing Horizontal and Vertical Lines

We learned in Section 3.8 that equations in the form $x = h$ and $y = k$ make vertical and horizontal lines. But perhaps you will one day find yourself not remembering which is which. Making a table and plotting points can quickly remind you which type of equation makes which type of line. Let's build a table for $y = 2$ and another one for $x = -3$:

x-value	y-value	Point
0	2	$(0, 2)$
1	2	$(1, 2)$

Figure 3.9.18: Table of values for $y = 2$

x-value	y-value	Point
-3	0	$(-3, 0)$
-3	1	$(-3, 1)$

Figure 3.9.19: Table of values for $x = -3$

With two points on each line, we can graph them:

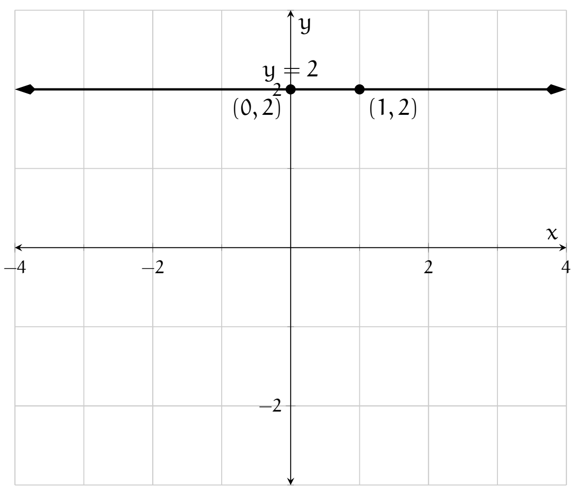


Figure 3.9.20: Graphing $y = 2$

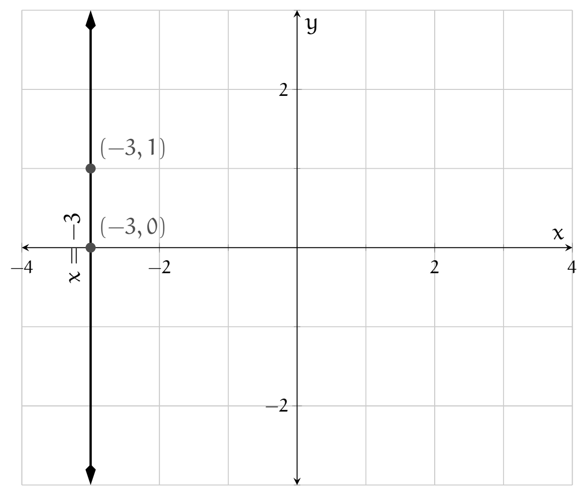


Figure 3.9.21: Graphing $x = -3$

3.9.5 Exercises

Graphing by Table

1. Use a table to make a plot of $y = 4x + 3$.

2. Use a table to make a plot of $y = -5x - 1$.

3. Use a table to make a plot of $y = -\frac{3}{4}x - 1$.

4. Use a table to make a plot of $y = \frac{5}{3}x + 3$.

Graphing Standard Form Equations

5. First find the x- and y-intercepts of the line with equation $6x + 5y = -90$. Then find one other point on the line. Use your results to graph the line.

6. First find the x- and y-intercepts of the line with equation $2x - 3y = -6$. Then find one other point on the line. Use your results to graph the line.

7. First find the x- and y-intercepts of the line with equation $3x + y = -9$. Then find one other point on the line. Use your results to graph the line.

8. First find the x- and y-intercepts of the line with equation $-15x + 3y = -3$. Then find one other point on the line. Use your results to graph the line.

9. First find the x- and y-intercepts of the line with equation $4x + 3y = -3$. Then find one other point on the line. Use your results to graph the line.

10. First find the x- and y-intercepts of the line with equation $-4x - 5y = 5$. Then find one other point on the line. Use your results to graph the line.

11. First find the x- and y-intercepts of the line with equation $5x - 3y = 0$. Then find one other point on the line. Use your results to graph the line.

12. First find the x- and y-intercepts of the line with equation $2x + 9y = 0$. Then find one other point on the line. Use your results to graph the line.

Graphing Slope-Intercept Equations

13. Use the slope and y-intercept from the line $y = -5x$ to plot the line. Use slope triangles.

14. Use the slope and y-intercept from the line $y = 3x - 6$ to plot the line. Use slope triangles.

15. Use the slope and y-intercept from the line $y = -\frac{2}{5}x + 2$ to plot the line. Use slope triangles.

16. Use the slope and y-intercept from the line $y = \frac{10}{3}x - 3$ to plot the line. Use slope triangles.

Graphing Horizontal and Vertical Lines

17. Plot the line $y = 1$.

18. Plot the line $y = -4$.

19. Plot the line $x = -8$.

20. Plot the line $x = 5$.

Choosing the Best Method to Graph Lines

21. Use whatever method you think best to plot $y = 2x + 2$.

22. Use whatever method you think best to plot $y = -3x + 6$.

23. Use whatever method you think best to plot $y = -\frac{3}{4}x - 1$.

24. Use whatever method you think best to plot $y = \frac{5}{3}x - 3$.

25. Use whatever method you think best to plot $y = -\frac{3}{4}(x - 5) + 2$.

26. Use whatever method you think best to plot $y = \frac{2}{5}(x + 1) - 3$.

27. Use whatever method you think best to plot $3x + 2y = 6$.

28. Use whatever method you think best to plot $5x - 4y = 8$.

29. Use whatever method you think best to plot $3x - 4y = 0$.

30. Use whatever method you think best to plot $9x + 6y = 0$.

31. Use whatever method you think best to plot $x = -3$.

32. Use whatever method you think best to plot $x = 2$.

33. Use whatever method you think best to plot $y = -7$.

34. Use whatever method you think best to plot $y = 5$.

3.10 Graphing Lines Chapter Review

3.10.1 Cartesian Coordinates

In Section 3.1 we covered the definition of the Cartesian Coordinate System and how to plot points using the x- and y-axes.

Example 3.10.1 On paper, sketch a Cartesian coordinate system with units, and then plot the following points: $(3,2), (-5,-1), (0,-3), (4,0)$.

Explanation.

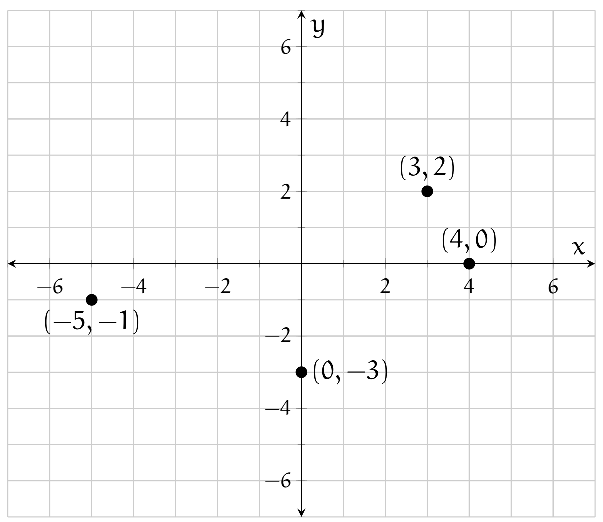


Figure 3.10.2: A Cartesian grid with the four points plotted.

3.10.2 Graphing Equations

In Section 3.2 we covered how to plot solutions to equations to produce a graph of the equation.

Example 3.10.3 Graph the equation $y = -2x + 5$.

Explanation.

x	$y = -2x + 5$	Point
-2		
-1		
0		
1		
2		

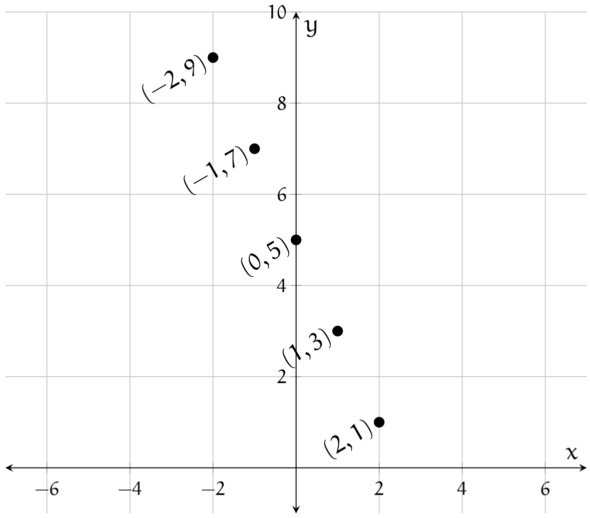
(a) Set up the table

x	$y = -2x + 5$	Point
-2	$-2(-2) + 5 = 9$	$(-2, 9)$
-1	$-2(-1) + 5 = 7$	$(-1, 7)$
0	$-2(0) + 5 = 5$	$(0, 5)$
1	$-2(1) + 5 = 3$	$(1, 3)$
2	$-2(2) + 5 = 1$	$(2, 1)$

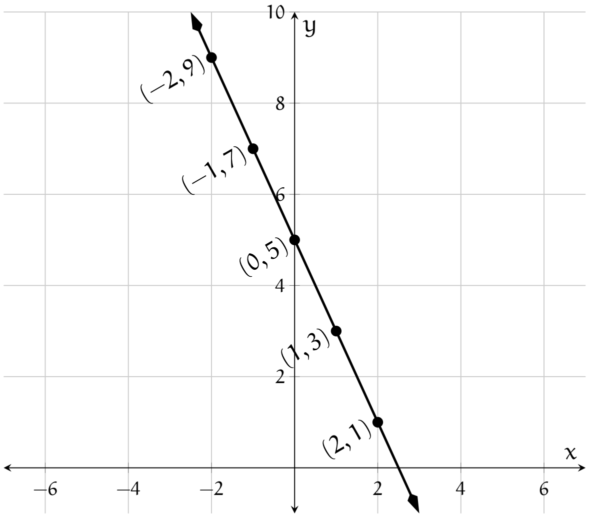
(b) Complete the table

Figure 3.10.4: Making a table for $y = -2x + 5$

We use points from the table to graph the equation. First, plot each point carefully. Then, connect the points with a smooth curve. Here, the curve is a straight line. Lastly, we can communicate that the graph extends further by sketching arrows on both ends of the line.



(a) Use points from the table



(b) Connect the points in whatever pattern is apparent

Figure 3.10.5: Graphing the Equation $y = -2x + 5$

3.10.3 Exploring Two-Variable Data and Rate of Change

In Section 3.3 we covered how to find patterns in tables of data and how to calculate the rate of change between points in data.

Example 3.10.6
Write an equation in the form $y = \ldots$ suggested by the pattern in the table.

x	y
0	−4
1	−6
2	−8
3	−10

Figure 3.10.7: A table of linear data.

Explanation.

We consider how the values change from one row to the next. From row to row, the x-value increases by 1. Also, the y-value decreases by 2 from row to row.

	x	y	
	0	−4	
$+1 \rightarrow$	1	−6	$\leftarrow -2$
$+1 \rightarrow$	2	−8	$\leftarrow -2$
$+1 \rightarrow$	3	−10	$\leftarrow -2$

Since row-to-row change is always 1 for x and is always -2 for y, the rate of change from one row to another row is always the same: -2 units of y for every 1 unit of x.

We know that the output for $x = 0$ is $y = -4$. And our observation about the constant rate of change tells us that if we increase the input by x units from 0, the output should decrease by $\overbrace{(-2) + (-2) + \cdots + (-2)}^{x \text{ times}}$, which is $-2x$. So the output would be $-4 - 2x$.

So the equation is $y = -2x - 4$.

3.10.4 Slope

In Section 3.4 we covered the definition of slope and how to use slope triangles to calculate slope. There is also the slope formula (3.4.3) which helps find the slope through any two points.

Example 3.10.8 Find the slope of the line in the following graph.

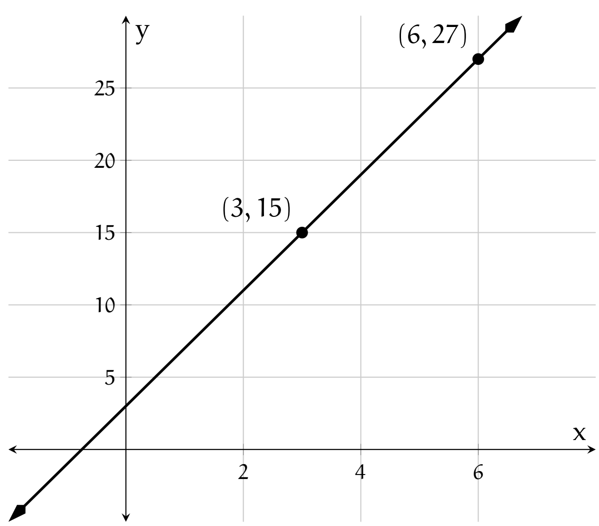


Figure 3.10.9: The line with two points indicated.

Explanation.

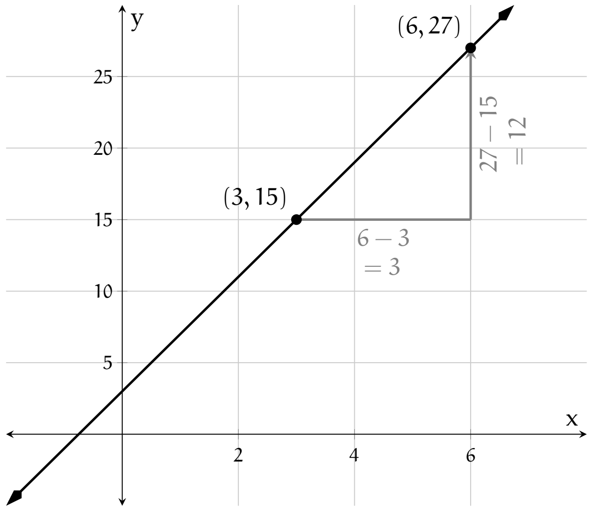


Figure 3.10.10: The line with a slope triangle drawn.

We picked two points on the line, and then drew a slope triangle. Next, we will do:

$$\text{slope} = \frac{12}{3} = 4$$

The line's slope is 4.

Example 3.10.11 Finding a Line's Slope by the Slope Formula. Use the slope formula (3.4.3) to find the slope of the line that passes through the points $(-5, 25)$ and $(4, -2)$.

Explanation.

$$\begin{aligned}\text{slope} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{-2 - (25)}{4 - (-5)}\\ &= \frac{-27}{9}\\ &= -3\end{aligned}$$

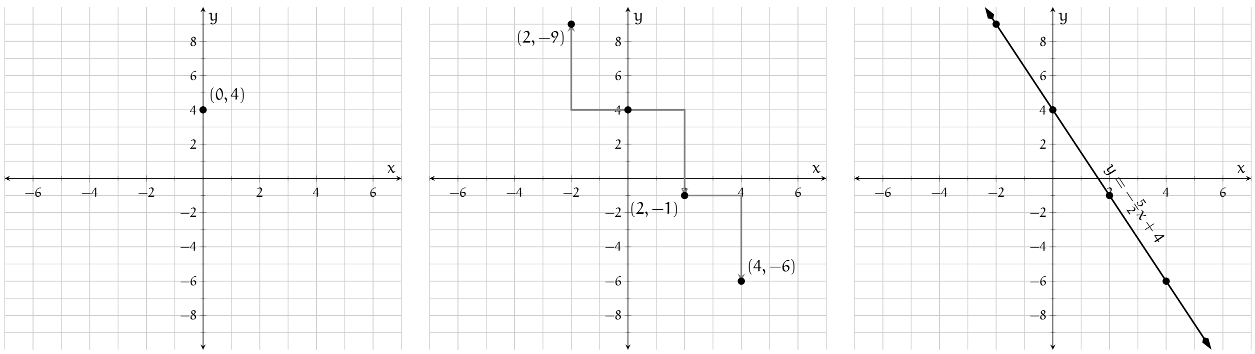
The line's slope is -3.

3.10.5 Slope-Intercept Form

In Section 3.5 we covered the definition of slope intercept-form and both wrote equations in slope-intercept form and graphed lines given in slope-intercept form.

Example 3.10.12 Graph the line $y = -\frac{5}{2}x + 4$.

Explanation.



(a) First, plot the line's y-intercept, $(0, 4)$.

(b) The slope is $-\frac{5}{2} = \frac{-5}{2} = \frac{5}{-2}$. So we can try using a "run" of 2 and a "rise" of -5 or a "run" of -2 and a "rise" of 5.

(c) Arrowheads and labels are encouraged.

Figure 3.10.13: Graphing $y = -\frac{5}{2}x + 4$

Writing a Line's Equation in Slope-Intercept Form Based on Graph. Given a line's graph, we can identify its y-intercept, and then find its slope using a slope triangle. With a line's slope and y-intercept, we can write its equation in the form of $y = mx + b$.

Example 3.10.14
Find the equation of the line in the graph.

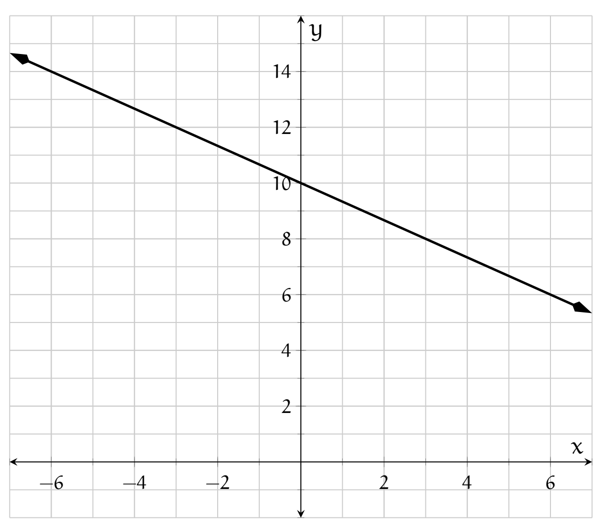


Figure 3.10.15: Graph of a line

Explanation.

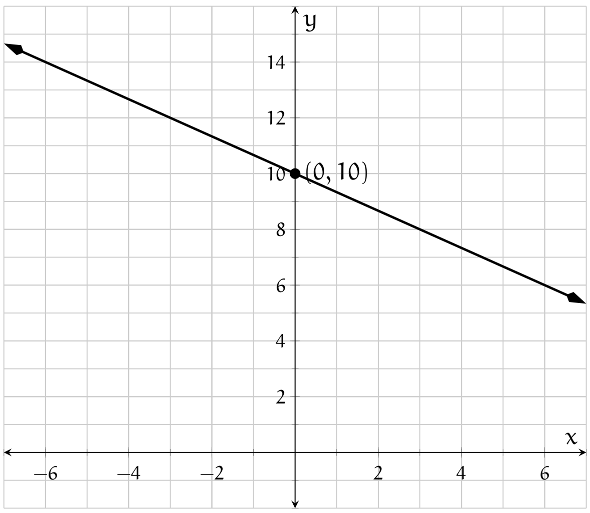


Figure 3.10.16: Identify the line's y-intercept, 10.

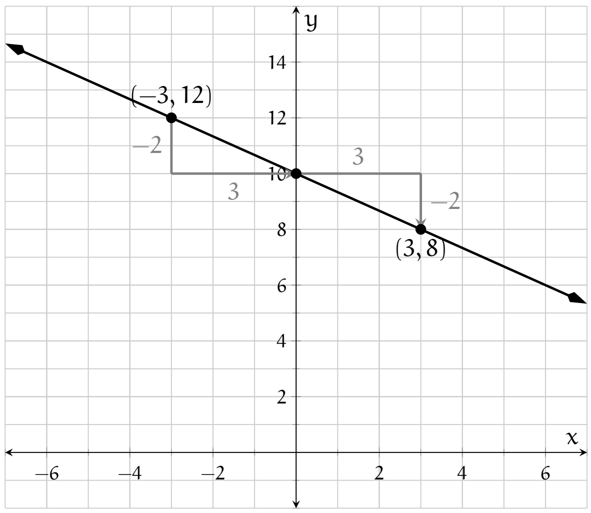


Figure 3.10.17: Identify the line's slope using a slope triangle. Note that we can pick any two points on the line to create a slope triangle. We would get the same slope: $-\frac{2}{3}$

With the line's slope $-\frac{2}{3}$ and y-intercept 10, we can write the line's equation in slope-intercept form: $y = -\frac{2}{3}x + 10$.

3.10.6 Point-Slope Form

In Section 3.6 we covered the definition of point-slope form and both wrote equations in point-slope form and graphed lines given in point-slope form.

Example 3.10.18 A line passes through $(-6, 0)$ and $(9, -10)$. Find this line's equation in point-slope form. .

Explanation. We will use the slope formula (3.4.3) to find the slope first. After labeling those two points as $(\overset{x_1}{-6}, \overset{y_1}{0})$ and $(\overset{x_2}{9}, \overset{y_2}{-10})$, we have:

$$\begin{aligned} \text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-10 - 0}{9 - (-6)} \\ &= \frac{-10}{15} \\ &= -\frac{2}{3} \end{aligned}$$

Now the point-slope equation looks like $y = -\frac{2}{3}(x - x_0) + y_0$. Next, we will use $(9, -10)$ and substitute x_0 with 9 and y_0 with -10, and we have:

$$y = -\frac{2}{3}(x - x_0) + y_0$$

$$\begin{aligned} y &= -\frac{2}{3}(x-9)+(-10) \\ y &= -\frac{2}{3}(x-9)-10 \end{aligned}$$

3.10.7 Standard Form

In Section 3.7 we covered the definition of standard form of a linear equation. We converted equations from standard form to slope-intercept form and vice versa. We also graphed lines from standard form by finding the intercepts of the line.

Example 3.10.19

a. Convert $2x+3y=6$ into slope-intercept form.

b. Convert $y=-\frac{4}{7}x-3$ into standard form.

Explanation.

a.

$$\begin{aligned} 2x+3y &= 6 \\ 2x+3y-2x &= 6-2x \\ 3y &= -2x+6 \\ \frac{3y}{3} &= \frac{-2x+6}{3} \\ y &= \frac{-2x}{3}+\frac{6}{3} \\ y &= -\frac{2}{3}x+2 \end{aligned}$$

The line's equation in slope-intercept form is $y=-\frac{2}{3}x+2$.

b.

$$\begin{aligned} y &= -\frac{4}{7}x-3 \\ 7\cdot y &= 7\cdot\left(-\frac{4}{7}x-3\right) \\ 7y &= 7\cdot\left(-\frac{4}{7}x\right)-7\cdot 3 \\ 7y &= -4x-21 \\ 7y+4x &= -4x-21+4x \\ 4x+7y &= -21 \end{aligned}$$

The line's equation in standard form is $4x+7y=-21$.

To graph a line in standard form, we could first change it to slope-intercept form, and then graph the line by its y-intercept and slope triangles. A second method is to graph the line by its x-intercept and y-intercept.

Example 3.10.20 Graph $2x-3y=-6$ using its intercepts. And then use the intercepts to calculate the line's slope.

Explanation. We calculate the line's x-intercept by substituting $y=0$ into the equation

$$\begin{aligned} 2x-3y &= -6 \\ 2x-3(0) &= -6 \\ 2x &= -6 \\ x &= -3 \end{aligned}$$

So the line's x-intercept is $(-3,0)$.

Similarly, we substitute $x=0$ into the equation to calculate the y-intercept:

$$\begin{aligned} 2x-3y &= -6 \\ 2(0)-3y &= -6 \end{aligned}$$

$$-3y = -6$$
$$y = 2$$

So the line's y-intercept is $(0, 2)$.

With both intercepts' coordinates, we can graph the line:

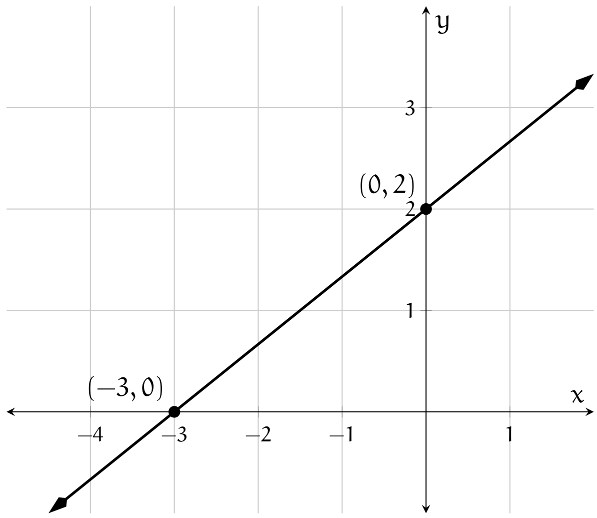


Figure 3.10.21: Graph of $2x - 3y = -6$

Now that we have graphed the line we can read the slope. The rise is 2 units and the run is 3 units so the slope is $\frac{2}{3}$.

3.10.8 Horizontal, Vertical, Parallel, and Perpendicular Lines

In Section 3.8 we studied horizontal and vertical lines. We also covered the relationships between the slopes of parallel and perpendicular lines.

Example 3.10.22 Line m's equation is $y = -2x + 20$. Line n is parallel to m, and line n also passes the point $(4, -3)$. Find an equation for line n in point-slope form.

Explanation. Since parallel lines have the same slope, line n's slope is also -2. Since line n also passes the point $(4, -3)$, we can write line n's equation in point-slope form:

$$y = m(x - x_1) + y_1$$
$$y = -2(x - 4) + (-3)$$
$$y = -2(x - 4) - 3$$

Two lines are perpendicular if and only if the product of their slopes is -1.

Example 3.10.23 Line m's equation is $y = -2x + 20$. Line n is perpendicular to m, and line q also passes the point $(4, -3)$. Find an equation for line q in slope-intercept form.

Explanation. Since line m and q are perpendicular, the product of their slopes is -1. Because line m's slope is given as -2, we can find line q's slope is $\frac{1}{2}$.

Since line q also passes the point $(4, -3)$, we can write line q's equation in point-slope form:

$$y = m(x - x_1) + y_1$$
$$y = \frac{1}{2}(x - 4) + (-3)$$
$$y = \frac{1}{2}(x - 4) - 3$$

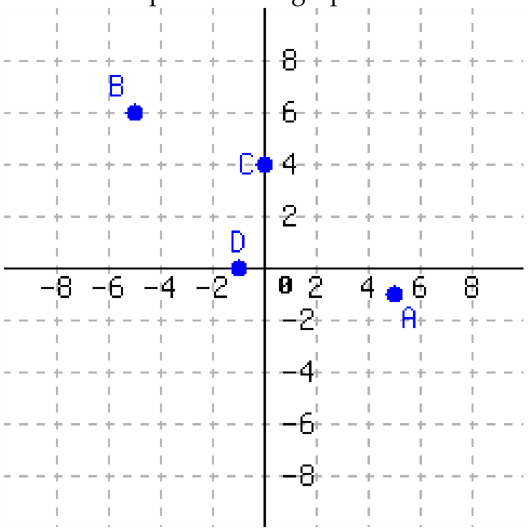
We can now convert this equation to slope-intercept form:

$$y = \frac{1}{2}(x - 4) - 3$$
$$y = \frac{1}{2}x - 2 - 3$$
$$y = \frac{1}{2}x - 5$$

3.10.9 Exercises

1. Sketch the points $(8, 2)$, $(5, 5)$, $(-3, 0)$, $\left(0, -\frac{14}{3}\right)$, $(3, -2.5)$, and $(-5, 7)$ on a Cartesian plane.

2. Locate each point in the graph:



Write each point's position as an ordered pair, like $(1, 2)$.

$A =$ ______ $B =$ ______
$C =$ ______ $D =$ ______

3. Consider the equation
$y = -\frac{7}{8}x - 4$
Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.
☐ $(0, -4)$ ☐ $(-8, 8)$ ☐ $(8, -8)$
☐ $(-24, 17)$

4. Consider the equation
$y = -\frac{3}{2}x - 1$
Which of the following ordered pairs are solutions to the given equation? There may be more than one correct answer.
☐ $(-2, 2)$ ☐ $(10, -13)$
☐ $(0, -1)$ ☐ $(-8, 13)$

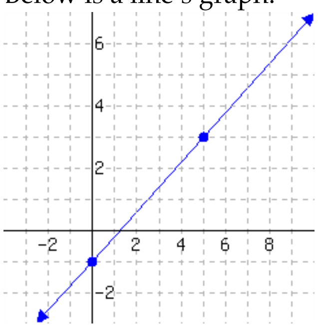
5. Write an equation in the form $y = \ldots$ suggested by the pattern in the table.

x	y
0	-4
1	-2
2	0
3	2

6. Write an equation in the form $y = \ldots$ suggested by the pattern in the table.

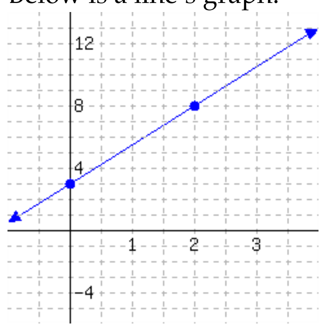
x	y
0	3
1	-2
2	-7
3	-12

7. Below is a line's graph.



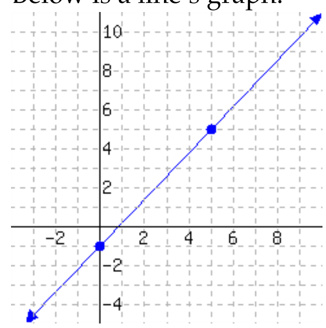
The slope of this line is ______.

8. Below is a line's graph.



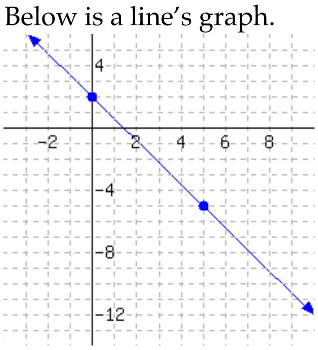
The slope of this line is ______.

9. Below is a line's graph.



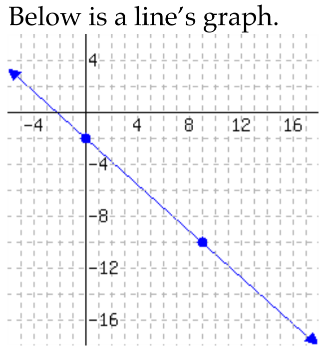
The slope of this line is ______.

10. Below is a line's graph.



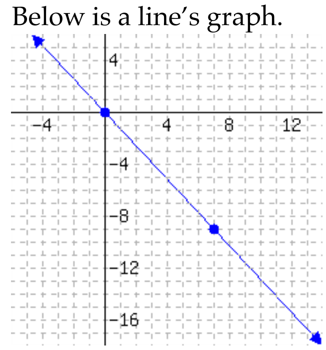
The slope of this line is ______.

11. Below is a line's graph.



The slope of this line is ______.

12. Below is a line's graph.



The slope of this line is ______.

13. A line passes through the points $(-8, 15)$ and $(4, 6)$. Find this line's slope.

14. A line passes through the points $(-6, 6)$ and $(2, -6)$. Find this line's slope.

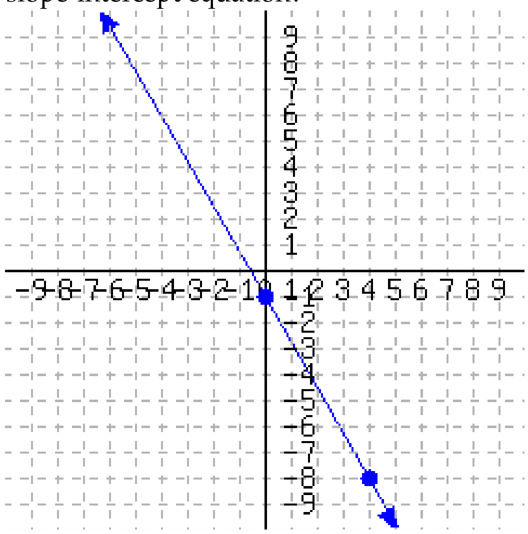
15. A line passes through the points $(4, -4)$ and $(-2, -4)$. Find this line's slope.

16. A line passes through the points $(2, -2)$ and $(-5, -2)$. Find this line's slope.

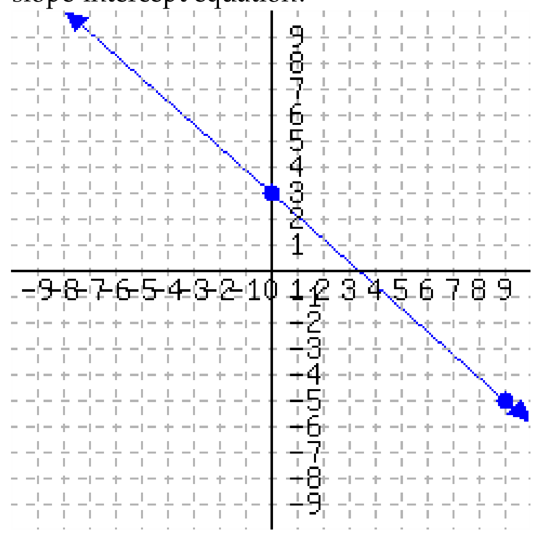
17. A line passes through the points $(1, -1)$ and $(1, 3)$. Find this line's slope.

18. A line passes through the points $(3, -2)$ and $(3, 1)$. Find this line's slope.

19. A line's graph is given. What is this line's slope-intercept equation?



20. A line's graph is given. What is this line's slope-intercept equation?



21. Find the line's slope and y-intercept.
A line has equation $3x - 8y = -32$.

This line's slope is ______.

This line's y-intercept is ______.

22. Find the line's slope and y-intercept.
A line has equation $3x - 2y = 4$.

This line's slope is ______.

This line's y-intercept is ______.

23. A line passes through the points $(9, -6)$ and $(-18, -12)$. Find this line's equation in point-slope form.
Using the point $(9, -6)$, this line's point-slope form equation is

______.

Using the point $(-18, -12)$, this line's point-slope form equation is

______.

24. A line passes through the points $(12, 2)$ and $(8, -1)$. Find this line's equation in point-slope form.
Using the point $(12, 2)$, this line's point-slope form equation is

______.

Using the point $(8, -1)$, this line's point-slope form equation is

______.

25. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way. Each minute, they lose 3 grams. Ten minutes since the experiment started, the remaining gas had a mass of 102 grams.

Let x be the number of minutes that have passed since the experiment started, and let y be the mass of the gas in grams at that moment. Use a linear equation to model the weight of the gas over time.

a. This line's slope-intercept equation is ______.

b. 31 minutes after the experiment started, there would be ______ grams of gas left.

c. If a linear model continues to be accurate, ______ minutes since the experiment started, all gas in the container will be gone.

26. Scientists are conducting an experiment with a gas in a sealed container. The mass of the gas is measured, and the scientists realize that the gas is leaking over time in a linear way. Each minute, they lose 8.8 grams. Seven minutes since the experiment started, the remaining gas had a mass of 334.4 grams.

Let x be the number of minutes that have passed since the experiment started, and let y be the mass of the gas in grams at that moment. Use a linear equation to model the weight of the gas over time.

a. This line's slope-intercept equation is ______.

b. 31 minutes after the experiment started, there would be ______ grams of gas left.

c. If a linear model continues to be accurate, ______ minutes since the experiment started, all gas in the container will be gone.

27. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$6x + 5y = -90$$

	x-value	y-value	Location (as an ordered pair)
y-intercept	______	______	______
x-intercept	______	______	______

28. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$6x + 3y = -18$$

	x-value	y-value	Location (as an ordered pair)
y-intercept	______	______	______
x-intercept	______	______	______

29. Find the line's slope and y-intercept.
A line has equation $-9x + y = 7$.

This line's slope is ______.

This line's y-intercept is ______.

30. Find the line's slope and y-intercept.
A line has equation $-x - y = 10$.

This line's slope is ______.

This line's y-intercept is ______.

31. Find the line's slope and y-intercept.
A line has equation $3x + 6y = 1$.

This line's slope is ______.

This line's y-intercept is ______.

32. Find the line's slope and y-intercept.
A line has equation $8x + 20y = 1$.

This line's slope is ______.

This line's y-intercept is ______.

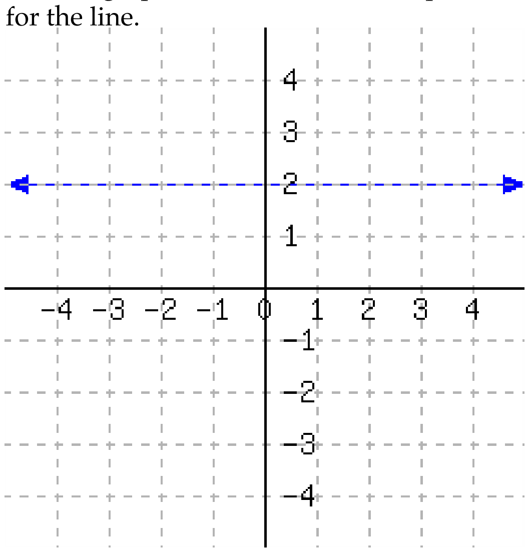
33. Fill out this table for the equation $x = -7$. The first row is an example.

x	y	Points
-7	-3	$(-7, -3)$
	-2	
	-1	
	0	
	1	
	2	

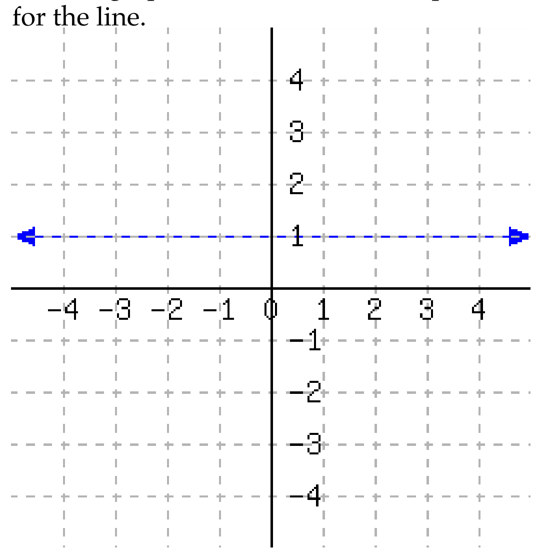
34. Fill out this table for the equation $x = -6$. The first row is an example.

x	y	Points
-6	-3	$(-6, -3)$
	-2	
	-1	
	0	
	1	
	2	

35. A line's graph is shown. Write an equation for the line.



36. A line's graph is shown. Write an equation for the line.



37. Line m passes points $(5, -10)$ and $(5, 9)$.
Line n passes points $(4, 10)$ and $(4, -4)$.
These two lines are (☐ parallel ☐ perpendicular ☐ neither parallel nor perpendicular) .

38. Line m passes points $(7, 4)$ and $(7, -3)$.
Line n passes points $(-1, 10)$ and $(-1, 7)$.
These two lines are (☐ parallel ☐ perpendicular ☐ neither parallel nor perpendicular) .

39. Line k's equation is $y = \frac{9}{4}x - 4$.
Line ℓ is perpendicular to line k and passes through the point $(-9, 9)$.
Find an equation for line ℓ in both slope-intercept form and point-slope forms.
An equation for ℓ in slope-intercept form is: ______.
An equation for ℓ in point-slope form is: ______.

40. Line k's equation is $y = -\frac{2}{9}x + 4$.
Line ℓ is perpendicular to line k and passes through the point $(4, 23)$.
Find an equation for line ℓ in both slope-intercept form and point-slope forms.
An equation for ℓ in slope-intercept form is: ______.
An equation for ℓ in point-slope form is: ______.

41. Graph the linear inequality $y > \frac{4}{3}x + 1$.

42. Graph the linear inequality $y \leq -\frac{1}{2}x - 3$.

43. Graph the linear inequality $y \geq 3$.

44. Graph the linear inequality $3x + 2y < -6$.

Chapter 4

Systems of Linear Equations

4.1 Solving Systems of Linear Equations by Graphing

We have learned how to graph a line given its equation. In this section, we will learn what a *system* of *two* linear equations is, and how to use graphing to solve such a system.

4.1.1 Solving Systems of Equations by Graphing

Example 4.1.2
Fabiana and David are running at constant speeds in parallel lanes on a track. David starts out ahead of Fabiana, but Fabiana is running faster. We want to determine when Fabiana will catch up with David. Let's start by looking at the graph of each runner's distance over time, in Figure 4.1.3.
Each of the two lines has an equation, as discussed in Chapter 3. The line representing David appears to have y-intercept $(0, 4)$ and slope $\frac{4}{3}$, so its equation is $y = \frac{4}{3}t + 4$. The line representing Fabiana appears to have y-intercept $(0, 0)$ and slope 2, so its equation is $y = 2t$.

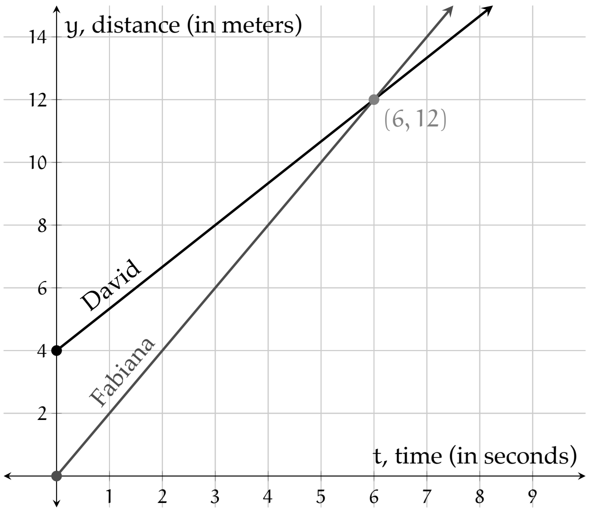


Figure 4.1.3: David and Fabiana's distances.

When these two equations are together as a package, we have what is called a "system of linear equations":

$$\begin{cases} y = \frac{4}{3}t + 4 \\ y = 2t \end{cases}$$

The large left brace indicates that this is a collection of two distinct equations, not one equation that was somehow algebraically manipulated into an equivalent equation.

As we can see in Figure 4.1.3, the graphs of the two equations cross at the point $(6, 12)$. We refer to the point $(6, 12)$ as the solution to this system of linear equations. To denote the solution set, we write $\{(6, 12)\}$. It's more valuable to interpret these numbers in context whenever possible: it took 6 seconds for the two runners to meet up, and when they met they were 12 meters up the track.

Definition 4.1.4 System of Linear Equations. A **system of linear equations** is any pairing of two (or more) linear equations. A **solution to a system of linear equations** is any point that is a solution for all of the equations in the system. The **solution set to a system of linear equations** is the collection of all solutions to the system. ◊

Remark 4.1.5 In Example 4.1.2, we stated that the solution was the point $(6, 12)$. It makes sense to write this as an ordered pair when we're given a graph. In some cases when we have no graph, particularly when our variables are not x and y, it might not be clear which variable "comes first" and we won't be able to write an ordered pair. Nevertheless, given the context we can write meaningful summary statements.

Example 4.1.6 Determine the solution to the system of equations graphed in Figure 4.1.7.

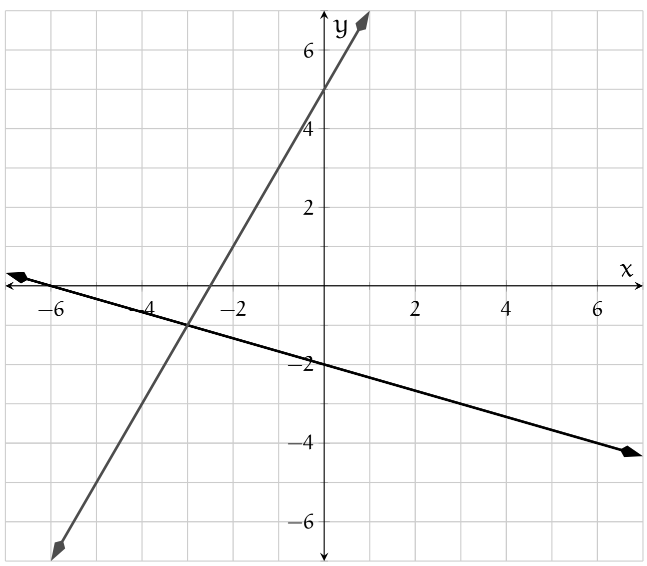
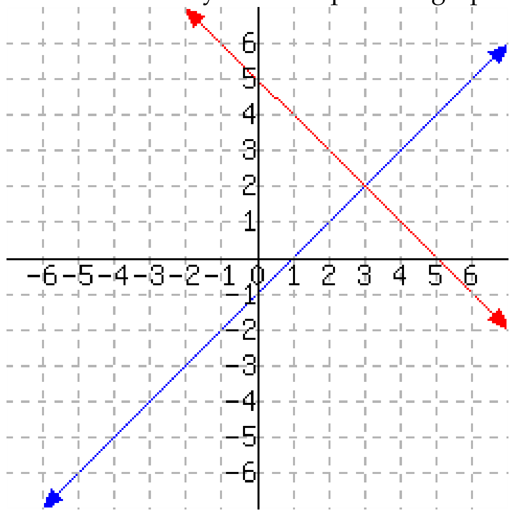


Figure 4.1.7: Graph of a System of Equations

Explanation. The two lines intersect where $x = -3$ and $y = -1$, so the solution is the point $(-3, -1)$. We write the solution set as $\{(-3, -1)\}$.

Checkpoint 4.1.8 Determine the solution to the system of equations graphed below.



The solution is the point ______.

Explanation. The two lines intersect where $x = 3$ and $y = 2$, so the solution is the point $(3, 2)$. We write the solution set as $\{(3, 2)\}$.

Now let's look at an example where *we* need to make a graph to find the solution.

Example 4.1.9 Solve the following system of equations by graphing:

$$\begin{cases} y = \frac{1}{2}x + 4 \\ y = -x - 5 \end{cases}$$

Notice that each of these equations is written in slope-intercept form. The first equation, $y = \frac{1}{2}x + 4$, is a linear equation with a slope of $\frac{1}{2}$ and a y-intercept of $(0, 4)$. The second equation, $y = -x-5$, is a linear equation with a slope of -1 and a y-intercept of $(0, -5)$. We'll use this information to graph both lines.

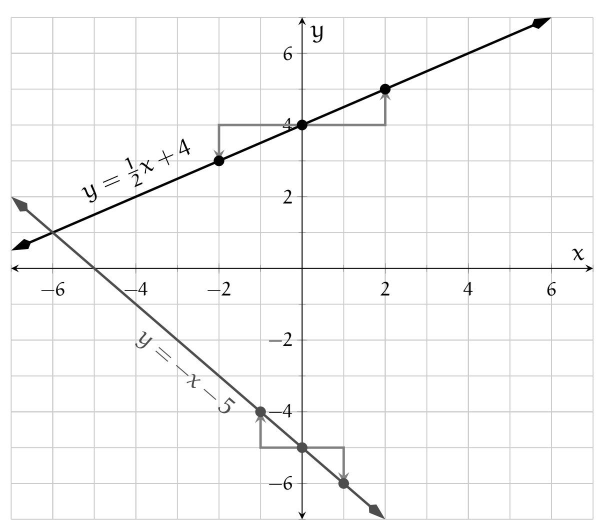


Figure 4.1.10: $y = \frac{1}{2}x + 4$ and $y = -x - 5$.

It appears that the two lines intersect where $x = -6$ and $y = 1$, so the solution of the system of equations would be the point $(-6, 1)$. We should check this with the two original equations.

$$\begin{aligned} y &= \frac{1}{2}x + 4 & \qquad y &= -x - 5 \\ 1 &\stackrel{?}{=} \frac{1}{2}(-6) + 4 & 1 &\stackrel{?}{=} -(-6) - 5 \\ 1 &\stackrel{\checkmark}{=} -3 + 4 & 1 &\stackrel{\checkmark}{=} 6 - 5 \end{aligned}$$

This verifies that $(-6, 1)$ is the solution, and we write the solution set as $\{(-6, 1)\}$.

Example 4.1.11 Solve the following system of equations by graphing:

$$\begin{cases} x - 3y = -12 \\ 2x + 3y = \quad 3 \end{cases}$$

Explanation. Since both line equations are given in standard form, we'll graph each one by finding the intercepts. Recall that to find the x-intercept of each equation, replace y with 0 and solve for x. Similarly, to find the y-intercept of each equation, replace x with 0 and solve for y.

For our first linear equation, we have:

$$\begin{aligned} x - 3(0) &= -12 \\ x &= -12 \end{aligned} \qquad \begin{aligned} 0 - 3y &= -12 \\ -3y &= -12 \\ y &= 4. \end{aligned}$$

So the intercepts are $(-12, 0)$ and $(0, 4)$.

For our second linear equation, we have:

$$\begin{aligned} 2x + 3(0) &= 3 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned} \qquad \begin{aligned} 2(0) + 3y &= 3 \\ 3y &= 3 \\ y &= 1. \end{aligned}$$

So the intercepts are $\left(\frac{3}{2}, 0\right)$ and $(0, 1)$.

Now we can graph each line by plotting the intercepts and connecting these points:

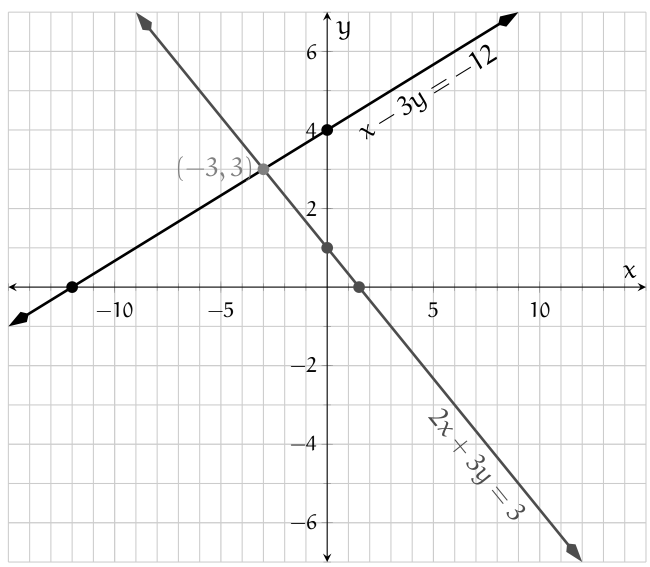


Figure 4.1.12: Graphs of $x - 3y = -12$ and $2x + 3y = 3$

It appears that the solution of the system of equations is the point of intersection of those two lines, which is $(-3, 3)$. It's important to check this is correct, because when making a hand-drawn graph, it would be easy to be off by a little bit. To check, we can substitute the values of x and y from the point $(-3, 3)$ into each equation:

$$\begin{aligned} x - 3y &= -12 \\ -3 - 3(3) &\stackrel{?}{=} -12 \\ -3 - 9 &\stackrel{\checkmark}{=} -12 \end{aligned} \qquad \begin{aligned} 2x + 3y &= 3 \\ 2(-3) + 3(3) &\stackrel{?}{=} 3 \\ -6 + 9 &\stackrel{\checkmark}{=} 3 \end{aligned}$$

So we have checked that $(-3, 3)$ is indeed the solution for the system. We write the solution set as $\{(-3, 3)\}$.

Example 4.1.13 A college has a north campus and a south campus. The north campus has 18,000 students, and the south campus has 4,000 students. In the past five years, the north campus lost 4,000 students, and the south campus gained 3,000 students. If these trends continue, in how many years would the two campuses have the same number of students? Write and solve a system of equations modeling this problem.

Explanation. Since all the given student counts are in the thousands, we make the decision to measure student population in thousands. So for instance, the north campus starts with a student population of 18 (thousand students).

The north campus lost 4 thousand students in 5 years. So it is losing students at a rate of $\frac{4 \text{ thousand}}{5 \text{ year}}$, or $\frac{4}{5} \frac{\text{thousand}}{\text{year}}$. This rate of change should be interpreted as a negative number, because the north campus is losing students over time. So we have a linear model with starting value 18 thousand students, and a slope of $-\frac{4}{5}$ thousand students per year. In other words,

$$y = -\frac{4}{5}t + 18,$$

where y stands for the number of students in thousands, and t stands for the number of years into the future.

Similarly, the number of students at the south campus can be modeled by $y = \frac{3}{5}t + 4$. Now we have a system of equations:

$$\begin{cases} y = -\frac{4}{5}t + 18 \\ y = \frac{3}{5}t + 4 \end{cases}$$

We will graph both lines using their slopes and y-intercepts.

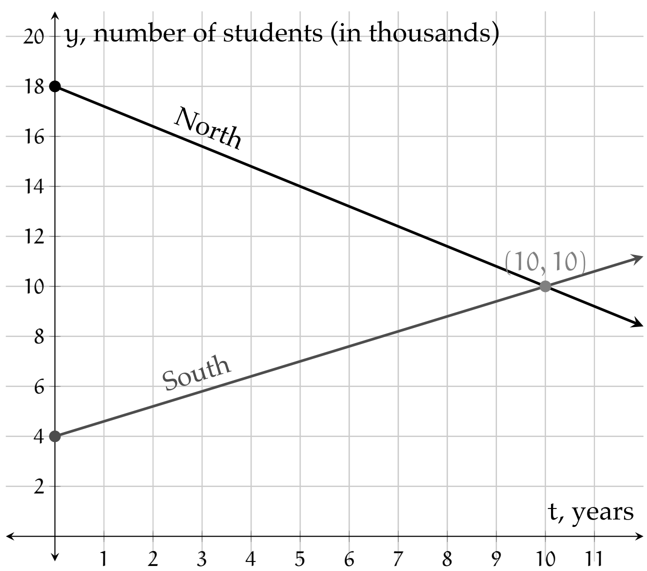


Figure 4.1.14: Number of Students at the South Campus and North Campus

According to the graph, the lines intersect at $(10, 10)$. So if the trends continue, both campuses will have 10,000 students 10 years from now.

Example 4.1.15 Solve the following system of equations by graphing:

$$\begin{cases} y = \phantom{-\frac{1}{2}}3(x-2)+1 \\ y = -\frac{1}{2}(x+1)-1 \end{cases}$$

Explanation. Since both line equations are given in point-slope form, we can start by graphing the point indicated in each equation and use the slope to determine the rest of the line.

For our first equation, $y = 3(x-2)+1$, the point indicated in the equation is $(2,1)$ and the slope is 3.

For our second equation, $y = -\frac{1}{2}(x+1)-1$, the point indicated in the equation is $(-1,-1)$ and the slope is $-\frac{1}{2}$.

Now we can graph each line by plotting the points and using their slopes.

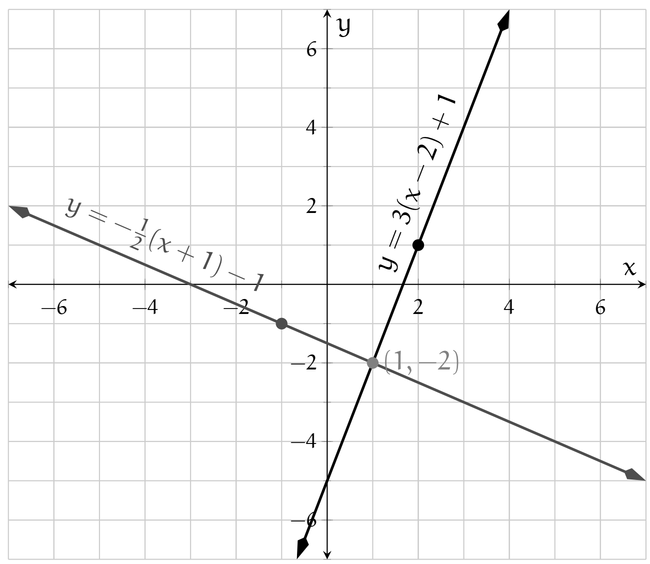


Figure 4.1.16: Graphs of $y = 3(x-2)+1$ and $y = -\frac{1}{2}(x+1)-1$

It appears that the solution of the system of equations is the point of intersection of those two lines, which is $(1,-2)$. It's important to check this is correct, because when making a hand-drawn graph, it would be easy to be off by a little bit. To check, we can substitute the values of x and y from the point $(1,-2)$ into each equation:

$$\begin{aligned} y &= 3(x-2)+1 \\ -2 &\stackrel{?}{=} 3(1-2)+1 \\ -2 &\stackrel{?}{=} 3(-1)+1 \\ -2 &\stackrel{\checkmark}{=} -3+1 \end{aligned}$$

$$\begin{aligned} y &= -\frac{1}{2}(x+1)-1 \\ -2 &\stackrel{?}{=} -\frac{1}{2}(1+1)-1 \\ -2 &\stackrel{?}{=} -\frac{1}{2}(2)-1 \\ -2 &\stackrel{\checkmark}{=} -1-1 \end{aligned}$$

So we have checked that $(2,-1)$ is indeed the solution for the system. We write the solution set as $\{(2,-1)\}$.

4.1.2 Special Systems of Equations

Recall that when we solved linear equations in one variable, there were two special cases discussed in detail in Section 2.4. In one special case, like with the equation $x = x + 1$, there is no solution. And in the another case, like with the equation $x = x$, there are infinitely many solutions. When solving systems of equations in two variables, we have similar special cases to consider.

Example 4.1.17 Parallel Lines. Let's look at the graphs of two lines with the same slope, $y = 2x - 4$ and $y = 2x + 1$:

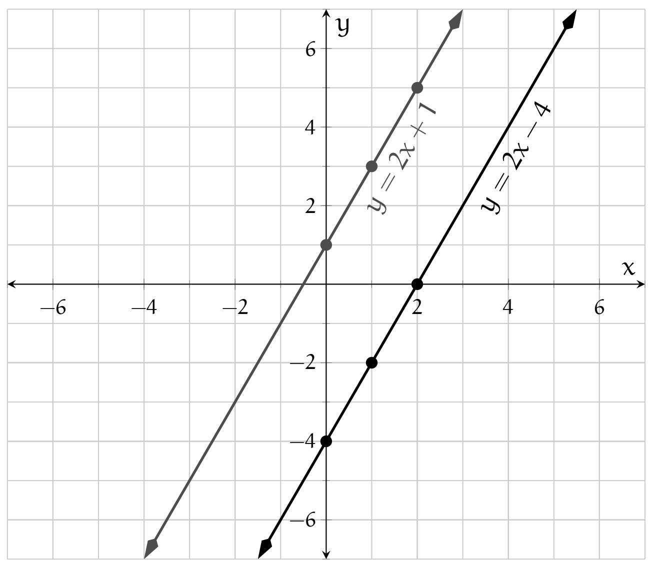


Figure 4.1.18: Graphs of $y = 2x - 4$ and $y = 2x + 1$

For this system of equations, what is the solution? Since the two lines have the same slope they are **parallel lines** and will never intersect. This means that there is *no solution* to this system of equations. We write the solution set as $\emptyset$.

The symbol $\emptyset$ is a special symbol that represents the **empty set**, a *set* that has no numbers in it which can also be written simply as $\{\}$. This symbol is *not* the same thing as the number zero. The *number* of eggs in an empty egg carton is zero whereas the empty carton itself could represent the empty set. The symbols for the empty set and the number zero may look similar depending on how you write the number zero, so try to keep the concepts separate.

Example 4.1.19 Coinciding Lines. Next we'll look at the other special case. Let's start with this system of equations:

$$\begin{cases} \quad y = 2x - 4 \\ 6x - 3y = \quad 12 \end{cases}$$

To solve this system of equations, we want to graph each line. The first equation is in slope-intercept form and can be graphed easily using its slope of 2 and its y-intercept of $(0, -4)$.

The second equation, $6x - 3y = 12$, can either be graphed by solving for y and using the slope-intercept form or by finding the intercepts. If we use the intercept method, we'll find that this line has an x-intercept of $(2, 0)$ and a y-intercept of $(0, -4)$. When we graph both lines we get Figure 4.1.20.

Now we can see these are actually the *same* line, or **coinciding lines**. To determine the solution to this system, we'll note that they overlap everywhere. This means that we have an infinite number of solutions: *all* points that fall on the line. It may be enough to report that there are infinitely many solutions. In order to be more specific, all we can do is say that any ordered pair (x, y) satisfying the line equation is a solution. In set-builder notation, we would write $\{(x, y) \mid y = 2x - 4\}$.

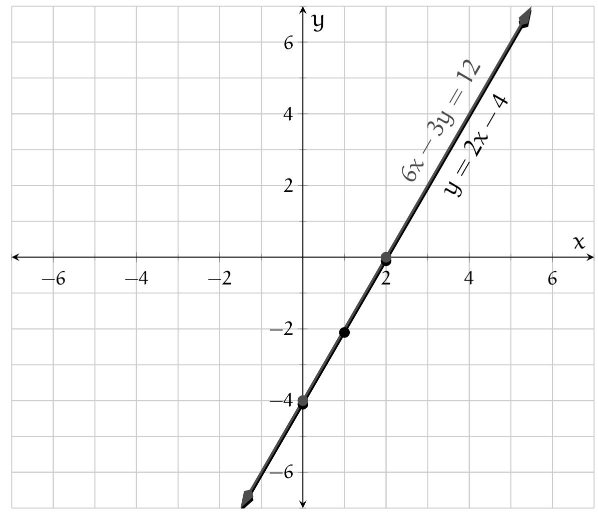


Figure 4.1.20: Graphs of $y = 2x - 4$ and $6x - 3y = 12$

Remark 4.1.21 In Example 4.1.19, what would have happened if we had decided to convert the second line equation into slope-intercept form?

$$\begin{aligned} 6x - 3y &= 12 \\ 6x - 3y - 6x &= 12 - 6x \\ -3y &= -6x + 12 \\ -\frac{1}{3} \cdot (-3y) &= -\frac{1}{3} \cdot (-6x + 12) \\ y &= 2x - 4 \end{aligned}$$

This is the literally the same as the first equation in our system. This is a different way to show that these two equations are equivalent and represent the same line. Any time we try to solve a system where the equations are equivalent, we'll have an infinite number of solutions.

Warning 4.1.22 Notice that for a system of equations with infinite solutions like Example 4.1.19, we didn't say that *every* point was a solution. Rather, *every point that falls on that line* is a solution. It would be incorrect to state this solution set as "all real numbers" or as "all ordered pairs."

List 4.1.23: A summary of the three types of systems of equations and their solution sets.

Intersecting Lines: If two linear equations make lines with different slopes, the system has one solution.

Parallel Lines: If two linear equations make lines with the same slope but different y-intercepts, the system has no solution.

Coinciding Lines: If two linear equations make lines with the same slope and the same y-intercept (in other words, they make the same line), the system has infinitely many solutions. This solution set consists of all ordered pairs on that line.

4.1.3 Reading Questions

1. What is the purpose of the one big left brace in a system of two equations?
2. When you find a solution to a system of two linear equations in two variables, why should you check the solution? Would it be good enough to only substitute the numbers into *one* of the original two equations?
3. Suppose you have a system of two linear equations, and you know the system has exactly one solution. What can you say about the slopes of the two lines that the two equations define?

4.1.4 Exercises

Warmup and Review Find the line's slope and y-intercept.

1. A line has equation $y = 3x + 5$.
   This line's slope is ______.
   This line's y-intercept is ______.
2. A line has equation $y = 4x + 1$.
   This line's slope is ______.
   This line's y-intercept is ______.
3. A line has equation $y = -x - 1$.
   This line's slope is ______.
   This line's y-intercept is ______.
4. A line has equation $y = -x + 1$.
   This line's slope is ______.
   This line's y-intercept is ______.
5. A line has equation $y = -\frac{6x}{5} + 7$.
   This line's slope is ______.
   This line's y-intercept is ______.
6. A line has equation $y = -\frac{8x}{7} + 4$.
   This line's slope is ______.
   This line's y-intercept is ______.
7. A line has equation $y = \frac{x}{10} - 4$.
   This line's slope is ______.
   This line's y-intercept is ______.
8. A line has equation $y = \frac{x}{10} + 10$.
   This line's slope is ______.
   This line's y-intercept is ______.

9. Graph the equation $y = -3x$.

10. Graph the equation $y = \frac{1}{4}x$.

11. Graph the equation $y = \frac{2}{3}x + 4$.

12. Graph the equation $y = -2x + 5$.

Solve the linear equation for y.

13. $16x + 2y = -8$

14. $6x - 2y = 28$

15. $8x + 4y = 40$

16. $3x + 5y = 45$

Checking Solutions for System of Equations

17. Decide whether $(1, -3)$ is a solution to the system of equations:

$$\begin{cases} x - 2y = 7 \\ 2x + 4y = -10 \end{cases}$$

The point $(1, -3)$ (☐ is ☐ is not) a solution.

18. Decide whether $(2, 4)$ is a solution to the system of equations:

$$\begin{cases} -5x - 5y = -30 \\ 2x - 2y = -4 \end{cases}$$

The point $(2, 4)$ (☐ is ☐ is not) a solution.

19. Decide whether $(3, 1)$ is a solution to the system of equations:

$$\begin{cases} -x + 3y = 0 \\ y = -2x + 7 \end{cases}$$

The point $(3, 1)$ (☐ is ☐ is not) a solution.

20. Decide whether $(4, -3)$ is a solution to the system of equations:

$$\begin{cases} 4x + 2y = 13 \\ y = -4x + 13 \end{cases}$$

The point $(4, -3)$ (☐ is ☐ is not) a solution.

21. Decide whether $\left(\frac{8}{5}, \frac{4}{5}\right)$ is a solution to the system of equations:

$$\begin{cases} -10x + 5y = -12 \\ 10x + 15y = 28 \end{cases}$$

The point $\left(\frac{8}{5}, \frac{4}{5}\right)$ (☐ is ☐ is not) a solution.

22. Decide whether $\left(\frac{7}{2}, \frac{9}{2}\right)$ is a solution to the system of equations:

$$\begin{cases} 2x - 2y = -2 \\ -6x - 4y = -39 \end{cases}$$

The point $\left(\frac{7}{2}, \frac{9}{2}\right)$ (☐ is ☐ is not) a solution.

Using a Graph to Solve a System Use a graph to solve the system of equations.

23. $\begin{cases} y = -\dfrac{5}{2}x - 4 \\ y = 2x + 5 \end{cases}$

24. $\begin{cases} y = \dfrac{2}{3}x + 5 \\ y = -2x - 11 \end{cases}$

25. $\begin{cases} y = 12x + 7 \\ 3x + y = -8 \end{cases}$

26. $\begin{cases} y = -3x + 5 \\ 4x + y = 8 \end{cases}$

27. $\begin{cases} x + y = 0 \\ 3x - y = 8 \end{cases}$

28. $\begin{cases} 4x - 2y = 4 \\ x + 2y = 6 \end{cases}$

29. $\begin{cases} y = 4x - 5 \\ y = -1 \end{cases}$

30. $\begin{cases} 3x - 4y = 12 \\ y = 3 \end{cases}$

31. $\begin{cases} x + y = -1 \\ x = 2 \end{cases}$

32. $\begin{cases} x - 2y = -4 \\ x = -4 \end{cases}$

33. $\begin{cases} y = 2(x+3) - 5 \\ y = -\frac{4}{3}(x-4) - 1 \end{cases}$

34. $\begin{cases} y = -\frac{2}{3}(x-6) - 2 \\ y = -\frac{1}{2}(x-1) + 2 \end{cases}$

35. $\begin{cases} y = -\frac{1}{2}(x-6) + 4 \\ y = 4(x+1) - 6 \end{cases}$

36. $\begin{cases} y = \frac{5}{6}(x-6) + 4 \\ y = 2(x+1) + 4 \end{cases}$

37. $\begin{cases} y = -\frac{4}{5}x + 8 \\ 4x + 5y = -35 \end{cases}$

38. $\begin{cases} 2x - 7y = 28 \\ y = \frac{2}{7}x - 3 \end{cases}$

39. $\begin{cases} -10x + 15y = 60 \\ 6x - 9y = 36 \end{cases}$

40. $\begin{cases} 6x - 8y = 32 \\ 9x - 12y = 12 \end{cases}$

41. $\begin{cases} y = -\frac{3}{5}x + 7 \\ 9x + 15y = 105 \end{cases}$

42. $\begin{cases} 9y - 12x = 18 \\ y = \frac{4}{3}x + 2 \end{cases}$

Determining the Number of Solutions in a System of Equations

43. Simply by looking at this system of equations, decide the number of solutions it has.

$$\begin{cases} y = \frac{4}{5}x + 2 \\ y = \frac{4}{5}x + 4 \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

44. Simply by looking at this system of equations, decide the number of solutions it has.

$$\begin{cases} y = -2x + 1 \\ y = -2x - 2 \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

45. Without graphing this system of equations, decide the number of solutions it has.

$$\begin{cases} y = -\frac{6}{5}x + 1 \\ 27x - 15y = 30 \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

46. Without graphing this system of equations, decide the number of solutions it has.

$$\begin{cases} y = \frac{5}{2}x + 3 \\ 21x - 6y = 12 \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

47. Without graphing this system of equations, decide the number of solutions it has.

$$\begin{cases} 3x - 3y = 0 \\ 4x - 2y = -8 \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

48. Without graphing this system of equations, decide the number of solutions it has.

$$\begin{cases} 3x - 12y = 36 \\ 2x - 8y = 24 \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

49. Simply by looking at this system of equations, decide the number of solutions it has.

$$\begin{cases} x = -5 \\ y = 1 \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

50. Simply by looking at this system of equations, decide the number of solutions it has.

$$\begin{cases} x = 2 \\ y = -5 \end{cases}$$

The system has (☐ no solution ☐ one solution ☐ infinitely many solutions) .

4.2 Substitution

In Section 4.1, we focused on solving systems of equations by graphing. In addition to being time consuming, graphing can be an awkward method to determine the exact solution when the solution has large numbers, fractions, or decimals. There are two symbolic methods for solving systems of linear equations, and in this section we will use one of them: substitution.

4.2.1 Solving Systems of Equations Using Substitution

Example 4.2.2 The Interview. In 2014, the New York Times[1] posted the following about the movie, "The Interview":

> "The Interview" generated roughly \$15 million in online sales and rentals during its first four days of availability, Sony Pictures said on Sunday.
>
> Sony did not say how much of that total represented \$6 digital rentals versus \$15 sales. The studio said there were about two million transactions overall.

A few days later, Joey Devilla cleverly pointed out in his blog[2], that there is enough information given to find the amount of sales versus rentals. Using algebra, we can write a system of equations and solve it to find the two quantities.[3]

First, we will define variables. We need two variables, because there are two unknown quantities: how many sales there were and how many rentals there were. Let r be the number of rental transactions and let s be the number of sales transactions.

If you are unsure how to write an equation from the background information, use the units to help you. The units of each term in an equation must match because we can only add like quantities. Both r and s are in transactions. The article says that the total number of transactions is 2 million. So our first equation will add the total number of rental and sales transactions and set that equal to 2 million. Our equation is:

$$(r\text{ transactions}) + (s\text{ transactions}) = 2{,}000{,}000\text{ transactions}$$

Without the units:

$$r + s = 2{,}000{,}000$$

The price of each rental was \$6. That means the problem has given us a *rate* of $6\,\frac{\text{dollars}}{\text{transaction}}$ to work with. The rate unit suggests this should be multiplied by something measured in transactions. It makes sense to multiply by r, and then the number of dollars generated from rentals was $6r$. Similarly, the price of each sale was \$15, so the revenue from sales was $15s$. The total revenue was \$15 million, which we can represent with this equation:

$$\left(6\,\frac{\text{dollars}}{\text{transaction}}\right)(r\text{ transactions}) + \left(15\,\frac{\text{dollars}}{\text{transaction}}\right)(s\text{ transactions}) = \$15{,}000{,}000$$

Without the units:

$$6r + 15s = 15{,}000{,}000$$

Here is our system of equations:

$$\begin{cases} r + s = 2{,}000{,}000 \\ 6r + 15s = 15{,}000{,}000 \end{cases}$$

To solve the system, we will use the **substitution** method. The idea is to use *one* equation to find an expression that is equal to r but, cleverly, does not use the variable "r." Then, substitute this for r into the *other* equation. This leaves you with *one* equation that only has *one* variable.

The first equation from the system is an easy one to solve for r:

$$\begin{aligned} r + s &= 2{,}000{,}000 \\ r &= 2{,}000{,}000 - s \end{aligned}$$

This tells us that the expression $2{,}000{,}000 - s$ is equal to r, so we can *substitute* it for r in the second equation:

$$\begin{aligned} 6r + 15s &= 15{,}000{,}000 \\ 6(2{,}000{,}000 - s) + 15s &= 15{,}000{,}000 \end{aligned}$$

Now we have an equation with only one variable, s, which we will solve for:

$$\begin{aligned} 6(2{,}000{,}000 - s) + 15s &= 15{,}000{,}000 \\ 12{,}000{,}000 - 6s + 15s &= 15{,}000{,}000 \\ 12{,}000{,}000 + 9s &= 15{,}000{,}000 \\ 9s &= 3{,}000{,}000 \\ \frac{9s}{9} &= \frac{3{,}000{,}000}{9} \\ s &= 333{,}333.\overline{3} \end{aligned}$$

At this point, we know that $s = 333{,}333.\overline{3}$. This tells us that out of the 2 million transactions, roughly 333,333 were from online sales. Recall that we solved the first equation for r, and found $r = 2{,}000{,}000 - s$.

$$\begin{aligned} r &= 2{,}000{,}000 - s \\ r &= 2{,}000{,}000 - 333{,}333.\overline{3} \\ r &= 1{,}666{,}666.\overline{6} \end{aligned}$$

To check our answer, we will see if $s = 333{,}333.\overline{3}$ and $r = 1{,}666{,}666.\overline{6}$ make the original equations true:

$$\begin{aligned} r + s &= 2{,}000{,}000 & 6r + 15s &= 15{,}000{,}000 \\ 1{,}666{,}666.\overline{6} + 333{,}333.\overline{3} &\stackrel{?}{=} 2{,}000{,}000 & 6\left(1{,}666{,}666.\overline{6}\right) + 15\left(333{,}333.\overline{3}\right) &\stackrel{?}{=} 15{,}000{,}000 \\ 2{,}000{,}000 &\stackrel{\checkmark}{=} 2{,}000{,}000 & 10{,}000{,}000 + 5{,}000{,}000 &\stackrel{\checkmark}{=} 15{,}000{,}000 \end{aligned}$$

In summary, there were roughly 333,333 copies sold and roughly 1,666,667 copies rented.

Remark 4.2.3 In Example 4.2.2, we *chose* to solve the equation $r + s = 2{,}000{,}000$ for r. We could just as easily have instead solved for s and substituted that result into the second equation instead. The summary conclusion would have been the same.

Remark 4.2.4 In Example 4.2.2, we rounded the solution values because only whole numbers make sense in the context of the problem. It was OK to round, because the original information we had to work with were rounded. In fact, it would be OK to round even more to $s = 330{,}000$ and $r = 1{,}700{,}000$, as long as we

[1](nyti.ms/2pupebT)
[2]http://www.joeydevilla.com/2014/12/31/
[3]Although since the given information uses approximate values, the solutions we will find will only be approximations too.

communicate clearly that we rounded and our values are rough.

In other exercises where there is no context and nothing suggests the given numbers are approximations, it is not OK to round and all answers should be communicated with their exact values.

Example 4.2.5 Solve the system of equations using substitution:

$$\begin{cases} x + 2y = 8 \\ 3x - 2y = 8 \end{cases}$$

Explanation. To use substitution, we need to solve for *one* of the variables in *one* of our equations. Looking at both equations, it will be easiest to solve for x in the first equation:

$$\begin{aligned} x + 2y &= 8 \\ x &= 8 - 2y \end{aligned}$$

Next, we replace x in the second equation with $8 - 2y$, giving us a linear equation in only one variable, y, that we may solve:

$$\begin{aligned} 3x - 2y &= 8 \\ 3(8 - 2y) - 2y &= 8 \\ 24 - 6y - 2y &= 8 \\ 24 - 8y &= 8 \\ -8y &= -16 \\ y &= 2 \end{aligned}$$

Now that we have the value for y, we need to find the value for x. We have already solved the first equation for x, so that is the easiest equation to use.

$$\begin{aligned} x &= 8 - 2y \\ x &= 8 - 2(2) \\ x &= 8 - 4 \\ x &= 4 \end{aligned}$$

To check this solution, we replace x with 4 and y with 2 in each equation:

$$\begin{aligned} x + 2y &= 8 \\ 4 + 2(2) &\stackrel{?}{=} 8 \\ 4 + 4 &\stackrel{\checkmark}{=} 8 \end{aligned} \qquad\qquad \begin{aligned} 3x - 2y &= 8 \\ 3(4) - 2(2) &\stackrel{?}{=} 8 \\ 12 - 4 &\stackrel{\checkmark}{=} 8 \end{aligned}$$

We conclude then that this system of equations is true when $x = 4$ and $y = 2$. Our solution is the point $(4, 2)$ and we write the solution set as $\{(4, 2)\}$.

Checkpoint 4.2.6 Solve the following system of equations.

$$\begin{cases} x + 3y = -60 \\ \quad 40 = 5x - 5y \end{cases}$$

Explanation. These equations have no fractions; let's try to keep it that way.

$$\begin{cases} x + 3y = -60 \\ \quad 40 = 5x - 5y \end{cases}$$

Since one of the coefficients of x is 1, it is wise to solve for the x in terms of the other variable and then use substitution to complete the problem.

$$x = -3y - 60 \quad \text{(from the first equation)}$$

which we can substitute in for x into the second equation:

$$\begin{aligned} 40 &= 5(-3y - 60) - 5y \quad \text{(from the second equation)} \\ 40 &= -20y - 300 \\ 20y &= -340 \\ y &= -17 \end{aligned}$$

We can substitute this for y back into the first equation to find x.

$$\begin{aligned} x &= -3\,(-17) - 60 \quad \text{(from the first equation, after we had solved for } x \text{ in terms of} y) \\ x &= 51 - 60 \\ x &= -9 \end{aligned}$$

So the solution is $x = -9, y = -17$.

Example 4.2.7 Solve this system of equations using substitution:

$$\begin{cases} 3x - 7y = 5 \\ -5x + 2y = 11 \end{cases}$$

Explanation. We need to solve for *one* of the variables in *one* of our equations. Looking at both equations, it will be easiest to solve for y in the second equation. The coefficient of y in that equation is smallest.

$$\begin{aligned} -5x + 2y &= 11 \\ 2y &= 11 + 5x \\ \frac{2y}{2} &= \frac{11 + 5x}{2} \\ y &= \frac{11}{2} + \frac{5}{2}x \end{aligned}$$

Note that in this example, there are fractions once we solve for y. We should take care with the steps that follow that the fraction arithmetic is correct.

Replace y in the first equation with $\frac{11}{2} + \frac{5}{2}x$, giving us a linear equation in only one variable, x, that we may solve:

$$\begin{aligned}
3x - 7y &= 5 \\
3x - 7\left(\frac{11}{2} + \frac{5}{2}x\right) &= 5 \\
3x - 7 \cdot \frac{11}{2} - 7 \cdot \frac{5}{2}x &= 5 \\
3x - \frac{77}{2} - \frac{35}{2}x &= 5 \\
\frac{6}{2}x - \frac{77}{2} - \frac{35}{2}x &= 5 \\
-\frac{29}{2}x - \frac{77}{2} &= 5 \\
-\frac{29}{2}x &= \frac{10}{2} + \frac{77}{2} \\
-\frac{29}{2}x &= \frac{87}{2} \\
-\frac{2}{29} \cdot \left(-\frac{29}{2}x\right) &= -\frac{2}{29} \cdot \left(\frac{87}{2}\right) \\
x &= -3
\end{aligned}$$

Now that we have the value for x, we need to find the value for y. We have already solved the second equation for y, so that is the easiest equation to use.

$$\begin{aligned}
y &= \frac{11}{2} + \frac{5}{2}x \\
y &= \frac{11}{2} + \frac{5}{2}(-3) \\
y &= \frac{11}{2} - \frac{15}{2} \\
y &= -\frac{4}{2} \\
y &= -2
\end{aligned}$$

To check this solution, we replace x with -3 and y with -2 in each equation:

$$\begin{aligned}
3x - 7y &= 5 & -5x + 2y &= 11 \\
3(-3) - 7(-2) &\stackrel{?}{=} 5 & -5(-3) + 2(-2) &\stackrel{?}{=} 11 \\
-9 + 14 &\stackrel{\checkmark}{=} 5 & 15 - 4 &\stackrel{\checkmark}{=} 11
\end{aligned}$$

We conclude then that this system of equations is true when $x = -3$ and $y = -2$. Our solution is the point $(-3, -2)$ and we write the solution set as $\{(-3, -2)\}$.

Example 4.2.8 Clearing Fraction Denominators Before Solving. Solve the system of equations using the substitution method:

$$\begin{cases} \frac{x}{3} - \frac{1}{2}y = \frac{5}{6} \\ \frac{1}{4}x = \frac{y}{2} + 1 \end{cases}$$

Explanation. When a system of equations has fraction coefficients, it can be helpful to take steps that replace the fractions with whole numbers. With each equation, we may multiply each side by the least common multiple of all the denominators.

In the first equation, the least common multiple of the denominators is 6, so:

$$\begin{aligned} \frac{x}{3} - \frac{1}{2}y &= \frac{5}{6} \\ 6 \cdot \left(\frac{x}{3} - \frac{1}{2}y\right) &= 6 \cdot \frac{5}{6} \\ 6 \cdot \frac{x}{3} - 6 \cdot \frac{1}{2}y &= 6 \cdot \frac{5}{6} \\ 2x - 3y &= 5 \end{aligned}$$

In the second equation, the least common multiple of the denominators is 4, so:

$$\begin{aligned} \frac{1}{4}x &= \frac{y}{2} + 1 \\ 4 \cdot \frac{1}{4}x &= 4 \cdot \frac{y}{2} + 4 \cdot 1 \\ 4 \cdot \frac{1}{4}x &= 4 \cdot \frac{y}{2} + 4 \cdot 1 \\ x &= 2y + 4 \end{aligned}$$

Now we have this system that is equivalent to the original system of equations, but there are no fraction coefficients:

$$\begin{cases} 2x - 3y = 5 \\ x = 2y + 4 \end{cases}$$

The second equation is already solved for x, so we will substitute x in the first equation with $2y + 4$, and we have:

$$\begin{aligned} 2x - 3y &= 5 \\ 2(2y + 4) - 3y &= 5 \\ 4y + 8 - 3y &= 5 \\ y + 8 &= 5 \\ y &= -3 \end{aligned}$$

And we have solved for y. To find x, we know $x = 2y + 4$, so we have:

$$\begin{aligned} x &= 2y + 4 \\ x &= 2(-3) + 4 \\ x &= -6 + 4 \\ x &= -2 \end{aligned}$$

The solution is $(-2, -3)$. Checking this solution is left as an exercise.

Checkpoint 4.2.9 Try a similar exercise.

Solve the following system of equations.

$$\begin{cases} -3 = -m + \frac{1}{2}r \\ -m + \frac{4}{3} = -r \end{cases}$$

Explanation. If an equation involves fractions, it is helpful to clear denominators by multiplying both sides of the equation by a common multiple of the denominators.

$$\begin{cases} 2(-3) = 2\left(-m + \frac{1}{2}r\right) \\ 3\left(-m + \frac{4}{3}\right) = 3(-r) \end{cases}$$

$$\begin{cases} -6 = -2m + r \\ -3m + 4 = -3r \end{cases}$$

Since one of the coefficients of r is 1, it is wise to solve for r in terms of the other variable and then use substitution to complete the problem.

$$2m - 6 = r \quad \text{(from the first equation)}$$

which we can substitute in for r into the second equation:

$$\begin{aligned} 4 - 3m &= -3(2m - 6) \quad \text{(from the second equation)} \\ 4 - 3m &= 18 - 6m \\ 3m &= 14 \\ m &= \frac{14}{3} \end{aligned}$$

We can substitute this back for m into the first equation to find r.

$$\begin{aligned} 2\left(\frac{14}{3}\right) - 6 &= r \quad \text{(from the first equation, after we had solved for } r \text{ in terms of } m\text{)} \\ \frac{28}{3} + (-6) &= r \\ \frac{10}{3} &= r \end{aligned}$$

So the solution is $m = \frac{14}{3}, r = \frac{10}{3}$.

For summary reference, here is the general procedure.

Process 4.2.10 Solving Systems of Equations by Substitution. *To solve a system of equations by substitution,*

1. *Solve one of the equations for one of the variables.*
2. *Substitute that expression into the* other *equation. There should now only be one variable in that equation.*
3. *Solve that equation for the one remaining variable.*
4. *Substitute that value into an earlier equation and solve for the other variable.*
5. *Verify your solution in the original equations.*

4.2.2 Applications of Systems of Equations

Example 4.2.11 Two Different Interest Rates. Notah made some large purchases with his two credit cards one month and took on a total of \$8,400 in debt from the two cards. He didn't make any payments the first month, so the two credit card debts each started to accrue interest. That month, his Visa card charged 2% interest and his Mastercard charged 2.5% interest. Because of this, Notah's total debt grew by \$178. How much money did Notah charge to each card?

Explanation. To start, we will define two variables based on our two unknowns. Let v be the amount charged to the Visa card (in dollars) and let m be the amount charged to the Mastercard (in dollars).

To determine our equations, notice that we are given two different totals. We will use these to form our two equations. The total amount charged is \$8,400 so we have:

$$(v\,\text{dollars}) + (m\,\text{dollars}) = \$8400$$

Or without units:

$$v + m = 8400$$

The other total we were given is the total amount of interest, \$178, which is also in dollars. The Visa had v dollars charged to it and accrues 2% interest. So $0.02v$ is the dollar amount of interest that comes from using this card. Similarly, $0.025m$ is the dollar amount of interest from using the Mastercard. Together:

$$0.02(v\,\text{dollars}) + 0.025(m\,\text{dollars}) = \$178$$

Or without units:

$$0.02v + 0.025m = 178$$

As a system, we write:

$$\begin{cases} v + m = 8400 \\ 0.02v + 0.025m = 178 \end{cases}$$

To solve this system by substitution, notice that it will be easier to solve for one of the variables in the first equation. We'll solve that equation for v:

$$\begin{aligned} v + m &= 8400 \\ v &= 8400 - m \end{aligned}$$

Now we will substitute $8400 - m$ for v in the second equation:

$$\begin{aligned} 0.02v + 0.025m &= 178 \\ 0.02(8400 - m) + 0.025m &= 178 \\ 168 - 0.02m + 0.025m &= 178 \\ 168 + 0.005m &= 178 \\ 0.005m &= 10 \\ \frac{0.005m}{0.005} &= \frac{10}{0.005} \\ m &= 2000 \end{aligned}$$

Lastly, we can determine the value of v by using the earlier equation where we isolated v:

$$\begin{aligned} v &= 8400 - m \\ v &= 8400 - 2000 \\ v &= 6400 \end{aligned}$$

In summary, Notah charged \$6400 to the Visa and \$2000 to the Mastercard. We should check that these numbers work as solutions to our original system *and* that they make sense in context. (For instance, if one of these numbers were negative, or was something small like \$0.50, they wouldn't make sense as credit card debt.)

The next two examples are called **mixture problems**, because they involve mixing two quantities together to form a combination and we want to find out how much of each quantity to mix.

Example 4.2.12 Mixing Solutions with Two Different Concentrations. LaVonda is a meticulous bartender and she needs to serve 600 milliliters of Rob Roy, an alcoholic cocktail that is 34% alcohol by volume. The main ingredients are scotch that is 42% alcohol and vermouth that is 18% alcohol. How many milliliters of each ingredient should she mix together to make the concentration she needs?

Explanation. The two unknowns are the quantities of each ingredient. Let s be the amount of scotch (in mL) and let v be the amount of vermouth (in mL).

One quantity given to us in the problem is 600 mL. Since this is the total volume of the mixed drink, we must have:

$$(s\,\text{mL}) + (v\,\text{mL}) = 600\,\text{mL}$$

Or without units:

$$s + v = 600$$

To build the second equation, we have to think about the alcohol concentrations for the scotch, vermouth, and Rob Roy. It can be tricky to think about percentages like these correctly. One strategy is to focus on the *amount* (in mL) of *alcohol* being mixed. If we have s milliliters of scotch that is 42% alcohol, then $0.42s$ is the actual *amount* (in mL) of alcohol in that scotch. Similarly, $0.18v$ is the amount of alcohol in the vermouth. And the final cocktail is 600 mL of liquid that is 34% alcohol, so it has $0.34(600) = 204$ milliliters of alcohol. All this means:

$$0.42(s\,\text{mL}) + 0.18(v\,\text{mL}) = 204\,\text{mL}$$

Or without units:

$$0.42s + 0.18v = 204$$

So our system is:

$$\begin{cases} s + v = 600 \\ 0.42s + 0.18v = 204 \end{cases}$$

To solve this system, we'll solve for s in the first equation:

$$\begin{aligned} s + v &= 600 \\ s &= 600 - v \end{aligned}$$

And then substitute s in the second equation with $600 - v$:

$$\begin{aligned}
0.42s + 0.18v &= 204 \\
0.42(600 - v) + 0.18v &= 204 \\
252 - 0.42v + 0.18v &= 204 \\
252 - 0.24v &= 204 \\
-0.24v &= -48 \\
\frac{-0.24v}{-0.24} &= \frac{-48}{-0.24} \\
v &= 200
\end{aligned}$$

As a last step, we will determine s using the equation where we had isolated s:

$$\begin{aligned}
s &= 600 - v \\
s &= 600 - 200 \\
s &= 400
\end{aligned}$$

In summary, LaVonda needs to combine 400 mL of scotch with 200 mL of vermouth to create 600 mL of Rob Roy that is 34% alcohol by volume.

As a check for Example 4.2.12, we will use **estimation** to see that our solution is reasonable. Since LaVonda is making a 34% solution, she would need to use more of the 42% concentration than the 18% concentration, because 34% is closer to 42% than to 18%. This agrees with our answer because we found that she needed 400 mL of the 42% solution and 200 mL of the 18% solution. This is an added check that we have found reasonable answers.

Example 4.2.13 Mixing a Coffee Blend. Desi owns a coffee shop and they want to mix two different types of coffee beans to make a blend that sells for \$12.50 per pound. They have some coffee beans from Columbia that sell for \$9.00 per pound and some coffee beans from Honduras that sell for \$14.00 per pound. How many pounds of each should they mix to make 30 pounds of the blend?

Explanation. Before we begin, it may be helpful to try to estimate the solution. Let's compare the three prices. Since \$12.50 is between the prices of \$9.00 and \$14.00, this mixture is possible. Now we need to estimate the amount of each type needed. The price of the blend (\$12.50 per pound) is closer to the higher priced beans (\$14.00 per pound) than the lower priced beans (\$9.00 per pound). So we will need to use more of that type. Keeping in mind that we need a total of 30 pounds, we roughly estimate 20 pounds of the \$14.00 Honduran beans and 10 pounds of the \$9.00 Columbian beans. How good is our estimate? Next we will solve this exercise exactly.

To set up our system of equations we define variables, letting C be the amount of Columbian coffee beans (in pounds) and H be the amount of Honduran coffee beans (in pounds).

The equations in our system will come from the total amount of beans and the total cost. The equation for the total amount of beans can be written as:

$$(C\,\text{lb}) + (H\,\text{lb}) = 30\,\text{lb}$$

Or without units:

$$C + H = 30$$

To build the second equation, we have to think about the cost of all these beans. If we have C pounds of Columbian beans that cost \$9.00 per pound, then $9C$ is the cost of those beans in dollars. Similarly, $14H$

is the cost of the Honduran beans. And the total cost is for 30 pounds of beans priced at \$12.50 per pound, totaling $12.5(30) = 37.5$ dollars. All this means:

$$\left(9\,\frac{\text{dollars}}{\text{lb}}\right)(C\,\text{lb}) + \left(14\,\frac{\text{dollars}}{\text{lb}}\right)(H\,\text{lb}) = \left(12.50\,\frac{\text{dollars}}{\text{lb}}\right)(30\,\text{lb})$$

Or without units and carrying out the multiplication on the right:

$$9C + 14H = 37.5$$

Now our system is:

$$\begin{cases} C + H = 30 \\ 9C + 14H = 37.50 \end{cases}$$

To solve the system, we'll solve the first equation for C:

$$\begin{aligned} C + H &= 30 \\ C &= 30 - H \end{aligned}$$

Next, we'll substitute C in the second equation with $30 - H$:

$$\begin{aligned} 9C + 14H &= 375 \\ 9(30 - H) + 14H &= 375 \\ 270 - 9H + 14H &= 375 \\ 270 + 5H &= 375 \\ 5H &= 105 \\ H &= 21 \end{aligned}$$

Since $H = 21$, we can conclude that $C = 9$.

In summary, Desi needs to mix 21 pounds of the Honduran coffee beans with 9 pounds of the Columbian coffee beans to create this blend. Our estimate at the beginning was pretty close, so we feel this answer is reasonable.

4.2.3 Solving Special Systems of Equations with Substitution

Remember the two special cases we encountered when solving by graphing in Subsection 4.1.2? If the two lines represented by a system of equations have the same slope, then they might be separate lines that never meet, meaning the system has no solutions. Or they might coincide as the same line, in which case there are infinitely many solutions represented by all the points on that line. Let's see what happens when we use the substitution method on each of the special cases.

Example 4.2.14 A System with No Solution. Solve the system of equations using the substitution method:

$$\begin{cases} y = 2x - 1 \\ 4x - 2y = 3 \end{cases}$$

Explanation. Since the first equation is already solved for y, we will substitute $2x - 1$ for y in the second equation, and we have:

$$\begin{aligned} 4x - 2y &= 3 \\ 4x - 2(2x - 1) &= 3 \\ 4x - 4x + 2 &= 3 \\ 2 &= 3 \end{aligned}$$

Even though we were only intending to substitute away y, we ended up with an equation where there are no variables at all. This will happen whenever the lines have the same slope. This tells us the system represents either parallel or coinciding lines. Since $2 = 3$ is false no matter what values x and y might be, there can be no solution to the system. So the lines are parallel and *distinct*. We write the solution set using the empty set symbol: the solution set is $\emptyset$.

To verify this, re-write the second equation, $4x - 2y = 3$, in slope-intercept form:

$$\begin{aligned} 4x - 2y &= 3 \\ -2y &= -4x + 3 \\ \frac{-2y}{-2} &= \frac{-4x + 3}{-2} \\ y &= \frac{-4x}{-2} + \frac{3}{-2} \\ y &= 2x - \frac{3}{2} \end{aligned}$$

So the system is equivalent to:

$$\begin{cases} y = 2x - 1 \\ y = 2x - \dfrac{3}{2} \end{cases}$$

Now it is easier to see that the two lines have the same slope but different y-intercepts. They are parallel and distinct lines, so the system has no solution.

Example 4.2.15 A System with Infinitely Many Solutions. Solve the system of equations using the substitution method:

$$\begin{cases} y = 2x - 1 \\ 4x - 2y = 2 \end{cases}$$

Explanation. Since $y = 2x - 1$, we will substitute $2x - 1$ for y in the second equation and we have:

$$\begin{aligned} 4x - 2y &= 2 \\ 4x - 2(2x - 1) &= 2 \\ 4x - 4x + 2 &= 2 \\ 2 &= 2 \end{aligned}$$

Even though we were only intending to substitute away y, we ended up with an equation where there are no variables at all. This will happen whenever the lines have the same slope. This tells us the system represents either parallel or coinciding lines. Since $2 = 2$ is true no matter what values x and y might be, the system equations are true no matter what x is, as long as $y = 2x - 1$. So the lines *coincide*. We write the solution set as $\{(x, y) \mid y = 2x - 1\}$.

To verify this, re-write the second equation, $4x - 2y = 2$, in slope-intercept form:

$$\begin{aligned} 4x - 2y &= 2 \\ -2y &= -4x + 2 \\ \frac{-2y}{-2} &= \frac{-4x}{-2} + \frac{2}{-2} \\ y &= 2x - 1 \end{aligned}$$

The system looks like:

$$\begin{cases} y = 2x - 1 \\ y = 2x - 1 \end{cases}$$

Now it is easier to see that the two equations represent the same line. Every point on the line is a solution to the system, so the system has infinitely many solutions. The solution set is $\{(x, y) \mid y = 2x - 1\}$.

4.2.4 Reading Questions

1. Give an example of a system of two equations in x and y where it would be nicer to solve the system using substitution than by graphing the two lines that the equations define. Explain why substitution would be nicer than graphing for your example system.

2. What might be a good first step if you have a system of two linear equations in two variables where there are fractions appearing in the equations?

3. In an application problem, thinking about the $\boxed{}$ can help you understand how to set up equations.

4.2.5 Exercises

Review and Warmup Solve the equation.

1. $\frac{5}{8} - 9a = 6$ **2.** $\frac{7}{4} - 7c = 6$ **3.** $\frac{3}{10} - \frac{1}{10}A = 4$ **4.** $\frac{7}{8} - \frac{1}{8}C = 1$

Solve the linear equation for y.

5. $-40x - 5y = -50$ **6.** $5y - 15x = -5$ **7.** $25x + 5y = -60$

8. $12x - 2y = -34$ **9.** $6x - y = -16$ **10.** $3x - y = -2$

Solving System of Equations Using Substitution Solve the following system of equations.

11. $\begin{cases} -4 + r = 0 \\ 6 = -2r + 2c \end{cases}$ **12.** $\begin{cases} 4y = -4x \\ 4 = -x \end{cases}$ **13.** $\begin{cases} -2y = -8 \\ -8 - 4x = -y \end{cases}$

14. $\begin{cases} -40-3y-4x=0 \\ 0=-20-5y \end{cases}$

15. $\begin{cases} y=-5x+21 \\ y=2x+7 \end{cases}$

16. $\begin{cases} y=-7-3x \\ y=4x+21 \end{cases}$

17. $\begin{cases} a=-3c+5 \\ 2a+c=-10 \end{cases}$

18. $\begin{cases} b=-30+4m \\ 3b+3m=-45 \end{cases}$

19. $\begin{cases} m=-20-5b \\ -m+5b=30 \end{cases}$

20. $\begin{cases} t=12-2B \\ 2B-5t=-12 \end{cases}$

21. $\begin{cases} x=2y-18 \\ 3y+3x=18 \end{cases}$

22. $\begin{cases} x=-18+5y \\ -3y-3x=36 \end{cases}$

23. $\begin{cases} x+y=-15 \\ 15-4x=-y \end{cases}$

24. $\begin{cases} 0=x+3y \\ y-5x+32=0 \end{cases}$

25. $\begin{cases} y=x+\dfrac{1}{2} \\ y=x-\dfrac{1}{2} \end{cases}$

26. $\begin{cases} C=2-5q \\ C=5-5q \end{cases}$

27. $\begin{cases} 2y=-8-B \\ -5B-2y=-8 \end{cases}$

28. $\begin{cases} -46+2a+5A=0 \\ 5a=-A \end{cases}$

29. $\begin{cases} -5m-4=0 \\ -3m=-2+5A \end{cases}$

30. $\begin{cases} 5y=-3x-2 \\ 5x=5 \end{cases}$

31. $\begin{cases} -5x=3+y \\ -5x=-3y+4 \end{cases}$

32. $\begin{cases} 2y+2=3x \\ 5x+y=3 \end{cases}$

33. $\begin{cases} 0=4-y-4x \\ -2x=4y+2 \end{cases}$

34. $\begin{cases} -x+3y=-2 \\ -2x+2y=1 \end{cases}$

35. $\begin{cases} c=1 \\ -2p+c=\dfrac{1}{4} \end{cases}$

36. $\begin{cases} -3p-4A-5=0 \\ -1+A=0 \end{cases}$

37. $\begin{cases} 0=-4r-t-\dfrac{5}{2} \\ 0=-2+\dfrac{2}{5}r \end{cases}$

38. $\begin{cases} -\dfrac{2}{3}=-n+\dfrac{3}{2}C \\ 0=1-n \end{cases}$

39. $\begin{cases} x+\dfrac{5}{2}+\dfrac{5}{3}y=0 \\ \dfrac{3}{4}x-\dfrac{1}{4}=2y \end{cases}$

40. $\begin{cases} \dfrac{4}{3}x+2=y \\ x+2y=\dfrac{2}{3} \end{cases}$

41. $\begin{cases} 0 = -\frac{5}{3}y + \frac{5}{2} + x \\ 0 = 3x + \frac{4}{3} - y \end{cases}$

42. $\begin{cases} -\frac{3}{2}x - y = -\frac{1}{5} \\ -y = -\frac{3}{5}x + 1 \end{cases}$

43. $\begin{cases} -3y - 5x = -3 \\ 2y + 2x = -2 \end{cases}$

44. $\begin{cases} 5x - 3y = -1 \\ -3y - 4x = -4 \end{cases}$

45. $\begin{cases} x + 4y = \frac{49}{10} \\ -2x - y = -\frac{77}{40} \end{cases}$

46. $\begin{cases} -5x + y = -\frac{71}{12} \\ -4x + 5y = -\frac{19}{12} \end{cases}$

47. $\begin{cases} \frac{1}{5}x - \frac{1}{2}y = -\frac{97}{60} \\ \frac{1}{5}x + \frac{1}{4}y = \frac{121}{120} \end{cases}$

48. $\begin{cases} -\frac{1}{5}x + \frac{1}{2}y = \frac{31}{100} \\ -\frac{1}{3}x + \frac{1}{2}y = \frac{3}{20} \end{cases}$

49. $\begin{cases} 2x + 3y = 36 \\ 5x + 4y = 55 \end{cases}$

50. $\begin{cases} 4x + 2y = 26 \\ 5x + y = 28 \end{cases}$

51. $\begin{cases} -x + 6y = -32 \\ 5x + 4y = 24 \end{cases}$

52. $\begin{cases} 4x - 5y = -5 \\ 4x + 3y = 67 \end{cases}$

53. $\begin{cases} -x - 3y = 3 \\ -x - 2y = 5 \end{cases}$

54. $\begin{cases} -4x - 5y = 53 \\ -x - 5y = 32 \end{cases}$

55. $\begin{cases} 3x + 4y = 24 \\ 3x = -12 \end{cases}$

56. $\begin{cases} -3x + y = 7 \\ 3x = -6 \end{cases}$

57. $\begin{cases} 3x + 2y = 6 \\ -9x - 6y = 6 \end{cases}$

58. $\begin{cases} 4x + 5y = 6 \\ -8x - 10y = 6 \end{cases}$

59. $\begin{cases} 4x + 3y = 5 \\ 12x + 9y = 15 \end{cases}$

60. $\begin{cases} 5x + y = 5 \\ 20x + 4y = 20 \end{cases}$

Applications

61. A rectangle's length is 5 feet shorter than five times its width. The rectangle's perimeter is 590 feet. Find the rectangle's length and width.

The rectangle's length is ________ feet, and its width is ________ feet.

62. A school fund raising event sold a total of 191 tickets and generated a total revenue of \$586.25. There are two types of tickets: adult tickets and child tickets. Each adult ticket costs \$6.95, and each child ticket costs \$1.50. Write and solve a system of equations to answer the following questions.

________ adult tickets and ________ child tickets were sold.

63. Phone Company A charges a monthly fee of \$43.80, and \$0.02 for each minute of talk time. Phone Company B charges a monthly fee of \$35.00, and \$0.06 for each minute of talk time. Write and solve a system equation to answer the following questions.

These two companies would charge the same amount on a monthly bill when the talk time was [ ] minutes.

64. Company A's revenue this fiscal year is \$878,000, but its revenue is decreasing by \$17,000 each year. Company B's revenue this fiscal year is \$488,000, and its revenue is increasing by \$13,000 each year. Write and solve a system of equations to answer the following question.

After [ ] years, Company B will catch up with Company A in revenue.

65. A test has 25 problems, which are worth a total of 100 points. There are two types of problems in the test. Each multiple-choice problem is worth 3 points, and each short-answer problem is worth 8 points. Write and solve a system equation to answer the following questions.

This test has [ ] multiple-choice problems and [ ] short-answer problems.

66. Candi invested a total of \$7,500 in two accounts. One account pays 3% interest annually; the other pays 4% interest annually. At the end of the year, Candi earned a total of \$255 in interest. Write and solve a system of equations to find how much money Candi invested in each account.

Candi invested [ ] in the 3% account and [ ] in the 4% account.

67. Katherine invested a total of \$13,000 in two accounts. After a year, one account lost 8.8%, while the other account gained 4.6%. In total, Katherine lost \$608. Write and solve a system of equations to find how much money Katherine invested in each account.

Katherine invested [ ] in the account with 8.8% loss and [ ] in the account with 4.6% gain.

68. Town A and Town B were located close to each other, and recently merged into one city. Town A had a population with 8% Asians. Town B had a population with 10% Asians. After the merge, the new city has a total of 4000 residents, with 9.3% Asians. Write and solve a system of equations to find how many residents Town A and Town B used to have.

Town A used to have [ ] residents, and Town B used to have [ ] residents.

69. You poured some 6% alcohol solution and some 12% alcohol solution into a mixing container. Now you have 600 grams of 10% alcohol solution. How many grams of 6% solution and how many grams of 12% solution did you pour into the mixing container?

Write and solve a system equation to answer the following questions.

You mixed [ ] grams of 6% solution with [ ] grams of 12% solution.

70. Briana invested a total of \$10,000 in two accounts. One account pays 7% interest annually; the other pays 2% interest annually. At the end of the year, Briana earned a total of \$400 in interest. How much money did Briana invest in each account?

Briana invested [ ] in the 7% account and [ ] in the 2% account.

71. Blake invested a total of \$52,000 in two accounts. One account pays 5.5% interest annually; the other pays 4.6% interest annually. At the end of the year, Blake earned a total interest of \$2,734. How much money did Blake invest in each account?

Blake invested [ ] in the 5.5% account and [ ] in the 4.6% account.

72. Renee invested a total of \$9,000 in two accounts. One account pays 3% interest annually; the other pays 6% interest annually. At the end of the year, Renee earned the same amount of interest from both accounts. How much money did Renee invest in each account?

Renee invested [ ] in the 3% account and [ ] in the 6% account.

73. Stephen invested a total of \$40,000 in two accounts. One account pays 6.6% interest annually; the other pays 3.4% interest annually. At the end of the year, Stephen earned the same amount of interest from both accounts. How much money did Stephen invest in each account?

Stephen invested [ ] in the 6.6% account [ ] in the 3.4% account.

74. Hannah invested a total of \$12,000 in two accounts. After a year, one account had *earned* 11.7%, while the other account had *lost* 7.6%. In total, Hannah had a net gain of \$535.50. How much money did Hannah invest in each account?

Hannah invested [ ] in the account that grew by 11.7% and [ ] in the account that fell by 7.6%.

75. You've poured some 12% (by mass) alcohol solution and some 8% alcohol solution into a large glass mixing container. Now you have 800 grams of 10% alcohol solution. How many grams of 12% solution and how many grams of 8% solution did you pour into the mixing container?

You poured [ ] grams of 12% solution and [ ] grams of 8% solution into the mixing container.

76. A store has some beans selling for \$1.70 per pound, and some vegetables selling for \$3.10 per pound. The store plans to use them to produce 14 pounds of mixture and sell for \$2.57 per pound. How many pounds of beans and how many pounds of vegetables should be used?

To produce 14 pounds of mixture, the store should use [ ] pounds of beans and [ ] pounds of vegetables.

77. Town A and Town B were located close to each other, and recently merged into one city. Town A had a population with 6% African Americans. Town B had a population with 10% African Americans. After the merge, the new city has a total of 4000 residents, with 8.8% African Americans. How many residents did Town A and Town B used to have?

Town A used to have [ ] residents, and Town B used to have [ ] residents.

4.3 Elimination

We learned how to solve a system of linear equations using substitution in Section 4.2. In this section, we will learn a second symbolic method for solving systems of linear equations.

4.3.1 Solving Systems of Equations by Elimination

Example 4.3.2 Alicia has \$1000 to give to her two grandchildren for New Year's. She would like to give the older grandchild \$120 more than the younger grandchild, because that is the cost of the older grandchild's college textbooks this term. How much money should she give to each grandchild?

To answer this question, we will demonstrate a new technique. You may have a very good way for finding how much money Alicia should give to each grandchild, but right now we will try to see this new method.

Let A be the dollar amount she gives to her older grandchild, and B be the dollar amount she gives to her younger grandchild. (As always, we start solving a word problem like this by defining the variables, including their units.) Since the total she has to give is \$1000, we can say that $A + B = 1000$. And since she wants to give \$120 more to the older grandchild, we can say that $A - B = 120$. So we have the system of equations:

$$\begin{cases} A + B = 1000 \\ A - B = 120 \end{cases}$$

We could solve this system by substitution as we learned in Section 4.2, but there is an easier method. If we add together the *left* sides from the two equations, it should equal the sum of the *right* sides:

$$\begin{array}{r} A + B \\ + A - B \end{array} = \begin{array}{r} 1000 \\ +120 \end{array}$$

So we have:

$$2A = 1120$$

Note that the variable B is eliminated. This happened because the "$+B$" and the "$-B$" perfectly cancel each other out when they are added. With only one variable left, it doesn't take much to finish:

$$\begin{aligned} 2A &= 1120 \\ A &= 560 \end{aligned}$$

To finish solving this system of equations, we need the value of B. For now, an easy way to find B is to substitute in our value of A into one of the original equations:

$$\begin{aligned} A + B &= 1000 \\ 560 + B &= 1000 \\ B &= 440 \end{aligned}$$

To check our work, substitute $A = 560$ and $B = 440$ into the original equations:

$$\begin{aligned} A + B &= 1000 \\ 560 + 440 &\stackrel{?}{=} 1000 \\ 1000 &\stackrel{\checkmark}{=} 1000 \end{aligned} \qquad \begin{aligned} A - B &= 120 \\ 560 - 440 &\stackrel{?}{=} 120 \\ 120 &\stackrel{\checkmark}{=} 120 \end{aligned}$$

This confirms that our solution is correct. In summary, Alicia should give \$560 to her older grandchild, and \$440 to her younger grandchild.

This method for solving the system of equations in Example 4.3.2 worked because B and $-B$ add to zero. Once the B-terms were eliminated we were able to solve for A. This method is called the **elimination method**. Some people call it the **addition method**, because we added the corresponding sides from the two equations to eliminate a variable.

If neither variable can be immediately eliminated, we can still use this method but it will require that we first adjust one or both of the equations. Let's look at an example where we need to adjust one of the equations.

Example 4.3.3 Scaling One Equation. Solve the system of equations using the elimination method.

$$\begin{cases} 3x - 4y = 2 \\ 5x + 8y = 18 \end{cases}$$

Explanation. To start, we want to see whether it will be easier to eliminate x or y. We see that the coefficients of x in each equation are 3 and 5, and the coefficients of y are -4 and 8. Because 8 is a multiple of 4 and the coefficients already have opposite signs, the y variable will be easier to eliminate.

To eliminate the y terms, we will multiply each side of the first equation by 2 so that we will have $-8y$. We can call this process scaling the first equation by 2.

$$\begin{cases} 2 \cdot (3x - 4y) = 2 \cdot (2) \\ 5x + 8y = 18 \end{cases}$$

$$\begin{cases} 6x - 8y = 4 \\ 5x + 8y = 18 \end{cases}$$

We now have an equivalent system of equations where the y-terms can be eliminated:

$$\begin{array}{r} 6x - 8y \\ + 5x + 8y \end{array} = \begin{array}{r} 4 \\ + 18 \end{array}$$

So we have:

$$\begin{aligned} 11x &= 22 \\ x &= 2 \end{aligned}$$

To solve for y, we can substitute 2 for x into either of the original equations or the new one. We use the first original equation, $3x - 4y = 2$:

$$\begin{aligned} 3x - 4y &= 2 \\ 3(2) - 4y &= 2 \\ 6 - 4y &= 2 \\ -4y &= -4 \\ y &= 1 \end{aligned}$$

Our solution is $x = 2$ and $y = 1$. We will check this in both of the original equations:

$$\begin{aligned} 5x + 8y &= 18 \\ 5(2) + 8(1) &\stackrel{?}{=} 18 \\ 10 + 8 &\stackrel{\checkmark}{=} 18 \end{aligned} \qquad \begin{aligned} 3x - 4y &= 2 \\ 3(2) - 4(1) &\stackrel{?}{=} 2 \\ 6 - 4 &\stackrel{\checkmark}{=} 2 \end{aligned}$$

The solution to this system is $(2, 1)$ and the solution set is $\{(2, 1)\}$.

Checkpoint 4.3.4 Solve the following system of equations.

$$\begin{cases} 5x + 4y = -7 \\ 5x + 2y = -1 \end{cases}$$

Explanation.

1. We subtract the two equations, which will cancel the terms in involving x and give $4y - 2y = -7 - (-1)$.
2. This gives $y = -3$.
3. Now that we have y, we find x using either equation. Let's use the first: $5x - 12 = -7$, so $x = 1$.
4. The solution to the system is $(1, -3)$. It is left as an exercise to check. Please also note that you may have solved this problem a different way.

Here's an example where we have to scale both equations.

Example 4.3.5 Scaling Both Equations. Solve the system of equations using the elimination method.

$$\begin{cases} 2x + 3y = 10 \\ -3x + 5y = -15 \end{cases}$$

Explanation. Considering the coefficients of x (2 and -3) and the coefficients of y (3 and 5) we see that we cannot eliminate the x or the y variable by scaling a single equation. We will need to scale *both*.

The x-terms already have opposite signs, so we choose to eliminate x. The least common multiple of 2 and 3 is 6. We can scale the first equation by 3 and the second equation by 2 so that the equations have terms $6x$ and $-6x$, which will cancel when added.

$$\begin{cases} 3 \cdot (2x + 3y) = 3 \cdot (10) \\ 2 \cdot (-3x + 5y) = 2 \cdot (-15) \end{cases}$$

$$\begin{cases} 6x + 9y = 30 \\ -6x + 10y = -30 \end{cases}$$

At this point we can add the corresponding sides from the two equations and solve for y:

$$\begin{array}{r} 6x + 9y \\ -\,6x + 10y \end{array} = \begin{array}{r} 30 \\ -\,30 \end{array}$$

So we have:

$$\begin{aligned} 19y &= 0 \\ y &= 0 \end{aligned}$$

To solve for x, we'll replace y with 0 in $2x + 3y = 10$:

$$\begin{aligned} 2x + 3y &= 10 \\ 2x + 3(0) &= 10 \\ 2x &= 10 \\ x &= 5 \end{aligned}$$

We'll check the system using $x = 5$ and $y = 0$ in each of the original equations:

$$\begin{aligned} 2x + 3y &= 10 \\ 2(5) + 3(0) &\stackrel{?}{=} 10 \\ 10 + 0 &\stackrel{\checkmark}{=} 10 \end{aligned} \qquad \begin{aligned} -3x + 5y &= -15 \\ -3(5) + 5(0) &\stackrel{?}{=} -15 \\ -15 + 0 &\stackrel{\checkmark}{=} -15 \end{aligned}$$

So the system's solution is $(5, 0)$ and the solution set is $\{(5, 0)\}$.

Checkpoint 4.3.6 Try a similar exercise.

Solve the following system of equations.

$$\begin{cases} 3x + 4y = -26 \\ 5x + 5y = -40 \end{cases}$$

Explanation.

1. Let's multiply the *first* equation by 5 and the *second* equation by 3

$$\begin{aligned} 15x + 20y &= -130 \\ 15x + 15y &= -120 \end{aligned}$$

2. Subtracting these two equations gives $20y - 15y = -10$, so $y = -2$.
3. Now that we have y, we can use either equation to find x; let's use the first one:

$$3x + (4) \cdot (-2) = -26$$

so $x = -6$.

4. The solution to the system is $(-6, -2)$. It is left as an exercise to check. Please also note that you may have solved this problem a different way.

Example 4.3.7 Meal Planning. Javed is on a meal plan and needs to consume 600 calories and 20 grams of fat for breakfast. A small avocado contains 300 calories and 30 grams of fat. He has bagels that contain 400 calories and 8 grams of fat. Write and solve a system of equations to determine how much bagel and avocado would combine to make his target calories and fat.

Explanation. To write this system of equations, we first need to define our variables. Let A be the number of avocados consumed and let B be the number of bagels consumed. Both A and B might be fractions. For our first equation, we will count calories from the avocados and bagels:

$$\left(300\,\frac{\text{calories}}{\text{avocado}}\right)(A\text{ avocados}) + \left(400\,\frac{\text{calories}}{\text{bagel}}\right)(B\text{ bagel}) = 600\text{ calories}$$

Or, without the units:

$$300A + 400B = 600$$

Similarly, for our second equation, we will count the grams of fat:

$$\left(30\,\frac{\text{g fat}}{\text{avocado}}\right)(A\text{ avocados}) + \left(8\,\frac{\text{g fat}}{\text{bagel}}\right)(B\text{ bagel}) = 20\text{ g fat}$$

Or, without the units:

$$30A + 8B = 20$$

So the system of equations is:

$$\begin{cases} 300A + 400B = 600 \\ 30A + 8B = 20 \end{cases}$$

Since none of the coefficients are equal to 1, it will be easier to use the elimination method to solve this system. Looking at the terms $300A$ and $30A$, we can eliminate the A variable if we multiply the second equation by -10 to get $-300A$:

$$\begin{cases} 300A + 400B = 600 \\ -10 \cdot (30A + 8B) = -10 \cdot (20) \end{cases}$$

$$\begin{cases} 300A + 400B = 600 \\ -300A + (-80B) = -200 \end{cases}$$

When we add the corresponding sides from the two equations together we have:

$$\begin{array}{r} 300A + 400B \\ -\,300A - 80B \end{array} = \begin{array}{r} 600 \\ -\,200 \end{array}$$

So we have:

$$\begin{aligned} 320B &= 400 \\ \frac{320B}{320} &= \frac{400}{320} \end{aligned}$$

$$B = \frac{5}{4}$$

We now know that Javed should eat $\frac{5}{4}$ bagels (or one and one-quarter bagels). To determine the number of avocados, we will substitute B with $\frac{5}{4}$ in either of our original equations.

$$\begin{aligned} 300A + 400B &= 600 \\ 300A + 400\left(\frac{5}{4}\right) &= 600 \\ 300A + 500 &= 600 \\ 300A &= 100 \\ \frac{300A}{300} &= \frac{100}{300} \\ A &= \frac{1}{3} \end{aligned}$$

To check this result, try using $B = \frac{5}{4}$ and $A = \frac{1}{3}$ in each of the original equations:

$$\begin{aligned} 300A + 400B &= 600 \\ 300\left(\frac{1}{3}\right) + 400\left(\frac{5}{4}\right) &\stackrel{?}{=} 600 \\ 100 + 500 &\stackrel{\checkmark}{=} 600 \end{aligned} \qquad \begin{aligned} 30A + 8B &= 20 \\ 30\left(\frac{1}{3}\right) + 8\left(\frac{5}{4}\right) &\stackrel{?}{=} 20 \\ 10 + 10 &\stackrel{\checkmark}{=} 20 \end{aligned}$$

In summary, Javed can eat $\frac{5}{4}$ of a bagel (so one and one-quarter bagel) and $\frac{1}{3}$ of an avocado in order to consume exactly 600 calories and 20 grams of fat.

For summary reference, here is the general procedure.

Process 4.3.8 Solving Systems of Equations by Elimination. *To solve a system of equations by elimination,*

1. *Algebraically manipulate both equations into standard form, unless the variables are already aligned (e.g. if both equations are in slope intercept form already).*
2. *Scale one or both of the equations to force one of the variables to have equal but opposite sign in the two equations.*
3. *Add the corresponding sides of the two equations together which should have the effect that one variable cancels out entirely.*
4. *Solve the resulting equation for the one remaining variable.*
5. *Substitute that value into either of the* original *equations to find the other variable.*
6. *Verify your solution.*

4.3.2 Solving Special Systems of Equations with Elimination

Remember the two special cases we encountered when solving by graphing and substitution? Sometimes a system of equations has no solutions at all, and sometimes the solution set is infinite with all of the points on one line satisfying the equations. Let's see what happens when we use the elimination method on each of the special cases.

Example 4.3.9 A System with Infinitely Many Solutions. Solve the system of equations using the elimination method.

$$\begin{cases} 3x + 4y = 5 \\ 6x + 8y = 10 \end{cases}$$

Explanation. To eliminate the x-terms, we multiply each term in the first equation by -2, and we have:

$$\begin{cases} -2 \cdot (3x + 4y) = -2 \cdot 5 \\ 6x + 8y = 10 \end{cases}$$

$$\begin{cases} -6x + -8y = -10 \\ 6x + 8y = 10 \end{cases}$$

We might notice that the equations look very similar. Adding the respective sides of the equation, we have:

$$0 = 0$$

Both of the variables have been eliminated. Since the statement $0 = 0$ is true no matter what x and y are, the solution set is infinite. Specifically, you just need any (x, y) satisfying *one* of the two equations, since the two equations represent the same line. We can write the solution set as $\{(x, y) \mid 3x + 4y = 5\}$.

Example 4.3.10 A System with No Solution. Solve the system of equations using the elimination method.

$$\begin{cases} 10x + 6y = 9 \\ 25x + 15y = 4 \end{cases}$$

Explanation. To eliminate the x-terms, we will scale the first equation by -5 and the second by 2:

$$\begin{cases} -5 \cdot (10x + 6y) = -5 \cdot (9) \\ 2 \cdot (25x + 15y) = 2 \cdot (4) \end{cases}$$

$$\begin{cases} -50x + (-30y) = -45 \\ 50x + 30y = 8 \end{cases}$$

Adding the respective sides of the equation, we have:

$$0 = -37$$

Both of the variables have been eliminated. In this case, the statement $0 = -37$ is just false, no matter what x and y are. So the system has no solution.

4.3.3 Deciding to Use Substitution versus Elimination

In every example so far from this section, both equations were in standard form, $Ax + By = C$. And all of the coefficients were integers . If none of the coefficients are equal to 1 then it is usually easier to use the elimination method, because otherwise you will probably have some fraction arithmetic to do in the middle of the substitution method. If there *is* a coefficient of 1, then it is a matter of preference.

Example 4.3.11 A college used to have a north campus with 6000 students and a south campus with 15,000 students. The percentage of students at the north campus who self-identify as LGBTQ was three times the percentage at the south campus. After the merge, 5.5% of students identify as LGBTQ. What percentage of students on each campus identified as LGBTQ before the merge?

Explanation. We will define N as the percentage (as a decimal) of students at the north campus and S as the percentage (as a decimal) of students at the south campus that identified as LGBTQ. Since the percentage of students at the north campus was three times the percentage at the south campus, we have:

$$N = 3S$$

For our second equation, we will count LGBTQ students at the various campuses. At the north campus, multiply the population, 6000, by the percentage N to get 6000N. This must be the actual number of LGBTQ students. Similarly, the south campus has 15000S LGBTQ students, and the combined school has $21000(0.055) = 1155$. When we combine the two campuses, we have:

$$6000N + 15000S = 1155$$

We write the system as:

$$\begin{cases} N = 3S \\ 6000N + 15000S = 1155 \end{cases}$$

Because the first equation is already solved for N, this is a good time to *not* use the elimination method. Instead we can substitute N in our second equation with 3S and solve for S:

$$\begin{aligned} 6000N + 15000S &= 1155 \\ 6000(3S) + 15000S &= 1155 \\ 18000S + 15000S &= 1155 \\ 33000S &= 1155 \\ \frac{33000S}{33000} &= \frac{1155}{33000} \\ S &= 0.035 \end{aligned}$$

We can determine N using the first equation:

$$\begin{aligned} N &= 3S \\ N &= 3(0.035) \\ N &= 0.105 \end{aligned}$$

Before the merge, 10.5% of the north campus students self-identified as LGBTQ, and 3.5% of the south campus students self-identified as LGBTQ.

If you need to solve a system, and one of the equations is not in standard form, substitution may be easier. But you also may find it easier to convert the equations into standard form. Additionally, if the system's coefficients are fractions or decimals, you may take an additional step to scale the equations so that they only have integer coefficients.

Example 4.3.12 Solve the system of equations using the method of your choice.

$$\begin{cases} -\frac{1}{3}y = \frac{1}{15}x + \frac{1}{5} \\ \frac{5}{2}x - y = 6 \end{cases}$$

Explanation. First, we can cancel the fractions by using the least common multiple of the denominators in each equation, similarly to the topic of Section 2.3. We have:

$$\begin{cases} 15 \cdot -\frac{1}{3}y = 15 \cdot \left(\frac{1}{15}x + \frac{1}{5}\right) \\ 2 \cdot \left(\frac{5}{2}x - y\right) = 2 \cdot (6) \end{cases}$$

$$\begin{cases} -5y = x + 3 \\ 5x - 2y = 12 \end{cases}$$

We could put convert the first equation into standard form by subtracting x from both sides, and then use elimination. However, the x-variable in the first equation has a coefficient of 1, so the substitution method may be faster. Solving for x in the first equation we have:

$$\begin{aligned} -5y &= x + 3 \\ -5y - 3 &= x + 3 - 3 \\ -5y - 3 &= x \end{aligned}$$

Substituting $-5y - 3$ for x in the second equation we have:

$$\begin{aligned} 5(-5y - 3) - 2y &= 12 \\ -25y - 15 - 2y &= 12 \\ -27y - 15 &= 12 \\ -27y &= 27 \\ y &= -1 \end{aligned}$$

Using the equation where we isolated x and substituting -1 for y, we have:

$$\begin{aligned} -5(-1) - 3 &= x \\ 5 - 3 &= x \\ 2 &= x \end{aligned}$$

The solution is $(2, -1)$. Checking the solution is left as an exercise.

Example 4.3.13 A penny is made by combining copper and zinc. A chemistry reference source says copper has a density of $9\,\frac{\text{g}}{\text{cm}^3}$ and zinc has a density of $7.1\,\frac{\text{g}}{\text{cm}^3}$. A penny's mass is $2.5\,\text{g}$ and its volume is $0.35\,\text{cm}^3$. How many cm^3 each of copper and zinc go into one penny?

Explanation. Let c be the volume of copper and z be the volume of zinc in one penny, both measured in

cm^3. Since the total volume is $0.35\,\mathrm{cm}^3$, one equation is:

$$\left(c\,\mathrm{cm}^3\right) + \left(z\,\mathrm{cm}^3\right) = 0.35\,\mathrm{cm}^3$$

Or without units:

$$c + z = 0.35.$$

For the second equation, we will examine the masses of copper and zinc. Since copper has a density of $9\,\frac{\mathrm{g}}{\mathrm{cm}^3}$ and we are using c to represent the volume of copper, the mass of copper is $9c$. Similarly, the mass of zinc is 7.1. Since the total mass is $2.5\,\mathrm{g}$, we have the equation:

$$\left(9\,\tfrac{\mathrm{g}}{\mathrm{cm}^3}\right)\left(c\,\mathrm{cm}^3\right) + \left(7.1\,\tfrac{\mathrm{g}}{\mathrm{cm}^3}\right)\left(z\,\mathrm{cm}^3\right) = 2.5\,\mathrm{g}$$

Or without units:

$$9c + 7.1z = 2.5.$$

So we have a system of equations:

$$\begin{cases} c + z = 0.35 \\ 9c + 7.1z = 2.5 \end{cases}$$

Since the coefficient of c (or z) in the first equation is 1, we could solve for one of these variables and use substitution to complete the problem. Some decimal arithmetic would be required. Alternatively, we can scale the equations by the right power of 10 to make all the coefficients integers:

$$\begin{cases} 100 \cdot (c + z) = 100 \cdot (0.35) \\ 10 \cdot (9c + 7.1z) = 10 \cdot (2.5) \end{cases}$$

$$\begin{cases} 100c + 100z = 35 \\ 90c + 71z = 25 \end{cases}$$

Now to set up elimination, scale each equation again to eliminate c:

$$\begin{cases} 9 \cdot (100c + 100z) = 9 \cdot (35) \\ -10 \cdot (90c + 71z) = -10 \cdot (25) \end{cases}$$

$$\begin{cases} 900c + 900z = 315 \\ -900c + (-710z) = -250 \end{cases}$$

Adding the corresponding sides from the two equations gives

$$190z = 65,$$

from which we find $z = \frac{65}{190} \approx 0.342$. So there is about $0.342\,\mathrm{cm}^3$ of zinc in a penny.

To solve for c, we can use one of the original equations:

$$\begin{aligned} c + z &= 0.35 \\ c + 0.342 &\approx 0.35 \\ c &\approx 0.008 \end{aligned}$$

Therefore there is about $0.342\,\mathrm{cm}^3$ of zinc and $0.008\,\mathrm{cm}^3$ of copper in a penny.

To summarize, if a variable is already isolated or has a coefficient of 1, consider using the substitution method. If both equations are in standard form or none of the coefficients are equal to 1, we suggest using the elimination method. Either way, if you have fraction or decimal coefficients, it may help to scale your equations so that only integer coefficients remain.

4.3.4 Reading Questions

1. What is another name that the “elimination method” goes by?

2. To use the elimination method, usually the first step is to $\boxed{}$ at least one equation.

3. Describe a good situation to use the substitution method instead of the elimination method for solving a system of two linear equations in two variables.

4.3.5 Exercises

Review and Warmup Solve the equation.

1. $\frac{3}{4} - 10x = 6$

2. $\frac{7}{10} - 8r = 6$

3. $\frac{5}{6} - \frac{1}{6}a = 5$

4. $\frac{5}{2} - \frac{1}{2}b = 2$

5. $\frac{8A}{9} - 9 = -\frac{89}{9}$

6. $\frac{4B}{5} - 7 = -\frac{43}{5}$

Solving System of Equations by Elimination Solve the following system of equations.

7. $\begin{cases} 4x + 2y = 4 \\ 3x + y = 7 \end{cases}$

8. $\begin{cases} x + y = 13 \\ 2x + 3y = 32 \end{cases}$

9. $\begin{cases} 4x - 2y = 42 \\ 5x + 2y = 48 \end{cases}$

10. $\begin{cases} -5x + 3y = 18 \\ 4x + 6y = -90 \end{cases}$

11. $\begin{cases} -4x - 4y = 8 \\ -4x - 5y = 3 \end{cases}$

12. $\begin{cases} -2x - y = 10 \\ -4x - 3y = 22 \end{cases}$

13. $\begin{cases} -3x + 2y = -14 \\ -2x = 4 \end{cases}$

14. $\begin{cases} x - y = -4 \\ -2x = 0 \end{cases}$

15. $\begin{cases} 4x + 2y = -4 \\ -8x - 4y = -4 \end{cases}$

16. $\begin{cases} 4x + y = -4 \\ 12x + 3y = -4 \end{cases}$

17. $\begin{cases} 5x + 4y = -5 \\ 20x + 16y = -20 \end{cases}$

18. $\begin{cases} 5x + 2y = -5 \\ -10x - 4y = 10 \end{cases}$

19. $\begin{cases} 6 + C = -2q \\ 6 - 4C = 5q \end{cases}$

20. $\begin{cases} -y = -57 + 4B \\ -5y = B - 38 \end{cases}$

21. $\begin{cases} -5y - x = 2 \\ y + 4x = 4 \end{cases}$

22. $\begin{cases} -4x + 5y = -4 \\ y - 2x = 1 \end{cases}$

23. $\begin{cases} -y = -4x - 30 \\ -30 + 3x + 3y = 0 \end{cases}$

24. $\begin{cases} 2x = 5y + 34 \\ 3x + 51 = -y \end{cases}$

25. $\begin{cases} y + 3 = 4x \\ -1 - 4y - 3x = 0 \end{cases}$

26. $\begin{cases} -3 = 2y + x \\ 3x = -2y \end{cases}$

27. $\begin{cases} 3x - 4y = -3 \\ 1 - x + 4y = 0 \end{cases}$

28. $\begin{cases} -5n + 2c = -2 \\ -n + 5 = 2c \end{cases}$

29. $\begin{cases} -3p + \frac{1}{2} - 3c = 0 \\ -\frac{4}{5} + p = -\frac{1}{3}c \end{cases}$

30. $\begin{cases} q = \frac{3}{5} - y \\ 0 = \frac{1}{3}q - \frac{5}{2} + \frac{5}{4}y \end{cases}$

31. $\begin{cases} -C = 5 + 2n \\ \frac{1}{2} = -\frac{3}{4}C - \frac{3}{2}n \end{cases}$

32. $\begin{cases} -y - 4 - \frac{3}{4}x = 0 \\ 2y - 2x = \frac{4}{5} \end{cases}$

33. $\begin{cases} 3x + 13 = 5y \\ -12x = -20y + 52 \end{cases}$

34. $\begin{cases} 0 = 1 + 4x + 4y \\ x + 1 + y = 0 \end{cases}$

35. $\begin{cases} x + y = \frac{17}{6} \\ 2x - 4y = \frac{2}{3} \end{cases}$

36. $\begin{cases} -x + y = -\frac{13}{12} \\ -3x - 3y = -\frac{67}{4} \end{cases}$

37. $\begin{cases} \frac{1}{2}x - \frac{1}{3}y = -\frac{13}{18} \\ \frac{1}{5}x - \frac{1}{2}y = -\frac{217}{180} \end{cases}$

38. $\begin{cases} -\frac{1}{5}x + \frac{1}{3}y = \frac{59}{200} \\ -\frac{1}{2}x + \frac{1}{4}y = \frac{13}{160} \end{cases}$

Applications

39. A test has 17 problems, which are worth a total of 72 points. There are two types of problems in the test. Each multiple-choice problem is worth 3 points, and each short-answer problem is worth 6 points. Write and solve a system of equations to answer the following questions.

This test has ______ multiple-choice problems and ______ short-answer problems.

40. Kandace invested a total of \$7,000 in two accounts. One account pays 5% interest annually; the other pays 4% interest annually. At the end of the year, Kandace earned a total of \$305 in interest. Write and solve a system of equations to find how much money Kandace invested in each account.

Kandace invested ______ in the 5% account and ______ in the 4% account.

41. Adrian invested a total of \$12,000 in two accounts. After a year, one account lost 6.8%, while the other account gained 6.9%. In total, Adrian lost \$473.50. Write and solve a system of equations to find how much money Adrian invested in each.

Adrian invested ______ in the account with 6.8% loss and ______ in the account with 6.9% gain.

42. Town A and Town B were located close to each other, and recently merged into one city. Town A had a population with 12% whites. Town B had a population with 10% whites. After the merge, the new city has a total of 4000 residents, with 10.7% whites. Write and solve a system of equations to find how many residents Town A and Town B used to have.

Town A used to have ______ residents, and Town B used to have ______ residents.

43. You poured some 6% alcohol solution and some 10% alcohol solution into a mixing container. Now you have 600 grams of 8.8% alcohol solution. Write and solve a system of equations to find how many grams of 6% solution and how many grams of 10% solution you poured into the mixing container.

You mixed ______ grams of 6% solution with ______ grams of 10% solution.

44. You will purchase some CDs and DVDs. If you purchase 15 CDs and 14 DVDs, it will cost you \$165.60; if you purchase 14 CDs and 15 DVDs, it will cost you \$166.45. Write and solve a system of equations to answer the following questions.

Each CD costs ______ and each DVD costs ______.

45. A school fund raising event sold a total of 235 tickets and generated a total revenue of \$1,085.60. There are two types of tickets: adult tickets and child tickets. Each adult ticket costs \$5.80, and each child ticket costs \$3.90. Write and solve a system of equations to answer the following questions.

______ adult tickets and ______ child tickets were sold.

46. Phone Company A charges a monthly fee of \$42.00, and \$0.04 for each minute of talk time. Phone Company B charges a monthly fee of \$30.00, and \$0.10 for each minute of talk time. Write and solve a system equation to answer the following questions.

These two companies would charge the same amount on a monthly bill when the talk time was ______ minutes.

47. Company A's revenue this fiscal year is \$843,000, but its revenue is decreasing by \$8,000 each year. Company B's revenue this fiscal year is \$634,000, and its revenue is increasing by \$11,000 each year. Write and solve a system of equations to answer the following question.

After ______ years, Company B will catch up with Company A in revenue.

48. If a boat travels from Town A to Town B, it has to travel 1190 mi along a river. A boat traveled from Town A to Town B along the river's current with its engine running at full speed. This trip took 42.5 hr. Then the boat traveled back from Town B to Town A, again with the engine at full speed, but this time against the river's current. This trip took 119 hr. Write and solve a system of equations to answer the following questions.

The boat's speed in still water with the engine running at full speed is ______.

The river current's speed was ______.

49. A small fair charges different admission for adults and children. It charges \$2.50 for adults, and \$1.50 for children. On a certain day, the total revenue is \$3,764 and the fair admits 2000 people. How many adults and children were admitted?

There were ______ adults and ______ children at the fair.

Challenge

50. Find the value of b so that the system of equations has an infinite number of solutions.

$$\begin{cases} -16x + 28y = -4 \\ 4x - by = 1 \end{cases}$$

4.4 Systems of Linear Equations Chapter Review

4.4.1 Solving Systems of Linear Equations by Graphing

In Section 4.1 we covered the definition of system of linear equations and how a solution to a system of linear equation is a point where the graphs of the two equations cross. We also considered special systems of equations where the lines they define coincide or never cross.

Example 4.4.1 Solving Systems of Linear Equations by Graphing. Solve the following system of equations by graphing:

$$\begin{cases} y = -\dfrac{2}{3}x - 4 \\ y = -4x - 14 \end{cases}$$

Explanation.

The first equation, $y = -\frac{2}{3}x - 4$, is a linear equation in slope-intercept form with a slope of $-\frac{2}{3}$ and a y-intercept of $(0, -4)$. The second equation, $y = -4x - 14$, is a linear equation in slope-intercept form with a slope of -4 and a y-intercept of $(0, -14)$. We'll use this information to graph both lines in Figure 4.4.2.

The two lines intersect where $x = -3$ and $y = -2$, so the solution of the system of equations is the point $(-3, -2)$. We write the solution set as $\{(-3, -2)\}$.

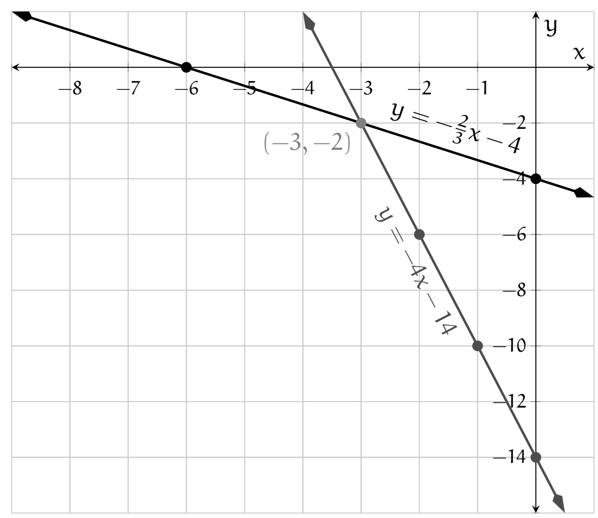


Figure 4.4.2: Graphs of $y = -\frac{2}{3}x - 4$ and $y = -4x - 14$.

Example 4.4.3 Special Systems of Equations. Solve the following system of equations by graphing:

$$\begin{cases} y = \dfrac{3}{2}(x - 1) + 4 \\ 3x - 2y = 4 \end{cases}$$

Explanation. The first equation, $y = \frac{3}{2}(x - 1) + 4$, is a linear equation in point-slope form with a slope of $\frac{3}{2}$ that passes through the point $(1, 4)$. The second equation, $3x - 2y = 4$, is a linear equation in standard form To graph this line, we either need to find the intercepts or put the equation into slope-intercept form. Just for practice, we will put the line in slope-intercept form.

$$\begin{aligned} 3x - 2y &= 4 \\ -2y &= -3x + 4 \end{aligned}$$

$$\frac{-2y}{-2} = \frac{-3x}{-2} + \frac{4}{-2}$$
$$y = \frac{3}{2}x - 2$$

We'll use this information to graph both lines:

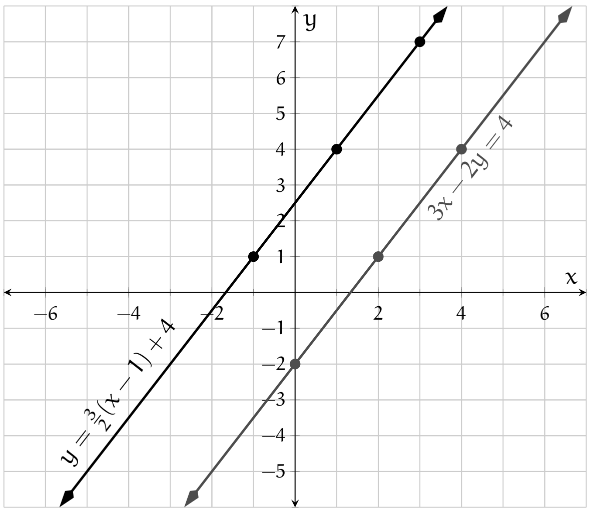


Figure 4.4.4: Graphs of $y = \frac{3}{2}(x-1)+4$ and $3x-2y=4$.

The two lines never intersect: they are parallel. So there are no solutions to the system of equations. We write the solution set as $\emptyset$.

4.4.2 Substitution

In Section 4.2, we covered the substitution method of solving systems of equations. We isolated one variable in one equation and then substituted into the other equation to solve for one variable.

Example 4.4.5 Solving Systems of Equations Using Substitution. Solve this system of equations using substitution:

$$\begin{cases} -5x + 6y = -10 \\ 4x - 3y = -1 \end{cases}$$

Explanation. We need to solve for *one* of the variables in *one* of our equations. Looking at both equations, it will be best to solve for y in the second equation. The coefficient of y in that equation is smallest.

$$4x - 3y = -1$$
$$-3y = -1 - 4x$$
$$\frac{-3y}{-3} = \frac{-1}{-3} - \frac{4x}{-3}$$
$$y = \frac{1}{3} + \frac{4}{3}x$$

Replace y in the first equation with $\frac{1}{3} + \frac{4}{3}x$, giving us a linear equation in only one variable, x, that we may solve:

$$\begin{aligned} -5x + 6y &= -10 \\ -5x + 6\left(\frac{1}{3} + \frac{4}{3}x\right) &= -10 \\ -5x + 2 + 8x &= -10 \\ 3x + 2 &= -10 \\ 3x &= -12 \\ x &= -4 \end{aligned}$$

Now that we have the value for x, we need to find the value for y. We have already solved the second equation for y, so that is the easiest equation to use.

$$\begin{aligned} y &= \frac{1}{3} + \frac{4}{3}x \\ y &= \frac{1}{3} + \frac{4}{3}(-4) \\ y &= \frac{1}{3} - \frac{16}{3} \\ y &= -\frac{15}{3} \\ y &= -5 \end{aligned}$$

To check this solution, we replace x with -4 and y with -5 in each equation:

$$\begin{aligned} -5x + 6y &= -10 \\ -5(-4) + 6(-5) &\stackrel{?}{=} -10 \\ 20 - 30 &\stackrel{\checkmark}{=} -10 \end{aligned} \qquad \begin{aligned} 4x - 3y &= -1 \\ 4(-4) - 3(-4) &\stackrel{?}{=} -1 \\ -16 + 15 &\stackrel{\checkmark}{=} -1 \end{aligned}$$

We conclude then that this system of equations is true when $x = -4$ and $y = -5$. Our solution is the point $(-4, -5)$ and we write the solution set as $\{(-4, -5)\}$.

Example 4.4.6 Applications of Systems of Equations. The Rusk Ranch Nature Center[1] in south-western Oregon is a volunteer run nonprofit that exists to promote the wellbeing of the local communities and conserve local nature with an emphasis on native butterflies. They sell admission tickets: \$6 for adults and \$4 for children. Amanda, who was working at the front desk, counted that one day she sold a total of 79 tickets and had \$384 in the register from those ticket sales. She didn't bother to count how many were adult tickets and how many were child tickets because she knew she could use math to figure it out at the end of the day. So, how many of the 79 tickets were adult and how many were child?

Explanation. Let's let a represent the number of adult tickets sold and c represent the number of child tickets sold. We need to build two equations to solve a system for both variables.

The first equation we will build relates to the fact that there were 79 total tickets sold. If we combine both the number of adult tickets and child tickets, the total is 79. This fact becomes:

$$a + c = 79$$

For the second equation we need to use the per-ticket dollar amounts to generate the total cost of \$384. The amount of money that was made from adult tickets is found my multiplying the number of adult tickets sold, a, by the price per ticket, \$6. Similarly, the amount of money from child tickets is found my multiplying the number of child tickets sold, c, by the price per ticket, \$4. These two amounts will add to be \$384. This fact becomes:

$$6a + 4c = 384$$

And so, our system is

$$\begin{cases} a + c = 79 \\ 6a + 4c = 384 \end{cases}$$

To solve, we will use the substitution method and solve the first equation for the variable a.

$$\begin{aligned} a + c &= 79 \\ a &= 79 - c \end{aligned}$$

Now we will substitute $79 - c$ for a in the second equation.

$$\begin{aligned} 6a + 4c &= 384 \\ 6(79 - c) + 4c &= 384 \\ 474 - 6c + 4c &= 384 \\ 474 - 2c &= 384 \\ -2c &= -90 \\ c &= 45 \end{aligned}$$

Last, we will solve for a by substituting 45 in for c in the equation $a = 79 - c$.

$$\begin{aligned} a &= 79 - c \\ a &= 79 - 45 \\ a &= 34 \end{aligned}$$

Our conclusion is that Amanda sold 34 adult tickets and 45 child tickets.

Example 4.4.7 Solving Special Systems of Equations with Substitution. Solve the systems of linear equations using substitution.

a.
$$\begin{cases} 3x - 5y = 9 \\ x = \frac{5}{3}y + 3 \end{cases}$$

b.
$$\begin{cases} y + 7 = 4x \\ 2y - 8x = 7 \end{cases}$$

Explanation. To solve the systems using substitution, we first need to solve for one variable in one equation, then substitute into the *other* equation.

a. In this case, x is already solved for in the second equation so we can substitute $\frac{5}{3}y + 3$ everywhere we see x in the first equation. Then simplify and solve for y.

$$\begin{aligned} 3x - 5y &= 9 \\ 3\left(\frac{5}{3}y + 3\right) - 5y &= 9 \\ 5y + 9 - 5y &= 9 \\ 9 &= 9 \end{aligned}$$

We will stop here since we have eliminated all of the variables in the equation and ended with a *true* statement. Since 9 always equals 9, no matter what, then any value of y must make the original equation, $3\left(\frac{5}{3}y + 3\right) - 5y = 9$ true. If you recall from the section on substitution, this means that both lines $3x - 5y = 9$ and $x = \frac{5}{3}y + 3$ are in fact the same line. Since a solution to a system of linear equations is any point where the lines touch, *all* points along both lines are solutions. We can say this

[1] ruskranchnaturecenter.org

in English as, "The solutions are all points on the line $3x - 5y = 9$," or in math as, "The solution set is $\{(x, y) \mid 3x - 5y = 9\}$."

b. We will first solve the top equation for y.

$$\begin{aligned} y + 7 &= 4x \\ y &= 4x - 7 \end{aligned}$$

Now we can substitute $4x - 7$ wherever we see y in the second equation.

$$\begin{aligned} 2y - 8x &= 7 \\ 2(4x - 7) - 8x &= 7 \\ 8x - 14 - 8x &= 7 \\ -14 &= 7 \end{aligned}$$

We will stop here since we have eliminated all of the variables in the equation and ended with a *false* statement. Since -14 never equals 7, then no values of x and y can make the original system true. If you recall from the section on substitution, this means that the lines $y + 7 = 4x$ and $2y - 8x = 7$ are parallel. Since a solution to a system of linear equations is any point where the lines touch, and parallel lines never touch, *no* points are solutions. We can say this in English as, "There are no solutions," or in math as, "The solution set is $\emptyset$."

4.4.3 Elimination

In Section 4.3, we explored a third way of solving systems of linear equations called elimination where we add two equations together to cancel a variable.

Example 4.4.8 Solving Systems of Equations by Elimination. Solve the system using elimination.

$$\begin{cases} 4x - 6y = 13 \\ 5x + 4y = -1 \end{cases}$$

Explanation. To solve the system using elimination, we first need to scale one or both of the equations so that one variable has equal but opposite coefficients in the system. In this case, we will choose to make y have opposite coefficients because the signs are already opposite for that variable in the system.

We need to multiply the first equation by 2 and the second equation by 3.

$$\begin{cases} 4x - 6y = 13 \\ 5x + 4y = -1 \end{cases}$$

$$\begin{cases} 2 \cdot (4x - 6y) = 2 \cdot (13) \\ 3 \cdot (5x + 4y) = 3 \cdot (-1) \end{cases}$$

$$\begin{cases} 8x - 12y = 26 \\ 15x + 12y = -3 \end{cases}$$

We now have an equivalent system of equations where the y-terms can be eliminated:

$$\begin{array}{r} 8x - 12y \\ +\ 15x + 12y \end{array} = \begin{array}{r} 26 \\ +\ (-3) \end{array}$$

So we have:

$$\begin{aligned} 23x &= 23 \\ x &= 1 \end{aligned}$$

To solve for y, we will substitute 1 for x into either of the original equations. We will use the first equation, $4x - 6y = 13$:

$$\begin{aligned} 4x - 6y &= 13 \\ 4(1) - 6y &= 13 \\ 4 - 6y &= 13 \\ -6y &= 9 \\ \frac{-6y}{-6} &= \frac{9}{-6} \\ y &= -\frac{3}{2} \end{aligned}$$

To verify this, we substitute the x and y values into both of the original equations.

$$\begin{aligned} 4x - 6y &= 13 \\ 4(1) - 6\left(-\frac{3}{2}\right) &\stackrel{?}{=} 13 \\ 4 + 9 &\stackrel{\checkmark}{=} 13 \end{aligned} \qquad \begin{aligned} 5x + 4y &= -1 \\ 5(1) + 4\left(-\frac{3}{2}\right) &\stackrel{?}{=} -1 \\ 5 - 6 &\stackrel{\checkmark}{=} -1 \end{aligned}$$

So the solution is the point $\left(-\frac{3}{2}, 1\right)$ and the solution set is $\left\{\left(-\frac{3}{2}, 1\right)\right\}$.

Example 4.4.9 Solving Special Systems of Equations with Elimination. Solve the system of equations using the elimination method.

$$\begin{cases} 24x + 6y = 9 \\ 8x + 2y = 2 \end{cases}$$

Explanation. To eliminate the x-terms, we will scale the second equation by -3.

$$\begin{cases} 24x + 6y = 9 \\ -3 \cdot (8x + 2y) = -3 \cdot (2) \end{cases}$$

$$\begin{cases} 24x + 6y = 9 \\ -24x - 6y = -6 \end{cases}$$

Adding the respective sides of the equation, we have:

$$0 = 3$$

Both of the variables have been eliminated. In this case, the statement $0 = 3$ is just false, no matter what x and y are. So the system has no solution. The solution set is $\emptyset$.

Example 4.4.10 Deciding to Use Substitution versus Elimination. Decide which method would be easiest to solve the systems of linear equations: substitution or elimination.

a.
$$\begin{cases} 2x + 3y = -11 \\ 5x - 6y = 13 \end{cases}$$

b.
$$\begin{cases} x - 7y = 10 \\ 9x - 16y = -4 \end{cases}$$

c.
$$\begin{cases} 6x + 30y = 15 \\ 4x + 20y = 10 \end{cases}$$

d.
$$\begin{cases} y = 3x - 2 \\ y = 7x + 6 \end{cases}$$

Explanation.

a. Elimination is probably easiest here. Multiply the first equation by 2 and eliminate the y variables. The solution to this one is $(-1, -3)$ if you want to solve it for practice.

b. Substitution is probably easiest here. Solve the first equation for x and substitute it into the second equation. We *could* use elimination if we multiplied the first equation by -9 and eliminate the x variable, but it's probably a little more work than substitution. The solution to this one is $(-4, -2)$ if you want to solve it for practice.

c. Elimination is probably easiest here. Multiply the first equation by 2 and the second equation by -3. Doing this will eliminate both variables and leave you with $0 = 0$. This should mean that all points on the line are solutions. So the solution set is $\{(x, y) \mid 6x + 30y = 15\}$.

d. Substitution is definitely easiest here. Substituting y from one equation into y in the other equation gives you $3x - 2 = 7x + 6$. Solve this and find then find y and you should get the solution to the system to be $(-2, -8)$ if you want to solve it for practice.

4.4.4 Exercises

Solving Systems of Linear Equations by Graphing Use a graph to solve the system of equations.

1.
$$\begin{cases} x + y = 0 \\ 3x - y = 8 \end{cases}$$

2.
$$\begin{cases} 4x - 2y = 4 \\ x + 2y = 6 \end{cases}$$

3.
$$\begin{cases} x + y = -1 \\ x = 2 \end{cases}$$

4.
$$\begin{cases} x - 2y = -4 \\ x = -4 \end{cases}$$

5.
$$\begin{cases} -10x + 15y = 60 \\ 6x - 9y = 36 \end{cases}$$

6.
$$\begin{cases} 6x - 8y = 32 \\ 9x - 12y = 12 \end{cases}$$

7.
$$\begin{cases} y = -\frac{3}{5}x + 7 \\ 9x + 15y = 105 \end{cases}$$

8.
$$\begin{cases} 9y - 12x = 18 \\ y = \frac{4}{3}x + 2 \end{cases}$$

Substitution Solve the following system of equations.

9.
$$\begin{cases} y = -2 - 4x \\ 5x + y = 3 \end{cases}$$

10.
$$\begin{cases} y = -4x - 13 \\ 3x + 4y = 0 \end{cases}$$

11.
$$\begin{cases} 5x + 3y = 11 \\ 5x + 5y = -5 \end{cases}$$

12. $\begin{cases} 2x + 2y = 30 \\ 4x + 3y = 54 \end{cases}$

13. $\begin{cases} -4x + 5y = 31 \\ 6x + 6y = -60 \end{cases}$

14. $\begin{cases} 2x - 4y = 22 \\ 6x + 2y = -60 \end{cases}$

15. $\begin{cases} 2x + 4y = 5 \\ -6x - 12y = -15 \end{cases}$

16. $\begin{cases} 2x + 2y = 5 \\ -8x - 8y = -20 \end{cases}$

17. A rectangle's length is 2 feet shorter than twice its width. The rectangle's perimeter is 176 feet. Find the rectangle's length and width.

The rectangle's length is ______ feet, and its width is ______ feet.

18. A school fund raising event sold a total of 202 tickets and generated a total revenue of \$806.10. There are two types of tickets: adult tickets and child tickets. Each adult ticket costs \$6.65, and each child ticket costs \$2.70. Write and solve a system of equations to answer the following questions.

______ adult tickets and ______ child tickets were sold.

19. A test has 19 problems, which are worth a total of 97 points. There are two types of problems in the test. Each multiple-choice problem is worth 4 points, and each short-answer problem is worth 7 points. Write and solve a system equation to answer the following questions.

This test has ______ multiple-choice problems and ______ short-answer problems.

20. A test has 21 problems, which are worth a total of 110 points. There are two types of problems in the test. Each multiple-choice problem is worth 5 points, and each short-answer problem is worth 6 points. Write and solve a system equation to answer the following questions.

This test has ______ multiple-choice problems and ______ short-answer problems.

21. Subin invested a total of \$10,000 in two accounts. One account pays 5% interest annually; the other pays 2% interest annually. At the end of the year, Subin earned a total of \$320 in interest. Write and solve a system of equations to find how much money Subin invested in each account.

Subin invested ______ in the 5% account and ______ in the 2% account.

22. Holli invested a total of \$10,000 in two accounts. After a year, one account lost 6.5%, while the other account gained 4.6%. In total, Holli lost \$483.50. Write and solve a system of equations to find how much money Holli invested in each account.

Holli invested ______ in the account with 6.5% loss and ______ in the account with 4.6% gain.

23. Town A and Town B were located close to each other, and recently merged into one city. Town A had a population with 12% African Americans. Town B had a population with 8% African Americans. After the merge, the new city has a total of 5000 residents, with 9.12% African Americans. Write and solve a system of equations to find how many residents Town A and Town B used to have.

Town A used to have ______ residents, and Town B used to have ______ residents.

24. You poured some 6% alcohol solution and some 12% alcohol solution into a mixing container. Now you have 600 grams of 10.8% alcohol solution. How many grams of 6% solution and how many grams of 12% solution did you pour into the mixing container?

Write and solve a system equation to answer the following questions.

You mixed ______ grams of 6% solution with ______ grams of 12% solution.

Elimination Solve the following system of equations.

25. $\begin{cases} 4x + 4y = -32 \\ 4x + 5y = -37 \end{cases}$

26. $\begin{cases} x + 3y = 27 \\ 4x + 2y = 18 \end{cases}$

27. $\begin{cases} -3x + 4y = 2 \\ 6x + 2y = 16 \end{cases}$

28. $\begin{cases} 6x - 3y = 42 \\ 3x + 5y = -18 \end{cases}$

29. $\begin{cases} 5x + 5y = 1 \\ 15x + 15y = 1 \end{cases}$

30. $\begin{cases} 5x + 3y = 1 \\ -20x - 12y = 1 \end{cases}$

31. $\begin{cases} x + y = 1 \\ -2x - 2y = -2 \end{cases}$

32. $\begin{cases} x + 5y = 1 \\ 3x + 15y = 3 \end{cases}$

33. A test has 17 problems, which are worth a total of 86 points. There are two types of problems in the test. Each multiple-choice problem is worth 3 points, and each short-answer problem is worth 8 points. Write and solve a system of equations to answer the following questions.

This test has ______ multiple-choice problems and ______ short-answer problems.

34. Irene invested a total of \$7,000 in two accounts. One account pays 3% interest annually; the other pays 4% interest annually. At the end of the year, Irene earned a total of \$255 in interest. Write and solve a system of equations to find how much money Irene invested in each account.

Irene invested ______ in the 3% account and ______ in the 4% account.

35. Town A and Town B were located close to each other, and recently merged into one city. Town A had a population with 12% whites. Town B had a population with 8% whites. After the merge, the new city has a total of 5000 residents, with 8.96% whites. Write and solve a system of equations to find how many residents Town A and Town B used to have.

Town A used to have ______ residents, and Town B used to have ______ residents.

36. You poured some 12% alcohol solution and some 6% alcohol solution into a mixing container. Now you have 640 grams of 9% alcohol solution. Write and solve a system of equations to find how many grams of 12% solution and how many grams of 6% solution you poured into the mixing container.

You mixed ______ grams of 12% solution with ______ grams of 6% solution.

37. If a boat travels from Town A to Town B, it has to travel 5002.5 mi along a river. A boat traveled from Town A to Town B along the river's current with its engine running at full speed. This trip took 172.5 hr. Then the boat traveled back from Town B to Town A, again with the engine at full speed, but this time against the river's current. This trip took 217.5 hr. Write and solve a system of equations to answer the following questions.

The boat's speed in still water with the engine running at full speed is $\boxed{}$.

The river current's speed was $\boxed{}$.

38. A small fair charges different admission for adults and children. It charges \$2.25 for adults, and \$0.25 for children. On a certain day, the total revenue is \$4,615 and the fair admits 4100 people. How many adults and children were admitted?

There were $\boxed{}$ adults and $\boxed{}$ children at the fair.

Part II

Preparation for STEM

Chapter 5

Exponents and Polynomials

5.1 Adding and Subtracting Polynomials

A polynomial is a particular type of algebraic expression.

- A company's sales, s (in millions of dollars), can be modeled by $2.2t + 5.8$, where t stands for the number of years since 2010.
- The height of an object from the ground, h (in feet), launched upward from the top of a building can be modeled by $-16t^2 + 32t + 300$, where t represents the amount of time (in seconds) since the launch.
- The volume of an open-top box with a square base, V (in cubic inches), can be calculated by $30s^2 - \frac{1}{2}s^2$, where s stands for the length of the square base, and the box sides have to be cut from a certain square piece of metal.

All of the expressions above are polynomials. In this section, we will learn some basic vocabulary relating to polynomials and we'll then learn how to add and subtract polynomials.

5.1.1 Polynomial Vocabulary

There is a lot of vocabulary associated with polynomials. We start this section with a flood of vocabulary terms and some examples of how to use them.

Definition 5.1.2 A **polynomial** is an expression with one or more terms summed together. A term of a polynomial must either be a plain number or the product of a number and one or more variables raised to natural number powers. The expression 0 is also considered a polynomial, with zero terms. $\Diamond$

Example 5.1.3

- Here are three polynomials: $x^2 - 5x + 2$, $t^3 - 1$, $7y$.
- The expression $3x^4y^3 + 7xy^2 - 12xy$ is an example of a polynomial in more than one variable.
- The polynomial $x^2 - 5x + 3$ has three terms: x^2, $-5x$, and 3.
- The polynomial $3x^4 + 7xy^2 - 12xy$ also has three terms.

- The polynomial $t^3 - 1$ has two terms.

Remark 5.1.4 A polynomial will never have a variable in the denominator of a fraction or under a radical.

Definition 5.1.5 The **coefficient** (or numerical coefficient) of a term in a polynomial is the numerical factor in the term. ◊

Example 5.1.6

- The coefficient of the term $\frac{4}{3}x^6$ is $\frac{4}{3}$.
- The coefficient of the second term of the polynomial $x^2 - 5x + 3$ is -5.
- The coefficient of the term $\frac{y^7}{4}$ is $\frac{1}{4}$.

Checkpoint 5.1.7 Identify which of the following are polynomials and which are not.

a. The expression $-2x^9 - \frac{7}{13}x^3 - 1$ (☐ is ☐ is not) a polynomial.

b. The expression $5x^{-2} - 5x^2 + 3$ (☐ is ☐ is not) a polynomial.

c. The expression $\sqrt{2}x - \frac{3}{5}$ (☐ is ☐ is not) a polynomial.

d. The expression $5x^3 - 5^{-5}x - x^4$ (☐ is ☐ is not) a polynomial.

e. The expression $\frac{25}{x^2} + 23 - x$ (☐ is ☐ is not) a polynomial.

f. The expression $37x^6 - x + 8^{\frac{4}{3}}$ (☐ is ☐ is not) a polynomial.

g. The expression $\sqrt{7x} - 4x^3$ (☐ is ☐ is not) a polynomial.

h. The expression $6x^{\frac{3}{2}} + 1$ (☐ is ☐ is not) a polynomial.

i. The expression $6^x - 3x^6$ (☐ is ☐ is not) a polynomial.

Explanation.

a. The expression $-2x^9 - \frac{7}{13}x^3 - 1$ is a polynomial.

b. The expression $5x^{-2} - 5x^2 + 3$ is not a polynomial because it has a negative exponent on a variable.

c. The expression $\sqrt{2}x - \frac{3}{5}$ is a polynomial. Note that *coefficients* can have radicals even though variables cannot, and the square root here is *only* applied to the 2.

d. The expression $5x^3 - 5^{-5}x - x^4$ is a polynomial. Note that *coefficients* can have negative exponents even though variables cannot.

e. The expression $\frac{25}{x^2} + 23 - x$ is not a polynomial because it has a variable in a denominator.

f. The expression $37x^6 - x + 8^{\frac{4}{3}}$ is a polynomial. Note that *coefficients* can have fractional exponents even though variables cannot.

g. The expression $\sqrt{7x} - 4x^3$ is not a polynomial because it has a variable inside a radical.

h. The expression $6x^{\frac{3}{2}} + 1$ is not a polynomial because a variable has a fractional exponent.

i. The expression $6^x - 3x^6$ is not a polynomial because it has a variable in an exponent.

Definition 5.1.8 A term in a polynomial with no variable factor is called a **constant term**. ♢

Example 5.1.9 The constant term of the polynomial $x^2 - 5x + 3$ is 3.

Definition 5.1.10 The **degree** of a term is one way to measure how "large" it is. When a term only has one variable, its degree is the exponent on that variable. When a term has more than on variable, its degree is the sum of the exponents on the variables. A constant term has degree 0. ♢

Example 5.1.11

- The degree of $5x^2$ is 2.
- The degree of $-\frac{4}{7}y^5$ is 5.
- The degree of $-4x^2y^3$ is 5.
- The degree of 17 is 0. Constant terms always have 0 degree.

Definition 5.1.12 The **degree of a nonzero polynomial** is the greatest degree that appears amongst its terms. ♢

Definition 5.1.13 The **leading term** of a polynomial is the term with the greatest degree (assuming there is no tie). The coefficient of a polynomial's leading term is called the polynomial's **leading coefficient**. ♢

Example 5.1.14 The degree of the polynomial $4x^2 - 5x + 3$ is 2 because the terms have degrees 2, 1, and 0, respectively, and 2 is the largest. Its leading term is $4x^2$, and its leading coefficient is 4.

Remark 5.1.15 To help us recognize a polynomial's degree, the standard convention at this level is to write a polynomial's terms in order from highest degree to lowest degree. When a polynomial is written in this order, it is written in **standard form**. For example, it is standard practice to write $7 - 4x - x^2$ as $-x^2 - 4x + 7$ since $-x^2$ is the leading term. By writing the polynomial in standard form, we can look at the first term to determine both the polynomial's degree and leading term.

There are special names for polynomials with a small number of terms, and for polynomials with certain degrees.

monomial A polynomial with one term, such as $3x^5$, is called a monomial.

binomial A polynomial with two terms, such as $3x^5 + 2x$, is called a binomial.

trinomial A polynomial with three terms, such as $x^2 - 5x + 3$, is called a trinomial.

constant polynomial A zeroth-degree polynomial is called a constant polynomial. An example is the polynomial 7, which has degree zero.

linear polynomial A first-degree polynomial is called a linear polynomial. An example is $-2x + 7$.

quadratic polynomial A second-degree polynomial is called a quadratic polynomial. An example is $4x^2 - 2x + 7$.

cubic polynomial A third-degree polynomial is called a cubic polynomial. An example is $x^3 + 4x^2 - 2x + 7$.

Fourth-degree and fifth-degree polynomials are called quartic and quintic polynomials, respectively. If the degree of the polynomial, n, is greater than five, we'll simply call it an nth-degree polynomial. For example, the polynomial $5x^8 - 4x^5 + 1$ is an 8th-degree polynomial.

5.1.2 Adding and Subtracting Polynomials

Example 5.1.16 Production Costs. Bayani started a company that makes one product: one-gallon ketchup jugs for industrial kitchens. The company's production expenses only come from two things: supplies and labor. The cost of supplies, S (in thousands of dollars), can be modeled by $S = 0.05x^2 + 2x + 30$, where x is number of thousands of jugs of ketchup produced. The labor cost for his employees, L (in thousands of dollars), can be modeled by $0.1x^2 + 4x$, where x again represents the number of jugs they produce (in thousands of jugs). Find a model for the company's total production costs.

Since Bayani's company only has these two costs, we can find a model for the total production costs, C (in thousands of dollars), by adding the supply costs and the labor costs:

$$C = \left(0.05x^2 + 2x + 30\right) + \left(0.1x^2 + 4x\right)$$

To finish simplifying our total production cost model, we'll combine the like terms:

$$\begin{aligned} C &= 0.05x^2 + 0.1x^2 + 2x + 4x + 30 \\ &= 0.15x^2 + 6x + 30 \end{aligned}$$

This simplified model can now calculate Bayani's total production costs C (in thousands of dollars) when the company produces x thousand jugs of ketchup.

In short, the process of adding two or more polynomials involves recognizing and then combining the like terms.

Checkpoint 5.1.17 Add the polynomials.
$\left(6x^2 + 4x\right) + \left(4x^2 - 5x\right)$

Explanation. We combine like terms as follows

$$\begin{aligned} \left(6x^2 + 4x\right) + \left(4x^2 - 5x\right) &= \left(6x^2 + 4x^2\right) + (4x - 5x) \\ &= 10x^2 - x \end{aligned}$$

Example 5.1.18 Simplify the expression $\left(\frac{1}{2}x^2 - \frac{2}{3}x - \frac{3}{2}\right) + \left(\frac{3}{2}x^2 + \frac{7}{2}x - \frac{1}{4}\right)$.

Explanation.

$$\begin{aligned} &\left(\frac{1}{2}x^2 - \frac{2}{3}x - \frac{3}{2}\right) + \left(\frac{3}{2}x^2 + \frac{7}{2}x - \frac{1}{4}\right) \\ &= \left(\frac{1}{2}x^2 + \frac{3}{2}x^2\right) + \left(-\frac{2}{3}x + \frac{7}{2}x\right) + \left(-\frac{3}{2} + \left(-\frac{1}{4}\right)\right) \\ &= \left(\frac{4}{2}x^2\right) + \left(-\frac{4}{6}x + \frac{21}{6}x\right) + \left(-\frac{6}{4} + \left(-\frac{1}{4}\right)\right) \\ &= (2x^2) + \frac{17}{6}x + \left(-\frac{7}{4}\right) \\ &= 2x^2 + \frac{17}{6}x - \frac{7}{4} \end{aligned}$$

Example 5.1.19 Profit, Revenue, and Costs. From Example 5.1.16, we know Bayani's ketchup company's production costs, C (in thousands of dollars), for producing x thousand jugs of ketchup is modeled by $C = 0.15x^2 + 6x + 30$. The revenue, R (in thousands of dollars), from selling the ketchup can be modeled by

$R = 13x$, where x stands for the number of thousands of jugs of ketchup sold. The company's net profit can be calculated using the concept:

$$\text{net profit} = \text{revenue} - \text{costs}$$

Assuming all products produced will be sold, a polynomial to model the company's net profit, P (in thousands of dollars) is:

$$\begin{aligned} P &= R - C \\ &= (13x) - \left(0.15x^2 + 6x + 30\right) \\ &= 13x - 0.15x^2 - 6x - 30 \\ &= -0.15x^2 + (13x + (-6x)) - 30 \\ &= -0.15x^2 + 7x - 30 \end{aligned}$$

The key distinction between the addition and subtraction of polynomials is that when we subtract a polynomial, we must subtract each term in that polynomial.

Notice that our first step in simplifying the expression in Example 5.1.19 was to subtract *every* term in the second expression. We can also think of this as distributing a factor of -1 across the second polynomial, $0.15x^2 + 6x + 30$, and then adding these terms as follows:

$$\begin{aligned} P &= R - C \\ &= (13x) - \left(0.15x^2 + 6x + 30\right) \\ &= 13x + (-1)(0.15x^2) + (-1)(6x) + (-1)(30) \\ &= 13x - 0.15x^2 - 6x - 30 \\ &= -0.15x^2 + (13x + (-6x)) - 30 \\ &= -0.15x^2 + 7x - 30 \end{aligned}$$

Example 5.1.20 Subtract $\left(5x^3 + 4x^2 - 6x\right) - \left(-3x^2 + 9x - 2\right)$.

Explanation. We must first subtract every term in $\left(-3x^2 + 9x - 2\right)$ from $\left(5x^3 + 4x^2 - 6x\right)$. Then we can combine like terms.

$$\begin{aligned} &\left(5x^3 + 4x^2 - 6x\right) - \left(-3x^2 + 9x - 2\right) \\ &= 5x^3 + 4x^2 - 6x + 3x^2 - 9x + 2 \\ &= 5x^3 + \left(4x^2 + 3x^2\right) + (-6x + (-9x)) + 2 \\ &= 5x^3 + 7x^2 - 15x + 2 \end{aligned}$$

Checkpoint 5.1.21 Subtract the polynomials.
$(3x - 10) - (-5x + 7)$

Explanation. We combine like terms as follows

$$\begin{aligned} (3x - 10) - (-5x + 7) &= (3x - (-5x)) + (-10 - 7) \\ &= 8x - 17 \end{aligned}$$

Let's look at one last example where the polynomial has multiple variables. Remember that like terms must have the same variable(s) with the same exponent.

Example 5.1.22 Subtract $\left(3x^2y + 8xy^2 - 17y^3\right) - \left(2x^2y + 11xy^2 + 4y^2\right)$.

Explanation. Again, we'll begin by subtracting each term in $\left(2x^2y + 11xy^2 + 4y^2\right)$. Once we've done this, we'll need to identify and combine like terms.

$$\begin{aligned}
&\left(3x^2y + 8xy^2 - 17y^3\right) - \left(2x^2y + 11xy^2 + 4y^2\right)\\
&= 3x^2y + 8xy^2 - 17y^3 - 2x^2y - 11xy^2 - 4y^2\\
&= \left(3x^2y - 2x^2y\right) + \left(8xy^2 - 11xy^2\right) + \left(-17y^3 - 4y^2\right)\\
&= x^2y - 3xy^2 - 17y^3 - 4y^2
\end{aligned}$$

5.1.3 Evaluating Polynomial Expressions

Evaluating expressions was introduced in Section 1.1, and involves replacing the variable(s) in an expression with specific numbers and calculating the result. Here, we will look at evaluating polynomial expressions.

Example 5.1.23 Evaluate the expression $-12y^3 + 4y^2 - 9y + 2$ for $y = -5$.

Explanation. We will replace y with -5 and simplify the result:

$$\begin{aligned}
-12y^3 + 4y^2 - 9y + 2 &= -12(-5)^3 + 4(-5)^2 - 9(-5) + 2\\
&= -12(-125) + 4(25) + 45 + 2\\
&= 1647
\end{aligned}$$

Recall that in Subsection 1.1.4 and Example 1.1.15 we discussed how $(-5)^2$ and -5^2 are not the same expressions. The first expression, $(-5)^2$, represents the number -5 squared, and is $(-5)(-5) = 25$. The second expression, -5^2, is the *opposite* of the number that you get after you square 5, and is $-5^2 = -(5 \cdot 5) = -25$.

Example 5.1.24 Evaluate the expression $C = 0.15x^2 + 6x + 30$ from Example 5.1.16 for $x = 10$ and explain what this means in context.

Explanation. We will replace x with 10:

$$\begin{aligned}
C &= 0.15x^2 + 6x + 30\\
&= 0.15(10)^2 + 6(10) + 30\\
&= 0.15(100) + 60 + 30\\
&= 15 + 90\\
&= 105
\end{aligned}$$

The context was that x represents so many thousands of jugs of ketchup, and C represents the total cost, in thousands of dollars, to produce that many jugs. So in context, we can interpret this as it costing $105,000 to produce 10,000 jugs of ketchup.

Checkpoint 5.1.25

a. Evaluate $(-y)^2$ when $y = -2$.

b. Evaluate $(-y)^3$ when $y = -2$.

Explanation.

a. $$\begin{aligned}
(-y)^2 &= \big(-1(-2)\big)^2\\
&= (2)^2\\
&= 4
\end{aligned}$$

b. $(-y)^3 = \big(-1(-2)\big)^3$
$= (2)^3$
$= 8$

5.1.4 Reading Questions

1. What are the names for a polynomial with one term? With two terms? With three terms? Care to take a guess at the name of a polynomial with four terms?

2. Adding and subtracting polynomials is mostly about combining $\boxed{}$ terms.

3. What should you be careful with when evaluating a polynomial for a negative number?

5.1.5 Exercises

Review and Warmup

1. List the terms in each expression.

a. $6s - 0.9z + 3.5$

b. $-y - 6.5z^2$

c. $1.6y^2 + 6.7z - 0.2x - 2.2t^2$

d. $-0.5x + 6.2z - 2 + 3.4y^2$

2. List the terms in each expression.

a. $7.6s^2 + 7.3s^2 + 1.5$

b. $-2.2z^2$

c. $-4.1x - 2.5t^2$

d. $-3.9t + 8.1z^2 + 6.7z^2 + 7.4x$

3. List the terms in each expression.

a. $-8.9s - 2.6 - 0.6x + 2.8y$

b. $-2.6t - 6.8x^2 + 6.4$

c. $5.8z^2 + 4.7s^2 + 3.4$

d. $-8.4y^2$

4. List the terms in each expression.

a. $-7.3t^2 + 5.6t - 2.5 - 4.7z^2$

b. $5.9y^2 + 3.1t^2$

c. $-8.5s^2$

d. $-6.8t^2 + 4.7t^2 + 6.4 + 3.6s^2$

Simplify each expression, if possible, by combining like terms.

5. a. $-6t - 5t - 5t - 6t$

b. $2t - 3t + 8s + 4t$

c. $9s^2 + 2s^2 + 7z^2$

d. $-4y^2 + 7y^2 - 5z$

6. a. $-4t^2 + 4y^2$

b. $-6s + 3 + 6s^2$

c. $5s^2 - 4t + 3s + 6$

d. $7z^2 + 5y^2 - 6x^2$

7. a. $-\frac{3}{8}t - \frac{1}{9}t$

b. $-\frac{2}{3}x + 7s - 2s$

c. $-\frac{7}{8}z - \frac{5}{7}z + \frac{7}{8}z$

d. $-\frac{6}{7}s^2 - \frac{1}{5}t^2$

8. a. $\frac{1}{3}t^2 - \frac{2}{3}y$

b. $-\frac{2}{9}s^2 + \frac{8}{9} - \frac{1}{9}z^2 + 2$

c. $-\frac{6}{5}z^2 - \frac{3}{8}s^2 - s^2$

d. $\frac{3}{5}s^2 + \frac{1}{2}y - \frac{1}{8}s^2 + \frac{1}{3}$

Vocabulary Questions Is the following expression a monomial, binomial, or trinomial?

9. $4y^{12} - 13y^3$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree ☐.

10. $-11r^7 + 4r^2$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree ☐.

11. 40 is a (☐ monomial ☐ binomial ☐ trinomial) of degree ☐

12. 5 is a (☐ monomial ☐ binomial ☐ trinomial) of degree ☐

13. $-18y^{11} - 9y^7 - 20y^6$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree ☐

14. $-20r^{10} - 2r^9 - 10r^2$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree ☐

15. $8x^3 + 17x^7 + 6x$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree ☐

16. $-14x^7 - 16x^8 + 6x$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree ☐

17. $13y^{11}$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree ☐

18. $-2y^{19}$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree ☐

Find the degree of the following polynomial.

19. $2x^8y^6 - 16x^2y^4 - 6x^2 + 13$

20. $6x^7y^9 + 11x^3y + 11x^2 + 1$

Simplifying Polynomials Add the polynomials.

21. $(6x - 2) + (-7x - 5)$

22. $(8x - 9) + (2x + 10)$

23. $\left(10x^2 + 5x\right) + \left(-10x^2 + 4x\right)$

24. $\left(-8x^2 - 3x\right) + \left(-x^2 - 2x\right)$

25. $\left(-3x^2 - 9x + 1\right) + \left(4x^2 - 7x + 3\right)$

26. $\left(4x^2 + 6x - 9\right) + \left(7x^2 + 8x + 2\right)$

27. $\left(4y^3 - 7y^2 - 4\right) + \left(-3y^3 - 4y^2 - 7\right)$

28. $\left(-10r^3 - 4r^2 + 6\right) + \left(-4r^3 + 7r^2 + 1\right)$

29. $\left(7r^6 - 2r^4 - 4r^2\right) + \left(4r^6 - 10r^4 - r^2\right)$

30. $\left(4t^6 - 8t^4 + 6t^2\right) + \left(-5t^6 + 3t^4 - 9t^2\right)$

Add the polynomials.

31. $\left(0.8t^5 - 0.3t^4 - 0.1t^2 - 0.4\right) + \left(0.4t^5 - 0.9t^3 + 0.4\right)$

32. $\left(0.2t^5 + 0.5t^4 - 0.6t^2 - 0.1\right) + \left(0.5t^5 + 0.2t^3 - 0.8\right)$

33. $\left(-2x^3 + 3x^2 - 5x + \dfrac{7}{6}\right) + \left(3x^3 - 10x^2 + 3x + \dfrac{5}{4}\right)$

34. $\left(3x^3 + 7x^2 + 5x + \dfrac{7}{10}\right) + \left(-5x^3 - 7x^2 + 2x + \dfrac{1}{4}\right)$

Subtract the polynomials.

35. $(-4x+1)-(-10x-3)$

36. $(-x-6)-(-x-8)$

37. $(x^2+7x)-(7x^2-3x)$

38. $(3x^2-5x)-(x^2-3x)$

39. $(-5x^9-3x^4)-(-5x^3-6)$

40. $(10x^{10}-4x^8)-(6x-4)$

41. $(-10x^2+6x-4)-(-4x^2+8x+2)$

42. $(-2x^2-10x-4)-(-8x^2+3x-10)$

43. $(-8x^6-3x^4-5x^2)-(7x^6-3x^4-5x^2)$

44. $(5y^6-9y^4+5y^2)-(7y^6+6y^4+8y^2)$

45. $(-5x^3+3x^2-5x-5)-(-8x^2-6x+7)$

46. $(6x^3-7x^2-5x+5)-(9x^2+2x-7)$

Add or subtract the given polynomials as indicated.

47. $\left[4r^{16}-10r^{15}+r^{14}-(-8r^{16}+3r^{15}-2r^{14})\right]-(-9r^{16}-7r^{15}-8r^{14})$

48. $\left[t^9+8t^8-(-8t^9-10t^8)\right]-(-5t^9-4t^8)$

49. $\left[7t^{13}+5t^{12}-(-9t^{13}-10t^{12})\right]-\left[-10t^{13}+2t^{12}+(-10t^{13}-7t^{12})\right]$

50. $\left[4t^{14}-8t^{13}+3t^5-(-9t^{14}+8t^{13}-9t^5)\right]-\left[-7t^{14}-8t^{13}+7t^5+(-4t^{14}-10t^{13}-8t^5)\right]$

Add or subtract the given polynomials as indicated.

51. $(2x^7y^3+8xy)+(-3x^7y^3+4xy)$

52. $(9x^4y^3-10xy)+(2x^4y^3+9xy)$

53. $(10x^9y^7+6xy+8)+(9x^9y^7+9xy+9)$

54. $(5x^8y^3-10xy-6)+(3x^8y^3-6xy+3)$

55. $(6x^8y^9+5x^5y^3+9xy)+$
$(-3x^8y^9-7x^5y^3+8xy)$

56. $(-7x^7y^8+8x^3y^4+6xy)+$
$(3x^7y^8-2x^3y^4+7xy)$

57. $(8x^6-3xy+5y^9)-(-2x^6-9xy+2y^9)$

58. $(9x^5+7xy-8y^6)-(2x^5+10xy-5y^6)$

59. $(-10x^7y^6+2x^2y^4+2xy)-$
$(-2x^7y^6-10x^2y^4+5xy)$

60. $(2x^8y^9-6x^3y^4-3xy)-$
$(-4x^8y^9+10x^3y^4-8xy)$

61. $(3x^4-9y^2)-$
$(-2x^4+3x^2y^2+7x^4y^2-2y^2)$

62. $(-4x^9+4y^4)-$
$(2x^9-10x^8y^4+4x^9y^4-10y^4)$

63. Subtract $-4y^{18}-8y^7-6y^5$ from the sum of $9y^{18}-4y^7+7y^5$ and $-10y^{18}+3y^7-7y^5$.

64. Subtract $-10r^{11}-6r^5-2r^3$ from the sum of $5r^{11}-9r^5+4r^3$ and $-10r^{11}+7r^5-7r^3$.

65. Subtract $-7x^3y^7-6xy$ from $-2x^3y^7-9xy$

66. Subtract $8x^3y^8+10xy$ from $10x^3y^8+7xy$

Evaluating Polynomials

67. Evaluate the expression t^2:

a. For $t=5$.

b. For $t=-4$.

68. Evaluate the expression t^2:

a. For $t=2$.

b. For $t=-8$.

69. Evaluate the expression $-x^2$:

a. For $x = 4$.

b. For $x = -2$.

70. Evaluate the expression $-x^2$:

a. For $x = 3$.

b. For $x = -4$.

71. Evaluate the expression y^3:

a. For $y = 2$.

b. For $y = -2$.

72. Evaluate the expression y^3:

a. For $y = 4$.

b. For $y = -3$.

73. a. Evaluate $(-2r)^2$ when $r = -1$.

b. Evaluate $(-2r)^3$ when $r = -1$.

74. a. Evaluate $(-r)^2$ when $r = -3$.

b. Evaluate $(-r)^3$ when $r = -3$.

75. Evaluate the expression $\frac{1}{3}(x+3)^2 - 7$ when $x = -6$.

76. Evaluate the expression $\frac{1}{5}(x+4)^2 - 4$ when $x = -9$.

77. Evaluate the expression $\frac{1}{3}(x+4)^2 - 9$ when $x = -7$.

78. Evaluate the expression $\frac{1}{4}(x+1)^2 - 6$ when $x = -5$.

79. Evaluate the expression $-16t^2 + 64t + 128$ when $t = -3$.

80. Evaluate the expression $-16t^2 + 64t + 128$ when $t = -5$.

81. Evaluate the expression $-16t^2 + 64t + 128$ when $t = -4$.

82. Evaluate the expression $-16t^2 + 64t + 128$ when $t = 2$.

Applications of Simplifying Polynomials The formula

$$y = \frac{1}{2}at^2 + v_0 t + y_0$$

gives the vertical position of an object, at time t, thrown with an initial velocity v_0, from an initial position y_0 in a place where the acceleration of gravity is a. The acceleration of gravity on earth is $-9.8\ \frac{\text{m}}{\text{s}^2}$. It is negative, because we consider the upward direction as positive in this situation, and gravity pulls down.

83. What is the height of a baseball thrown with an initial velocity of $v_0 = 82\ \frac{\text{m}}{\text{s}}$, from an initial position of $y_0 = 94$ m, and at time $t = 1$ s?
One seconds after the baseball was thrown, it was $\boxed{}$ high in the air.

84. What is the height of a baseball thrown with an initial velocity of $v_0 = 87\ \frac{\text{m}}{\text{s}}$, from an initial position of $y_0 = 76$ m, and at time $t = 9$ s?
Nine seconds after the baseball was thrown, it was $\boxed{}$ high in the air.

85. An auto company's sales volume can be modeled by $6.1x^2 + 6.9x + 4$, and its cost can be modeled by $4.6x^2 + 3.7x + 4$, where x represents the number of cars produced, and y stands for money in thousand dollars. We can calculate the company's net profit by subtracting cost from sales. Find the polynomial which models the company's sales in thousands of dollars.

The company's profit can be modeled by $\boxed{}$ dollars.

86. An auto company's sales volume can be modeled by $8.4x^2 + 1.2x + 3.9$, and its cost can be modeled by $4.9x^2 - 2.2x + 3.9$, where x represents the number of cars produced, and y stands for money in thousand dollars. We can calculate the company's net profit by subtracting cost from sales. Find the polynomial which models the company's sales in thousands of dollars.

The company's profit can be modeled by [ ] dollars.

87. A handyman is building two pig pens sharing the same side. Assume the length of the shared side is x meters. The cost of building one pen would be $34x^2 - 4x + 22$ dollars, and the cost of building the other pen would be $22x^2 + 4x + 46.5$ dollars. What's the total cost of building those two pens?

A polynomial representing the total cost of building those two pens is [ ] dollars.

88. A handyman is building two pig pens sharing the same side. Assume the length of the shared side is x meters. The cost of building one pen would be $23.5x^2 + 8x + 20.5$ dollars, and the cost of building the other pen would be $25.5x^2 - 8x + 19$ dollars. What's the total cost of building those two pens?

A polynomial representing the total cost of building those two pens is [ ] dollars.

89. A farmer is building fence around a triangular area. The cost of building the shortest side is $35x$ dollars, where x stands for the length of the side in feet. The cost of building the other two sides can be modeled by $8x^2 + 0.5x + 40$ dollars and $4x^3 - 4x + 35$ dollars, respectively. What's the total cost of building fence for all three sides?

The cost of building fence for all three sides would be [ ] dollars.

90. A farmer is building fence around a triangular area. The cost of building the shortest side is $40x$ dollars, where x stands for the length of the side in feet. The cost of building the other two sides can be modeled by $5x^2 + 4.5x + 30$ dollars and $4x^3 + 1.5x + 35$ dollars, respectively. What's the total cost of building fence for all three sides?

The cost of building fence for all three sides would be [ ] dollars.

91. An architect is designing a house on an empty plot. The area of the plot can be modeled by the polynomial $4x^4 + 6x^2 - 2.5x$, and the area of the house's base can be modeled by $6x^3 - 2.5x + 15$. The rest of the plot is the yard. What's the yard's area?

The area of the yard can be modeled by the polynomial [ ].

92. An architect is designing a house on an empty plot. The area of the plot can be modeled by the polynomial $5x^4 + 16x^2 + 6x$, and the area of the house's base can be modeled by $4x^3 + 6x + 15$. The rest of the plot is the yard. What's the yard's area?

The area of the yard can be modeled by the polynomial [ ].

5.2 Introduction to Exponent Rules

In this section, we will look at some rules or properties we use when simplifying expressions that involve multiplication and exponents.

5.2.1 Exponent Basics

Before we discuss any exponent rules, we need to quickly remind ourselves of some important concepts and vocabulary.

When working with expressions with exponents, we have the following vocabulary:

$$\text{base}^{\text{exponent}} = \text{power}$$

For example, when we calculate $8^2 = 64$, the **base** is 8, the **exponent** is 2, and the expression 8^2 is called the 2nd **power** of 8.

The foundational understanding of exponents is that when the exponent is a positive integer, the power can be rewritten as repeated multiplication of the base. For example, the 4th power of 3 can be written as 4 factors of 3 like so:

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3$$

5.2.2 Exponent Rules

Product Rule. If we write out $3^5 \cdot 3^2$ without using exponents, we'd have:

$$3^5 \cdot 3^2 = (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3)$$

If we then count how many 3s are being multiplied together, we find we have $5 + 2 = 7$, a total of seven 3s. So $3^5 \cdot 3^2$ simplifies like this:

$$\begin{aligned} 3^5 \cdot 3^2 &= 3^{5+2} \\ &= 3^7 \end{aligned}$$

Example 5.2.2 Simplify $x^2 \cdot x^3$.

To simplify $x^2 \cdot x^3$, we write this out in its expanded form, as a product of x's, we have

$$\begin{aligned} x^2 \cdot x^3 &= (x \cdot x)(x \cdot x \cdot x) \\ &= x \cdot x \cdot x \cdot x \cdot x \\ &= x^5 \end{aligned}$$

Note that we obtained the exponent of 5 by adding 2 and 3.

This demonstrates our first exponent rule, the **Product Rule**: when multiplying two expressions that have the same base, we can simplify the product by adding the exponents.

$$x^m \cdot x^n = x^{m+n} \tag{5.2.1}$$

Checkpoint 5.2.3 Use the properties of exponents to simplify the expression.
$x^{16} \cdot x^{9}$

Explanation. We *add* the exponents as follows:

$$\begin{aligned} x^{16} \cdot x^{9} &= x^{16+9} \\ &= x^{25} \end{aligned}$$

Recall that $x = x^1$. It helps to remember this when multiplying certain expressions together.

Example 5.2.4 Multiply $x(x^3 + 2)$ by using the distributive property.
According to the distributive property,

$$x(x^3 + 2) = x \cdot x^3 + x \cdot 2$$

How can we simplify that term $x \cdot x^3$? It's really the same as $x^1 \cdot x^3$, so according to the Product Rule, it is x^4. So we have:

$$\begin{aligned} x(x^3 + 2) &= x \cdot x^3 + x \cdot 2 \\ &= x^4 + 2x \end{aligned}$$

Power to a Power Rule. If we write out $\left(3^5\right)^2$ without using exponents, we'd have 3^5 multiplied by itself:

$$\begin{aligned} \left(3^5\right)^2 &= \left(3^5\right) \cdot \left(3^5\right) \\ &= (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \end{aligned}$$

If we again count how many 3s are being multiplied, we have a total of two groups each with five 3s. So we'd have $2 \cdot 5 = 10$ instances of a 3. So $\left(3^5\right)^2$ simplifies like this:

$$\begin{aligned} \left(3^5\right)^2 &= 3^{2 \cdot 5} \\ &= 3^{10} \end{aligned}$$

Example 5.2.5 Simplify $\left(x^2\right)^3$.
To simplify $\left(x^2\right)^3$, we write this out in its expanded form, as a product of x's, we have

$$\begin{aligned} \left(x^2\right)^3 &= \left(x^2\right) \cdot \left(x^2\right) \cdot \left(x^2\right) \\ &= (x \cdot x) \cdot (x \cdot x) \cdot (x \cdot x) \\ &= x^6 \end{aligned}$$

Note that we obtained the exponent of 6 by multiplying 2 and 3.

This demonstrates our second exponent rule, the **Power to a Power Rule**: when a base is raised to an exponent and that expression is raised to another exponent, we multiply the exponents.

$$\left(x^m\right)^n = x^{m \cdot n}$$

Checkpoint 5.2.6 Use the properties of exponents to simplify the expression.
$\left(r^{11}\right)^2$

Explanation. We *multiply* the exponents as follows:

$$\begin{aligned}\left(r^{11}\right)^2 &= r^{11\cdot 2}\\ &= r^{22}\end{aligned}$$

Product to a Power Rule. The third exponent rule deals with having multiplication inside a set of parentheses and an exponent outside the parentheses. If we write out $(3t)^5$ without using an exponent, we'd have $3t$ multiplied by itself five times:

$$(3t)^5 = (3t)(3t)(3t)(3t)(3t)$$

Keeping in mind that there is multiplication between every 3 and t, and multiplication between all of the parentheses pairs, we can reorder and regroup the factors:

$$\begin{aligned}(3t)^5 &= (3\cdot t)\cdot(3\cdot t)\cdot(3\cdot t)\cdot(3\cdot t)\cdot(3\cdot t)\\ &= (3\cdot 3\cdot 3\cdot 3\cdot 3)\cdot(t\cdot t\cdot t\cdot t\cdot t)\\ &= 3^5t^5\end{aligned}$$

We could leave it written this way if 3^5 feels especially large. But if you are able to evaluate $3^5 = 243$, then perhaps a better final version of this expression is $243t^5$.

We essentially applied the outer exponent to each factor inside the parentheses. It is important to see how the exponent 5 applied to *both* the 3 *and* the t, not just to the t.

Example 5.2.7 Simplify $(xy)^5$.

To simplify $(xy)^5$, we write this out in its expanded form, as a product of x's and y's, we have

$$\begin{aligned}(xy)^5 &= (x\cdot y)\cdot(x\cdot y)\cdot(x\cdot y)\cdot(x\cdot y)\cdot(x\cdot y)\\ &= (x\cdot x\cdot x\cdot x\cdot x)\cdot(y\cdot y\cdot y\cdot y\cdot y)\\ &= x^5y^5\end{aligned}$$

Note that the exponent on xy can simply be applied to both x and y.

This demonstrates our third exponent rule, the **Product to a Power Rule**: when a product is raised to an exponent, we can apply the exponent to each factor in the product.

$$(x\cdot y)^n = x^n\cdot y^n$$

Checkpoint 5.2.8 Use the properties of exponents to simplify the expression.

$(4x)^4$

Explanation. We *multiply* the exponents and apply the rule $(ab)^m = a^m\cdot b^m$ as follows:

$$\begin{aligned}(4x)^4 &= (4)^4x^4\\ &= 256x^4\end{aligned}$$

List 5.2.9: Summary of the Rules of Exponents for Multiplication

If a and b are real numbers, and m and n are positive integers, then we have the following rules:

Product Rule $a^m \cdot a^n = a^{m+n}$

Power to a Power Rule $(a^m)^n = a^{m \cdot n}$

Product to a Power Rule $(ab)^m = a^m \cdot b^m$

Many examples will make use of more than one exponent rule. In deciding which exponent rule to work with first, it's important to remember that the order of operations still applies.

Example 5.2.10 Simplify the following expressions.

a. $\left(3^7 r^5\right)^4$

b. $\left(t^3\right)^2 \cdot \left(t^4\right)^5$

Explanation.

a. Since we cannot simplify anything inside the parentheses, we'll begin simplifying this expression using the Product to a Power Rule. We'll apply the outer exponent of 4 to each factor inside the parentheses. Then we'll use the Power to a Power Rule to finish the simplification process.

$$\begin{aligned}\left(3^7 r^5\right)^4 &= \left(3^7\right)^4 \cdot \left(r^5\right)^4\\ &= 3^{7\cdot 4} \cdot r^{5\cdot 4}\\ &= 3^{28} r^{20}\end{aligned}$$

Note that 3^{28} is too large to actually compute, even with a calculator, so we leave it written as 3^{28}.

b. According to the order of operations, we should first simplify any exponents before carrying out any multiplication. Therefore, we'll begin simplifying this by applying the Power to a Power Rule and then finish using the Product Rule.

$$\begin{aligned}\left(t^3\right)^2 \cdot \left(t^4\right)^5 &= t^{3\cdot 2} \cdot t^{4\cdot 5}\\ &= t^6 \cdot t^{20}\\ &= t^{6+20}\\ &= t^{26}\end{aligned}$$

Remark 5.2.11 We cannot simplify an expression like x^2y^3 using the Product Rule, as the factors x^2 and y^3 do not have the same base.

5.2.3 Reading Questions

1. How many exponent rules are discussed in this section? Write an example of each rule in action.

2. The order of operations say that operations inside parentheses should get the highest priority. But with $(5x)^3$, you cannot actually do anything with the 5 and the x. Which exponent rule allows you to sidestep the order of operations and still simplify this expression a little?

5.2.4 Exercises

Review and Warmup

1. Evaluate the following.
a. 2^2
b. 3^3
c. $(-4)^2$
d. $(-3)^3$

2. Evaluate the following.
a. 2^2
b. 5^3
c. $(-2)^2$
d. $(-5)^3$

3. Evaluate the following.
a. 1^6
b. $(-1)^{13}$
c. $(-1)^{14}$
d. 0^{20}

4. Evaluate the following.
a. 1^7
b. $(-1)^{11}$
c. $(-1)^{16}$
d. 0^{19}

5. Evaluate the following.
a. $(-5)^2$
b. -2^2

6. Evaluate the following.
a. $(-3)^2$
b. -4^2

7. Evaluate the following.
a. $(-2)^3$
b. -4^3

8. Evaluate the following.
a. $(-1)^3$
b. -4^3

Exponent Rules Use the properties of exponents to simplify the expression.

9. $9 \cdot 9^7$

10. $2 \cdot 2^3$

11. $3^9 \cdot 3^7$

12. $4^6 \cdot 4^2$

13. $y^9 \cdot y^5$

14. $t^{11} \cdot t^{17}$

15. $r^{13} \cdot r^{11} \cdot r^7$

16. $y^{15} \cdot y^4 \cdot y^{15}$

17. $\left(18^9\right)^3$

18. $\left(20^5\right)^6$

19. $\left(t^2\right)^2$

20. $\left(y^3\right)^{10}$

21. $(3t)^2$

22. $(2x)^3$

23. $(4ry)^3$

24. $(3xy)^3$

25. $\left(3t^{10}\right)^2$

26. $\left(5r^{11}\right)^3$

27. $\left(-4r^{20}\right) \cdot \left(5r^7\right)$

28. $\left(8r^3\right) \cdot \left(-3r^{20}\right)$

29. $\left(-9r^3\right)^3$

30. $\left(-6y^5\right)^2$

31. $\left(-2y^9\right) \cdot \left(7y^{19}\right) \cdot \left(-2y^{10}\right)$

32. $\left(-6r^{11}\right) \cdot \left(-2r^{12}\right) \cdot \left(2r^5\right)$

33. a. $\left(-10t^3\right)^2$
b. $-\left(10t^3\right)^2$

34. a. $\left(-8m^5\right)^2$
b. $-\left(8m^5\right)^2$

Use the properties of exponents to simplify the expression.

35. $\left(-\frac{y^{17}}{7}\right) \cdot \left(\frac{y^{11}}{4}\right)$

36. $\left(\frac{y^{20}}{3}\right) \cdot \left(-\frac{y^5}{3}\right)$

37. Use the distributive property to write an equivalent expression to $-3x(6x-1)$ that has no grouping symbols.

38. Use the distributive property to write an equivalent expression to $-6r(3r+8)$ that has no grouping symbols.

39. Use the distributive property to write an equivalent expression to $-9t^4(t-4)$ that has no grouping symbols.

40. Use the distributive property to write an equivalent expression to $-3b^3(b+4)$ that has no grouping symbols.

41. Use the distributive property to simplify $3+5c(2+6c)$ completely.

42. Use the distributive property to simplify $6+2y(2+10y)$ completely.

43. Use the distributive property to simplify $8m-2m(7-9m^3)$ completely.

44. Use the distributive property to simplify $5n-7n(1-10n^4)$ completely.

45. Use the distributive property to simplify $2q^2-3q^2(-5-10q^3)$ completely.

46. Use the distributive property to simplify $8x^2-9x^2(-10-10x^4)$ completely.

47. Simplify the following expressions if possible.

a. r^2+r^2

b. $(r^2)(r^2)$

c. r^2+r^3

d. $(r^2)(r^3)$

48. Simplify the following expressions if possible.

a. $t+t$

b. $(t)(t)$

c. $t+t^2$

d. $(t)(t^2)$

49. Simplify the following expressions if possible.

a. b^3+b^3

b. $(b^3)(b^3)$

c. b^3+b^2

d. $(b^3)(b^2)$

50. Simplify the following expressions if possible.

a. c^2+c^2

b. $(c^2)(c^2)$

c. c^2+c^4

d. $(c^2)(c^4)$

51. Simplify the following expressions if possible.

a. $-2q^2+q^2$

b. $(-2q^2)(q^2)$

c. $-2q^2+4q^3$

d. $(-2q^2)(4q^3)$

52. Simplify the following expressions if possible.

a. m^4+2m^4

b. $(m^4)(2m^4)$

c. m^4-m^3

d. $(m^4)(-m^3)$

53. Simplify the following expressions if possible.

a. $-2n^4-3n^4$

b. $(-2n^4)(-3n^4)$

c. $-2n^4-4n^2$

d. $(-2n^4)(-4n^2)$

54. Simplify the following expressions if possible.

a. $4q^4+3q^4$

b. $(4q^4)(3q^4)$

c. $4q^4+4q^3$

d. $(4q^4)(4q^3)$

55. Simplify the following expressions if possible.

a. $x^2+4x+4x^2$

b. $(x^2)(4x)(4x^2)$

56. Simplify the following expressions if possible.

a. $-2r^4 - r - 2r^4$

b. $(-2r^4)(-r)(-2r^4)$

Simplify the following expression.

57. $-3t^5\left(-4t^5\right)^2$

58. $3b^3\left(-2b^2\right)^2$

59. $3c^4r^3\left(2c^2r^4\right)^3$

60. $5nx^2\left(-2n^5x\right)^2$

61. $(-3m^4)(5m^3) + (3m^4)(-3m^3)$

62. $(n^3)(3n^4) + (5n)(5n^6)$

63. $(-4q^4)\left(2q^4\right)^4 + (3q^2)(-2q^6)$

64. $(3x^2)\left(x^3\right)^2 + (5x^2)(-4x^3)$

65. $(2r^3)\left(r^2\right)^2 + (-5r)^2(4r^5)$

66. $(3t^5)\left(2t^3\right)^4 + \left(3t^2\right)^2(-t^{13})$

67. $(-2b)^4 q^{10} - 3\left(b^2q^5\right)^2$

68. $\left(4c^3\right)^2 x^{10} - 5\left(c^3x^5\right)^2$

Challenge

69.

a. Let $x^{11} \cdot x^a = x^{28}$. Let's say that a is a natural number. How many possibilities are there for a?

b. Let $x^b \cdot x^c = x^{90}$. Let's say that b and c are natural numbers. How many possibilities are there for b?

c. Let $x^d \cdot x^e = x^{1450}$. Let's say that d and e are natural numbers. How many possibilities are there for d?

70. Choose the correct inequality or equal sign to make the relation true.
3^{400} $(\square <\quad \square >\quad \square =)$ 4^{300}

71. Fill in the blanks with algebraic expressions that make the equation true. You may not use 0 or 1 in any of the blank spaces.

Here is an example: $? + ? = 8x$.

One possible answer is: $3x + 5x = 8x$.

There are infinitely many correct answers to this problem. *Be creative.* After finding a correct answer, see if you can come up with a different answer that is also correct.

a. $\boxed{} + \boxed{} = -13x$

b. $\boxed{} + \boxed{} = -15x^{25}$

c. $\boxed{} \cdot \boxed{} \cdot \boxed{} = 5x^{55}$

5.3 Dividing by a Monomial

We learned how to *add* and *subtract* polynomials in Section 5.1. Then in Section 5.2, we learned how to *multiply* monomials together (but not yet how to multiply general polynomials together). In this section we learn how to *divide* a general polynomial by a monomial.

5.3.1 Quotient of Powers Rule

When we *multiply* the same base raised to powers, we *add* the exponents, as in $2^2 \cdot 2^3 = 2^5$. What happens when we *divide* the same base raised to powers?

Example 5.3.2 Simplify $\frac{x^5}{x^2}$ by first writing out what each power means.

Explanation. Without knowing a rule for simplifying this quotient of powers, we can write the expressions without exponents and simplify.

$$\begin{aligned}\frac{x^5}{x^2} &= \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}\\ &= \frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot 1}\\ &= \frac{x \cdot x \cdot x}{1}\\ &= x^3\end{aligned}$$

Notice that the difference of the exponents of the numerator and the denominator (5 and 2, respectively) is 3, which is the exponent of the simplified expression.

When we divide as we've just done, we end up canceling factors from the numerator and denominator one-for-one. These common factors cancel to give us factors of 1. The general rule for this is:

Fact 5.3.3 Quotient of Powers Rule. *For any non-zero real number* a *and integers* m *and* n *where* $m > n$,

$$\frac{a^m}{a^n} = a^{m-n}$$

This rule says that when you're dividing two expressions that have the same base, you can simplify the quotient by subtracting the exponents. In Example 5.3.2, this means that we can directly compute $\frac{x^5}{x^2}$:

$$\begin{aligned}\frac{x^5}{x^2} &= x^{5-2}\\ &= x^3\end{aligned}$$

Now we can update the list of exponent rules from Section 5.2.

List 5.3.4: Summary of the Rules of Exponents (Thus Far)

If a and b are real numbers, and m and n are positive integers, then we have the following rules:

Product Rule $a^m \cdot a^n = a^{m+n}$

Power to a Power Rule $(a^m)^n = a^{m \cdot n}$

Product to a Power Rule $(ab)^m = a^m \cdot b^m$

Quotient of Powers Rule $\frac{a^m}{a^n} = a^{m-n}$ (when $m > n$)

5.3.2 Dividing a Polynomial by a Monomial

Recall that dividing by a number c is the same as multiplying by the reciprocal $\frac{1}{c}$. For example, whether you divide 8 by 2 or multiply 8 by $\frac{1}{2}$, the result is 4 either way. In symbols,

$$\frac{8}{2} = \frac{1}{2} \cdot 8 \qquad \text{(both work out to 4)}$$

If we apply this idea to a polynomial being divided by a monomial, say with $\frac{a+b}{c}$, we can see that the distributive law works for this kind of division as well as with multiplciation:

$$\begin{aligned} \frac{a+b}{c} &= \frac{1}{c} \cdot (a+b) \\ &= \frac{1}{c} \cdot a + \frac{1}{2} \cdot b \\ &= \frac{a}{c} + \frac{b}{c} \end{aligned}$$

In the end, the c has been "distributed" into the a and the b. Once we recognize that division by a monomial is distributive, we are left with individual monomial pairs that we can divide.

Example 5.3.5 Simplify $\dfrac{2x^3 + 4x^2 - 10x}{2}$.

We recognize that the 2 we're dividing by can be divided into each and every term of the numerator. Once we recognize that, we will simply perform those divisions.

$$\begin{aligned} \frac{2x^3 + 4x^2 - 10x}{2} &= \frac{2x^3}{2} + \frac{4x^2}{2} + \frac{-10x}{2} \\ &= x^3 + 2x^2 - 5x \end{aligned}$$

Example 5.3.6 Simplify $\dfrac{15x^4 - 9x^3 + 12x^2}{3x^2}$.

Explanation. We recognize that each term in the numerator can be divided by $3x^2$. To actually carry out that division we'll need to use the Quotient of Powers Rule. This is going to cause a change in each coefficient and exponent.

$$\begin{aligned}\frac{15x^4 - 9x^3 + 12x^2}{3x^2} &= \frac{15x^4}{3x^2} + \frac{-9x^3}{3x^2} + \frac{12x^2}{3x^2}\\ &= 5x^2 - 3x + 4\end{aligned}$$

Remark 5.3.7 Once you become comfortable with this process, you might leave out the step where we wrote out the distribution. You will do the distribution in your head and this will become a one-step exercise. Here's how Example 5.3.6 would be visualized:

$$\frac{15x^4 - 9x^3 + 12x^2}{3x^2} = \square x^{\square} - \square x^{\square} + \square x^{\square}$$

And when calculated, we'd get:

$$\frac{15x^4 - 9x^3 + 12x^2}{3x^2} = 5x^2 - 3x + 4$$

(With the last term, note that $\frac{x^2}{x^2}$ reduces to 1.)

Example 5.3.8 Simplify $\dfrac{20x^3y^4 + 30x^2y^3 - 5x^2y^2}{-5xy^2}$.

Explanation.

$$\begin{aligned}\frac{20x^3y^4 + 30x^2y^3 - 5x^2y^2}{-5xy^2} &= \frac{20x^3y^4}{-5xy^2} + \frac{30x^2y^3}{-5xy^2} + \frac{-5x^2y^2}{-5xy^2}\\ &= -4x^2y^2 - 6xy + x\end{aligned}$$

Checkpoint 5.3.9 Simplify the following expression

$$\frac{18r^{20} + 18r^{16} - 54r^{14}}{-6r^2}$$

Explanation. We divide each term by $-6r^2$ as follows.

$$\begin{aligned}\frac{18r^{20} + 18r^{16} - 54r^{14}}{-6r^2} &= \frac{18r^{20}}{-6r^2} + \frac{18r^{16}}{-6r^2} + \frac{-54r^{14}}{-6r^2}\\ &= -\frac{18}{6}r^{18} - \frac{18}{6}r^{14} + \frac{54}{6}r^{12}\\ &= -3r^{18} - 3r^{14} + 9r^{12}\end{aligned}$$

Example 5.3.10 The density of an object, ρ (pronounced "rho"), can be calculated by the formula

$$\rho = \frac{m}{V}$$

where m is the object's mass, and V is its volume. The mass of a certain cancerous growth can be modeled by $4t^3 - 6t^2 + 8t$ grams, where t is the number of days since the growth began. If its volume is $2t$ cubic centimeters, find the growth's density.

Explanation. We have:

$$\begin{aligned}\rho &= \frac{m}{V}\\ &= \frac{4t^3 - 6t^2 + 8t}{2t}\ \frac{g}{cm^3}\\ &= \frac{4t^3}{2t} - \frac{6t^2}{2t} + \frac{8t}{2t}\quad \frac{g}{cm^3}\\ &= 2t^2 - 3t + 4\quad \frac{g}{cm^3}\end{aligned}$$

The growth's density can be modeled by $2t^2 - 3t + 4\ \frac{g}{cm^3}$.

5.3.3 Reading Questions

1. How is dividing a polynomial by a monomial similar to distributing multiplication over a polynomial? For example, how is the process of simplifying $\frac{15x^3+5x^2+10x}{5x}$ similar to simplifying $5x\left(15x^3 + 5x^2 + 10x\right)$?

5.3.4 Exercises

Quotient of Powers Rule Use the properties of exponents to simplify the expression.

1. $\dfrac{y^3}{y}$

2. $\dfrac{t^5}{t^4}$

3. $\dfrac{-25y^{20}}{5y^{13}}$

4. $\dfrac{21t^{15}}{7t^9}$

5. $\dfrac{8r^5}{48r^2}$

6. $\dfrac{10r^{12}}{40r^2}$

7. $\dfrac{33y^{18}}{11y^{10}}$

8. $\dfrac{-78t^{13}}{13t^7}$

9. $\dfrac{15r^{17}}{60r^5}$

10. $\dfrac{2r^6}{4r}$

11. $\dfrac{r^5}{r^4}$

12. $\dfrac{x^7}{x^3}$

13. $\dfrac{13^{11}}{13^5}$

14. $\dfrac{14^{18}}{14^{15}}$

15. $\dfrac{16^{12} \cdot 14^{10}}{16^4 \cdot 14^8}$

16. $\dfrac{17^6 \cdot 13^{12}}{17^4 \cdot 13^9}$

17. $\dfrac{-85x^9y^{10}z^{18}}{17x^6y^3z^{17}}$

18. $\dfrac{76x^{18}y^8z^9}{19x^{10}y^4z^7}$

19. $\dfrac{4x^{12}y^9}{2x^6y^3}$

20. $\dfrac{-20x^7y^{18}}{4x^4y^7}$

Dividing Polynomials by Monomials Simplify the following expression

21. $\dfrac{-20y^{17} + 65y^8}{5}$

22. $\dfrac{-117y^7 + 45y^4}{9}$

23. $\dfrac{16y^{14} + 16y^9 - 8y^7}{4y^3}$

24. $\dfrac{40r^{14} + 96r^{12} + 80r^{10}}{-8r^3}$

25. $\dfrac{65r^{13} + 50r^7}{5r}$

26. $\dfrac{27t^{20} + 18t^9}{9t}$

27. $\dfrac{54t^{15} - 117t^{11} - 99t^9 - 117t^8}{-9t^4}$

28. $\dfrac{-60x^{21} + 55x^{10} + 25x^9 + 30x^8}{-5x^4}$

29. $\dfrac{80x^2y^2 - 16xy + 80xy^2}{-8xy}$

30. $\dfrac{63x^2y^2 - 18xy + 54xy^2}{-9xy}$

31. $\dfrac{-28x^{14}y^{22} + 70x^{11}y^{17} + 91x^{12}y^{20}}{7x^5y^2}$

32. $\dfrac{-3x^{25}y^{14} + 3x^{14}y^{13} - 9x^{22}y^9}{3x^5y^2}$

33. $\dfrac{-40r^{19} - 25r^{14} + 5r^6}{5r^2}$

34. $\dfrac{-72r^9 - 64r^6 - 24r^4}{-8r^2}$

Application Problems

35. A rectangular prism's volume can be calculated by the formula $V = Bh$, where V stands for volume, B stands for base area, and h stands for height. A certain rectangular prism's volume can be modeled by $30x^5 - 35x^3 - 15x^2$ cubic units. If its height is $5x$ units, find the prism's base area.

$B =$ $\boxed{}$ square units

36. A rectangular prism's volume can be calculated by the formula $V = Bh$, where V stands for volume, B stands for base area, and h stands for height. A certain rectangular prism's volume can be modeled by $35x^6 + 40x^4 + 15x$ cubic units. If its height is $5x$ units, find the prism's base area.

$B =$ $\boxed{}$ square units

37. A cylinder's volume can be calculated by the formula $V = Bh$, where V stands for volume, B stands for base area, and h stands for height. A certain cylinder's volume can be modeled by $18\pi x^6 + 12\pi x^5 - 6\pi x^3$ cubic units. If its base area is $2\pi x^2$ square units, find the cylinder's height.

$h =$ $\boxed{}$ units

38. A cylinder's volume can be calculated by the formula $V = Bh$, where V stands for volume, B stands for base area, and h stands for height. A certain cylinder's volume can be modeled by $12\pi x^5 - 20\pi x^4 + 6\pi x^3$ cubic units. If its base area is $2\pi x^2$ square units, find the cylinder's height.

$h =$ $\boxed{}$ units

5.4 Multiplying Polynomials

Previously in Section 5.2, we learned to multiply two monomials together (such as $4xy \cdot 3x^2$). And in Section 5.1, we learned how to add and subtract polynomials even when there is more than one term (such as $(4x^2 - 3x) + (5x^2 + x - 2)$). In this section, we will learn how to multiply polynomials with more than one term.

Example 5.4.2 Revenue. Avery owns a local organic jam company that currently sells about 1500 jars a month at a price of \$13 per jar. Avery has found that for each time they would raise the price of a jar by 25 cents, they will sell 50 fewer jars of jam per month.

In general, this company's revenue can be calculated by multiplying the cost per jar by the total number of jars of jam sold. If we let x represent the number of times the price was raised by 25 cents, then the price will be $13 + 0.25x$.

Conversely, the number of jars the company will sell will be the 1500 they currently sell each month, minus 50 times x. This gives us the expression $1500 - 50x$ to represent how many jars the company will sell after raising the price x times.

Combining these expressions, we can write a formula for the revenue model:

$$\begin{aligned} \text{revenue} &= (\text{price per item}) \times (\text{number of items sold}) \\ R &= (13 + 0.25x)(1500 - 50x) \end{aligned}$$

To simplify the expression $(13 + 0.25x)(1500 - 50x)$, we'll need to multiply $13 + 0.25x$ by $1500 - 50x$. In this section, we learn how to do that.

5.4.1 Review of the Distributive Property

Polynomial multiplication relies on the distributive property , and may also rely on the rules of exponents. When we multiply a monomial with a binomial, we apply this property by distributing the monomial to each term in the binomial. For example,

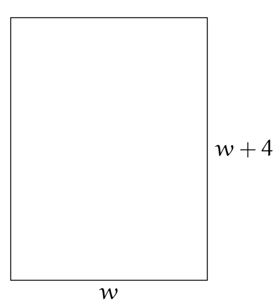
$$\begin{aligned} -4x(3x^2 + 5) &= (-4x) \cdot \left(3x^2\right) + (-4x) \cdot (5) \\ &= -12x^3 - 20x \end{aligned}$$

Remark 5.4.3 We can use the distributive property when multiplying on either the left or the right. This means that $a(b + c) = ab + ac$, but also $(b + c)a = ba + ca$.

Example 5.4.4 A rectangle's length is 4 meters longer than its width. Assume its width is w meters. Use a simplified polynomial to model the rectangle's area in terms of w as the only variable.

Explanation.

Since the rectangle's length is 4 meters longer than its width, we can model its length by $w + 4$ meters.



The rectangle's area would be:

$$\begin{aligned} A &= \ell w \\ &= (w + 4)w \\ &= w^2 + 4w \end{aligned}$$

The rectangle's area can be modeled by $w^2 + 4w$ square meters.

In the second line of work above, we should recognize that $(w+4)w$ is equivalent to $w(w+4)$. Whether the w is written before or after the binomial, we are still able to use distribution to simplify the product.

Checkpoint 5.4.5 A rectangle's length is 3 feet shorter than twice its width. If we use w to represent the rectangle's width, use a polynomial to represent the rectangle's area in expanded form.

area = $\boxed{}$ square feet

Explanation. The rectangle's width is w feet. Since the rectangle's length is 3 feet shorter than twice its width, its length is $2w-3$ feet. A rectangle's area formula is:

area = (length) · (width)

After substitution, we have:

$$\begin{aligned} \text{area} &= (\text{length}) \cdot (\text{width}) \\ &= (2w-3)w \\ &= 2w^2 - 3w \end{aligned}$$

The rectangle's area is $2w^2 - 3w$ square feet.

The distributive property can be understood visually with a **generic rectangle**.

	$3x$	4
$2x$	$2x \cdot 3x$	$2x \cdot 4$

Figure 5.4.6: A Generic Rectangle Modeling $2x(3x+4)$

The big rectangle consists of two smaller rectangles. The big rectangle's area is $2x(3x+4)$, and the sum of those two smaller rectangles is $2x \cdot 3x + 2x \cdot 4$. Since the sum of the areas of those two smaller rectangles is the same as the bigger rectangle's area, we have:

$$\begin{aligned} 2x(3x+4) &= 2x \cdot 3x + 2x \cdot 4 \\ &= 6x^2 + 8x \end{aligned}$$

Generic rectangles can be used to visualize multiplying polynomials.

5.4.2 Multiplying Binomials

Multiplying Binomials Using Distribution. Whether we're multiplying a monomial with a polynomial or two larger polynomials together, the first step is still based on the distributive property . We'll start with multiplying two binomials and then move on to larger polynomials.

We know we can distribute the 3 in $(x+2)3$ to obtain $(x+2) \cdot 3 = x \cdot 3 + 2 \cdot 3$. We can actually distribute *anything* across $(x+2)$ if it is multiplied. For example:

$$(x+2)😺 = x \cdot 😺 + 2 \cdot 😺$$

With this in mind, we can multiply $(x+2)(x+3)$ by distributing the $(x+3)$ across $(x+2)$:

$$(x+2)(x+3) = x(x+3) + 2(x+3)$$

To finish multiplying, we'll continue by distributing again, but this time across $(x+3)$:

$$(x+2)(x+3) = x(x+3) + 2(x+3)$$

$$\begin{aligned} &= x \cdot x + x \cdot 3 + 2 \cdot x + 2 \cdot 3 \\ &= x^2 + 3x + 2x + 6 \\ &= x^2 + 5x + 6 \end{aligned}$$

To multiply a binomial by another binomial, we simply had to repeat the step of distribution and simplify the resulting terms. In fact, multiplying any two polynomials will rely upon these same steps.

Multiplying Binomials Using FOIL. While multiplying two binomials requires two applications of the distributive property, people often remember this distribution process using the acronym FOIL. FOIL refers to the pairs of terms from each binomial that end up distributed to each other.

If we take another look at the example we just completed, $(x+2)(x+3)$, we can highlight how the FOIL process works. FOIL is the acronym for "First, Outer, Inner, Last".

$$\begin{aligned} (x+2)(x+3) &= \overbrace{(x \cdot x)}^{\text{F}} + \overbrace{(3 \cdot x)}^{\text{O}} + \overbrace{(2 \cdot x)}^{\text{I}} + \overbrace{(2 \cdot 3)}^{\text{L}} \\ &= x^2 + 3x + 2x + 6 \\ &= x^2 + 5x + 6 \end{aligned}$$

F: x^2 The x^2 term was the result of the product of *first* terms from each binomial.

O: $3x$ The $3x$ was the result of the product of the *outer* terms from each binomial. This was from the x in the front of the first binomial and the 3 in the back of the second binomial.

I: $2x$ The $2x$ was the result of the product of the *inner* terms from each binomial. This was from the 2 in the back of the first binomial and the x in the front of the second binomial.

L: 6 The constant term 6 was the result of the product of the *last* terms of each binomial.

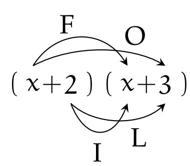


Figure 5.4.7: Using FOIL Method to multiply $(x+2)(x+3)$

Multiplying Binomials Using Generic Rectangles. We can also approach this same example using the generic rectangle method. To use generic rectangles, we treat $x+2$ as the base of a rectangle, and $x+3$ as the height. Their product, $(x+2)(x+3)$, represents the rectangle's area. The next diagram shows how to set up generic rectangles to multiply $(x+2)(x+3)$.

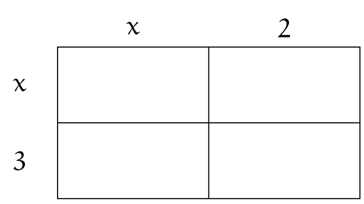


Figure 5.4.8: Setting up Generic Rectangles to Multiply $(x+2)(x+3)$

The big rectangle consists of four smaller rectangles. We will find each small rectangle's area in the next diagram by the formula area = base · height.

	x	2
x	x^2	$2x$
3	$3x$	6

Figure 5.4.9: Using Generic Rectangles to Multiply $(x + 2)(x + 3)$

To finish finding this product, we need to add the areas of the four smaller rectangles:

$$\begin{aligned}(x+2)(x+3) &= x^2 + 3x + 2x + 6\\ &= x^2 + 5x + 6\end{aligned}$$

Notice that the areas of the four smaller rectangles are exactly the same as the four terms we obtained using distribution, which are also the same four terms that came from the FOIL method. Both the FOIL method and generic rectangles approach are different ways to represent the distribution that is occurring.

Example 5.4.10 Multiply $(2x - 3y)(4x - 5y)$ using distribution.

Explanation. To use the distributive property to multiply those two binomials, we'll first distribute the second binomial across $(2x - 3y)$. Then we'll distribute again, and simplify the terms that result.

$$\begin{aligned}(2x-3y)(4x-5y) &= 2x(4x-5y) - 3y(4x-5y)\\ &= 8x^2 - 10xy - 12xy + 15y^2\\ &= 8x^2 - 22xy + 15y^2\end{aligned}$$

Example 5.4.11 Multiply $(2x - 3y)(4x - 5y)$ using FOIL.

Explanation. First, Outer, Inner, Last: Either with arrows on paper or mentally in our heads, we'll pair up the four pairs of monomials and multiply those pairs together.

$$\begin{aligned}(2x-3y)(4x-5y) &= \overbrace{(2x\cdot 4x)}^{\text{F}} + \overbrace{(2x\cdot(-5y))}^{\text{O}} + \overbrace{(-3y\cdot 4x)}^{\text{I}} + \overbrace{(-3y\cdot(-5y))}^{\text{L}}\\ &= 8x^2 - 10xy - 12xy + 15y^2\\ &= 8x^2 - 22xy + 15y^2\end{aligned}$$

Example 5.4.12 Multiply $(2x - 3y)(4x - 5y)$ using generic rectangles.

Explanation. We begin by drawing four rectangles and marking their bases and heights with terms in the given binomials:

	$2x$	$-3y$
$4x$		
$-5y$		

Figure 5.4.13: Setting up Generic Rectangles to Multiply $(2x-3y)(4x-5y)$

Next, we calculate each rectangle's area by multiplying its base with its height:

	$2x$	$-3y$
$4x$	$8x^2$	$-12xy$
$-5y$	$-10xy$	$15y^2$

Figure 5.4.14: Using Generic Rectangles to Multiply $(2x-3y)(4x-5y)$

Finally, we add up all rectangles' area to find the product:

$$\begin{aligned}(2x-3y)(4x-5y) &= 8x^2-10xy-12xy+15y^2\\ &= 8x^2-22xy+15y^2\end{aligned}$$

Example 5.4.15 Multiply and simplify the formula for Avery's organic jam revenue, R (in dollars), from Example 5.4.2 where $R=(13+0.25x)(1500-50x)$ and x represents the number of times they raised the price by 25 cents.

Explanation. To multiply this, we'll use FOIL:

$$\begin{aligned}R &= (13+0.25x)(1500-50x)\\ &= (13\cdot 1500)-(13\cdot 50x)+(0.25x\cdot 1500)-(0.25x\cdot 50x)\\ &= 19500-650x+375x-12.5x^2\\ &= -12.5x^2-275x+19500\end{aligned}$$

Example 5.4.16 Tyrone is an artist and he sells each of his paintings for \$200. Currently, he can sell 100 paintings per year. So his annual revenue from selling paintings is $\$200\cdot 100=\$20000$. He plans to raise the price. However, for each \$20 price increase per painting, his customers will buy 5 fewer paintings annually.

Assume Tyrone would raise the price of his paintings x times, each time by \$20. Use an expanded polynomial to represent his new revenue per year.

Explanation. Currently, each painting costs \$200. After raising the price x times, each time by \$20, each painting's new price would be $200+20x$ dollars.

Currently, Tyrone sells 100 paintings per year. After raising the price x times, each time selling 5 fewer paintings, he would sell $100-5x$ paintings per year.

His annual revenue can be calculated by multiplying each painting's price by the number of paintings he would sell:

$$\begin{aligned}\text{annual revenue} &= (\text{price})(\text{number of sales})\\ &= (200+20x)(100-5x)\\ &= 200(100)+200(-5x)+20x(100)+20x(-5x)\\ &= 20000-1000x+2000x-100x^2\\ &= -100x^2+1000x+20000\end{aligned}$$

After raising the price x times, each time by \$20, Tyrone's annual income from paintings would be $-100x^2 + 1000x + 20000$ dollars.

5.4.3 Multiplying Polynomials Larger Than Binomials

The foundation for multiplying any pair of polynomials is distribution and monomial multiplication. Whether we are working with binomials, trinomials, or larger polynomials, the process is fundamentally the same.

Example 5.4.17 Multiply $(x+5)\left(x^2-4x+6\right)$.

We can approach this product using either distribution generic rectangles. We cannot directly use the FOIL method, although it can be helpful to draw arrows to the six pairs of products that will occur.

$$(x+5)\left(x^2-4x+6\right)$$

Figure 5.4.18: Multiply Each Term by Each Term

Using the distributive property, we begin by distributing across $\left(x^2-4x+6\right)$, perform a second step of distribution, and then combine like terms.

$$\begin{aligned}(x+5)\left(x^2-4x+6\right) &= x\left(x^2-4x+6\right)+5\left(x^2-4x+6\right)\\ &= x\cdot x^2 - x\cdot 4x + x\cdot 6 + 5\cdot x^2 - 5\cdot 4x + 5\cdot 6\\ &= x^3-4x^2+6x+5x^2-20x+30\\ &= x^3+x^2-14x+30\end{aligned}$$

With the foundation of monomial multiplication and understanding how distribution applies in this context, we are able to find the product of any two polynomials.

Checkpoint 5.4.19 Multiply the polynomials.
$(a-3b)(a^2+7ab+9b^2)$

Explanation. We multiply the polynomials by using the terms from $a-3b$ successively.

$$\begin{aligned}(a-3b)\left(a^2+7ab+9b^2\right) &= aa^2 + a\cdot 7ab + a\cdot 9b^2 - 3ba^2 - 3b\cdot 7ab - 3b\cdot 9b^2\\ &= a^3+4a^2b-12ab^2-27b^3\end{aligned}$$

5.4.4 Reading Questions

1. Describe three ways you can go about multiplying $(x+3)(2x+5)$.

2. If you multiplied out $(a+b+c)(d+e+f+g)$, how many terms would there be? (Try to answer without actually writing them all down.)

5.4.5 Exercises

Review and Warmup Use the properties of exponents to simplify the expression.

1. $x^{10} \cdot x^9$

2. $r^{13} \cdot r^3$

3. $(10y^{15}) \cdot (-2y^{15})$

4. $(6r^{17}) \cdot (3r^8)$

5. $(-3y^{12})^2$

6. $(-8x^2)^3$

7. Count the number of terms in each expression.

a. $5x$

b. $-8x^2 - 3$

c. $-y^2 - 6x^2 + 8x - 7x^2$

d. $y^2 - 8y + 3x^2 - 2$

8. Count the number of terms in each expression.

a. $7x^2 - 2z$

b. $-9t + 8y^2 + x + 7t^2$

c. x^2

d. $t + 7 + 2y$

9. List the terms in each expression.

a. $8.2x + 6.3s^2$

b. $7.9y + 6z - 3.4z^2$

c. $-6.5y - 3.6t + 3.1$

d. $2.8t$

10. List the terms in each expression.

a. $-8.3x^2 - 3.6y^2 + 5.9s$

b. $-6.7z^2 + 6.2$

c. $-4.7t + 7.1t + 4.9s - 6.3$

d. $-5.5s + 5.5t^2 - 6.2z - 3.3s$

11. List the terms in each expression.

a. $-6.7x + 4.6t + 3.9z$

b. $3.5z^2 + 6.5t + 6.5 - 5.5y$

c. $-4.5z^2 + 8.9s^2$

d. $8.6t - 3s^2 - 0.9s^2$

12. List the terms in each expression.

a. $-5.1s^2 - 5.3x + 1.9y$

b. $2.3z + 2.5y^2 + 0.3s$

c. $-7.8z + 6.1x^2 + 7.9 - 0.9z$

d. $4.5t^2 - 0.5s^2$

13. Simplify each expression, if possible, by combining like terms.

a. $-4s + 3x^2 + 2t + 6t$

b. $-7x - 6x$

c. $4s^2 - 6x^2$

d. $8s^2 - 6y^2$

14. Simplify each expression, if possible, by combining like terms.

a. $-2s^2 - 7s^2 - 2s^2 - 2x^2$

b. $-9z + 3y^2 - 4z - 6z$

c. $5y^2 - 3 - 4t^2$

d. $9z + 6z - 5z$

15. Simplify each expression, if possible, by combining like terms.

a. $-4s - s$

b. $\frac{4}{3}x - \frac{8}{5}s + 7t$

c. $-\frac{2}{5}t - 1 - 7t^2 - \frac{5}{9}x^2$

d. $-t - \frac{2}{3}t - \frac{8}{9}x$

16. Simplify each expression, if possible, by combining like terms.

a. $\frac{1}{4}s^2 - \frac{5}{7}t^2$

b. $-\frac{4}{3}x^2 - \frac{5}{7}y^2$

c. $4z^2 + 2 - \frac{9}{2}x$

d. $-4y^2 + \frac{1}{2}t + \frac{4}{3}t^2$

Multiplying Monomials with Binomials Multiply the polynomials.

17. $-5x(x+9)$

18. $-3x(x-5)$

19. $-6x(-7x-10)$

20. $7x(-2x+10)$

21. $4x^2(x-5)$

22. $6x^2(x+3)$

23. $-8t^2\left(5t^2-4t\right)$

24. $5x^2\left(2x^2-8x\right)$

25. $-2x^2\left(9x^2-3x+5\right)$

26. $8x^2\left(6x^2-8x+8\right)$

27. $(-5x^{12}y^5)(-10x^5-9y^3)$

28. $(-6x^{14}y^{13})(5x^{10}+9y^{11})$

Multiply the polynomials.

29. $(7a^{16}b^{20})(9a^{16}b^{19}-8a^{19}b^{18})$

30. $(-8a^{17}b^9)(-4a^3b^9+8a^{13}b^{18})$

31. $(9a^5)(-7a^9-8a^4b^9+10b^7)$

32. $(10a^8)(2a^5+8a^9b^6-10b^7)$

Applications of Multiplying Monomials with Binomials

33. A rectangle's length is 1 feet shorter than 3 times its width. If we use w to represent the rectangle's width, use a polynomial to represent the rectangle's area in expanded form.
area = ______
square feet

34. A rectangle's length is 2 feet shorter than 5 times its width. If we use w to represent the rectangle's width, use a polynomial to represent the rectangle's area in expanded form.
area = ______
square feet

35. A triangle's height is 4 feet longer than 4 times its base. If we use b to represent the triangle's base, use a polynomial to represent the triangle's area in expanded form. A triangle's area can be calculated by $A = \frac{1}{2}bh$, where b stands for base, and h stands for height.
area = ______
square feet

36. A triangle's height is 4 feet longer than twice its base. If we use b to represent the triangle's base, use a polynomial to represent the triangle's area in expanded form. A triangle's area can be calculated by $A = \frac{1}{2}bh$, where b stands for base, and h stands for height.
area = ______
square feet

37. A trapezoid's top base is 5 feet longer than its height, and its bottom base is 9 feet longer than its height. If we use h to represent the trapezoid's height, use a polynomial to represent the trapezoid's area in expanded form. A trapezoid's area can be calculated by $A = \frac{1}{2}(a+b)h$, where a stands for the top base, b stands for the bottom base, and h stands for height.

area = ______ square feet

38. A trapezoid's top base is 6 feet longer than its height, and its bottom base is 2 feet longer than its height. If we use h to represent the trapezoid's height, use a polynomial to represent the trapezoid's area in expanded form. A trapezoid's area can be calculated by $A = \frac{1}{2}(a+b)h$, where a stands for the top base, b stands for the bottom base, and h stands for height.

area = ______ square feet

Multiplying Binomials Multiply the polynomials.

39. $(r+2)(r+10)$

40. $(t+9)(t+5)$

41. $(6t+9)(t+4)$

42. $(3x+1)(x+7)$

43. $(x+9)(x-4)$

44. $(x+5)(x-10)$

45. $(y-9)(y-6)$

46. $(y-3)(y-1)$

47. $(4r+7)(r+4)$

48. $(2r+8)(3r+6)$

49. $(5t-9)(t-5)$

50. $(3t-5)(6t-5)$

51. $(2x-1)(x-5)$

52. $(8x-6)(x-7)$

53. $(4x-2)(x+1)$

54. $(10y-8)(y+5)$

55. $(5y-4)\left(3y^2-6\right)$

56. $(3r-10)\left(2r^2-6\right)$

57. $\left(10r^3+5\right)\left(r^2+10\right)$

58. $\left(7t^3+10\right)\left(t^2+7\right)$

59. $\left(3t^2-7\right)\left(t^2-7\right)$

60. $\left(6x^2-3\right)\left(x^2-7\right)$

61. $(a+3b)(a+3b)$

62. $(a-4b)(a+7b)$

63. $(a+9b)(5a-4b)$

64. $(a-6b)(6a-10b)$

65. $(7a+9b)(3a-4b)$

66. $(8a-4b)(9a+4b)$

67. $(9ab-7)(6ab-4)$

68. $(10ab+2)(3ab+4)$

69. $5(x+2)(x-5)$

70. $-3(x+9)(x-4)$

71. $x(x-9)(x-4)$

72. $2y(y+5)(y-6)$

73. $-(2y-3)(y-5)$

74. $-5(4r-3)(r-3)$

Applications of Multiplying Binomials

75. An artist sells his paintings at \$17.00 per piece. Currently, he can sell 140 paintings per year. Thus, his annual income from paintings is $17 \cdot 140 = 2380$ dollars. He plans to raise the price. However, for each \$5.00 of price increase per painting, his customers would buy 10 fewer paintings annually. Assume the artist would raise the price of his painting x times, each time by \$5.00. Use an expanded polynomial to represent his new income per year.

new annual income = ______ dollars

76. An artist sells his paintings at \$18.00 per piece. Currently, he can sell 110 paintings per year. Thus, his annual income from paintings is $18 \cdot 110 = 1980$ dollars. He plans to raise the price. However, for each \$3.00 of price increase per painting, his customers would buy 8 fewer paintings annually. Assume the artist would raise the price of his painting x times, each time by \$3.00. Use an expanded polynomial to represent his new income per year.

new annual income = ______ dollars

77. A rectangle's base can be modeled by $x + 9$ meters, and its height can be modeled by $x - 9$ meters. Use a polynomial to represent the rectangle's area in expanded form.
area = $\boxed{}$
square meters

78. A rectangle's base can be modeled by $x + 10$ meters, and its height can be modeled by $x - 4$ meters. Use a polynomial to represent the rectangle's area in expanded form.
area = $\boxed{}$
square meters

Multiplying Larger Polynomials Multiply the polynomials.

79. $(-2x+4)\left(x^2-2x-2\right)$

80. $(2x+2)\left(x^2+2x+3\right)$

81. $(3x-3)\left(-2x^3-3x^2-4x+4\right)$

82. $(-3x+5)\left(2x^3+3x^2+5x+3\right)$

83. $\left(x^2-4x+3\right)\left(x^2+4x+3\right)$

84. $\left(x^2-4x-3\right)\left(x^2-4x-5\right)$

85. $(a-8b)(a^2-3ab-2b^2)$

86. $(a+9b)(a^2+7ab+2b^2)$

87. $(a+b+10)(a+b-10)$

88. $(a+b-2)(a+b+2)$

Challenge

89. Fill in the blanks with algebraic expressions that make the equation true. You may not use 0 or 1 in any of the blank spaces. An example is $?+?=8x$, where one possible answer is $3x+5x=8x$. There are infinitely many correct answers to this problem. *Be creative.* After finding a correct answer, see if you can come up with a different answer that is also correct.

a. $\boxed{} + \boxed{} = -15xy$

b. $\boxed{} + \boxed{} = -13x^{30}y^2$

c. $\boxed{} \cdot \boxed{} \cdot \boxed{} \cdot \boxed{} \cdot \boxed{} = 2x^{60}y^{50}$

5.5 Special Cases of Multiplying Polynomials

Since we are now able to multiply polynomials together in general, we will look at a few special patterns with polynomial multiplication where there are some shortcuts worth knowing about.

5.5.1 Squaring a Binomial

Example 5.5.2 To "square a binomial" is to take a binomial and multiply it by itself. In the same way that $4^2 = 4 \cdot 4$, it's also true that $(x+4)^2 = (x+4)(x+4)$. To expand this expression, we'll simply distribute $(x+4)$ across $(x+4)$:

$$\begin{aligned}(x+4)^2 &= (x+4)(x+4)\\ &= x(x+4)+4(x+4)\\ &= x^2+4x+4x+16\\ &= x^2+8x+16\end{aligned}$$

Similarly, to expand $(y-7)^2$, we'll have:

$$\begin{aligned}(y-7)^2 &= (y-7)(y-7)\\ &= y(y-7)-7(y-7)\\ &= y^2-7y-7y+49\\ &= y^2-14y+49\end{aligned}$$

These two examples might look like any other example of multiplying binomials, but looking closely we can see that something *special* happened. Focusing on the original expression and the simplified one, we can see that a specific pattern occurred in each:

$$(x+4)^2 = x^2+2(4x)+4^2$$

And:

$$(y-7)^2 = y^2-2(7y)+7^2$$

Either way, we have:

$$(\text{first})^2 \pm 2(\text{first})(\text{second}) + (\text{second})^2$$

and the choice of $+$ or $-$ matches the original binomial.

What we're seeing is a pattern relating two things. The left side is the **square of a binomial**, and the result on the right is called a **perfect square trinomial**, a trinomial that was born from something getting squared.

The general way this pattern is established is by squaring each of the two most general binomials, $(a+b)$ and $(a-b)$. Once we have done so, we can substitute anything in place of a and b and rely upon the general pattern to simplify squared binomials.

We can write $(a+b)^2$ as $(a+b)(a+b)$ and then multiply those binomials:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2+ab+ba+b^2\end{aligned}$$

$$= a^2 + 2ab + b^2$$

Notice the final simplification step was to add $ab + ba$. Since these are like terms, we can combine them into $2ab$.

Similarly, we can find a general formula for $(a - b)^2$:

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a^2 - ab - ba + b^2\\ &= a^2 - 2ab + b^2\end{aligned}$$

Fact 5.5.3 Binomial Squared Formulas. *If a and b are real numbers or variable expressions, then we have the following formulas:*

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

These formulas will allow us to multiply this type of special product more quickly.

Remark 5.5.4 Notice that when both $(a + b)^2$ and $(a - b)^2$ are expanded, the last term is *adding* b^2 either way. This is because any number or expression, regardless of its sign, is positive after it is squared.

Some students will prefer to memorize the Binomial Squared Formulas and apply them by substuting expressions in for a and b. An alternative visualization is presented in Figure 5.5.5.

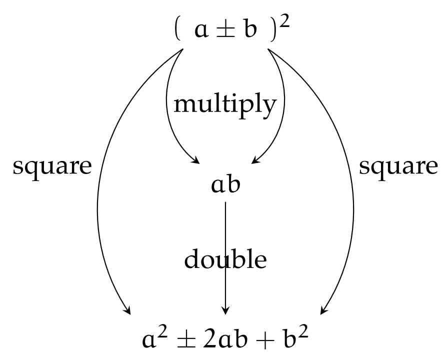


Figure 5.5.5: Visualizing the Squaring of a Binomial

Example 5.5.6 Expand $(2x - 3)^2$ using the Binomial Squared Formulas.

To apply the formula for squaring a binomial, we take $a = 2x$ and $b = 3$. Expanding this, we have:

$$\begin{aligned}(2x-3)^2 &= (2x)^2 - 2(2x)(3) + (3)^2\\ &= 4x^2 \quad - 12x \quad + 9\end{aligned}$$

Checkpoint 5.5.7 Expand the following using the Binomial Squared Formula.

a. $(5xy + 1)^2$

b. $4(3x - 7)^2$

Explanation.

a. $$\begin{aligned}(5xy+1)^2 &= (5xy)^2 + 2(5xy)(1) + 1^2\\ &= 25x^2y^2 + 10xy + 1\end{aligned}$$

b. With this expression, we will first note that the factor of 4 is *outside* the portion of the expression that is squared. Using the order of operations, we will first expand $(3x-7)^2$ and then multiply that expression by 4:

$$\begin{aligned}4(3x-7)^2 &= 4\left((3x)^2 - 2(3x)(7) + 7^2\right)\\ &= 4\left(9x^2 - 42x + 49\right)\\ &= 36x^2 - 168x + 196\end{aligned}$$

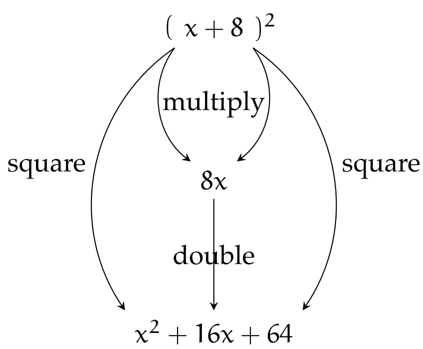
Example 5.5.8 Use the visualization in Figure 5.5.5 to expand these binomials squared.

a. $(x+8)^2$

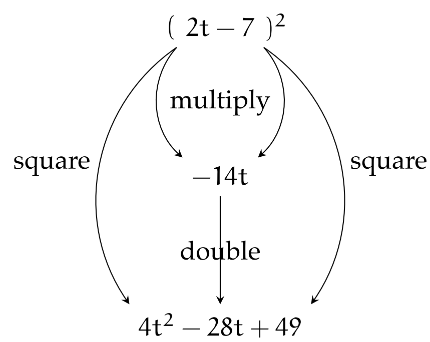
b. $(2t-7)^2$

Explanation.

a. Diagramming the process:



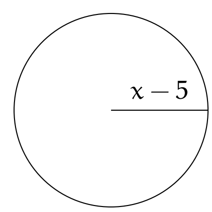
b. Diagramming the process:



Example 5.5.9
A circle's area can be calculated using the formula

$$A = \pi r^2$$

where A stands for area, and r stands for radius. If a certain circle's radius can be modeled by $x-5$ feet, use an expanded polynomial to model the circle's area.



Explanation. The circle's area would be:

$$\begin{aligned}A &= \pi r^2\\ &= \pi(x-5)^2 && \text{Now use a method for squaring this binomial}\ldots\\ &= \pi\left(x^2 - 10x + 25\right)\\ &= \pi x^2 - 10\pi x + 25\pi\end{aligned}$$

The circle's area can be modeled by $\pi x^2 - 10\pi x + 25\pi$ square feet.

Checkpoint 5.5.10 Expand $\left(y^3-12\right)^2$.

Explanation.

$$\begin{aligned}\left(y^3-12\right)^2 &= \left(y^3\right)^2 - 2\left(y^3\right)(12) + 12^2\\ &= y^6 - 24y^3 + 144\end{aligned}$$

Warning 5.5.11 Common Mistakes. Now we know how to expand $(a+b)^2$ and $(a-b)^2$. It is a common mistake to think that these are equal to a^2+b^2 and a^2-b^2, respectively, as if you could just "distribute" the exponent. Now we know that actually you get $a^2+2ab+b^2$ and $a^2-2ab+b^2$.

5.5.2 The Product of the Sum and Difference of Two Terms

To motivate the next "special case" for multiplying polynomials, we'll look at a couple of examples.

Example 5.5.12 Multiply the following binomials:

a. $(x+5)(x-5)$

b. $(y-8)(y+8)$

Explanation. We can approach these as using distribution, FOIL, or generic rectangles, and obtain the following:

a. $$\begin{aligned}(x+5)(x-5) &= x^2-5x+5x-25\\ &= x^2-25\end{aligned}$$

b. $$\begin{aligned}(y+8)(y-8) &= y^2-8y+8y-64\\ &= y^2-64\end{aligned}$$

Notice that for each of these products, we multiplied the sum of two terms by the difference of the *same* two terms. Notice also in these three examples that once these expressions were multiplied, the two middle terms were opposites and thus canceled to zero.

These pairs, generally written as $(a+b)$ and $(a-b)$, are known as **conjugates**. If we multiply $(a+b)(a-b)$, we can see this general pattern more clearly:

$$\begin{aligned}(a+b)(a-b) &= a^2-ab+ab-b^2\\ &= a^2-b^2\end{aligned}$$

As with the square of a binomial producing a perfect suqare trinomial, this pattern also has two things we can give a name to. The left side is the **product of a sum and its conjugate**, and the result on the right is a **difference of squares**.

Fact 5.5.13 The Product of a Sum and Its Conjugate Formula. *If a and b are real numbers or variable expressions, then we have the following formula:*

$$(a+b)(a-b)=a^2-b^2$$

Example 5.5.14 Multiply the following using Fact 5.5.13.

a. $(4x-7y)(4x+7y)$

b. $-2(3x+1)(3x-1)$

Explanation. The first step to using this method is to identify the values of a and b.

a. In this instance, $a=4x$ and $b=7y$. Using the formula,

$$\begin{aligned}(4x-7y)(4x+7y) &= (4x)^2-(7y)^2\\ &= 16x^2-49y^2\end{aligned}$$

b. In this instance, we have a constant factor as well as a product in the form $(a+b)(a-b)$. We will first expand $(3x+1)(3x-1)$ by identifying $a=3x$ and $b=1$ and using the formula. Then we will multiply the factor of -2 through this expression. So,

$$\begin{aligned} -2(3x+1)(3x-1) &= -2\left((3x)^2 - 1^2\right) \\ &= -2\left(9x^2 - 1\right) \\ &= -18x^2 + 2 \end{aligned}$$

Checkpoint 5.5.15 Expand $(4x+2)(4x-2)$.

Explanation.

$$\begin{aligned} (4x+2)(4x-2) &= (4x)^2 - 2^2 \\ &= 16x^2 - 4 \end{aligned}$$

Checkpoint 5.5.16 Expand $\left(x^7+9\right)\left(x^7-9\right)$.

Explanation.

$$\begin{aligned} \left(x^7+9\right)\left(x^7-9\right) &= \left(x^7\right)^2 - 9^2 \\ &= x^{14} - 81 \end{aligned}$$

5.5.3 Binomials Raised to Other Powers

Example 5.5.17 Simplify the expression $(x+5)^3$ into an expanded polynomial.

Before we start expanding this expression, it is important to recognize that $(x+5)^3 \neq x^3 + 5^3$, similar to the message in Warning 5.5.11. To be sure, we can see that if we evaluate at $x = 1$, we get different results.

$$\begin{aligned} (1+5)^3 &= 6^3 \\ &= 216 \end{aligned} \qquad\qquad \begin{aligned} 1^3 + 5^3 &= 1 + 125 \\ &= 126 \end{aligned}$$

We will need to rely on distribution to expand this expression. The first step in expanding $(x+5)^3$ is to remember that the exponent of 3 indicates that

$$(x+5)^3 = \overbrace{(x+5)(x+5)(x+5)}^{\text{3 times}}$$

Once we rewrite this in an expanded form, we next multiply the two binomials on the left and then finish by multiplying that result by the remaining binomial:

$$\begin{aligned} (x+5)^3 &= \overbrace{(x+5)(x+5)}^{\text{a binomial squared}}(x+5) \\ &= \left(x^2 + 10x + 25\right)(x+5) \\ &= x^3 + 5x^2 + 10x^2 + 50x + 25x + 125 \\ &= x^3 + 15x^2 + 75x + 125 \end{aligned}$$

Checkpoint 5.5.18 Expand $(2y-6)^3$.

Explanation.

$$\begin{aligned}(2y-6)^3 &= \overbrace{(2y-6)(2y-6)}^{\text{a binomial squared}}(2y-6)\\ &= \left(4y^2-24y+36\right)(2y-6)\\ &= 8y^3-24y^2-48y^2+144y+72y-216\\ &= 8y^3-72y^2+216y-216\end{aligned}$$

Generalizing, if we want to expand a binomial raised to a high whole number power, we can start by rewriting the expression without an exponent. Then it will help some to use the formula for the square of a binomial.

Example 5.5.19 To multiply $(x-3)^4$, we'd start by rewriting $(x-3)^4$ in expanded form as:

$$(x-3)^4 = \overbrace{(x-3)(x-3)(x-3)(x-3)}^{4\text{ times}}$$

We will then multiply pairs of polynomials from the left to the right.

$$\begin{aligned}(x-3)^4 &= \overbrace{(x-3)(x-3)}^{\text{a perfect square}}\overbrace{(x-3)(x-3)}^{\text{a perfect square}}\\ &= \left(x^2-6x+9\right)\left(x^2-6x+9\right)\\ &= x^4-6x^3+9x^2-6x^3+36x^2-54x+9x^2-54x+81\\ &= x^4-12x^3+54x^2-108x+81\end{aligned}$$

5.5.4 Reading Questions

1. How many special patterns should you be on the lookout for when multiplying and/or squaring binomials?

2. Do you prefer to memorize the formula for the square of a binomial or to visualize the process?

5.5.5 Exercises

Review and Warmup Use the properties of exponents to simplify the expression.

1. $\left(3y^{11}\right)^4$ **2.** $\left(5x^{12}\right)^3$ **3.** $(2x)^2$ **4.** $(4r)^2$

5. $\left(-10y^5\right)^2$ **6.** $\left(-7x^6\right)^3$ **7.** $-3\left(-3t^8\right)^3$ **8.** $-4\left(-8t^9\right)^2$

Simplify each expression, if possible, by combining like terms.

9. a. $-9s+2$ b. $7x+2$ c. $9y^2+3y$ d. $-x+8s$

10. a. $-2t+7x$ b. $-y-2s$ c. $-9x^2+5y$ d. $t-8y$

11. a. $7t^2+7-5t^2-4$ b. $-t^2-2t^2$ c. $3y+6y-2x$ d. $-4s-3z+7z$

12. a. $t+6t$ b. $9z-6t+2+7z$ c. $6x-5x-4z+2s^2$ d. $3z^2+2$

Perfect Square Trinomial Formula Expand the square of a ***bi***nomial.

13. $(x+6)^2$ **14.** $(y+3)^2$ **15.** $(9y+2)^2$ **16.** $(6r+3)^2$

17. $(r-9)^2$ **18.** $(t-2)^2$ **19.** $(6t-2)^2$ **20.** $(3t-8)^2$

21. $\left(8x^2-4\right)^2$ **22.** $\left(5x^2-10\right)^2$ **23.** $\left(y^7-11\right)^2$ **24.** $\left(y^{10}+6\right)^2$

25. $(6a-5b)^2$ **26.** $(7a+2b)^2$ **27.** $(8ab-8)^2$ **28.** $(9ab+5)^2$

29. $(x^2+10y^2)^2$ **30.** $(x^2+2y^2)^2$

Difference of Squares Formula Multiply the polynomials.

31. $(x-5)(x+5)$ **32.** $(y+12)(y-12)$ **33.** $(5y+4)(5y-4)$

34. $(3r-9)(3r+9)$ **35.** $(10+6r)(10-6r)$ **36.** $(6+10t)(6-10t)$

37. $\left(t^5-8\right)\left(t^5+8\right)$ **38.** $\left(t^9+10\right)\left(t^9-10\right)$ **39.** $\left(4x^9-9\right)\left(4x^9+9\right)$

40. $\left(2x^7+3\right)\left(2x^7-3\right)$ **41.** $\left(1-12y^5\right)\left(1+12y^5\right)$ **42.** $\left(1-8y^3\right)\left(1+8y^3\right)$

43. $(6x+3y)(6x-3y)$ **44.** $(7x-8y)(7x+8y)$ **45.** $(ab-8)(ab+8)$

46. $(ab-9)(ab+9)$ **47.** $4(t+6)(t-6)$ **48.** $5(x-3)(x+3)$

49. $2(2x-3)(2x+3)$ **50.** $6(5y+5)(5y-5)$ **51.** $3(y+4)^2$

52. $5(r+10)^2$ **53.** $7(7r+1)^2$ **54.** $6(4t+5)^2$

Multiply the polynomials.

55. $(x^2+9y^2)(x^2-9y^2)$ **56.** $(x^2+10y^2)(x^2-10y^2)$

57. $(2x^8+4y^3)(2x^8-4y^3)$ **58.** $(3x^6-10y^3)(3x^6+10y^3)$

59. $(4x^4y^8+7y^3)(4x^4y^8-7y^3)$ **60.** $(5x^2y^4+3y^3)(5x^2y^4-3y^3)$

Binomials Raised to Other Powers Simplify the given expression into an expanded polynomial.

61. $(r+6)^3$ **62.** $(r+4)^3$ **63.** $(r-2)^3$ **64.** $(t-6)^3$

65. $(4t+2)^3$ **66.** $(2x+4)^3$ **67.** $(5x-2)^3$ **68.** $(4y-5)^3$

69. Determine if the following statements are true or false.

a. $(a-b)^2 = a^2 - b^2$

(☐ True ☐ False)

b. $(a+b)^2 = a^2 + b^2$

(☐ True ☐ False)

c. $(a+b)(a-b) = a^2 - b^2$

(☐ True ☐ False)

70. Determine if the following statements are true or false.

a. $(2(a-b))^2 = 4(a-b)^2$

(☐ True ☐ False)

b. $2(a+b)^2 = 2a^2 + 2b^2$

(☐ True ☐ False)

c. $2(a+b)(a-b) = 2a^2 - 2b^2$

(☐ True ☐ False)

5.6 More Exponent Rules

5.6.1 Review of Exponent Rules for Products and Exponents

In Section 5.2, we introduced three basic rules involving products and exponents. Then in Section 5.3, we introduced one more. We begin this section with a recap of these four exponent rules.

List 5.6.2: Summary of Exponent Rules (Thus Far)

Product Rule When multiplying two expressions that have the same base, simplify the product by adding the exponents.

$$x^m \cdot x^n = x^{m+n}$$

Power to a Power Rule When a base is raised to an exponent and that expression is raised to another exponent, multiply the exponents.

$$(x^m)^n = x^{m \cdot n}$$

Product to a Power Rule When a product is raised to an exponent, apply the exponent to each factor in the product.

$$(x \cdot y)^n = x^n \cdot y^n$$

Quotient of Powers Rule When dividing two expressions that have the same base, simplify the quotient by subtracting the exponents.

$$\frac{x^m}{x^n} = x^{m-n}$$

For now, we only know this rule when $m > n$.

Checkpoint 5.6.3

a. Simplify $r^{16} \cdot r^5$.

b. Simplify $\left(x^{11}\right)^{10}$.

c. Simplify $(3r)^4$.

d. Simplify $\frac{3y^7}{y^3}$.

Explanation.

a. We *add* the exponents because this is a product of powers with the same base:

$$\begin{aligned} r^{16} \cdot r^5 &= r^{16+5} \\ &= r^{21} \end{aligned}$$

b. We *multiply* the exponents because this is a power being raised to a power:

$$\begin{aligned} \left(x^{11}\right)^{10} &= x^{11 \cdot 10} \\ &= x^{110} \end{aligned}$$

c. We apply the power to each factor in the product:

$$\begin{aligned}(3r)^4 &= 3^4r^4\\ &= 81r^4\end{aligned}$$

d. We *subtract* the exponents because this expression is dividing powers with the same base:

$$\begin{aligned}\frac{3y^7}{y^3} &= \frac{3}{1}\frac{y^7}{y^3}\\ &= 3y^{7-3}\\ &= 3y^4\end{aligned}$$

5.6.2 Quotient to a Power Rule

One rule we have learned is the product to a power rule, as in $(2x)^3 = 2^3x^3$. When two factors are multiplied and the product is raised to a power, we may apply the exponent to each of those factors individually. We can use the rules of fractions to extend this property to a *quotient* raised to a power.

Example 5.6.4 Let y be a real number, where $y \neq 0$. Find another way to write $\left(\frac{5}{y}\right)^4$.

Explanation. Writing the expression without an exponent and then simplifying, we have:

$$\begin{aligned}\left(\frac{5}{y}\right)^4 &= \left(\frac{5}{y}\right)\left(\frac{5}{y}\right)\left(\frac{5}{y}\right)\left(\frac{7}{y}\right)\\ &= \frac{5\cdot5\cdot5\cdot5}{y\cdot y\cdot y\cdot y}\\ &= \frac{5^4}{y^4}\\ &= \frac{625}{y^4}\end{aligned}$$

Similar to the product to a power rule, we essentially applied the outer exponent to the "factors" inside the parentheses—to factors of the numerator *and* factors of the denominator. The general rule is:

Fact 5.6.5 Quotient to a Power Rule. *For real numbers a and b (with $b \neq 0$) and natural number m,*

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

This rule says that when you raise a fraction to a power, you may separately raise the numerator and denominator to that power. In Example 5.6.4, this means that we can directly calculate $\left(\frac{5}{y}\right)^4$:

$$\begin{aligned}\left(\frac{5}{y}\right)^4 &= \frac{5^4}{y^4}\\ &= \frac{625}{y^4}\end{aligned}$$

Checkpoint 5.6.6

a. Simplify $\left(\frac{p}{2}\right)^6$.

b. Simplify $\left(\frac{5^6w^7}{5^2w^4}\right)^9$. If you end up with a large power of a specific number, leave it written that way.

c. Simplify $\frac{\left(2r^5\right)^7}{\left(2^2r^8\right)^3}$. If you end up with a large power of a specific number, leave it written that way.

Explanation.

a. We can use the quotient to a power rule:

$$\begin{aligned}\left(\frac{p}{2}\right)^6 &= \frac{p^6}{2^6}\\ &= \frac{p^6}{64}\end{aligned}$$

b. If we stick closely to the order of operations, we should first simplify inside the parentheses and then work with the outer exponent. Going this route, we will first use the quotient rule:

$$\begin{aligned}\left(\frac{5^6w^7}{5^2w^4}\right)^9 &= \left(5^{6-2}w^{7-4}\right)^9\\ &= \left(5^4w^3\right)^9\\ &= \left(5^4\right)^9 \cdot \left(w^3\right)^9\\ &= 5^{4\cdot 9} \cdot w^{3\cdot 9}\\ &= 5^{36} \cdot w^{27}\end{aligned}$$

c. According to the order of operations, we should simplify inside parentheses first, then apply exponents, then divide. Since we cannot simplify inside the parentheses, we must apply the outer exponents to each factor inside the respective set of parentheses first:

$$\begin{aligned}\frac{\left(2r^5\right)^7}{\left(2^2r^8\right)^3} &= \frac{2^7\left(r^5\right)^7}{\left(2^2\right)^3\left(r^8\right)^3}\\ &= \frac{2^7r^{5\cdot 7}}{2^{2\cdot 3}r^{8\cdot 3}}\\ &= \frac{2^7r^{35}}{2^6r^{24}}\\ &= 2^{7-6}r^{35-24}\\ &= 2^1r^{11}\\ &= 2r^{11}\end{aligned}$$

5.6.3 Zero as an Exponent

So far, we have been working with exponents that are natural numbers $(1, 2, 3, \ldots)$. By the end of this section, we will expand our understanding to include exponents that are any integer, as with 5^0 and 12^{-2}. As a first step, let's explore how 0 should behave as an exponent by considering the pattern of decreasing powers of 2 in Figure 5.6.7.

Power		Product		Result	
2^4	$=$	$2 \cdot 2 \cdot 2 \cdot 2$	$=$	16	
2^3	$=$	$2 \cdot 2 \cdot 2$	$=$	8	(divide by 2)
2^2	$=$	$2 \cdot 2$	$=$	4	(divide by 2)
2^1	$=$	2	$=$	2	(divide by 2)
2^0	$=$	$?$	$=$	$?$	

Figure 5.6.7: Descending Powers of 2

As we move down from one row to the row below it, we reduce the exponent in the power by 1 and we remove a factor of 2 from the product. The result in one row is half of the result of the previous row. The question is, what happens when the exponent gets down to 0 and you remove the last remaining factor of 2? Following that pattern with the final results, moving from 2^1 to 2^0 should meant the result of 2 is divided by 2, leaving 1. So we have:

$$2^0 = 1$$

Fact 5.6.8 The Zero Exponent Rule. *For any non-zero real number* a,

$$a^0 = 1$$

We exclude the case where $a = 0$ from this rule, because our reasoning for this rule with the table had us dividing by the base, and we cannot divide by 0.

Checkpoint 5.6.9 Simplify the following expressions. Assume all variables represent non-zero real numbers.

a. $\left(173x^4y^{251}\right)^0$ b. $(-8)^0$ c. -8^0 d. $3x^0$

Explanation. To simplify any of these expressions, it is critical that we remember an exponent only applies to what it is touching or immediately next to.

a. In the expression $\left(173x^4y^{251}\right)^0$, the exponent 0 applies to everything inside the parentheses.

$$\left(173x^4y^{251}\right)^0 = 1$$

b. In the expression $(-8)^0$ the exponent applies to everything inside the parentheses, -8.

$$(-8)^0 = 1$$

c. In contrast to the previous example, the exponent only applies to the 8. The exponent has a higher priority than negation in the order of operations. We should consider that $-8^0 = -\left(8^0\right)$, and so:

$$\begin{aligned} -8^0 &= -\left(8^0\right) \\ &= -1 \end{aligned}$$

d. In the expression $3x^0$, the exponent 0 only applies to the x:

$$\begin{aligned} 3x^0 &= 3 \cdot x^0 \\ &= 3 \cdot 1 \\ &= 3 \end{aligned}$$

5.6.4 Negative Exponents

We understand what it means for a variable to have a natural number exponent. For example, x^5 means $\overbrace{x \cdot x \cdot x \cdot x \cdot x}^{\text{five times}}$. Now we will try to give meaning to an exponent that is a negative integer, like in x^{-5}.

To consider what it could possibly mean to have a negative integer exponent, let's extend the pattern we examined in Figure 5.6.7. In that table, each time we move down a row, we reduce the power by 1 and we divide the value by 2. We can continue this pattern in the power and value columns, going all the way down into when the exponent is negeative.

Power	Result	
2^3	8	
2^2	4	(divide by 2)
2^1	2	(divide by 2)
2^0	1	(divide by 2)
2^{-1}	$1/2 = 1/2^1$	(divide by 2)
2^{-2}	$1/4 = 1/2^2$	(divide by 2)
2^{-3}	$1/8 = 1/2^3$	(divide by 2)

Figure 5.6.10: Negative Powers of 2

We are seeing a pattern where $2^{\text{negative number}}$ is equal to $\frac{1}{2^{\text{positive number}}}$. Note that the choice of base 2 was arbitrary, and this pattern works for all bases except 0, since we cannot divide by 0 in moving from one row to the next.

Fact 5.6.11 The Negative Exponent Rule. *For any non-zero real number* a *and any natural number* n*,*

$$a^{-n} = \frac{1}{a^n}$$

If we take reciprocals of both sides, we have another helpful fact:

$$\frac{1}{a^{-n}} = a^n.$$

Taken together, these facts tell us that a power in the numerator with a negative exponent belongs in the *denominator* (with a *positive* exponent). And similarly, a power in the denominator with a negative exponent belongs in the *numerator* (with a *positive* exponent). In other words, you can view a negative exponent as telling you to move something to/from the numerator/denominator of an expression, changing the sign of the exponent at the same time.

You may be expected to simplify expressions so that they do not have any negative exponents. This can always be accomplished using the negative exponent rule. Try it with these exercises.

Checkpoint 5.6.12

a. Write $4y^{-6}$ without using negative exponents.

b. Write $\frac{3x^{-4}}{yz^{-2}}$ without using negative exponents.

c. Simplify $\left(-5x^{-5}\right)\left(-8x^4\right)$ and write it without using negative exponents.

Explanation.

a. An exponent only applies to whatever it is "touching". In the expression $4y^{-6}$, only the y is affected by the exponent.

$$\begin{aligned} 4y^{-6} &= 4 \cdot \frac{1}{y^6} \\ &= \frac{4}{y^6} \end{aligned}$$

b. Negative exponents tell us to move some variables between the numerator and denominator to make the exponents positive. The x^{-4} in the numerator should become x^4 in the denominator. The z^{-2} in the denominator should become z^2 in the numerator.

$$\frac{3x^{-4}}{yz^{-2}} = \frac{3z^2}{yx^4}$$

Notice that the factors of 3 and y did not move, as both of those factors had positive exponents.

c. The product of powers rule still applies, and we can add exponents even when one or both are negative:

$$\begin{aligned} \left(-5x^{-5}\right)\left(-8x^4\right) &= (-5)(-8)x^{-5}x^4 \\ &= 40x^{-1} \\ &= \frac{40}{x} \end{aligned}$$

5.6.5 Summary of Exponent Rules

Now that we have some new exponent rules beyond those from Section 5.2 and Section 5.3, let's summarize.

List 5.6.13: Summary of the Rules of Exponents for Multiplication and Division

If a and b are real numbers, and m and n are integers, then we have the following rules:

Product Rule $a^m \cdot a^n = a^{m+n}$

Power to a Power Rule $(a^m)^n = a^{m \cdot n}$

Product to a Power Rule $(ab)^m = a^m \cdot b^m$

Quotient Rule $\frac{a^m}{a^m} = a^{m-n}$, as long as $a \neq 0$

Quotient to a Power Rule $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, as long as $b \neq 0$

Zero Exponent Rule $a^0 = 1$ for $a \neq 0$

Negative Exponent Rule $a^{-m} = \frac{1}{a^m}$

Negative Exponent Reciprocal Rule $\frac{1}{a^{-m}} = a^m$

Remark 5.6.14 Why we have "$a \neq 0$" and "$b \neq 0$" for some rules. We have to be careful to make sure the rules we state don't suggest that it would ever be OK to divide by zero. Dividing by zero leads us to expressions that have no meaning. For example, both $\frac{9}{0}$ and $\frac{0}{0}$ are *undefined*, meaning no one has defined what it means to divide a number by 0. Also, we established that $a^0 = 1$ using repeated division by a in table rows, so that reasoning doesn't work if $a = 0$.

Warning 5.6.15 A Common Mistake. It may be tempting to apply the rules of exponents to expressions containing addition or subtraction. However, none of the Summary of the Rules of Exponents for Multiplication and Division involve addition or subtraction in the initial expression. Because whole number exponents mean repeated multiplication, not repeated addition or subtraction, trying to apply exponent rules in situations that do not use multiplication simply doesn't work.

Can we say something like $a^m + a^n = a^{m+n}$? How would that work out when $a = 2$, $m = 3$, and $n = 4$?

$$\begin{aligned} 2^3 + 2^4 &\stackrel{?}{=} 2^{3+4} \\ 8 + 16 &\stackrel{?}{=} 2^7 \\ 24 &\stackrel{\text{no}}{=} 128 \end{aligned}$$

As we can see, that's not even close. This attempt at a "sum rule" falls apart. In fact, without knowing values for a, n, and m, there's no way to simplify the expression $a^n + a^m$.

Checkpoint 5.6.16 Decide whether each statements is true or false.

a. $(7+8)^3 = 7^3 + 8^3$
(☐ true ☐ false)

b. $(xy)^3 = x^3y^3$
(☐ true ☐ false)

c. $2x^3 \cdot 4x^2 \cdot 5x^6 = (2 \cdot 4 \cdot 5)x^{3+2+6}$
(☐ true ☐ false)

d. $\left(x^3y^5\right)^4 = x^{3+4}y^{5+4}$
(☐ true ☐ false)

e. $2\left(x^2y^5\right)^3 = 8x^6y^{15}$
(☐ true ☐ false)

f. $x^2 + x^3 = x^5$
(☐ true ☐ false)

g. $x^3 + x^3 = 2x^3$
(☐ true ☐ false)

h. $x^3 \cdot x^3 = 2x^6$
(☐ true ☐ false)

i. $3^2 \cdot 2^3 = 6^5$
(☐ true ☐ false)

j. $3^{-2} = -\frac{1}{9}$
(☐ true ☐ false)

Explanation.

a. False, $(7+8)^3 \neq 7^3 + 8^3$. Following the order of operations, on the left $(7+8)^3$ would simplify as 15^3, which is 3375. However, on the right side, we have

$$\begin{aligned} 7^3 + 8^3 &= 343 + 512 \\ &= 855 \end{aligned}$$

Since $3375 \neq 855$, the equation is false.

b. True. As the cube applies to the product of x and y, $(xy)^3 = x^3y^3$.

c. True. The coefficients do get multiplied together and the exponents added when the expressions are multiplied, so $2x^3 \cdot 4x^2 \cdot 5x^6 = (2 \cdot 4 \cdot 5)x^{3+2+6}$.

d. False, $\left(x^3y^5\right)^4 \neq x^{3+4}y^{5+4}$. When we have a power to a power, we multiply the exponents rather than adding them. So

$$\left(x^3y^5\right)^4 = x^{3\cdot 4}y^{5\cdot 4}$$

e. False, $2\left(x^2y^5\right)^3 \neq 8x^6y^{15}$. The exponent of 3 applies to x^2 and y^5, but does not apply to the 2. So

$$\begin{aligned} 2\left(x^2y^5\right)^3 &= 2x^{2\cdot 3}6y^{5\cdot 3} \\ &= 2x^6y^{15} \end{aligned}$$

f. False, $x^2 + x^3 \neq x^5$. The two terms on the left hand side are not like terms and there is no way to combine them.

g. True. The terms x^3 and x^3 are like terms, so $x^3 + x^3 = 2x^3$.

h. False, $x^3 \cdot x^3 \neq 2x^6$. When x^3 and x^3 are multiplied, their coefficients are each 1. So the coefficient of their product is still 1, and we have $x^3 \cdot x^3 = x^6$.

i. False, $3^2 \cdot 2^3 \neq 6^5$. Note that neither the bases nor the exponents are the same. Following the order of operations, on the left $3^2 \cdot 2^3$ would simplify as $9 \cdot 8$, which is 72. However, on the right side, we have $6^5 = 7776$. Since $72 \neq 7776$, the equation is false.

j. False, $3^{-2} \neq -\frac{1}{9}$. The exponent of -2 on the number 3 does not result in a negative number. Instead, $3^{-2} = \frac{1}{3^2}$, which is $\frac{1}{9}$.

As we mentioned before, many situations we'll come across will require us to use more than one exponent rule. In these situations, we'll have to decide which rule to use first. There are often different, correct approaches we could take. But if we rely on order of operations, we will have a straightforward approach to simplify the expression correctly. To bring it all together, try these exercises.

Checkpoint 5.6.17

a. Simplify $\dfrac{6x^3}{2x^7}$ and write it without using negative exponents.

b. Simplify $4\left(\frac{1}{5}tv^{-4}\right)^2$ and write it without using negative exponents.

c. Simplify $\left(\dfrac{3^0y^4 \cdot y^5}{6y^2}\right)^3$ and write it without using negative exponents.

d. Simplify $\left(7^4x^{-6}t^2\right)^{-5}\left(7x^{-2}t^{-7}\right)^4$ and write it without using negative exponents. Leave larger numbers (such as 7^{10}) in exponent form.

Explanation.

a. In the expression $\frac{6x^3}{2x^7}$, the coefficients reduce using the properties of fractions. One way to simplify the variable powers is:

$$\begin{aligned}\frac{6x^3}{2x^7} &= \frac{6}{2} \cdot \frac{x^3}{x^7} \\ &= 3 \cdot x^{3-7} \\ &= 3 \cdot x^{-4} \\ &= 3 \cdot \frac{1}{x^4} \\ &= \frac{3}{x^4}\end{aligned}$$

b. In the expression $4\left(\frac{1}{5}tv^{-4}\right)^2$, the exponent 2 applies to each factor inside the parentheses.

$$\begin{aligned}4\left(\frac{1}{5}tv^{-4}\right)^2 &= 4\left(\frac{1}{5}\right)^2 (t)^2 \left(v^{-4}\right)^2 \\ &= 4\left(\frac{1}{25}\right)\left(t^2\right)\left(v^{-4\cdot 2}\right) \\ &= 4\left(\frac{1}{25}\right)\left(t^2\right)\left(v^{-8}\right) \\ &= 4\left(\frac{1}{25}\right)\left(t^2\right)\left(\frac{1}{v^8}\right) \\ &= \frac{4t^2}{25v^8}\end{aligned}$$

c. To follow the order of operations in the expression $\left(\frac{3^0y^4\cdot y^5}{6y^2}\right)^3$, the numerator inside the parentheses should be dealt with first. After that, we'll simplify the quotient inside the parentheses. As a final step, we'll apply the exponent to that simplified expression:

$$\begin{aligned}\left(\frac{3^0y^4\cdot y^5}{6y^2}\right)^3 &= \left(\frac{1\cdot y^{4+5}}{6y^2}\right)^3 \\ &= \left(\frac{y^9}{6y^2}\right)^3 \\ &= \left(\frac{y^{9-2}}{6}\right)^3 \\ &= \left(\frac{y^7}{6}\right)^3 \\ &= \frac{\left(y^7\right)^3}{6^3} \\ &= \frac{y^{7\cdot 3}}{216} \\ &= \frac{y^{21}}{216}\end{aligned}$$

d. We'll again rely on the order of operations, and look to simplify anything inside parentheses first and then apply exponents. In this example, we will begin by applying the product to a power rule, followed by the power to a power rule.

$$\begin{aligned}\left(7^4x^{-6}t^2\right)^{-5}\left(7x^{-2}t^{-7}\right)^4 &= \left(7^4\right)^{-5}\left(x^{-6}\right)^{-5}\left(t^2\right)^{-5}\cdot(7)^4\left(x^{-2}\right)^4\left(t^{-7}\right)^4\\ &= 7^{-20}x^{30}t^{-10}\cdot 7^4x^{-8}t^{-28}\\ &= 7^{-20+4}x^{30-8}t^{-10-28}\\ &= 7^{-16}x^{22}t^{-38}\\ &= \frac{x^{22}}{7^{16}t^{38}}\end{aligned}$$

5.6.6 Reading Questions

1. When you are considering using the exponent rule $a^m \cdot a^n = a^{m+n}$, are m and n allowed to be negative integers?

2. What are the differences between these three expressions?

$$x+0 \qquad 0x \qquad x^0$$

3. If you rearrange $\frac{xy^{-3}}{a^2b^8c}$ so that it is written without negative exponents, how many factors will you have "moved?"

5.6.7 Exercises

Review and Warmup

1. Evaluate the following.

a. 3^2

b. 2^3

c. $(-4)^2$

d. $(-2)^3$

2. Evaluate the following.

a. 3^2

b. 5^3

c. $(-4)^2$

d. $(-5)^3$

Use the properties of exponents to simplify the expression.

3. $6 \cdot 6^7$

4. $7 \cdot 7^4$

5. $7^{10} \cdot 7^8$

6. $8^7 \cdot 8^2$

Simplifying Products and Quotients Involving Exponents Use the properties of exponents to simplify the expression.

7. $r^{20} \cdot r^6$

8. $t^3 \cdot t^{18}$

9. $\left(y^4\right)^7$

10. $\left(t^5\right)^3$

11. $\left(2r^6\right)^4$

12. $\left(4y^7\right)^3$

13. $\left(-2y^{13}\right)\cdot\left(9y^4\right)$

14. $\left(6r^{15}\right)\cdot\left(-8r^{16}\right)$

15. $\left(-\frac{r^{18}}{8}\right)\cdot\left(\frac{r^{10}}{8}\right)$

16. $\left(-\frac{r^{20}}{4}\right)\cdot\left(-\frac{r^3}{7}\right)$

17. $-2\left(-4r^2\right)^3$

18. $-3\left(-10r^3\right)^2$

19. $(-36)^0$

20. $(-31)^0$

21. -27^0

22. -32^0

23. $37^0 + (-37)^0$

24. $43^0 + (-43)^0$

25. $48q^0$

26. $5B^0$

27. $(-649t)^0$

28. $(-428p)^0$

29. $\left(\frac{x^7}{5}\right)^3$

30. $\left(\frac{x^3}{6}\right)^2$

31. $\left(\frac{-7}{10x^6}\right)^2$

32. $\left(\frac{-7}{6x^{10}}\right)^3$

33. $\left(\frac{5x^9}{6}\right)^2$

34. $\left(\frac{9x^{10}}{2}\right)^3$

35. $\left(\frac{x^6}{2y^3z^{10}}\right)^2$

36. $\left(\frac{x^3}{2y^7z^6}\right)^2$

37. $\left(\frac{-5x^4}{8y^9}\right)^2$

38. $\left(\frac{-7x^5}{8y^2}\right)^2$

Rewrite the expression simplified and using only positive exponents.

39. $\left(\frac{1}{6}\right)^{-2}$

40. $\left(\frac{1}{7}\right)^{-3}$

41. $\frac{7^{-2}}{4^{-3}}$

42. $\frac{7^{-3}}{2^{-2}}$

43. $10^{-1} - 8^{-1}$

44. $2^{-1} - 5^{-1}$

45. $5x^{-4}$

46. $15x^{-5}$

47. $\frac{9}{x^{-6}}$

48. $\frac{20}{x^{-8}}$

49. $\frac{14x^{-9}}{x}$

50. $\frac{9x^{-10}}{x}$

51. $\frac{18x^{-17}}{x^{-35}}$

52. $\frac{9x^{-19}}{x^{-23}}$

53. $\frac{16x^{-4}}{17x^{-7}}$

54. $\frac{6x^{-7}}{7x^{-24}}$

55. $\frac{y^{-12}}{r^{-8}}$

56. $\frac{y^{-20}}{x^{-3}}$

57. $\frac{y^{-8}}{t^{17}}$

58. $\frac{r^{-16}}{y^{12}}$

59. $\frac{1}{8r^{-5}}$

60. $\frac{1}{40t^{-13}}$

61. $\frac{t^2}{t^{24}}$

62. $\frac{x^4}{x^7}$

63. $\frac{9x^{33}}{3x^{38}}$

64. $\frac{-18y^7}{9y^{22}}$

65. $\frac{-11y^4}{3y^5}$

66. $\frac{-7y^5}{3y^{36}}$

67. $\frac{r^4}{(r^6)^2}$

68. $\frac{r^2}{(r^3)^8}$

69. $\frac{t^{-4}}{(t^9)^6}$

70. $\frac{t^{-5}}{(t^6)^3}$

71. $x^{-20} \cdot x^5$

72. $x^{-14} \cdot x^9$

73. $(5y^{-7}) \cdot (7y^2)$

74. $(2y^{-19}) \cdot (2y^9)$

75. $\left(\frac{10}{9}\right)^{-2}$

76. $\left(\frac{3}{2}\right)^{-2}$

77. $(-8)^{-3}$

78. $(-9)^{-3}$

79. $\frac{1}{(-10)^{-2}}$

80. $\frac{1}{(-2)^{-3}}$

81. $\frac{-6}{(-2)^{-2}}$

82. $\frac{3}{(-2)^{-3}}$

83. 5^{-3}

84. 6^{-2}

85. $7^{-1} + 2^{-1}$

86. $8^{-1} + 6^{-1}$

87. $\frac{1}{9^{-2}}$

88. $\frac{1}{10^{-2}}$

89. -2^{-3}

90. -3^{-2}

91. $\frac{(5y^3)^3}{y^{22}}$

92. $\frac{(5y^9)^3}{y^{29}}$

93. $\frac{(5y^6)^2}{y^{-13}}$

94. $\frac{(5r^{12})^3}{r^{-9}}$

95. $\left(\frac{r^{14}}{r^4}\right)^{-3}$

96. $\left(\frac{t^7}{t^5}\right)^{-2}$

97. $\left(\frac{20t^{19}}{5t^2}\right)^{-4}$

98. $\left(\frac{10x^{13}}{5x^7}\right)^{-3}$

99. $\left(-5x^{-7}\right)^{-3}$

100. $\left(-2y^{-18}\right)^{-2}$

101. $\left(4y^{-12}\right)^{-3}$

102. $\left(3y^{-6}\right)^{-2}$

103. $\frac{7r^9 \cdot 6r^6}{5r^2}$

104. $\frac{5r^6 \cdot 5r^{10}}{7r^{13}}$

105. $\left(t^4\right)^3 \cdot t^{-7}$

106. $\left(t^{13}\right)^5 \cdot t^{-16}$

107. $\left(3x^8\right)^2 \cdot x^{-15}$

108. $\left(3x^4\right)^3 \cdot x^{-8}$

109. $\frac{\left(y^9\right)^2}{\left(y^8\right)^4}$

110. $\frac{\left(y^6\right)^3}{\left(y^{15}\right)^5}$

111. $\left(y^4\right)^{-5}$

112. $\left(r^{15}\right)^{-3}$

113. $\left(r^{12}y^6\right)^{-3}$

114. $\left(t^4x^3\right)^{-3}$

115. $\left(t^{-10}r^{12}\right)^{-3}$

116. $\left(x^{-15}y^8\right)^{-3}$

117. $\left(\frac{x^{15}}{2}\right)^{-3}$

118. $\left(\frac{y^{10}}{4}\right)^{-4}$

119. $\left(\frac{y^{11}}{x^7}\right)^{-3}$

120. $\left(\frac{y^{11}}{t^7}\right)^{-3}$

121. $\frac{\left(r^3x^{-5}\right)^{-3}}{\left(r^{-3}x^5\right)^{-2}}$

122. $\frac{\left(r^5x^{-7}\right)^{-3}}{\left(r^{-8}x^3\right)^{-4}}$

Rewrite the expression simplified and using only positive exponents.

123. $8x^{-5}y^3z^{-3}\left(3x^8\right)^{-3}$

124. $10x^{-4}y^4z^{-8}\left(3x^4\right)^{-4}$

125. $\left(\frac{x^4y^5z^4}{x^{-2}y^{-5}z^{-4}}\right)^{-3}$

126. $\left(\frac{x^4y^8z^5}{x^{-5}y^{-7}z^{-8}}\right)^{-2}$

Challenge

127. Consider the exponential expression $\frac{x^a \cdot x^b}{x^c}$ where $a > 0, b < 0$, and $c > 0$.

a. Are there values for a, b, and c so that the expression equals x^7? If so, fill in the blanks below with possible values for a, b, and c. If not, fill in the blanks below with the word none.

$a =$ ______, $b =$ ______, and $c =$ ______

b. Are there values for a, b, and c so that the exponential expression equals $\frac{1}{x^6}$? If so, fill in the blanks below with possible values for a, b, and c. If not, fill in the blanks below with the word none.

$a =$ ______, $b =$ ______, and $c =$ ______

128. Consider the exponential expression $\frac{x^a \cdot x^b}{x^c}$ where $a < 0, b < 0$, and $c > 0$.

a. Are there values for a, b, and c so that the expression equals x^6? If so, fill in the blanks below with possible values for a, b, and c. If not, fill in the blanks below with the word none.

$a =$ ______, $b =$ ______, and $c =$ ______

b. Are there values for a, b, and c so that the expression equals $\frac{1}{x^7}$? If so, fill in the blanks below with possible values for a, b, and c. If not, fill in the blanks below with the word none.

$a =$ ______, $b =$ ______, and $c =$ ______

129. Consider the exponential expression $\dfrac{x^a \cdot x^b}{x^c}$ where $a > 0$, $b > 0$, and $c < 0$.

a. Are there values for a, b, and c so that the expression equals x^7? If so, fill in the blanks below with possible values for a, b, and c. If not, fill in the blanks below with the word none.

$a =$ ______, $b =$ ______, and $c =$ ______

b. Are there values for a, b, and c so that the expression equals $\frac{1}{x^7}$? If so, fill in the blanks below with possible values for a, b, and c. If not, fill in the blanks below with the word none.

$a =$ ______, $b =$ ______, and $c =$ ______

5.7 Exponents and Polynomials Chapter Review

5.7.1 Adding and Subtracting Polynomials

In Section 5.1 we covered the definitions of a polynomial, a coefficient of a term, the degree of a term, the degree of a polynomial, theleading term of a polynomial, a constant term, monomials, binomials, and trinomials, and how to write a polynomial in standard form.

Example 5.7.1 Polynomial Vocabulary. Decide if the following statements are true or false.

a. The expression $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is a polynomial.

b. The expression $4x^6 - 3x^{-2} - x + 1$ is a polynomial.

c. The degree of the polynomial $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is 10.

d. The degree of the term $5x^2y^4$ is 6.

e. The leading coefficient of $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is $\frac{3}{5}$.

f. There are 4 terms in the polynomial $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$.

g. The polynomial $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is in standard form.

Explanation.

a. True. The expression $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is a polynomial.

b. False. The expression $4x^6 - 3x^{-2} - x + 1$ is *not* a polynomial. Variables are only allowed to have whole number exponents in polynomials and the second term has a -2 exponent.

c. False. The degree of the polynomial $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is *not* 10. It is 7, which is the highest power of any variable in the expression.

d. True. The degree of the term $5x^2y^4$ is 6.

e. False. The leading coefficient of $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is *not* $\frac{3}{5}$.The leading coefficient comes from the degree 7 term which is $-\frac{1}{5}$.

f. True. There are 4 terms in the polynomial $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$.

g. False. The polynomial $\frac{3}{5}x^2 - \frac{1}{5}x^7 + \frac{x}{2} - 4$ is *not* in standard form. The exponents have to be written from highest to lowest, i.e. $-\frac{1}{5}x^7 + \frac{3}{5}x^2 + \frac{x}{2} - 4$.

Example 5.7.2 Adding and Subtracting Polynomials. Simplify the expression $\left(\frac{2}{9}x - 4x^2 - 5\right) + \left(6x^2 - \frac{1}{6}x - 3\right)$.

Explanation. First identify like terms and group them either physically or mentally. Then we will look for common denominators for these like terms and combine appropriately.

$$\begin{aligned}
&\left(\frac{2}{9}x - 4x^2 - 5\right) + \left(6x^2 - \frac{1}{6}x - 3\right)\\
&= \frac{2}{9}x - 4x^2 - 5 + 6x^2 - \frac{1}{6}x - 3
\end{aligned}$$

$$\begin{aligned}
&= \left(-4x^2 + 6x^2\right) + \left(\frac{2}{9}x - \frac{1}{6}x\right) + (-3-5) \\
&= 2x^2 + \left(\frac{4}{18}x - \frac{3}{18}x\right) - 8 \\
&= 2x^2 + \frac{1}{18}x - 8
\end{aligned}$$

5.7.2 Introduction to Exponent Rules

In Section 5.2 we covered the rules of exponents for multiplication.

Rules of Exponents. Let x, and y represent real numbers, variables, or algebraic expressions, and let m and n represent positive integers . Then the following properties hold:

Product of Powers $x^m \cdot x^n = x^{m+n}$

Power to Power $(x^m)^n = x^{m \cdot n}$

Product to Power $(xy)^n = x^n \cdot y^n$

Example 5.7.3 Simplify the following expressions using the rules of exponents:
a. $-2t^3 \cdot 4t^5$ b. $5\left(v^4\right)^2$ c. $-(3u)^2$ d. $(-3z)^2$

Explanation.

a. $-2t^3 \cdot 4t^5 = -8t^8$ b. $5\left(v^4\right)^2 = 5v^8$ c. $-(3u)^2 = -9u^2$ d. $(-3z)^2 = 9z^2$

5.7.3 Dividing by a Monomial

In Section 5.3 we covered how you can split a fraction up into multiple terms if there is a sum or difference in the numerator. Mathematically, this happens using the rule $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$. This formula can be used for any number of terms in the numerator, and for both sums and differences.

Example 5.7.4 Simplify the expression $\frac{12x^5+2x^3-4x^2}{4x^2}$.
Explanation.

$$\begin{aligned}
\frac{12x^5 + 2x^3 - 4x^2}{4x^2} &= \frac{12x^5}{4x^2} + \frac{2x^3}{4x^2} - \frac{4x^2}{4x^2} \\
&= 3x^3 + \frac{x}{2} - 1
\end{aligned}$$

5.7.4 Multiplying Polynomials

In Section 5.4 we covered how to multiply two polynomials together using distribution, FOIL, and generic rectangles.

Example 5.7.5 Multiplying Binomials. Expand the expression $(5x-6)(3+2x)$ using the binomial multiplication method of your choice: distribution, FOIL, or generic rectangles.

Explanation. We will show work using the FOIL method.

$$\begin{aligned}(5x-6)(3-2x) &= (5x\cdot 3)+(5x\cdot(-2x))+(-6\cdot 3)+(-6\cdot(-2x))\\ &= 15x-10x^2-18+12x\\ &= -10x^2+27x-18\end{aligned}$$

Example 5.7.6 Multiplying Polynomials Larger than Binomials. Expand the expression $(3x-2)\left(4x^2-2x+5\right)$ by multiplying every term in the first factor with every term in the second factor.

Explanation. $(3x-2)\left(4x^2-2x+5\right)$

$$\begin{aligned}&= 3x\cdot 4x^2+3x\cdot(-2x)+3x\cdot 5+(-2)\cdot 4x^2+(-2)\cdot(-2x)+(-2)\cdot 5\\ &= 12x^3-6x^2+15x-8x^2+4x-10\\ &= 12x^3-14x^2+19x-10\end{aligned}$$

5.7.5 Special Cases of Multiplying Polynomials

In Section 5.5 we covered how to square a binomial and how to find the product of the sum or difference of two terms.

Example 5.7.7 Squaring a Binomial. Recall that Fact 5.5.3 gives formulas that help square a binomial.

Simplify the expression $(2x+3)^2$.

Explanation. Remember that you *can* use FOIL to do these problems, but in the interest of understanding concepts at a higher level for use in later chapters, we will use the relevant formula from Fact 5.5.3. In this case, since we have a sum of two terms being squared, we will use $(a+b)^2=a^2+2ab+b^2$.

First identify a and b. In this case, $a=2x$ and $b=3$. So, we have:

$$\begin{aligned}(a+b)^2 &= (a)^2+2(a)(b)+(b)^2\\ (2x+3)^2 &= (2x)^2+2(2x)(3)+(3)^2\\ &= 4x^2+12x+9\end{aligned}$$

Example 5.7.8 The Product of the Sum and Difference of Two Terms. Recall that Fact 5.5.13 gives a formula to help multiply things that look like $(a+b)(a-b)$.

Simplify the expression $(7x+4)(7x-4)$.

Explanation. Remember that you *can* use FOIL to do these problems, but in the interest of understanding concepts at a higher level for use in later chapters, we will use the formula from Fact 5.5.13. In this case, that means we will use $(a+b)(a-b)=a^2-b^2$.

First identify a and b. In this case, $a=7x$ and $b=4$. So, we have:

$$\begin{aligned}(a+b)(a-b) &= (a)^2-(b)^2\\ (7x+4)(7x-4) &= (7x)^2-(4)^2\\ &= 49x^2-16\end{aligned}$$

Example 5.7.9 Binomials Raised to Other Powers. To raise binomials to powers higher than 2, we start by expanding the expression and multiplying all factors together from left to right.

Expand the expression $(2x-5)^3$.

Explanation.

$$
\begin{aligned}
&(2x-5)^3\\
&= (2x-5)(2x-5)(2x-5)\\
&= \left[(2x)^2 - 2(2x)(5) + 5^2\right](2x-5)\\
&= \left[4x^2 - 20x + 25\right](2x-5)\\
&= \left[4x^2\right](2x) + \left[4x^2\right](-5) + [-20x](2x) + [-20x](-5) + [25](2x) + [25](-5)\\
&= 8x^3 - 20x^2 - 40x^2 + 100x + 50x - 125\\
&= 8x^3 - 60x^2 + 150x - 125
\end{aligned}
$$

5.7.6 More Exponent Rules

In Section 5.6 we covered the exponent rules and how to use them.

Example 5.7.10 Quotients and Exponents. Let t and q be real numbers, where $q \neq 0$ and $t \neq 0$. Find another way to write $\left(\frac{q^9}{t \cdot q^3}\right)^2$.

Explanation. We first use the Quotient Rule, then the Quotient to a Power Rule, then the Power to a Power Rule.

$$
\begin{aligned}
\left(\frac{q^9}{t \cdot q^3}\right)^2 &= \left(\frac{q^{9-3}}{t}\right)^2\\
&= \left(\frac{q^6}{t}\right)^2\\
&= \frac{q^{6\cdot 2}}{t^2}\\
&= \frac{q^{12}}{t^2}
\end{aligned}
$$

Example 5.7.11 The Zero Exponent. Recall that the Zero Exponent Rule says that any real number raised to the 0-power is 1. Using this, and the other exponent rules, find another way to write -9^0.

Explanation. Remember that in expressions like -9^0, the exponent only applies to what it is directly next to! In this case, the 0 only applies to the 9 and not the negative sign. So,

$$-9^0 = -1$$

Example 5.7.12 Negative Exponents. Write $5x^{-3}$ without any negative exponents.

Explanation. Recall that the Negative Exponent Rule says that a factor in the numerator with a negative exponent can be flipped into the denominator. So

$$5x^{-3} = \frac{5}{x^3}$$

Note that the 5 does not move to the denominator because the -3 exponent *only applies* to the x to which it is directly attached.

Example 5.7.13 Summary of Exponent Rules. Use the exponent rules in List 5.6.13 to write the expressions in a different way. Reduce and simplify when possible. Always find a way to write your final simplification without any negative exponents.

a. $\dfrac{24p^3}{20p^{12}}$ b. $\left(\dfrac{2v^5}{4g^{-2}}\right)^4$ c. $12n^7\left(m^0 \cdot n^2\right)^2$ d. $\dfrac{k^5}{k^{-4}}$

Explanation.

a.
$$\begin{aligned}\frac{24p^3}{20p^{12}} &= \frac{24}{20}\cdot\frac{p^3}{p^{12}}\\ &= \frac{6}{5}\cdot p^{3-12}\\ &= \frac{6}{5}\cdot p^{-9}\\ &= \frac{6}{5}\cdot\frac{1}{p^9}\\ &= \frac{6}{5p^9}\end{aligned}$$

b.
$$\begin{aligned}\left(\frac{2v^5}{4g^{-2}}\right)^4 &= \left(\frac{v^5}{2g^{-2}}\right)^4\\ &= \left(\frac{v^5g^2}{2}\right)^4\\ &= \frac{v^{5\cdot 4}g^{2\cdot 4}}{2^4}\\ &= \frac{v^{20}g^8}{16}\end{aligned}$$

c.
$$\begin{aligned}&12n^7\left(m^0\cdot n^2\right)^2\\ &= 12n^7\left(1\cdot n^2\right)^2\\ &= 12n^7\left(n^2\right)^2\\ &= 12n^7n^{2\cdot 2}\\ &= 12n^7n^4\\ &= 12n^{7+4}\\ &= 12n^{11}\end{aligned}$$

d.
$$\begin{aligned}\frac{k^5}{k^{-4}} &= k^5\cdot k^4\\ &= k^{5+4}\\ &= k^9\end{aligned}$$

5.7.7 Exercises

Adding and Subtracting Polynomials Is the following expression a monomial, binomial, or trinomial?

1. $-2r^{12} + 12r^9$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree ______.

2. $-16r^7 - 8r^4 - 12r^3$ is a (☐ monomial ☐ binomial ☐ trinomial) of degree ______

Find the degree of the following polynomial.

3. $12x^6y^9 + 11xy^4 + 5x^2 - 19$

4. $17x^6y^7 - 4xy^2 - 19x^2 + 10$

Add the polynomials.

5. $\left(-2x^2 - 6x - 7\right) + \left(-10x^2 - 8x - 4\right)$

6. $\left(3x^2 - 9x - 7\right) + \left(-5x^2 + 2x + 9\right)$

7. $\left(-5x^6 - 10x^4 + 9x^2\right) + \left(5x^6 - 9x^4 + x^2\right)$

8. $\left(2y^6 - 7y^4 - 2y^2\right) + \left(6y^6 + 3y^4 - 6y^2\right)$

Add the polynomials.

9. $\left(6x^3 - 3x^2 + 3x + \dfrac{5}{4}\right) + \left(-5x^3 + 9x^2 - 10x + \dfrac{1}{2}\right)$

10. $\left(-7x^3 - 6x^2 - 3x + \dfrac{7}{8}\right) + \left(8x^3 + 6x^2 - 9x + \dfrac{3}{2}\right)$

Subtract the polynomials.

11. $\left(4x^2+10x\right)-\left(10x^2+7x\right)$

12. $\left(6x^2+3x\right)-\left(-2x^2+x\right)$

13. $\left(-10x^2-9x+9\right)-\left(10x^2-7x-3\right)$

14. $\left(2x^2+3x-9\right)-\left(5x^2+7x-1\right)$

15. $\left(7x^6-8x^4+8x^2\right)-\left(2x^6-4x^4+2x^2\right)$

16. $\left(-4x^6-5x^4-2x^2\right)-\left(-3x^6-7x^4-6x^2\right)$

Add or subtract the given polynomials as indicated.

17. $\left(5x^3-6xy+9y^9\right)-\left(2x^3+4xy+2y^9\right)$

18. $\left(6x^9+9xy-3y^8\right)-\left(-2x^9+5xy-6y^8\right)$

19. A handyman is building two pig pens sharing the same side. Assume the length of the shared side is x meters. The cost of building one pen would be $26x^2+4x-49.5$ dollars, and the cost of building the other pen would be $37x^2-4x+9.5$ dollars. What's the total cost of building those two pens?
A polynomial representing the total cost of building those two pens is __________ dollars.

20. A handyman is building two pig pens sharing the same side. Assume the length of the shared side is x meters. The cost of building one pen would be $45.5x^2-4.5x+49.5$ dollars, and the cost of building the other pen would be $40.5x^2+4.5x-18.5$ dollars. What's the total cost of building those two pens?
A polynomial representing the total cost of building those two pens is __________ dollars.

Introduction to Exponent Rules Use the properties of exponents to simplify the expression.

21. $8\cdot 8^6$

22. $9\cdot 9^3$

23. $t^2\cdot t^{17}$

24. $y^4\cdot y^{10}$

25. $t^6\cdot t^4\cdot t^{17}$

26. $r^8\cdot r^{16}\cdot r^6$

27. $\left(10^5\right)^7$

28. $\left(12^2\right)^2$

29. $\left(y^9\right)^9$

30. $\left(t^{10}\right)^6$

31. $(2y)^4$

32. $(4r)^2$

33. $(4r^4)\cdot(9r^8)$

34. $(-6t^6)\cdot(-8t^{20})$

35. $\left(-2x^5\right)^3$

36. $\left(-7t^7\right)^2$

Use the properties of exponents to simplify the expression.

37. $\left(\frac{t^{12}}{9}\right)\cdot\left(\frac{t^{19}}{3}\right)$

38. $\left(-\frac{x^{14}}{3}\right)\cdot\left(-\frac{x^{13}}{8}\right)$

Dividing by a Monomial Simplify the following expression

39. $\frac{-63t^{14}-108t^{11}}{9}$

40. $\frac{55t^4+35t^3}{5}$

41. $\frac{3x^{21}-3x^{12}+18x^7}{3x^3}$

42. $\frac{64x^{19}-88x^{10}+64x^7}{-8x^3}$

43. $\frac{90x^{10}+108x^8}{9x}$

44. $\frac{42y^{16}+35y^7}{7y}$

Multiplying Polynomials Multiply the polynomials.

45. $-x\,(x-3)$

46. $x\,(x+9)$

47. $6r^2\left(9r^2+8r+6\right)$

48. $-3t^2\left(7t^2-4t-3\right)$ **49.** $(8t+9)(t+3)$ **50.** $(5x+3)(x+1)$

51. $(x+1)(x-4)$ **52.** $(x+8)(x-10)$ **53.** $(3y-6)(2y-5)$

54. $(2y-5)(4y-9)$ **55.** $3(x+2)(x+3)$ **56.** $-3(x+2)(x+3)$

57. $x(x-2)(x+2)$ **58.** $-x(x+2)(x+3)$

Multiply the polynomials.

59. $(a-2b)(a^2+10ab+6b^2)$ **60.** $(a+3b)(a^2-5ab-6b^2)$

61. A rectangle's length is 3 feet shorter than 4 times its width. If we use w to represent the rectangle's width, use a polynomial to represent the rectangle's area in expanded form.
area = ______
square feet

62. A rectangle's length is 4 feet shorter than twice its width. If we use w to represent the rectangle's width, use a polynomial to represent the rectangle's area in expanded form.
area = ______
square feet

Special Cases of Multiplying Polynomials Expand the square of a ***bi***nomial.

63. $(10y+7)^2$ **64.** $(6r+1)^2$ **65.** $(r-8)^2$

66. $(t-2)^2$ **67.** $(9a-6b)^2$ **68.** $(10a+3b)^2$

Multiply the polynomials.

69. $(x+9)(x-9)$ **70.** $(x-1)(x+1)$ **71.** $(2-10y)(2+10y)$

72. $(8+5y)(8-5y)$ **73.** $\left(4r^8+8\right)\left(4r^8-8\right)$ **74.** $\left(2r^5-7\right)\left(2r^5+7\right)$

Simplify the given expression into an expanded polynomial.

75. $(t+5)^3$ **76.** $(t+3)^3$

More Exponent Rules Use the properties of exponents to simplify the expression.

77. $\left(3r^{12}\right)^2$ **78.** $\left(5x^3\right)^4$ **79.** $(5t^6)\cdot(4t^8)$

80. $(8t^8)\cdot(-3t^{20})$ **81.** $\left(-\frac{t^{10}}{3}\right)\cdot\left(\frac{t^{14}}{4}\right)$ **82.** $\left(-\frac{x^{12}}{6}\right)\cdot\left(-\frac{x^7}{3}\right)$

83. $(-18)^0$ **84.** $(-13)^0$ **85.** -45^0

86. -50^0 **87.** $\left(\frac{-3}{8x^8}\right)^2$ **88.** $\left(\frac{-3}{4x^2}\right)^2$

89. $\frac{6x^{11}}{36x^2}$ **90.** $\frac{8x^{19}}{32x^{17}}$ **91.** $\left(\frac{x^3}{2y^4z^7}\right)^2$

92. $\left(\frac{x^9}{2y^8z^5}\right)^2$

Rewrite the expression simplified and using only positive exponents.

93. $\left(\frac{1}{8}\right)^{-2}$

94. $\left(\frac{1}{9}\right)^{-2}$

95. $17x^{-12}$

96. $11x^{-3}$

97. $\frac{6}{x^{-5}}$

98. $\frac{16}{x^{-6}}$

99. $\frac{15x^{-10}}{x^{-13}}$

100. $\frac{6x^{-12}}{x^{-3}}$

101. $\frac{r^{-5}}{(r^{9})^{9}}$

102. $\frac{t^{-5}}{(t^{6})^{7}}$

103. $t^{-20} \cdot t^{7}$

104. $t^{-14} \cdot t^{10}$

105. $(8x^{-8}) \cdot (-7x^{2})$

106. $(6x^{-20}) \cdot (-3x^{12})$

107. $\left(-5y^{-13}\right)^{-3}$

108. $\left(-2y^{-7}\right)^{-2}$

109. $\left(3r^{14}\right)^{4} \cdot r^{-37}$

110. $\left(4r^{10}\right)^{2} \cdot r^{-8}$

111. $\left(t^{11}x^{4}\right)^{-5}$

112. $\left(t^{13}x^{13}\right)^{-2}$

113. $\left(t^{-9}r^{10}\right)^{-5}$

114. $\left(x^{-6}t^{12}\right)^{-2}$

115. $\left(\frac{x^{13}}{4}\right)^{-2}$

116. $\left(\frac{y^{8}}{4}\right)^{-4}$

Chapter 6

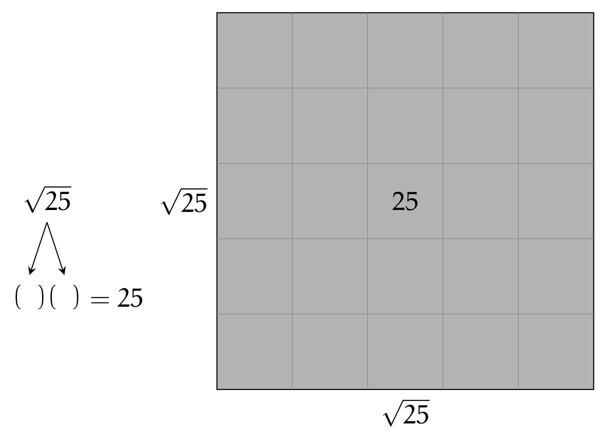
Radical Expressions and Equations

6.1 Square and nth Root Properties

In this section, we learn what expressions like $\sqrt{25}$ and $\sqrt[3]{27}$ mean, and some of the properties they have that allow for simplification.

6.1.1 Square Roots

Consider the non-negative number 25. You can ask the question "what number multiplies by itself to make 25?" We use the symbol $\sqrt{25}$ to represent the answer to this question, whether or not you know the "answer".
A geometric visualization of the same idea is to imagine a square with 25 units of area inside it. What would a side length have to be? We use $\sqrt{25}$ to represent that side length.



Definition 6.1.2 Square Root. Given a non-negative number x, if $r \cdot r = x$ for some positive number r, then r is called the **square root** of x, and we can write $\sqrt{x}$ instead of r. The $\sqrt{}$ symbol is called the **radical** or the **root**. We call expressions with the $\sqrt{}$ symbol **radical expressions**. The number inside the radical is called the **radicand**. ◊

For example, if you are confronted with the expression $\sqrt{16}$, you should think about the equation $r \cdot r = 16$ (or if you prefer, $r^2 = 16$) and ask yourself if you know a positive value for r that solves that equation. Of course, 4 is a non-negative solution. So we an say $\sqrt{16} = 4$.

To demonstrate more of the vocabulary, both $\sqrt{2}$ and $3\sqrt{2}$ are radical expressions. In both expressions, the number 2 is the radicand.

The word "radical" means something like "on the fringes" when used in politics, sports, and other places. It actually has that same meaning in math, when you consider a square with area A as in Figure 6.1.3.

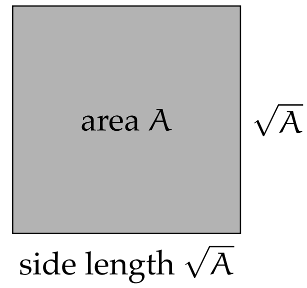


Figure 6.1.3: "Radical" means "off to the side."

The one-digit multiplication times table has special numbers along the diagonal. They are known as **perfect squares**. And for working with square roots, it will be helpful if you can memorize these first few perfect square numbers. For example, the times table tells us that $7 \cdot 7 = 49$. Just knowing that fact from memory lets us know that $\sqrt{49} = 7$. It's advisable to memorize the following:

$$\begin{array}{ccc} \sqrt{0}=0 & \sqrt{1}=1 & \sqrt{4}=2 \\ \sqrt{9}=3 & \sqrt{16}=4 & \sqrt{25}=5 \\ \sqrt{36}=6 & \sqrt{49}=7 & \sqrt{64}=8 \\ \sqrt{81}=9 & \sqrt{100}=10 & \sqrt{121}=11 \\ \sqrt{144}=12 & \sqrt{225}=15 & \sqrt{256}=16 \end{array}$$

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Figure 6.1.4: Multiplication table with squares

6.1.2 Square Root Decimal Values

Most square roots have decimal places that go on forever. Take $\sqrt{5}$ as an example. The number 5 is between two perfect squares, 4 and 9. Therefore, as demonstrated in Figure 6.1.5, $\sqrt{4} < \sqrt{5} < \sqrt{9}$. In other words,

$$2 < \sqrt{5} < 3$$

So $\sqrt{5}$ has a decimal value somewhere between 2 and 3.

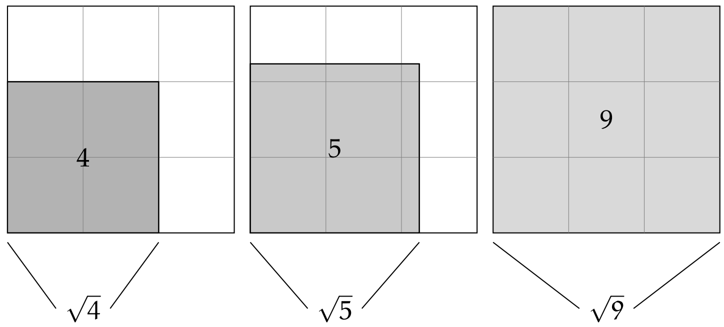


Figure 6.1.5: $2 < \sqrt{5} < 3$

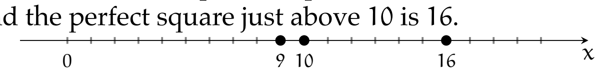
With a calculator, we can see:

$$\sqrt{5} \approx 2.236$$

Actually the decimal will not terminate, and that is why we used the $\approx$ symbol instead of an equals sign. To get 2.236 we rounded down slightly from the true value of $\sqrt{5}$. With a calculator, we can check that $2.236^2 = 4.999696$, a little shy of 5.

When the radicand is a perfect square, its square root is a rational number. If the radicand is not a perfect square, the square root is irrational. (It has a decimal that goes on forever without any pattern that is easy to see.) We want to be able to estimate square roots without using a calculator.

Example 6.1.6 To estimate $\sqrt{10}$ without a calculator, we can find the nearest perfect squares that are whole numbers on either side of 10. Recall that the perfect squares are $1, 4, 9, 16, 25, 36, 49, 64, \ldots$ The perfect square that is just below 10 is 9 and the perfect square just above 10 is 16.



This tells us that $\sqrt{10}$ is between $\sqrt{9}$ and $\sqrt{16}$, or between 3 and 4. We can also say that $\sqrt{10}$ is much closer to 3 than 4 because 10 is closer to 9, so we think 3.1 or 3.2 would be a good estimate.

To check our estimates (3.1 or 3.2) we can square them and see if the result is close to 10. We find $3.1^2 = 9.61$ and $3.2^2 = 10.24$, so our estimates are pretty good.

Checkpoint 6.1.7 Estimate $\sqrt{19}$ without a calculator.

Explanation. The radicand, 19, is between 16 and 25, so $\sqrt{19}$ is between $\sqrt{16}$ and $\sqrt{25}$, or between 4 and 5. We notice that 19 is in the middle between 16 and 25 but closer to 16. We estimate $\sqrt{19}$ to be about 4.4.

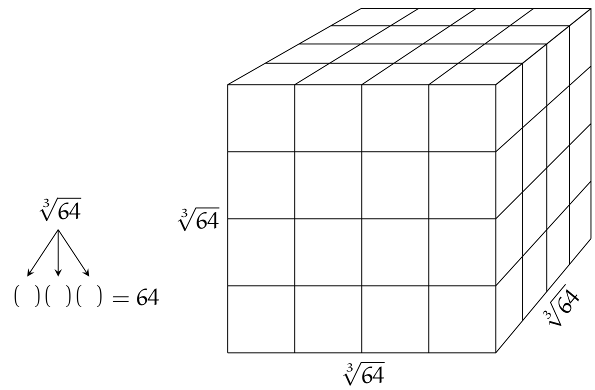
We can check our estimate by calculating:

$$4.4^2 = 19.36$$

So 4.4^2 is close to 19, and 4.4 is close to $\sqrt{19}$.

6.1.3 Cube Root and Higher Order Roots

The concept behind a square root easily extends to a **cube root**. What if we have a number in mind, like 64, and we would like to know what number can be multiplied with itself *three* times to make 64? A geometric visualization of the same idea is to imagine a cube with 64 units of area inside it. What would an edge length have to be? We use $\sqrt[3]{64}$ to represent that edge length.



Definition 6.1.8 nth Root. Given a number x, if $\overbrace{r \cdot r \cdot \cdots \cdot r}^{n \text{ times}} = x$ for some number r, then r is called an n**th root** of x. (Or if you prefer, when $r^n = x$.)

- When n is odd, there is always exactly one real number nth root for any x, and we can write $\sqrt[n]{x}$ to mean that one nth root.
- When n is even and x is positive, there are two real number nth roots, one of which is positive and the other of which is negative. We can write $\sqrt[n]{x}$ to mean the positive nth root.
- When n is even and x is negative, there aren't any real number nth roots, and we say that $\sqrt[n]{x}$ is "undefined" or "does not exist".

The $\sqrt[n]{}$ symbol is called the n**th radical** or the n**th root**. We call expressions with the $\sqrt[n]{}$ symbol **radical expressions**. The number inside the radical is called the **radicand**. The **index** of a radical is the number n in $\sqrt[n]{}$. ◊

As noted earlier, when we have $\sqrt[3]{}$, we can say "cube root" instead of "3rd root". Also, when we have $\sqrt[2]{}$, we can say "square root" instead of "2nd root" and we can simply write $\sqrt{x}$ instead.

For some examples of nth roots:

- $\sqrt[3]{8} = 2$, because $\overbrace{2 \cdot 2 \cdot 2}^{3 \text{ instances}} = 8$.
- $\sqrt[4]{81} = 3$, because $\overbrace{3 \cdot 3 \cdot 3 \cdot 3}^{4 \text{ instances}} = 81$.
- $\sqrt[5]{-32} = -2$, because $\overbrace{(-2) \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2)}^{5 \text{ instances}} = -32$.

As with square roots, in general an nth root's decimal value is a decimal that goes on forever. For example, $\sqrt[3]{20} \approx 2.714\ldots$. For practical applications, we may want to use a calculator to find a decimal approximation to an nth root. Some calculators will do this for you directly, and some will not.

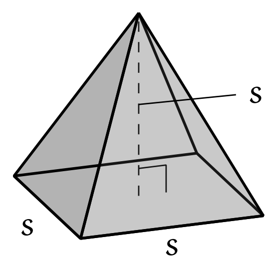
- Maybe your calculator has a button that looks like $\sqrt[x]{y}$. Then you should be able to type something like `3` $\sqrt[x]{y}$ `20` to get $\sqrt[3]{20} \approx 2.714\ldots$.
- Maybe your calculator has a button that looks like $\sqrt[n]{}$. Then you should be able to type something like `3` $\sqrt[n]{}$ `20` to get $\sqrt[3]{20} \approx 2.714\ldots$.

- Maybe your calculator allows you to type letters, parentheses, and commas, and you can type `root(3,20)` to get $\sqrt[3]{20} \approx 2.714\ldots$.
- Maybe your calculator allows you to type letters, parentheses, and commas, but the syntax for an nth root is reversed from the last example, and you can type `root(20,3)` to get $\sqrt[3]{20} \approx 2.714\ldots$.
- If your calculator has none of the above options, then you should be able to type `20^(1/3)` as a way to get $\sqrt[3]{20} \approx 2.714\ldots$. This is technically using mathematics we will learn in Section 6.3.

Try using your own calculator to calculate $\sqrt[3]{20}$ so that you can become familiar with whatever method it uses.

Example 6.1.9

A pyramid has a square base, and its height is equal to one side length of the square at its base. In this situation, the volume V of the pyramid, in in^3, is given by $V = \frac{1}{3}s^3$, where s is the pyramid's base side length in in. Archimedes droppped the pyramid in a bathtub, and judging by how high the water level rose, the volume of the pyramid is $243\,\text{in}^3$ (a little more than 1 gal). How tall is the pyramid?



The equation tells us that:

$$243 = \frac{1}{3}s^3$$

We can multiply on both sides by 3 and:

$$729 = s^3$$

This means that s is $\sqrt[3]{729}$. A calculator tells us that this is 9. So the pyramid's height is 9 inches.

6.1.4 Roots of Negative Numbers

Can we find the square root of a negative number, such as $\sqrt{-64}$? How about its cube root, $\sqrt[3]{-64}$?

As noted in Defintion 6.1.8, when the index of an nth root is odd, there will always be a real number nth root even when the radicand is negative. For example, $\sqrt[3]{-64}$ is -4, because $(-4)(-4)(-4) = -64$.

When the index of an nth root is even, it is a problem to have a negative radicand. For example, to find $\sqrt{-64}$, you would need to find a value r so that $r \cdot r = -64$. But whether r is positive or negative, multiplying it by itself will give a positive result. It could never be -64. So there is no way to have a real number square root of a negative number. And the same thing is true for *any* even index nth root with a negative radicand, such as $\sqrt[4]{-64}$. An even-indexed root of a negative number is not a real number.

If you are confronted with an expression like $\sqrt{-25}$ or $\sqrt[4]{-16}$ (any square root or even-indexed root of a negative number), you can state that the expression "is not real" or that it is "not defined" (as a real number). Don't get carried away though. Expressions like $\sqrt[3]{-27}$ and $\sqrt[5]{-1}$ *are* defined, because the index is odd.

Imaginary Numbers. Mathematicians imagined a new type of number, neither positive nor negative, that would multiply by itself to make a negative result. But that is beyond the scope of this section.

6.1.5 Radical Rules and Exponent Rules

In an earlier chapter, we learned some algebra rules for exponents. A summary of these rules is in List 5.6.13. There are a couple of very similar rules for radicals, presented here without motivation. (These rules are easier to explain once we study Section 6.3.)

List 6.1.10: Rules of Radicals for Multiplication and Division

If a and b are positive real numbers, and m is a positive integer , then we have the following rules:

Root of a Product Rule $\sqrt[m]{a \cdot b} = \sqrt[m]{a} \cdot \sqrt[m]{b}$

Root of a Quotient Rule $\sqrt[m]{\frac{a}{b}} = \frac{\sqrt[m]{a}}{\sqrt[m]{b}}$ as long as $b \neq 0$

Knowing these algebra rules helps to make complicated radical expressions look simpler.

Example 6.1.11 Simplify $\sqrt{18}$. Anything we can do to make the radicand a smaller simpler number is helpful. Note that $18 = 9 \cdot 2$, so we can write

$$\begin{aligned} \sqrt{18} &= \sqrt{9 \cdot 2} \\ &= \sqrt{9} \cdot \sqrt{2} && \text{according to the Root of a Product Rule} \\ &= 3\sqrt{2} \end{aligned}$$

This expression $3\sqrt{2}$ is considered "simpler" than $\sqrt{18}$ because the radicand is so much smaller.

Checkpoint 6.1.12 Simplify $\sqrt{72}$.

Explanation. As with the previous example, it will help if 72 can be written as a product of a perfect square. In this case, 4 divides 72, and $72 = 4 \cdot 18$. So

$$\begin{aligned} \sqrt{72} &= \sqrt{4 \cdot 18} \\ &= \sqrt{4} \cdot \sqrt{18} \\ &= 2\sqrt{18} \end{aligned}$$

But we aren't done. Can 18 can be written as a product of a perfect square? Yes, because $18 = 9 \cdot 2$. So

$$\begin{aligned} \sqrt{72} &= 2\sqrt{18} \\ &= 2\sqrt{9 \cdot 2} \\ &= 2\sqrt{9} \cdot \sqrt{2} \\ &= 2 \cdot 3\sqrt{2} \\ &= 6\sqrt{2} \end{aligned}$$

This is as simple as we can make this expression.

Example 6.1.13 Simplify $\sqrt[3]{80}$. Anything we can do to make the radicand a smaller simpler number is helpful. With lessons learned from the previous examples, maybe there is a way to rewrite 80 as the product of two numbers in a helpful way. Since we have a *cube* root, writing 80 as a product of a perfect *cube* would be helpful. We can write $80 = 8 \cdot 10$, where 8 is a perfect cube.

$$\sqrt[3]{80} = \sqrt[3]{8 \cdot 10}$$

$$\begin{aligned} &= \sqrt[3]{8} \cdot \sqrt[3]{10} && \text{according to the Root of a Product Rule} \\ &= 2\sqrt[3]{10} \end{aligned}$$

This expression $2\sqrt[3]{10}$ is considered "simpler" than $\sqrt[3]{80}$ because the radicand is so much smaller.

Checkpoint 6.1.14 Simplify $\sqrt[4]{48}$.

Explanation. As with the previous example, it will help if 48 can be written as a product of a 4th power. In this case, 16 divides 48, and $48 = 16 \cdot 3$. So

$$\begin{aligned} \sqrt[4]{48} &= \sqrt[4]{16 \cdot 3} \\ &= \sqrt[4]{16}\sqrt[4]{3} \\ &= 2\sqrt[4]{3} \end{aligned}$$

This is as simple as we can make this expression.

When a radical is applied to a fraction, the Root of a Quotient Rule is useful.

Example 6.1.15 Simplify $\sqrt{\frac{8}{25}}$. According to the Root of a Quotient Rule,

$$\begin{aligned} \sqrt{\frac{8}{25}} &= \frac{\sqrt{8}}{\sqrt{25}} \\ &= \frac{\sqrt{8}}{5} \\ &= \frac{\sqrt{4 \cdot 2}}{5} \\ &= \frac{\sqrt{4} \cdot \sqrt{2}}{5} \\ &= \frac{2\sqrt{2}}{5} \end{aligned}$$

This is as simple as we can make this expression, unless you prefer to write it as $\frac{2}{5}\sqrt{2}$.

Checkpoint 6.1.16

a. $\sqrt{\frac{1}{36}}$ b. $\sqrt[3]{\frac{8}{27}}$ c. $\sqrt[3]{\frac{81}{125}}$

Explanation.

a.
$$\begin{aligned}\sqrt{\frac{1}{25}} &= \frac{\sqrt{1}}{\sqrt{25}}\\ &= \frac{1}{5}\end{aligned}$$

b.
$$\begin{aligned}\sqrt[3]{\frac{8}{27}} &= \frac{\sqrt[3]{8}}{\sqrt[3]{27}}\\ &= \frac{2}{3}\end{aligned}$$

c.
$$\begin{aligned}\sqrt[3]{\frac{81}{125}} &= \frac{\sqrt[3]{81}}{\sqrt[3]{125}}\\ &= \frac{\sqrt[3]{81}}{5}\\ &= \frac{\sqrt[3]{27\cdot 3}}{5}\\ &= \frac{\sqrt[3]{27}\,\sqrt[3]{3}}{5}\\ &= \frac{3\sqrt[3]{3}}{5}\end{aligned}$$

6.1.6 Multiplying Square Root Expressions

We can use the Root of a Product Rule and the Root of a Quotient Rule to multiply and divide square root expressions. We want to simplify each radical first to keep the radicands as small as possible.

Example 6.1.17 Multiply $\sqrt{8}\cdot\sqrt{54}$.

Explanation. We will simplify each radical first, and then multiply them together. We do not want to multiply $8\cdot 54$ because we will end up with a larger number that is harder to factor.

$$\begin{aligned}\sqrt{8}\cdot\sqrt{54} &= \sqrt{4\cdot 2}\cdot\sqrt{9\cdot 6}\\ &= \sqrt{4}\cdot\sqrt{2}\cdot\sqrt{9}\cdot\sqrt{6}\\ &= 2\sqrt{2}\cdot 3\sqrt{6}\\ &= 6\sqrt{2}\cdot\sqrt{6}\\ &= 6\sqrt{12}\\ &= 6\sqrt{4\cdot 3}\\ &= 6\sqrt{4}\cdot\sqrt{3}\\ &= 6\cdot 2\sqrt{3}\\ &= 12\sqrt{3}\end{aligned}$$

It is worth noting that this is considered as simple as we canmake it, because the radicand of 3 is so small.

Checkpoint 6.1.18 Multiply $2\sqrt{7}\cdot 3\sqrt{21}$.

Explanation. First multiply the non-radical factors together and the radical factors together. Then look for further simplifications.

$$\begin{aligned}2\sqrt{7} \cdot 3\sqrt{21} &= 2 \cdot 3 \cdot \sqrt{7} \cdot \sqrt{21} \\ &= 6 \cdot \sqrt{7} \cdot \sqrt{7 \cdot 3} \\ &= 6\sqrt{49 \cdot 3} \\ &= 6 \cdot 7 \cdot \sqrt{3} \\ &= 42\sqrt{3}\end{aligned}$$

Example 6.1.19 Multiply $\sqrt{\frac{6}{5}} \cdot \sqrt{\frac{3}{5}}$.

Explanation. First multiply the fractions together under the radical. Then look for further simplifications.

$$\begin{aligned}\sqrt{\frac{6}{5}} \cdot \sqrt{\frac{3}{5}} &= \sqrt{\frac{6}{5} \cdot \frac{3}{5}} \\ &= \sqrt{\frac{18}{25}} \\ &= \frac{\sqrt{18}}{\sqrt{25}} \\ &= \frac{\sqrt{9 \cdot 2}}{5} \\ &= \frac{3\sqrt{2}}{5}\end{aligned}$$

6.1.7 Adding and Subtracting Square Root Expressions

We learned the Root of a Product Rule previously and applied this to multiplication of square roots, but we cannot apply this property to the operations of addition or subtraction. Here are two examples to demonstrate why not.

$$\begin{aligned}\sqrt{9} + \sqrt{16} &\stackrel{?}{=} \sqrt{9+16} \\ 3 + 4 &\stackrel{?}{=} \sqrt{25} \\ 7 &\stackrel{\text{no}}{=} 5\end{aligned} \qquad \begin{aligned}\sqrt{169} - \sqrt{25} &\stackrel{?}{=} \sqrt{169-25} \\ 13 - 5 &\stackrel{?}{=} \sqrt{144} \\ 8 &\stackrel{\text{no}}{=} 12\end{aligned}$$

We do not get the same result if we combine radical sums and differences in the same way we can combine radical products and quotiens.

To add and subtract radical expressions, we need to recognize that we can only add and subtract like terms. In this case, we will call them **like radicals**. Adding like radicals will work just like adding like terms. In the same way that $x + 3x = 4x$ combines two like terms, $\sqrt{5} + 3\sqrt{5} = 4\sqrt{5}$ combines two like radicals.

Example 6.1.20 Simplify $\sqrt{2} + \sqrt{8}$.

Explanation. First, simplify each radical. Simplifying is the best way to understand whether or not we even have two like radicals that could be combined.

$$\begin{aligned}\sqrt{2} + \sqrt{8} &= \sqrt{2} + \sqrt{4 \cdot 2} \\ &= \sqrt{2} + 2\sqrt{2}\end{aligned}$$

$$= 3\sqrt{2}$$

Checkpoint 6.1.21 Simplify $2\sqrt{3} - 3\sqrt{48}$.

Explanation. First we will simplify the radical term where 48 is the radicand, and we may see that we then have like radicals.

$$\begin{aligned}
2\sqrt{3} - 3\sqrt{48} &= 2\sqrt{3} - 3\sqrt{16 \cdot 3} \\
&= 2\sqrt{3} - 3 \cdot 4\sqrt{3} \\
&= 2\sqrt{3} - 12\sqrt{3} \\
&= -10\sqrt{3}
\end{aligned}$$

Example 6.1.22 Simplify $\sqrt{2} + \sqrt{27}$.

Explanation.

$$\begin{aligned}
\sqrt{2} + \sqrt{27} &= \sqrt{2} + \sqrt{9 \cdot 3} \\
&= \sqrt{2} + 3\sqrt{3}
\end{aligned}$$

We cannot simplify the expression further because $\sqrt{2}$ and $\sqrt{3}$ are not like radicals.

Example 6.1.23 Simplify $\sqrt{6} - \sqrt{18} \cdot \sqrt{12}$.

Explanation. In this example, we should multiply the latter two square roots first (after simplifying them) and then see if we have like radicals.

$$\begin{aligned}
\sqrt{6} - \sqrt{18} \cdot \sqrt{12} &= \sqrt{6} - \sqrt{9 \cdot 2} \cdot \sqrt{4 \cdot 3} \\
&= \sqrt{6} - 3\sqrt{2} \cdot 2\sqrt{3} \\
&= \sqrt{6} - 6\sqrt{2} \cdot \sqrt{3} \\
&= \sqrt{6} - 6\sqrt{6} \\
&= -5\sqrt{6}
\end{aligned}$$

6.1.8 Distributing with Square Roots

In Section 5.4, we learned how to multiply polynomials like $2(x+3)$ and $(x+2)(x+3)$. All the methods we learned there apply when we multiply square root expressions. We will look at a few examples done with different methods.

Example 6.1.24 Multiply $\sqrt{5}\left(\sqrt{3} - \sqrt{2}\right)$.

Explanation. We will use the distributive property to do this problem:

$$\begin{aligned}
\sqrt{5}\left(\sqrt{3} - \sqrt{2}\right) &= \sqrt{5} \cdot \sqrt{3} - \sqrt{5} \cdot \sqrt{2} \\
&= \sqrt{15} - \sqrt{10}
\end{aligned}$$

Example 6.1.25 Multiply $\left(\sqrt{6} + \sqrt{12}\right)\left(\sqrt{3} - \sqrt{2}\right)$.

Explanation. We will use the FOIL Method to expand the product. This time, there is an opportunity to simplify some of the radicals after multiplying.

$$\begin{aligned}\left(\sqrt{6}+\sqrt{12}\right)\left(\sqrt{3}-\sqrt{2}\right) &= \sqrt{6}\cdot\sqrt{3}-\sqrt{6}\cdot\sqrt{2}+\sqrt{12}\cdot\sqrt{3}-\sqrt{12}\cdot\sqrt{2}\\ &= \sqrt{18}-\sqrt{12}+\sqrt{36}-\sqrt{24}\\ &= 3\sqrt{2}-2\sqrt{3}+6-2\sqrt{6}\end{aligned}$$

Example 6.1.26 Expand $\left(\sqrt{3}-\sqrt{2}\right)^2$.

Explanation. We will use the FOIL method to expand this expression:

$$\begin{aligned}\left(\sqrt{3}-\sqrt{2}\right)^2 &= \left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)\\ &= \left(\sqrt{3}\right)^2-\sqrt{3}\cdot\sqrt{2}-\sqrt{2}\cdot\sqrt{3}+\left(\sqrt{2}\right)^2\\ &= 3-\sqrt{6}-\sqrt{6}+2\\ &= 5-2\sqrt{6}\end{aligned}$$

Example 6.1.27 Multiply $\left(\sqrt{5}-\sqrt{7}\right)\left(\sqrt{5}+\sqrt{7}\right)$.

Explanation. We can once again use the FOIL method to expand this expression. (But it is worth noting that this expression is in the special form $(a-b)(a+b)$ and will simplify to a^2-b^2.)

$$\begin{aligned}\left(\sqrt{5}-\sqrt{7}\right)\left(\sqrt{5}+\sqrt{7}\right) &= \left(\sqrt{5}\right)^2+\sqrt{5}\cdot\sqrt{7}-\sqrt{7}\cdot\sqrt{5}-\left(\sqrt{7}\right)^2\\ &= 5+\sqrt{35}-\sqrt{35}-7\\ &= -2\end{aligned}$$

6.1.9 Reading Questions

1. Is there a difference between $\sqrt[3]{2}$ and $3\sqrt{2}$? Explain.

2. Choose one of the radical rules from List 6.1.10. Then find its counterpart in the exponent rules from List 5.6.13.

3. Describe a way you can visualize $\sqrt{81}$ in a geometric shape. Describe a way you can visualize $\sqrt[3]{27}$ in a geometric shape.

6.1.10 Exercises

Review and Warmup Which of the following are square numbers? There may be more than one correct answer.

1.

☐ 117 ☐ 54 ☐ 100 ☐ 64 ☐ 49 ☐ 3

2.

☐ 1 ☐ 125 ☐ 16 ☐ 115 ☐ 121 ☐ 138

Evaluate the following.

3. a. $\sqrt{144}$ b. $\sqrt{121}$ c. $\sqrt{36}$

4. a. $\sqrt{4}$ b. $\sqrt{64}$ c. $\sqrt{144}$

5. a. $\sqrt{\dfrac{9}{25}}$ b. $\sqrt{-\dfrac{36}{49}}$

6. a. $\sqrt{\dfrac{16}{121}}$ b. $\sqrt{-\dfrac{81}{100}}$

7. Do not use a calculator.
a. $\sqrt{36}$ b. $\sqrt{0.36}$ c. $\sqrt{3600}$

8. Do not use a calculator.
a. $\sqrt{64}$ b. $\sqrt{0.64}$ c. $\sqrt{6400}$

9. Do not use a calculator.
a. $\sqrt{64}$ b. $\sqrt{6400}$ c. $\sqrt{640000}$

10. Do not use a calculator.
a. $\sqrt{100}$ b. $\sqrt{10000}$ c. $\sqrt{1000000}$

11. Do not use a calculator.
a. $\sqrt{121}$ b. $\sqrt{1.21}$ c. $\sqrt{0.0121}$

12. Do not use a calculator.
a. $\sqrt{144}$ b. $\sqrt{1.44}$ c. $\sqrt{0.0144}$

13. Without using a calculator, estimate the value of $\sqrt{18}$:
(☐ 3.76 ☐ 3.24 ☐ 4.76 ☐ 4.24)

14. Without using a calculator, estimate the value of $\sqrt{24}$:
(☐ 4.10 ☐ 4.90 ☐ 5.10 ☐ 5.90)

Simplify Radical Expressions Evaluate the following.

15. $\sqrt{\dfrac{16}{49}}$

16. $\sqrt{\dfrac{25}{36}}$

17. $-\sqrt{64}$

18. $-\sqrt{81}$

19. $\sqrt{-100}$

20. $\sqrt{-121}$

21. $\sqrt{-\dfrac{121}{144}}$

22. $\sqrt{-\dfrac{4}{25}}$

23. $-\sqrt{\dfrac{9}{121}}$

24. $-\sqrt{\dfrac{16}{81}}$

25. a. $\sqrt{100}-\sqrt{36}$ b. $\sqrt{100-36}$

26. a. $\sqrt{100}-\sqrt{64}$ b. $\sqrt{100-64}$

Simplify the radical expression or state that it is `not a real number`.

27. $\dfrac{\sqrt{125}}{\sqrt{5}}$

28. $\dfrac{\sqrt{75}}{\sqrt{3}}$

29. $\dfrac{\sqrt{6}}{\sqrt{216}}$

30. $\dfrac{\sqrt{4}}{\sqrt{144}}$

31. $\sqrt{8}$

32. $\sqrt{147}$

33. $\sqrt{980}$

34. $\sqrt{216}$

35. $\sqrt{231}$ **36.** $\sqrt{70}$

Multiplying Square Root Expressions Simplify the expression.

37. $8\sqrt{3}\cdot 3\sqrt{11}$
38. $8\sqrt{7}\cdot 8\sqrt{2}$
39. $9\sqrt{7}\cdot 5\sqrt{25}$
40. $2\sqrt{13}\cdot 2\sqrt{121}$
41. $2\sqrt{5}\cdot 5\sqrt{40}$
42. $3\sqrt{15}\cdot 3\sqrt{30}$
43. $\sqrt{2}\cdot 3\sqrt{32}$
44. $\sqrt{4}\cdot 4\sqrt{16}$
45. $\sqrt{\frac{2}{7}}\cdot\sqrt{\frac{1}{7}}$
46. $\sqrt{\frac{1}{8}}\cdot\sqrt{\frac{3}{8}}$
47. $\sqrt{\frac{30}{19}}\cdot\sqrt{\frac{6}{19}}$
48. $\sqrt{\frac{18}{13}}\cdot\sqrt{\frac{6}{13}}$

Adding and Subtracting Square Root Expressions Simplify the expression.

49. $10\sqrt{15}-11\sqrt{15}$
50. $12\sqrt{11}-13\sqrt{11}$
51. $13\sqrt{11}-13\sqrt{11}+15\sqrt{11}$
52. $14\sqrt{5}-20\sqrt{5}+11\sqrt{5}$
53. $\sqrt{80}+\sqrt{45}$
54. $\sqrt{45}+\sqrt{125}$
55. $\sqrt{343}-\sqrt{63}$
56. $\sqrt{275}-\sqrt{539}$
57. $\sqrt{28}+\sqrt{175}+\sqrt{8}+\sqrt{18}$
58. $\sqrt{32}+\sqrt{8}+\sqrt{125}+\sqrt{20}$
59. $\sqrt{98}-\sqrt{8}-\sqrt{12}-\sqrt{27}$
60. $\sqrt{75}-\sqrt{27}-\sqrt{8}-\sqrt{50}$

Distributing with Square Roots Expand and simplify the expression.

61. $\sqrt{2}\left(\sqrt{19}+\sqrt{17}\right)$
62. $\sqrt{7}\left(\sqrt{11}+\sqrt{5}\right)$
63. $\left(3+\sqrt{11}\right)\left(7+\sqrt{11}\right)$
64. $\left(9+\sqrt{11}\right)\left(10+\sqrt{11}\right)$
65. $\left(6-\sqrt{7}\right)\left(7-3\sqrt{7}\right)$
66. $\left(3-\sqrt{7}\right)\left(4-5\sqrt{7}\right)$
67. $\left(1+\sqrt{6}\right)^2$
68. $\left(2+\sqrt{3}\right)^2$
69. $\left(\sqrt{2}-3\right)^2$
70. $\left(\sqrt{6}-4\right)^2$
71. $\left(\sqrt{15}-\sqrt{5}\right)^2$
72. $\left(\sqrt{35}+\sqrt{5}\right)^2$
73. $\left(8-5\sqrt{7}\right)^2$
74. $\left(5-3\sqrt{7}\right)^2$
75. $\left(10-\sqrt{13}\right)\left(10+\sqrt{13}\right)$
76. $\left(7-\sqrt{5}\right)\left(7+\sqrt{5}\right)$
77. $\left(\sqrt{5}+\sqrt{6}\right)\left(\sqrt{5}-\sqrt{6}\right)$
78. $\left(\sqrt{6}+\sqrt{13}\right)\left(\sqrt{6}-\sqrt{13}\right)$
79. $\left(4\sqrt{5}+5\sqrt{7}\right)\left(4\sqrt{5}-5\sqrt{7}\right)$
80. $\left(5\sqrt{6}+3\sqrt{11}\right)\left(5\sqrt{6}-3\sqrt{11}\right)$

Higher Index Roots

81. Simplify $\sqrt[3]{125}$.
82. Simplify $\sqrt[3]{27}$.
83. Simplify $\sqrt[4]{16}$.
84. Simplify $\sqrt[4]{81}$.
85. Simplify $\sqrt[5]{32}$.
86. Simplify $\sqrt[3]{8}$.

87. Simplify $\sqrt[5]{-32}$.

88. Simplify $\sqrt[4]{-81}$.

89. Simplify $\sqrt[4]{-16}$.

90. Simplify $\sqrt[4]{-81}$.

91. Simplify $\sqrt[3]{-27}$.

92. Simplify $\sqrt[6]{-64}$.

93. Simplify $\sqrt[3]{16}$.

94. Simplify $\sqrt[3]{162}$.

95. Simplify $\sqrt[3]{192}$.

96. Simplify $\sqrt[3]{54}$.

97. Simplify $\sqrt[3]{80}$.

98. Simplify $\sqrt[4]{192}$.

99. Simplify $\sqrt[3]{\frac{3}{64}}$.

100. Simplify $\sqrt[5]{\frac{11}{32}}$.

101. Simplify $\sqrt[3]{\frac{189}{125}}$.

102. Simplify $\sqrt[3]{\frac{88}{27}}$.

103. Simplify $\sqrt[3]{\frac{54}{125}}$.

104. Simplify $\sqrt[3]{\frac{56}{27}}$.

6.2 Rationalizing the Denominator

A radical expression typically has several equivalent forms. For example, $\frac{\sqrt{2}}{3}$ and $\frac{2}{\sqrt{6}}$ are the same number. Mathematics has a preference for one of these forms over the other, and this section is about how to convert a given radical expression to that form.

6.2.1 Rationalizing the Denominator

To simplify radical expressions, we have seen that it helps to make the radicand as small as possible. Another helpful principle is to not leave any irrational numbers, such as $\sqrt{3}$ or $2\sqrt{5}$, in the denominator of a fraction. In other words, we want the denominator to be rational. The process of dealing with such numbers in the denominator is called **rationalizing the denominator**.

Let's see how we can replace $\frac{1}{\sqrt{5}}$ with an equivalent expression that has no radical expressions in its denominator. If we multiply a radical by itself, the result is the radicand, by Definition 6.1.2. As an example:

$$\sqrt{5} \cdot \sqrt{5} = 5$$

With $\frac{1}{\sqrt{5}}$, we may multiply both the numerator and denominator by the same non-zero number and have an equivalent expression. If we multiply the numerator and denominator by $\sqrt{5}$, we have:

$$\begin{aligned}\frac{1}{\sqrt{5}} &= \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} \\ &= \frac{1 \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} \\ &= \frac{\sqrt{5}}{5}\end{aligned}$$

And voilà, we have an expression with no radical in its denominator. We can use a calculator to verify that $\frac{1}{\sqrt{5}} \approx 0.4472$, and also $\frac{\sqrt{5}}{5} \approx 0.4472$. They are equal.

Example 6.2.2 Rationalize the denominator of the expressions.

a. $\frac{3}{\sqrt{6}}$

b. $\frac{\sqrt{5}}{\sqrt{72}}$

Explanation.

a. To rationalize the denominator of $\frac{3}{\sqrt{6}}$, we multiply both the numerator and denominator by $\frac{\sqrt{6}}{\sqrt{6}}$.

$$\begin{aligned}\frac{3}{\sqrt{6}} &= \frac{3}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} \\ &= \frac{3\sqrt{6}}{6} \\ &= \frac{\sqrt{6}}{2}\end{aligned}$$

Note that we reduced a fraction $\frac{3}{6}$ whose numerator and denominator were no longer inside the radical.

b. To rationalize the denominator of $\frac{\sqrt{5}}{\sqrt{72}}$, we *could* multiply both the numerator and denominator by $\sqrt{72}$, and it would be effective; however, we should note that the $\sqrt{72}$ in the denominator can be *reduced* first. Doing this will simplify the arithmetic because there will be smaller numbers to work with.

$$\begin{aligned}\frac{\sqrt{5}}{\sqrt{72}} &= \frac{\sqrt{5}}{\sqrt{36\cdot 2}}\\ &= \frac{\sqrt{5}}{\sqrt{36}\cdot\sqrt{2}}\\ &= \frac{\sqrt{5}}{6\cdot\sqrt{2}}\end{aligned}$$

Now all that remains is to multiply the numerator and denominator by $\sqrt{2}$.

$$\begin{aligned}&= \frac{\sqrt{5}}{6\cdot\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{10}}{6\cdot 2}\\ &= \frac{\sqrt{10}}{12}\end{aligned}$$

Checkpoint 6.2.3 Rationalize the denominator in $\frac{2}{\sqrt{10}}$.

Explanation. We will rationalize the denominator by multiplying the numerator and denominator by $\sqrt{10}$:

$$\begin{aligned}\frac{2}{\sqrt{10}} &= \frac{2}{\sqrt{10}}\cdot\frac{\sqrt{10}}{\sqrt{10}}\\ &= \frac{2\cdot\sqrt{10}}{\sqrt{10}\cdot\sqrt{10}}\\ &= \frac{2\sqrt{10}}{10}\\ &= \frac{\sqrt{10}}{5}\end{aligned}$$

Again note that the fraction was simplified in the last step.

Example 6.2.4 Rationalize the denominator in $\sqrt{\frac{2}{7}}$.

Explanation. This example is slightly different. The entire fraction, including its denominator, is within a radical. Having a denominator within a radical is just as undesirable as having a radical in a denominator. So we want to do something to change the expression.

$$\begin{aligned}\sqrt{\frac{2}{7}} &= \frac{\sqrt{2}}{\sqrt{7}}\\ &= \frac{\sqrt{2}}{\sqrt{7}}\cdot\frac{\sqrt{7}}{\sqrt{7}}\end{aligned}$$

$$= \frac{\sqrt{2} \cdot \sqrt{7}}{\sqrt{7} \cdot \sqrt{7}}$$
$$= \frac{\sqrt{14}}{7}$$

6.2.2 Rationalize the Denominator Using the Difference of Squares Formula

Conside the number $\frac{1}{\sqrt{2}+1}$. Its denominator is irrational, approximately $2.414\ldots$. Can we rewrite this as an equivalent expression where the denominator is rational? Let's try multiplying the numerator and denominator by $\sqrt{2}$:

$$\begin{aligned}
\frac{1}{\sqrt{2}+1} &= \frac{1}{\left(\sqrt{2}+1\right)} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{\sqrt{2}}{\sqrt{2} \cdot \sqrt{2} + 1 \cdot \sqrt{2}} \\
&= \frac{\sqrt{2}}{2 + \sqrt{2}}
\end{aligned}$$

We removed one radical from the denominator, but created another. We need to find another method. The difference of squares formula will help:

$$(a+b)(a-b) = a^2 - b^2$$

Those two squares in $a^2 - b^2$ can be used as a tool to annihilate radicals. Take $\frac{1}{\sqrt{2}+1}$, and multiply both the numerator and denominator by $\sqrt{2} - 1$:

$$\begin{aligned}
\frac{1}{\sqrt{2}+1} &= \frac{1}{\left(\sqrt{2}+1\right)} \cdot \frac{\left(\sqrt{2}-1\right)}{\left(\sqrt{2}-1\right)} \\
&= \frac{\sqrt{2}-1}{\left(\sqrt{2}\right)^2 - (1)^2} \\
&= \frac{\sqrt{2}-1}{2-1} \\
&= \frac{\sqrt{2}-1}{1} \\
&= \sqrt{2}-1
\end{aligned}$$

Example 6.2.5 Rationalize the denominator in $\frac{\sqrt{7}-\sqrt{2}}{\sqrt{5}+\sqrt{3}}$.

Explanation. To address the radicals in the denominator, we multiply both numerator and denominator by $\sqrt{5} - \sqrt{3}$.

$$\frac{\sqrt{7}-\sqrt{2}}{\sqrt{5}+\sqrt{3}} = \frac{\sqrt{7}-\sqrt{2}}{\sqrt{5}+\sqrt{3}} \cdot \frac{\left(\sqrt{5}-\sqrt{3}\right)}{\left(\sqrt{5}-\sqrt{3}\right)}$$

$$= \frac{\sqrt{7}\cdot\sqrt{5} - \sqrt{7}\cdot\sqrt{3} - \sqrt{2}\cdot\sqrt{5} - \sqrt{2}\cdot -\sqrt{3}}{\left(\sqrt{5}\right)^2 - \left(\sqrt{3}\right)^2}$$
$$= \frac{\sqrt{35} - \sqrt{21} - \sqrt{10} + \sqrt{6}}{5-3}$$
$$= \frac{\sqrt{35} - \sqrt{21} - \sqrt{10} + \sqrt{6}}{2}$$

Checkpoint 6.2.6 Rationalize the denominator in $\frac{\sqrt{3}}{3-2\sqrt{3}}$.

Explanation. To remove the radical in $3-2\sqrt{3}$ with the difference of squares formula, we multiply it with $3+2\sqrt{3}$.

$$\begin{aligned}
\frac{\sqrt{3}}{3-2\sqrt{3}} &= \frac{\sqrt{3}}{(3-2\sqrt{3})}\cdot\frac{(3+2\sqrt{3})}{(3+2\sqrt{3})} \\
&= \frac{3\cdot\sqrt{3} + 2\sqrt{3}\cdot\sqrt{3}}{(3)^2 - \left(2\sqrt{3}\right)^2} \\
&= \frac{3\sqrt{3} + 2\cdot 3}{9 - 2^2\left(\sqrt{3}\right)^2} \\
&= \frac{3\sqrt{3}+6}{9-4(3)} \\
&= \frac{3\left(\sqrt{3}+2\right)}{9-12} \\
&= \frac{3\left(\sqrt{3}+2\right)}{-3} \\
&= \frac{\sqrt{3}+2}{-1} \\
&= -\sqrt{3}-2
\end{aligned}$$

6.2.3 Reading Questions

1. To rationalize a denominator in an expression like $\frac{3}{\sqrt{5}}$, explain the first step you will take.

2. What is the special pattern from Section 5.5 that helps to rationalize the denominator in an expression like $\frac{3}{2+\sqrt{5}}$?

6.2.4 Exercises

Review and Warmup Rationalize the denominator and simplify the expression.

1. $\frac{1}{\sqrt{6}}$

2. $\frac{1}{\sqrt{6}}$

3. $\frac{7}{\sqrt{7}}$

4. $\frac{40}{\sqrt{10}}$

5. $\dfrac{1}{\sqrt{180}}$ **6.** $\dfrac{1}{\sqrt{8}}$ **7.** $\dfrac{2}{\sqrt{252}}$ **8.** $\dfrac{9}{\sqrt{180}}$

Rationalizing the Denominator Evaluate the following.

9. $\dfrac{3}{\sqrt{4}}$ **10.** $\dfrac{5}{\sqrt{64}}$

Rationalize the denominator and simplify the expression.

11. $\dfrac{1}{\sqrt{6}}$ **12.** $\dfrac{1}{\sqrt{7}}$ **13.** $\dfrac{7}{\sqrt{10}}$ **14.** $\dfrac{7}{\sqrt{10}}$

15. $\dfrac{5}{8\sqrt{2}}$ **16.** $\dfrac{7}{3\sqrt{3}}$ **17.** $\dfrac{6}{\sqrt{10}}$ **18.** $\dfrac{20}{\sqrt{14}}$

19. $\dfrac{18}{\sqrt{6}}$ **20.** $\dfrac{12}{\sqrt{6}}$ **21.** $\dfrac{1}{\sqrt{175}}$ **22.** $\dfrac{1}{\sqrt{180}}$

23. $\dfrac{2}{\sqrt{72}}$ **24.** $\dfrac{6}{\sqrt{32}}$ **25.** $\sqrt{\dfrac{7}{9}}$ **26.** $\sqrt{\dfrac{3}{16}}$

27. $\sqrt{\dfrac{9}{2}}$ **28.** $\sqrt{\dfrac{81}{2}}$ **29.** $\sqrt{\dfrac{11}{2}}$ **30.** $\sqrt{\dfrac{13}{15}}$

31. $\sqrt{\dfrac{108}{7}}$ **32.** $\sqrt{\dfrac{72}{5}}$ **33.** $\dfrac{4}{\sqrt{x}}$ **34.** $\dfrac{2}{\sqrt{y}}$

35. $\sqrt{\dfrac{5}{2}}$ **36.** $\sqrt{\dfrac{6}{11}}$ **37.** $\sqrt{\dfrac{11}{48}}$ **38.** $\sqrt{\dfrac{11}{175}}$

Rationalizing the Denominator Using the Difference of Squares Formula Rationalize the denominator and simplify the expression.

39. $\dfrac{7}{\sqrt{15}+7}$ **40.** $\dfrac{2}{\sqrt{22}+5}$ **41.** $\dfrac{8}{\sqrt{22}+9}$ **42.** $\dfrac{2}{\sqrt{14}+9}$

43. $\dfrac{\sqrt{2}-6}{\sqrt{11}+4}$ **44.** $\dfrac{\sqrt{5}-8}{\sqrt{13}+10}$ **45.** $\dfrac{\sqrt{3}-9}{\sqrt{7}+8}$ **46.** $\dfrac{\sqrt{2}-10}{\sqrt{11}+5}$

6.3 Radical Expressions and Rational Exponents

Recall that in Subsection 6.1.3, we learned to evaluate the cube root of a number, say $\sqrt[3]{8}$, we can type `8^(1/3)` into a calculator. This suggests that $\sqrt[3]{8} = 8^{1/3}$. In this section, we will learn why this is true, and how to simplify expressions with rational exponents.

Many learners will find a review of exponent rules to be helpful before continuing with the current section. Section 5.2 covers an introduction to exponent rules, and there is more in Section 5.6. The basic rules are summarized in List 5.6.13. These rules are still true and we can use them throughout this section whenever they might help.

6.3.1 Radical Expressions and Rational Exponents

Compare the following calculations:

$$\begin{aligned}\sqrt{9} \cdot \sqrt{9} &= 3 \cdot 3\\ &= 9\end{aligned} \qquad\qquad \begin{aligned}9^{1/2} \cdot 9^{1/2} &= 9^{1/2+1/2}\\ &= 9^1\\ &= 9\end{aligned}$$

If we rewrite the above calculations with exponents, we have:

$$\left(\sqrt{9}\right)^2 = 9 \qquad\qquad \left(9^{1/2}\right)^2 = 9$$

Since $\sqrt{9}$ and $9^{1/2}$ are both positive, and squaring either of them generates the same number, we conclude that:

$$\sqrt{9} = 9^{1/2}$$

We can verify this result by entering `9^(1/2)` into a calculator, and we get `3`. In general for any non-negative real number a, we have:

$$\sqrt{a} = a^{1/2}$$

Similarly, when a is non-negative all of the following are true:

$$\sqrt[2]{a} = a^{1/2} \qquad \sqrt[3]{a} = a^{1/3} \qquad \sqrt[4]{a} = a^{1/4} \qquad \sqrt[5]{a} = a^{1/5} \qquad \cdots$$

For example, when we see $16^{1/4}$, that is equal to $\sqrt[4]{16}$, which we know is 2 because $\overbrace{2 \cdot 2 \cdot 2 \cdot 2}^{\text{four times}} = 16$. How can we relate this to the exponential expression $16^{1/4}$? In a sense, we are cutting up 16 into 4 equal parts. But not parts that you *add* together, rather parts that you *multiply* together.

Let's summarize this information with a new exponent rule.

Fact 6.3.2 Radicals and Rational Exponents Rule. *If m is any natural number, and a is any non-negative real number, then*

$$a^{1/m} = \sqrt[m]{a}.$$

Additionally, if m is an odd *natural number, then even when a is negative, we still have $a^{1/m} = \sqrt[m]{a}$.*

Warning 6.3.3 Exponents on Negative Bases. Some computers and calculators follow different conventions when there is an exponent on a negative base. To see an example of this, visit *WolframAlpha* and try entering `cuberoot(-8)`, and then try `(-8)^(1/3)`, and you will get different results. `cuberoot(-8)` will come out as

-2, but `(-8)^(1/3)` will come out as a certain non-real complex number. Most likely, any calculator you are using *does* behave as in Fact 6.3.2, but you should confirm this.

With the Radicals and Rational Exponents Rule, we can re-write radical expressions as expressions with rational exponents.

Example 6.3.4 Write the radical expression $\sqrt[3]{6}$ as an expression with a rational exponent. Then use a calculator to find its decimal approximation.

According to the Radicals and Rational Exponents Rule, $\sqrt[3]{6} = 6^{1/3}$. A calculator tells us that `6^(1/3)` works out to approximately 1.817.

For many examples that follow, we will not need a calculator. We will, however, need to recognize the roots in Figure 6.3.5.

Square Roots	Cube Roots	4^{th}-Roots	5^{th}-Roots	Roots of Powers of 2
$\sqrt{1}=1$	$\sqrt[3]{1}=1$	$\sqrt[4]{1}=1$	$\sqrt[5]{1}=1$	
$\sqrt{4}=2$	$\sqrt[3]{8}=2$	$\sqrt[4]{16}=2$	$\sqrt[5]{32}=2$	$\sqrt{4}=2$
$\sqrt{9}=3$	$\sqrt[3]{27}=3$	$\sqrt[4]{81}=3$		$\sqrt[3]{8}=2$
$\sqrt{16}=4$	$\sqrt[3]{64}=4$			$\sqrt[4]{16}=2$
$\sqrt{25}=5$	$\sqrt[3]{125}=5$			$\sqrt[5]{32}=2$
$\sqrt{36}=6$				$\sqrt[6]{64}=2$
$\sqrt{49}=7$				$\sqrt[7]{128}=2$
$\sqrt{64}=8$				$\sqrt[8]{256}=2$
$\sqrt{81}=9$				$\sqrt[9]{512}=2$
$\sqrt{100}=10$				$\sqrt[10]{1024}=2$
$\sqrt{121}=11$				
$\sqrt{144}=12$				

Figure 6.3.5: Small Roots of Appropriate Natural Numbers

Example 6.3.6 Write the expressions in radical form using the Radicals and Rational Exponents Rule and simplify the results.

a. $4^{1/2}$

b. $(-9)^{1/2}$

c. $-16^{1/4}$

d. $64^{-1/3}$

e. $(-27)^{1/3}$

f. $3^{1/2} \cdot 3^{1/2}$

Explanation.

a. $$\begin{aligned} 4^{1/2} &= \sqrt{4} \\ &= 2 \end{aligned}$$

b. $(-9)^{1/2} = \sqrt{-9}$ This value is non-real.

c. Without parentheses around -16, the negative sign in this problem should be left out of the radical.

$$\begin{aligned} -16^{1/4} &= -\sqrt[4]{16} \\ &= -2 \end{aligned}$$

d. Here we will use the Negative Exponent Rule.

$$\begin{aligned} 64^{-1/3} &= \frac{1}{64^{1/3}} \\ &= \frac{1}{\sqrt[3]{64}} \\ &= \frac{1}{4} \end{aligned}$$

e. $$\begin{aligned} (-27)^{1/3} &= \sqrt[3]{-27} \\ &= -3 \end{aligned}$$

f. $$\begin{aligned} 3^{1/2} \cdot 3^{1/2} &= \sqrt{3} \cdot \sqrt{3} \\ &= \sqrt{3 \cdot 3} \\ &= \sqrt{9} \\ &= 3 \end{aligned}$$

The Radicals and Rational Exponents Rule applies to variables in expressions just as much as it does to numbers.

Example 6.3.7 Write the expressions as simplified as they can be using radicals.

a. $2x^{-1/2}$ b. $(5x)^{1/3}$ c. $\left(-27x^{12}\right)^{1/3}$ d. $\left(\frac{16x}{81y^8}\right)^{1/4}$

Explanation.

a. Note that in this example the exponent is only applied to the x. Making this type of observation should be our first step for each of these exercises.

$$\begin{aligned} 2x^{-1/2} &= \frac{2}{x^{1/2}} && \text{by the Negative Exponent Rule} \\ &= \frac{2}{\sqrt{x}} && \text{by the Radicals and Rational Exponents Rule} \end{aligned}$$

b. In this exercise, the exponent applies to both the 5 and x.

$$(5x)^{1/3} = \sqrt[3]{5x} \qquad \text{by the Radicals and Rational Exponents Rule}$$

c. We start out as with the previous exercise. As in the previous exercise, we have a choice as to how to simplify this expression. Here we should note that we *do* know what the cube root of -27 is, so we will take the path to splitting up the expression, using the Product to a Power Rule, before applying the root.

$$\left(-27x^{12}\right)^{1/3} = \sqrt[3]{-27x^{12}}$$

Here we notice that -27 has a nice cube root, so it is good to break up the radical.

$$\begin{aligned} &= \sqrt[3]{-27}\,\sqrt[3]{x^{12}} \\ &= -3\,\sqrt[3]{x^{12}} \end{aligned}$$

Can this be simplified more? There are two ways to think about that. One way is to focus on the cube root and see that x^4 cubes to make x^{12}, and the other way is to convert the cube root back to a fraction exponent and use exponent rules.

$$= -3\sqrt[3]{x^4x^4x^4} \qquad\qquad = -3\left(x^{12}\right)^{1/3}$$
$$= -3x^4 \qquad\qquad = -3x^{12\cdot 1/3}$$
$$= -3x^4$$

d. We'll use the exponent rule for a fraction raised to a power.

$$\begin{aligned}
\left(\frac{16x}{81y^8}\right)^{1/4} &= \frac{(16x)^{1/4}}{(81y^8)^{1/4}} && \text{by the Quotient to a Power Rule}\\
&= \frac{16^{1/4}\cdot x^{1/4}}{81^{1/4}\cdot (y^8)^{1/4}} && \text{by the Product to a Power Rule}\\
&= \frac{16^{1/4}\cdot x^{1/4}}{81^{1/4}\cdot y^2}\\
&= \frac{\sqrt[4]{16}\cdot \sqrt[4]{x}}{\sqrt[4]{81}\cdot y^2} && \text{by the Radicals and Rational Exponents Rule}\\
&= \frac{2\sqrt[4]{x}}{3y^2}
\end{aligned}$$

Remark 6.3.8 In general, it is easier to do algebra with rational exponents on variables than with radicals of variables. You should use Radicals and Rational Exponents Rule to convert from rational exponents to radicals on variables *only as a last step* in simplifying.

The Radicals and Rational Exponents Rule describes what can be done when there is a fractional exponent and the numerator is a 1. The numerator doesn't have to be a 1 though and we need guidance for that situation.

Fact 6.3.9 Full Radicals and Rational Exponents Rule. *If m and n are natural numbers such that $\frac{m}{n}$ is a reduced fraction, and a is any non-negative real number, then*

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.$$

Additionally, if n is an odd *natural number, then even when a is negative, we still have $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$.*

Example 6.3.10 Guitar Frets. On a guitar, there are 12 frets separating a note and the same note one octave higher. By moving from one fret to another that is five frets away, the frequency of the note changes by a factor of $2^{5/12}$. Use the Full Radicals and Rational Exponents Rule to write this number as a radical expression. And use a calculator to find this number as a decimal.

Explanation. According to the Full Radicals and Rational Exponents Rule,

$$\begin{aligned}
2^{5/12} &= \sqrt[12]{2^5}\\
&= \sqrt[12]{32}
\end{aligned}$$

A calculator says $2^{5/12} \approx 1.334\cdots$. The fact that this is very close to $\frac{4}{3} \approx 1.333\ldots$ is important. It is part of the explanation for why two notes that are five frets apart on the same string would sound good to human ears when played together as a chord (known as a "fourth," in music).

Remark 6.3.11 By the Full Radicals and Rational Exponents Rule, there are two ways to express $a^{m/n}$ as a radical expression:

$$a^{m/n} = \sqrt[n]{a^m} \qquad \text{and} \qquad a^{m/n} = \left(\sqrt[n]{a}\right)^m$$

There are different times to use each formula. In general, use $a^{m/n} = \sqrt[n]{a^m}$ for variables and $a^{m/n} = \left(\sqrt[n]{a}\right)^m$ for numbers.

Example 6.3.12

a. Consider the expression $27^{4/3}$. Use both versions of the Full Radicals and Rational Exponents Rule to explain why Remark 6.3.11 says that with numbers, $a^{m/n} = \left(\sqrt[n]{a}\right)^m$ is preferred.

b. Consider the expression $x^{4/3}$. Use both versions of the Full Radicals and Rational Exponents Rule to explain why Remark 6.3.11 says that with variables, $a^{m/n} = \sqrt[n]{a^m}$ is preferred.

Explanation.

a. The expression $27^{4/3}$ can be evaluated in the following two ways.

$$\begin{aligned} 27^{4/3} &= \sqrt[3]{27^4} && \text{by the first part of the Full Radicals and Rational Exponents Rule} \\ &= \sqrt[3]{531441} \\ &= 81 \end{aligned}$$

or

$$\begin{aligned} 27^{4/3} &= \left(\sqrt[3]{27}\right)^4 && \text{by the second part of the Full Radicals and Rational Exponents Rule} \\ &= 3^4 \\ &= 81 \end{aligned}$$

The calculation using $a^{m/n} = \left(\sqrt[n]{a}\right)^m$ worked with smaller numbers and can be done without a calculator. This is why we made the general recommendation in Remark 6.3.11.

b. The expression $x^{4/3}$ can be evaluated in the following two ways.

$$x^{4/3} = \sqrt[3]{x^4} \qquad \text{by the first part of Full Radicals and Rational Exponents Rule}$$

or

$$x^{4/3} = \left(\sqrt[3]{x}\right)^4 \qquad \text{by the second part of the Full Radicals and Rational Exponents Rule}$$

In this case, the simplification using $a^{m/n} = \sqrt[n]{a^m}$ is just shorter looking and easier to write. This is why we made the general recommendation in Remark 6.3.11.

Example 6.3.13 Simplify the expressions using Fact 6.3.9.

a. $8^{2/3}$

b. $(64x)^{-2/3}$

c. $\left(-\frac{27}{64}\right)^{2/3}$

Explanation.

a. We will use the second part of the Full Radicals and Rational Exponents Rule, since this expression only involves a number base (not variable).

$$\begin{aligned}8^{2/3} &= \left(\sqrt[3]{8}\right)^2\\ &= 2^2\\ &= 4\end{aligned}$$

b. $$\begin{aligned}(64x)^{-2/3} &= \frac{1}{(64x)^{2/3}}\\ &= \frac{1}{64^{2/3}x^{2/3}}\\ &= \frac{1}{\left(\sqrt[3]{64}\right)^2\ \sqrt[3]{x^2}}\\ &= \frac{1}{4^2\sqrt[3]{x^2}}\\ &= \frac{1}{16\sqrt[3]{x^2}}\end{aligned}$$

c. In this problem the negative can be associated with either the numerator or the denominator, but not both. We choose the numerator.

$$\begin{aligned}\left(-\frac{27}{64}\right)^{2/3} &= \left(\sqrt[3]{-\frac{27}{64}}\right)^2 && \text{by the second part of the Full Radicals and Rational Exponents Rule}\\ &= \left(\frac{\sqrt[3]{-27}}{\sqrt[3]{64}}\right)^2\\ &= \left(\frac{-3}{4}\right)^2\\ &= \frac{(-3)^2}{(4)^2}\\ &= \frac{9}{16}\end{aligned}$$

6.3.2 More Expressions with Rational Exponents

To recap, here is a "complete" list of exponent and radical rules.

List 6.3.14: Complete List of Exponent Rules

Product Rule $a^n \cdot a^m = a^{n+m}$

Power to a Power Rule $(a^n)^m = a^{n \cdot m}$

Product to a Power Rule $(ab)^n = a^n \cdot b^n$

Quotient Rule $\dfrac{a^n}{a^m} = a^{n-m}$, as long as $a \neq 0$

Quotient to a Power Rule $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$, as long as $b \neq 0$

Zero Exponent Rule $a^0 = 1$ for $a \neq 0$

Negative Exponent Rule $a^{-n} = \frac{1}{a^n}$

Negative Exponent Reciprocal Rule $\frac{1}{a^{-n}} = a^n$

Negative Exponent on Fraction Rule $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$

Radical and Rational Exponent Rule $x^{1/n} = \sqrt[n]{x}$

Radical and Rational Exponent Rule $x^{m/n} = \left(\sqrt[n]{x}\right)^m$, usually for numbers

Radical and Rational Exponent Rule $x^{m/n} = \sqrt[n]{x^m}$, usually for variables

Example 6.3.15 Convert the following radical expressions into expressions with rational exponents, and simplify them if possible.

a. $\dfrac{1}{\sqrt{x}}$

b. $\dfrac{1}{\sqrt[3]{25}}$

Explanation.

a.

$$\begin{aligned} \frac{1}{\sqrt{x}} &= \frac{1}{x^{1/2}} && \text{by the Radicals and Rational Exponents Rule} \\ &= x^{-1/2} && \text{by the Negative Exponent Rule} \end{aligned}$$

b.

$$\begin{aligned} \frac{1}{\sqrt[3]{25}} &= \frac{1}{25^{1/3}} && \text{by the Radicals and Rational Exponents Rule} \\ &= \frac{1}{\left(5^2\right)^{1/3}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{5^{2\cdot 1/3}} && \text{by the Power to a Power Rule}\\
&= \frac{1}{5^{2/3}} \\
&= 5^{-2/3} && \text{by the Negative Exponent Rule}
\end{aligned}
$$

Learners of these simplifications often find it challenging, so we now include a many examples of varying difficulty.

Example 6.3.16 Use exponent properties in List 6.3.14 to simplify the expressions, and write all final versions using radicals.

a. $2w^{7/8}$

b. $\frac{1}{2}y^{-1/2}$

c. $(27b)^{2/3}$

d. $\left(-8p^6\right)^{5/3}$

e. $\sqrt{x^3} \cdot \sqrt[4]{x}$

f. $h^{1/3} + h^{1/3} + h^{1/3}$

g. $\frac{\sqrt{z}}{\sqrt[3]{z}}$

h. $\sqrt{\sqrt[4]{q}}$

i. $3\left(c^{1/2} + d^{1/2}\right)^2$

j. $3\left(4k^{2/3}\right)^{-1/2}$

Explanation.

a.

$$2w^{7/8} = 2\sqrt[8]{w^7} \qquad \text{by the Full Radicals and Rational Exponents Rule}$$

b.

$$
\begin{aligned}
\frac{1}{2}y^{-1/2} &= \frac{1}{2}\frac{1}{y^{1/2}} && \text{by the Negative Exponent Rule}\\
&= \frac{1}{2}\frac{1}{\sqrt{y}} && \text{by the Full Radicals and Rational Exponents Rule}\\
&= \frac{1}{2\sqrt{y}}
\end{aligned}
$$

c.

$$
\begin{aligned}
(27b)^{2/3} &= (27)^{2/3} \cdot (b)^{2/3} && \text{by the Product to a Power Rule}\\
&= \left(\sqrt[3]{27}\right)^2 \cdot \sqrt[3]{b^2} && \text{by the Full Radicals and Rational Exponents Rule}\\
&= 3^2 \cdot \sqrt[3]{b^2}\\
&= 9\sqrt[3]{b^2}
\end{aligned}
$$

d.

$$
\begin{aligned}
\left(-8p^6\right)^{5/3} &= (-8)^{5/3} \cdot \left(p^6\right)^{5/3} && \text{by the Product to a Power Rule}\\
&= (-8)^{5/3} \cdot p^{6\cdot 5/3} && \text{by the Power to a Power Rule}\\
&= \left(\sqrt[3]{-8}\right)^5 \cdot p^{10} && \text{by the Full Radicals and Rational Exponents Rule}\\
&= (-2)^5 \cdot p^{10}\\
&= -32p^{10}
\end{aligned}
$$

e.

$$\begin{aligned}
\sqrt{x^3}\cdot\sqrt[4]{x} &= x^{3/2}\cdot x^{1/4} && \text{by the Full Radicals and Rational Exponents Rule}\\
&= x^{3/2+1/4} && \text{by the Product Rule}\\
&= x^{6/4+1/4}\\
&= x^{7/4}\\
&= \sqrt[4]{x^7} && \text{by the Full Radicals and Rational Exponents Rule}
\end{aligned}$$

f.

$$\begin{aligned}
h^{1/3}+h^{1/3}+h^{1/3} &= 3h^{1/3}\\
&= 3\sqrt[3]{h} && \text{by the Radicals and Rational Exponents Rule}
\end{aligned}$$

g.

$$\begin{aligned}
\frac{\sqrt{z}}{\sqrt[3]{z}} &= \frac{z^{1/2}}{z^{1/3}} && \text{by the Radicals and Rational Exponents Rule}\\
&= z^{1/2-1/3} && \text{by the Quotient Rule}\\
&= z^{3/6-2/6}\\
&= z^{1/6}\\
&= \sqrt[6]{z} && \text{by the Radicals and Rational Exponents Rule}
\end{aligned}$$

h.

$$\begin{aligned}
\sqrt{\sqrt[4]{q}} &= \sqrt{q^{1/4}} && \text{by the Radicals and Rational Exponents Rule}\\
&= \left(q^{1/4}\right)^{1/2} && \text{by the Radicals and Rational Exponents Rule}\\
&= q^{1/4\cdot 1/2} && \text{by the Power to a Power Rule}\\
&= q^{1/8}\\
&= \sqrt[8]{q} && \text{by the Radicals and Rational Exponents Rule}
\end{aligned}$$

i.

$$\begin{aligned}
3\left(c^{1/2}+d^{1/2}\right)^2 &= 3\left(c^{1/2}+d^{1/2}\right)\left(c^{1/2}+d^{1/2}\right)\\
&= 3\left(\left(c^{1/2}\right)^2+2c^{1/2}\cdot d^{1/2}+\left(d^{1/2}\right)^2\right)\\
&= 3\left(c^{1/2\cdot 2}+2c^{1/2}\cdot d^{1/2}+d^{1/2\cdot 2}\right)\\
&= 3\left(c+2c^{1/2}\cdot d^{1/2}+d\right)\\
&= 3\left(c+2(cd)^{1/2}+d\right) && \text{by the Product to a Power Rule}\\
&= 3\left(c+2\sqrt{cd}+d\right) && \text{by the Radicals and Rational Exponents Rule}\\
&= 3c+6\sqrt{cd}+3d
\end{aligned}$$

j.

$$3\left(4k^{2/3}\right)^{-1/2} = \frac{3}{\left(4k^{2/3}\right)^{1/2}} \qquad \text{by the Negative Exponent Rule}$$

$$
\begin{aligned}
&= \frac{3}{4^{1/2}\left(k^{2/3}\right)^{1/2}} && \text{by the Product to a Power Rule} \\
&= \frac{3}{4^{1/2}k^{2/3 \cdot 1/2}} && \text{by the Power to a Power Rule} \\
&= \frac{3}{4^{1/2}k^{1/3}} \\
&= \frac{3}{\sqrt{4} \cdot \sqrt[3]{k}} && \text{by the Radicals and Rational Exponents Rule} \\
&= \frac{3}{2\sqrt[3]{k}}
\end{aligned}
$$

We will end a with a short application of rational exponents. Kepler's Laws of Orbital Motion[1] describe how planets orbit stars and how satellites orbit planets. In particular, his third law has a rational exponent, which we will now explore.

Example 6.3.17 Kepler and the Satellite. Kepler's third law of motion says that for objects with a roughly circular orbit that the time (in hours) that it takes to make one full revolution around the planet, T, is proportional to three-halves power of the distance (in kilometers) from the center of the planet to the satellite, r. For the Earth, it looks like this:

$$T = \frac{2\pi}{\sqrt{G \cdot M_E}} r^{3/2}$$

In this case, both G and M_E are constants. G stands for the universal gravitational constant[2] where G is about $8.65 \times 10^{-13}\ \frac{\text{km}^3}{\text{kg} \cdot \text{h}^2}$ and M_E stands for the mass of the Earth[3] where M_E is about 5.972×10^{24} kg. Inputting these values into this formula yields a simplified version that looks like this:

$$T \approx 2.76 \times 10^{-6} r^{3/2}$$

Most satellites orbit in what is called low Earth orbit[4], including the international space station which orbits at about 340 km above from Earth's surface. The Earth's average radius is about 6380 km. Find the period of the international space station.

Explanation. The formula has already been identified, but the input takes just a little thought. The formula uses r as the distance from the center of the Earth to the satellite, so to find r we need to combine the radius of the Earth and the distance to the satellite above the surface of the Earth.

$$
\begin{aligned}
r &= 340 + 6380 \\
&= 6720
\end{aligned}
$$

Now we can input this value into the formula and evaluate.

$$
\begin{aligned}
T &\approx 2.76 \cdot 10^{-6} r^{3/2} \\
&\approx 2.76 \cdot 10^{-6} (6720)^{3/2} \\
&\approx 2.76 \cdot 10^{-6} \left(\sqrt{6720}\right)^{3} \\
&\approx 1.52
\end{aligned}
$$

The formula tells us that it takes a little more than an hour and a half for the ISS to orbit the Earth! That works out to 15 or 16 sunrises per day.

[1] en.wikipedia.org/wiki/Kepler%27s_laws_of_planetary_motion

6.3.3 Reading Questions

1. Raising a number to a reciprocal power (like $\frac{1}{2}$ or $\frac{1}{5}$) is the same as doing what other thing to that number?

2. When the exponent on an expression is a fraction like $\frac{3}{5}$, which part of the fraction is essentially the index of a radical?

6.3.4 Exercises

Review and Warmup Use the properties of exponents to simplify the expression.

1. $x^{13} \cdot x^{17}$ **2.** $y^{15} \cdot y^{11}$ **3.** $\left(t^{11}\right)^3$ **4.** $\left(y^{12}\right)^{10}$

5. $\left(\frac{7x^2}{2}\right)^2$ **6.** $\left(\frac{3x^3}{8}\right)^2$ **7.** $\left(-6y^4\right)^3$ **8.** $\left(-2x^6\right)^2$

9. $\frac{y^{11}}{y}$ **10.** $\frac{t^{13}}{t^9}$

Rewrite the expression simplified and using only positive exponents.

11. $r^{-9} \cdot r^3$ **12.** $t^{-3} \cdot t^2$ **13.** $(9t^{-14}) \cdot (10t^2)$ **14.** $(6x^{-8}) \cdot (5x^4)$

Calculations Without using a calculator, evaluate the expression.

15. a. $9^{\frac{1}{2}}$ b. $(-9)^{\frac{1}{2}}$ c. $-9^{\frac{1}{2}}$

16. a. $16^{\frac{1}{2}}$ b. $(-16)^{\frac{1}{2}}$ c. $-16^{\frac{1}{2}}$

17. a. $8^{\frac{1}{3}}$ b. $(-8)^{\frac{1}{3}}$ c. $-8^{\frac{1}{3}}$

18. a. $27^{\frac{1}{3}}$ b. $(-27)^{\frac{1}{3}}$ c. $-27^{\frac{1}{3}}$

19. $9^{-\frac{3}{2}}$ **20.** $125^{-\frac{1}{3}}$ **21.** $\left(\frac{1}{81}\right)^{-\frac{3}{4}}$ **22.** $\left(\frac{1}{9}\right)^{-\frac{3}{2}}$

23. $\sqrt[2]{9^3}$ **24.** $\sqrt[2]{81^3}$ **25.** $\sqrt[5]{1024}$ **26.** $\sqrt[3]{64}$

27. a. $\sqrt[3]{8}$ b. $\sqrt[3]{-8}$ c. $-\sqrt[3]{8}$

28. a. $\sqrt[3]{27}$ b. $\sqrt[3]{-27}$ c. $-\sqrt[3]{27}$

29. a. $\sqrt[4]{16}$ b. $\sqrt[4]{-16}$ c. $-\sqrt[4]{16}$

30. a. $\sqrt[4]{81}$ b. $\sqrt[4]{-81}$ c. $-\sqrt[4]{81}$

31. $\sqrt[3]{-\frac{27}{125}}$ **32.** $\sqrt[3]{-\frac{27}{125}}$ **33.** $\sqrt[3]{-\frac{1}{64}}$ **34.** $\sqrt[3]{-\frac{27}{125}}$

Use a calculator to evaluate the expression as a decimal to four significant digits.

35. $\sqrt[3]{9^2}$ **36.** $\sqrt[5]{11^3}$ **37.** $\sqrt[3]{13^2}$ **38.** $\sqrt[5]{18^3}$

[2] en.wikipedia.org/wiki/Gravitational_constant
[3] en.wikipedia.org/wiki/Earth_mass
[4] en.wikipedia.org/wiki/Low_Earth_orbit

39. On a guitar, there are 12 frets separating a note and the same note one octave higher. By moving from one fret to another that is seven frets away, the frequency of the note changes by a factor of $2^{7/12}$. Use a calculator to find this number as a decimal. This decimal shows you that $2^{7/12}$ is very close to a "nice" fraction with small numerator and denominator. Notes with this frequency ratio form a "perfect fifth" in music. What is that fraction?

40. On a guitar, there are 12 frets separating a note and the same note one octave higher. By moving from one fret to another that is four frets away, the frequency of the note changes by a factor of $2^{4/12}$. Use a calculator to find this number as a decimal. This decimal shows you that $2^{4/12}$ is very close to a "nice" fraction with small numerator and denominator. Notes with this frequency ratio form a "major third" in music. What is that fraction?

Convert Radicals to Fractional Exponents Use rational exponents to write the expression.

41. $\sqrt[9]{x}$

42. $\sqrt[6]{y}$

43. $\sqrt[3]{4z+6}$

44. $\sqrt{9t+10}$

45. $\sqrt[6]{r}$

46. $\sqrt[3]{m}$

47. $\dfrac{1}{\sqrt[8]{n^3}}$

48. $\dfrac{1}{\sqrt[5]{b^4}}$

Convert Fractional Exponents to Radicals Convert the expression to radical notation.

49. $c^{\frac{2}{3}}$

50. $x^{\frac{5}{6}}$

51. $y^{\frac{5}{9}}$

52. $r^{\frac{2}{3}}$

53. $15^{\frac{1}{6}}t^{\frac{5}{6}}$

54. $4^{\frac{1}{4}}r^{\frac{3}{4}}$

55. Convert $m^{\frac{2}{3}}$ to a radical expression.

56. Convert $n^{\frac{5}{6}}$ to a radical expression.

57. Convert $b^{-\frac{3}{5}}$ to a radical expression.

58. Convert $c^{-\frac{2}{7}}$ to a radical expression.

59. Convert $2^{\frac{1}{5}}x^{\frac{4}{5}}$ to a radical expression.

60. Convert $7^{\frac{1}{7}}y^{\frac{3}{7}}$ to a radical expression.

Simplifying Expressions with Rational Exponents Simplify the expression, answering with rational exponents and not radicals.

61. $\sqrt[11]{z}\,\sqrt[11]{z}$

62. $\sqrt[9]{t}\,\sqrt[9]{t}$

63. $\sqrt[5]{32r^2}$

64. $\sqrt[3]{27m^5}$

65. $\dfrac{\sqrt[3]{27n}}{\sqrt[6]{n^5}}$

66. $\dfrac{\sqrt{36b}}{\sqrt[6]{b^5}}$

67. $\dfrac{\sqrt{4c^3}}{\sqrt[10]{c}}$

68. $\dfrac{\sqrt{49x}}{\sqrt[6]{x^5}}$

69. $\sqrt[5]{y}\cdot\sqrt[10]{y^3}$

70. $\sqrt{z}\cdot\sqrt[6]{z^5}$

71. $\sqrt{\sqrt[3]{t}}$

72. $\sqrt[4]{\sqrt{r}}$

73. $\sqrt{b}\,\sqrt[7]{b}$

74. $\sqrt{r}\,\sqrt[8]{r}$

6.4 Solving Radical Equations

In this section, we will learn how to solve equations involving radicals. The basic strategy to solve radical equations is to isolate the radical on one side of the equation and then raise to a power on both sides to cancel the radical.

6.4.1 Solving Radical Equations

Definition 6.4.2 Radical Equation. A radical equation is an equation in which there is a variable inside at least one radical. ◊

Examples include the equations $\sqrt{x-2} = 3 + x$ and $1 + \sqrt[3]{2-x} = x$.

Example 6.4.3 The formula $T = 2\pi\sqrt{\frac{L}{g}}$ is used to calculate the period of a pendulum and is attributed to the scientist Christiaan Huygens[1]. In the formula, T stands for the pendulum's period (how long one back-and-forth oscillation takes) in seconds, L stands for the pendulum's length in meters, and g is approximately $9.8\ \frac{\text{m}}{\text{s}^2}$ which is the gravitational acceleration constant on Earth.

An engineer is designing a pendulum. Its period must be 10 seconds. How long should the pendulum's length be?

We will substitute 10 into the formula for T and also the value of g, and then solve for L:

$$
\begin{aligned}
10 &= 2\pi\sqrt{\frac{L}{9.8}} \\
\frac{1}{2\pi}\cdot 10 &= \frac{1}{2\pi}\cdot 2\pi\sqrt{\frac{L}{9.8}} \\
\frac{5}{\pi} &= \sqrt{\frac{L}{9.8}} \\
\left(\frac{5}{\pi}\right)^2 &= \left(\sqrt{\frac{L}{9.8}}\right)^2 && \text{canceling square root by squaring both sides} \\
\frac{25}{\pi^2} &= \frac{L}{9.8} \\
9.8\cdot\frac{25}{\pi^2} &= 9.8\cdot\frac{L}{9.8} \\
24.82 &\approx L
\end{aligned}
$$

To build a pendulum with a period of 10 seconds, its length should be approximately 24.82 meters.

Remark 6.4.4 Squaring both sides of an equation is "dangerous," as it could create **extraneous solutions**, which will not make the equation true. For example, if we square both sides of $1 = -1$, we have:

$$
\begin{aligned}
1 &= -1 && \text{false} \\
(1)^2 &= (-1)^2 && \text{square both sides}\ldots \\
1 &= 1 && \text{true}
\end{aligned}
$$

By squaring both sides of an equation, we can sometimes turn a false equation into a true one. This is why

[1] en.wikipedia.org/wiki/Christiaan_Huygens#Pendulums

we *must check solutions* when we square both sides of an equation.

Example 6.4.5 Solve the equation $1+\sqrt{y-1}=4$ for y.
Explanation. We will isolate the radical first, and then square both sides.

$$\begin{aligned} 1+\sqrt{y-1} &= 4 \\ \sqrt{y-1} &= 3 \\ \left(\sqrt{y-1}\right)^2 &= 3^2 \\ y-1 &= 9 \\ y &= 10 \end{aligned}$$

Because we squared both sides of an equation, we must check the solution.

$$\begin{aligned} 1+\sqrt{10-1} &\stackrel{?}{=} 4 \\ 1+\sqrt{9} &\stackrel{?}{=} 4 \\ 1+3 &\stackrel{\checkmark}{=} 4 \end{aligned}$$

So, 10 is the solution to the equation $1+\sqrt{y-1}=4$.

Example 6.4.6 Solve the equation $5+\sqrt{q}=3$ for q.
Explanation. First, isolate the radical and square both sides.

$$\begin{aligned} 5+\sqrt{q} &= 3 \\ \sqrt{q} &= -2 \\ \left(\sqrt{q}\right)^2 &= (-2)^2 \\ q &= 4 \end{aligned}$$

Because we squared both sides of an equation, we must check the solution.

$$\begin{aligned} 5+\sqrt{4} &\stackrel{?}{=} 3 \\ 5+2 &\stackrel{?}{=} 3 \\ 7 &\stackrel{\text{no}}{=} 3 \end{aligned}$$

Thus, the potential solution -2 is actually extraneous and we have no real solutions to the equation $5+\sqrt{q}=3$. The solution set is the empty set, $\emptyset$.

Remark 6.4.7 In the previous example, it would be legitimate to observe that there are no solutions at earlier stages. From the very beginning, how could 5 plus a positive quantity result in 3? Or at the second step, since square roots are non-negative, how could a square root equal -2?

You do not have to be able to make these observations. If you follow the general steps for solving radical equations *and* you remember to check the possible solutions you find, then that will be enough.

Sometimes, we need to square both sides of an equation *twice* before finding the solutions, like in the next example.

Example 6.4.8 Solve the equation $\sqrt{p-5} = 5 - \sqrt{p}$ for p.

Explanation. We cannot isolate two radicals, so we will simply square both sides, and later try to isolate the remaining radical.

$$\begin{aligned}
\sqrt{p-5} &= 5 - \sqrt{p} \\
\left(\sqrt{p-5}\right)^2 &= (5 - \sqrt{p})^2 \\
p - 5 &= 25 - 10\sqrt{p} + p && \text{after expanding the binomial squared} \\
-5 &= 25 - 10\sqrt{p} \\
-30 &= -10\sqrt{p} \\
3 &= \sqrt{p} \\
3^2 &= (\sqrt{p})^2 \\
9 &= p
\end{aligned}$$

Because we squared both sides of an equation, we must check the solution.

$$\begin{aligned}
\sqrt{9-5} &\stackrel{?}{=} 5 - \sqrt{9} \\
\sqrt{4} &\stackrel{?}{=} 5 - 3 \\
2 &\stackrel{\checkmark}{=} 2
\end{aligned}$$

So 9 is the solution. The solution set is $\{9\}$.

Let's look at an example of solving an equation with a cube root. There is very little difference between solving an equation with one cube root and solving an equation with one square root. Instead of *squaring* both sides, you *cube* both sides.

Example 6.4.9 Solve for q in $\sqrt[3]{2-q} + 2 = 5$.

Explanation.

$$\begin{aligned}
\sqrt[3]{2-q} + 2 &= 5 \\
\sqrt[3]{2-q} &= 3 \\
\left(\sqrt[3]{2-q}\right)^3 &= 3^3 \\
2 - q &= 27 \\
-q &= 25 \\
q &= -25
\end{aligned}$$

Unlike squaring both sides of an equation, raising both sides of an equation to the 3rd power will not create extraneous solutions. It's still good practice to check solution, though. This part is left as exercise.

For summary reference, here is the general procedure for solving a radical equation.

Process 6.4.10 Solving Radical Equations. *A basic strategy to solve radical equations is to take the following steps:*

1. *Isolate a radical on one side of the equation.*

2. *Raise both sides of the equation to a power to cancel the radical.*

3. *If there is still a radical in the equation, repeat the isolation and raising to a power.*
4. *Once the remaining equation has no radicals, solve it.*
5. *Check any and all solutions. Be aware that there may be "extraneous solutions".*

6.4.2 Solving a Radical Equation with More Than One Variable

We also need to be able to solve radical equations with other variables, like in the next example. The strategy is the same: isolate the radical, and then raise both sides to a certain power to cancel the radical.

Example 6.4.11 Solve for L in the formula $T = 2\pi\sqrt{\frac{L}{g}}$. (This is the formula for a the period T of a swinging pendulum whose length is L, on earth where the acceleration from earth's gravity is g.)
Explanation.

$$\begin{aligned} T &= 2\pi\sqrt{\frac{L}{g}} \\ \frac{1}{2\pi}\cdot T &= \frac{1}{2\pi}\cdot 2\pi\sqrt{\frac{L}{g}} \\ \frac{T}{2\pi} &= \sqrt{\frac{L}{g}} \\ \left(\frac{T}{2\pi}\right)^2 &= \left(\sqrt{\frac{L}{g}}\right)^2 \\ \frac{T^2}{4\pi^2} &= \frac{L}{g} \\ g\cdot\frac{T^2}{4\pi^2} &= g\cdot\frac{L}{g} \\ \frac{T^2 g}{4\pi^2} &= L \end{aligned}$$

Example 6.4.12 The study of black holes has resulted in some interesting mathematics. One fundamental concept about black holes is that there is a distance close enough to the black hole that not even light can escape, called the Schwarzschild radius[2] or the event horizon radius. To find the Schwarzschild radius, R_s, we set the formula for the escape velocity equal to the speed of light, c, and we get $c = \sqrt{\frac{2GM}{R_s}}$ which we need to solve for R_s. Note that G is a constant, and M is the mass of the black hole.

Explanation. We will start by taking the equation $c = \sqrt{\frac{2GM}{R_s}}$ and applying our standard radical-equation-solving techniques. Isolate the radical and square both sides:

$$\begin{aligned} c &= \sqrt{\frac{2GM}{R_s}} \\ c^2 &= \left(\sqrt{\frac{2GM}{R_s}}\right)^2 \end{aligned}$$

$$\begin{aligned} c^2 &= \frac{2GM}{R_s} \\ R_s \cdot c^2 &= R_s \cdot \frac{2GM}{R_s} \\ R_s c^2 &= 2GM \\ \frac{R_s c^2}{c^2} &= \frac{2GM}{c^2} \\ R_s &= \frac{2GM}{c^2} \end{aligned}$$

So, the Schwarzschild radius can be found using the formula $R_s = \frac{2GM}{c^2}$.

6.4.3 Reading Questions

1. What is the basic approach to solving a radical equation?

2. What is it called when doing algebra leads you to find a number that *could* be a solution to an equation, but is not actually a solution?

6.4.4 Exercises

Review and Warmup Solve the equation.

1. $-9n + 8 = -n - 8$

2. $-8p + 4 = -p - 24$

3. $18 = -3(8 - 2x)$

4. $66 = -2(2 - 5y)$

5. $15 = 8 - 7(t - 8)$

6. $144 = 4 - 10(a - 8)$

7. $(x - 1)^2 = 4$

8. $(x + 2)^2 = 81$

9. $x^2 + x - 20 = 0$

10. $x^2 + 19x + 84 = 0$

11. $x^2 + 13x + 12 = -18$

12. $x^2 - 17x + 59 = -1$

Solving Radical Equations Solve the equation.

13. $\sqrt{x} = 12$

14. $\sqrt{x} = 8$

15. $\sqrt{2y} = 8$

16. $\sqrt{5y} = 10$

17. $4\sqrt{r} = 16$

18. $2\sqrt{r} = 10$

19. $-5\sqrt{t} = 15$

20. $-4\sqrt{t} = 20$

21. $-5\sqrt{-5 - x} + 2 = -8$

22. $3\sqrt{3 - x} + 2 = 29$

23. $\sqrt{x - 12} = \sqrt{x} - 2$

24. $\sqrt{y + 3} = \sqrt{y} + 1$

25. $\sqrt{y + 9} = -1 - \sqrt{y}$

26. $\sqrt{r + 9} = -1 - \sqrt{r}$

27. $\sqrt{2r} = 8$

28. $\sqrt{8t} = 3$

29. $\sqrt[3]{t - 5} = 7$

30. $\sqrt[3]{x - 2} = 10$

31. $\sqrt{8x + 5} + 4 = 10$

32. $\sqrt{4x + 9} + 2 = 8$

33. $\sqrt[3]{y - 12} = -5$

34. $\sqrt[3]{y - 8} = 3$

[2] en.wikipedia.org/wiki/Schwarzschild_radius

Solving Radical Equations with Variables

35. Solve the equation for R. Assume that R is positive.

$$Z = \sqrt{L^2 + R^2}$$

R

36. According to the Pythagorean Theorem, the length c of the hypothenuse of a rectangular triangle can be found through the following equation:

$$c = \sqrt{a^2 + b^2}$$

Solve the equation for the length a of one of the triangle's legs.
a

37. In an electric circuit, resonance occurs when the frequency f, inductance L, and capacitance C fulfill the following equation:

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Solve the equation for the inductance L.
The frequency is measured in Hertz, the inductance in Henry, and the capacitance in Farad.
L

38. A pendulum has the length L. The time period T that it takes to once swing back and forth can be found with the following formula:

$$T = 2\pi\sqrt{\frac{L}{32}}$$

Solve the equation for the length L.
The length is measured in feet and the time period in seconds.
L

Radical Equation Applications According to the Pythagorean Theorem, the length c of the hypothenuse of a rectangular triangle can be found through the following equation.

$$c = \sqrt{a^2 + b^2}$$

39. If a rectangular triangle has a hypothenuse of 5 ft and one leg is 4 ft long, how long is the third side of the triangle?
The third side of the triangle is [ ] long.

40. If a rectangular triangle has a hypothenuse of 5 ft and one leg is 4 ft long, how long is the third side of the triangle?
The third side of the triangle is [ ] long.

In a coordinate system, the distance r of a point (x, y) from the origin $(0, 0)$ is given by the following equation.

$$r = \sqrt{x^2 + y^2}$$

41. If a point in a coordinate system is 13 cm away from the origin and its x coordinate is 12 cm, what is its y coordinate? Assume that y is positive.
y

42. If a point in a coordinate system is 13 cm away from the origin and its x coordinate is 12 cm, what is its y coordinate? Assume that y is positive.
y

A pendulum has the length L ft. The time period T that it takes to once swing back and forth is 6 s. Use the following formula to find its length.

$$T = 2\pi\sqrt{\frac{L}{32}}$$

43. The pendulum is $\boxed{}$ long.

44. The pendulum is $\boxed{}$ long.

Challenge Solve for x.

45.

$$\sqrt{1+\sqrt{7}} = \sqrt{2+\sqrt{\frac{1}{\sqrt{x}}}-1}$$

46.

$$\sqrt{1+\sqrt{8}} = \sqrt{2+\sqrt{\frac{1}{\sqrt{x}}}-1}$$

6.5 Radical Expressions and Equations Chapter Review

6.5.1 Square and nth Root Properties

In Section 6.1 we defined the square root $\sqrt{x}$ and nth root $\sqrt[n]{x}$ radicals. When x is positive, the expression $\sqrt[n]{x}$ means a positive number r, where $\overbrace{r \cdot r \cdot \cdots \cdot r}^{n \text{ times}} = x$. The square root $\sqrt{x}$ is just the case where $n = 2$.

When x is negative, $\sqrt[n]{x}$ might not be defined. It depends on whether or not n is an even number. When x is negative and n is odd, $\sqrt[n]{x}$ is a negative number where $\overbrace{r \cdot r \cdot \cdots \cdot r}^{n \text{ times}} = x$.

There are two helpful rules for simplifying radicals.

List 6.5.1: Rules of Radicals for Multiplication and Division

If a and b are positive real numbers, and m is a positive integer , then we have the following rules:

Root of a Product Rule $\sqrt[m]{a \cdot b} = \sqrt[m]{a} \cdot \sqrt[m]{b}$

Root of a Quotient Rule $\sqrt[m]{\frac{a}{b}} = \frac{\sqrt[m]{a}}{\sqrt[m]{b}}$ as long as $b \neq 0$

Checkpoint 6.5.2

a. Simplify $\sqrt{72}$.

b. Simplify $\sqrt[3]{72}$.

c. Simplify $\sqrt{\frac{72}{25}}$.

Explanation.

a.

$$\begin{aligned}\sqrt{72} &= \sqrt{4 \cdot 18}\\ &= \sqrt{4} \cdot \sqrt{18}\\ &= 2\sqrt{18}\\ &= 2\sqrt{9 \cdot 2}\\ &= 2\sqrt{9} \cdot \sqrt{2}\\ &= 2 \cdot 3\sqrt{2}\\ &= 6\sqrt{2}\end{aligned}$$

b.

$$\begin{aligned}\sqrt[3]{72} &= \sqrt[3]{8 \cdot 9}\\ &= \sqrt[3]{8} \cdot \sqrt[3]{9}\\ &= 2\sqrt[3]{9}\end{aligned}$$

c.

$$\begin{aligned}\sqrt{\frac{72}{25}} &= \frac{\sqrt{72}}{\sqrt{25}}\\ &= \frac{6\sqrt{2}}{5}\end{aligned}$$

6.5.2 Rationalizing the Denominator

In Section 6.2 we covered how to rationalize the denominator when it contains a single square root or a binomial with a square root term.

Example 6.5.3 Rationalize the denominator of the expressions.

a. $\frac{12}{\sqrt{3}}$

b. $\frac{\sqrt{5}}{\sqrt{75}}$

Explanation.

a.

$$\begin{aligned}\frac{12}{\sqrt{3}} &= \frac{12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{12\sqrt{3}}{3}\\ &= 4\sqrt{3}\end{aligned}$$

b. First we will simplify $\sqrt{75}$.

$$\begin{aligned}\frac{\sqrt{5}}{\sqrt{75}} &= \frac{\sqrt{5}}{\sqrt{25\cdot 3}}\\ &= \frac{\sqrt{5}}{\sqrt{25}\cdot\sqrt{3}}\\ &= \frac{\sqrt{5}}{5\sqrt{3}}\end{aligned}$$

Now we can rationalize the denominator by multiplying the numerator and denominator by $\sqrt{3}$.

$$\begin{aligned}&= \frac{\sqrt{5}}{5\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{\sqrt{15}}{5\cdot 3}\\ &= \frac{\sqrt{15}}{15}\end{aligned}$$

Example 6.5.4 Rationalize Denominator Using the Difference of Squares Formula. Rationalize the denominator in $\frac{\sqrt{6}-\sqrt{5}}{\sqrt{3}+\sqrt{2}}$.

Explanation. To remove radicals in $\sqrt{3}+\sqrt{2}$ with the difference of squares formula, we multiply it with $\sqrt{3}-\sqrt{2}$.

$$\begin{aligned}\frac{\sqrt{6}-\sqrt{5}}{\sqrt{3}+\sqrt{2}} &= \frac{\sqrt{6}-\sqrt{5}}{\sqrt{3}+\sqrt{2}} \cdot \frac{\left(\sqrt{3}-\sqrt{2}\right)}{\left(\sqrt{3}-\sqrt{2}\right)}\\ &= \frac{\sqrt{6}\cdot\sqrt{3}-\sqrt{6}\cdot\sqrt{2}-\sqrt{5}\cdot\sqrt{3}-\sqrt{5}\cdot-\sqrt{2}}{\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2}\\ &= \frac{\sqrt{18}-\sqrt{12}-\sqrt{15}+\sqrt{10}}{9-4}\\ &= \frac{3\sqrt{2}-2\sqrt{3}-\sqrt{15}+\sqrt{10}}{5}\end{aligned}$$

6.5.3 Radical Expressions and Rational Exponents

In Section 6.3 we learned the rational exponent rule and added it to our list of exponent rules.

Example 6.5.5 Radical Expressions and Rational Exponents. Simplify the expressions using Fact 6.3.2 or Fact 6.3.9.

a. $100^{1/2}$
b. $(-64)^{-1/3}$
c. $-81^{3/4}$
d. $\left(-\frac{1}{27}\right)^{2/3}$

Explanation.

a. $$\begin{aligned} 100^{1/2} &= \left(\sqrt{100}\right) \\ &= 10 \end{aligned}$$

b. $$\begin{aligned} (-64)^{-1/3} &= \frac{1}{(-64)^{1/3}} \\ &= \frac{1}{\left(\sqrt[3]{(-64)}\right)} \\ &= \frac{1}{-4} \end{aligned}$$

c. $$\begin{aligned} -81^{3/4} &= -\left(\sqrt[4]{81}\right)^3 \\ &= -3^3 \\ &= -27 \end{aligned}$$

d. In this problem the negative can be associated with either the numerator or the denominator, but not both. We choose the numerator.

$$\begin{aligned} \left(-\frac{1}{27}\right)^{2/3} &= \left(\sqrt[3]{-\frac{1}{27}}\right)^2 \\ &= \left(\frac{\sqrt[3]{-1}}{\sqrt[3]{27}}\right)^2 \\ &= \left(\frac{-1}{3}\right)^2 \\ &= \frac{(-1)^2}{(3)^2} \\ &= \frac{1}{9} \end{aligned}$$

Example 6.5.6 More Expressions with Rational Exponents. Use exponent properties in List 6.3.14 to simplify the expressions, and write all final versions using radicals.

a. $7z^{5/9}$

b. $\frac{5}{4}x^{-2/3}$

c. $\left(-9q^5\right)^{4/5}$

d. $\sqrt{y^5} \cdot \sqrt[4]{y^2}$

e. $\frac{\sqrt{t^3}}{\sqrt[3]{t^2}}$

f. $\sqrt{\sqrt[3]{x}}$

g. $5\left(4 + a^{1/2}\right)^2$

h. $-6\left(2p^{-5/2}\right)^{3/5}$

Explanation.

a. $7z^{5/9} = 7\sqrt[9]{z^5}$

b. $$\begin{aligned} \frac{5}{4}x^{-2/3} &= \frac{5}{4} \cdot \frac{1}{x^{2/3}} \\ &= \frac{5}{4} \cdot \frac{1}{\sqrt[3]{x^2}} \\ &= \frac{5}{4\sqrt[3]{x^2}} \end{aligned}$$

c. $$\begin{aligned} \left(-9q^5\right)^{4/5} &= (-9)^{4/5} \cdot \left(q^5\right)^{4/5} \\ &= (-9)^{4/5} \cdot q^{5 \cdot 4/5} \\ &= \left(\sqrt[5]{-9}\right)^4 \cdot q^4 \\ &= \left(q\sqrt[5]{-9}\right)^4 \end{aligned}$$

d. $$\begin{aligned} \sqrt{y^5} \cdot \sqrt[4]{y^2} &= y^{5/2} \cdot y^{2/4} \\ &= y^{5/2 + 2/4} \\ &= y^{10/4 + 1/4} \\ &= x^{11/4} \\ &= \sqrt[4]{x^{11}} \end{aligned}$$

e. $$\begin{aligned} \frac{\sqrt{t^3}}{\sqrt[3]{t^2}} &= \frac{t^{3/2}}{t^{2/3}} \\ &= t^{3/2 - 2/3} \\ &= t^{9/6 - 4/6} \\ &= t^{5/6} \\ &= \sqrt[6]{t^5} \end{aligned}$$

f. $$\begin{aligned} \sqrt{\sqrt[3]{x}} &= \sqrt{x^{1/3}} \\ &= \left(x^{1/3}\right)^{1/2} \\ &= x^{1/3 \cdot 1/2} \\ &= x^{1/6} \\ &= \sqrt[6]{x} \end{aligned}$$

g. $$\begin{aligned} 5\left(4 + a^{1/2}\right)^2 &= 5\left(4 + a^{1/2}\right)\left(4 + a^{1/2}\right) \\ &= 5\left(4^2 + 2 \cdot 4 \cdot a^{1/2} + \left(a^{1/2}\right)^2\right) \\ &= 5\left(16 + 8a^{1/2} + a^{1/2 \cdot 2}\right) \\ &= 5\left(16 + 8a^{1/2} + a\right) \\ &= 5\left(16 + 8\sqrt{a} + a\right) \\ &= 80 + 40\sqrt{a} + 5a \end{aligned}$$

h. $$\begin{aligned} -6\left(2p^{-5/2}\right)^{3/5} &= -6 \cdot 2^{3/5} \cdot p^{-5/2 \cdot 3/5} \\ &= -6 \cdot 2^{3/5} \cdot p^{-3/2} \\ &= -\frac{6 \cdot 2^{3/5}}{p^{3/2}} \\ &= -\frac{6\sqrt[5]{2^3}}{\sqrt{p^3}} \\ &= -\frac{6\sqrt[5]{8}}{\sqrt{p^3}} \end{aligned}$$

6.5.4 Solving Radical Equations

In Section 6.4 we covered solving equations that contain a radical. We learned about extraneous solutions and the need to check our solutions.

Example 6.5.7 Solving Radical Equations. Solve for r in $r = 9 + \sqrt{r+3}$.

Explanation. We will isolate the radical first, and then square both sides.

$$\begin{aligned}
r &= 9 + \sqrt{r+3} \\
r - 9 &= \sqrt{r+3} \\
(r-9)^2 &= \left(\sqrt{r+3}\right)^2 \\
r^2 - 18r + 81 &= r + 3 \\
r^2 - 19r + 78 &= 0 \\
(r-6)(r-13) &= 0
\end{aligned}$$

$$\begin{aligned}
r - 6 &= 0 && \text{or } r - 13 && = 0 \\
r &= 6 && \text{or } r && = 13
\end{aligned}$$

Because we squared both sides of an equation, we must check both solutions.

$$\begin{aligned}
6 &\stackrel{?}{=} 9 + \sqrt{6+3} & \qquad 13 &\stackrel{?}{=} 9 + \sqrt{13+3} \\
6 &\stackrel{?}{=} 9 + \sqrt{9} & 13 &\stackrel{?}{=} 9 + \sqrt{16} \\
6 &\stackrel{\text{no}}{=} 9 + 3 & 13 &\stackrel{\checkmark}{=} 9 + 4
\end{aligned}$$

It turns out 6 is an extraneous solution and 13 is a valid solution. So the equation has one solution: 13. The solution set is $\{13\}$.

Example 6.5.8 Solving Radical Equations that Require Squaring Twice. Solve the equation $\sqrt{t+9} = -1 - \sqrt{t}$ for t.

Explanation. We cannot isolate two radicals, so we will simply square both sides, and later try to isolate the remaining radical.

$$\begin{aligned}
\sqrt{t+9} &= -1 - \sqrt{t} \\
\left(\sqrt{t+9}\right)^2 &= \left(-1 - \sqrt{t}\right)^2 \\
t + 9 &= 1 + 2\sqrt{t} + t && \text{after expanding the binomial squared} \\
9 &= 1 + 2\sqrt{t} \\
8 &= 2\sqrt{t} \\
4 &= \sqrt{t} \\
(4)^2 &= \left(\sqrt{t}\right)^2 \\
16 &= t
\end{aligned}$$

Because we squared both sides of an equation, we must check the solution by substituting 16 into $\sqrt{t+9} = -1 - \sqrt{t}$, and we have:

$$\sqrt{t+9} = -1 - \sqrt{t}$$

$$\sqrt{16+9} \stackrel{?}{=} -1 - \sqrt{16}$$
$$\sqrt{25} \stackrel{?}{=} -1 - 4$$
$$5 \stackrel{\text{no}}{=} -5$$

Our solution did not check so there is no solution to this equation. The solution set is the empty set, which can be denoted { } or $\emptyset$.

6.5.5 Exercises

Square Root and nth Root Evaluate the following.

1. $\sqrt{\frac{1}{100}}$ **2.** $\sqrt{\frac{4}{121}}$ **3.** $-\sqrt{16}$ **4.** $-\sqrt{25}$

Simplify the radical expression or state that it is `not a real number`.

5. $\frac{\sqrt{48}}{\sqrt{3}}$ **6.** $\frac{\sqrt{32}}{\sqrt{2}}$ **7.** $\sqrt{250}$ **8.** $\sqrt{99}$

Simplify the expression.

9. $9\sqrt{13} \cdot 9\sqrt{121}$ **10.** $9\sqrt{3} \cdot 7\sqrt{4}$ **11.** $\sqrt{\frac{5}{2}} \cdot \sqrt{\frac{7}{2}}$ **12.** $\sqrt{\frac{7}{3}} \cdot \sqrt{\frac{1}{3}}$

Simplify the expression.

13. $13\sqrt{10} - 14\sqrt{10}$ **14.** $14\sqrt{5} - 15\sqrt{5}$ **15.** $\sqrt{180} + \sqrt{45}$ **16.** $\sqrt{80} + \sqrt{125}$

17. Simplify $\sqrt[6]{64}$. **18.** Simplify $\sqrt[3]{64}$. **19.** Simplify $\sqrt[3]{-8}$. **20.** Simplify $\sqrt[3]{-8}$.

21. Simplify $\sqrt[4]{-16}$. **22.** Simplify $\sqrt[4]{-81}$. **23.** Simplify $\sqrt[4]{144}$. **24.** Simplify $\sqrt[3]{135}$.

25. Simplify $\sqrt[3]{\frac{11}{8}}$. **26.** Simplify $\sqrt[6]{\frac{9}{64}}$. **27.** Simplify $\sqrt[3]{\frac{40}{27}}$. **28.** Simplify $\sqrt[3]{\frac{56}{125}}$.

Rationalizing the Denominator Rationalize the denominator and simplify the expression.

29. $\frac{2}{\sqrt{252}}$ **30.** $\frac{6}{\sqrt{112}}$ **31.** $\sqrt{\frac{2}{27}}$ **32.** $\sqrt{\frac{5}{112}}$

33. $\frac{6}{\sqrt{15}+8}$ **34.** $\frac{7}{\sqrt{7}+4}$ **35.** $\frac{\sqrt{5}-13}{\sqrt{13}+3}$ **36.** $\frac{\sqrt{3}-14}{\sqrt{7}+10}$

Radical Expressions and Rational Exponents Without using a calculator, evaluate the expression.

37. $125^{-\frac{2}{3}}$ **38.** $8^{-\frac{5}{3}}$ **39.** $\left(\frac{1}{81}\right)^{-\frac{3}{4}}$ **40.** $\left(\frac{1}{9}\right)^{-\frac{3}{2}}$

41. $\sqrt[3]{125^2}$ **42.** $\sqrt[4]{81^3}$ **43.** $\sqrt[5]{1024}$ **44.** $\sqrt[3]{64}$

Use rational exponents to write the expression.

45. $\sqrt[5]{b}$ **46.** $\sqrt{c}$ **47.** $\sqrt[5]{8x+7}$ **48.** $\sqrt[4]{5z+1}$

Convert the expression to radical notation.

49. $t^{\frac{2}{3}}$ **50.** $r^{\frac{4}{5}}$ **51.** $m^{\frac{5}{4}}$ **52.** $r^{\frac{2}{3}}$ **53.** $5^{\frac{1}{5}}a^{\frac{4}{5}}$ **54.** $13^{\frac{1}{4}}b^{\frac{3}{4}}$

Simplify the expression, answering with rational exponents and not radicals.

55. $\sqrt[11]{c}\ \sqrt[11]{c}$ **56.** $\sqrt[9]{x}\ \sqrt[9]{x}$ **57.** $\sqrt[5]{32z^2}$

58. $\sqrt[3]{125t^5}$ **59.** $\dfrac{\sqrt{16r}}{\sqrt[10]{r^3}}$ **60.** $\dfrac{\sqrt{36m}}{\sqrt[10]{m^3}}$

61. $\sqrt{n}\cdot\sqrt[6]{n^5}$ **62.** $\sqrt{a}\cdot\sqrt[10]{a^3}$

Solving Radical Equations Solve the equation.

63. $t=\sqrt{t-3}+5$ **64.** $t=\sqrt{t-1}+3$ **65.** $\sqrt{x+9}=\sqrt{x}+1$ **66.** $\sqrt{x+8}=\sqrt{x}+2$

67. $\sqrt{y}+110=y$ **68.** $\sqrt{y}+56=y$ **69.** $r=\sqrt{r+4}+16$ **70.** $r=\sqrt{r+2}+88$

71. $\sqrt{52-t}=t+4$ **72.** $\sqrt{17-t}=t+3$

According to the Pythagorean Theorem, the length c of the hypothenuse of a rectangular triangle can be found through the following equation.

$$c=\sqrt{a^2+b^2}$$

73. If a rectangular triangle has a hypothenuse of 41 ft and one leg is 40 ft long, how long is the third side of the triangle?
The third side of the triangle is ______ long.

74. If a rectangular triangle has a hypothenuse of 17 ft and one leg is 15 ft long, how long is the third side of the triangle?
The third side of the triangle is ______ long.

75. A pendulum has the length L ft. The time period T that it takes to once swing back and forth is 2 s. Use the following formula to find its length.

$$T=2\pi\sqrt{\frac{L}{32}}$$

The pendulum is ______ long.

76. A pendulum has the length L ft. The time period T that it takes to once swing back and forth is 4 s. Use the following formula to find its length.

$$T=2\pi\sqrt{\frac{L}{32}}$$

The pendulum is ______ long.

Chapter 7

Solving Quadratic Equations

7.1 Solving Quadratic Equations by Using a Square Root

In this section, we will learn how to solve some specific types of quadratic equations using the square root property. We will also learn how to use the Pythagorean Theorem to find the length of one side of a right triangle when the other two lengths are known.

7.1.1 Solving Quadratic Equations Using the Square Root Property

When we learned how to solve linear equations, we used inverse operations to isolate the variable. For example, we use subtraction to remove an unwanted term that is added to one side of a linear equation. We can't quite do the same thing with squaring and using square roots, but we can do something very similar. Taking the square root is the inverse of squaring *if you happen to know the original number was positive*. In general, we have to remember that the original number may have been negative, and that usually leads to *two* solutions to a quadratic equation.

For example, if $x^2 = 9$, we can think of undoing the square with a square root, and $\sqrt{9} = 3$. However, there are *two* numbers that we can square to get 9: -3 and 3. So we need to include both solutions. This brings us to the Square Root Property.

Fact 7.1.2 Square Root Property. *If k is positive, and $x^2 = k$ then*

$$x = -\sqrt{k} \qquad \textit{or} \qquad x = \sqrt{k}.$$

It is common to write $x = \pm\sqrt{k}$ for short, but it is important to remember that this means x could possibly be one of two things, not that x is two things at the same time. The positive solution, $\sqrt{k}$, is called the ***principal root*** *of k.*

Example 7.1.3 Solve for y in $y^2 = 49$.

Explanation.

$$\begin{aligned} y^2 &= 49 \\ y &= \pm\sqrt{49} \end{aligned}$$

$$y = \pm 7$$

$$y = -7 \qquad \text{or} \qquad y = 7$$

To check these solutions, we will substitute -7 and 7 for y in the original equation:

$$\begin{aligned} y^2 &= 49 \\ (-7)^2 &\stackrel{?}{=} 49 \\ 49 &\stackrel{\checkmark}{=} 49 \end{aligned} \qquad\qquad \begin{aligned} y^2 &= 49 \\ (7)^2 &\stackrel{?}{=} 49 \\ 49 &\stackrel{\checkmark}{=} 49 \end{aligned}$$

The solution set is $\{-7, 7\}$.

Remark 7.1.4 Every solution to a quadratic equation can be checked, as shown in Example 7.1.3. In general, the process of checking is omitted from this section.

Checkpoint 7.1.5 Solve for z in $4z^2 - 81 = 0$.

Explanation. Before we use the square root property we need to isolate the squared quantity.

$$\begin{aligned} 4z^2 - 81 &= 0 \\ 4z^2 &= 81 \\ z^2 &= \frac{81}{4} \\ z &= \pm\sqrt{\frac{81}{4}} \\ z &= \pm\frac{9}{2} \end{aligned}$$

$$z = -\frac{9}{2} \quad \text{or} \quad z = \frac{9}{2}$$

The solution set is $\left\{-\frac{9}{2}, \frac{9}{2}\right\}$.

We can also use the square root property to solve an equation that has a squared expression (as opposed to just having a squared variable).

Example 7.1.6 Solve for p in $50 = 2(p-1)^2$.

Explanation. It's important here to suppress any urge you may have to expand the squared binomial. We begin by isolating the squared expression.

$$\begin{aligned} 50 &= 2(p-1)^2 \\ \frac{50}{2} &= \frac{2(p-1)^2}{2} \\ 25 &= (p-1)^2 \end{aligned}$$

Now that we have the squared expression isolated, we can use the square root property.

$$\begin{aligned} p - 1 &= \pm\sqrt{25} \\ p - 1 &= \pm 5 \end{aligned}$$

$$p = \pm 5 + 1$$

$$\begin{array}{ccc} p = -5 + 1 & \text{or} & p = 5 + 1 \\ p = -4 & \text{or} & p = 6 \end{array}$$

The solution set is $\{-4, 6\}$.

This method of solving quadratic equations is not limited to equations that have rational solutions, or when the radicands are perfect squares. Here are a few examples where the solutions are irrational numbers.

Checkpoint 7.1.7 Solve for q in $(q+2)^2 - 12 = 0$.

Explanation. It's important here to suppress any urge you may have to expand the squared binomial.

$$\begin{aligned} (q+2)^2 - 12 &= 0 \\ (q+2)^2 &= 12 \\ q + 2 &= \pm\sqrt{12} \\ q + 2 &= \pm\sqrt{4 \cdot 3} \\ q + 2 &= \pm 2\sqrt{3} \\ q &= \pm 2\sqrt{3} - 2 \end{aligned}$$

$$q = -2\sqrt{3} - 2 \quad \text{or} \quad q = 2\sqrt{3} - 2$$

The solution set is $\left\{-2\sqrt{3} - 2, 2\sqrt{3} - 2\right\}$.

To check the solution, we would replace q with each of $-2\sqrt{3} - 2$ and $2\sqrt{3} - 2$ in the original equation, as shown here:

$$\begin{aligned} \left(\left(-2\sqrt{3} - 2\right) + 2\right)^2 - 12 &\stackrel{?}{=} 0 & \left(\left(2\sqrt{3} - 2\right) + 2\right)^2 - 12 &\stackrel{?}{=} 0 \\ \left(-2\sqrt{3}\right)^2 - 12 &\stackrel{?}{=} 0 & \left(2\sqrt{3}\right)^2 - 12 &\stackrel{?}{=} 0 \\ (-2)^2\left(\sqrt{3}\right)^2 - 12 &\stackrel{?}{=} 0 & (2)^2\left(\sqrt{3}\right)^2 - 12 &\stackrel{?}{=} 0 \\ (4)(3) - 12 &\stackrel{?}{=} 0 & (4)(3) - 12 &\stackrel{?}{=} 0 \\ 12 - 12 &\stackrel{\checkmark}{=} 0 & 12 - 12 &\stackrel{\checkmark}{=} 0 \end{aligned}$$

Note that these simplifications relied on exponent rules and the multiplicative property of square roots.

Remember that if a square root is in a denominator then we may be expected to rationalize it as in Section 6.2. We will rationalize the denominator in the next example.

Example 7.1.8 Solve for n in $2n^2 - 3 = 0$.

Explanation.

$$\begin{aligned} 2n^2 - 3 &= 0 \\ 2n^2 &= 3 \\ n^2 &= \frac{3}{2} \end{aligned}$$

$$n = \pm\sqrt{\frac{3}{2}}$$
$$n = \pm\sqrt{\frac{6}{4}}$$
$$n = \pm\frac{\sqrt{6}}{2}$$

$$n = -\frac{\sqrt{6}}{2} \qquad \text{or} \qquad n = \frac{\sqrt{6}}{2}$$

The solution set is $\left\{-\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}\right\}$.

When the radicand is a negative number, there is no real solution. Here is an example of an equation with no real solution.

Example 7.1.9 Solve for x in $x^2 + 49 = 0$.
Explanation.

$$x^2 + 49 = 0$$
$$x^2 = -49$$

Since $\sqrt{-49}$ is not a real number, we say the equation has no real solution.

7.1.2 The Pythagorean Theorem

Right triangles have an important property called the **Pythagorean Theorem**.

Theorem 7.1.10 The Pythagorean Theorem. *For any right triangle, the lengths of the three sides have the following relationship:* $a^2 + b^2 = c^2$. *The sides* a *and* b *are called* ***legs*** *and the longest side* c *is called the* ***hypotenuse***.

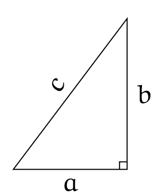


Figure 7.1.11: In a right triangle, the length of its three sides satisfy the equation $a^2 + b^2 = c^2$

Example 7.1.12 Keisha is designing a wooden frame in the shape of a right triangle, as shown in Figure 7.1.13. The legs of the triangle are 3 ft and 4 ft. How long should she make the diagonal side? Use the Pythagorean Theorem to find the length of the hypotenuse.

According to Pythagorean Theorem, we have:

$$\begin{aligned} c^2 &= a^2 + b^2 \\ c^2 &= 3^2 + 4^2 \\ c^2 &= 9 + 16 \\ c^2 &= 25 \end{aligned}$$

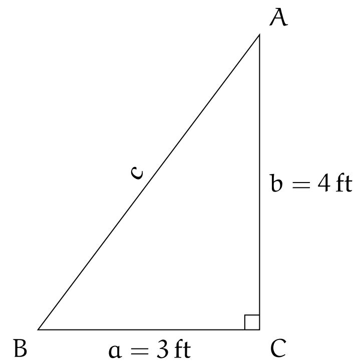


Figure 7.1.13

Now we have a quadratic equation that we need to solve. We need to find the number that has a square of 25. That is what the square root operation does.

$$\begin{aligned} c &= \sqrt{25} \\ c &= 5 \end{aligned}$$

The diagonal side Keisha will cut is 5 ft long.

Note that -5 is also a solution of $c^2 = 25$ because $(-5)^2 = 25$ but a length cannot be a negative number. We will need to include both solutions when they are relevant.

Example 7.1.14 A 16.5ft ladder is leaning against a wall. The distance from the base of the ladder to the wall is 4.5 feet. How high on the wall does the ladder reach?
The Pythagorean Theorem says:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 4.5^2 + b^2 &= 16.5^2 \\ 20.25 + b^2 &= 272.25 \end{aligned}$$

Now we need to isolate b^2 in order to solve for b:

$$\begin{aligned} 20.25 + b^2 - 20.25 &= 272.25 - 20.25 \\ b^2 &= 252 \end{aligned}$$

We use the square root property. Because this is a geometric situation we only need to use the principal root:

$$b = \sqrt{252}$$

Now simplify this radical and then approximate it:

$$\begin{aligned} b &= \sqrt{36 \cdot 7} \\ b &= 6\sqrt{7} \\ b &\approx 15.87 \end{aligned}$$

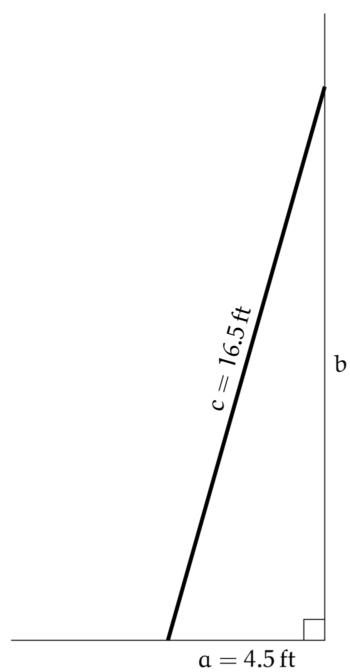


Figure 7.1.15: Leaning Ladder

The ladder reaches about 15.87 feet high on the wall.

Here are some more examples using the Pythagorean Theorem to find sides of triangles. Note that in many contexts, only the principal root will be relevant.

Example 7.1.16 Find the missing length in this right triangle.

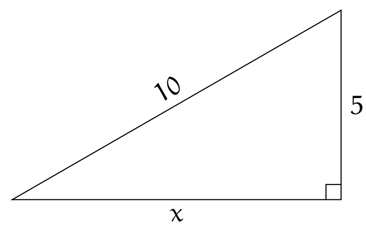


Figure 7.1.17: A Right Triangle

Explanation. We will use the Pythagorean Theorem to solve for x:

$$\begin{aligned} 5^2 + x^2 &= 10^2 \\ 25 + x^2 &= 100 \\ x^2 &= 75 \\ x &= \sqrt{75} && \text{(no need to consider } -\sqrt{75} \text{ in this context)} \\ x &= \sqrt{25 \cdot 3} \\ x &= 5\sqrt{3} \end{aligned}$$

The missing length is $x = 5\sqrt{3}$.

Example 7.1.18 Sergio is designing a 50-inch TV, which implies the diagonal of the TV's screen will be 50 inches long. He needs the screen's width to height ratio to be 4 : 3. Find the TV screen's width and height.

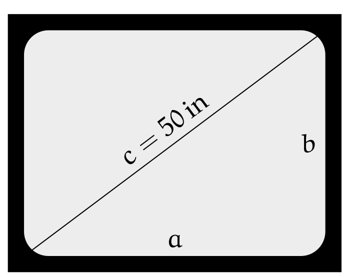


Figure 7.1.19: Pythagorean Theorem Problem

Explanation. Let's let x represent the height of the screen, in inches. Since the screen's width to height ratio will be 4 : 3, then the width is $\frac{4}{3}$ times as long as the height, or $\frac{4}{3}x$ inches. We will draw a diagram.

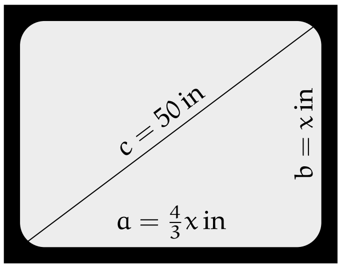


Figure 7.1.20: Pythagorean Theorem Problem

Now we can use the Pythagorean Theorem to write and solve an equation:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ \left(\frac{4}{3}x\right)^2 + x^2 &= 50^2 \\ \frac{16}{9}x^2 + \frac{9}{9}x^2 &= 2500 \\ \frac{25}{9}x^2 &= 2500 \\ \frac{9}{25} \cdot \frac{25}{9}x^2 &= \frac{9}{25} \cdot 2500 \\ x^2 &= 900 \\ x &= 30 \end{aligned}$$

Since the screen's height is 30 inches, its width is $\frac{4}{3}x = \frac{4}{3}(30) = 40$ inches.

Example 7.1.21 Luca wanted to make a bench.

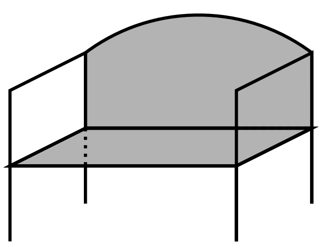


Figure 7.1.22: Sketch of a Bench with Highlighted Back

He wanted the top of the bench back to be a perfect portion of a circle, in the shape of an arc, as in Figure 7.1.23. (Note that this won't be a half-circle, just a small portion of a circular edge.) He started with a rectangular board 6 inches wide and 48 inches long, and a piece of string, like a compass, to draw a circular arc on the board. How long should the string be so that it can be swung round to draw the arc?

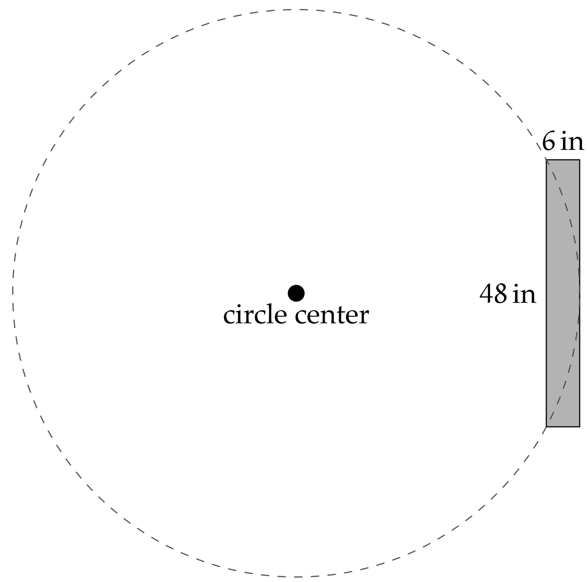


Figure 7.1.23: Bench Back Board

Explanation. Let's first define x to be the radius of the circle in question, in inches. The circle should go through the bottom corners of the board and just barely touch the top of the board. That means that the line from the middle of the bottom of the board to the center of the circle will be 6 inches shorter than the radius.

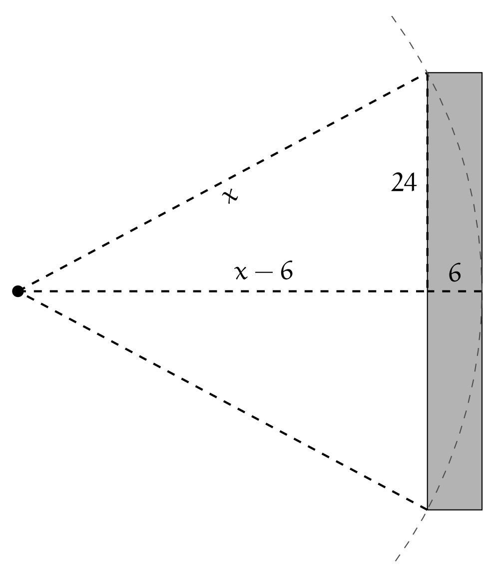


Figure 7.1.24: Bench Back Board Diagram

Now we can set up the Pythagorean Theorem based on the scenario. The equation $a^2 + b^2 = c^2$ turns into…

$$(x-6)^2 + 24^2 = x^2$$
$$x^2 - 12x + 36 + 576 = x^2$$

$$-12x + 612 = 0$$

Note that at this point the equation is no longer quadratic! Solve the linear equation by isolating x

$$-12x = -612$$
$$x = 51$$

So, the circle radius required is 51 inches. Luca found a friend to stand on the string end and drew a circular segment on the board to great effect.

7.1.3 Reading Questions

1. Typically, how many solutions can there be with a quadratic equation?

2. When you see a $\pm$ sign, as in $x = \pm 2$, is that saying that x is both -2 and 2?

3. Have you memorized the Pythagorean Theorem? State the formula.

7.1.4 Exercises

Solving Quadratic Equations with the Square Root Property Solve the equation.

1. $x^2 = 25$

2. $x^2 = 36$

3. $x^2 = \frac{1}{64}$

4. $x^2 = \frac{1}{81}$

5. $x^2 = 12$

6. $x^2 = 20$

7. $x^2 = 67$

8. $x^2 = 5$

9. $3x^2 = 27$

10. $4x^2 = 100$

11. $x^2 = \frac{64}{9}$

12. $x^2 = \frac{25}{64}$

13. $4x^2 = 121$

14. $36x^2 = 49$

15. $7x^2 - 59 = 0$

16. $59x^2 - 67 = 0$

17. $2 - 7x^2 = -3$

18. $4 - 7x^2 = 2$

19. $53x^2 + 17 = 0$

20. $61x^2 + 23 = 0$

21. $(x+1)^2 = 9$

22. $(x+3)^2 = 100$

23. $(2x+8)^2 = 49$

24. $(8x+10)^2 = 9$

25. $9 - 5(t+1)^2 = 4$

26. $10 - 3(x+1)^2 = -2$

27. $(x-10)^2 = 11$

28. $(x+4)^2 = 17$

29. $(y+2)^2 = 45$

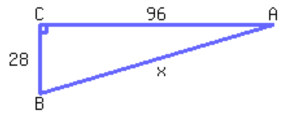
30. $(r-4)^2 = 98$

31. $-4 = 8 - (r-4)^2$

32. $-1 = 62 - (t+5)^2$

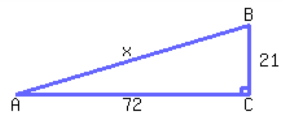
Pythagorean Theorem Applications

33.



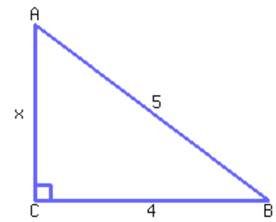
Find the value of x.

34.



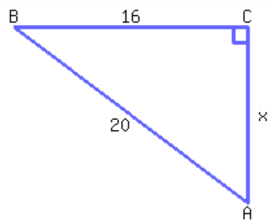
Find the value of x.

35.



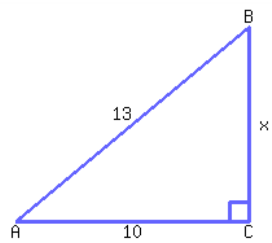
Find the value of x.

36.



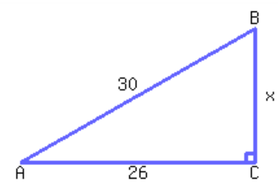
Find the value of x.

37.



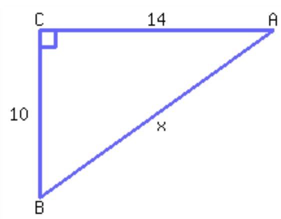
Find the value of x, accurate to at least two decimal places.

38.



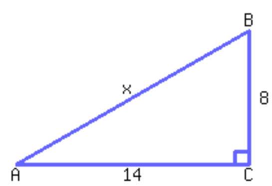
Find the value of x, accurate to at least two decimal places.

39.



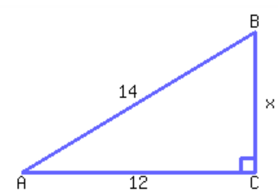
Find the exact value of x.

40.



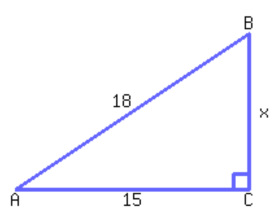
Find the exact value of x.

41.



Find the exact value of x.

42.



Find the exact value of x.

43. Kandace is designing a rectangular garden. The garden's diagonal must be 37.7 feet, and the ratio between the garden's base and height must be 12 : 5. Find the length of the garden's base and height.

The garden's base is ________ feet and its height is ________.

44. Brandon is designing a rectangular garden. The garden's diagonal must be 30.6 feet, and the ratio between the garden's base and height must be 15 : 8. Find the length of the garden's base and height.

The garden's base is ____ feet and its height is ____.

45. Peter is designing a rectangular garden. The garden's base must be 8.4 feet, and the ratio between the garden's hypotenuse and height must be 13 : 5. Find the length of the garden's hypotenuse and height.

The garden's hypotenuse is ____ feet and its height is ____.

46. Gustav is designing a rectangular garden. The garden's base must be 54 feet, and the ratio between the garden's hypotenuse and height must be 17 : 8. Find the length of the garden's hypotenuse and height.

The garden's hypotenuse is ____ feet and its height is ____.

Challenge

47. Imagine that you are in Math Land, where roads are perfectly straight, and Mathlanders can walk along a perfectly straight line between any two points. One day, you bike 4 miles west, 3 miles north, and 6 miles east. Then, your bike gets a flat tire and you have to walk home. How far do you have to walk?You have to walk ____ miles home.

7.2 The Quadratic Formula

We have learned how to solve certain quadratic equations using the square root property. In this section, we will learn another method, the quadratic formula.

7.2.1 Solving Quadratic Equations with the Quadratic Formula

The standard form for a quadratic equation is

$$ax^2 + bx + c = 0$$

where a is some nonzero number.

When $b = 0$ and the equation's form is $ax^2 + c = 0$, then we can simply use the square root property to solve it. For example, $x^2 - 4 = 0$ leads to $x^2 = 4$, which leads to $x = \pm 2$, a solution set of $\{-2, 2\}$.

But can we solve equations where $b \neq 0$? A general method for solving a quadratic equation is to use what is known as the quadratic formula.

Fact 7.2.2 The Quadratic Formula. *For any quadratic equation* $ax^2 + bx + c = 0$ *where* $a \neq 0$, *the solutions are given by*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

As we have seen from solving quadratic equations, there can be at most two solutions. Both of the solutions are included in the quadratic formula with the $\pm$ symbol. We could write the two solutions separately:

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{or} \qquad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

This method for solving quadratic equations will work to solve *every* quadratic equation. It is most helpful when $b \neq 0$.

Example 7.2.3 Linh is in a physics class that launches a tennis ball from a rooftop that is 90.2 feet above the ground. They fire it directly upward at a speed of 14.4 feet per second and measure the time it takes for the ball to hit the ground below. We can model the height of the tennis ball, h, in feet, with the quadratic equation $h = -16x^2 + 14.4x + 90.2$, where x represents the time in seconds after the launch. According to the model, when should the ball hit the ground? Round the time to one decimal place.

The ground has a height of 0 feet. Substituting 0 for h in the equation, we have this quadratic equation:

$$0 = -16x^2 + 14.4x + 90.2$$

We cannot solve this equation with the square root property, so we will use the quadratic formula. First we will identify that $a = -16$, $b = 14.4$ and $c = 90.2$, and substitute them into the formula:

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-(14.4) \pm \sqrt{(14.4)^2 - 4(-16)(90.2)}}{2(-16)} \\
x &= \frac{-14.4 \pm \sqrt{207.36 - (-5772.8)}}{-32}
\end{aligned}$$

$$x = \frac{-14.4 \pm \sqrt{207.36 + 5772.8}}{-32}$$
$$x = \frac{-14.4 \pm \sqrt{5980.16}}{-32}$$

These are the exact solutions but because we have a context we want to approximate the solutions with decimals.

$$x \approx -1.966 \text{ or } x \approx 2.866$$

We don't use the negative solution because a negative time does not make sense in this context. The ball will hit the ground approximately 2.9 seconds after it is launched.

The quadratic formula can be used to solve any quadratic equation, but it requires that you don't make *any* slip-up with remembering the formula, that you correctly identify a, b, and c, and that you don't make any arithmetic mistakes when you calculate and simplify. We recommend that you always check whether you can use the square root property before using the quadratic formula. Here is another example.

Example 7.2.4 Solve for x in $2x^2 - 9x + 5 = 0$.

Explanation. First, we check and see that we cannot use the square root property (because $b \neq 0$) so we will use the quadratic formula. Next we identify that $a = 2$, $b = -9$ and $c = 5$. We substitute them into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(2)(5)}}{2(2)}$$
$$x = \frac{9 \pm \sqrt{81 - 40}}{4}$$
$$x = \frac{9 \pm \sqrt{41}}{4}$$

This is fully simplified because we cannot simplify $\sqrt{41}$ or reduce the fraction. The solution set is $\left\{\frac{9-\sqrt{22}}{4}, \frac{9+\sqrt{22}}{4}\right\}$. We do not have a context here so we leave the solutions in their exact form.

When a quadratic equation is not in standard form we must convert it before we can identify the values of a, b and c. We will show that in the next example.

Example 7.2.5 Solve for x in $x^2 = -10x - 3$.

Explanation. First, we convert the equation into standard form by adding $10x$ and 3 to each side of the equation:

$$x^2 + 10x + 3 = 0$$

Next, we check that we cannot use the square root property so we will use the quadratic formula. We identify that $a = 1$, $b = 10$ and $c = 3$. We substitute them into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-10 \pm \sqrt{(10)^2 - 4(1)(3)}}{2(1)}$$
$$x = \frac{-10 \pm \sqrt{100 - 12}}{2}$$
$$x = \frac{-10 \pm \sqrt{88}}{2}$$

We notice that the radical can be simplified:

$$x = \frac{-10 \pm 2\sqrt{22}}{2}$$
$$x = \frac{-10}{2} \pm \frac{2\sqrt{22}}{2}$$
$$x = -5 \pm \sqrt{22}$$

The solution set is $\{-5 - \sqrt{22}, -5 + \sqrt{22}\}$.

Remark 7.2.6 The irrational solutions to quadratic equations can be checked, although doing so can sometimes involve a lot of simplification and is not shown throughout this section. As an example, to check the solution of $-5 + \sqrt{22}$ from Example 7.2.5, we would replace x with $-5 + \sqrt{22}$ and check that the two sides of the equation are equal. This check is shown here:

$$\begin{aligned} x^2 &= -10x - 3 \\ (-5 + \sqrt{22})^2 &\stackrel{?}{=} -10(-5 + \sqrt{22}) - 3 \\ (-5)^2 + 2(-5)(\sqrt{22}) + (\sqrt{22})^2 &\stackrel{?}{=} -10(-5 + \sqrt{22}) - 3 \\ 25 - 10\sqrt{22} + 22 &\stackrel{?}{=} 50 - 10\sqrt{22} - 3 \\ 47 - 10\sqrt{22} &\stackrel{\checkmark}{=} 47 - 10\sqrt{22} \end{aligned}$$

When the radicand from the quadratic formula, $b^2 - 4ac$, which is called the **discriminant**, is a negative number, the quadratic equation has no real solution. Example 7.2.7 shows what happens in this case.

Example 7.2.7 Solve for y in $y^2 - 4y + 8 = 0$.

Explanation. Identify that $a = 1$, $b = -4$ and $c = 8$. We will substitute them into the quadratic formula:

$$\begin{aligned} y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(8)}}{2(1)} \\ &= \frac{4 \pm \sqrt{16 - 32}}{2} \\ &= \frac{4 \pm \sqrt{-16}}{2} \end{aligned}$$

The square root of a negative number is not a real number, so we will simply state that this equation has no real solutions.

Sometimes a radical equation gives rise to a quadratic equation, and the quadratic formula is useful.

Example 7.2.8 Solve for z in $\sqrt{z}+2=z$.
Explanation. We will isolate the radical first, and then square both sides.

$$\begin{aligned}
\sqrt{z}+2 &= z\\
\sqrt{z} &= z-2\\
\left(\sqrt{z}\right)^2 &= (z-2)^2\\
z &= z^2-4z+4\\
0 &= z^2-5z+4\\
z &= \frac{5\pm\sqrt{(-5)^2-4(1)(4)}}{2}\\
&= \frac{5\pm\sqrt{25-16}}{2}\\
&= \frac{5\pm\sqrt{9}}{2}\\
&= \frac{5\pm 3}{2}
\end{aligned}$$

$$\begin{aligned}
z &= \frac{5-3}{2} &\quad \text{or} \quad && z &= \frac{5+3}{2}\\
z &= 1 &\quad \text{or} \quad && z &= 4
\end{aligned}$$

Because we squared both sides of an equation, we must check both solutions.

$$\begin{aligned}
\sqrt{1}+2 &\stackrel{?}{=} 1 &\qquad \sqrt{4}+2 &\stackrel{?}{=} 4\\
1+2 &\stackrel{\text{no}}{=} 1 &\qquad 2+2 &\stackrel{\checkmark}{=} 4
\end{aligned}$$

It turned out that 1 is an extraneous solution, but 4 is a valid solution. So the equation has one solution: 4. The solution set is $\{4\}$.

Example 7.2.9 Solve the equation $\sqrt{2n-6}=1+\sqrt{n-2}$ for n.
Explanation. We cannot isolate two radicals, so we will simply square both sides, and later try to isolate the remaining radical.

$$\begin{aligned}
\sqrt{2n-6} &= 1+\sqrt{n-2}\\
\left(\sqrt{2n-6}\right)^2 &= \left(1+\sqrt{n-2}\right)^2\\
2n-6 &= 1^2+2\sqrt{n-2}+\left(\sqrt{n-2}\right)^2\\
2n-6 &= 1+2\sqrt{n-2}+n-2\\
2n-6 &= 2\sqrt{n-2}+n-1\\
n-5 &= 2\sqrt{n-2}
\end{aligned}$$

Note here that we can leave the factor of 2 next to the radical. We will square the 2 also.

$$\begin{aligned}(n-5)^2 &= \left(2\sqrt{n-2}\right)^2\\ n^2-10n+25 &= 4(n-2)\\ n^2-10n+25 &= 4n-8\\ n^2-14n+33 &= 0\end{aligned}$$

$$\begin{aligned}n &= \frac{14\pm\sqrt{14^2-4(1)(33)}}{2}\\ &= \frac{14\pm\sqrt{196-132}}{2}\\ &= \frac{14\pm\sqrt{64}}{2}\\ &= \frac{14\pm 8}{2}\end{aligned}$$

$$\begin{aligned}n &= \frac{14-8}{2} &&\text{or} && n=\frac{14+8}{2}\\ n &= 3 &&\text{or} && n=11\end{aligned}$$

So our two potential solutions are 3 and 11. We should now verify that they truly are solutions.

$$\begin{aligned}\sqrt{2(3)-6} &\stackrel{?}{=} 1+\sqrt{3-2} & \sqrt{2(11)-6} &\stackrel{?}{=} 1+\sqrt{11-2}\\ \sqrt{6-6} &\stackrel{?}{=} 1+\sqrt{1} & \sqrt{22-6} &\stackrel{?}{=} 1+\sqrt{9}\\ \sqrt{0} &\stackrel{?}{=} 1+1 & \sqrt{16} &\stackrel{?}{=} 1+3\\ 0 &\stackrel{\text{no}}{=} 2 & 4 &\stackrel{\checkmark}{=} 4\end{aligned}$$

So, 11 is the only solution. The solution set is $\{11\}$.

7.2.2 Reading Questions

1. What is the formula for the discriminant? (The part of the quadratic formula inside the radical.)

2. Are there any kinds of quadratic equations where the quadratic formula is not the best tool to use?

3. Given a quadratic euqation, will the quadratic formula always show you two real solutions?

7.2.3 Exercises

Review and Warmup

1. Evaluate $\dfrac{-5A+5B+7}{6A-9B}$ for $A=2$ and $B=-4$.

2. Evaluate $\dfrac{-6C-5c+9}{-6C-3c}$ for $C=-10$ and $c=-9$.

3. Evaluate the expression $\frac{1}{3}(x+4)^2 - 2$ when $x = -7$.

4. Evaluate the expression $\frac{1}{3}(x+4)^2 - 7$ when $x = -7$.

5. Evaluate the expression $-16t^2 + 64t + 128$ when $t = 3$.

6. Evaluate the expression $-16t^2 + 64t + 128$ when $t = -5$.

7. Evaluate the expression x^2:

a. For $x = 7$.

b. For $x = -2$.

8. Evaluate the expression y^2:

a. For $y = 4$.

b. For $y = -6$.

9. Evaluate each algebraic expression for the given value(s):
$\frac{\sqrt{x}}{y} - \frac{y}{x}$, for $x = 25$ and $y = 10$:

10. Evaluate each algebraic expression for the given value(s):
$\frac{y}{4x} - \frac{\sqrt{x}}{3y}$, for $x = 25$ and $y = -4$:

Solve Quadratic Equations Using the Quadratic Formula Solve the equation.

11. $x^2 + 7x + 1 = 0$

12. $x^2 + 8x + 11 = 0$

13. $20x^2 + 56x + 15 = 0$

14. $10x^2 + 39x + 35 = 0$

15. $x^2 = x + 1$

16. $x^2 = 5x - 5$

17. $x^2 + 3x - 9 = 0$

18. $x^2 - 9x + 9 = 0$

19. $2x^2 + 3x - 1 = 0$

20. $3x^2 - x - 1 = 0$

21. $4x^2 - 10x - 5 = 0$

22. $7x^2 - 2x - 1 = 0$

23. $5x^2 - 9x + 6 = 0$

24. $3x^2 + 3x + 3 = 0$

Solve Quadratic Equations Using an Appropriate Method Solve the equation.

25. $3x^2 - 27 = 0$

26. $4x^2 - 16 = 0$

27. $25x^2 - 81 = 0$

28. $36x^2 - 25 = 0$

29. $4 - 7r^2 = 1$

30. $0 - 3r^2 = -7$

31. $x^2 + 5x = 24$

32. $x^2 + 4x = 60$

33. $(x-9)^2 = 64$

34. $(x-7)^2 = 16$

35. $x^2 = -9x - 16$

36. $x^2 = -3x + 2$

37. $3x^2 = x + 1$

38. $2x^2 = -(5x + 1)$

39. $22 - 4(r+5)^2 = 6$

40. $23 - 2(t-8)^2 = 5$

Radical Equations That Give Rise to Quadratic Equations Solve the equation.

41. $\sqrt{t + 72} = t$

42. $\sqrt{2x + 15} = x$

43. $\sqrt{x} + 2 = x$

44. $\sqrt{y} + 56 = y$

45. $y = \sqrt{y + 9} + 3$

46. $r = \sqrt{r + 1} + 5$

47. $\sqrt{r} + 90 = r$

48. $\sqrt{r} + 42 = r$

49. $t = \sqrt{t + 3} + 9$

50. $t = \sqrt{t + 1} + 89$

51. $\sqrt{51 - x} = x + 5$

52. $\sqrt{148 - x} = x + 8$

Quadratic Formula Applications

53. Two numbers' sum is -1, and their product is -42. Find these two numbers.
These two numbers are ______.

54. Two numbers' sum is -13, and their product is 42. Find these two numbers.
These two numbers are ______.

55. Two numbers' sum is 7.7, and their product is -25.5. Find these two numbers.

These two numbers are ______.
(Use a comma to separate your numbers.)

56. Two numbers' sum is 4.7, and their product is 3.96. Find these two numbers.

These two numbers are ______.
(Use a comma to separate your numbers.)

57. A rectangle's base is 6 cm longer than its height. The rectangle's area is 112 cm^2. Find this rectangle's dimensions.

The rectangle's height is ______.

The rectangle's base is ______.

58. A rectangle's base is 9 cm longer than its height. The rectangle's area is 162 cm^2. Find this rectangle's dimensions.

The rectangle's height is ______.

The rectangle's base is ______.

59. A rectangle's base is 3 in shorter than four times its height. The rectangle's area is 85 in^2. Find this rectangle's dimensions.

The rectangle's height is ______.

The rectangle's base is ______.

60. A rectangle's base is 1 in shorter than twice its height. The rectangle's area is 15 in^2. Find this rectangle's dimensions.

The rectangle's height is ______.

The rectangle's base is ______.

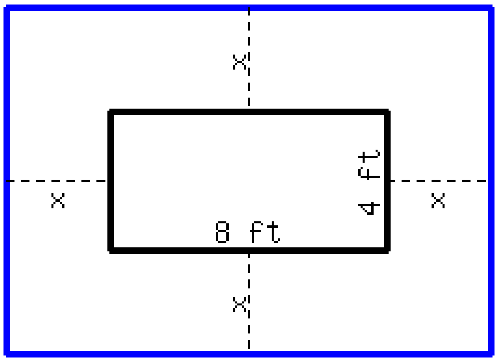
61. You will build a rectangular sheep pen next to a river. There is no need to build a fence along the river, so you only need to build three sides.
You have a total of 510 feet of fence to use, and the area of the pen must be 31900 square feet. Find the dimensions of the pen.
There should be two solutions:When the width is ______ feet, the length is ______ feet.

When the width is ______ feet, the length is ______ feet.

62. You will build a rectangular sheep pen next to a river. There is no need to build a fence along the river, so you only need to build three sides.
You have a total of 470 feet of fence to use, and the area of the pen must be 27500 square feet. Find the dimensions of the pen.
There should be two solutions:When the width is ______ feet, the length is ______ feet.

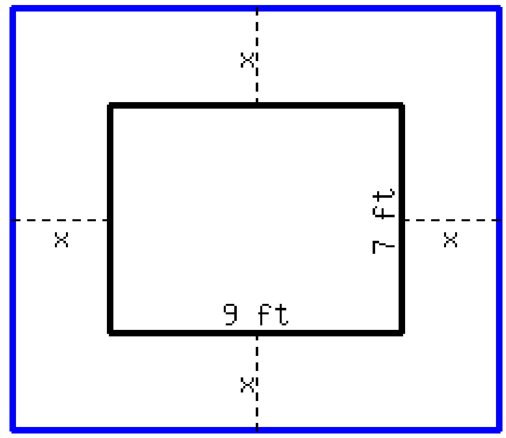
When the width is ______ feet, the length is ______ feet.

63. There is a rectangular lot in the garden, with 8 ft in length and 4 ft in width. You plan to expand the lot by an equal length around its four sides, and make the area of the expanded rectangle 140 ft^2. How long should you expand the original lot in four directions?



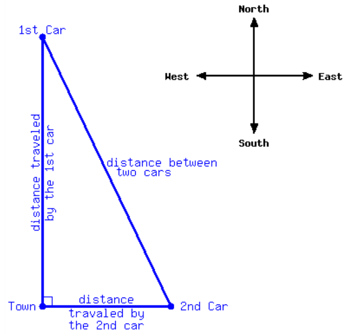
You should expand the original lot by ______ in four directions.

64. There is a rectangular lot in the garden, with 9 ft in length and 7 ft in width. You plan to expand the lot by an equal length around its four sides, and make the area of the expanded rectangle 195 ft^2. How long should you expand the original lot in four directions?



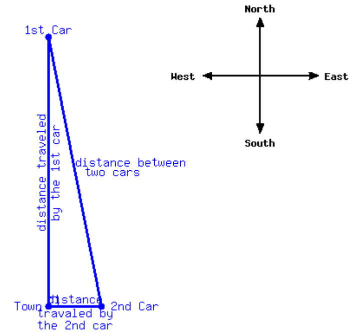
You should expand the original lot by ______ in four directions.

65. One car started at Town A, and traveled due north at 60 miles per hour. 2 hours later, another car started at the same spot and traveled due east at 55 miles per hour. Assume both cars don't stop, after how many hours since the second car starts would the distance between them be 338 miles? Round your answer to two decimal places if needed.



Approximately ______ hours since the second car starts, the distance between those two cars would be 338 miles.

66. One car started at Town A, and traveled due north at 65 miles per hour. 3 hours later, another car started at the same spot and traveled due east at 40 miles per hour. Assume both cars don't stop, after how many hours since the second car starts would the distance between them be 358 miles? Round your answer to two decimal places if needed.



Approximately ______ hours since the second car starts, the distance between those two cars would be 358 miles.

67. An object is launched upward at the height of 370 meters. Its height can be modeled by

$$h = -4.9t^2 + 90t + 370,$$

where h stands for the object's height in meters, and t stands for time passed in seconds since its launch. The object's height will be 380 meters twice before it hits the ground. Find how many seconds since the launch would the object's height be 380 meters. Round your answers to two decimal places if needed.
The object's height would be 380 meters the first time at ______ seconds, and then the second time at ______ seconds.

68. An object is launched upward at the height of 400 meters. Its height can be modeled by

$$h = -4.9t^2 + 70t + 400,$$

where h stands for the object's height in meters, and t stands for time passed in seconds since its launch. The object's height will be 430 meters twice before it hits the ground. Find how many seconds since the launch would the object's height be 430 meters. Round your answers to two decimal places if needed.
The object's height would be 430 meters the first time at ______ seconds, and then the second time at ______ seconds.

69. Currently, an artist can sell 280 paintings every year at the price of \$60.00 per painting. Each time he raises the price per painting by \$5.00, he sells 5 fewer paintings every year.
Assume he will raise the price per painting x times, then he will sell $280 - 5x$ paintings every year at the price of $60 + 5x$ dollars. His yearly income can be modeled by the equation:

$$i = (60 + 5x)(280 - 5x)$$

where i stands for his yearly income in dollars. If the artist wants to earn \$22,500.00 per year from selling paintings, what new price should he set?
To earn \$22,500.00 per year, the artist could sell his paintings at two different prices.
The lower price is ______ per painting, and the higher price is ______ per painting.

70. Currently, an artist can sell 210 paintings every year at the price of \$130.00 per painting. Each time he raises the price per painting by \$10.00, he sells 5 fewer paintings every year.
Assume he will raise the price per painting x times, then he will sell $210 - 5x$ paintings every year at the price of $130 + 10x$ dollars. His yearly income can be modeled by the equation:

$$i = (130 + 10x)(210 - 5x)$$

where i stands for his yearly income in dollars. If the artist wants to earn \$33,300.00 per year from selling paintings, what new price should he set?
To earn \$33,300.00 per year, the artist could sell his paintings at two different prices.
The lower price is ______ per painting, and the higher price is ______ per painting.

71. Solve for x in the equation $mx^2 + nx + p = 0$.

7.3 Complex Solutions to Quadratic Equations

7.3.1 Imaginary Numbers

Let's take a closer look at a square root with a negative radicand. Remember that $\sqrt{16} = 4$ because $4 \cdot 4 = 16$. So what about $\sqrt{-16}$? There is no real number that we can square to get -16, because when you square a real number, the result is either positive or 0. You might think about 4 and -4, but:

$$4 \cdot 4 = 16 \text{ and } (-4)(-4) = 16$$

so neither of those could be $\sqrt{-16}$. To handle this situation, mathematicians separate a factor of $\sqrt{-1}$ and represent it with the letter i.

Definition 7.3.2 Imaginary Numbers. The **imaginary unit**, i, is defined by $i = \sqrt{-1}$. The imaginary unit[1] satisfies the equation $i^2 = -1$. A real number times i, such as $4i$, is called an **imaginary number**. ♢

Now we can simplify square roots with negative radicands like $\sqrt{-16}$.

$$\begin{aligned}\sqrt{-16} &= \sqrt{-1 \cdot 16}\\ &= \sqrt{-1} \cdot \sqrt{16}\\ &= i \cdot 4\\ &= 4i\end{aligned}$$

Imaginary numbers are used in electrical engineering, physics, computer science, and advanced mathematics. Let's look some more examples.

Example 7.3.3 Simplify $\sqrt{-2}$.
Explanation.

$$\begin{aligned}\sqrt{-2} &= \sqrt{-1 \cdot 2}\\ &= \sqrt{-1} \cdot \sqrt{2}\\ &= i\sqrt{2}\end{aligned}$$

We write the i in front of the radical because it can be easy to mix up $\sqrt{2}i$ and $\sqrt{2i}$, if you don't draw the radical very carefully.

Example 7.3.4 Simplify $\sqrt{-72}$.
Explanation.

$$\begin{aligned}\sqrt{-72} &= \sqrt{-1 \cdot 36 \cdot 2}\\ &= \sqrt{-1} \cdot \sqrt{36} \cdot \sqrt{2}\\ &= 6i\sqrt{2}\end{aligned}$$

[1]en.wikipedia.org/wiki/Imaginary_number

7.3.2 Solving Quadratic Equations with Imaginary Solutions

Back in Example 7.1.9, we examined an equation that had no real solution. Let's revisit that example now that we are aware of imaginary numbers.

Example 7.3.5 Solve for x in $x^2 + 49 = 0$, where x might not be a real number.

Explanation. There is no x term so we will use the square root method.

$$\begin{aligned} x^2 + 49 &= 0 \\ x^2 &= -49 \\ x &= \pm\sqrt{-49} \\ x &= \pm\sqrt{-1 \cdot 49} \\ x &= \pm\sqrt{-1} \cdot \sqrt{49} \\ x &= \pm i \cdot 7 \end{aligned}$$

$$x = -7i \qquad \text{or} \qquad x = 7i$$

The solution set is $\{-7i, 7i\}$.

Example 7.3.6 Solve for p in $p^2 + 75 = 0$, where p might not be a real number.

Explanation. There is no p term so we will use the square root method.

$$\begin{aligned} p^2 + 75 &= 0 \\ p^2 &= -75 \\ p &= \pm\sqrt{-75} \\ p &= \pm\sqrt{-1 \cdot 25 \cdot 3} \\ p &= \pm\sqrt{-1} \cdot \sqrt{25} \cdot \sqrt{3} \\ p &= \pm i \cdot 5\sqrt{3} \end{aligned}$$

$$p = -5i\sqrt{3} \qquad \text{or} \qquad p = 5i\sqrt{3}$$

The solution set is $\left\{-5i\sqrt{3}, 5i\sqrt{3}\right\}$.

7.3.3 Solving Quadratic Equations with Complex Solutions

Sometimes we need to work with a sum of a real number and an imaginary number, like $3 + 2i$ or $-4 - 8i$. These combinations are called "complex numbers".

Definition 7.3.7 Complex Number. A **complex number** is a number that can be expressed in the form $a + bi$, where a and b are real numbers and i is the imaginary unit. In this expression, a is the **real part** and b (not bi) is the **imaginary part** of the complex number[2]. ◊

[2] en.wikipedia.org/wiki/Complex_number

Example 7.3.8 In an advanced math course, you might study the relationship between a lynx polulation (or any generic predator) and a hare population (or any generic prey) as time passes. For example, if the predator population is high, they will eat many prey. But then the prey population will become low, so the predators will go hungry and have fewer offspring. With time, the predator population will decline, and that will lead to a rebound in the prey population. Then prey will be plentiful, and the preadator population will rebound, and the whole situaiton starts over. This cycle may take years or even decades to play out.

Strange as it may seem, to understand this phenomenon mathematically, you will need to solve equations similar to:

$$(1-t)(3-t)+10=0$$

Let's practice solving this equation.

$$\begin{aligned} (1-t)(3-t)+10&=0\\ 3-t-3t+t^2+10&=0\\ t^2-4t+13&=0 \end{aligned}$$

We can try the quadratic formula.

$$\begin{aligned} t&=\frac{4\pm\sqrt{(-4)^2-4(1)(13)}}{2(1)}\\ &=\frac{4\pm\sqrt{16-52}}{2}\\ &=\frac{4\pm\sqrt{-36}}{2}\\ &=\frac{4\pm\sqrt{-1}\cdot\sqrt{36}}{2}\\ &=\frac{4\pm i\cdot 6}{2}\\ &=2\pm 3i \end{aligned}$$

These two solutions, $2-3i$ and $2+3i$ have implications for how fast the predator and prey populations rise and fall over time, but an explanation is beyond the scope of basic algebra.

Here are some more examples of equations that have complex number solutions.

Example 7.3.9 Solve for m in $(m-1)^2+18=0$, where m might not be a real number.

Explanation. This equation has a squared expression so we will use the square root method.

$$\begin{aligned} (m-1)^2+18&=0\\ (m-1)^2&=-18\\ m-1&=\pm\sqrt{-18}\\ m-1&=\pm\sqrt{-1\cdot 9\cdot 2}\\ m-1&=\pm\sqrt{-1}\cdot\sqrt{9}\cdot\sqrt{2}\\ m-1&=\pm i\cdot 3\sqrt{2}\\ m&=1\pm i\cdot 3\sqrt{2} \end{aligned}$$

$$m = 1 - 3i\sqrt{2} \qquad \text{or} \qquad m = 1 + 3i\sqrt{2}$$

The solution set is $\left\{1 - 3i\sqrt{2}, 1 + 3i\sqrt{2}\right\}$.

Example 7.3.10 Solve for y in $y^2 - 4y + 13 = 0$, where y might not be a real number.

Explanation. Note that there is a y term, so the square root method is not available. We will use the quadratic formula. We identify that $a = 1$, $b = -4$ and $c = 13$ and substitute them into the quadratic formula.

$$\begin{aligned}
y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} \\
&= \frac{4 \pm \sqrt{16 - 52}}{2} \\
&= \frac{4 \pm \sqrt{-36}}{2} \\
&= \frac{4 \pm \sqrt{-1} \cdot \sqrt{36}}{2} \\
&= \frac{4 \pm 6i}{2} \\
&= 2 \pm 3i
\end{aligned}$$

The solution set is $\{2 - 3i, 2 + 3i\}$.

Note that in Example 7.3.10, the expressions $2 + 3i$ and $2 - 3i$ are fully simplified. In the same way that the terms 2 and $3x$ cannot be combined, the terms 2 and $3i$ can not be combined.

Remark 7.3.11 Each complex solution can be checked, just as every real solution can be checked. For example, to check the solution of $2 + 3i$ from Example 7.3.10, we would replace y with $2 + 3i$ and check that the two sides of the equation are equal. In doing so, we will need to use the fact that $i^2 = -1$. This check is shown here:

$$\begin{aligned}
y^2 - 4y + 13 &= 0 \\
(2 + 3i)^2 - 4(2 + 3i) + 13 &\stackrel{?}{=} 0 \\
(2^2 + 2(3i) + 2(3i) + (3i)^2) - 4 \cdot 2 - 4 \cdot (3i) + 13 &\stackrel{?}{=} 0 \\
4 + 6i + 6i + 9i^2 - 8 - 12i + 13 &\stackrel{?}{=} 0 \\
4 + 9(-1) - 8 + 13 &\stackrel{?}{=} 0 \\
4 - 9 - 8 + 13 &\stackrel{?}{=} 0 \\
0 &\stackrel{\checkmark}{=} 0
\end{aligned}$$

7.3.4 Reading Questions

1. What is $\left(i^2\right)^2$?

2. A number like $4i$ is called a $\boxed{}$ number. A number like $3+4i$ is called a $\boxed{}$ number.

7.3.5 Exercises

Simplifying Square Roots with Negative Radicands Simplify the radical and write it as a complex number using i.

1. $\sqrt{-30}$ **2.** $\sqrt{-30}$ **3.** $\sqrt{-24}$

4. $\sqrt{-56}$ **5.** $\sqrt{-270}$ **6.** $\sqrt{-240}$

Quadratic Equations with Imaginary and Complex Solutions Solve the quadratic equation. Solutions could be complex numbers.

7. $x^2 = -100$ **8.** $x^2 = -49$ **9.** $5y^2 - 6 = -86$

10. $3y^2 - 6 = -306$ **11.** $-2r^2 - 9 = 3$ **12.** $-5r^2 - 7 = 8$

13. $-3r^2 - 10 = 140$ **14.** $-3t^2 - 4 = 131$ **15.** $-8(t-10)^2 - 8 = 64$

16. $-6(x+5)^2 - 8 = 478$ **17.** $x^2 + 2x + 5 = 0$ **18.** $y^2 + 4y + 5 = 0$

19. $y^2 + 4y + 11 = 0$ **20.** $r^2 - 8r + 19 = 0$

7.4 Solving Equations in General

In your algebra studies, you have learned how to solve linear equations, quadratic equations, and radical equations. In this section, we examine some similarities among the processes for solving these equations. Understanding these similarities can improve your general equation solving ability, even into the future with new equations that are not of these three types.

7.4.1 Equations Where the Variable Appears Once

Here are some examples of equations that all have something in common: the variable only appears once.

$$2\overset{\downarrow}{x}+1=7 \qquad (\overset{\downarrow}{x}+4)^2=36 \qquad \sqrt{2\overset{\downarrow}{x}-3}=3$$

For equations like this, there is a strategy for solving them that will keep you from overcomplicating things. In each case, according to the order of operations, the variable is having some things "done" to it in a specific order.

With $2x+1=7$,

1. x is multiplied by 2
2. then that result is added to 1
3. and this result is a number, 7

With $(x+4)^2=36$,

1. x is added to 4
2. then that result is squared
3. and this result is a number, 36

With $\sqrt{2x-3}=3$,

1. x is multiplied by 2
2. then that result has 3 subtracted from it
3. then that result has a square root applied
4. and this result is a number, 3

Because there is just one instance of the variable, and then things happen to that value in a specific order according to the order of operations, then there is a good strategy to solve these equations. We can just *undo each step in the opposite order*.

Example 7.4.2 Solve the equation $2x+1=7$.

Explanation. The actions that happen to x are *multiply* by 2, and then *add* 1. So we will do the opposite actions in the opposite order to each side of the equation. We will *subtract* 1 and then *divide* by 2.

$$\begin{aligned} 2x+1&=7 && \text{now subtract 1 from each side}\\ 2x+1-1&=7-1\\ 2x&=6 && \text{now divide by 2 on each side}\\ \frac{2x}{2}&=\frac{6}{2}\\ x&=3 \end{aligned}$$

You should check this solution by substituting it into the original equation.

Example 7.4.3 Solve the equation $(x+4)^2=36$.

Explanation. The actions that happen to x are *add* 4, and then *square*. So we will do the opposite actions in the opposite order to each side of the equation. We will apply the Square Root Property and then *subtract* 4.

$$\begin{aligned}
(x+4)^2 &= 36 && \text{now apply the Square Root Property}\\
x+4 &= \pm\sqrt{36} \\
x+4 &= \pm 6 && \text{now subtract 4 on each side}\\
x+4-4 &= \pm 6 - 4 \\
x &= \pm 6 - 4
\end{aligned}$$

$$\begin{aligned}
x &= -6-4 && \text{or} && x = 6-4\\
x &= -10 && \text{or} && x = 2
\end{aligned}$$

You should check these solutions by substituting them into the original equation.

Example 7.4.4 Solve the equation $\sqrt{2x-3} = 3$.

Explanation. The actions that happen to x are *multiply* by 2, and then *subtract* 3, and then apply the square root. So we will do the opposite actions in the opposite order to each side of the equation. We will *square* both sides, *add* 3 and then *divide* by 2.

$$\begin{aligned}
\sqrt{2x-3} &= 3 && \text{now square both sides}\\
2x-3 &= 9 && \text{now add 3 to each side}\\
2x-3+3 &= 9+3 \\
2x &= 12 && \text{now divide by 2 on each side}\\
\frac{2x}{2} &= \frac{12}{2} \\
x &= 6
\end{aligned}$$

You should check this solution by substituting it into the original equation.

7.4.2 Equations With More Than One Instance of the Variable

Now consider equations like

$$5\overset{\downarrow}{x}+1 = 3\overset{\downarrow}{x}+2 \qquad \overset{\downarrow}{x}^2 + 6\overset{\downarrow}{x} = -8 \qquad \sqrt{\overset{\downarrow}{x}-3} = \sqrt{\overset{\downarrow}{x}}-1$$

In these examples, the variable appears more than once. We can't exactly dive in to the strategy of undoing each step in the opposite order. For each of these equations, remind yourself that you can apply any operation you want, as long as you apply it to both sides of the equation. In many cases, you will find that there is some basic algebra move you can take that will turn the equation into something more "standard" that you know how to work with.

With $5x+1 = 3x+2$, we have a linear equation. If we can simply reorganize the terms to combine like terms, a solution will be apparent.

With $x^2 + 6x = -8$, adding 8 to both sides would give us a quadratic equation in standard form. And then the quadratic formula can be used.

With $\sqrt{x-3} = \sqrt{x} - 1$, the complication is those two radicals. We can take any action we like as long as we apply it to both sides, and *squaring* both sides would remove at least one radical. Maybe after that we will have a simpler equation.

Example 7.4.5 Solve the equation $5x + 1 = 3x + 2$.

Explanation. We'll use basic algebra to rearrange the terms.

$$\begin{aligned} 5x + 1 &= 3x + 2 && \text{now subtract 1 from each side} \\ 5x &= 3x + 1 && \text{now subtract } 3x \text{ from each side} \\ 2x &= 1 && \text{now divide by 2 on each side} \\ x &= \frac{1}{2} \end{aligned}$$

You should check this solution by substituting it into the original equation.

Example 7.4.6 Solve the equation $x^2 + 6x = -8$.

Explanation. Adding 8 to each side will give us a quadratic equation in standard form, and then we may apply The Quadratic Formula.

$$\begin{aligned} x^2 + 6x &= -8 && \text{now add 8 to each side} \\ x^2 + 6x + 8 &= 0 && \text{now apply The Quadratic Formula} \end{aligned}$$

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{6^2 - 4(1)(8)}}{2(1)} \\ &= \frac{-6 \pm \sqrt{36-32}}{2} \\ &= \frac{-6 \pm \sqrt{4}}{2} \\ &= \frac{-6 \pm 2}{2} \end{aligned}$$

$$\begin{aligned} x &= \frac{-6-2}{2} && \text{or} && x = \frac{-6+2}{2} \\ x &= \frac{-8}{2} && \text{or} && x = \frac{-4}{2} \\ x &= -4 && \text{or} && x = -2 \end{aligned}$$

You should check these solutions by substituting them into the original equation.

Example 7.4.7 Solve the equation $\sqrt{x-3} = \sqrt{x} - 1$.

Explanation. Hoping to obtain a simpler equation, we will square each side. This will eliminate at least one radical, which may help.

$$\begin{aligned}
\sqrt{x-3} &= \sqrt{x}-1 && \text{now square both sides}\\
\left(\sqrt{x-3}\right)^2 &= \left(\sqrt{x}-1\right)^2\\
x-3 &= \left(\sqrt{x}\right)^2 - 2\sqrt{x}+1\\
x-3 &= x-2\sqrt{x}+1 && \text{now note that there are some like terms}\\
-3 &= -2\sqrt{x}+1 && \text{now we have an equation with only one instance of the variable}\\
-4 &= -2\sqrt{x}\\
2 &= \sqrt{x}\\
2^2 &= x\\
x &= 4
\end{aligned}$$

You should check this solution by substituting it into the original equation. It is *especially* important to do this when the equation was a radical equation. At one point, we squared both sides, and this can introduce extraneous solutions (see Remark 6.4.4).

7.4.3 Solving For a Variable in Terms of Other Variables

In the examples so far in this section, there has been one variable (but possibly more than one instance of that variable). This leaves out important situations in science applications where you have a formula with *multiple* variables, and you need to isolate *one* of them. Fortunately these situations are not more difficult than what we have explored so far, as long as you can keep track of which variable you are trying to solve for.

Example 7.4.8 In physics, there is a formula for converting a Celsius temperature to Fahrenheit:

$$F = \frac{9}{5}C + 32$$

Solve this equation for C in terms of F.

Explanation. The variable we are after is C, and that variable only appears once. So we will apply the strategy of undoing the things that are happening to C. First C is multiplied by $\frac{9}{5}$, and then it is added to 32. So we will undo these actions in the opposite order: *subtract* 32 and then multiply by $\frac{5}{9}$ (or *divide* by $\frac{9}{5}$ if you prefer).

$$\begin{aligned}
F &= \frac{9}{5}\overset{\downarrow}{C} + 32\\
F-32 &= \frac{9}{5}\overset{\downarrow}{C} + 32 - 32\\
F-32 &= \frac{9}{5}\overset{\downarrow}{C}\\
\frac{5}{9}\cdot(F-32) &= \frac{5}{9}\cdot\frac{9}{5}\overset{\downarrow}{C}\\
\frac{5}{9}(F-32) &= \overset{\downarrow}{C}
\end{aligned}$$

$$C = \frac{5}{9}(F - 32)$$

We are satisfied, because we have isolated C in terms of F.

Example 7.4.9 In physics, when an object of mass m is moving with a speed v, its "kinetic energy" E is given by:

$$E = \frac{1}{2}mv^2$$

Solve this equation for v in terms of the other variables.

Explanation. The variable we are after is v, and that variable only appears once. So we will apply the strategy of undoing the things that are happening to v. First v is squared, then it is multiplied by m and by $\frac{1}{2}$. So we will undo these actions in the opposite order: *multiply* by 2, *divide* by m, and apply the square root.

$$\begin{aligned}
E &= \frac{1}{2}mv^2 \\
2 \cdot E &= 2 \cdot \frac{1}{2}mv^2 \\
2E &= mv^2 \\
\frac{2E}{m} &= \frac{mv^2}{m} \\
\frac{2E}{m} &= v^2 \\
\pm\sqrt{\frac{2E}{m}} &= v \\
v &= \sqrt{\frac{2E}{m}}
\end{aligned}$$

At the very end, we chose the positive square root, since a speed v cannot be negative. We are satisfied, because we have isolated v in terms of E and m.

7.4.4 Reading Questions

1. When there is only one instance of a variable in an equation, describe a strategy for solving the equation.
2. You can do whatever algebra you like to the sides of an equation, as long as you do what?

7.4.5 Exercises

Solve the equation.

1. $x + 3 = -8$	**2.** $9x + 5 = 4$	**3.** $7x - 2 = -4x$
4. $9x + 6 = -9x$	**5.** $-9x - 6 = 5x - 9$	**6.** $-7x + 4 = -3x + 5$
7. $-8x^2 = -200$	**8.** $3x^2 = 108$	**9.** $6(x - 8)^2 = 294$

10. $-3(x+19)^2 = -192$

11. $2x^2 - 162 = 0$

12. $-4x^2 + 484 = 0$

13. $49x^2 = 144$

14. $x^2 = 4$

15. $3x^2 = 2$

16. $13x^2 = 5$

17. $6(x+5)^2 = 5$

18. $5(x-3)^2 = 2$

19. $2x^2 + 6x - 9 = 0$

20. $-x^2 + 3x + 3 = 0$

21. $6x^2 + 8x + 7 = 9$

22. $8x^2 + 6x + 3 = 4$

23. $x^2 + 4x + 1 = 7x + 8$

24. $2x^2 + x + 7 = 3x + 8$

25. $x^2 + 9x + 6 = 6x^2 + 8x + 5$

26. $4x^2 + 9x + 7 = 5x^2 + 2x + 4$

27. $\sqrt{-9x-2} = 8$

28. $\sqrt{2x+4} = 6$

29. $\sqrt[3]{4x-3} = -5$

30. $\sqrt[3]{6x-9} = 3$

31. $\sqrt[4]{8x+3} = -9$

32. $\sqrt[4]{-9x-4} = -1$

33. $\sqrt{x-7} = \sqrt{x+9} + 7$

34. $\sqrt{x-4} = \sqrt{x+2} - 4$

35. $\sqrt{9x+8} = \sqrt{6x-8} + 4$

36. $\sqrt{7x-4} = \sqrt{8x+6} - 3$

Solve an Equation for a Variable

37. Solve the equation $A = bh$ for b.

38. Solve the equation $A = bh$ for h.

39. Solve the equation $P = 2(\ell + w)$ for l.

40. Solve the equation $P = 2(\ell + w)$ for w.

41. Solve the equation $A = \frac{1}{2}bh$ for b.

42. Solve the equation $A = \frac{1}{2}bh$ for h.

43. Solve the equation $y = mx + b$ for m.

44. Solve the equation $y = mx + b$ for x.

45. Solve the equation $y = mx + b$ for b.

46. Solve the equation $y = m(x-h) + k$ for k.

47. Solve the equation $y = m(x-h) + k$ for h.

48. Solve the equation $y = m(x-h) + k$ for x.

49. Solve the equation $c = 2\pi r$ for r.

50. Solve the equation $c = \pi d$ for d.

51. Solve the equation $A = s^2$ for s. Assume $s > 0$.

52. Solve the equation $A = \pi r^2$ for r. Assume $r > 0$.

53. Solve the equation $V = \pi r^2 h$ for r. Assume $r > 0$.

54. Solve the equation $V = \frac{1}{3}s^2 h$ for s. Assume $s > 0$.

55. Solve the equation $V = \pi r^2 h$ for h.

56. Solve the equation $V = \frac{1}{3}s^2 h$ for h.

57. Solve the equation $V = s^3$ for s.

58. Solve the equation $V = \frac{4}{3}\pi r^3$ for r.

59. Solve the equation $S = 6s^2$ for s. Assume $s > 0$.

60. Solve the equation $S = 4\pi r^2$ for r. Assume $r > 0$.

61. Solve the equation $v = \frac{d}{t}$ for d.

62. Solve the equation $v = \frac{d}{t}$ for t.

63. Solve the equation $p = \frac{1}{2}gt^2 + vt + d$ for t.

64. Solve the equation $y = ax^2 + bx + c$ for x.

65. Solve the equation $F = ma$ for m.

66. Solve the equation $F = ma$ for a.

67. Solve the equation $a = \frac{v^2}{r}$ for v. Assume $v > 0$.

68. Solve the equation $K = \frac{1}{2}mv^2$ for v. Assume $v > 0$.

69. Solve the equation $T = 2\pi\sqrt{\frac{\ell}{g}}$ for l.

70. Solve the equation $T = 2\pi\sqrt{\frac{\ell}{g}}$ for g.

7.5 Solving Quadratic Equations Chapter Review

7.5.1 Solving Quadratic Equations by Using a Square Root

In Section 7.1 we covered how to solve quadratic equations using the square root property and how to use the Pythagorean Theorem.

Example 7.5.1 Solving Quadratic Equations Using the Square Root Property. Solve for w in $3(2-w)^2-24=0$.

Explanation. It's important here to suppress any urge you may have to expand the squared binomial. We begin by isolating the squared expression.

$$\begin{aligned}3(2-w)^2-24&=0\\3(2-w)^2&=24\\(2-w)^2&=8\end{aligned}$$

Now that we have the squared expression isolated, we can use the square root property.

$$\begin{aligned}2-w&=-\sqrt{8} &&\text{or}&& 2-w=\sqrt{8}\\2-w&=-\sqrt{4\cdot 2} &&\text{or}&& 2-w=\sqrt{4\cdot 2}\\2-w&=-\sqrt{4}\cdot\sqrt{2} &&\text{or}&& 2-w=\sqrt{4}\cdot\sqrt{2}\\2-w&=-2\sqrt{2} &&\text{or}&& 2-w=2\sqrt{2}\\-w&=-2\sqrt{2}-2 &&\text{or}&& -w=2\sqrt{2}-2\\w&=2\sqrt{2}+2 &&\text{or}&& w=-2\sqrt{2}+2\end{aligned}$$

The solution set is $\left\{2\sqrt{2}+2, -2\sqrt{2}+2\right\}$.

Example 7.5.2 The Pythagorean Theorem. Faven was doing some wood working in her garage. She needed to cut a triangular piece of wood for her project that had a hypotenuse of 16 inches, and the sides of the triangle should be equal in length. How long should she make her sides?

Explanation. Let's start by representing the length of the triangle, measured in inches, by the letter x. That would also make the other side x inches long.

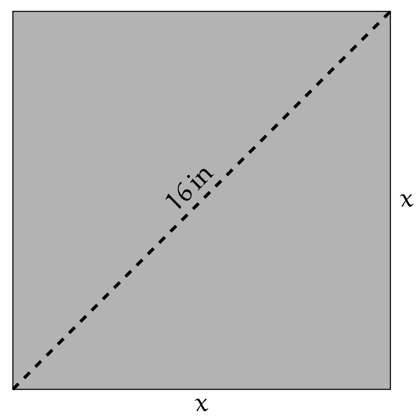


Figure 7.5.3: Piece of wood with labels for Faven

Faven should now set up the Pythagorean theorem regarding the picture. That would be

$$x^2 + x^2 = 16^2$$

Solving this equation, we have:

$$\begin{aligned} x^2 + x^2 &= 16^2 \\ 2x^2 &= 256 \\ x^2 &= 128 \\ \sqrt{x^2} &= \sqrt{128} \\ x &= \sqrt{64 \cdot 2} \\ x &= \sqrt{64} \cdot \sqrt{2} \\ x &= 8\sqrt{2} \\ x &\approx 11.3 \end{aligned}$$

Faven should make the sides of her triangle about 11.3 inches long to force the hypotenuse to be 16 inches long.

7.5.2 The Quadratic Formula

In Section 7.2 we covered how to use the quadratic formula to solve any quadratic equation.

Example 7.5.4 Solving Quadratic Equations with the Quadratic Formula. Solve the equations using the quadratic formula.

a. $x^2 + 4x = 6$

b. $5x^2 - 2x + 1 = 0$

Explanation.

a. First we should change the equation into standard form.

$$x^2 + 4x = 6$$

$$x^2 + 4x - 6 = 0$$

Next, we check and see that we cannot factor the left side or use the square root property so we must use the quadratic formula. We identify that $a = 1$, $b = 4$, and $c = -6$. We will substitute them into the quadratic formula:

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-6)}}{2(1)} \\
&= \frac{-4 \pm \sqrt{16 + 24}}{2} \\
&= \frac{-4 \pm \sqrt{40}}{2} \\
&= \frac{-4 \pm \sqrt{4 \cdot 10}}{2} \\
&= \frac{-4 \pm \sqrt{4} \cdot \sqrt{10}}{2} \\
&= \frac{-4 \pm 2\sqrt{10}}{2} \\
&= -\frac{4}{2} \pm \frac{2\sqrt{10}}{2} \\
&= -2 \pm \sqrt{10}
\end{aligned}$$

So the solution set is $\left\{-2 + \sqrt{10}, -2 - \sqrt{10}\right\}$.

b. Since the equation $5x^2 - 2x + 1 = 0$ is already in standard form, we check and see that we cannot factor the left side or use the square root property so we must use the quadratic formula. We identify that $a = 5$, $b = -2$, and $c = 1$. We will substitute them into the quadratic formula:

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(5)(1)}}{2(5)} \\
&= \frac{2 \pm \sqrt{4 - 20}}{10} \\
&= \frac{2 \pm \sqrt{-16}}{10}
\end{aligned}$$

Since the solutions have square roots of negative numbers, we must conclude that there are no real solutions.

7.5.3 Complex Solutions to Quadratic Equations

In Section 7.3 we covered what both imaginary numbers and complex numbers are, as well as how to solve quadratic equations where the solutions are imaginary numbers or complex numbers.

Example 7.5.5 Imaginary Numbers. Simplify the expression $\sqrt{-12}$ using the imaginary number, i.

Explanation. Start by splitting the -1 from the 12 and by looking for the largest perfect-square factor of -12, which happens to be 4.

$$\begin{aligned}\sqrt{-12} &= \sqrt{4 \cdot -1 \cdot 3}\\ &= \sqrt{4} \cdot \sqrt{-1} \cdot \sqrt{3}\\ &= 2i\sqrt{3}\end{aligned}$$

Example 7.5.6 Solving Quadratic Equations with Imaginary Solutions. Solve for m in $2m^2+16=0$, where p is an imaginary number.

Explanation. There is no m term so we will use the square root method.

$$\begin{aligned}2m^2+16 &= 0\\ 2m^2 &= -16\\ m^2 &= -8\end{aligned}$$

$$\begin{aligned}m &= -\sqrt{-8} &\quad\text{or}\quad&& m &= \sqrt{-8}\\ m &= -\sqrt{4} \cdot \sqrt{-1} \cdot \sqrt{2} &\quad\text{or}\quad&& m &= \sqrt{4} \cdot \sqrt{-1} \cdot \sqrt{2}\\ m &= -2i\sqrt{2} &\quad\text{or}\quad&& m &= 2i\sqrt{2}\end{aligned}$$

The solution set is $\left\{-2i\sqrt{2}, 2i\sqrt{2}\right\}$.

Example 7.5.7 Solving Quadratic Equations with Complex Solutions. Solve the equation $3(v-2)^2+54=0$, where v is a complex number.

Explanation.

$$\begin{aligned}3(v-2)^2+54 &= 0\\ 3(v-2)^2 &= -54\\ (v-2)^2 &= -18\end{aligned}$$

$$\begin{aligned}v-2 &= -\sqrt{-18} &\quad\text{or}\quad&& v-2 &= \sqrt{-18}\\ v-2 &= -\sqrt{9 \cdot -1 \cdot 2} &\quad\text{or}\quad&& v-2 &= \sqrt{9 \cdot -1 \cdot 2}\\ v-2 &= -\sqrt{9} \cdot \sqrt{-1} \cdot \sqrt{2} &\quad\text{or}\quad&& v-2 &= \sqrt{9} \cdot \sqrt{-1} \cdot \sqrt{2}\\ v-2 &= -3i\sqrt{2} &\quad\text{or}\quad&& v-2 &= 3i\sqrt{2}\\ v &= 2-3i\sqrt{2} &\quad\text{or}\quad&& v &= 2+3i\sqrt{2}\end{aligned}$$

So, the solution set is $\left\{2+3i\sqrt{2}, 2-3i\sqrt{2}\right\}$.

7.5.4 Solving Equations in General

In Section 2.1 we learned how to solve linear equations. In Section 6.4 we learned how to solve radical equations. In Section 7.1 and Section, we learned how to solve quadratic equations.

Then in Section 7.4 we looked at a few strategies to solve equations in general, often relying on those earlier specific techniques.

Example 7.5.8 Equations where the Variable Appears Once. Solve the equations using an effective method.

a. $(x-4)^2-2=0$ b. $\sqrt{3x+2}-2=5$ c. $3(5x-6)-7=2$

Explanation.

a. Since the variable x only appears once, we can apply steps one at a time to undo all of the operations that are done to x and eventually isolate it.

$$\begin{aligned}(x-4)^2-2&=0\\(x-4)^2&=2\\x-4&=\pm\sqrt{2}\\x&=4\pm\sqrt{2}\end{aligned}$$

So the solution set is $\left\{4+\sqrt{2},4-\sqrt{2}\right\}$

b. Since the variable x only appears once, we can apply steps one at a time to undo all of the operations that are done to x and eventually isolate it.

$$\begin{aligned}\sqrt{3x+2}-2&=5\\\sqrt{3x+2}&=7\\\left(\sqrt{3x+2}\right)^2&=7^2\\3x+2&=49\\3x&=47\\x&=\frac{47}{3}\end{aligned}$$

At this point $\frac{47}{3}$ is only a potential solution. We may have introduced an extraneous solution at the point where we squared both sides. So we should check it.

$$\begin{aligned}\sqrt{3\cdot\frac{47}{3}+2}-2&\stackrel{?}{=}5\\\sqrt{47+2}&\stackrel{?}{=}7\\\sqrt{49}&\stackrel{\checkmark}{=}7\end{aligned}$$

So, the solution set is $\left\{\frac{47}{3}\right\}$.

c. Since the variable x only appears once, we can apply steps one at a time to undo all of the operations that are done to x and eventually isolate it.

$$3(5x-6)-7=2$$

$$\begin{aligned}3(5x-6)&=9\\5x-6&=3\\5x&=9\\x&=\frac{9}{5}\end{aligned}$$

The solution set is $\left\{\frac{9}{5}\right\}$.

Example 7.5.9 Equations With More Than One Instance of the Variable. Recognize that these equations have more than one instance of the variable, so it is not immediately possible to isolate the variable by undoing the operations that are done to it. Instead, call upon a special technique to solve the equation.

a. $(x-4)^2+2x=0$ b. $16x-2(3x-1)=7$ c. $\sqrt{x+2}=x-4$

Explanation.

a. To solve the equation $(x-4)^2+2x=0$, note that it is a quadratic equation, and we can write it in standard form.

$$\begin{aligned}(x-4)^2+2x&=0\\x^2-8x+16+2x&=0\\x^2-6x+16&=0\end{aligned}$$

Now we may use the quadratic formula 7.2.2.

$$\begin{aligned}x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\x&=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(16)}}{2(1)}\\&=\frac{6\pm\sqrt{36-48}}{2}\\&=\frac{6\pm\sqrt{-12}}{2}\end{aligned}$$

At this point, we notice that the solutions are complex. Continue to simplify until they are completely reduced.

$$\begin{aligned}x&=\frac{6\pm\sqrt{4\cdot-1\cdot3}}{2}\\&=\frac{6\pm\sqrt{4}\cdot\sqrt{-1}\cdot\sqrt{3}}{2}\\&=\frac{6\pm2i\sqrt{3}}{2}\\&=\frac{6}{2}\pm\frac{2i\sqrt{3}}{2}\\&=3\pm i\sqrt{3}\end{aligned}$$

So the solution set is $\left\{3-i\sqrt{3},3+i\sqrt{3}\right\}$.

b. To solve the equation $16x-2(3x-1)=3$ we first we first note that it is linear. Since it is linear, we just need to follow the steps outlined in Process 2.1.4.

$$\begin{aligned}16x-2(3x-1)&=7\\16x-6x+3&=7\\10x+3&=7\\10x&=4\\x&=\frac{4}{10}\\x&=\frac{2}{5}\end{aligned}$$

So, the solution set is $\left\{\frac{2}{5}\right\}$.

c. Since the equation $\sqrt{x+2}=x-4$ is a radical equation, we should isolate the radical (which it already is) and square both sides of the equation.

$$\begin{aligned}\sqrt{x+2}&=x-4\\\left(\sqrt{x+2}\right)^2&=(x-4)^2\\x+2&=x^2-8x+16\\0&=x^2-9x+14\end{aligned}$$

Since the equation is now quadratic, we may use the quadratic formula 7.2.2 to solve it.

$$\begin{aligned}x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\x&=\frac{-(-9)\pm\sqrt{(-9)^2-4(1)(14)}}{2(1)}\\&=\frac{9\pm\sqrt{81-56}}{2}\\&=\frac{9\pm\sqrt{25}}{2}\\&=\frac{9\pm5}{2}\end{aligned}$$

$$\begin{array}{lcl}x=\dfrac{9-5}{2} & \text{or} & x=\dfrac{9+5}{2}\\x=\dfrac{4}{2} & \text{or} & x=\dfrac{14}{2}\\x=2 & \text{or} & x=7\end{array}$$

Since this is a radical equation, we should verify our solutions and look out for "extraneous solutions".

$$\begin{array}{ccc}\sqrt{2+2}\stackrel{?}{=}2-4 & \text{or} & \sqrt{7+2}\stackrel{?}{=}7-4\\\sqrt{4}\stackrel{\text{no}}{=}-2 & \text{or} & \sqrt{9}\stackrel{\checkmark}{=}3\end{array}$$

So the solution set is $\{7\}$.

Example 7.5.10 Solving For a Variable in Terms of Other Variables. Often in science classes, you are given a formula that needs to be rearranged to be useful to a situation. Below are a few equations from physics that describe the natural world.

a. Solve the equation $v^2 = v_0^2 + 2ax$ for x. (This equation describes the motion of objects that are accelerating.)

b. Solve the equation $c\ell = \ell_0\sqrt{c^2 - v^2}$ for v. (This equation describes the size of things moving at very fast speeds.)

c. Solve the equation $y = \frac{\alpha t^2}{2} + vt$ for t. (This is another equation that describes the motion of objects that are accelerating.)

Explanation.

a. Since x only appears once in the euqation, we only need to undo the operations that are done to it.

$$\begin{aligned} v^2 &= v_0^2 + 2ax \\ v^2 - v_0^2 &= 2ax \\ \frac{v^2 - v_0^2}{2a} &= x \end{aligned}$$

So we find $x = \frac{v^2 - v_0^2}{2a}$.

b. Since v only appears once in the euqation, we only need to undo the operations that are done to it. According to the order of operations, on the right side of the equation,

(a) v is squared.

(b) The result is negated.

(c) The result is added to c^2.

(d) The result has a square root applied.

(e) The result is multiplied by ℓ_0.

So we do all of the opposite things in the opposite order.

$$\begin{aligned} c\ell &= \ell_0\sqrt{c^2 - v^2} \\ \frac{c \cdot \ell}{\ell_0} &= \sqrt{c^2 - v^2} \\ \left(\frac{c \cdot \ell}{\ell_0}\right)^2 &= \left(\sqrt{c^2 - v^2}\right)^2 \\ \left(\frac{c \cdot \ell}{\ell_0}\right)^2 &= c^2 - v^2 \\ \left(\frac{c \cdot \ell}{\ell_0}\right)^2 - c^2 &= -v^2 \\ -\left(\frac{c \cdot \ell}{\ell_0}\right)^2 + c^2 &= v^2 \end{aligned}$$

$$\pm\sqrt{-\left(\frac{c\cdot\ell}{\ell_0}\right)^2+c^2}=v$$
$$\pm\sqrt{c^2-\left(\frac{c\cdot\ell}{\ell_0}\right)^2}=v$$

So, we find $v=\pm\sqrt{c^2-\left(\frac{c\cdot\ell}{\ell_0}\right)^2}$.

c. This is a quadratic equtaion when we view t as the variable. First, we should rearrange the equation to standard form.

$$y=\frac{\alpha t^2}{2}+vt$$
$$0=\frac{\alpha}{2}t^2+vt-y$$

It is helpful with many equations to "clear denominators". In this case, that means multiplying each side of the equation by 2.

$$0=\alpha t^2+2vt-2y$$

Now, we may apply the quadratic formula 7.2.2.

$$t=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$t=\frac{-2v\pm\sqrt{(2v)^2-4\alpha(-2y)}}{2\alpha}$$
$$t=\frac{-2v\pm\sqrt{4v^2+8\alpha y}}{2\alpha}$$
$$t=\frac{-2v\pm\sqrt{4\left(v^2+2\alpha y\right)}}{2\alpha}$$
$$t=\frac{-2v\pm 2\sqrt{v^2+2\alpha y}}{2\alpha}$$
$$t=\frac{-v\pm\sqrt{v^2+2\alpha y}}{\alpha}$$

So we find $t=\frac{-v\pm\sqrt{v^2+2\alpha y}}{\alpha}$.

7.5.5 Exercises

Solving Quadratic Equations by Using a Square Root Solve the equation.

1. $x^2=27$

2. $x^2=63$

3. $64x^2=9$

4. $4x^2=81$

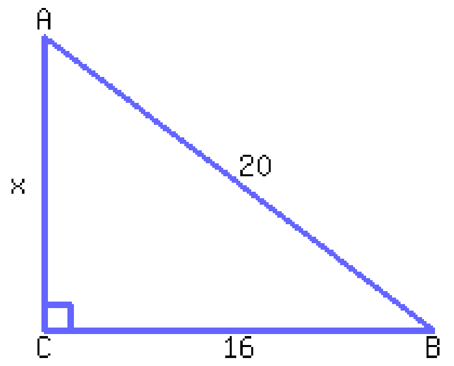
5. $(x+6)^2=36$

6. $(x+9)^2=4$

7. $-4-5(x-9)^2=-9$

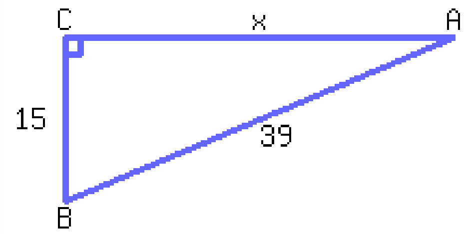
8. $18-3(x-9)^2=6$

9.



Find the value of x.

10.



Find the value of x.

11. Devon is designing a rectangular garden. The garden's diagonal must be 64.6 feet, and the ratio between the garden's base and height must be 15 : 8. Find the length of the garden's base and height.

The garden's base is ______ feet and its height is ______.

12. Tammy is designing a rectangular garden. The garden's diagonal must be 13.5 feet, and the ratio between the garden's base and height must be 4 : 3. Find the length of the garden's base and height.

The garden's base is ______ feet and its height is ______.

The Quadratic Formula Solve the equation.

13. $28x^2 + 29x + 6 = 0$

14. $24x^2 + 29x - 4 = 0$

15. $x^2 = -7x - 11$

16. $x^2 = 7x - 11$

17. $4x^2 + 4x + 6 = 0$

18. $2x^2 + 7x + 7 = 0$

19. $x^2 - 26x = 0$

20. $x^2 - 6x = 0$

21. $x^2 - 7x = 18$

22. $x^2 - x = 20$

23. $x^2 = 9x - 19$

24. $x^2 = -5x - 5$

25. An object is launched upward at the height of 200 meters. Its height can be modeled by

$$h = -4.9t^2 + 70t + 200,$$

where h stands for the object's height in meters, and t stands for time passed in seconds since its launch. The object's height will be 240 meters twice before it hits the ground. Find how many seconds since the launch would the object's height be 240 meters. Round your answers to two decimal places if needed.

The object's height would be 240 meters the first time at ______ seconds, and then the second time at ______ seconds.

26. An object is launched upward at the height of 220 meters. Its height can be modeled by

$$h = -4.9t^2 + 50t + 220,$$

where h stands for the object's height in meters, and t stands for time passed in seconds since its launch. The object's height will be 230 meters twice before it hits the ground. Find how many seconds since the launch would the object's height be 230 meters. Round your answers to two decimal places if needed.

The object's height would be 230 meters the first time at ______ seconds, and then the second time at ______ seconds.

Complex Solutions to Quadratic Equations Simplify the radical and write it as a complex number using i.

27. $\sqrt{-40}$

28. $\sqrt{-56}$

Solve the quadratic equation. Solutions could be complex numbers.

29. $-2y^2 - 5 = 1$

30. $3r^2 + 8 = 2$

31. $-5(r+4)^2 + 5 = 85$

32. $3(t-6)^2 + 5 = -295$

Solving Equations in General Solve the equation.

33. $\sqrt{t} + 72 = t$

34. $\sqrt{x} + 30 = x$

35. $5 + 10(y-5) = -6 - (7 - 2y)$

36. $3 + 8(t-3) = 1 - (2 - 3t)$

37. $x^2 + 5x = 24$

38. $x^2 + 8x = 9$

39. $-8 - 8A + 2 = -A + 12 - 7A$

40. $-6 - 10C + 6 = -C + 13 - 9C$

41. $x^2 + 8x + 3 = 0$

42. $x^2 - 6x - 9 = 0$

43. $-7 - 2(x+2)^2 = -9$

44. $30 - 6(x+2)^2 = 6$

45. $14 = \frac{t}{5} + \frac{t}{2}$

46. $3 = \frac{a}{3} + \frac{a}{6}$

47. $y = \sqrt{y+4} + 86$

48. $r = \sqrt{r+2} + 40$

49. $3x^2 + 41 = 0$

50. $43x^2 + 47 = 0$

51. $5x^2 = -42x - 49$

52. $2x^2 = -21x - 10$

53. $x = \sqrt{x-1} + 7$

54. $x = \sqrt{x+7} - 1$

Chapter 8

Quantities in the Physical World

8.1 Scientific Notation

Very large and very small numbers can be awkward to write and calculate with. These kinds of numbers can show in the sciences. For example in biology, a human hair might be as thick as 0.000181 meters. And the closest that Mars gets to the sun is 206620000 meters. Keeping track of the decimal places and extra zeros raises the potential for mistakes to be made. In this section, we discuss a format used for very large and very small numbers called **scientific notation** that helps alleviate the issues with these numbers.

8.1.1 The Basics of Scientific Notation

An October 3, 2016 CBS News headline[1] read:

> Federal Debt in FY 2016 Jumped $1,422,827,047,452.46—that's $12,036 Per Household.

The article also later states:

> By the close of business on Sept. 30, 2016, the last day of fiscal 2016, it had climbed to $19,573,444,713,936.79.

When presented in this format, trying to comprehend the value of these numbers can be overwhelming. More commonly, such numbers would be presented in a descriptive manner:

- The federal debt climbed by 1.42 trillion dollars in 2016.
- The federal debt was 19.6 trillion dollars at the close of business on Sept. 30, 2016.

In science, government, business, and many other disciplines, it's not uncommon to deal with very large numbers like these. When numbers get this large, it can be hard to discern when a number has eleven digits and when it has twelve.

We have descriptive language for all numbers based on the place value of the different digits: ones, tens, thousands, ten thousands, etc. We tend to rely upon this language more when we start dealing with larger numbers. Here's a chart for some of the most common numbers we see and use in the world around us:

[1] http://www.cnsnews.com/news/article/terence-p-jeffrey/federal-debt-fy-2016-jumped-142282704745246

Number	US English Name	Power of 10
1	one	10^0
10	ten	10^1
100	hundred	10^2
1,000	one thousand	10^3
10,000	ten thousand	10^4
100,000	one hundred thousand	10^5
1,000,000	one million	10^6
1,000,000,000	one billion	10^9

Figure 8.1.2: Whole Number Powers of 10

Each number above has a corresponding power of ten and this power of ten will be important as we start to work with the content in this section. This descriptive language also covers even larger numbers: trillion, quadrillion, quintillion, sextillion, septillion, and so on. There's also corresponding language to describe very small numbers, such as thousandth, millionth, billionth, trillionth, etc.

Through centuries of scientific progress, humanity became increasingly aware of very large numbers and very small measurements. As one example, the star that is nearest to our sun is Proxima Centauri[2]. Proxima Centauri is about 25,000,000,000,000 miles from our sun. Again, many will find the descriptive language easier to read: Proxima Centauri is about 25 trillion miles from our sun.

To make computations involving such numbers more manageable, a standardized notation called "scientific notation" was established. The foundation of scientific notation is the fact that multiplying or dividing by a power of 10 will move the decimal point of a number so many places to the right or left, respectively. So first, let's take a moment to review that level of basic arithmetic.

Checkpoint 8.1.3 Perform the following operations:

a. Multiply 5.7 by 10.

b. Multiply 3.1 by 10000.

Explanation.

a. $5.7 \times 10 = 57$

$10 = 10^1$ and multiplying by 10^1 moved the decimal point one place to the right.

b. $3.1 \times 10000 = 31000$

$10000 = 10^4$ and multiplying by 10^4 moved the decimal point four places to the right.

Multiplying a number by 10^n where n is a positive integer had the effect of moving the decimal point n places to the right.

Every number can be written as a product of a number between 1 and 10 and a power of 10. For example, $650 = 6.5 \times 100$. Since $100 = 10^2$, we can also write

$$650 = 6.5 \times 10^2$$

and this is our first example of writing a number in scientific notation.

Definition 8.1.4 A positive number is written in **scientific notation** when it has the form $a \times 10^n$ where n is an integer and $1 \leq a < 10$. In other words, a has precisely one non-zero digit to the left of the decimal place. The exponent n used here is called the number's **order of magnitude**. The number a is sometimes

[2]imagine.gsfc.nasa.gov/features/cosmic/nearest_star_info.html

called the **significand** or the **mantissa**.

Some conventions do not require a to be between 1 and 10, excluding both values, but that is the convention used in this book.

Some calculators and computer readouts cannot display exponents in superscript. In some cases, these devices will display scientific notation in the form `6.5E2` instead of 6.5×10^2. ◊

8.1.2 Scientific Notation for Large Numbers

To write a number larger than 10 in scientific notation, like 89412, first write the number with the decimal point right after its first digit, like 8.9412. Now count how many places there are between where the decimal point originally was and where it is now.

$$8.\overbrace{9412}^{4}$$

Use that count as the power of 10. In this example, we have

$$89412 = 8.9412 \times 10^4$$

Scientific notation communicates the "essence" of the number (8.9412) and then its size, or order of magnitude (10^4).

Example 8.1.5 To get a sense of how scientific notation works, let's consider familiar lengths of time converted to seconds.

Length of Time	Length in Seconds	Scientific Notation
one second	1 second	1×10^0 second
one minute	60 seconds	6×10^1 seconds
one hour	3600 seconds	3.6×10^3 seconds
one month	2,628,000 seconds	2.628×10^6 seconds
ten years	315,400,000 seconds	3.154×10^8 seconds
79 years (about a lifetime)	2,491,000,000 seconds	2.491×10^9 seconds

Note that roughly 2.6 *million* seconds is one month, while roughly 2.5 *billion* seconds is an entire lifetime.

Checkpoint 8.1.6 Write each of the following in scientific notation.

a. The federal debt at the close of business on Sept. 30, 2016: about 19,600,000,000,000 dollars.

b. The world's population in 2016: about 7,418,000,000 people.

Explanation.

a. To convert the federal debt to scientific notation, we will count the number of digits after the first non-zero digit (which happens to be a 1 here). Since there are 13 places after the first non-zero digit, we write:

$$1\overbrace{9{,}600{,}000{,}000{,}000}^{13 \text{ places}} \text{ dollars} = 1.96 \times 10^{13} \text{ dollars}$$

b. Since there are nine places after the first non-zero digit of 7, the world's population in 2016 was about

$$7,\overbrace{418{,}000{,}000}^{9 \text{ places}} \text{ people} = 7.418 \times 10^9 \text{ people}$$

Checkpoint 8.1.7 Convert each of the following from scientific notation to decimal notation (without any exponents).

a. The earth's diameter is about 1.27×10^7 meters.

b. As of 2019, there are 3.14×10^{13} known digits of π.

Explanation.

a. To convert this number to decimal notation we will move the decimal point after the digit 1 seven places to the right, including zeros where necessary. The earth's diameter is:

$$1.27 \times 10^7 \text{ meters} = 1\overbrace{2{,}700{,}000}^{7 \text{ places}} \text{ meters.}$$

b. As of 2019 there are

$$3.14 \times 10^{13} = 3\overbrace{1{,}400{,}000{,}000{,}000}^{13 \text{ places}}$$

known digits of π.

8.1.3 Scientific Notation for Small Numbers

Scientific notation can also be useful when working with numbers smaller than 1. As we saw in Figure 8.1.2, we can represent thousands, millions, billions, trillions, etc., with positive integer exponents on 10. We can similarly represent numbers smaller than 1 (which are written as tenths, hundredths, thousandths, millionths, billionths, trillionths, etc.), with *negative* integer exponents on 10. This relationship is outlined in Figure 8.1.8.

Number	English Name	Power of 10
1	one	10^0
0.1	one tenth	$\frac{1}{10} = 10^{-1}$
0.01	one hundredth	$\frac{1}{100} = 10^{-2}$
0.001	one thousandth	$\frac{1}{1{,}000} = 10^{-3}$
0.0001	one ten thousandth	$\frac{1}{10{,}000} = 10^{-4}$
0.00001	one hundred thousandth	$\frac{1}{100{,}000} = 10^{-5}$
0.000001	one millionth	$\frac{1}{1{,}000{,}000} = 10^{-6}$
0.000000001	one billionth	$\frac{1}{1{,}000{,}000{,}000} = 10^{-9}$

Figure 8.1.8: Negative Integer Powers of 10

To see how this works with a digit other than 1, let's look at 0.005. When we state 0.005 as a number, we say "5 thousandths." Thus $0.005 = 5 \times \frac{1}{1000}$. The fraction $\frac{1}{1000}$ can be written as $\frac{1}{10^3}$, which we know is equivalent to 10^{-3}. Using negative exponents, we can then rewrite 0.005 as 5×10^{-3}. This is the scientific notation for 0.005.

In practice, we won't generally do that much computation. To write a small number in scientific notation we start as we did before and place the decimal point behind the first non-zero digit. We then count the number of decimal places between where the decimal had originally been and where it now is. Keep in mind that negative powers of ten are used to help represent very small numbers (smaller than 1) and positive powers of ten are used to represent very large numbers (larger than 1). So to convert 0.005 to scientific notation, we have:

$$0\overbrace{.005}^{3} = 5 \times 10^{-3}$$

Example 8.1.9 In quantum mechanics, there is an important value called Planck's Constant[3]. Written as a decimal, the value of Planck's constant (rounded to six significant digits) is

$$0.000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,662\,607.$$

In scientific notation, this number will be $6.62607 \times 10^{?}$. To determine the exponent, we need to count the number of places from where the decimal originally is to where we will move it (following the first "6"):

$$0\overbrace{.000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,6}^{34 \text{ places}}62\,607$$

So in scientific notation, Planck's Constant is 6.62607×10^{-34}. It will be much easier to use 6.62607×10^{-34} in a calculation, and an added benefit is that scientific notation quickly communicates both the value and the order of magnitude of Planck's Constant.

Checkpoint 8.1.10 Write each of the following in scientific notation.

a. The weight of a single grain of long grain rice is about 0.029 grams.

b. The gate pitch of a microprocessor is 0.000 000 014 meters

Explanation.

a. To convert this weight to scientific notation, we must first move the decimal behind the first non-zero digit to obtain 2.9, which requires that we move the decimal point 2 places. Thus we have:

$$0\overbrace{.02}^{2}9 \text{ grams} = 2.9 \times 10^{-2} \text{ grams}$$

b. The gate pitch of a microprocessor is:

$$0\overbrace{.000\,000\,01}^{8 \text{ places}}4 \text{ meters} = 1.4 \times 10^{-8} \text{ meters}$$

Checkpoint 8.1.11 Convert each of the following from scientific notation to decimal notation (without any exponents).

a. A download speed of 7.53×10^{-3} Gigabyte per second.

b. The weight of a poppy seed is about 3×10^{-7} kilograms

Explanation.

a. To convert a download speed of 7.53×10^{-3} Gigabyte per second to decimal notation, we will move

[3] en.wikipedia.org/wiki/Planck_constant

the decimal point 3 places to the left and include the appropriate number of zeros:

$$7.53 \times 10^{-3} \text{ Gigabyte per second} = 0\overbrace{.007}^{3}53 \text{ Gigabyte per second}$$

b. The weight of a poppy seed is about:

$$3 \times 10^{-7} \text{ kilograms} = 0\overbrace{.0000003}^{7 \text{ places}} \text{ kilograms}$$

Checkpoint 8.1.12 Decide if the numbers are written in scientific notation or not. Use Definition 8.1.4.

a. The number $7 \times 10^{1.9}$ (☐ is ☐ is not) in scientific notation.

b. The number 2.6×10^{-31} (☐ is ☐ is not) in scientific notation.

c. The number 10×7^4 (☐ is ☐ is not) in scientific notation.

d. The number 0.93×10^3 (☐ is ☐ is not) in scientific notation.

e. The number 4.2×10^0 (☐ is ☐ is not) in scientific notation.

f. The number 12.5×10^{-6} (☐ is ☐ is not) in scientific notation.

Explanation.

a. The number $7 \times 10^{1.9}$ *is not* in scientific notation. The exponent on the 10 is required to be an integer and 1.9 is not.

b. The number 2.6×10^{-31} *is* in scientific notation.

c. The number 10×7^4 *is not* in scientific notation. The base must be 10, not 7.

d. The number 0.93×10^3 *is not* in scientific notation. The coefficient of the 10 must be between 1 (inclusive) and 10.

e. The number 4.2×10^0 *is* in scientific notation.

f. The number 12.5×10^{-6} *is not* in scientific notation. The coefficient of the 10 must be between 1 (inclusive) and 10.

8.1.4 Multiplying and Dividing Using Scientific Notation

One main reason for having scientific notation is to make calculations involving immensely large or small numbers easier to perform. By having the order of magnitude separated out in scientific notation, we can separate any calculation into two components.

Example 8.1.13 On Sept. 30th, 2016, the US federal debt was about \$19,600,000,000,000 and the US population was about 323,000,000. What was the average debt per person that day?

a. Calculate the answer using the numbers provided, which are not in scientific notation.

b. First, confirm that the given values in scientific notation are 1.96×10^{13} and 3.23×10^8. Then calculate the answer using scientific notation.

Explanation. We've been asked to answer the same question, but to perform the calculation using two different approaches. In both cases, we'll need to divide the debt by the population.

a. We may need to use a calculator to handle such large numbers and we have to be careful that we type the correct number of `0`s.

$$\frac{19600000000000}{323000000} \approx 60681.11$$

b. To perform this calculation using scientific notation, our work would begin by setting up the quotient as $\frac{1.96 \times 10^{13}}{3.23 \times 10^{8}}$. Dividing this quotient follows the same process we did with variable expressions of the same format, such as $\frac{1.96w^{13}}{3.23w^{8}}$. In both situations, we'll divide the coefficients and then use exponent rules to simplify the powers.

$$\begin{aligned}\frac{1.96 \times 10^{13}}{3.23 \times 10^{8}} &= \frac{1.96}{3.23} \times \frac{10^{13}}{10^{8}}\\ &\approx 0.6068111 \times 10^{5}\\ &\approx 60681.11\end{aligned}$$

The federal debt per capita in the US on September 30th, 2016 was about \$60,681.11 per person. Both calculations give us the same answer, but the calculation relying upon scientific notation has less room for error and allows us to perform the calculation as two smaller steps.

Whenever we multiply or divide numbers that are written in scientific notation, we must separate the calculation for the coefficients from the calculation for the powers of ten, just as we simplified earlier expressions using variables and the exponent rules.

Example 8.1.14

a. Multiply $\left(2 \times 10^{5}\right)\left(3 \times 10^{4}\right)$.

b. Divide $\frac{8 \times 10^{17}}{4 \times 10^{2}}$.

Explanation. We will simplify the significand/mantissa parts as one step and then simplify the powers of 10 as a separate step.

a. $$\begin{aligned}\left(2 \times 10^{5}\right)\left(3 \times 10^{4}\right) &= (2 \times 3) \times \left(10^{5} \times 10^{4}\right)\\ &= 6 \times 10^{9}\end{aligned}$$

b. $$\begin{aligned}\frac{8 \times 10^{17}}{4 \times 10^{2}} &= \frac{8}{4} \times \frac{10^{17}}{10^{2}}\\ &= 2 \times 10^{15}\end{aligned}$$

Often when we multiply or divide numbers in scientific notation, the resulting value will not be in scientific notation. Suppose we were multiplying $\left(9.3 \times 10^{17}\right)\left(8.2 \times 10^{-6}\right)$ and need to state our answer using scientific notation. We would start as we have previously:

$$\begin{aligned}\left(9.3 \times 10^{17}\right)\left(8.2 \times 10^{-6}\right) &= (9.3 \times 8.2) \times \left(10^{17} \times 10^{-6}\right)\\ &= 76.26 \times 10^{11}\end{aligned}$$

While this is a correct value, it is not written using scientific notation. One way to convert this answer into scientific notation is to turn just the coefficient into scientific notation and momentarily ignore the power of ten:

$$\begin{aligned} &= 76.26 \times 10^{11} \\ &= 7.626 \times 10^{1} \times 10^{11} \end{aligned}$$

Now that the coefficient fits into the proper format, we can combine the powers of ten and have our answer written using scientific notation.

$$\begin{aligned} &= 7.626 \times 10^{1} \times 10^{11} \\ &= 7.626 \times 10^{12} \end{aligned}$$

Example 8.1.15 Multiply or divide as indicated. Write your answer using scientific notation.

a. $\left(8 \times 10^{21}\right)\left(2 \times 10^{-7}\right)$

b. $\dfrac{2 \times 10^{-6}}{8 \times 10^{-19}}$

Explanation. Again, we'll separate out the work for the significand / mantissa from the work for the powers of ten. If the resulting coefficient is not between 1 and 10, we'll need to adjust that coefficient to put it into scientific notation.

a. $$\begin{aligned} \left(8 \times 10^{21}\right)\left(2 \times 10^{-7}\right) &= (8 \times 2) \times \left(10^{21} \times 10^{-7}\right) \\ &= 16 \times 10^{14} \\ &= 1.6 \times 10^{1} \times 10^{14} \\ &= 1.6 \times 10^{15} \end{aligned}$$

We need to remember to apply the product rule for exponents to the powers of ten.

b. $$\begin{aligned} \frac{2 \times 10^{-6}}{8 \times 10^{-19}} &= \frac{2}{8} \times \frac{10^{-6}}{10^{-19}} \\ &= 0.25 \times 10^{13} \\ &= 2.5 \times 10^{-1} \times 10^{13} \\ &= 2.5 \times 10^{12} \end{aligned}$$

There are times where we will have to raise numbers written in scientific notation to a power. For example, suppose we have to find the area of a square whose radius is 3×10^{7} feet. To perform this calculation, we first remember the formula for the area of a square, $A = s^2$ and then substitute 3×10^7 for s: $A = \left(3 \times 10^{7}\right)^{2}$. To perform this calculation, we'll need to remember to use the product to a power rule and the power to a power rule:

$$\begin{aligned} A &= \left(3 \times 10^{7}\right)^{2} \\ &= (3)^{2} \times \left(10^{7}\right)^{2} \\ &= 9 \times 10^{14} \end{aligned}$$

8.1.5 Reading Questions

1. Which number is very large and which number is very small?

$$9.99 \times 10^{-47} \qquad 1.01 \times 10^{23}$$

2. Since some computer/calculator screens can't display an exponent, how might a computer/calculator display the number 2.318×10^{13}?

3. Why do we bother having scientific notation for numbers?

8.1.6 Exercises

Converting To and From Scientific Notation Write the following number in scientific notation.

1. 100000 **2.** 20000 **3.** 300 **4.** 400000
5. 0.005 **6.** 0.0006 **7.** 0.07 **8.** 0.008

Write the following number in decimal notation without using exponents.

9. 9×10^2 **10.** 1.1×10^5 **11.** 2.02×10^3
12. 3.02×10^2 **13.** 4.01×10^0 **14.** 5.01×10^0
15. 6×10^{-4} **16.** 7×10^{-2} **17.** 8×10^{-4}
18. 8.99×10^{-2}

Arithmetic with Scientific Notation Multiply the following numbers, writing your answer in scientific notation.

19. $(9 \times 10^2)(7 \times 10^2)$ **20.** $(2 \times 10^4)(4 \times 10^5)$ **21.** $(3 \times 10^2)(9 \times 10^4)$
22. $(4 \times 10^3)(6 \times 10^3)$ **23.** $(5 \times 10^5)(3 \times 10^5)$ **24.** $(6 \times 10^3)(9 \times 10^4)$

Divide the following numbers, writing your answer in scientific notation.

25. $\dfrac{4.2 \times 10^5}{7 \times 10^3}$ **26.** $\dfrac{2.4 \times 10^3}{8 \times 10^2}$ **27.** $\dfrac{7.2 \times 10^5}{9 \times 10^2}$

28. $\dfrac{5.4 \times 10^6}{9 \times 10^4}$ **29.** $\dfrac{6 \times 10^3}{2 \times 10^{-4}}$ **30.** $\dfrac{2.4 \times 10^5}{3 \times 10^{-2}}$

31. $\dfrac{2 \times 10^2}{4 \times 10^{-3}}$ **32.** $\dfrac{1 \times 10^4}{5 \times 10^{-2}}$ **33.** $\dfrac{4.8 \times 10^{-5}}{6 \times 10^2}$

34. $\dfrac{3.5 \times 10^{-3}}{7 \times 10^5}$ **35.** $\dfrac{1.6 \times 10^{-2}}{8 \times 10^4}$ **36.** $\dfrac{6.3 \times 10^{-4}}{9 \times 10^3}$

Simplify the following expression, writing your answer in scientific notation.

37. $(5 \times 10^5)^4$ **38.** $(2 \times 10^2)^2$ **39.** $(2 \times 10^8)^3$
40. $(3 \times 10^5)^2$ **41.** $(3 \times 10^{10})^3$ **42.** $(4 \times 10^7)^4$

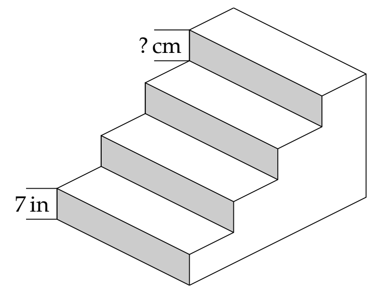
8.2 Unit Conversion

Unit Conversion References. This textbook provides unit conversions in Appendix B for your convenience. But you may also find unit conversion facts in many other places, including the internet.

Unit conversion is a systematic method for converting from one kind of unit of measurement to another. It is used extensively in chemistry and other health- or science-related fields. It is a valuable skill to learn, and necessary for success in many applications.

8.2.1 Unit Ratios

Example 8.2.1
When building a staircase, a step typically has a rise of 7 inches (7 in). An inch is a unit of length in the imperial unit system, used in the United States, Canada, the United Kingdom, and a few other places. Many parts of the world do not use this unit of measurement, and the people there do not have a sense of how long 7 inches is. Instead, much of the world would measure a length like this using centimeters (cm). How many centimeters is 7 inches?



To convert from one unit of measurement to another (like inches to centimeters), we use what are called unit ratios. A **unit ratio** is a ratio (or fraction) where the numerator and denominator are quantities *with units* that equal each other. They equal each other as measurements, but they are measured with different units. For example, Appendix B tells us that 1 inch is equal to 2.54 centimeters. Knowing that, we can build the unit ratios $\frac{1\,\text{in}}{2.54\,\text{cm}}$ and $\frac{2.54\,\text{cm}}{1\,\text{in}}$. Each of these unit ratios are equivalent to 1, because their numerator equals their denominator.

With a unit ratio, we can work out a conversion by taking what we would like to convert (7 in) and multiplying by a unit ratio in such a way that the "old" units cancel and the "new" units remain.

$$\begin{aligned} 7\,\text{in} &= \frac{7\,\text{in}}{1} && \text{We are about to do fraction-like multiplication.} \\ &= \frac{7\,\text{in}}{1} \cdot \frac{2.54\,\text{cm}}{1\,\text{in}} && \text{1 in equals 2.54 cm.} \\ &= \frac{7\,\cancel{\text{in}}}{1} \cdot \frac{2.54\,\text{cm}}{1\,\cancel{\text{in}}} && \text{Units may now cancel.} \\ &= \frac{7}{1} \cdot \frac{2.54\,\text{cm}}{1} \\ &= 7 \cdot 2.54\,\text{cm} \\ &= 17.78\,\text{cm} \end{aligned}$$

So 7 inches is equal to 17.78 centimeters. In practice, anyone talking about the rise of a stair might simply round to 18 cm.

Note there was another unit ratio, $\frac{1\,\text{in}}{2.54\,\text{cm}}$, but using that would not have been helpful, since it would not have arranged units such that the inches canceled.

Remark 8.2.2 When you are comfortable, you might do the steps from Example 8.2.1 on one line, like:

$$7\,\text{in} = \frac{7\,\cancel{\text{in}}}{1} \cdot \frac{2.54\,\text{cm}}{1\,\cancel{\text{in}}} = \frac{7}{1} \cdot \frac{2.54}{1}\,\text{cm} = 17.78\,\text{cm}$$

The examples in this section will continue to show the steps completely drawn out, to give a better sense of what you would write first, second, and so on.

Example 8.2.3 A canned beverage typically contains 12 fluid ounces (12 fl oz). A fluid ounce is a unit of volume used in the United States. (The United Kingdom also has a fluid ounce, but it is a slightly different amount.) In the rest of the world, people do not have a sense of how much 12 fluid ounces is. Most of the world would measure a canned beverage's volume using milliliters (mL). How many milliliters is 12 fluid ounces?

Appendix B tells us that 1 fl oz is (almost) equal to 29.57 mL. Knowing that, we can build the unit ratios $\frac{1\,\text{fl oz}}{29.57\,\text{mL}}$ and $\frac{29.57\,\text{mL}}{1\,\text{fl oz}}$. Each of these unit ratios are (almost) equivalent to 1, because their numerator (almost) equals their denominator.

Using the appropriate unit ratio to enable cancellation of fluid ounces:

$$\begin{aligned}
12\,\text{fl oz} &= \frac{12\,\text{fl oz}}{1} && \text{We are about to do fraction-like multiplication.}\\
&\approx \frac{12\,\text{fl oz}}{1}\cdot\frac{29.57\,\text{mL}}{1\,\text{fl oz}} && \text{1 fl oz approximately equals 29.57 mL.}\\
&= \frac{12\,\cancel{\text{fl oz}}}{1}\cdot\frac{29.57\,\text{mL}}{1\,\cancel{\text{fl oz}}} && \text{Units may now cancel.}\\
&= \frac{12}{1}\cdot\frac{29.57\,\text{mL}}{1}\\
&= 12\cdot 29.57\,\text{mL}\\
&\approx 354.8\,\text{mL}
\end{aligned}$$

So 12 fluid ounces is *approximately* equal to 354.8 milliliters. In practice, you might round to 355 mL.

Notice that each conversion fact from Appendix B gives two possible unit ratios. Deciding which one to use will depend on where units need to be placed in order to cancel the appropriate units. In unit conversion, we multiply ratios together and cancel common units the same way we can cancel common factors when multiplying fractions.

Example 8.2.4 It's 1760 feet (1760 ft) to walk from Jonah's house to where he works. How many miles is that?

Explanation. Since we are converting feet to miles, we use the conversion fact that there are 5280 feet in 1 mile. In this conversion, we need to use a unit ratio that will allow the feet units to cancel. So we need to use $\frac{1\,\text{mi}}{5280\,\text{ft}}$. This is different from previous examples in that the 1 is in the numerator this time. But the process is not all that different.

$$\begin{aligned}
1760\,\text{ft} &= \frac{1760\,\text{ft}}{1} && \text{We are about to do fraction-like multiplication.}\\
&= \frac{1760\,\text{ft}}{1}\cdot\frac{1\,\text{mi}}{5280\,\text{ft}} && \text{1 mi equals 5280 ft.}\\
&= \frac{1760\,\cancel{\text{ft}}}{1}\cdot\frac{1\,\text{mi}}{5280\,\cancel{\text{ft}}} && \text{Units may now cancel.}\\
&= \frac{1760}{1}\cdot\frac{1\,\text{mi}}{5280}\\
&= \frac{1760}{5280}\,\text{mi}
\end{aligned}$$

$$= \frac{1}{3}\,\text{mi} \approx 0.3333\,\text{mi}$$

So Jonah walks $\frac{1}{3}$ of a mile, or about 0.3333 mi, to get from his house to where he works.

Checkpoint 8.2.5 Convert 60 inches to feet.

Explanation. We start by writing what it is that we are converting as a ratio, by placing it over a 1. This is similar to writing a whole number as a fraction when we want to multiply it by a fraction. Next we multiply that ratio by a unit ratio, one that will have inches in the denominator so that inches will cancel. Multiply what's left just as we multiply fractions (multiply the numerators together and multiply the denominators together), including the units, and simplify by dividing.

$$\begin{aligned}
60\,\text{in} &= \frac{60\,\text{in}}{1} && \text{We are about to do fraction-like multiplication.}\\
&= \frac{60\,\text{in}}{1} \cdot \frac{1\,\text{ft}}{12\,\text{in}} && \text{1 ft equals 12 in.}\\
&= \frac{60\,\cancel{\text{in}}}{1} \cdot \frac{1\,\text{ft}}{12\,\cancel{\text{in}}} && \text{Units may now cancel.}\\
&= \frac{60}{1} \cdot \frac{1\,\text{ft}}{12}\\
&= \frac{60}{12}\,\text{ft}\\
&= 5\,\text{ft}
\end{aligned}$$

We find that 60 inches is equivalent to 5 feet.

Example 8.2.6 Why Do We Convert Units? Converting from one unit to another can be necessary when you are given information where the units don't quite match. Cassidy was driving at a speed of 32 mph for seven minutes. How far did they travel in that time span?

Normally, to find a distance traveled, you would multiply speed by how much time passed. For example if Cassidy had been driving 50 mph for two hours, we would find $50 \cdot 2 = 100$, and conclude they had driven 100 miles.

But in this example, Cassidy's speed is 32 miles per *hour*, but the time elapsed is seven *minutes*. The time units do not match. It will help to convert the 7 min into hours. So let's do that.

$$\begin{aligned}
7\,\text{min} &= \frac{7\,\text{min}}{1} && \text{We are about to do fraction-like multiplication.}\\
&= \frac{7\,\text{min}}{1} \cdot \frac{1\,\text{h}}{60\,\text{min}} && \text{1 h equals 60 min.}\\
&= \frac{7\,\cancel{\text{min}}}{1} \cdot \frac{1\,\text{h}}{60\,\cancel{\text{min}}} && \text{Units may now cancel.}\\
&= \frac{7}{1} \cdot \frac{1\,\text{h}}{60}\\
&= \frac{7}{60}\,\text{h}\\
&\approx 0.1167\,\text{h}
\end{aligned}$$

Now we can multiply Cassidy's speed (32 mph) by their elapsed time ($\frac{7}{60}$ h). We find $32 \cdot \frac{7}{60} \approx 3.733$, so Cassidy has traveled about 3.733 miles.

Actually we can do this multiplication *with units* and the units will cancel appropriately:

$$\begin{aligned}
32\,\frac{\text{mi}}{\text{h}} \cdot \frac{7}{60}\,\text{h} &= \frac{32\,\text{mi}}{1\,\text{h}} \cdot \frac{7\,\text{h}}{60} \\
&= \frac{32\,\text{mi}}{1\,\cancel{\text{h}}} \cdot \frac{7\,\cancel{\text{h}}}{60} \\
&= \frac{32 \cdot 7}{60}\,\text{mi} \\
&\approx 3.733\,\text{mi}
\end{aligned}$$

Checkpoint 8.2.7 The density of oil is 6.9 pounds per gallon. You have a 2.5-liter bottle of oil. How much does this much oil weigh? (To find weight, multiply density with volume when the units match.)

Explanation. The density is in pounds per *gallon*, but the volume is in *liters*. So first let's convert the 2.5 L to gallons.

$$\begin{aligned}
2.5\,\text{L} &= \frac{2.5\,\text{L}}{1} && \text{We are about to do fraction-like multiplication.} \\
&\approx \frac{2.5\,\text{L}}{1} \cdot \frac{1\,\text{gal}}{3.785\,\text{L}} && \text{1 gal approximately equals 3.785 L.} \\
&= \frac{2.5\,\cancel{\text{L}}}{1} \cdot \frac{1\,\text{gal}}{3.785\,\cancel{\text{L}}} && \text{Units may now cancel.} \\
&= \frac{2.5}{1} \cdot \frac{1\,\text{gal}}{3.785} \\
&= \frac{2.5}{3.785}\,\text{gal} \\
&\approx 0.6605\,\text{gal}
\end{aligned}$$

Now we can multiply the density (6.9 lb/gal) by the volume ($\approx$ 0.6605 gal). We find $6.9 \cdot 0.6605 \approx 4.557$, so the oil weighs about 4.557 pounds.

With units:

$$\begin{aligned}
6.9\,\frac{\text{lb}}{\text{gal}} \cdot 0.6605\,\text{gal} &= \frac{6.9\,\text{lb}}{1\,\text{gal}} \cdot \frac{0.6605\,\text{gal}}{1} \\
&= \frac{6.9\,\text{lb}}{1\,\cancel{\text{gal}}} \cdot \frac{0.6605\,\cancel{\text{gal}}}{1} \\
&= 6.9 \cdot 0.6605\,\text{lb} \\
&\approx 4.557\,\text{lb}
\end{aligned}$$

8.2.2 Using Multiple Unit Ratios

In previous examples, we used only one unit ratio to make a conversion. However, sometimes there is a need to use more than one unit ratio in a conversion. This may happen when your reference guide for conversions does not directly tell you how to convert from one unit to another. In those situations, we'll have to consider the conversion facts that are available and then make a plan.

Example 8.2.8 Convert 350 yards to miles.

Explanation. In Appendix B, there is not a conversion that relates yards to miles. But notice that we can convert yards to feet (using the fact that one yard is three feet) and then we can convert feet to miles (using the fact that one mile is 5280 feet). So we will use two unit ratios. The unit ratio $\frac{3\,\text{ft}}{1\,\text{yd}}$ can be used to cancel the yards in 350 yd. Then the unit ratio $\frac{1\,\text{mi}}{5280\,\text{ft}}$ can be used to cancel the feet that are left over from the first conversion.

$$\begin{aligned}
350\,\text{yd} &= \frac{350\,\text{yd}}{1} && \text{We are about to do fraction-like multiplication.}\\
&= \frac{350\,\text{yd}}{1}\cdot\frac{3\,\text{ft}}{1\,\text{yd}}\cdot\frac{1\,\text{mi}}{5280\,\text{ft}} && \text{Both unit ratios are needed.}\\
&= \frac{350\,\cancel{\text{yd}}}{1}\cdot\frac{3\,\cancel{\text{ft}}}{1\,\cancel{\text{yd}}}\cdot\frac{1\,\text{mi}}{5280\,\cancel{\text{ft}}} && \text{Units may now cancel.}\\
&= \frac{350}{1}\cdot\frac{3}{1}\cdot\frac{1\,\text{mi}}{5280}\\
&= \frac{350\cdot 3}{5280}\,\text{mi}\\
&\approx 0.1989\,\text{mi}
\end{aligned}$$

So 350 yards is about 0.1989 miles.

Checkpoint 8.2.9 Convert 4.5 months into hours.

Explanation. Notice that we can convert months to days (using the fact that one month is approximately 30 days) and then we can convert days to hours (using the fact that one day is 24 hours).

$$\begin{aligned}
4.5\,\text{mo} &= \frac{4.5\,\text{mo}}{1} && \text{We are about to do fraction-like multiplication.}\\
&\approx \frac{4.5\,\text{mo}}{1}\cdot\frac{30\,\text{d}}{1\,\text{mo}}\cdot\frac{24\,\text{h}}{1\,\text{d}} && \text{Two unit ratios are needed.}\\
&= \frac{4.5\,\cancel{\text{mo}}}{1}\cdot\frac{30\,\cancel{\text{d}}}{1\,\cancel{\text{mo}}}\cdot\frac{24\,\text{h}}{1\,\cancel{\text{d}}} && \text{Units may now cancel.}\\
&= \frac{4.5}{1}\cdot\frac{30}{1}\cdot\frac{24\,\text{h}}{1}\\
&= 4.5\cdot 30\cdot 24\,\text{h}\\
&= 3240\,\text{h}
\end{aligned}$$

So 4.5 months is about 3240 hours.

8.2.3 Converting Squared or Cubed Units

When calculating the area or volume of a geometric figure, units of measurement are multiplied together, resulting in squared units (when calculating area) or cubed units (when calculating volume). Thus, there may be circumstances where you may need to convert either squared or cubed units. For example, suppose you are carpeting a room in your home and you know the square footage of the room, but the carpet is sold in square yards. In that case, you would need to convert the square feet of the room into square yards.

Example 8.2.10 Jin's bedroom is 153 square feet ($153\,\text{ft}^2$). How many square yards is that?

We start the process the same as in the previous examples. That is, we write what we are converting in ratio form with a denominator of 1.

$$153\,\text{ft}^2 = \frac{153\,\text{ft}^2}{1} \qquad \text{We are about to do fraction-like multiplication.}$$

Now, we do want feet to be replaced with yards, so the unit ratio $\frac{1\,\text{yd}}{3\,\text{ft}}$ will be useful. But using it once is not enough:

$$\begin{aligned} 153\,\text{ft}^2 &= \frac{153\,\text{ft}^2}{1} \\ &= \frac{153\,\text{ft}^2}{1} \cdot \frac{1\,\text{yd}}{3\,\text{ft}} && \text{1 yd equals 3 feet.} \end{aligned}$$

The ft^2 in the first numerator do not fully cancel with the ft in the second denominator. We need to use this unit ratio *twice*.

$$\begin{aligned} 153\,\text{ft}^2 &= \frac{153\,\text{ft}^2}{1} \\ &= \frac{153\,\text{ft}^2}{1} \cdot \frac{1\,\text{yd}}{3\,\text{ft}} \cdot \frac{1\,\text{yd}}{3\,\text{ft}} && \text{1 yd equals 3 feet.} \end{aligned}$$

Now there is ft^2 in the overall numerator, and $\text{ft} \cdot \text{ft}$ in the overall denominator. They will fully cancel.

Here is the complete process from the beginning.

$$\begin{aligned} 153\,\text{ft}^2 &= \frac{153\,\text{ft}^2}{1} \\ &= \frac{153\,\text{ft}^2}{1} \cdot \frac{1\,\text{yd}}{3\,\text{ft}} \cdot \frac{1\,\text{yd}}{3\,\text{ft}} \\ &= \frac{153\,\cancel{\text{ft}^2}}{1} \cdot \frac{1\,\text{yd}}{3\,\cancel{\text{ft}}} \cdot \frac{1\,\text{yd}}{3\,\cancel{\text{ft}}} && \text{Units may now cancel.} \\ &= \frac{153}{1} \cdot \frac{1\,\text{yd}}{3} \cdot \frac{1\,\text{yd}}{3} \\ &= \frac{153}{9}\,\text{yd} \cdot \text{yd} \\ &= 17\,\text{yd}^2 \end{aligned}$$

So Jin's bedroom has 17 square yards of area.

Alternatively, we can set up conversions with squared or cubed units this way:

$$\begin{aligned} 153\,\text{ft}^2 &= \frac{153\,\text{ft}^2}{1} \\ &= \frac{153\,\text{ft}^2}{1} \cdot \left(\frac{1\,\text{yd}}{3\,\text{ft}}\right)^2 && \text{The ft in the denominator will be squared.} \\ &= \frac{153\,\text{ft}^2}{1} \cdot \frac{1\,\text{yd}^2}{9\,\text{ft}^2} && \text{Using Fact 5.6.5.} \\ &= \frac{153\,\cancel{\text{ft}^2}}{1} \cdot \frac{1\,\text{yd}^2}{9\,\cancel{\text{ft}^2}} && \text{Units may now cancel.} \end{aligned}$$

$$
\begin{aligned}
&= \frac{153}{1} \cdot \frac{1\,\text{yd}^2}{9} \\
&= \frac{153}{9}\,\text{yd}^2 \\
&= 17\,\text{yd}^2
\end{aligned}
$$

When using this setup where the unit ratio is raised to a power, you must be careful to remember that *everything* inside the parentheses is raised to that power: the units and the numbers alike.

Checkpoint 8.2.11 Convert 85 cubic inches into cubic centimeters.

Explanation.

$$
\begin{aligned}
85\,\text{in}^3 &= \frac{85\,\text{in}^3}{1} \\
&= \frac{85\,\text{in}^3}{1} \cdot \left(\frac{2.54\,\text{cm}}{1\,\text{in}}\right)^3 && \text{The inches in the denominator will be cubed.} \\
&= \frac{85\,\text{in}^3}{1} \cdot \frac{2.54^3\,\text{cm}^3}{1\,\text{in}^3} && \text{Using the quotient to a power rule.} \\
&= \frac{85\,\cancel{\text{in}^3}}{1} \cdot \frac{2.54^3\,\text{cm}^3}{1\,\cancel{\text{in}^3}} && \text{Units may now cancel.} \\
&= \frac{85}{1} \cdot \frac{2.54^3\,\text{cm}^3}{1} \\
&= 85 \cdot 2.54^3\,\text{cm}^3 \\
&\approx 1393\,\text{cm}^3
\end{aligned}
$$

So 85 cubic inches is about 1393 cubic centimeters.

8.2.4 Converting Rates

A rate unit has a numerator and a denominator. For example, speed is a rate, and speed can be measured in $\frac{\text{mi}}{\text{h}}$. The numerator unit is a mile and the denominator unit is an hour.

Suppose we wanted to convert a speed rate, such as $65\,\frac{\text{mi}}{\text{h}}$, into $\frac{\text{m}}{\text{s}}$. Or a concentration rate, such as $180\,\frac{\text{mg}}{\text{L}}$, into $\frac{\text{g}}{\text{dL}}$. We can use the same process that we've used before to do these conversions. That is, we start by writing what we want to convert as a ratio, which will have units in both the numerator and denominator, and then we multiply by unit ratios until both units have been converted into the units we want. It helps to focus on converting one unit at a time and to make sure that the units in our unit ratios are placed so that the proper units will cancel.

Example 8.2.12 Convert $65\,\frac{\text{mi}}{\text{h}}$ into $\frac{\text{m}}{\text{min}}$.

Explanation. We start by writing what we are converting, which is $65\,\frac{\text{mi}}{\text{h}}$, as a ratio. Then, our job is to convert the miles to meters and the hours to minutes, one at a time. It doesn't matter which unit ratio we use first, as long as the units line up to cancel appropriately.

$$
65\,\frac{\text{mi}}{\text{h}} = \frac{65\,\text{mi}}{1\,\text{h}} \qquad \text{Write the rate as a ratio.}
$$

$$\begin{aligned}
&\approx \frac{65\,\text{mi}}{1\,\text{h}} \cdot \frac{1.609\,\text{km}}{1\,\text{mi}} \cdot \frac{1000\,\text{m}}{1\,\text{km}} \cdot \frac{1\,\text{h}}{60\,\text{min}} && \text{Use unit ratios to make cancellations.}\\
&= \frac{65\,\cancel{\text{mi}}}{1\,\cancel{\text{h}}} \cdot \frac{1.609\,\cancel{\text{km}}}{1\,\cancel{\text{mi}}} \cdot \frac{1000\,\text{m}}{1\,\cancel{\text{km}}} \cdot \frac{1\,\cancel{\text{h}}}{60\,\text{min}} && \text{Units may now cancel.}\\
&= \frac{65}{1} \cdot \frac{1.609}{1} \cdot \frac{1000\,\text{m}}{1} \cdot \frac{1}{60\,\text{min}}\\
&= \frac{65 \cdot 1.609 \cdot 1000}{60}\,\frac{\text{m}}{\text{min}}\\
&\approx 1743\,\frac{\text{m}}{\text{min}}
\end{aligned}$$

Notice that the last unit ratio is used to convert the hours to minutes and the hour must be placed in the numerator to cancel the hour in the original rate that was in the denominator. Also, note that this will automatically cause minutes to end up in the denominator, which is where this unit should end up so that we end up with meters *per minute* for our final unit.

An important thing to keep in mind, as demonstrated in the previous example, as well as the next example, is that we avoid multiplying or dividing any numbers until the end, after the final units that we want have been obtained. Stopping partway through to multiply or divide some numbers could lead to confusion and mistakes.

Checkpoint 8.2.13 Convert 180 mg/L into g/dL, given that there are 10 deciliters in a liter.

Explanation. We start by writing what we are converting, which is 180 mg/L, as a ratio. Then, we need to convert the milligrams into grams and the liters into deciliters, converting one unit at a time. We will start by converting the milligrams into grams. Then, we will convert the liters to deciliters.

$$\begin{aligned}
180\,\frac{\text{mg}}{\text{L}} &= \frac{180\,\text{mg}}{1\,\text{L}} && \text{Write the rate as a ratio.}\\
&= \frac{180\,\text{mg}}{1\,\text{L}} \cdot \frac{1\,\text{g}}{1000\,\text{mg}} \cdot \frac{1\,\text{L}}{10\,\text{dL}} && \text{Use unit ratios to make cancellations.}\\
&= \frac{180\,\cancel{\text{mg}}}{1\,\cancel{\text{L}}} \cdot \frac{1\,\text{g}}{1000\,\cancel{\text{mg}}} \cdot \frac{1\,\cancel{\text{L}}}{10\,\text{dL}} && \text{Units may now cancel.}\\
&= \frac{180}{1} \cdot \frac{1\,\text{g}}{1000} \cdot \frac{1}{10\,\text{dL}}\\
&= \frac{180}{1000 \cdot 10}\,\frac{\text{g}}{\text{dL}}\\
&\approx 0.018\,\frac{\text{g}}{\text{dL}}
\end{aligned}$$

So for example if salt is mixed into water with a concentration of 180 mg/L, the concentration can also be described as 0.018 g/dL.

8.2.5 Reading Questions

1. Unit conversion is a lot like multiplying $\boxed{}$.

2. If you are using a unit ratio to convert inches to feet, how do you decide whether to use $\frac{1\,\text{ft}}{12\,\text{in}}$ or to use $\frac{12\,\text{in}}{1\,\text{ft}}$?

3. If you use a power of a unit ratio to make a unit conversion, what do you need to remember?

8.2.6 Exercises

Review and Warmup

1. Multiply: $\frac{5}{9} \cdot \frac{5}{8}$

2. Multiply: $\frac{4}{9} \cdot \frac{4}{7}$

3. Multiply: $\frac{14}{11} \cdot \frac{13}{6}$

4. Multiply: $\frac{15}{7} \cdot \frac{4}{15}$

5. Multiply: $10 \cdot \frac{1}{7}$

6. Multiply: $3 \cdot \frac{2}{5}$

Unit Conversions

7. Convert 7.8 min to seconds.

8. Convert 2.6 mi^2 to acres.

9. Convert 633 mi^2 to acres.

10. Convert 1.11 mi to feet.

11. Convert 49.7 mg to grams.

12. Convert 865 mg to grams.

13. Convert 3.42 m^2 to hectares.

14. Convert 7.9 mL to cubic centimeters.

15. Convert 16 mL to gallons.

16. Convert 5.4 B to kilobits.

17. Convert 91 T to ounces.

18. Convert 418 ns to milliseconds.

19. Convert 7.95 mm to hectometers.

20. Convert 26.3 km to hectometers.

21. Convert 649 ft^2 to square miles.

22. Convert 1.17 kg to milligrams.

23. Convert 45 m^3 to cubic yards.

24. Convert 25 hm^2 to square meters.

25. Convert 8.5 yd^3 to cubic feet.

26. Convert 65 dm^2 to square meters.

27. Convert 3.85 in^3 to cubic centimeters.

28. Convert 13.5 mm^2 to square meters.

29. Convert 785 km^3 to cubic meters.

30. Convert 5.35 mi^2 to square feet.

31. Convert 40.5 $\frac{\text{yd}}{\text{ms}}$ to meters per second.

32. Convert 29 $\frac{\text{m}}{\text{s}}$ to decimeters per millisecond.

33. Convert 64 $\frac{\text{acre}}{\text{wk}}$ to square miles per day.

34. Convert 1.4 $\frac{\text{mi}^2}{\text{ms}}$ to acres per second.

35. Convert 84 $\frac{\text{mL}}{\text{d}}$ to liters per hour.

36. Convert 8.78 $\frac{\text{cc}}{\text{wk}}$ to liters per day.

37. Convert 34.5 $\frac{\text{T}}{\text{wk}}$ to pounds per day.

38. Convert 77.3 $\frac{\text{g}}{\text{wk}}$ to kilograms per day.

39. Convert 1.99 $\frac{\text{kb}}{\text{h}}$ to bits per day.

40. Convert 57.7 $\frac{\text{kb}}{\text{h}}$ to megabits per minute.

41. Convert 94 $\frac{\text{oz}}{\text{in}^3}$ to pounds per gallon.

42. Convert 4.2 $\frac{\text{oz}}{\text{cc}}$ to pounds per milliliter.

Applications

43. Renee's bedroom has 124 ft^2 of floor. She would like to carpet the floor, but carpeting is sold by the square yard. How many square yards of carpeting will she need to get?

44. Charlotte's bedroom has 137 ft^2 of floor. She would like to carpet the floor, but carpeting is sold by the square yard. How many square yards of carpeting will she need to get?

45. Kenji is traveling in Europe and renting a car. He is used to thinking of gasoline amounts in gallons, but in Europe it is sold in liters. After filling the gas tank, he notices it took 39 L of gas. How many gallons is that?

46. Alisa is traveling in Europe and renting a car. She is used to thinking of gasoline amounts in gallons, but in Europe it is sold in liters. After filling the gas tank, she notices it took 42 L of gas. How many gallons is that?

47. Kara found a family recipe from the old country that uses 330 mL of soup stock. The recipe serves four, but Kara wants to scale it up to serve fourteen. And none of Kara's measuring devices use the metric system. How many cups of soup stock should she use?

48. Scot found a family recipe from the old country that uses 360 mL of soup stock. The recipe serves four, but Scot wants to scale it up to serve ten. And none of Scot's measuring devices use the metric system. How many cups of soup stock should he use?

49. Dawn was driving at a steady speed of 67 mph for 10 minutes. How far did she travel in that time?

50. Tien was driving at a steady speed of 25 mph for 23 minutes. How far did he travel in that time?

51. The algae in a pond is growing at a rate of $0.18\ \frac{\text{kg}}{\text{d}}$. How much algae is in the poind after 16 weeks?

52. The algae in a pond is growing at a rate of $0.22\ \frac{\text{kg}}{\text{d}}$. How much algae is in the poind after 9 weeks?

53. Brandon is downloading content at an average rate of 48 Mbps (megabits per second). After 189 minutes, how much has he downloaded? It is appropriate to express an amount of data like this in bytes, kilobytes, megabytes, gigabytes, or terabytes.

54. Sarah is downloading content at an average rate of 58 Mbps (megabits per second). After 124 minutes, how much has she downloaded? It is appropriate to express an amount of data like this in bytes, kilobytes, megabytes, gigabytes, or terabytes.

[1] https://cnx.org/contents/hAiMlVjM@8.4:caPlSDX_@6/Dimensional-Analysis
[2] https://cnx.org/contents/hAiMlVjM@8.4:8DuPvYyV@7/Converting-Between-the-Two-Systems-of-Measurement
[3] https://cnx.org/contents/hAiMlVjM@8.5:jRv6NP4J@7/Converting-Rates
[4] http://creativecommons.org/licenses/by/4.0

8.3 Geometry Formulas

In this section we will evaluate some formulas related to the geometry of two- and three-dimensional shapes.

8.3.1 Evaluating Perimeter and Area Formulas

Rectangles. The rectangle in Figure 8.3.2 has a length (as measured by the edges on the top and bottom) and a width (as measured by the edges on the left and right).

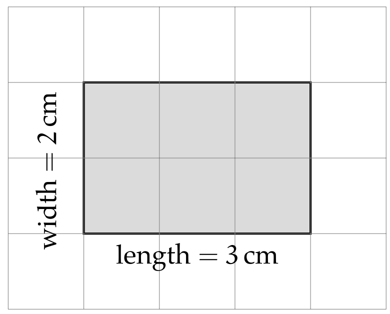


Figure 8.3.2: A Rectangle

Perimeter is the distance around the edge(s) of a two-dimensional shape. To calculate perimeter, start from a point on the shape (usually a corner), travel around the shape, and add up the total distance traveled. For the rectangle in Figure 8.3.2, if we travel around it, the total distance would be:

$$\begin{aligned} \text{rectangle perimeter} &= 3\,\text{cm} + 2\,\text{cm} + 3\,\text{cm} + 2\,\text{cm} \\ &= 10\,\text{cm}. \end{aligned}$$

Another way to compute a rectangle's perimeter would be to start at one corner, add up the edge length half-way around, and then double that. So we could have calculated the perimeter this way:

$$\begin{aligned} \text{rectangle perimeter} &= 2(3\,\text{cm} + 2\,\text{cm}) \\ &= 2(5\,\text{cm}) \\ &= 10\,\text{cm}. \end{aligned}$$

There is nothing special about this rectangle having length 3 cm and width 2 cm. With a generic rectangle, it has some length we can represent with the variable ℓ and some width we can represent with the variable w. We can use P to represent its perimeter, and then the perimeter of the rectangle will be given by:

$$P = 2(\ell + w).$$

Area is the number of 1×1 squares that fit inside a two-dimensional shape (possibly after morphing them into non-square shapes). If the edges of the squares are, say, 1 cm long, then the area is measured in "square cm," written cm^2. In Figure 8.3.2, the rectangle has six 1 cm × 1 cm squares, so its area is 6 square centimeters.

Note that we can find that area by multiplying the length and the width:

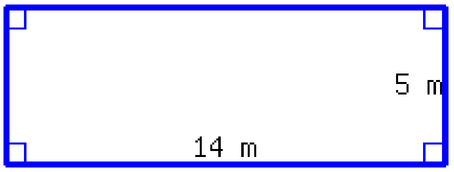
$$\text{rectangle area} = (3\,\text{cm}) \cdot (2\,\text{cm})$$

$$= 6\,\text{cm}^2$$

Again, there is nothing special about this rectangle having length 3 cm and width 2 cm. With a generic rectangle, it has some length we can represent with the variable ℓ and some width we can represent with the variable w. We can represent its area with the variable A, and then the area of the rectangle will be given by:

$$A = \ell \cdot w.$$

Checkpoint 8.3.3 Find the perimeter and area of the rectangle.



Its perimeter is ______ and its area is ______.

Explanation. Using the perimeter and area formulas for a rectangle, we have:

$$\begin{aligned} P &= 2(\ell + w) & A &= \ell \cdot w \\ &= 2(14 + 5) & &= 14 \cdot 5 \\ &= 2(19) & &= 70 \\ &= 38 \end{aligned}$$

Since length and width were measured in meters, we find that the perimeter is 38 meters and the area is 70 square meters.

Example 8.3.4
Imagine a rectangle with width 7.5 in and height 11.43 cm as in Figure 8.3.5.

a. Find the perimeter (in inches) of the rectangle.

b. Find the area (in square centimeters) of the rectangle.

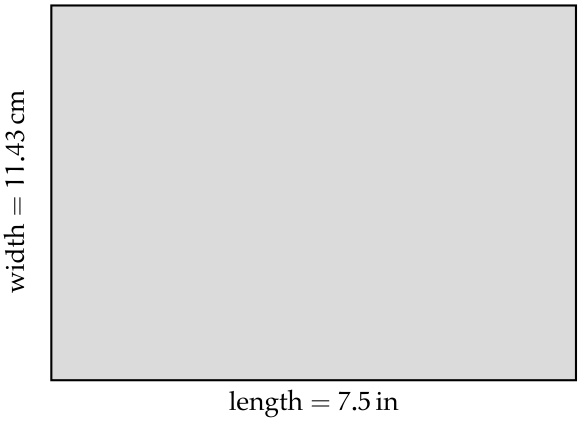


Figure 8.3.5: A Rectangle

Explanation.

a. To find the perimeter (in inches) of the rectangle, we should first convert all lengths into inches. By Appendix B, we know that 1 in = 2.54 cm. So, we have

$$11.43\,\text{cm} = \frac{11.43\,\text{cm}}{1} \cdot \frac{1\,\text{in}}{2.54\,\text{cm}}$$

$$\begin{aligned} &= \frac{11.43}{2.54}\text{ in} \\ &= 4.5\text{ in} \end{aligned}$$

So, the total perimeter is $2 \cdot 4.5\text{ in} + 2 \cdot 7.5\text{ in} = 24\text{ in}$.

b. To find the area (in square centimeters) of the rectangle, we should first convert all lengths into centimeters. So, we have

$$\begin{aligned} 7.5\text{ in} &= \frac{7.5\text{ in}}{1} \cdot \frac{2.54\text{ cm}}{1\text{ in}} \\ &= \frac{7.5}{2.54}\text{ cm} \\ &= 19.05\text{ cm} \end{aligned}$$

So, the total area is $19.05\text{ cm} \cdot 11.43\text{ cm} = 85.725\text{ cm}^2$.

Triangles. The perimeter of a general triangle has no special formula—all that is needed is to add the lengths of its three sides. The *area* of a triangle is a bit more interesting. In Figure 8.3.6, there are three triangles. From left to right, there is an acute triangle, a right triangle, and an obtuse triangle. Each triangle is drawn so that there is a "bottom" horizontal edge. This edge is referred to as the "base" of the triangle. With each triangle, a "height" that is perpendicular to the base is also illustrated.

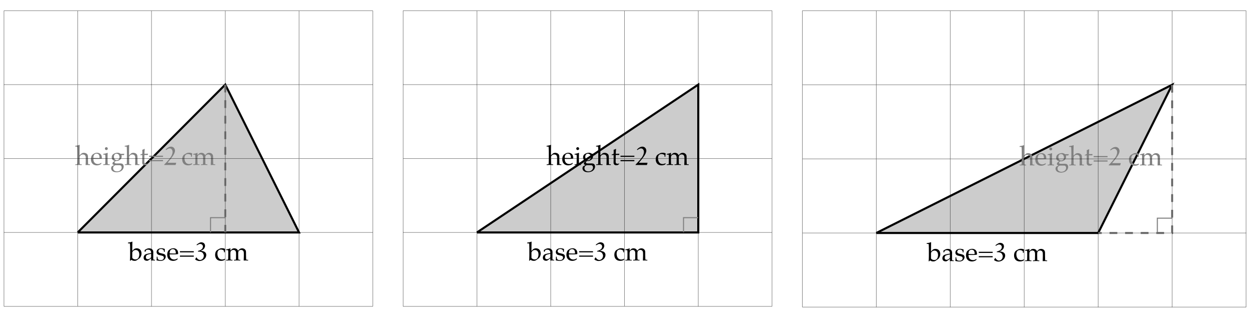


Figure 8.3.6: Triangles

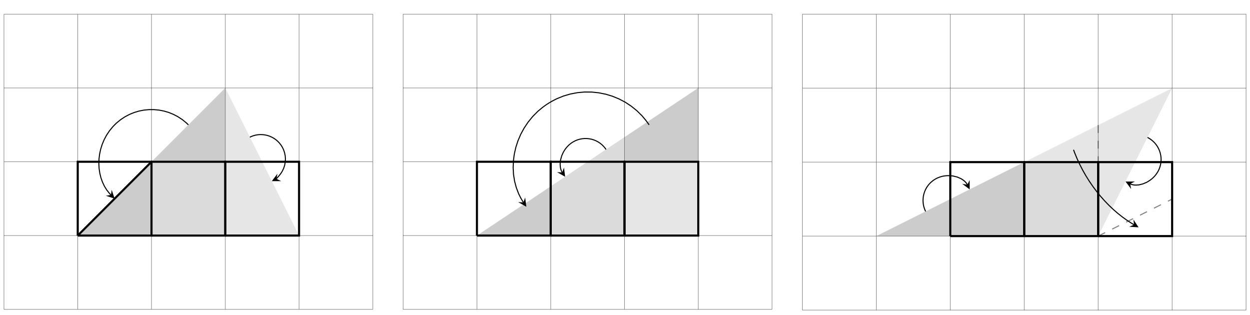


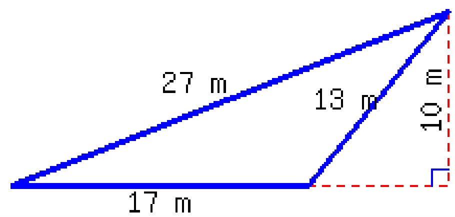
Figure 8.3.7: Triangles

Each of these triangles has the same base width, 3 cm, and the same height, 2 cm. Note that they each have the same area as well. Figure 8.3.7 illustrates how they each have an area of $3\,\text{cm}^2$.

As with the triangles in Figure 8.3.7, you can always rearrange little pieces of a triangle so that the resulting shape is a rectangle with the same base width, but with a height that's one-half of the triangle's height. With a generic rectangle, it has some base width we can represent with the variable b and some height we can represent with the variable h. We can represent its area with the variable A, and then the area of the triangle will be given by $A = b \cdot \left(\frac{1}{2}h\right)$, or more conventionally:

$$A = \frac{1}{2}bh.$$

Checkpoint 8.3.8 Find the perimeter and area of the triangle.



Its perimeter is ______ and its area is ______.

Explanation. For perimeter, we just add the three side lengths:

$$\begin{aligned} P &= 13 + 27 + 17 \\ &= 57 \end{aligned}$$

For area, we use the triangle area formula:

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(17)(10) \\ &= 5(17) \\ &= 85 \end{aligned}$$

Since length and width were measured in meters, we find that the perimeter is 57 meters and the area is 85 square meters.

Circles. To find formulas for the perimeter and area of a circle, it helps to first know that there is a special number called π (spelled "pi" and pronounced like "pie") that appears in many places in mathematics. The decimal value of π is about $3.14159265\ldots$, and it helps to memorize some of these digits. It also helps to understand that π is a little larger than 3. There are many definitions for π that can explain where it comes from and how you can find all its decimal places, but here we are just going to accept that it is a special number, and it is roughly $3.14159265\ldots$.

The perimeter of a circle is the distance around its edge. For circles, the perimeter has a special name: the **circumference**. Imagine wrapping a string around the circle and cutting it so that it makes one complete loop. If we straighten out that piece of string, we have a length that is just as long as the circle's circumference.

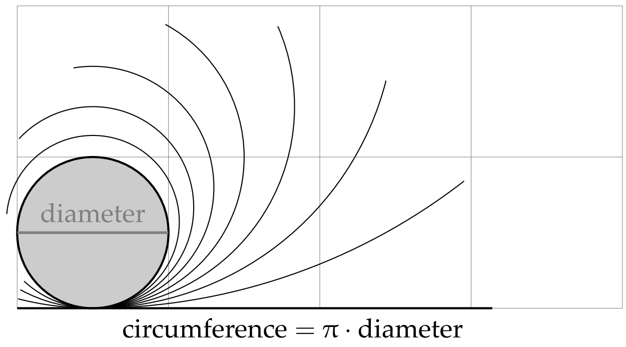


Figure 8.3.9: Circle Diameter and Circumference

As we can see in Figure 8.3.9, the circumference of a circle is a little more than three times as long as its diameter. (The diameter of a circle is the length of a straight line running from a point on the edge through the center to the opposite edge.) In fact, the circumference is actually exactly π times the length of the diameter. With a generic circle, it has some diameter we can represent with the variable d. We can represent its circumference with the variable c, and then the circumference of the circle will be given by:

$$c = \pi d.$$

Alternatively, we often prefer to work with a circle's **radius** instead of its diameter. The radius is the distance from any point on the circle's edge to its center. (Note that the radius is half the diameter.) From this perspective, we can see in Figure 8.3.10 that the circumference is a little more than 6 times the radius.

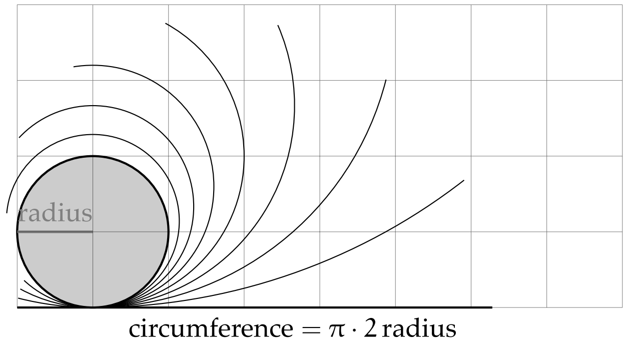


Figure 8.3.10: Circle Diameter and Circumference

This gives us another formula for a circle's circumerence that uses the variable r for its radius: $c = \pi \cdot 2r$. Or more conventionally,

$$c = 2\pi r.$$

There is also a formula for the *area* of a circle based on its radius. Figure 8.3.11 shows how three squares can be cut up and rearranged to fit inside a circle. This shows how the area of a circle of radius r is just a little larger than $3r^2$. Since π is just a little larger than 3, could it be that the area of a circle is given by πr^2?

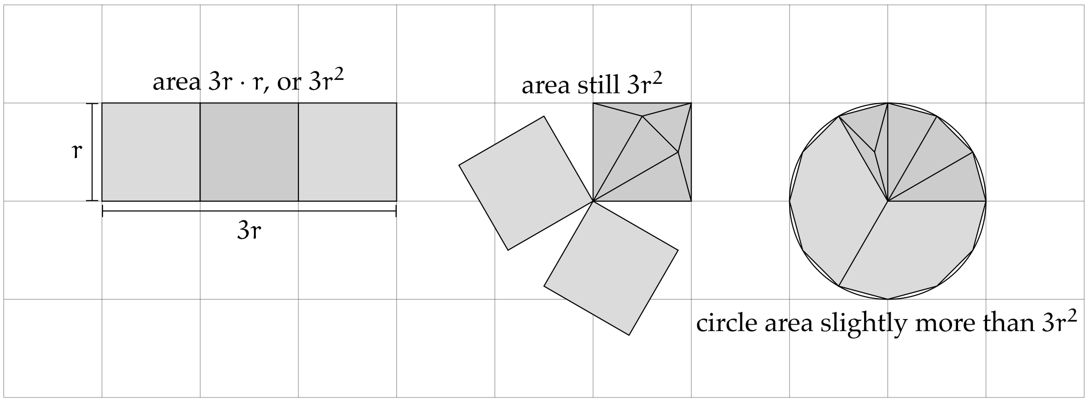


Figure 8.3.11: Circle area is slightly larger than $3r^2$.

One way to establish this formula is to imagine slicing up the circle into many pie slices as in Figure 8.3.12. Then you can rearrange the slices into a strange shape that is *almost* a rectangle with height equal to the radius of the original circle, and width equal to half the circumference of the original circle.

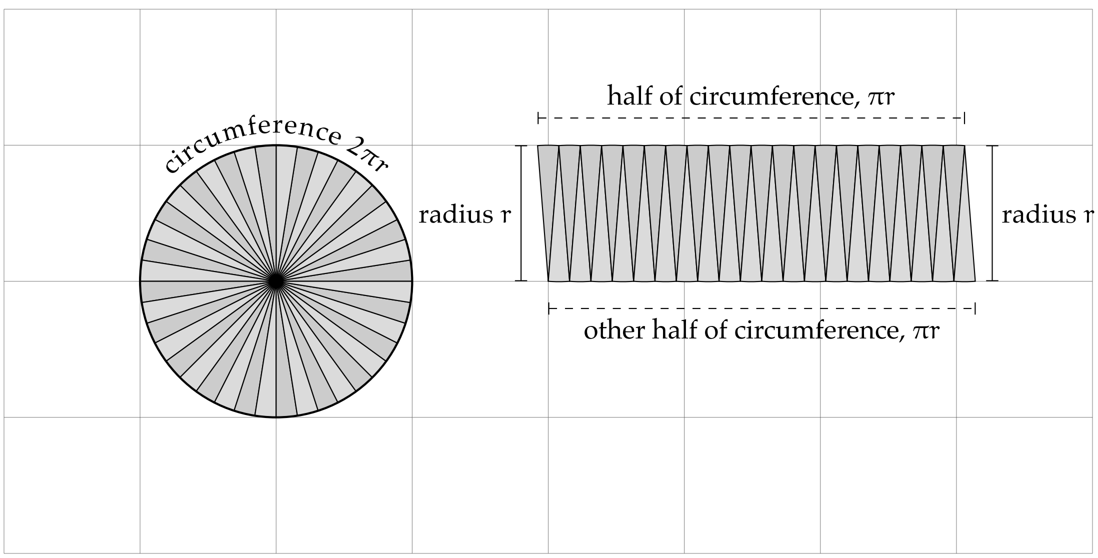


Figure 8.3.12: Reasoning the circle area formula.

Since the area of the circle is equal to the area of the almost-rectangular shape in Figure 8.3.12, we have the circle area formula:

$$A = \pi r^2.$$

Checkpoint 8.3.13 A circle's diameter is 6 m.

a. This circle's circumference, in terms of π, is ______.

b. This circle's circumference, rounded to the hundredth place, is ______.

c. This circle's area, in terms of π, is ______.

d. This circle's area, rounded to the hundredth place, is ______.

Explanation. We use r to represent radius and d to represent diameter. In this problem, it's given that the diameter is 6 m. A circle's radius is half as long as its diameter, so the radius is 3 m.

Throughout these computations, all quantities have units attached, but we only show them in the final step.

a. $$\begin{aligned} c &= \pi d \\ &= \pi \cdot 6 \\ &= 6\pi \text{ m} \end{aligned}$$

b. $$\begin{aligned} c &= \pi d \\ &\approx 3.1415926 \cdot 6 \\ &\approx 18.85 \text{ m} \end{aligned}$$

c. $$\begin{aligned} A &= \pi r^2 \\ &= \pi \cdot 3^2 \\ &= \pi \cdot 9 \\ &= 9\pi \text{ m}^2 \end{aligned}$$

d. $$\begin{aligned} A &= \pi r^2 \\ &\approx 3.1415926 \cdot 3^2 \\ &\approx 3.1415926 \cdot 9 \\ &\approx 28.27 \text{ m}^2 \end{aligned}$$

8.3.2 Volume

The **volume** of a three-dimensional object is the number of $1 \times 1 \times 1$ cubes that fit inside the object (possibly after morphing them into non-cube shapes). If the edges of the cubes are, say, 1 cm long, then the volume is measured in "cubic centimeters ," written cm^3.

Rectangular Prisms. The 3D shape in Figure 8.3.14 is called a rectangular prism.

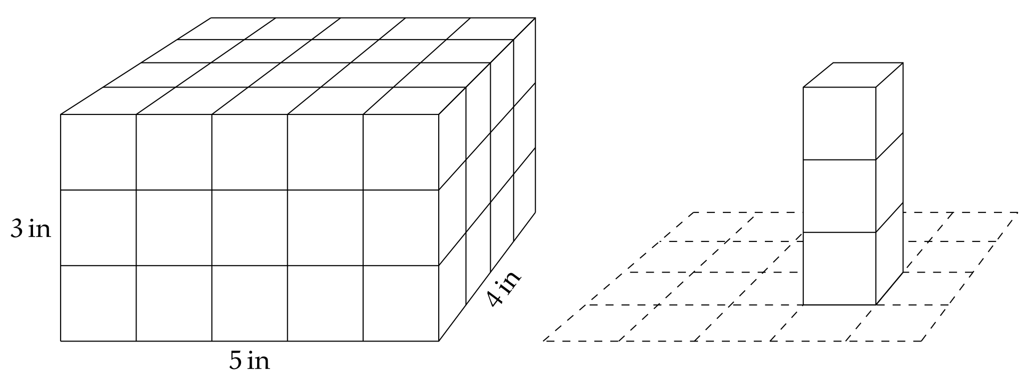


Figure 8.3.14: Volume of a Rectangular Prism

The rectangular prism in Figure 8.3.14 is composed of $1 \text{ in} \times 1 \text{ in} \times 1 \text{ in}$ unit cubes, with each cube's volume being 1 cubic inch (or in^3). The shape's volume is the number of such unit cubes. The bottom face has $5 \cdot 4 = 20$ unit squares. Since there are 3 layers of cubes, the shape has a total of $3 \cdot 20 = 60$ unit cubes. In other words, the shape's volume is 60 in^3 because it has sixty 1 in × 1 in × 1 in cubes inside it.

We found the number of unit squares in the bottom face by multiplying $5 \cdot 4 = 20$. Then to find the volume, we multiplied by 3 because there are three layers of cubes. So one formula for a prism's volume is

$$V = wdh$$

where V stands for volume, w for width, d for depth, and h for height.

Checkpoint 8.3.15 A masonry brick is in the shape of a rectangular prism and is 8 inches wide, 3.5 inches deep, and 2.25 inches high. What is its volume?

Explanation. Using the formula for the volume of a rectangular prism:

$$\begin{aligned} V &= wdh \\ &= 8(3.5)(2.25) \\ &= 63 \end{aligned}$$

So the brick's volume is 63 cubic inches.

Example 8.3.16
Imagine a rectangular prism with width 40 in, depth 4 ft, and height 2 yd as in Figure 8.3.17.

1. Find the volume (in cubic feet) of the prism.
2. Find the surface area (in square inches) of the prism.

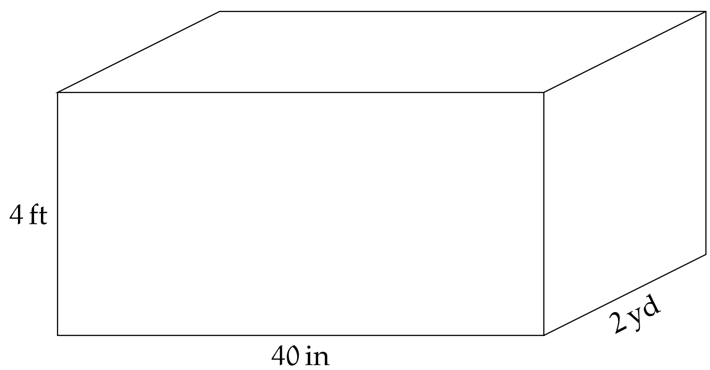


Figure 8.3.17: A Prism

Explanation.

1. To find the volume (in cubic feet) of the prism, we should first convert all lengths into feet. By Appendix B, we know that $1\text{ ft} = 12\text{ in}$ and that $1\text{ yd} = 3\text{ ft}$. So, we have

$$\begin{aligned} 40\text{ in} &= \frac{40\text{ in}}{1} \cdot \frac{1\text{ ft}}{12\text{ in}} \\ &= \frac{40}{12}\text{ ft} \\ &= \frac{10}{3}\text{ ft} \end{aligned}$$

and

$$\begin{aligned} 2\text{ yd} &= \frac{2\text{ yd}}{1} \cdot \frac{3\text{ ft}}{1\text{ yd}} \\ &= 2 \cdot 3\text{ ft} \\ &= 6\text{ ft} \end{aligned}$$

So, the total volume is $4\text{ ft} \cdot \frac{10}{3}\text{ ft} \cdot 6\text{ ft} = 80\text{ ft}^3$.

2. To find the surface area (in square inches) of the prism, we should first convert all lengths into inches. So, we have

$$\begin{aligned} 4\,\text{ft} &= \frac{4\,\text{ft}}{1} \cdot \frac{12\,\text{in}}{1\,\text{ft}} \\ &= 4 \cdot 12\,\text{in} \\ &= 48\,\text{in} \end{aligned}$$

and

$$\begin{aligned} 2\,\text{yd} &= \frac{2\,\text{yd}}{1} \cdot \frac{36\,\text{in}}{1\,\text{yd}} \\ &= 2 \cdot 36\,\text{in} \\ &= 72\,\text{in} \end{aligned}$$

To find the surface area, we should add up the six areas of the faces of the prism, each of which is a rectangle. Note that each face has a corresponding symmetrical face on the other side of the prism.

$$\begin{aligned} \text{Surface Area} &= \overbrace{2(40\,\text{in} \cdot 36\,\text{in})}^{\text{top and bottom}} + \overbrace{2(48\,\text{in} \cdot 36\,\text{in})}^{\text{left and right}} + \overbrace{2(40\,\text{in} \cdot 48\,\text{in})}^{\text{front and back}} \\ &= 10176\,\text{in}^2 \end{aligned}$$

Cylinders. A cylinder is not a prism, but it has some similarities. Instead of a square base, the base is a circle. Its volume can also be calculated in a similar way to how prism volume is calculated. Let's look at an example.

Example 8.3.18 Find the volume of a cylinder with a radius of 3 meters and a height of 2 meters.

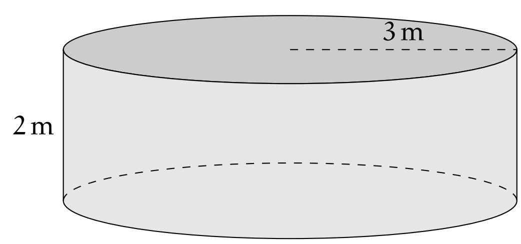


Figure 8.3.19: A Cylinder

Explanation. The base of the cylinder is a circle. We know the area of a circle is given by the formula $A = \pi r^2$, so the base area is $9\pi\,\text{m}^2$, or about $28.27\,\text{m}^2$. That means about 28.27 unit squares can fit into the base. One of them is drawn in Figure 8.3.20 along with two unit cubes above it.

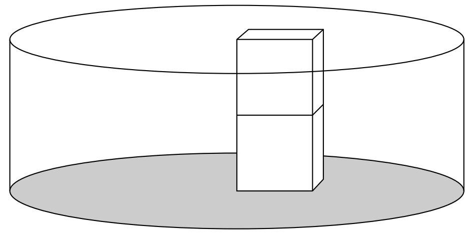


Figure 8.3.20: Finding Cylinder Volume

For each unit square in the base circle, there are two unit cubes of volume. So the volume is the base area times the height: $9\pi\,\mathrm{m}^2 \cdot 2\,\mathrm{m}$, which equals $18\pi\,\mathrm{m}^3$. Approximating π with a decimal value, this is about $56.55\,\mathrm{m}^3$.

Example 8.3.18 demonstrates that the volume of a cylinder can be calculated with the formula

$$V = \pi r^2 h$$

where r is the radius and h is the height.

Checkpoint 8.3.21 A soda can is basically in the shape of a cylinder with radius 1.3 inches and height 4.8 inches. What is its volume?

Its exact volume in terms of π is: ______.

As a decimal approximation rounded to four significant digits, its volume is: ______.

Explanation. Using the formula for the volume of a cylinder:

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(1.3)^2(4.8) \\ &= 8.112\pi \\ &\approx 25.48 \end{aligned}$$

So the can's volume is 8.112π cubic inches, which is about 25.48 cubic inches.

Note that the volume formulas for a rectangular prism and a cylinder have something in common: both formulas first find the area of the base (which is a rectangle for a prism and a circle for a cylinder) and then multiply by the height. So there is another formula

$$V = Bh$$

that works for both shapes. Here, B stands for the base area (which is wd for a prism and πr^2 for a cylinder.)

8.3.3 Summary

Here is a list of all the formulas we've learned in this section.

List 8.3.22: Geometry Formulas

Perimeter of a Rectangle $P = 2(\ell + w)$

Area of a Rectangle $A = \ell w$

Area of a Triangle $A = \frac{1}{2}bh$

Circumference of a Circle $c = 2\pi r$

Area of a Circle $A = \pi r^2$

Volume of a Rectangular Prism $V = wdh$

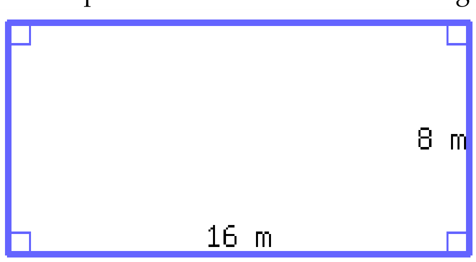
Volume of a Cylinder $V = \pi r^2 h$

Volume of a Rectangular Prism or Cylinder $V = Bh$

8.3.4 Exercises

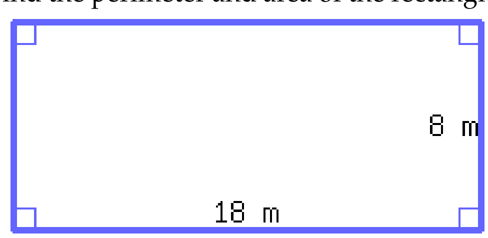
Perimeter and Area

1. Find the perimeter and area of the rectangle.



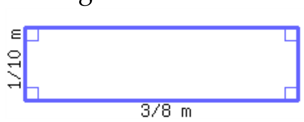
Its perimeter is and its area is .

2. Find the perimeter and area of the rectangle.

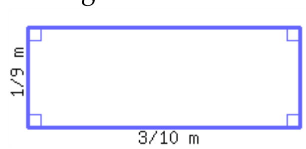


Its perimeter is and its area is .

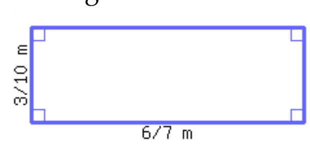
3. Find the perimeter of the rectangle below.



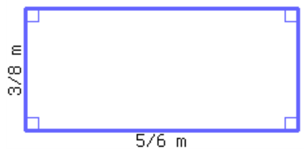
4. Find the perimeter of the rectangle below.



5. Find the area of the rectangle below.



6. Find the area of the rectangle below.



7. Find the perimeter and area of a rectangular table top with a length of 5.8 ft and a width of 29 in.

Its perimeter is ☐

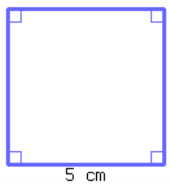
and its area is ☐.

8. Find the perimeter and area of a rectangular table top with a length of 6 ft and a width of 25 in.

Its perimeter is ☐

and its area is ☐.

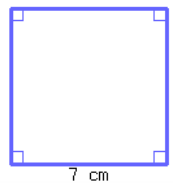
9. Find the perimeter and area of the square.



a. The perimeter is ☐.

b. The area is ☐.

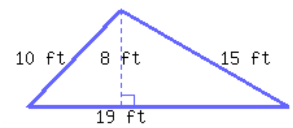
10. Find the perimeter and area of the square.



a. The perimeter is ☐.

b. The area is ☐.

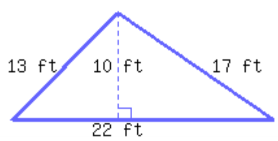
11. Find the perimeter and area of the triangle.



Its perimeter is ☐

and its area is ☐.

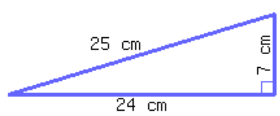
12. Find the perimeter and area of the triangle.



Its perimeter is ☐

and its area is ☐.

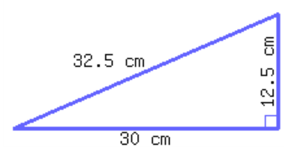
13. Find the perimeter and area of the right triangle.



Its perimeter is ☐

and its area is ☐.

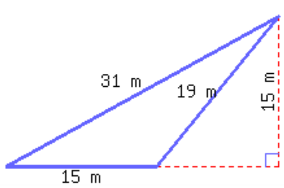
14. Find the perimeter and area of the right triangle.



Its perimeter is ☐

and its area is ☐.

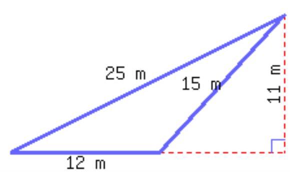
15. Find the perimeter and area of the triangle.



Its perimeter is ☐

and its area is ☐.

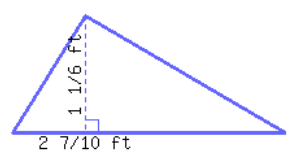
16. Find the perimeter and area of the triangle.



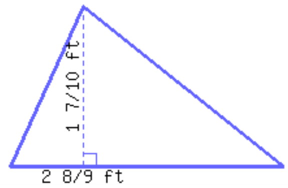
Its perimeter is ☐

and its area is ☐.

17. The area of the triangle below is ☐ square feet.



18. The area of the triangle below is ____ square feet.



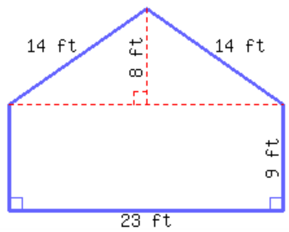
19. Find the area of a triangular flag with a base of 2.3 m and a height of 70 cm.

Its area is ____.

20. Find the area of a triangular flag with a base of 2.6 m and a height of 140 cm.

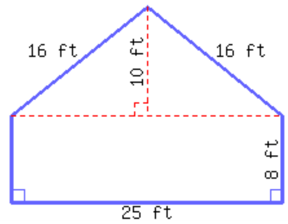
Its area is ____.

21. Find the perimeter and area of this polygon.



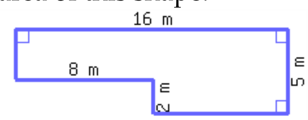
Its perimeter is ____ and its area is ____.

22. Find the perimeter and area of this polygon.



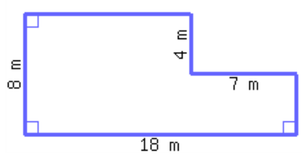
Its perimeter is ____ and its area is ____.

23. Find the perimeter and area of this shape.



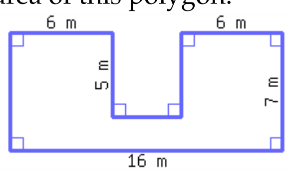
Its perimeter is ____ and its area is ____.

24. Find the perimeter and area of this shape.



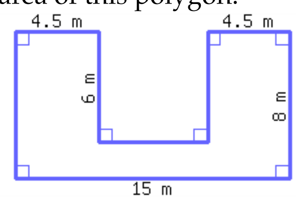
Its perimeter is ____ and its area is ____.

25. Find the perimeter and area of this polygon.



Its perimeter is ____ and its area is ____.

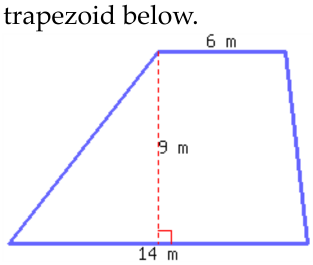
26. Find the perimeter and area of this polygon.



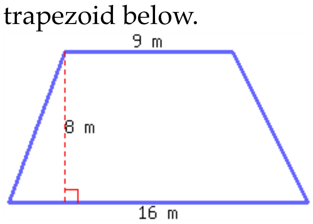
Its perimeter is ____ and its area is ____.

A trapezoid's area can be calculated by the formula $A = \frac{1}{2}(b_1 + b_2)h$, where A stands for area, b_1 for the first base's length, b_2 for the second base's length, and h for height.

27. Find the area of the trapezoid below.



28. Find the area of the trapezoid below.



The formula $A = \frac{1}{2}\,r\,n\,s$ gives the area of a regular polygon with side length s, number of sides n and, apothem r. (The *apothem* is the distance from the center of the polygon to one of its sides.)

29. What is the area of a regular pentagon with $s = 30$ in and $r = 54$ in?

30. What is the area of a regular 94-gon with $s = 42$ in and $r = 43$ in?

31. A circle's radius is 6 m.

a. The circumference, in terms of π, is ______.

b. This circle's circumference, rounded to the hundredths place, is ______.

c. This circle's area, in terms of π, is ______.

d. This circle's area, rounded to the hundredths place, is ______.

32. A circle's radius is 7 m.

a. The circumference, in terms of π, is ______.

b. This circle's circumference, rounded to the hundredths place, is ______.

c. This circle's area, in terms of π, is ______.

d. This circle's area, rounded to the hundredths place, is ______.

33. A circle's diameter is 16 m.

a. This circle's circumference, in terms of π, is ______.

b. This circle's circumference, rounded to the hundredths place, is ______.

c. This circle's area, in terms of π, is ______.

d. This circle's area, rounded to the hundredths place, is ______.

34. A circle's diameter is 18 m.

a. This circle's circumference, in terms of π, is ______.

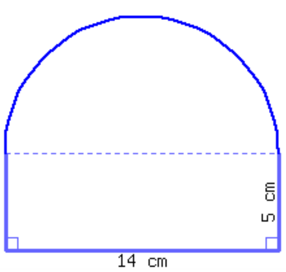
b. This circle's circumference, rounded to the hundredths place, is ______.

c. This circle's area, in terms of π, is ______.

d. This circle's area, rounded to the hundredths place, is ______.

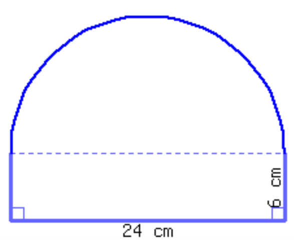
Find the perimeter and area of this shape, which is a semicircle on top of a rectangle.

35.



Its perimeter is ______ and its area is ______.

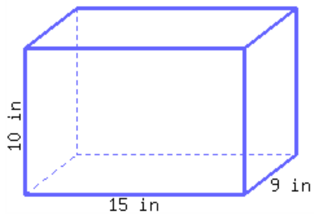
36.



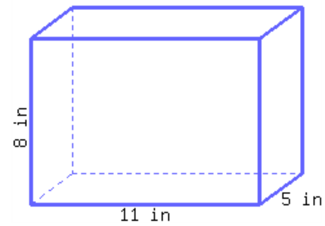
Its perimeter is ______ and its area is ______.

Volume

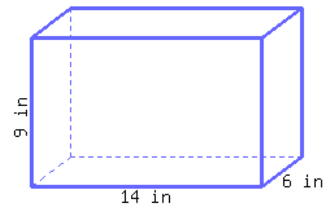
37. Find the volume of this rectangular prism.



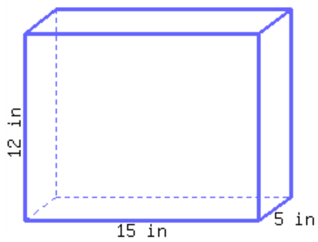
38. Find the volume of this rectangular prism.



39. Find the volume of this rectangular prism.



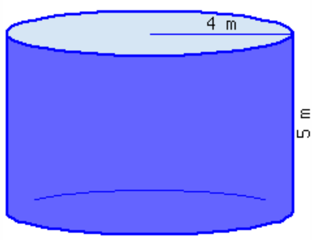
40. Find the volume of this rectangular prism.



41. A cube's side length is 7 cm. Its volume is ______.

42. A cube's side length is 8 cm. Its volume is ______.

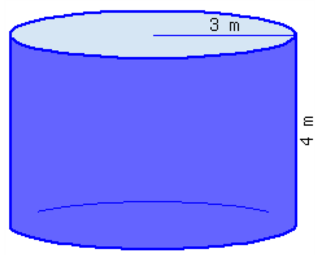
43. Find the volume of this cylinder.



a. This cylinder's volume, in terms of π, is ______.

b. This cylinder's volume, rounded to the hundredths place, is ______.

44. Find the volume of this cylinder.



a. This cylinder's volume, in terms of π, is ______.

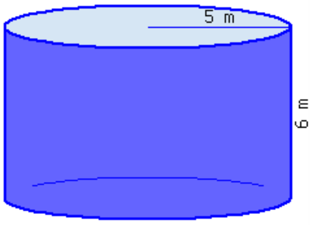
b. This cylinder's volume, rounded to the hundredths place, is ______.

45. Find the volume of this cylinder.

a. This cylinder's volume, in terms of π, is ______.

b. This cylinder's volume, rounded to the hundredths place, is ______.

46. Find the volume of this cylinder.



a. This cylinder's volume, in terms of π, is ____.

b. This cylinder's volume, rounded to the hundredths place, is ____.

47. A cylinder's base's diameter is 12 ft, and its height is 4 ft.

a. This cylinder's volume, in terms of π, is ____.

b. This cylinder's volume, rounded to the hundredths place, is ____.

48. A cylinder's base's diameter is 6 ft, and its height is 5 ft.

a. This cylinder's volume, in terms of π, is ____.

b. This cylinder's volume, rounded to the hundredths place, is ____.

The formula $V = \frac{1}{3} \cdot s^2 \cdot h$ gives the volume of a right square pyramid.

49. What is the volume of a right square pyramid with $s = 51$ in and $h = 82$ in?

50. What is the volume of a right square pyramid with $s = 63$ in and $h = 48$ in?

51. Fill out the table with various formulas as they were given in this section.

Rectangle Perimeter	____
Rectangle Area	____
Triangle Area	____
Circle Circumference	____
Circle Area	____
Rectangular Prism Volume	____
Cylinder Volume	____
Volume of either Rectangular Prism or Cylinder	____

8.4 Geometry Applications

8.4.1 Solving Equations for Geometry Problems

With geometry problems in algebra, it is really helpful to draw a picture to understand the scenario better. After drawing the shape and labeling the given information, we will choose the formula to use from the list in Subsection 8.3.3.

Example 8.4.1 An Olympic-size swimming pool is rectangular and 50 m in length. We don't know its width, but we do know that it required 150 m of painter's tape to outline the edge of the pool during recent renovations. Use this information to set up an equation and find the width of the pool.

Explanation.

The pool's shape is a rectangle, so it helps to sketch a rectangle representing the pool as in Figure 8.4.2. Since we know its length is 50 m, it is a good idea to label that in the sketch. The width is our unknown quantity, so we can use w as a variable to represent the pool's width in meters and label that too.

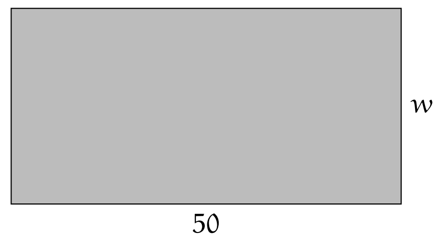


Figure 8.4.2: An Olympic-size pool

Since it required 150 m of painter's tape to outline the pool, we know the perimeter of the pool is 150 m. This suggests using the perimeter formula for a rectangle: $P = 2(\ell + w)$. (This formula was discussed in Subsection 8.3.1).

With this formula, we can substitute 150 in for P and 50 in for ℓ:

$$150 = 2(50 + w).$$

Now we can solve the equation for the width of the pool.

First, we will distribute on the right side, and then isolate w.

$$\begin{aligned} 150 &= 100 + 2w \\ 150 - 100 &= 100 - 100 + 2w \\ 50 &= 2w \\ \frac{50}{2} &= \frac{2w}{2} \\ 25 &= w. \end{aligned}$$

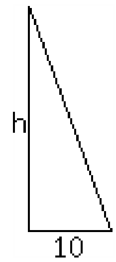
Checking the solution $w = 25$ meters:

$$\begin{aligned} 150 &= 2(50 + w) \\ 150 &\stackrel{?}{=} 2(50 + 25) \\ 150 &\stackrel{?}{=} 2(75) \\ 150 &\stackrel{\checkmark}{=} 150. \end{aligned}$$

We found that the width of the pool is 25 meters.

Checkpoint 8.4.3 One sail on a sail boat is approximately shaped like a triangle. If the base length is 10 feet and the total sail area is 125 square feet, we can wonder how tall is the sail. Set up an equation to model the sail's height.

Explanation. Since the sail's shape is (approximately) a triangle, it helps to sketch a triangle representing the sail. Since we know its base width is 10 feet, it is a good idea to label that in the sketch. The heigth is our unknown quantity, so we can use h as a variable to represent the sail's height in feet and label that too.



Since the total area is known to be 125 square feet, this suggests using the area formula for a triangle: $A = \frac{1}{2}bh$.

With this formula, we can substitute 125 in for A and 100 in for b:

$$125 = \frac{1}{2}(10)h$$

and this equation models the height of the pool.

Let's look at another example. In this one we need to use an algebraic expression for one of the sides of a rectangle.

Example 8.4.4 Azul is designing a rectangular garden and they have 40 meters of wood planking for the border. Their garden's length is 4 meters less than three times the width, and the perimeter must be 40 meters. Find the garden's length and width.

Explanation. Let Azul's garden width be w meters. We can then represent the length as $3w - 4$ meters since we are told that it is 4 meters less than three times the width. It's given that the perimeter is 40 meters. Substituting those values into the formula, we have:

$$\begin{aligned} P &= 2(\ell + w) \\ 40 &= 2(3w - 4 + w) \\ 40 &= 2(4w - 4) && \text{Like terms were combined.} \end{aligned}$$

The next step to solve this equation is to remove the parentheses by distribution.

$$\begin{aligned} 40 &= 2(4w - 4) \\ 40 &= 8w - 8 \\ 40 + 8 &= 8w - 8 + 8 \\ 48 &= 8w \\ \frac{48}{8} &= \frac{8w}{8} \\ 6 &= w. \end{aligned}$$

Checking the solution $w = 6$:

$$\begin{aligned} 40 &= 2(4w - 4) \\ 40 &\stackrel{?}{=} 2(4(6) - 4) \\ 40 &\stackrel{\checkmark}{=} 2(20). \end{aligned}$$

To determine the length, recall that this was represented by $3w - 4$, which is:

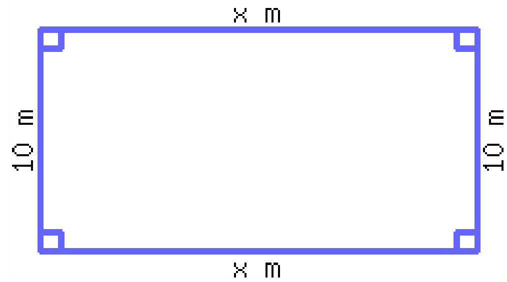
$$\begin{aligned} 3w - 4 &= 3(6) - 4 \\ &= 14. \end{aligned}$$

Thus, the width of Azul's garden is 6 meters and the length is 14 meters.

Checkpoint 8.4.5 A rectangle's perimeter is 56 m. Its width is 10 m. Use an equation to solve for the rectangle's length.

Its length is ______.

Explanation. When we deal with a geometric figure, it's always a good idea to sketch it to help us think. Let the length be x meters.



The perimeter is given as 56 m. Adding up the rectangle's 4 sides gives the perimeter. The equation is:

$$\begin{aligned} x+x+10+10 &= 56 \\ 2x+20 &= 56 \\ 2x+20-\mathbf{20} &= 56-\mathbf{20} \\ 2x &= 36 \\ \frac{2x}{2} &= \frac{36}{2} \\ x &= 18 \end{aligned}$$

So the rectangle's length is 18 m. Don't forget the unit m.

For triangle problems, we may need to use the Pythagorean Theorem that we learned in Subsection 7.1.2. If we know the lengths of two sides of a right triangle then we can find the length of the third side.

Example 8.4.6 Tan owns a road sign manufacturing company and he is producing triangular yield signs for the State of Oregon. The signs are equilateral triangles measuring 36 inches on each side as shown in Figure 8.4.7. Find the area of one sign in square feet to help Tan estimate the amount of material he needs to produce the signs.

Explanation. We will start by converting 36 inches to 3 feet, because the area needs to be in square feet. The area of a triangle is found using $A = \frac{1}{2}bh$, where A is the area, b is the width of the base, and h is the height. In this case the base is at the top of the triangle.

We know the width of the triangle is 3 feet, but we don't know the height. By drawing in the height we form two right triangles so we can use the Pythagorean Theorem to find the height. Half of the width is 1.5 feet, so we will substitute for b and c in the pythagorean theorem.

According to Pythagorean Theorem, we have:

$$\begin{aligned}
c^2 &= a^2 + b^2 \\
3^2 &= a^2 + 1.5^2 \\
9 &= a^2 + 2.25 \\
9 - 2.25 &= a^2 + 2.25 - 2.25 \\
6.75 &= a^2 \\
\sqrt{6.75} &= a \\
2.598 &\approx a
\end{aligned}$$

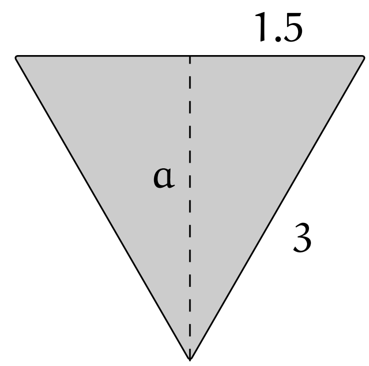


Figure 8.4.7

The height of the triangle is approximately 2.598 feet.

Now we can calculate the area of one sign.

$$\begin{aligned}
A &= \frac{1}{2}bh \\
&\approx \frac{1}{2}(3)(2.598) \\
&= 3.897
\end{aligned}$$

The area of one sign is approximately $3.897\,\text{ft}^2$.

Now we will look at an example that involves a circle. It can be difficult to measure the radius of a circle or cylinder. But if we can measure the circumference, then we can find the radius.

Example 8.4.8 Batula wants to order a custom replacement column for the front of her house and she needs to know the radius. She takes a string and wraps it around the old column. She measures the string and finds the circumference is 3 feet, 2.5 inches. What is the radius of the column?

Explanation. The formula for the circumference of a circle is $C = 2\pi r$, where C stands for the circumference and r stands for the radius.

We will let the radius of Batula's column be r inches. It's given that the circumference is 3 feet, 2.5 inches, so let's convert 3 feet into inches.

$$\begin{aligned}
3\,\text{ft} &= \frac{3\,\text{ft}}{1} \cdot \frac{12\,\text{in}}{1\,\text{ft}} \\
&= \frac{3\,\cancel{\text{ft}}}{1} \cdot \frac{12\,\text{in}}{1\,\cancel{\text{ft}}} \\
&= 3 \cdot 12\,\text{in} \\
&= 36\,\text{in}
\end{aligned}$$

Since 3 feet is 36 inches, we can add the 2.5 inches for a total of 38.5 inches. Substituting the circumference into the formula, we have:

$$\begin{aligned}
C &= 2\pi r \\
38.5 &= 2\pi r
\end{aligned}$$

The next step is to divide both sides by 2π.

$$\begin{aligned}38.5 &= 2\pi r\\ \frac{38.5}{2\pi} &= \frac{2\pi r}{2\pi}\\ 6.127 &\approx r.\end{aligned}$$

Checking the solution $r \approx 6.13$ inches:

$$\begin{aligned}38.5 &= 2\pi r\\ 38.5 &\stackrel{?}{\approx} 2\pi(6.127)\\ 38.5 &\stackrel{\checkmark}{\approx} 38.5.\end{aligned}$$

Therefore, Batula should order a column with a radius of 6.127 inches. A specific measurement like that may not be possible, but Batula could round to something like $6\,\frac{1}{8}$ inches, which is very close. If the manufacturer wanted the diameter instead, we would multiply that by 2 to get 12.25 or $6\,\frac{1}{4}$ inches.

Here is an example using volume.

Example 8.4.9 Mark is designing a cylindrical container for his ice cream business. He wants each container to be 15 centimeters tall and hold 1 gallon of ice cream. What dimension should Mark use for the radius of the container?

Explanation. The formula for the volume of a cylinder is $V = \pi r^2 h$, where V stands for the volume, r stands for the radius and h is the height.

Since the volume is in gallons and the dimensions are in centimeters, we need to convert 1 gallon to cubic centimeters.

$$\begin{aligned}1\text{ gal} &= \frac{1\text{ gal}}{1}\cdot\frac{231\text{ in}^3}{1\text{ gal}}\cdot\frac{2.54^3\text{ cm}^3}{1\text{ in}^3}\\ &= \frac{1\,\cancel{\text{gal}}}{1}\cdot\frac{231\,\cancel{\text{in}^3}}{1\,\cancel{\text{gal}}}\cdot\frac{2.54^3\text{ cm}^3}{1\,\cancel{\text{in}^3}}\\ &= 1\cdot 231\cdot 2.54^3\text{ cm}^3\\ &= 3785.41\text{ cm}^3\end{aligned}$$

Now we can substitute the volume and height into the formula:

$$\begin{aligned}V &= \pi r^2 h\\ 3785.41 &= \pi r^2(15)\end{aligned}$$

The next step is to divide both sides by 15π.

$$\begin{aligned}3785.41 &= 15\pi r^2\\ \frac{3785.41}{15\pi} &= \frac{15\pi r^2}{15\pi}\\ \frac{3785.41}{15\pi} &= r^2\\ \sqrt{\frac{3785.41}{15\pi}} &= r\\ 8.963 &\approx r.\end{aligned}$$

Checking the solution $r \approx 8.963$ centimeters:

$$\begin{aligned}3785.41 &= \pi r^2 h\\ 3785.41 &\stackrel{?}{\approx} \pi(8.963^2)(15)\\ 3785.41 &\stackrel{\checkmark}{\approx} 3783.18.\end{aligned}$$

Note that our check is approximate because we rounded our answer. Mark will want to make the radius of his container at least 8.963 centimeters. He should make it a little larger to have space at the top of the container.

8.4.2 Proportionality in Similar Triangles

Another appliction of geometry involves similar triangles. Two triangles are considered **similar** if they have the same angles and their side lengths are proportional, as shown in Figure 8.4.10:

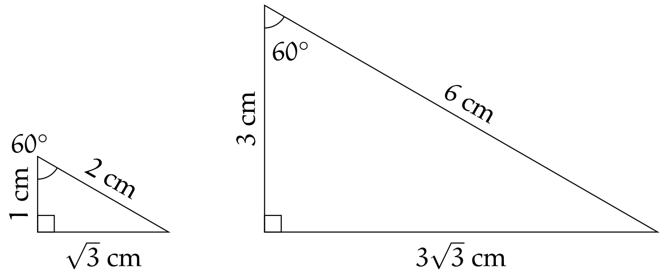


Figure 8.4.10: Similar Triangles

In the first triangle in Figure 8.4.10, the ratio of the left side length to the hypotenuse length is $\frac{1\text{ cm}}{2\text{ cm}}$; in the second triangle, the ratio of the left side length to the hypotenuse length is $\frac{3\text{ cm}}{6\text{ cm}}$. Since both reduce to $\frac{1}{2}$, we can write the following proportion:

$$\frac{1\text{ cm}}{2\text{ cm}} = \frac{3\text{ cm}}{6\text{ cm}}$$

If we extend this concept, we can use it to solve for an unknown side length. Consider the two similar triangles in the next example.

Example 8.4.11

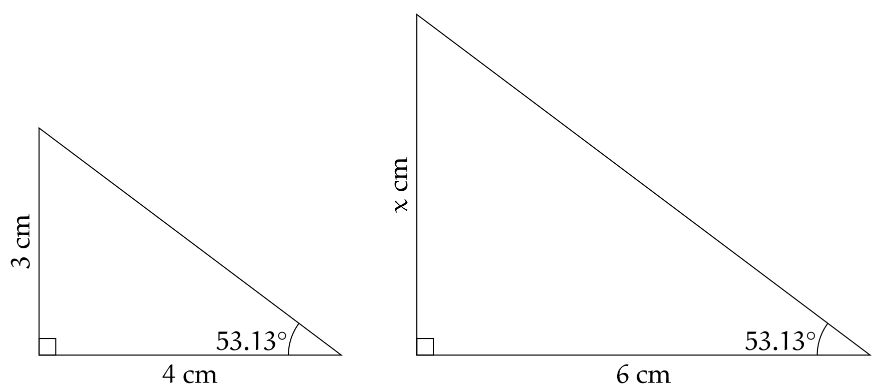


Figure 8.4.12: Similar Triangles

Since the two triangles are similar, we know that their side length should be proportional. To determine the unknown length, we can set up a proportion and solve for x:

$$\frac{\text{bigger triangle's left side length in cm}}{\text{bigger triangle's bottom side length in cm}} = \frac{\text{smaller triangle's left side length in cm}}{\text{smaller triangle's bottom side length in cm}}$$

$$\frac{x\text{ cm}}{6\text{ cm}} = \frac{3\text{ cm}}{4\text{ cm}}$$

$$\begin{aligned}
\frac{x}{6} &= \frac{3}{4}\\
6 \cdot \frac{x}{6} &= 6 \cdot \frac{3}{4}\\
x &= \frac{18}{4}\\
x &= \frac{9}{2} = 4.5
\end{aligned}$$

The unknown side length is then 4.5 cm.

Remark 8.4.13 Looking at the triangles in Figure 8.4.10, you may notice that there are many different proportions you could set up, such as:

$$\begin{aligned}
\frac{2\,\text{cm}}{1\,\text{cm}} &= \frac{6\,\text{cm}}{3\,\text{cm}}\\
\frac{2\,\text{cm}}{6\,\text{cm}} &= \frac{1\,\text{cm}}{3\,\text{cm}}\\
\frac{6\,\text{cm}}{2\,\text{cm}} &= \frac{3\,\text{cm}}{1\,\text{cm}}\\
\frac{3\sqrt{3}\,\text{cm}}{\sqrt{3}\,\text{cm}} &= \frac{3\,\text{cm}}{1\,\text{cm}}
\end{aligned}$$

This is often the case when we set up ratios and proportions.

If we take a second look at Figure 8.4.12, there are also several other proportions we could have used to find the value of x.

$$\frac{\text{bigger triangle's left side length}}{\text{smaller triangle's left side length}} = \frac{\text{bigger triangle's bottom side length}}{\text{smaller triangle's bottom side length}}$$

$$\frac{\text{smaller triangle's bottom side length}}{\text{bigger triangle's bottom side length}} = \frac{\text{smaller triangle's left side length}}{\text{bigger triangle's left side length}}$$

$$\frac{\text{bigger triangle's bottom side length}}{\text{smaller triangle's bottom side length}} = \frac{\text{bigger triangle's left side length}}{\text{smaller triangle's left side length}}$$

Written as algebraic proportions, these three equations would, respectively, be

$$\frac{x\,\text{cm}}{3\,\text{cm}} = \frac{6\,\text{cm}}{4\,\text{cm}}, \qquad \frac{4\,\text{cm}}{6\,\text{cm}} = \frac{3\,\text{cm}}{x\,\text{cm}}, \qquad \frac{6\,\text{cm}}{4\,\text{cm}} = \frac{x\,\text{cm}}{3\,\text{cm}}$$

While these are only a few of the possibilities, if we clear the denominators from any properly designed proportion, every one is equivalent to $x = 4.5$.

8.4.3 Exercises

1. A circle's circumference is 4π mm.

a. This circle's diameter is ______.

b. This circle's radius is ______.

2. A circle's circumference is 6π mm.

a. This circle's diameter is ______.

b. This circle's radius is ______.

3. A circle's circumference is 36 cm. Find the following values. Round your answer to at least 2 decimal places.

a. This circle's diameter is ______.

b. This circle's radius is ______.

4. A circle's circumference is 38 cm. Find the following values. Round your answer to at least 2 decimal places.

a. This circle's diameter is ______.

b. This circle's radius is ______.

5. A circle's circumference is 12π mm.

a. This circle's diameter is ______.

b. This circle's radius is ______.

6. A circle's circumference is 14π mm.

a. This circle's diameter is ______.

b. This circle's radius is ______.

7. A circle's circumference is 45 cm. Find the following values. Round your answer to at least 2 decimal places.

a. This circle's diameter is ______.

b. This circle's radius is ______.

8. A circle's circumference is 47 cm. Find the following values. Round your answer to at least 2 decimal places.

a. This circle's diameter is ______.

b. This circle's radius is ______.

9. A cylinder's base's radius is 4 m, and its volume is 160π m^3.
This cylinder's height is ______.

10. A cylinder's base's radius is 10 m, and its volume is 200π m^3.
This cylinder's height is ______.

11. A rectangle's area is 336 mm^2. Its height is 16 mm.
Its base is ______.

12. A rectangle's area is 276 mm^2. Its height is 12 mm.
Its base is ______.

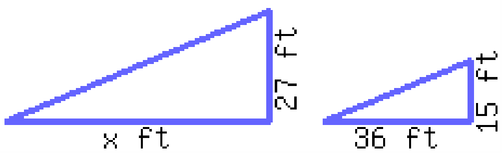
13. A rectangular prism's volume is 13224 ft^3. The prism's base is a rectangle. The rectangle's length is 24 ft and the rectangle's width is 19 ft.
This prism's height is ______.

14. A rectangular prism's volume is 5600 ft^3. The prism's base is a rectangle. The rectangle's length is 25 ft and the rectangle's width is 16 ft.
This prism's height is ______.

15. A triangle's area is 175.5 m^2. Its base is 27 m.
Its height is ______.

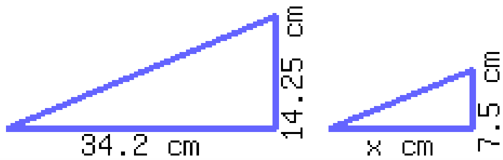
16. A triangle's area is 275.5 m^2. Its base is 29 m.
Its height is ______.

17. The following two triangles are similar to each other. Find the length of the missing side.



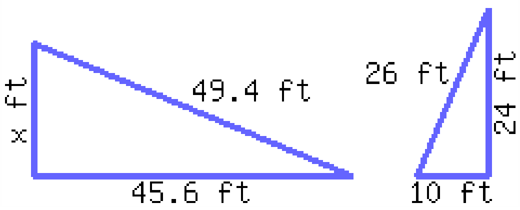
The missing side's length is ______

18. The following two triangles are similar to each other. Find the length of the missing side.



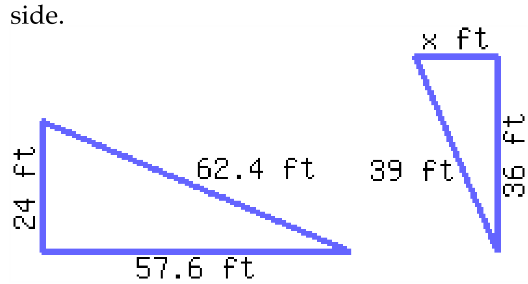
The missing side's length is ______

19. The following two triangles are similar to each other. Find the length of the missing side.



The missing side's length is ______

20. The following two triangles are similar to each other. Find the length of the missing side.



The missing side's length is ______

8.5 Quantities in the Physical World Chapter Review

8.5.1 Scientific Notation

In Section 8.1 we covered the definition of scientific notation, how to convert to and from scientific notation, and how to do some calculations in scientific notation.

Example 8.5.1 Scientific Notation for Large Numbers.

a. The distance to the star Betelgeuse is about 3,780,000,000,000,000 miles. Write this number in scientific notation.

b. The gross domestic product (GDP) of California in the year 2017 was about $\$2.746 \times 10^{13}$. Write this number in standard notation.

Explanation.

a. $3{,}780{,}000{,}000{,}000{,}000 = 3.78 \times 10^{15}$.

b. $\$2.746 \times 10^{13} = \$2{,}746{,}000{,}000{,}000$.

Example 8.5.2 Scientific Notation for Small Numbers.

a. Human DNA forms a double helix with diameter 2×10^{-9} meters. Write this number in standard notation.

b. A single grain of Forget-me-not (Myosotis) pollen is about 0.00024 inches in diameter. Write this number in scientific notation.

Explanation.

a. $2 \times 10^{-9} = 0.000000002$.

b. $0.00024 = 2.4 \times 10^{-4}$.

Example 8.5.3 Multiplying and Dividing Using Scientific Notation. The fastest spacecraft so far have traveled about 5×10^{6} miles per day.

a. If that spacecraft traveled at that same speed for 2×10^{4} days (which is about 55 years), how far would it have gone? Write your answer in scientific notation.

b. The nearest star to Earth, besides the Sun, is Proxima Centauri, about 2.5×10^{13} miles from Earth. How many days would you have to fly in that spacecraft at top speed to reach Proxima Centauri

Explanation.

a. Remember that you can find the distance traveled by multiplying the rate of travel times the time traveled: $d = r \cdot t$. So this problem turns into

$$d = r \cdot t$$
$$d = \left(5 \times 10^{6}\right) \cdot \left(2 \times 10^{4}\right)$$

Multiply coefficient with coefficient and power of 10 with power of 10.

$$= (5 \cdot 2)\left(10^{6} \times 10^{4}\right)$$

$$= 10 \times 10^{10}$$

Remember that this still isn't in scientific notation. So we convert like this:

$$\begin{aligned} &= 1.0 \times 10^{1} \times 10^{10} \\ &= 1.0 \times 10^{11} \end{aligned}$$

So, after traveling for 2×10^4 days (55 years), we will have traveled about 1.0×10^{11} miles. That's one-hundred million miles. I hope someone remembered the snacks.

b. Since we are looking for time, let's solve the equation $d = r \cdot t$ for t by dividing by r on both sides: $t = \frac{d}{r}$. So we have:

$$\begin{aligned} t &= \frac{d}{r} \\ t &= \frac{2.5 \times 10^{13}}{5 \times 10^{6}} \end{aligned}$$

Now we can divide coefficient by coefficient and power of 10 with power of 10.

$$\begin{aligned} t &= \frac{2.5}{5} \times \frac{10^{13}}{10^{6}} \\ t &= 0.5 \times 10^{7} \\ t &= 5 \times 10^{-1} \times 10^{7} \\ t &= 5 \times 10^{6} \end{aligned}$$

This means that to get to Proxima Centauri, even in our fastest spacecraft, would take 5×10^6 years. Converting to standard form, this is 5,000,000 years. I think we're going to need a faster ship.

8.5.2 Unit Conversion

Unit conversion is a particular process that uses unit ratios to convert units. You may refer to Appendix B to find unit conversion facts needed to do these conversions.

Example 8.5.4 Using Multiple Unit Ratios. How many grams are in 5 pounds?

$$5\,\text{lb} = \frac{5\,\text{lb}}{1} \qquad \text{Rewrite as a ratio.}$$

$$= \frac{5\,\text{lb}}{1} \cdot \frac{1\,\text{kg}}{2.205\,\text{lb}} \cdot \frac{1000\,\text{g}}{1\,\text{kg}} \qquad \text{Two unit ratios are needed.}$$

$$= \frac{5\,\cancel{\text{lb}}}{1} \cdot \frac{1\,\cancel{\text{kg}}}{2.205\,\cancel{\text{lb}}} \cdot \frac{1000\,\text{g}}{1\,\cancel{\text{kg}}} \qquad \text{Units may now cancel.}$$

$$= \frac{5}{1} \cdot \frac{1}{2.205} \cdot \frac{1000\,\text{g}}{1} \qquad \text{Only units of g remain.}$$

$$= \frac{5 \cdot 1000}{2.205}\,\text{g} \qquad \text{Multiply what's left and then divide.}$$

$$\approx 2268\,\text{g}$$

So 5 pounds is about 2268 grams.

Example 8.5.5 Converting Squared or Cubed Units. Convert 240 square inches into square centimeters.

$$\begin{aligned}
240\,\text{in}^2 &= \frac{240\,\text{in}^2}{1} && \text{Rewrite as a ratio.}\\
&= \frac{240\,\text{in}^2}{1}\cdot\left(\frac{2.54\,\text{cm}}{1\,\text{in}}\right)^2 && \text{The unit ratio needs to be squared.}\\
&= \frac{240\,\text{in}^2}{1}\cdot\frac{2.54^2\,\text{cm}^2}{1\,\text{in}^2} && \text{Everything inside the parentheses is squared.}\\
&= \frac{240\,\cancel{\text{in}^2}}{1}\cdot\frac{2.54^2\,\text{cm}^2}{1\,\cancel{\text{in}^2}} && \text{Units may now cancel.}\\
&= \frac{240}{1}\cdot\frac{2.54^2\,\text{cm}^2}{1} && \text{Only units of sq cm remain.}\\
&= 240\cdot 2.54^2\,\text{cm}^2 && \text{Multiply.}\\
&\approx 1548\,\text{cm}^2
\end{aligned}$$

So 240 square inches is approximately 1548 square centimeters.

Example 8.5.6 Converting Rates. Gold has a density of $19.3\,\frac{\text{g}}{\text{mL}}$. What is this density in ounces per cubic inch?

$$\begin{aligned}
19.3\,\frac{\text{g}}{\text{mL}} &= \frac{19.3\,\text{g}}{1\,\text{mL}} && \text{Write the rate as a ratio.}\\
&\approx \frac{19.3\,\text{g}}{1\,\text{mL}}\cdot\frac{16.39\,\text{mL}}{1\,\text{in}^3}\cdot\frac{1\,\text{oz}}{28.35\,\text{g}} && \text{Use unit ratios to make cancellations.}\\
&= \frac{19.3\,\cancel{\text{g}}}{1\,\cancel{\text{mL}}}\cdot\frac{16.39\,\cancel{\text{mL}}}{1\,\text{in}^3}\cdot\frac{1\,\text{oz}}{28.35\,\cancel{\text{g}}} && \text{Units may now cancel.}\\
&= \frac{19.3}{1}\cdot\frac{16.39}{1\,\text{in}^3}\cdot\frac{1\,\text{oz}}{28.35} && \text{Only oz per cubic inch remain.}\\
&= \frac{19.3\cdot 16.39}{28.35}\,\frac{\text{oz}}{\text{in}^3} && \text{Multiply what's left and then divide.}\\
&\approx 11.16\,\frac{\text{oz}}{\text{in}^3}
\end{aligned}$$

Notice that we did not need to raise any unit ratios to a power since there is a conversion fact that tells us that $1\,\text{in}^3 \approx 16.39\,\text{mL}$.

Thus, the density of gold is about $11.16\,\frac{\text{oz}}{\text{in}^3}$.

8.5.3 Geometry Formulas

In Section 8.3 we established the following formulas.

Perimeter of a Rectangle $P = 2(\ell + w)$

Area of a Rectangle $A = \ell w$

Area of a Triangle $A = \frac{1}{2}bh$

Circumference of a Circle $c = 2\pi r$

Area of a Circle $A = \pi r^2$

Volume of a Rectangular Prism $V = wdh$

Volume of a Cylinder $V = \pi r^2 h$

Volume of a Rectangular Prism or Cylinder $V = Bh$

8.5.4 Exercises

Scientific Notation Write the following number in scientific notation.

1. 350

2. 450000

3. 0.0055

4. 0.00065

Write the following number in decimal notation without using exponents.

5. 7.51×10^4

6. 8.51×10^3

7. 9.5×10^0

8. 1.49×10^0

9. 2.5×10^{-3}

10. 3.49×10^{-4}

Multiply the following numbers, writing your answer in scientific notation.

11. $(5 \times 10^3)(7 \times 10^2)$

12. $(5 \times 10^5)(4 \times 10^5)$

Divide the following numbers, writing your answer in scientific notation.

13. $\dfrac{5.4 \times 10^3}{6 \times 10^{-2}}$

14. $\dfrac{4.2 \times 10^4}{7 \times 10^{-2}}$

Unit Conversion

15. Convert 211 tbsp to teaspoons.

16. Convert 5.98 t to kilograms.

17. Convert 9.5 s to milliseconds.

18. Convert 43 hm to feet.

19. Convert 8.9 yd to centimeters.

20. Convert 2.7 pt to gallons.

21. Convert 48 cm^2 to square inches.

22. Convert 2.08 ft^3 to cubic miles.

23. Convert 52.8 $\frac{\text{dam}}{\text{s}}$ to meters per millisecond.

24. Convert 226 $\frac{\text{mi}^2}{\text{wk}}$ to acres per day.

25. Convert 7.6 $\frac{\text{T}}{\text{s}}$ to pounds per nanosecond.

26. Convert 46 $\frac{\text{B}}{\text{d}}$ to kilobytes per week.

27. Carly's bedroom has 107 ft^2 of floor. She would like to carpet the floor, but carpeting is sold by the square yard. How many square yards of carpeting will she need to get?

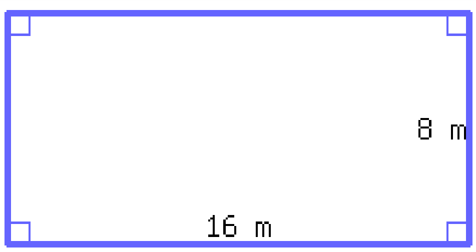
28. Jon is traveling in Europe and renting a car. He is used to thinking of gasoline amounts in gallons, but in Europe it is sold in liters. After filling the gas tank, he notices it took 32 L of gas. How many gallons is that?

29. Cody was driving at a steady speed of 39 mph for 11 minutes. How far did he travel in that time?

30. The algae in a pond is growing at a rate of $0.3\ \frac{\text{kg}}{\text{d}}$. How much algae is in the poind after 4 weeks?

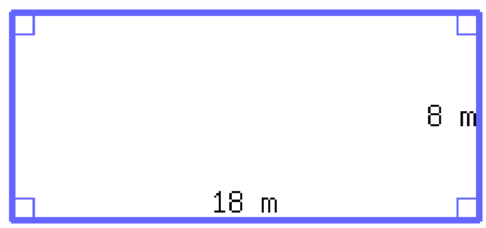
Geometry

31. Find the perimeter and area of the rectangle.



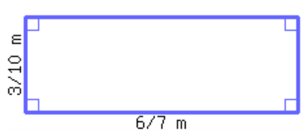
Its perimeter is ______ and its area is ______.

32. Find the perimeter and area of the rectangle.



Its perimeter is ______ and its area is ______.

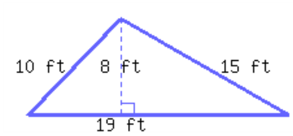
33. Find the area of the rectangle below.



34. Find the perimeter and area of a rectangular table top with a length of 5.9 ft and a width of 32 in.

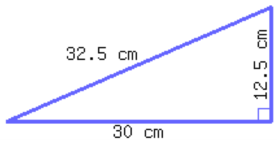
Its perimeter is ______ and its area is ______.

35. Find the perimeter and area of the triangle.



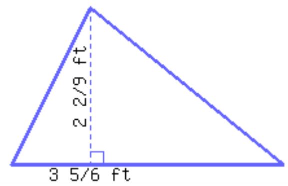
Its perimeter is ______ and its area is ______.

36. Find the perimeter and area of the right triangle.



Its perimeter is ______ and its area is ______.

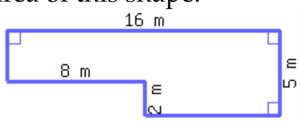
37. The area of the triangle below is ______ square feet.



38. Find the area of a triangular flag with a base of 2.7 m and a height of 70 cm.

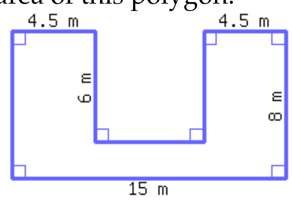
Its area is ______.

39. Find the perimeter and area of this shape.



Its perimeter is ______ and its area is ______.

40. Find the perimeter and area of this polygon.



Its perimeter is ______ and its area is ______.

41. The formula $A = \frac{1}{2}\, r\, n\, s$ gives the area of a regular polygon with side length s, number of sides n and, apothem r. (The *apothem* is the distance from the center of the polygon to one of its sides.)
What is the area of a regular pentagon with $s = 72$ in and $r = 96$ in?

42. A circle's radius is 9 m.

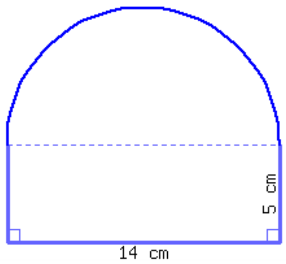
a. The circumference, in terms of π, is ______.

b. This circle's circumference, rounded to the hundredths place, is ______.

c. This circle's area, in terms of π, is ______.

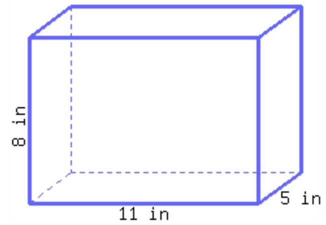
d. This circle's area, rounded to the hundredths place, is ______.

43. Find the perimeter and area of this shape, which is a semicircle on top of a rectangle.



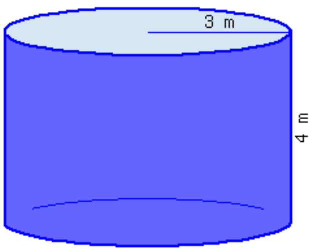
Its perimeter is ______ and its area is ______.

44. Find the volume of this rectangular prism.



45. A cube's side length is 3 cm. Its volume is ______.

46. Find the volume of this cylinder.



a. This cylinder's volume, in terms of π, is ______.

b. This cylinder's volume, rounded to the hundredths place, is ______.

47. A cylinder's base's diameter is 6 ft, and its height is 5 ft.

a. This cylinder's volume, in terms of π, is ______.

b. This cylinder's volume, rounded to the hundredths place, is ______.

48. A cylinder's base's diameter is 18 ft, and its height is 6 ft.

a. This cylinder's volume, in terms of π, is ______.

b. This cylinder's volume, rounded to the hundredths place, is ______.

49. Fill out the table with various formulas as they were given in this section.

Rectangle Perimeter	______
Rectangle Area	______
Triangle Area	______
Circle Circumference	______
Circle Area	______
Rectangular Prism Volume	______
Cylinder Volume	______
Volume of either Rectangular Prism or Cylinder	______

50. A circle's circumference is 16π mm.

a. This circle's diameter is ______.

b. This circle's radius is ______.

51. A circle's circumference is 47 cm. Find the following values. Round your answer to at least 2 decimal places.

a. This circle's diameter is ______.

b. This circle's radius is ______.

52. A circle's circumference is 49 cm. Find the following values. Round your answer to at least 2 decimal places.

a. This circle's diameter is ______.

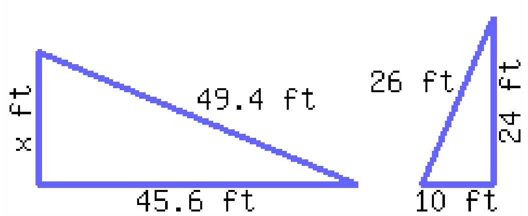
b. This circle's radius is ______.

53. A cylinder's base's radius is 2 m, and its volume is 8π m^3.
This cylinder's height is ______.

54. A rectangular prism's volume is 5355 ft^3. The prism's base is a rectangle. The rectangle's length is 21 ft and the rectangle's width is 17 ft.
This prism's height is ______.

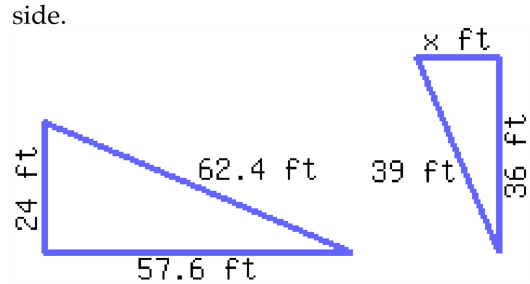
55. A triangle's area is 149.5 m^2. Its base is 23 m.
Its height is ______.

56. The following two triangles are similar to each other. Find the length of the missing side.



The missing side's length is ______

57. The following two triangles are similar to each other. Find the length of the missing side.



The missing side's length is ______

Chapter 9

Topics in Graphing

9.1 Review of Graphing

This section is a short review of the basics of graphing. The topics here are introduced in Sections 3.1 and 3.2 from Part I. Here we only briefly remind readers of the basics to warm up for the graphing topics in the rest of this chapter. Some readers may benefit from turning to those earlier sections instead of reviewing from this section.

9.1.1 Cartesian Coordinates

A Cartesian coordinate system is usually represented as a graph where there are two "axes" (straight lines extending infinitely), one horizontal and one vertical. The horizontal axis is usually labeled "x", and the vertical axis is usually labeled "y". The scale marks on the axes are used to define locations on the plane that they span. An address is a pair of "coordinates" written as in this example: $(3, 2)$. The first coordinate tells you where the location is with respect to the horizontal axis, and the second coordinate tells you a location with respect to the vertical axis. The point $(3, 2)$ is marked in Figure 9.1.2.

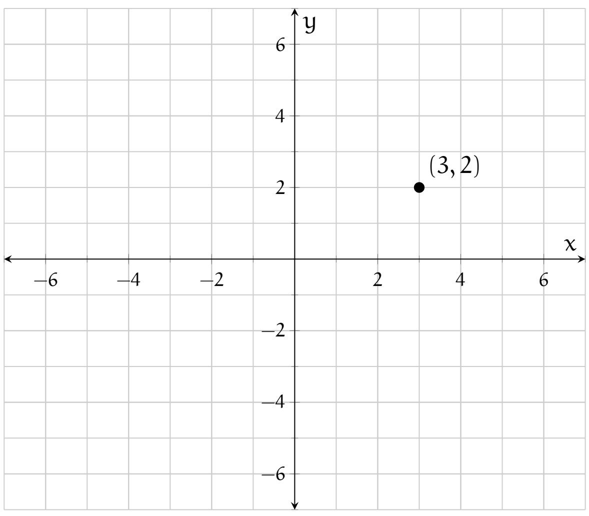


Figure 9.1.2

Example 9.1.3 On paper, sketch a Cartesian coordinate system with the axes scaled using regularly spaced ticks and labels, and then plot the following points: $(2, 3), (-5, 1), (0, -3), (4, 0)$.

Explanation.

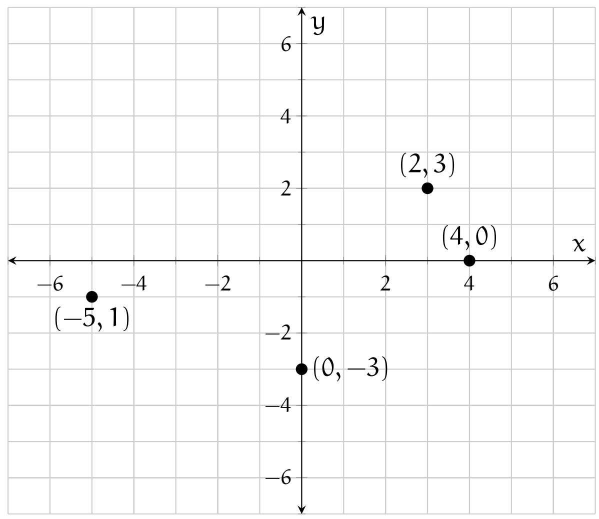


Figure 9.1.4: A Cartesian grid with the four points plotted.

Note that negative numbers in the first coordinate mean that a point is left of the y-axis. And similarly, negative numbers in the second coordinate mean that a point is below the x-axis.

9.1.2 Graphing Equations by Plotting Points

When you have an equation in the form

$$y = \text{expression in } x$$

it suggests that we could substitute in different values for x and get back different values for y. Pairing these x- and y-values together, we can plot points and create a "graph" of the equation. Creating a graph of a given equation in x and y is the basic objective of this section. Sometimes the equation has special features that give you a shortcut for creating a graph, for example as discussed in Section 3.5. However, here we want to focus on the universal approach of just substituting in values for x and seeing what comes out.

Example 9.1.5 Rheema helped plant a lovely Douglas Fir in a local park volunteering with Portland's Friends of Trees [1]. The tree they planted was 4 ft tall when they planted it. Rheema watched the tree grow over the next few years and noticed that every year, the tree grew about 1.5 ft. So, the height of the tree can be found by using the formula $y = 1.5x + 4$, where x-values represent the number of years since the tree was planted. Let's make a graph of this equation by making a table of values. The most straightforward method to graph any equation is to build a table of x- and y-values, and then plot the points.

x	$y = 1.5x + 4$	Point	Interpretation
0	$1.5(0) + 4$ $= 4$	$(0, 4)$	When the tree was planted, the tree was 4 ft tall.
2	$1.5(2) + 4$ $= 7$	$(2, 7)$	Two years after tree was planted, the tree was 7 ft tall.
4	$1.5(4) + 4$ $= 10$	$(4, 10)$	Four years after tree was planted, the tree was 10 ft tall.
6	$1.5(6) + 4$ $= 13$	$(6, 13)$	Six years after tree was planted, the tree was 13 ft tall.
8	$1.5(8) + 4$ $= 16$	$(8, 16)$	Eight years after tree was planted, the tree was 16 ft tall.

Figure 9.1.6: A table of values for $y = 1.5x + 4$

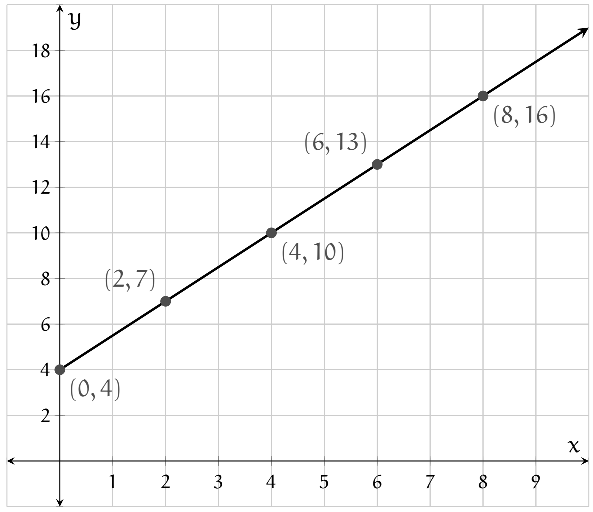


Figure 9.1.7: A graph of $y = 1.5x + 4$

Example 9.1.8 Make a graph of the linear equation $y = \frac{5}{2}x - 3$ by building a table of x- and y-values and plotting the points.

Explanation. To create an easy-to-graph table of values, we should examine the formula and notice that if all of the x-values were multiples of 2, then the fraction in the equation would cancel nicely and leave us with integer y-values.

x	$y = \frac{5}{2}x - 3$	Point
-4	$\frac{5}{2}(-4) - 3$ $= -13$	$(-4, -13)$
-2	$\frac{5}{2}(-2) - 3$ $= -8$	$(-2, -8)$
0	$\frac{5}{2}(0) - 3$ $= -3$	$(0, -3)$
2	$\frac{5}{2}(2) - 3$ $= 2$	$(2, 2)$
4	$\frac{5}{2}(4) - 3$ $= 7$	$(4, 7)$
6	$\frac{5}{2}(6) - 3$ $= 12$	$(6, 12)$

Figure 9.1.9: A table of values for $y = \frac{5}{2}x - 3$

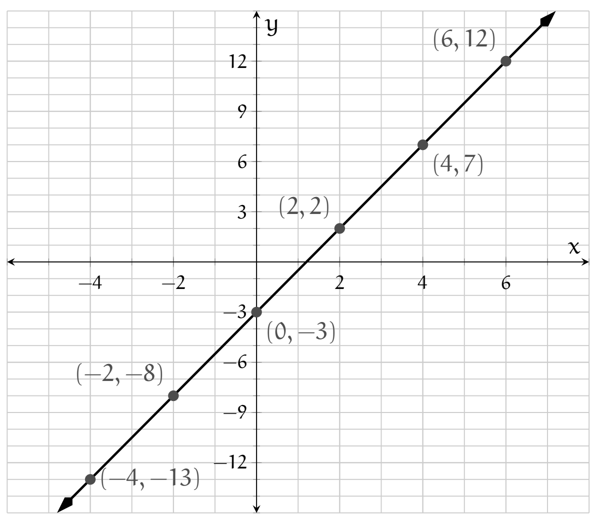


Figure 9.1.10: A graph of $y = \frac{5}{2}x - 3$

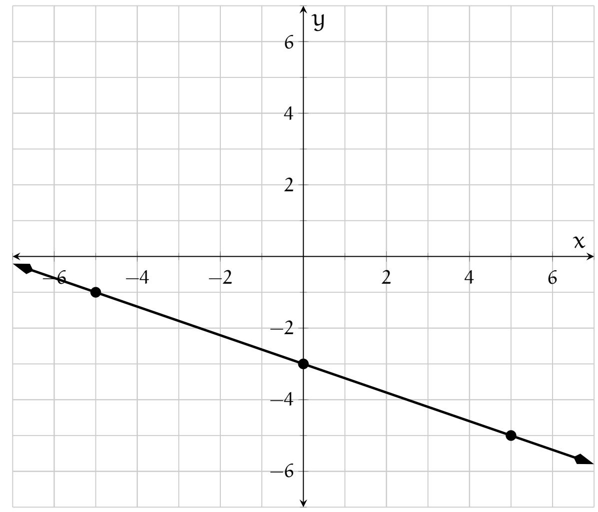
Example 9.1.11 Create a table of ordered pairs and then make a plot of the equation $y = -\frac{2}{5}x - 3$. Note that this equation is a linear equation, and we can see that the slope is negative. Therefore we should *expect* to see a downward sloping line as we view it from left to right. If we don't see that in the end, it suggests some

[1]friendsoftrees.org/

mistake was made.

Explanation. This time, with the slope having denominator 5, it is wise to use multiples of 5 as the x-values.

x	$y = -\frac{2}{5}x - 3$	Point
-5	$-\frac{2}{5}(-5) - 3$ $= -5$	$(-5, -5)$
0	$-\frac{2}{5}(0) - 3$ $= -3$	$(0, -3)$
5	$-\frac{2}{5}(5) - 3$ $= -1$	$(5, -1)$



Checkpoint 9.1.12 Make a table for the equation.

x	$y = \frac{11}{5}x - 8$
________	________
________	________
________	________
________	________
________	________

Explanation. Since this equation has a fractional coefficient for x with denominator 5, it would be wise to choose our own x-values that are multiples of 5. Then when we use them to solve for y, the denominator will be cleared, and we will not need to continue with fraction arithmetic.

This solution will use the x-values -5, 0, 5, 10 and 15. The choice to use these x-values is arbitrary, but they are small multiples of 5, which will make computation easier.

One at a time, we substitute these x-values into the equation $y = \frac{11}{5}x - 8$, and solve for y:

$$y = \frac{11}{5}(-5) - 8 \implies y = -19$$
$$y = \frac{11}{5}(0) - 8 \implies y = -8$$
$$y = \frac{11}{5}(5) - 8 \implies y = 3$$
$$y = \frac{11}{5}(10) - 8 \implies y = 14$$
$$y = \frac{11}{5}(15) - 8 \implies y = 25$$

So the table may be completed as:

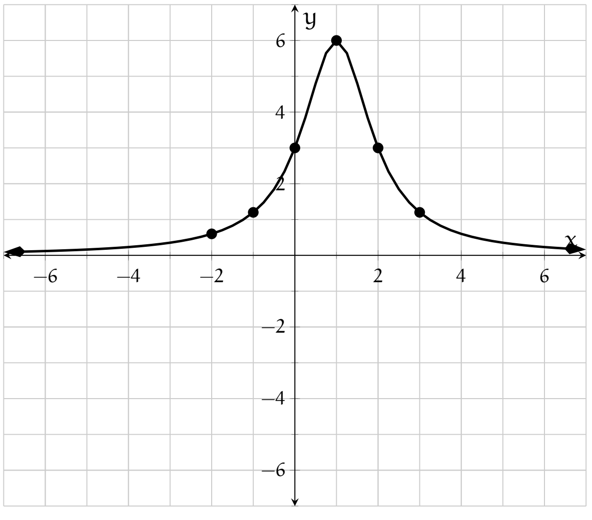
x	y
-5	-19
0	-8
5	3
10	14
15	25

Even when the equation is not a linear equation, this method (making a table of points) will work to help create a graph.

Example 9.1.13 Create a table of ordered pairs and then make a plot of the equation $y = \frac{6}{x^2-2x+2}$.

Explanation. The form of this equation is not one that we recognize yet. But the general approach for making a graph is still going to work out.

x	$y = \frac{6}{x^2-2x+2}$	Point
-2	$\frac{6}{(-2)^2-2(-2)+2}$ $= 0.6$	$(-2, 0.6)$
-1	$\frac{6}{(-1)^2-2(-1)+2}$ $= 1.2$	$(-1, 1.2)$
0	$\frac{6}{(0)^2-2(0)+2}$ $= 3$	$(0, 3)$
1	$\frac{6}{(1)^2-2(1)+2}$ $= 6$	$(1, 6)$
2	$\frac{6}{(2)^2-2(2)+2}$ $= 3$	$(2, 3)$
3	$\frac{6}{(3)^2-2(3)+2}$ $= 1.2$	$(3, 1.2)$



9.1.3 Graphing Lines Using Intercepts

As noted earlier, sometimes the form of an equation suggests an alternative way that we could graph it. In the case of a linear equation, an alternative to making a table is to find the line's "intercepts". These are the locations where the line crosses either the x-axis or the y-axis. In the case of a straight line, that is theoretically all you need to graph the complete line. Here, we review this approach. We also hope this example simply serves as a reminder of what intercepts are.

Recall that the standard form (3.7.1) of a line equation is $Ax + By = C$ where where A, B, and C are three numbers (each of which might be 0, although at least one of A and B must be nonzero). If a linear equation is given in standard form, we can relative easily find the line's x- and y-intercepts by substituting in $y = 0$ and $x = 0$, respectively.

Example 9.1.14 Find the intercepts of $3x + 5y = 60$, and then graph the equation given those intercepts.

To find the x-intercept, set $y = 0$ and solve for x.

$$\begin{aligned} 3x + 5(0) &= 60 \\ 3x &= 60 \\ x &= 20 \end{aligned}$$

So the x-intercept is the point $(20, 0)$.

To find the y-intercept, set $x = 0$ and solve for y.

$$\begin{aligned} 3(0) + 5y &= 60 \\ 5y &= 60 \\ y &= 12 \end{aligned}$$

So, the y-intercept is the point $(0, 12)$.

Next, we just plot these points and draw the line that runs through them.

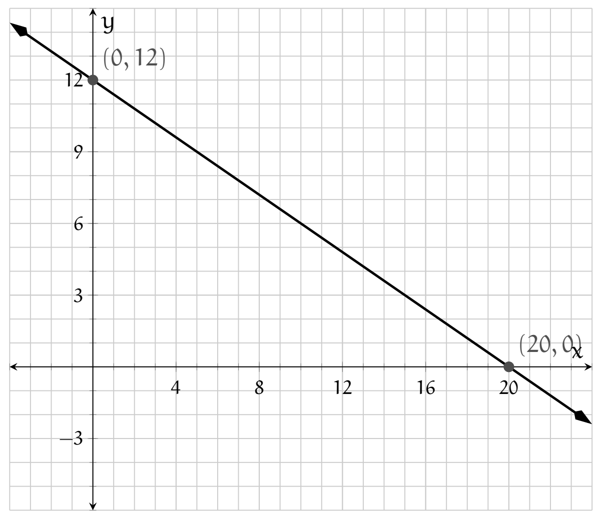


Figure 9.1.15: A graph of $3x + 5y = 60$

Checkpoint 9.1.16 Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$2x + 5y = -20$$

	x-value	y-value	Location (as an ordered pair)
y-intercept	______	______	____________
x-intercept	______	______	____________

Explanation. A line's y-intercept is on the y-axis, implying that its x-value must be 0. To find a line's y-intercept, we substitute in $x = 0$. In this problem we have:

$$\begin{aligned} 2x + 5y &= -20 \\ 2(0) + 5y &= -20 \\ 5y &= -20 \\ \frac{5y}{5} &= \frac{-20}{5} \\ y &= -4 \end{aligned}$$

This line's y-intercept is $(0, -4)$.

Next, a line's x-intercept is on the x-axis, implying that its y-value must be 0. To find a line's x-intercept, we substitute in $y = 0$. In this problem we have:

$$\begin{aligned} 2x + 5y &= -20 \\ 2x + 5(0) &= -20 \\ 2x &= -20 \\ \frac{2x}{2} &= \frac{-20}{2} \\ x &= -10 \end{aligned}$$

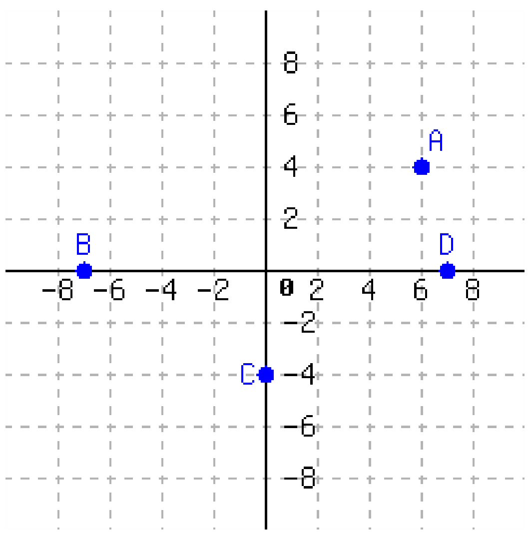
The line's x-intercept is $(-10, 0)$.
The entries for the table are:

	x-value	y-value	Location
y-intercept	0	-4	$(0, -4)$
x-intercept	-10	0	$(-10, 0)$

9.1.4 Exercises

Identifying Coordinates Locate each point in the graph:

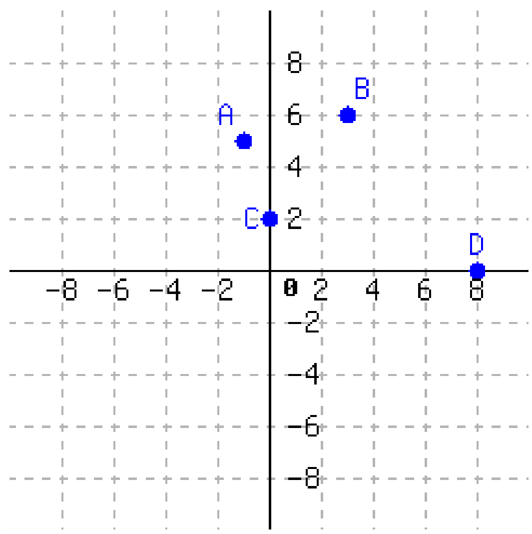
1.



Write each point's position as an ordered pair, like $(1, 2)$.

A = ______ B = ______
C = ______ D = ______

2.



Write each point's position as an ordered pair, like $(1, 2)$.

A = ______ B = ______
C = ______ D = ______

Plotting Points

3. Sketch the points $(8, 2)$, $(5, 5)$, $(-3, 0)$, and $(2, -6)$ on a Cartesian plane.

4. Sketch the points $(1, -4)$, $(-3, 5)$, $(0, 4)$, and $(-2, -6)$ on a Cartesian plane.

Tables for Equations Make a table for the equation.

5.

x	$y = 4x$

6.

x	$y = 8x$

7.

x	$y = 6x + 3$

8.

x	$y = 8x - 3$

9.

x	$y = \frac{10}{3}x + 10$

10.

x	$y = \frac{13}{10}x - 8$

11.

x	$y = -\frac{9}{8}x - 6$

12.

x	$y = \frac{9}{8}x - 4$

Graphs of Equations

13. Create a table of ordered pairs and then make a plot of the equation $y = 2x + 3$.

14. Create a table of ordered pairs and then make a plot of the equation $y = -x - 4$.

15. Create a table of ordered pairs and then make a plot of the equation $y = \frac{4}{3}x$.

16. Create a table of ordered pairs and then make a plot of the equation $y = -\frac{3}{4}x + 2$.

17. Create a table of ordered pairs and then make a plot of the equation $y = x^2 + 1$.

18. Create a table of ordered pairs and then make a plot of the equation $y = (x - 2)^2$. Use x-values from 0 to 4.

Lines and Intercepts

19. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$5x + 6y = 30$$

	x-value	y-value	Location (as an ordered pair)
y-intercept	________	________	________________
x-intercept	________	________	________________

20. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$6x + 3y = -36$$

	x-value	y-value	Location (as an ordered pair)
y-intercept	________	________	________________
x-intercept	________	________	________________

21. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$6x - 7y = -42$$

	x-value	y-value	Location (as an ordered pair)
y-intercept	________	________	________________
x-intercept	________	________	________________

22. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$x - 7y = -14$$

	x-value	y-value	Location (as an ordered pair)
y-intercept	________	________	________________
x-intercept	________	________	________________

23. Find the x- and y-intercepts of the line with equation $5x - 2y = 10$. Then find one other point on the line. Use your results to graph the line.

24. Find the x- and y-intercepts of the line with equation $5x - 6y = -90$. Then find one other point on the line. Use your results to graph the line.

25. Find the x- and y-intercepts of the line with equation $x + 5y = -15$. Then find one other point on the line. Use your results to graph the line.

26. Find the x- and y-intercepts of the line with equation $6x + y = -18$. Then find one other point on the line. Use your results to graph the line.

27. Make a graph of the line $-5x - y = -3$.

28. Make a graph of the line $x + 5y = 5$.

29. Make a graph of the line $20x - 4y = 8$.

30. Make a graph of the line $3x + 5y = 10$.

9.2 Key Features of Quadratic Graphs

In this section we will learn about quadratic graphs and their key features, including vertex, axis of symmetry and intercepts.

9.2.1 Properties of Quadratic Graphs

Hannah fired a toy rocket from the ground, which launched into the air with an initial speed of 64 feet per second. The height of the rocket can be modeled by the equation $y = -16t^2 + 64t$, where t is how many seconds had passed since the launch. To see the shape of the graph made by this equation, we make a table of values and plot the points.

t	$y = -16t^2 + 64t$	Point
0	$-16(0)^2 + 64(0)$ $= 0$	$(0, 0)$
1	$-16(1)^2 + 64(1)$ $= 48$	$(1, 48)$
2	$-16(2)^2 + 64(2)$ $= 64$	$(2, 64)$
3	$-16(3)^2 + 64(3)$ $= 48$	$(3, 48)$
4	$-16(4)^2 + 64(4)$ $= 0$	$(4, 0)$

Figure 9.2.2: Points for $y = -16t^2 + 64t$

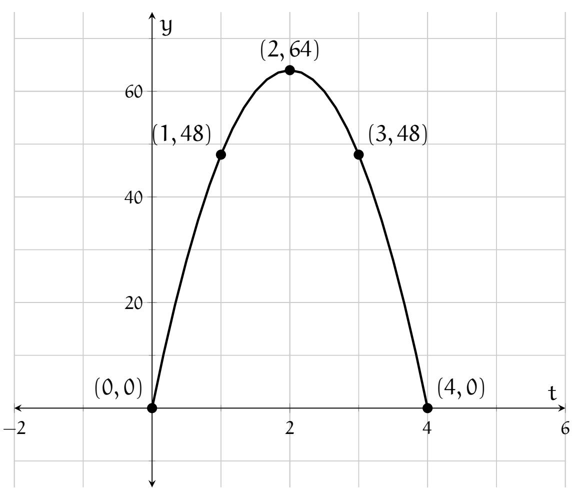


Figure 9.2.3: Graph of $y = -16t^2 + 64t$

A curve with the shape that we see in Figure 9.2.3 is called a **parabola**. Notice the symmetry in Figure 9.2.2, how the y-values in rows above the middle row match those below the middle row. Also notice the symmetry in the shape of the graph, how its left side is a mirror image of its right side.

The first feature that we will talk about is the *direction* that a parabola opens. All parabolas open either upward or downward. This parabola in the rocket example opens downward because a is negative. That means that for large values of t, the at^2 term will be large and negative, and the resulting y-value will be low on the y-axis. So the negative leading coefficient causes the arms of the parabola to point downward.

Here are some more quadratic graphs so we can see which way they open.

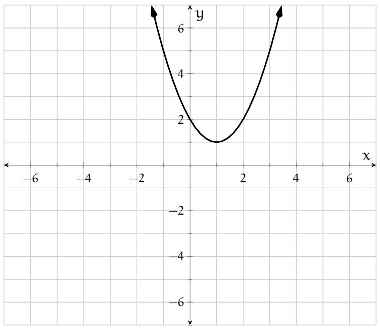


Figure 9.2.4: The graph of $y = x^2 - 2x + 2$ opens upward. Its leading coefficient is positive.

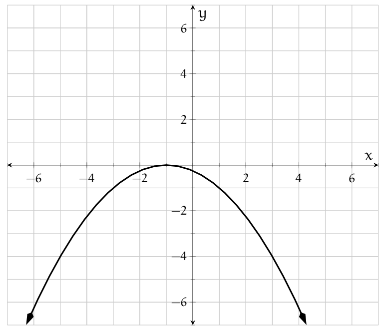


Figure 9.2.5: The graph of $y = -\frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{4}$ opens downward. Its leading coefficient is negative.

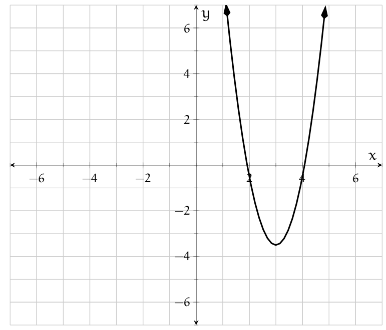


Figure 9.2.6: The graph of $y = 3x^2 - 18x + 23.5$ opens upward. Its leading coefficient is positive.

Fact 9.2.7 *The graph of a quadratic equation* $y = ax^2 + bx + c$ *opens upward or downward according to the sign of the leading coefficient* a*. If the leading coefficient is positive, the parabola opens upward. If the leading coefficient is negative, the parabola opens downward.*

Checkpoint 9.2.8 Determine whether each quadratic graph opens upward or downward.

a. The graph of $y = 3x^2 - 4x - 7$ opens (☐ upward ☐ downward) .

b. The graph of $y = -5x^2 + x$ opens (☐ upward ☐ downward) .

c. The graph of $y = 2 + 3x - x^2$ opens (☐ upward ☐ downward) .

d. The graph of $y = \frac{1}{3}x^2 - \frac{2}{5}x + \frac{1}{4}$ opens (☐ upward ☐ downward) .

Explanation.

a. The graph of $y = 3x^2 - 4x - 7$ opens upward as the leading coefficient is the positive number 3.

b. The graph of $y = -5x^2 + x$ opens downward as the leading coefficient is the negative number -5.

c. The graph of $y = 2 + 3x - x^2$ opens downward as the leading coefficient is -1. (Note that the leading coefficient is the coefficient on x^2.)

d. The graph of $y = \frac{1}{3}x^2 - \frac{2}{5}x + \frac{1}{4}$ opens upward as the leading coefficient is the positive number $\frac{1}{3}$.

The **vertex** of a quadratic graph is the highest or lowest point on the graph, depending on whether the graph opens downward or upward. In Figure 9.2.3, the vertex is $(2, 64)$. This tells us that Hannah's rocket reached its maximum height of 64 feet after 2 seconds. If the parabola opens downward, as in the rocket example, then the y-value of the vertex is the **maximum** y-value. If the parabola opens upward then the y-value of the vertex is the **minimum** y-value.

The **axis of symmetry** is a vertical line that passes through the vertex, cutting the quadratic graph into two symmetric halves. We write the axis of symmetry as an equation of a vertical line so it always starts with "$x =$." In Figure 9.2.3, the equation for the axis of symmetry is $x = 2$.

The **vertical intercept** is the point where the parabola crosses the vertical axis. The vertical intercept is the y-intercept if the vertical axis is labeled y. In Figure 9.2.3, the point $(0, 0)$ is the starting point of the rocket, and it is where the graph crosses the y-axis, so it is the vertical intercept. The y-value of 0 means the rocket was on the ground when the t-value was 0, which was when the rocket launched.

The **horizontal intercept(s)** are the points where the parabola crosses the horizontal axis. They are the x-intercepts if the horizontal axis is labeled x. The point $(0, 0)$ on the path of the rocket is also a horizontal intercept. The t-value of 0 indicates the time when the rocket was launched from the ground. There is another horizontal intercept at the point $(4, 0)$, which means the rocket came back to hit the ground after 4 seconds.

It is possible for a quadratic graph to have zero, one, or two horizontal intercepts. The figures below show an example of each.

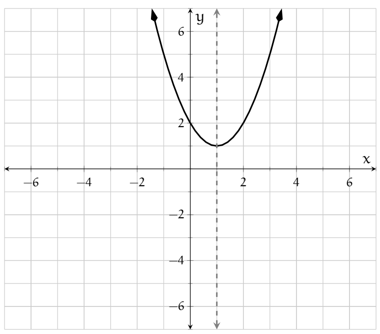


Figure 9.2.9: The graph of $y = x^2 - 2x + 2$ has no horizontal intercepts

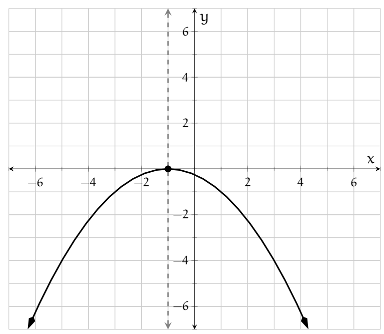


Figure 9.2.10: The graph of $y = -\frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{4}$ has one horizontal intercept

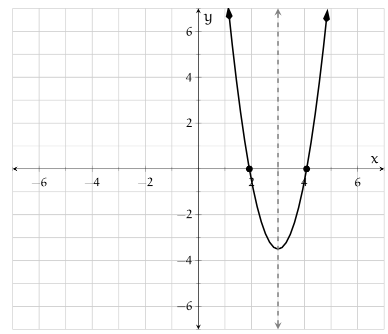


Figure 9.2.11: The graph of $y = 3x^2 - 18x + 23.5$ has two horizontal intercepts

Here is a summary of the key features of quadratic graphs.

List 9.2.12: Summary of Key Features of Quadratic Graphs

Consider a quadratic equation in the form $y = ax^2 + bx + c$ and the parabola that it makes when graphed.

Direction The parabola opens upward if a is positive and opens downward of a is negative.

Vertex The vertex of the parabola is the maximum or minimum point on the graph.

Axis of Symmetry The axis of symmetry is the vertical line that passes through the vertex.

Vertical Intercept The vertical intercept is the point where the graph intersects the vertical axis. There is exactly one vertical intercept.

Horizontal Intercept(s) The horizontal intercept(s) are the point(s) where a graph intersects the horizontal axis. The graph of a parabola can have zero, one, or two horizontal intercepts.

Example 9.2.13 Identify the key features of the quadratic graph of $y = x^2 - 2x - 8$ shown in Figure 9.2.14.

Explanation.

First, we see that this parabola opens upward because the leading coefficient is positive.

Then we locate the vertex which is the point $(1, -9)$. The axis of symmetry is the vertical line $x = 1$.

The vertical intercept or y-intercept is the point $(0, -8)$.

The horizontal intercepts are the points $(-2, 0)$ and $(4, 0)$.

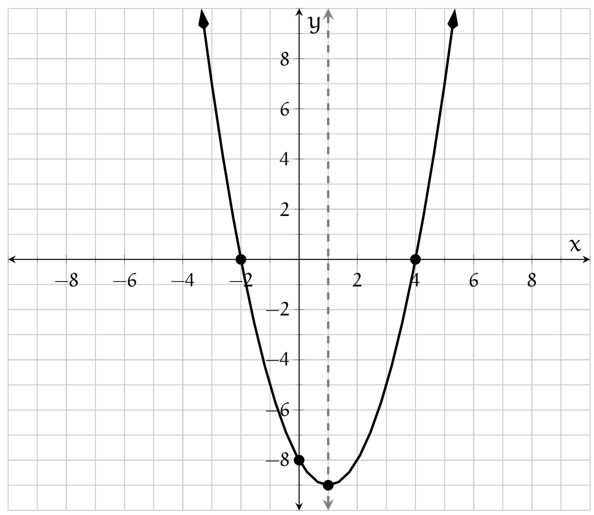
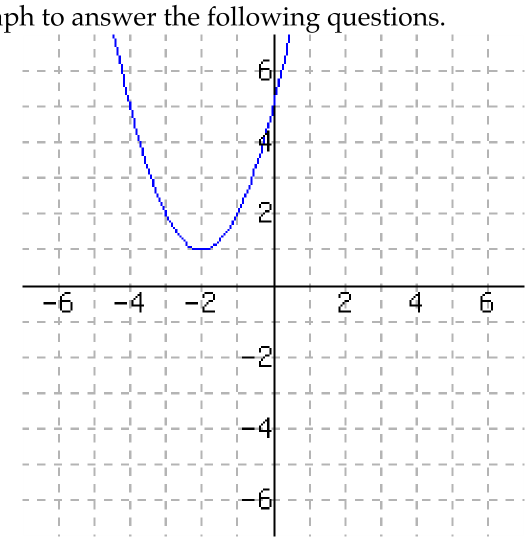


Figure 9.2.14: Graph of $y = x^2 - 2x - 8$

Checkpoint 9.2.15 Use the graph to answer the following questions.



a. What are the coordinates of the vertex?

b. What is the equation of the axis of symmetry?

c. What are the coordinates of the x-intercept(s)?

d. What are the coordinates of the y-intercept?

Explanation.

a. The vertex is at $(-2, 1)$.

b. The equation of the axis of symmetry is $x = -2$.

c. There are no x-intercepts. (Answer `None`.)

d. The y-intercept is at $(0, 5)$.

9.2.2 Finding the Vertex and Axis of Symmetry Algebraically

The coordinates of the vertex are not easy to identify on a graph if they are not integers . Another way to find the coordinates of the vertex is by using a formula.

Fact 9.2.16 *If we denote* (h, k) *as the coordinates of the vertex of a quadratic graph defined by* $y = ax^2 + bx + c$*, then* $h = -\frac{b}{2a}$*. Then we can find* k *by substituting* h *in for* x*.*

To understand why, we can look at the quadratic formula 7.2.2. The vertex is on the axis of symmetry, so it will always occur halfway between the two x-intercepts (if there are any). The quadratic formula shows that the x-intercepts happen at $-\frac{b}{2a}$ minus some number and at $-\frac{b}{2a}$ plus that same number. So $-\frac{b}{2a}$ is right in the middle, and it must be the horizontal coordinate of the vertex, h. If we have already memorized the quadratic formula, this new formula for h is not hard to remember:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 9.2.17 Determine the vertex and axis of symmetry of the parabola $y = x^2 - 4x - 12$.

We find the first coordinate of the vertex using the formula $h = -\frac{b}{2a}$, for $a = 1$ and $b = -4$.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{(-4)}{2(1)} \\ &= 2 \end{aligned}$$

Now we know the first coordinate of the vertex is 2, so we may substitute $x = 2$ to determine the second coordinate of the vertex:

$$\begin{aligned} k &= (2)^2 - 4(2) - 12 \\ &= 4 - 8 - 12 \\ &= -16 \end{aligned}$$

The vertex is the point $(2, -16)$ and the axis of symmetry is the line $x = 2$.

Example 9.2.18 Determine the vertex and axis of symmetry of the parabola $y = -3x^2 - 3x + 7$.

Explanation. Using the formula $h = -\frac{b}{2a}$ with $a = -3$ and $b = -3$, we have :

$$h = -\frac{b}{2a}$$

$$= -\frac{(-3)}{2(-3)}$$
$$= -\frac{1}{2}$$

Now that we've determined $h = -\frac{1}{2}$, we can substitute it for x to find the y-value of the vertex:

$$\begin{aligned}
k &= -3x^2 - 3x + 7 \\
&= -3\left(-\frac{1}{2}\right)^2 - 3\left(-\frac{1}{2}\right) + 7 \\
&= -3\left(\frac{1}{4}\right) + \frac{3}{2} + 7 \\
&= -\frac{3}{4} + \frac{3}{2} + 7 \\
&= -\frac{3}{4} + \frac{6}{4} + \frac{28}{4} \\
&= \frac{31}{4}
\end{aligned}$$

The vertex is the point $\left(-\frac{1}{2}, \frac{31}{4}\right)$ and the axis of symmetry is the line $x = -\frac{1}{2}$.

9.2.3 Graphing Quadratic Equations by Making a Table

When we learned how to graph lines, we could choose any x-values to build a table of values. For quadratic equations, we want to make sure the vertex is present in the table, since it is such a special point. So we find the vertex first and then choose our x-values surrounding it. We can use the property of symmetry to speed things up.

Example 9.2.19 Determine the vertex and axis of symmetry for the parabola $y = -x^2 - 2x + 3$. Then make a table of values and sketch the graph.

Explanation. To determine the vertex of $y = -x^2 - 2x + 3$, we want to find the x-value of the vertex first. We use $h = -\frac{b}{2a}$ with $a = -1$ and $b = -2$:

$$\begin{aligned}
h &= -\frac{(-2)}{2(-1)} \\
&= \frac{2}{-2} \\
&= -1
\end{aligned}$$

To find the y-coordinate of the vertex, we substitute $x = -1$ into the equation for our parabola.

$$\begin{aligned}
k &= -x^2 - 2x + 3 \\
&= -(-1)^2 - 2(-1) + 3 \\
&= -1 + 2 + 3 \\
&= 4
\end{aligned}$$

Now we know that our axis of symmetry is the line $x = -1$ and the vertex is the point $(-1, 4)$. We set up our table with two values on each side of $x = -1$. We choose $x = -3, -2, -1, 0,$ and 1 as shown in Figure 9.2.20.

Next, we determine the y-coordinates by replacing x with each value and we have the complete table as shown in Figure 9.2.21. Notice that each pair of y-values on either side of the vertex match. This helps us to check that our vertex and y-values are correct.

x	$y=-x^2-2x+3$	Point
-3		
-2		
-1		
0		
1		

Figure 9.2.20: Setting up the table

x	$y=-x^2-2x+3$		Point
-3	$-(-3)^2-2(-3)+3$	$=0$	$(-3,0)$
-2	$-(-2)^2-2(-2)+3$	$=3$	$(-2,3)$
-1	$-(-1)^2-2(-1)+3$	$=4$	$(-1,4)$
0	$-(0)^2-2(0)+3$	$=3$	$(0,3)$
1	$-(1)^2-2(1)+3$	$=0$	$(1,0)$

Figure 9.2.21: Values and points for $y=-x^2-2x+3$

Now that we have our table, we plot the points and draw in the axis of symmetry as shown in Figure 9.2.22. We complete the graph by drawing a smooth curve through the points and drawing an arrow on each end as shown in Figure 9.2.23.

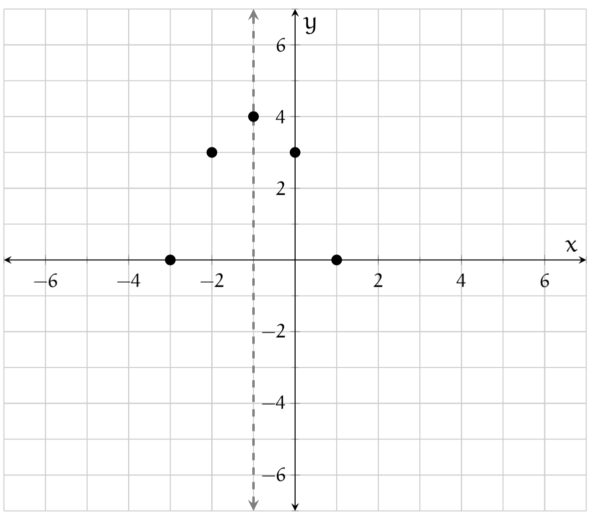


Figure 9.2.22: Plot of the points and axis of symmetry

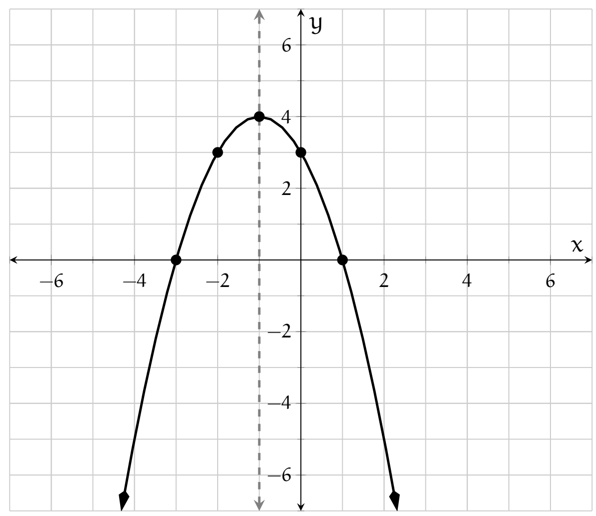


Figure 9.2.23: Graph of $y=-x^2-2x+3$

The method we used works best when the x-value of the vertex is an integer. We can still make a graph if that is not the case as we will demonstrate in the next example.

Example 9.2.24 Determine the vertex and axis of symmetry for the parabola $y=2x^2-3x-4$. Use this to create a table of values and sketch the graph.

Explanation. To determine the vertex of $y=2x^2-3x-4$, we find $h=-\frac{b}{2a}$ with $a=2$ and $b=-3$:

$$\begin{aligned} h &= -\frac{(-3)}{2(2)} \\ &= \frac{3}{4} \end{aligned}$$

Next, we determine the y-coordinate by replacing x with $\frac{3}{4}$ in $y = 2x^2 - 3x - 4$:

$$\begin{aligned} k &= 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) - 4 \\ &= 2\left(\frac{9}{16}\right) - \frac{9}{4} - 4 \\ &= \frac{9}{8} - \frac{18}{8} - \frac{32}{8} \\ &= -\frac{41}{8} \end{aligned}$$

Thus the vertex occurs at $\left(\frac{3}{4}, -\frac{41}{8}\right)$, or at $(0.75, -5.125)$. The axis of symmetry is then the line $x = \frac{3}{4}$, or $x = 0.75$. Now that we know the x-value of the vertex, we create a table. We choose x-values on both sides of $x = 0.75$, but we choose integers because it will be easier to find the y-values.

x	$y = 2x^2 - 3x - 4$	Point
-1	$2(-1)^2 - 3(-1) - 4$ $= 1$	$(-1, 1)$
0	$2(0)^2 - 3(0) - 4$ $= -4$	$(0, -4)$
0.75	$2(0.75)^2 - 3(-0.75) - 4$ $= -5.125$	$(0.75, -5.125)$
1	$2(1)^2 - 3(1) - 4$ $= -5$	$(1, -5)$
2	$2(2)^2 - 3(2) - 4$ $= -2$	$(2, -2)$

Figure 9.2.25: Values and points for $y = 2x^2 - 3x - 4$

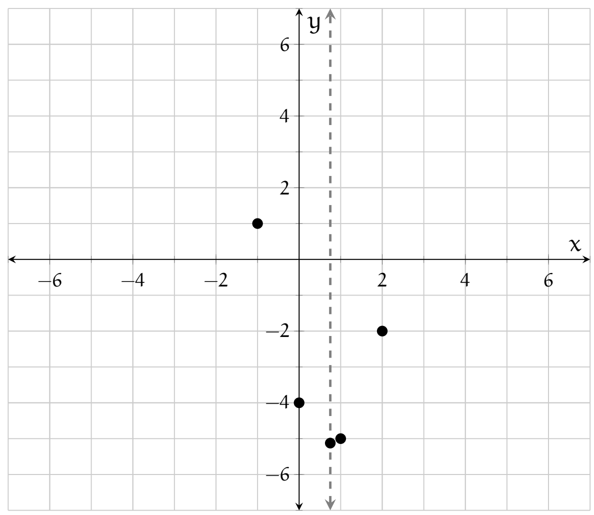


Figure 9.2.26: Plot of initial points

The points graphed in Figure 9.2.26 don't have the symmetry we'd expect from a parabola. This is because the vertex occurs at an x-value that is not an integer, and all of the chosen values in the table are integers. We can use the axis of symmetry to determine more points on the graph (as shown in Figure 9.2.27), which will give it the symmetry we expect. From there, we can complete the sketch of this graph.

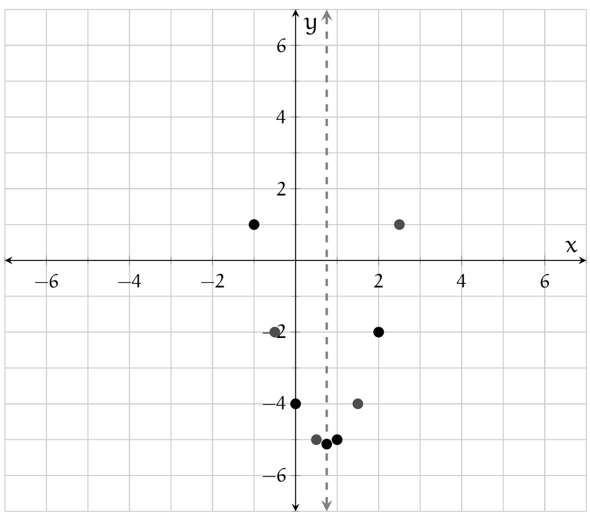


Figure 9.2.27: Plot of symmetric points

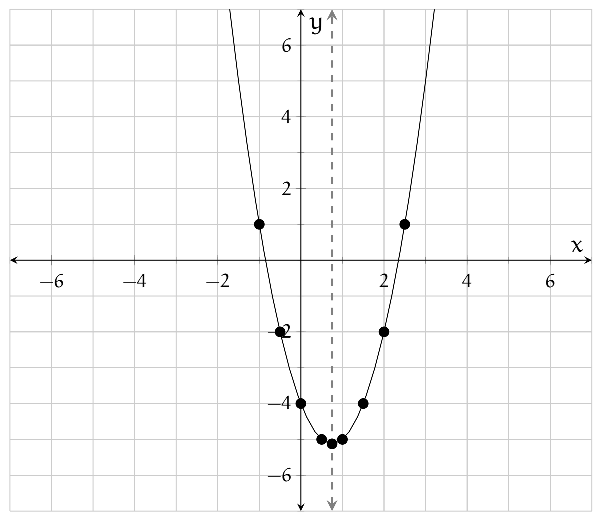


Figure 9.2.28: Graph of $y = 2x^2 - 3x - 4$

9.2.4 Applications of Quadratic Graphs Involving the Vertex.

We looked at the height of Hannah's toy rocket with respect to time at the beginning of this section and saw that it reached a maximum height of 64 feet after 2 seconds. Let's look at some more applications that involve finding the minimum or maximum y-value on a quadratic graph.

Example 9.2.29 Jae got a new air rifle for target practice. The first thing they did with it was some testing to find out how accurate the targeting cross-hairs were. In Olympic 10-meter air rifle shooting[1], the bulls-eye is a 0.5 mm diameter dot, about the size of the head of a pin, so accuracy is key. To test the accuracy, Jae stood at certain specific distances from a bullseye target, aimed the cross-hairs on the bullseye, and fired. Jae recorded how far above or how far below the pellet hit relative to the bullseye.

Distance to Target in Yards	5	10	20	30	35	40	50
Above/Below Bulls-eye	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\odot$	$\downarrow$	$\downarrow$
Distance Above/Below in Inches	0.1	0.6	1.1	0.6	0	0.8	3.2

Figure 9.2.30: Shooting Distance vs Pellet Rise/Fall

Make a graph of the height of the pellet relative to the bulls-eye at the shooting distances Jae used in Figure 9.2.30 and find the vertex. What does the vertex mean in this context?

Explanation.

Note that values measured below the bulls-eye should be graphed as negative y-values. Keep in mind that the units on the axes are different: along the x-axis, the units are yards, whereas on the y-axis, the units are inches.
Since the input values seem to be increasing by 5s or 10s, we scale the x-axis by 10s. The y-axis needs to be scaled by 1s.
From the graph we can see that the point $(20, 1.1)$ is our best guess for the vertex. This means the highest above the cross-hairs Jae hit was 1.1 inches when the target was 20 yards away.

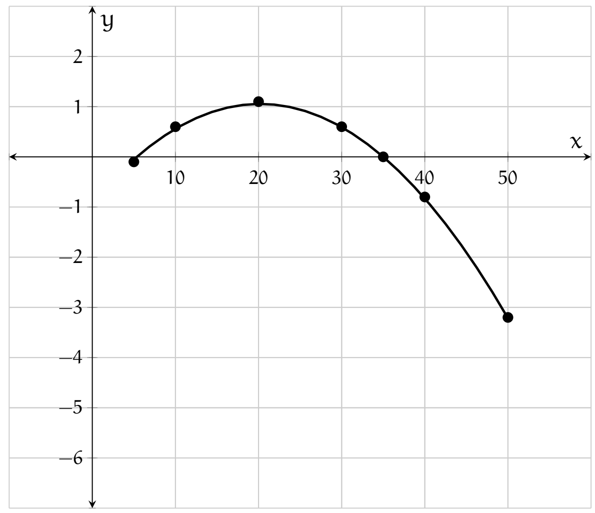


Figure 9.2.31: Graph of Target Data

Example 9.2.32 We looked at the quadratic equation $R = (13 + 0.25x)(1500 - 50x)$ in Example 5.4.2 of Section 5.4, where R was the revenue (in dollars) for x 25-cent price increases from an initial price of \$13. The expression simplified to

$$R = -12.5x^2 - 275x + 19500.$$

Find the vertex of this quadratic expression and explain what it means in the context of this model.

Explanation. Note that if we tried to use $R = (13+0.25x)(1500-50x)$, we would not be able to immediately identify the values of a and b needed to determine the vertex. Using the expanded form of $R = -12.5x^2 - 275x + 19500$, we see that $a = -12.5$ and $b = -275$, so the vertex occurs at:

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{-275}{2(-12.5)} \\ &= -11 \end{aligned}$$

And the second coordinate for the vertex is at:

$$\begin{aligned} k &= -12.5(-11)^2 - 275(-11) + 19500 \\ &= 21012.5 \end{aligned}$$

So the vertex occurs at $(-11, 21012.5)$.

Literally interpreting this, we can state that -11 of the 25-cent price increases result in a maximum revenue of \$21,012.50.

We can calculate "-11 of the 25-cent price increases" to be a decrease of \$2.75. The price was set at \$13 per jar, so the maximum revenue of \$21,012.50 would occur when Avery sets the price at \$10.25 per jar.

[1] en.wikipedia.org/wiki/ISSF_10_meter_air_rifle

Example 9.2.33 Kali has 500 feet of fencing and she needs to build a rectangular pen for her goats. What are the dimensions of the rectangle that would give her goats the largest area?

Explanation. We use ℓ for the length of the pen and w for the width, in feet. We know that the perimeter must be 500 feet so that gives us

$$2\ell + 2w = 500$$

First we solve for the length:

$$\begin{aligned} 2\ell + 2w &= 500 \\ 2\ell &= 500 - 2w \\ \ell &= 250 - w \end{aligned}$$

Now we can write a formula for the rectangle's area:

$$\begin{aligned} A &= \ell \cdot w \\ A &= (250 - w) \cdot w \\ A &= 250w - w^2 \\ A &= -w^2 + 250w \end{aligned}$$

The area is a quadratic expression so we can identify $a = -1$ and $b = 250$ and find the vertex:

$$\begin{aligned} h &= -\frac{(250)}{2(-1)} \\ &= \frac{250}{2} \\ &= 125 \end{aligned}$$

Since the width of the rectangle that will maximize area is 125 ft, we can find the length using our expression:

$$\begin{aligned} \ell &= 250 - w \\ &= 250 - 125 \\ &= 125 \end{aligned}$$

To find the maximum area we can either substitute the width into the area formula or multiply the length by the width:

$$\begin{aligned} A &= \ell \cdot w \\ A &= 125 \cdot 125 \\ A &= 15{,}625 \end{aligned}$$

The maximum area that Kali can get is 15,625 square feet if she builds her pen to be a square with a length and width of 125 feet.

9.2.5 Reading Questions

1. There are four key features of a quadratic graph discussed in this section. What are they?

2. Explain how the formula for the first coordinate of a parabola's vertex is similar to the quadratic formula.

3. If a parabola's vertex is at $(4, 6)$, and you know the coordinates of some points on the parabola where $x = 1, 2, 3$, at what other x-values do you know coordinates on the parabola?

9.2.6 Exercises

Review and Warmup Make a table for the equation.

1. The first row is an example.

x	$y = -x + 2$	Points
-3	5	$(-3, 5)$
-2	______	______
-1	______	______
0	______	______
1	______	______
2	______	______

2. The first row is an example.

x	$y = -x + 3$	Points
-3	6	$(-3, 6)$
-2	______	______
-1	______	______
0	______	______
1	______	______
2	______	______

3. The first row is an example.

x	$y = \frac{3}{10}x - 3$	Points
-30	-12	$(-30, -12)$
-20	______	______
-10	______	______
0	______	______
10	______	______
20	______	______

4. The first row is an example.

x	$y = \frac{5}{6}x + 9$	Points
-18	-6	$(-18, -6)$
-12	______	______
-6	______	______
0	______	______
6	______	______
12	______	______

5. Evaluate the expression $\frac{1}{5}(x+3)^2 - 2$ when $x = -8$.

6. Evaluate the expression $\frac{1}{3}(x+3)^2 - 8$ when $x = -6$.

7. Evaluate the expression $-16t^2 + 64t + 128$ when $t = 3$.

8. Evaluate the expression $-16t^2 + 64t + 128$ when $t = 2$.

Algebraically Determining the Vertex and Axis of Symmetry of Quadratic Equations Find the axis of symmetry and vertex of the quadratic function.

9. $y = 5x^2 + 20x - 5$

10. $y = -4x^2 + 8x - 1$

11. $y = 4 + 40x - 4x^2$

12. $y = -3 - 16x - 4x^2$

13. $y = -2 - x^2 + 2x$

14. $y = -4 - x^2 - 10x$

15. $y = 2x^2 + 8x$

16. $y = 3x^2 - 12x$

17. $y = 5 + 4x^2$

18. $y = 1 + 5x^2$

19. $y = -4x^2 + 12x - 3$

20. $y = -3x^2 - 15x + 1$

21. $y = -4x^2 - 4x - 5$

22. $y = -2x^2 + 6x - 1$

23. $y = 2x^2$

24. $y = 3x^2$

25. $y = 0.4x^2 + 2$

26. $y = 4x^2 - 4$

27. $y = 0.5(x+2)^2 - 2$

28. $y = -0.5(x+5)^2 - 1$

Graphing Quadratic Equations Using the Vertex and a Table For the given quadratic equation, find the vertex. Then create a table of ordered pairs centered around the vertex and make a graph.

29. $y = x^2 + 2$

30. $y = x^2 + 1$

31. $y = x^2 - 5$

32. $y = x^2 - 3$

33. $y = (x - 2)^2$

34. $y = (x - 4)^2$

35. $y = (x + 3)^2$

36. $y = (x + 2)^2$

Graphing Quadratic Equations Using the Vertex and a Table

37. For $y = 4x^2 - 8x + 5$, determine the vertex, create a table of ordered pairs, and then make a graph.

38. For $y = 2x^2 + 4x + 7$, determine the vertex, create a table of ordered pairs, and then make a graph.

39. For $y = -x^2 + 4x + 2$, determine the vertex, create a table of ordered pairs, and then make a graph.

40. For $y = -x^2 + 2x - 5$, determine the vertex, create a table of ordered pairs, and then make a graph.

41. For $y = x^2 - 5x + 3$, determine the vertex, create a table of ordered pairs, and then make a graph.

42. For $y = x^2 + 7x - 1$, determine the vertex, create a table of ordered pairs, and then make a graph.

43. For $y = -2x^2 - 5x + 6$, determine the vertex, create a table of ordered pairs, and then make a graph.

44. For $y = 2x^2 - 9x$, determine the vertex, create a table of ordered pairs, and then make a graph.

Finding Maximum and Minimum Values for Applications of Quadratic Equations

45. Consider two numbers where one number is 5 less than a second number. Find a pair of such numbers that has the least product possible. One approach is to let x represent the smaller number, and write a formula for a function of x that outputs the product of the two numbers. Then find its vertex and interpret it.

These two numbers are $\boxed{}$ and the least possible product is $\boxed{}$.

46. Consider two numbers where one number is 6 less than a second number. Find a pair of such numbers that has the least product possible. One approach is to let x represent the smaller number, and write a formula for a function of x that outputs the product of the two numbers. Then find its vertex and interpret it.

These two numbers are $\boxed{}$ and the least possible product is $\boxed{}$.

47. Consider two numbers where one number is 4 less than 4 times a second number. Find a pair of such numbers that has the least product possible. One approach is to let x represent the smaller number, and write a formula for a function of x that outputs the product of the two numbers. Then find its vertex and interpret it.

These two numbers are $\boxed{}$ and the least possible product is $\boxed{}$.

48. Consider two numbers where one number is 9 less than 4 times a second number. Find a pair of such numbers that has the least product possible. One approach is to let x represent the smaller number, and write a formula for a function of x that outputs the product of the two numbers. Then find its vertex and interpret it.

These two numbers are $\boxed{}$ and the least possible product is $\boxed{}$.

49. You will build a rectangular sheep enclosure next to a river. There is no need to build a fence along the river, so you only need to build on three sides. You have a total of 470 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that runs perpendicular to the river, and write a formula for a function of x that outputs the area of the enclosure. Then find its vertex and interpret it.

The length of the pen (parallel to the river) should be ______, the width (perpendicular to the river) should be ______, and the maximum possible area is ______.

50. You will build a rectangular sheep enclosure next to a river. There is no need to build a fence along the river, so you only need to build on three sides. You have a total of 480 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that runs perpendicular to the river, and write a formula for a function of x that outputs the area of the enclosure. Then find its vertex and interpret it.

The length of the pen (parallel to the river) should be ______, the width (perpendicular to the river) should be ______, and the maximum possible area is ______.

51. You will build a rectangular sheep enclosure next to a river. There is no need to build a fence along the river, so you only need to build on three sides. You have a total of 490 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that runs perpendicular to the river, and write a formula for a function of x that outputs the area of the enclosure. Then find its vertex and interpret it.

The length of the pen (parallel to the river) should be ______, the width (perpendicular to the river) should be ______, and the maximum possible area is ______.

52. You will build a rectangular sheep enclosure next to a river. There is no need to build a fence along the river, so you only need to build on three sides. You have a total of 500 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that runs perpendicular to the river, and write a formula for a function of x that outputs the area of the enclosure. Then find its vertex and interpret it.

The length of the pen (parallel to the river) should be ______, the width (perpendicular to the river) should be ______, and the maximum possible area is ______.

53. You will build two identical rectangular enclosures next to a each other, sharing a side. You have a total of 300 feet of fence to use. Find the dimensions of each pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that the two pens share, and write a formula for a function of x that outputs the total area of the enclosures. Then find its vertex and interpret it.

The length of each (along the wall that they share) should be ______, the width should be ______, and the maximum possible area of each pen is ______.

54. You will build two identical rectangular enclosures next to a each other, sharing a side. You have a total of 324 feet of fence to use. Find the dimensions of each pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that the two pens share, and write a formula for a function of x that outputs the total area of the enclosures. Then find its vertex and interpret it.

The length of each (along the wall that they share) should be ______, the width should be [ ], and the maximum possible area of each pen is [ ].

55. You plan to build four identical rectangular animal enclosures in a row. Each adjacent pair of pens share a fence between them. You have a total of 336 feet of fence to use. Find the dimensions of each pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that adjacent pens share, and write a formula for a function of x that outputs the total area. Then find its vertex and interpret it.

The length of each pen (along the walls that they share) should be [ ], the width (perpendicular to the river) should be [ ], and the maximum possible area of each pen is [ ].

56. You plan to build four identical rectangular animal enclosures in a row. Each adjacent pair of pens share a fence between them. You have a total of 352 feet of fence to use. Find the dimensions of each pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that adjacent pens share, and write a formula for a function of x that outputs the total area. Then find its vertex and interpret it.

The length of each pen (along the walls that they share) should be [ ], the width (perpendicular to the river) should be [ ], and the maximum possible area of each pen is [ ].

57. Currently, an artist can sell 240 paintings every year at the price of $90.00 per painting. Each time he raises the price per painting by $15.00, he sells 5 fewer paintings every year.

a. To obtain maximum income of [ ], the artist should set the price per painting at [ ].

b. To earn $43,875.00 per year, the artist could sell his paintings at two different prices. The lower price is [ ] per painting, and the higher price is [ ] per painting.

58. Currently, an artist can sell 270 paintings every year at the price of $150.00 per painting. Each time he raises the price per painting by $5.00, he sells 5 fewer paintings every year.

a. To obtain maximum income of [ ], the artist should set the price per painting at [ ].

b. To earn $43,700.00 per year, the artist could sell his paintings at two different prices. The lower price is [ ] per painting, and the higher price is [ ] per painting.

9.3 Graphing Quadratic Expressions

We have learned how to visually locate the key features of quadratic graphs and how to find the vertex algebraically. In this section we'll explore how to find the intercepts algebraically and use their coordinates to more precisely graph a quadratic equation. Then we will see how to interpret the key features in context and distinguish between quadratic and other graphs.

Let's start by looking at a quadratic equation that models the path of a baseball after it is hit by Ignacio, the batter. The height of the baseball, H, measured in feet, after t seconds is given by $H = -16t^2+75t+4.7$. We know the graph will have the shape of a parabola and we want to know the initial height, the maximum height, and the amount of time it takes for the ball to hit the ground if it is not caught. These important ideas correspond to the vertical intercept, the vertex, and one of the horizontal intercepts.

The graph of this equation is shown in Figure 9.3.2. We cannot easily read where the intercepts occur from the graph because they are not integers. We previously covered how to determine the vertex algebraically. In this section, we'll learn how to find the intercepts algebraically. Then we'll come back to this example and find the intercepts for the path of the baseball.

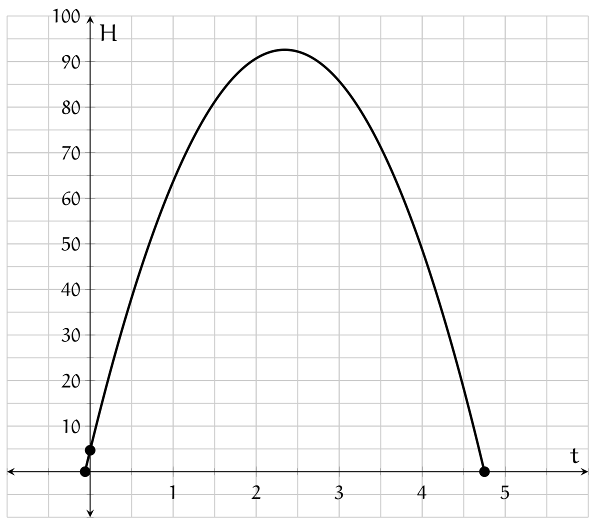


Figure 9.3.2: Graph of $H = -16t^2 + 75t + 4.7$

9.3.1 Finding the Vertical and Horizontal Intercepts Algebraically

In List 9.2.12, we identified that the **vertical intercept** occurs where the graph intersects the vertical axis. If we're using x and y as our variables, the x-value on the vertical axis is $x = 0$. We can substitute 0 for x to find the value of y.

The **horizontal intercepts** occur where the graph intersects the horizontal axis. If we're using x and y as our variables, the y-value on the horizontal axis is $y = 0$, so we can substitute 0 for y and find the value(s) of x.

Example 9.3.3 Find the intercepts for the quadratic equation $y = x^2 - 4x - 12$ using algebra.

To determine the y-intercept, we substitute $x = 0$ and find $y = 0^2 - 4(0) - 12 = -12$. So the y-intercept occurs where $y = -12$. On a graph, this is the point $(0, -12)$.

To determine the x-intercept(s), we set $y = 0$ and solve for x:

$$\begin{aligned} 0 &= x^2 - 4x - 12 \\ x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-12)}}{2(1)} \\ &= \frac{4 \pm \sqrt{16+48}}{2} \\ &= \frac{4 \pm \sqrt{64}}{2} \end{aligned}$$

$$= \frac{4 \pm 8}{2}$$

$$x = \frac{4-8}{2} \quad \text{or} \quad x = \frac{4+8}{2}$$
$$x = \frac{-4}{2} \quad \text{or} \quad x = \frac{12}{2}$$
$$x = -2 \quad \text{or} \quad x = 6$$

The x-intercepts occur where $x = -2$ and where $x = 6$. On a graph, these are the points $(-2, 0)$ and $(6, 0)$.

Notice in Example 9.3.3 that the y-intercept was $(0, -12)$ and the value of c was -12. When we substitute 0 for x we will always get the value of c.

Fact 9.3.4 *The vertical intercept of a quadratic equation occurs at the point* $(0, c)$ *where* c *is the constant term, because substituting* $x = 0$ *leaves only the constant term.*

Example 9.3.5 Algebraically determine any horizontal and vertical intercepts of the quadratic equation $y = -x^2 + 5x - 7$.

Explanation. To determine the vertical intercept, take the constant term -7, and recognize that the y-intercept is at the point $(0, -7)$.

To determine the horizontal intercepts, we'll set $y = 0$ and solve for x:

$$0 = -x^2 + 5x - 7$$
$$x = \frac{-5 \pm \sqrt{5^2 - 4(-1)(-7)}}{2(-1)}$$
$$x = \frac{-5 \pm \sqrt{-3}}{-2}$$

The radicand is negative so there are no real solutions to the equation. This means there are no horizontal intercepts.

9.3.2 Graphing Quadratic Equations Using Their Key Features

To graph a quadratic equation using its key features, we can use algebra to determine the following: whether the parabola opens upward or downward, the vertical intercept, the horizontal intercepts and the vertex. Then we can graph the points and connect them with a smooth curve.

Example 9.3.6 Graph the quadratic equation $y = 2x^2 + 10x + 8$ by algebraically determining its key features.

To start, we'll note that this parabola will open upward, since the leading coefficient is positive.

To find the y-intercept, we substitute $x = 0$ to find $2(0)^2 + 10(0) + 8 = 8$. The y-intercept is $(0, 8)$.

Next, we'll find the horizontal intercepts by setting $y = 0$ and solving for x:

$$2x^2 + 10x + 8 = 0$$

$$x = \frac{-10 \pm \sqrt{10^2 - 4(2)(8)}}{2(2)}$$

$$\begin{aligned} &= \frac{-10 \pm \sqrt{100-64}}{4} \\ &= \frac{-10 \pm \sqrt{36}}{4} \\ &= \frac{-10 \pm 6}{4} \end{aligned}$$

$$\begin{aligned} x &= \frac{-10-6}{4} & \text{or} \quad & x = \frac{-10+6}{4} \\ x &= \frac{-16}{4} & \text{or} \quad & x = \frac{-4}{4} \\ x &= -4 & \text{or} \quad & x = -1 \end{aligned}$$

The x-intercepts are $(-4, 0)$ and $(-1, 0)$.

Lastly, we'll determine the vertex. Noting that $a = 2$ and $b = 10$, we have:

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{10}{2(2)} \\ &= -2.5 \end{aligned}$$

Using this x-value to find the y-coordinate, we have:

$$\begin{aligned} k &= 2(-2.5)^2 + 10(-2.5) + 8 \\ &= 12.5 - 25 + 8 \\ &= -4.5 \end{aligned}$$

The vertex is the point $(-2.5, -4.5)$, and the axis of symmetry is the line $x = -2.5$.

We're now ready to graph this curve. We'll start by drawing and scaling the axes so all of our key features will be displayed as shown in Figure 9.3.7. Next, we'll plot these key points as shown in Figure 9.3.8. Finally, we'll note that this parabola opens upward and connect these points with a smooth curve, as shown in Figure 9.3.9.

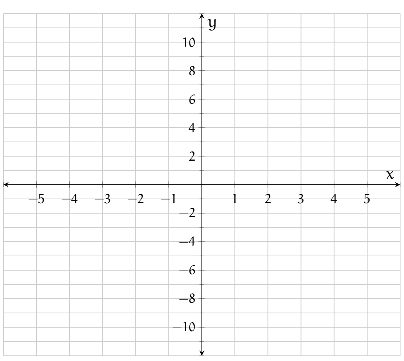


Figure 9.3.7: Setting up the grid.

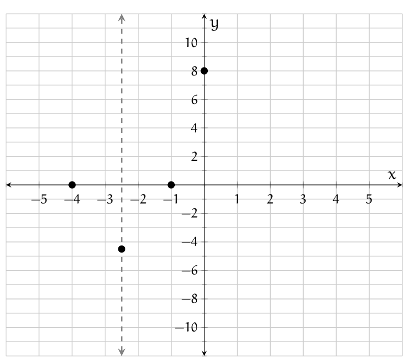


Figure 9.3.8: Marking key features.

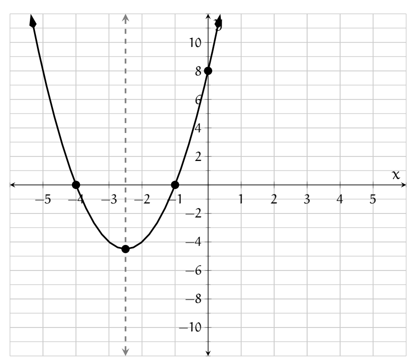


Figure 9.3.9: Completing the graph.

Example 9.3.10 Graph the quadratic equation $y = -x^2 + 4x - 5$ by algebraically determining its key features.

To start, we'll note that this parabola will open downward, as the leading coefficient is negative.

To find the y-intercept, we'll substitute x with 0:

$$\begin{aligned} y &= -(0)^2 + 4(0) - 5 \\ &= -5 \end{aligned}$$

The y-intercept is $(0, -5)$.

Next, we'll find the horizontal intercepts by setting $y = 0$ and solving for x.

$$-x^2 + 4x - 5 = 0$$

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{(4)^2 - 4(-1)(-5)}}{2(-1)} \\ &= \frac{-4 \pm \sqrt{16 - 20}}{-2} \\ &= \frac{-4 \pm \sqrt{-8}}{-2} \end{aligned}$$

The radicand is negative, so there are no real solutions to the equation. This is a parabola that does not have any horizontal intercepts.

To determine the vertex, we'll use $a = -1$ and $b = 4$:

$$\begin{aligned} h &= -\frac{4}{2(-1)} \\ &= 2 \end{aligned}$$

Using this x-value to find the y-coordinate, we have:

$$\begin{aligned} k &= -(2)^2 + 4(2) - 5 \\ &= -4 + 8 - 5 \\ &= -1 \end{aligned}$$

The vertex is the point $(2, -1)$, and the axis of symmetry is the line $x = 2$.

Plotting this information in an appropriate grid, we have:

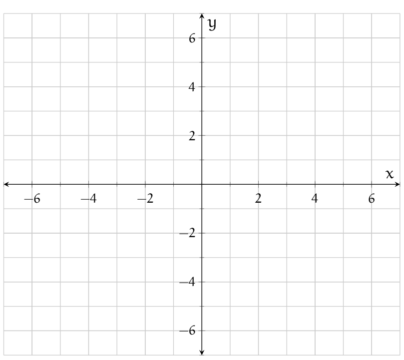


Figure 9.3.11: Setting up the grid.

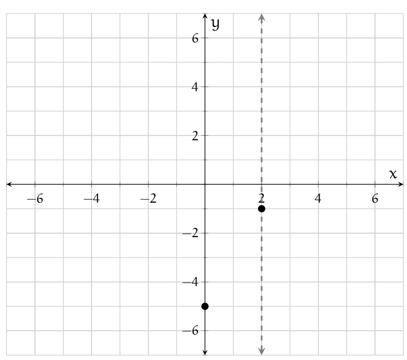


Figure 9.3.12: Marking key features.

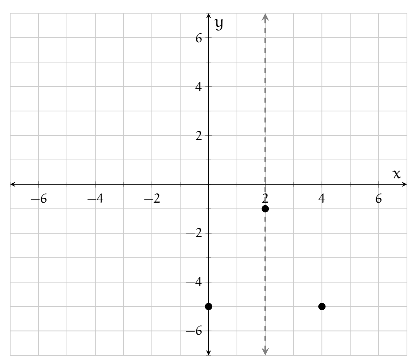


Figure 9.3.13: Using the axis of symmetry to determine one additional point.

Since we don't have any x-intercepts, we would like to have a few more points to graph. We make a table with a few more values around the vertex, plot these, and then draw a smooth curve. This is shown in Figure 9.3.14 and Figure 9.3.15.

x	$y = -x^2 + 4x - 5$	Point
0	$-(0)^2 + 4(0) - 5$ $= -5$	$(0, -5)$
1	$-(1)^2 + 4(1) - 5$ $= -2$	$(1, -2)$
2	$-(2)^2 + 4(2) - 5$ $= -1$	$(2, -1)$
3	$-(3)^2 + 4(3) - 5$ $= -2$	$(3, -2)$
4	$-(4)^2 + 4(4) - 5$ $= -5$	$(4, -5)$

Figure 9.3.14: Determine additional points to plot.

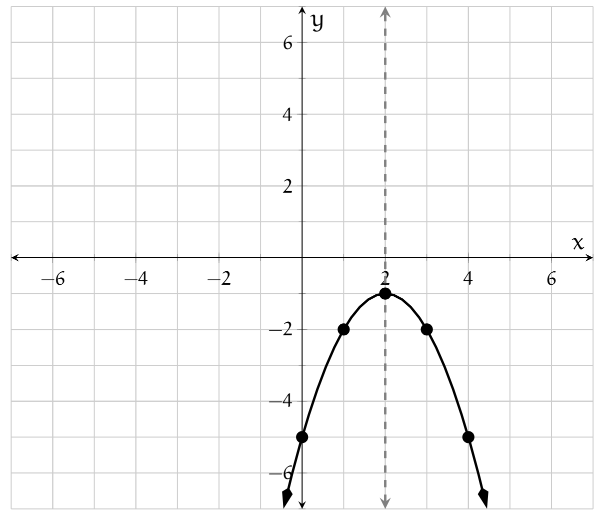


Figure 9.3.15: Completing the graph.

9.3.3 Applications of Quadratic Equations

Now we have learned how to find all the key features of a quadratic equation algebraically. Here are some applications of quadratic equations so we can learn how to identify and interpret the vertex, intercepts and additional points in context.

Example 9.3.16 Returning to the path of the baseball in Figure 9.3.2, the equation that represents the height of the baseball after Ignacio hit it, is $H = -16t^2 + 75t + 4.7$. The height is in feet and the time, t, is in seconds. Find and interpret the following, in context.

a. The vertical intercept.

b. The horizontal intercept(s).

c. The vertex.

d. The height of the baseball 1 second after it was hit.

e. The time(s) when the baseball is 80 feet above the ground.

Explanation.

a. To determine the vertical intercept, we'll substitute $t = 0$ to find $-16(0)^2 + 75(0) + 4.7 = 4.7$. The vertical intercept occurs at $(0, 4.7)$. This is the height of the baseball at time $t = 0$, so the initial height of the baseball was 4.7 feet.

b. To determine the horizontal intercepts, we'll solve $H = 0$.

$$\begin{aligned} H &= 0 \\ -16t^2 + 75t + 4.7 &= 0 \end{aligned}$$

$$\begin{aligned} t &= \frac{-75 \pm \sqrt{75^2 - 4(-16)(4.7)}}{2(-16)} \\ &= \frac{-75 \pm \sqrt{5925.8}}{-32} \end{aligned}$$

Rounding these two values with a calculator, we obtain:

$$\approx -0.06185, 4.749$$

The horizontal intercepts occur at approximately $(-0.06185, 0)$ and $(4.749, 0)$. If we assume that the ball was hit when $t = 0$, a negative time does not make sense. The second horizontal intercept tells us that the ball hit the ground after approximately 4.75 seconds.

c. The vertex occurs at $t = h = -\frac{b}{2a}$, and for this equation $a = -16$ and $b = 75$. So we have:

$$\begin{aligned} h &= -\frac{75}{2(-16)} \\ &= 2.34375 \end{aligned}$$

And then we can find the vertex's second coordinate:

$$\begin{aligned} k &= -16(2.34375)^2 + 75(2.34375) + 4.7 \\ &\approx 92.59 \end{aligned}$$

Thus the vertex is about $(2.344, 92.59)$.

The vertex tells us that the baseball reached a maximum height of approximately 92.6 feet about 2.3 seconds after Ignacio hit it.

d. To find the height of the baseball after 1 second, we can compute H when $t = 1$:

$$-16(1)^2 + 75(1) + 4.7 = 63.7$$

The height of the baseball was 63.7 feet after 1 second.

e. If we want to know when the baseball was 80 feet in the air, then we set $H = 80$ and we have:

$$\begin{aligned} H &= 80 \\ -16t^2 + 75t + 4.7 &= 80 \\ -16t^2 + 75t - 75.3 &= 0 \end{aligned}$$

$$t = \frac{-75 \pm \sqrt{75^2 - 4(-16)(-75.3)}}{2(-16)}$$

$$= \frac{-75 \pm \sqrt{805.8}}{-32}$$

Rounding these two values with a calculator, we obtain:

$$\approx 1.457, 3.231$$

The baseball was 80 feet above the ground at two times, at about 1.5 seconds on the way up and about 3.2 seconds on the way down.

Example 9.3.17 The profit that Keenan's manufacturing company makes for producing n refrigerators is given by $P = -0.01n^2 + 520n - 54000$, for $0 \leq n \leq 51{,}896$.

a. Determine the profit the company will make when they produce 1000 refrigerators.

b. Determine the maximum profit and the number of refrigerators produced that yields this profit.

c. How many refrigerators need to be produced in order for the company to "break even?" (In other words, for their profit to be \$0.)

d. How many refrigerators need to be produced in order for the company to make a profit of \$1,000,000?

Explanation.

a. This question is giving us an input value and asking for the output value. We substitute 1000 for n and we have:

$$\begin{aligned} P &= -0.01(1000)^2 + 520(1000) - 54000 \\ &= 366000 \end{aligned}$$

If Keenan's company sells 1000 refrigerators it will make a profit of \$366,000.

b. This question is asking for the maximum, so we need to find the vertex. This parabola opens downward so the vertex will tell us the maximum profit and the corresponding number of refrigerators that need to be produced. Using $a = -0.01$ and $b = 520$, we have:

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{520}{2(-0.01)} \\ &= 26000 \end{aligned}$$

Now we find the value of P when $n = 26000$:

$$\begin{aligned} k &= -0.01(26000)^2 + 520(26000) - 54000 \\ &= 6706000 \end{aligned}$$

The maximum profit is \$6,706,000, which occurs if 26,000 units are produced.

c. This question is giving a height of 0 and asking us to find the time(s). So we will be finding the horizontal intercept(s). We set $P = 0$ and solve for n using the quadratic formula:

$$0 = -0.01n^2 + 520n - 54000$$

$$\begin{aligned} n &= \frac{-520 \pm \sqrt{520^2 - 4(-0.01)(-54000)}}{2(-0.01)} \\ &= \frac{-520 \pm \sqrt{268240}}{-0.02} \\ &\approx 104, 51896 \end{aligned}$$

The company will break even if they produce about 104 refrigerators or 51,896 refrigerators. If the company produces more refrigerators than it can sell its profit will go down.

d. This question is giving us the profit value. We set $P = 1000000$ and solve for n using the quadratic formula:

$$\begin{aligned} 1000000 &= -0.01n^2 + 520n - 54000 \\ 0 &= -0.01n^2 + 520n - 1054000 \\ n &= \frac{-520 \pm \sqrt{520^2 - 4(-0.01)(-1054000)}}{2(-0.01)} \\ &= \frac{-520 \pm \sqrt{228240}}{-0.02} \\ &\approx 2113, 49887 \end{aligned}$$

The company will make $1,000,000 in profit if they produce about 2113 refrigerators or 49,887 refrigerators.

Example 9.3.18 Maia has a remote-controlled airplane and she is going to do a stunt dive where the plane dives toward the ground and back up along a parabolic path. The height of the plane after t seconds is given by $H = 0.7t^2 - 23t + 200$, for $0 \leq t \leq 30$. The height is measured in feet.

a. Determine the starting height of the plane as the dive begins.

b. Determine the height of the plane after 5 seconds.

c. Will the plane hit the ground, and if so, at what time?

d. If the plane does not hit the ground, what is the closest it gets to the ground, and at what time?

e. At what time(s) will the plane have a height of 50 feet?

Explanation.

a. This question is asking for the starting height which is the vertical intercept. So we find H when $t = 0$:

$$0.7(0)^2 - 23(0) + 200 = 200$$

When Maia begins the stunt, the plane has a height of 200 feet. Recall that we can also look at the value of $c = 200$ to determine the vertical intercept.

b. This question is telling us to use $t = 5$ and find H:

$$0.7(5)^2 - 23(5) + 200 = 102.5$$

After 5 seconds, the plane is 102.5 feet above the ground.

c. The ground has a height of 0 feet, so it is asking us to find the horizontal intercept(s) if there are any. We set $H = 0$ and solve for t using the quadratic formula:

$$\begin{aligned} H &= 0.7t^2 - 23t + 200 \\ 0 &= 0.7t^2 - 23t + 200 \\ t &= \frac{23 \pm \sqrt{(-23)^2 - 4(0.7)(200)}}{2(0.7)} \\ t &= \frac{23 \pm \sqrt{-31}}{1.4} \end{aligned}$$

The radicand is negative so there are no real solutions to the equation $H = 0$. That means the plane did not hit the ground.

d. This question is asking for the lowest point of the airplane so we should find the vertex. Using $a = 0.7$ and $b = -23$, we have:

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{(-23)}{2(0.7)} \\ &\approx 16.43 \end{aligned}$$

Now we can find the value of H when $t \approx 16.43$:

$$\begin{aligned} k &= 0.7(16.43)^2 - 23(16.43) + 200 \\ &\approx 11.07 \end{aligned}$$

The minimum height of the plane is about 11 feet, which occurs after about 16 seconds.

e. This question is giving us a height and asking for the corresponding time(s) so we set $H = 50$ and solve for t using the quadratic formula:

$$\begin{aligned} H &= 0.7t^2 - 23t + 200 \\ 50 &= 0.7t^2 - 23t + 200 \\ 0 &= 0.7t^2 - 23t + 150 \\ t &= \frac{23 \pm \sqrt{(-23)^2 - 4(0.7)(150)}}{2(0.7)} \\ &= \frac{23 \pm \sqrt{109}}{1.4} \\ &\approx 8.971, 23.89 \end{aligned}$$

Maia's plane will be 50 feet above the ground about 9 seconds and 24 seconds after the plane begins the stunt.

9.3.4 Distinguishing Quadratic Equations from Other Equations

So far, we've seen that the graphs of quadratic equations are parabolas and have a specific curved with a vertex. We've also seen that they have the algebraic form of $y = ax^2 + bx + c$. Here, we practice recognizing a quadratic equation so that we can call to mind that the equation has these features, which may be useful in some application.

Example 9.3.19 Determine if each equation is a quadratic equation.

a. $y + 5x^2 - 4 = 0$

b. $x^2 + y^2 = 9$

c. $y = -5x + 1$

d. $y = (x - 6)^2 + 3$

e. $y = \sqrt{x + 1} + 5$

Explanation.

a. As $y + 5x^2 - 4 = 0$ can be re-written as $y = -5x^2 + 4$, this equation is a quadratic equation.

b. The equation $x^2 + y^2 = 9$ cannot be re-written in the form $y = ax^2 + bx + c$ (due to the y^2 term), so this equation is not a quadratic equation.

c. The equation $y = -5x + 1$ is a linear equation, not a quadratic equation.

d. The equation $y = (x - 6)^2 + 3$ can be re-written as $y = x^2 - 12x + 39$, so this is a quadratic equation.

e. The equation $y = \sqrt{x + 1} + 5$ is not a quadratic equation as x is inside a radical, not squared.

Example 9.3.20 Determine if each graph *could* be the graph of a quadratic equation.

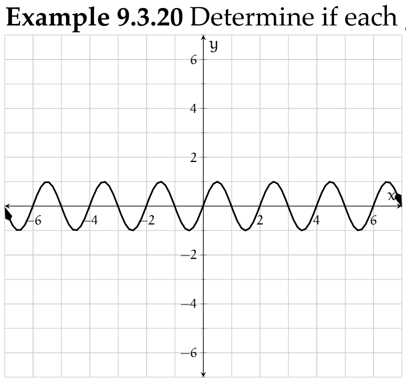


Figure 9.3.21

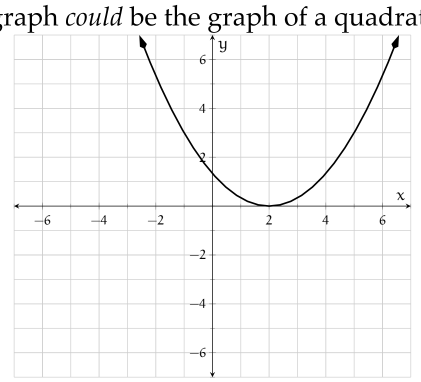


Figure 9.3.22

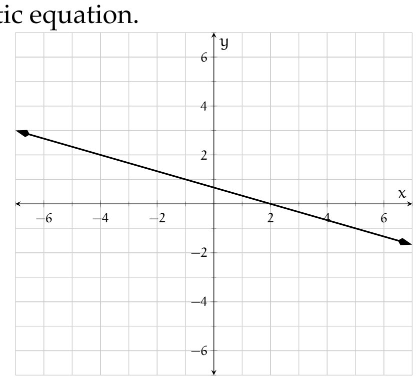


Figure 9.3.23

Explanation.

a. Since this graph has multiple maximum points and minimum points, it is not a parabola and it is not possible that it represents a quadratic equation.

b. This graph looks like a parabola, and it's possible that it represents a quadratic equation.

c. This graph does not appear to be a parabola, but looks like a straight line. It's not likely that it represents a quadratic equation.

9.3.5 Reading Questions

1. Explain how to find a parabola's y-intercept when you have the equation for the parabola.

2. Why does a parabola sometimes have zero x-intercepts, sometimes have one, and sometimes have two?

3. When you have the equation for a quadratic graph, what can you always try to use to find any horizontal intercepts?

9.3.6 Exercises

Review and Warmup Solve the equation.

1. $x^2 + 10x + 16 = 0$

2. $x^2 + 8x + 15 = 0$

3. $x^2 - 8x + 16 = 0$

4. $x^2 - 12x + 36 = 0$

5. $x^2 - 1 = 0$

6. $x^2 - 9 = 0$

7. $41x^2 - 47 = 0$

8. $17x^2 - 59 = 0$

9. $3x^2 - 8x - 4 = 0$

10. $7x^2 + 4x - 2 = 0$

11. $2x^2 + 3x + 6 = 0$

12. $5x^2 + 5x + 7 = 0$

Finding the Intercepts of Quadratic Equations Algebraically Find the y-intercept and any x-intercept(s) of the quadratic curve.

13. $y = x^2 + 4x + 3$

14. $y = -x^2 - 2x + 3$

15. $y = x^2 - 4$

16. $y = -x^2 + 9$

17. $y = x^2 - 4x$

18. $y = -x^2 + 5x$

19. $y = x^2 + 8x + 16$

20. $y = x^2 + x + 3$

21. $y = x^2 + 3x + 4$

22. $y = x^2 + 2x + 5$

23. $y = x^2 + x + 6$

24. $y = x^2 + 8x + 3$

25. $y = x^2 + 8x + 6$

26. $y = x^2 + 7x + 8$

27. $y = x^2 + 8x + 10$

28. $y = 2x^2 - 9x + 10$

29. $y = 4x^2 + 4x + 1$

30. $y = 4x^2 - 49$

31. $y = -11x - 7 - 4x^2$

32. $y = 5x - 4x^2$

Sketching Graphs of Quadratic Equations Graph each curve by algebraically determining its key features.

33. $y = x^2 - 7x + 12$

34. $y = x^2 + 5x - 14$

35. $y = -x^2 - x + 20$

36. $y = -x^2 + 4x + 21$

37. $y = x^2 - 8x + 16$

38. $y = x^2 + 6x + 9$

39. $y = x^2 - 4$

40. $y = x^2 - 9$

41. $y = x^2 + 6x$

42. $y = x^2 - 8x$

43. $y = -x^2 + 5x$

44. $y = -x^2 + 16$

45. $y = x^2 + 4x + 7$

46. $y = x^2 - 2x + 6$

47. $y = x^2 + 2x - 5$

48. $y = x^2 - 6x + 2$

49. $y = -x^2 + 4x - 1$

50. $y = -x^2 - x + 3$

51. $y = 2x^2 - 4x - 30$

52. $y = 3x^2 + 21x + 36$

Applications of Quadratic Equations

53. An object was shot up into the air with an initial vertical speed of 544 feet per second. Its height as time passes can be modeled by the quadratic equation $y = -16t^2 + 544t$. Here t represents the number of seconds since the object's release, and y represents the object's height in feet.

a. After ______, this object reached its maximum height of ______.

b. This object flew for [ ] before it landed on the ground.

c. This object was [ ] in the air 22 s after its release.

d. This object was 3600 ft high at two times: once [ ] after its release, and again later [ ] after its release.

54. An object was shot up into the air with an initial vertical speed of 576 feet per second. Its height as time passes can be modeled by the quadratic equation $y = -16t^2 + 576t$. Here t represents the number of seconds since the object's release, and y represents the object's height in feet.

a. After [ ], this object reached its maximum height of [ ].

b. This object flew for [ ] before it landed on the ground.

c. This object was [ ] in the air 11 s after its release.

d. This object was 5120 ft high at two times: once [ ] after its release, and again later [ ] after its release.

55. From an oceanside clifftop 200 m above sea level, an object was shot into the air with an initial vertical speed of 274.4 $\frac{m}{s}$. It fell into the ocean. Its height (above sea level) as time passes can be modeled by the quadratic equation $y = -4.9t^2 + 274.4t + 200$. Here t represents the number of seconds since the object's release, and y represents the object's height (above sea level) in meters.

a. After [ ], this object reached its maximum height of [ ].

b. This object flew for [ ] before it landed in the ocean.

c. This object was [ ] above sea level 20 s after its release.

d. This object was 3081.2 m above sea level twice: once [ ] after its release, and again later [ ] after its release.

56. From an oceanside clifftop 160 m above sea level, an object was shot into the air with an initial vertical speed of 294 $\frac{m}{s}$. It fell into the ocean. Its height (above sea level) as time passes can be modeled by the quadratic equation $y = -4.9t^2 + 294t + 160$. Here t represents the number of seconds since the object's release, and y represents the object's height (above sea level) in meters.

a. After [ ], this object reached its maximum height of [ ].

b. This object flew for [ ] before it landed in the ocean.

c. This object was [ ] above sea level 46 s after its release.

d. This object was 1747.6 m above sea level twice: once [ ] after its release, and again later [ ] after its release.

57. A remote control aircraft will perform a stunt by flying toward the ground and then up. Its height, in feet, can be modeled by the equation $h = 0.4t^2 - 2.4t + 0.6$, where t is in seconds. The plane (☐ will ☐ will not) hit the ground during this stunt.

58. A remote control aircraft will perform a stunt by flying toward the ground and then up. Its height, in feet, can be modeled by the equation $h = 1.1t^2 - 8.8t + 21.6$, where t is in seconds. The plane (☐ will ☐ will not) hit the ground during this stunt.

59. A submarine is traveling in the sea. Its depth, in meters, can be modeled by $d = -0.1t^2 + t - 1.5$, where t stands for time in seconds. The submarine (☐ will ☐ will not) hit the sea surface along this route.

60. A submarine is traveling in the sea. Its depth, in meters, can be modeled by $d = -0.8t^2 + 9.6t - 31.8$, where t stands for time in seconds. The submarine (☐ will ☐ will not) hit the sea surface along this route.

61. An object is launched upward at the height of 310 meters. Its height can be modeled by

$$h = -4.9t^2 + 100t + 310,$$

where h stands for the object's height in meters, and t stands for time passed in seconds since its launch. The object's height will be 360 meters twice before it hits the ground. Find how many seconds since the launch would the object's height be 360 meters. Round your answers to two decimal places if needed.

The object's height would be 360 meters the first time at ☐ seconds, and then the second time at ☐ seconds.

62. An object is launched upward at the height of 330 meters. Its height can be modeled by

$$h = -4.9t^2 + 80t + 330,$$

where h stands for the object's height in meters, and t stands for time passed in seconds since its launch. The object's height will be 350 meters twice before it hits the ground. Find how many seconds since the launch would the object's height be 350 meters. Round your answers to two decimal places if needed.

The object's height would be 350 meters the first time at ☐ seconds, and then the second time at ☐ seconds.

63. Currently, an artist can sell 280 paintings every year at the price of \$60.00 per painting. Each time he raises the price per painting by \$15.00, he sells 10 fewer paintings every year.

Assume he will raise the price per painting x times, then he will sell $280 - 10x$ paintings every year at the price of $60 + 15x$ dollars. His yearly income can be modeled by the equation:

$$i = (60 + 15x)(280 - 10x)$$

where i stands for his yearly income in dollars. If the artist wants to earn \$28,800.00 per year from selling paintings, what new price should he set?

To earn \$28,800.00 per year, the artist could sell his paintings at two different prices. The lower price is ☐ per painting, and the higher price is ☐ per painting.

64. Currently, an artist can sell 280 paintings every year at the price of \$80.00 per painting. Each time he raises the price per painting by \$10.00, he sells 5 fewer paintings every year.

Assume he will raise the price per painting x times, then he will sell $280 - 5x$ paintings every year at the price of $80 + 10x$ dollars. His yearly income can be modeled by the equation:

$$i = (80 + 10x)(280 - 5x)$$

where i stands for his yearly income in dollars. If the artist wants to earn \$29,150.00 per year from selling paintings, what new price should he set?

To earn \$29,150.00 per year, the artist could sell his paintings at two different prices. The lower price is ________ per painting, and the higher price is ________ per painting.

Challenge

65. Consider the equation $y = x^2 + nx + p$. Let n and p be real numbers. Give your answers as points.

a. Suppose the graph has two real x-intercepts. What are they?

b. What is its y-intercept?

c. What is its vertex?

9.4 Graphically Solving Equations and Inequalities

It is possible to solve equations and inequalities simply by reading a graph well. In this section, we take that approach to solving equations.

9.4.1 Solving Equations Using a Graph

To *algebraically* solve an equation like $-0.01x^2 + 0.7x - 18 = -0.04x^2 - 3.6x + 32$, we'd start by rearranging terms so that we could apply the quadratic formula. That would be a lot of pencil-and-paper work, and a lot of opportunity to make human errors. An alternative is to *graphically* solve this equation. We start by graphing both

$$y = -0.01x^2 + 0.7x - 18 \qquad \text{and} \qquad y = -0.04x^2 - 3.6x + 32.$$

It happens that we learned how to graph equations like these by hand in Section 9.3, but we will "cheat" in this section and use graphing technology to just make the graphs for us.

Example 9.4.2 Solve the equation $-0.01x^2 + 0.7x - 18 = -0.04x^2 - 3.6x + 32$ graphically.

Explanation.

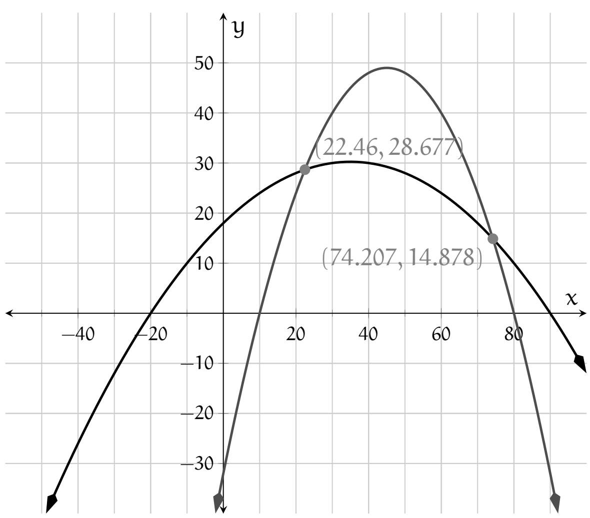


Figure 9.4.3: $y = -0.01x^2 + 0.7x - 18$ and $y = -0.04x^2 - 3.6x + 32$

There are two points of intersection where the curves cross each other: $(22.46, 28.677)$ and $(74.207, 14.878)$. Each one of them tells you a solution to the equation we started with. The point $(22.46, 28.677)$ means that when x is about 22.46, both $-0.01x^2 + 0.7x - 18$ and $-0.04x^2 - 3.6x + 32$ work out to the same result. That result is about 28.677, but that really doesn't matter right now. Its the x-value, about 22.46, that matters. That is one solution to the equation.

The second point of intersection similarly shows us that 74.207 is another approximate solution. We can conclude that the solution set to the equation is approximately $\{22.46, 74.207\}$.

Example 9.4.4 Graphically solve the equation $-0.01(x - 90)(x + 20) = 25$.

Explanation. Start by graphing two curves on the same plot: $y = \text{left}$ and $y = \text{right}$. Specifically for this example, $y = -0.01(x - 90)(x + 20)$ and $y = 25$.

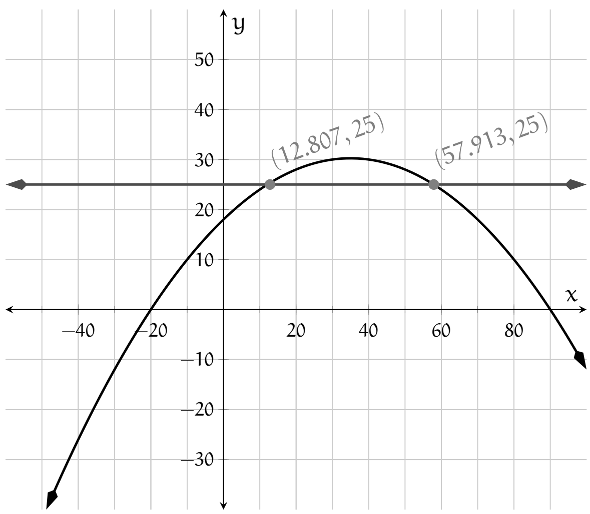


Figure 9.4.5: $y = -0.01(x-90)(x+20)$ and $y = 25$

The points of intersection are $(12.807, 25)$ and $(57.913, 25)$, which tells us that the solutions are approximately 12.807 and 57.913. The solution set is approximately $\{12.807, 57.913\}$.

One excellent thing about solving equations graphically is that it doesn't really matter what "kind" of equation it is. The equation can have mathematics in it that you haven't specifically studied, but as long as something (like a computer or your teacher) provides you with the graphs, you can still solve the equation.

Example 9.4.6 Graphically solve the equation $|x+5| = \frac{1}{x+1}$.

Explanation. If you've only been learning algebra from this textbook, this equation has some unfamiliar bits and pieces. The vertical bars in $|x+5|$ represent the basic math concept of absolute value, which you can brush up on in Appendix A.3. On the other side of the equation there is the expression $\frac{1}{x+1}$, with a variable in the denominator. This textbook hasn't discussed such things yet.

Even though we don't yet have general knowledge for these kinds of math expressions and their graphs, we can still trust some source to provide the graphs for us.

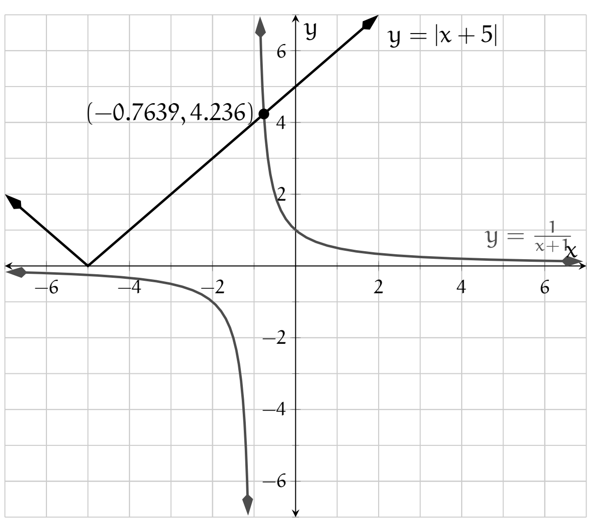


Figure 9.4.7: $y = |x+5|$ and $y = \frac{1}{x+1}$

It appears there is only one point of intersection at about $(-0.7639, 4.236)$. So the solution set is approximately $\{-0.7639\}$.

If we are solving graphically and something is already providing you with the graph, it's not even necessary to have math expressions for the two curves.

Example 9.4.8
In Figure 9.4.9, there are two curves plotted. The horizontal axis represents years, one curve represents the population of California, and the other curve represents the population of New York. In what year did the population of California equal the population of New York?
It appears there is only one point of intersection at about $(1963, 17.5)$. So the solution set is approximately $\{1963\}$. But in context, this says that 1963 is the year when California's population equaled New York's.

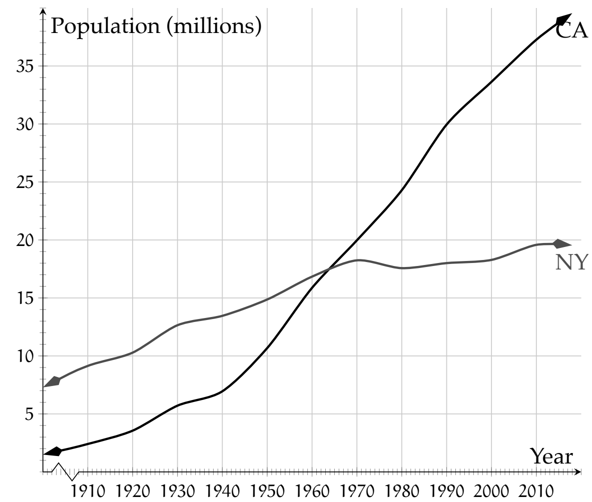


Figure 9.4.9: Populations of California and New York

9.4.2 Solving Inequalities Using a Graph

In Part I of this book, we learn how to solve linear inequalities such as $2x + 1 < 5$ using algebra. By using graphs instead of symbolic algebra, we can solve inequalities with more complicated math expressions, as well as inequalities in context that may not even have math expressions.

Example 9.4.10
In Figure 9.4.11, there are two curves plotted. The horizontal axis represents years, one curve represents the percent of US women ages 25–34 years old participating in the workforce, and the other curve represents the percent of US women ages 45–54 years old participating in the workforce. When was the percent from the 25–34 group more than the percent from the 45–54 group?
The curve for women 25–34 appears to rise above the other curve between the years 1975 and 1997. So the solution set is the interval $(1975, 1997)$. But in context, this means that in between 1975 and 1997, the percentage of women 25–34 in the workforce was greater than the percentage of women 45–54 in the workforce.

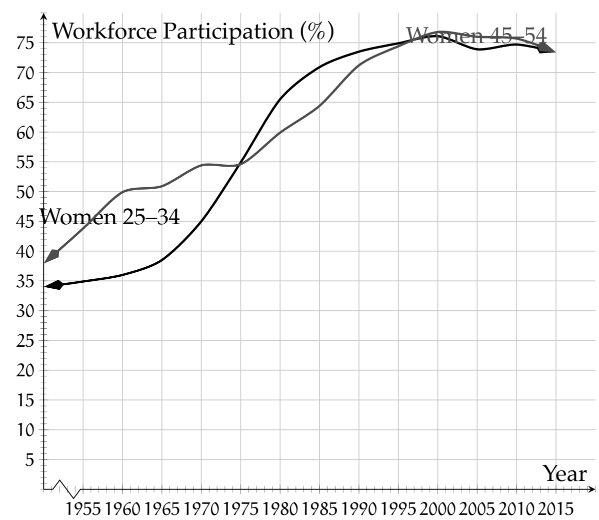


Figure 9.4.11: Women in the Workforce

It is helpful to take another look at this graph, with some annotations. We wanted the 25–34 curve to be greater than the 45–54 curve. Visually, we lock sights onto the indicated region. The solution set we are looking for is the *years* that this happened, which are down on the horizontal axis. So we have to project the region we've identified down onto the horizontal axis. After we've done this, the interval we see on the horizontal axis is the solution set.

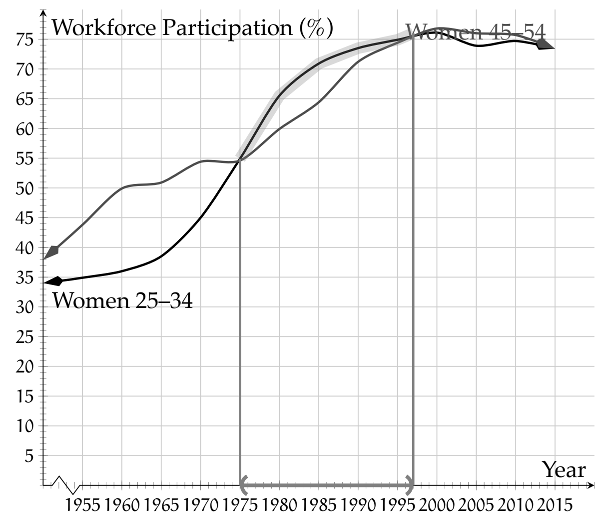


Figure 9.4.12: Women in the Workforce

Example 9.4.13 Graphically solve the following inequalities.

a. $-20t^2 - 70t + 300 \geq -5t + 300$

b. $-20t^2 - 70t + 300 < -5t + 300$

Explanation.

For both parts of this example, we start by graphing the equations $y = -20t^2 - 70t + 300$ and $y = -5t + 300$ and determining the points of intersection. You may use some piece of technology to do this, or perhaps you find yourself provided with these graphs, with the intersection points clearly marked or easy to determine.

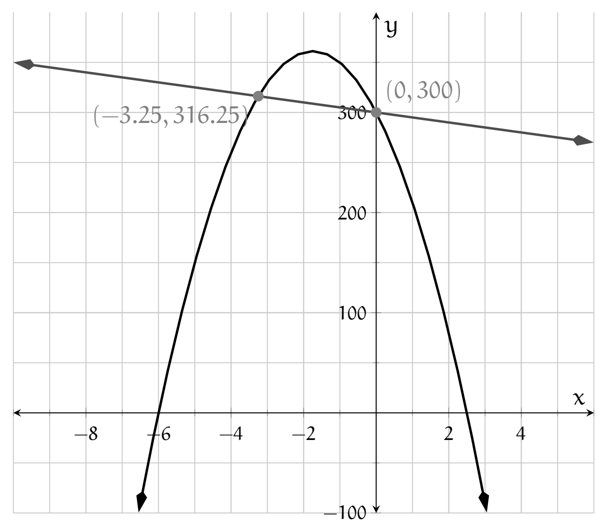


Figure 9.4.14: Points of intersection for $y = -20t^2 - 70t + 300$ and $y = -5t + 300$

a.

To solve $-20t^2 - 70t + 300 \geq -5t + 300$, we need to determine where the y-values of the parabola are higher than (or equal to) those of the line. This region is highlighted in Figure 9.4.15.

We can see that this region includes all values of t between, and including, $t = -3.25$ and $t = 0$. So the solutions to this inequality include all values of t for which $-3.25 \leq t \leq 0$. We can write this solution set in interval notation as $[-3.25, 0]$ or in set-builder notation as $\{t \mid -3.25 \leq t \leq 0\}$.

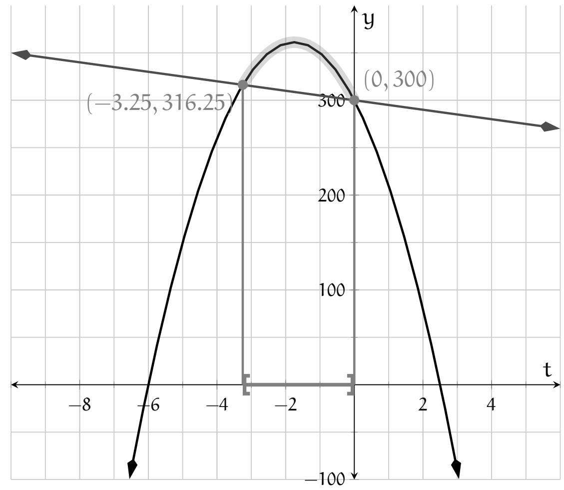


Figure 9.4.15

b.

To now solve $-20t^2 - 70t + 300 < -5t + 300$, we will need to determine where the y-values of the parabola are *less* than those of the line. This region is highlighted in Figure 9.4.16.

We can see that $-20t^2 - 70t + 300 < -5t + 300$ for all values of t where $t < -3.25$ or $t > 0$. We can write this solution set in interval notation as $(-\infty, -3.25) \cup (0, \infty)$ or in set-builder notation as $\{t \mid t < -3.25 \text{ or } t > 0\}$.

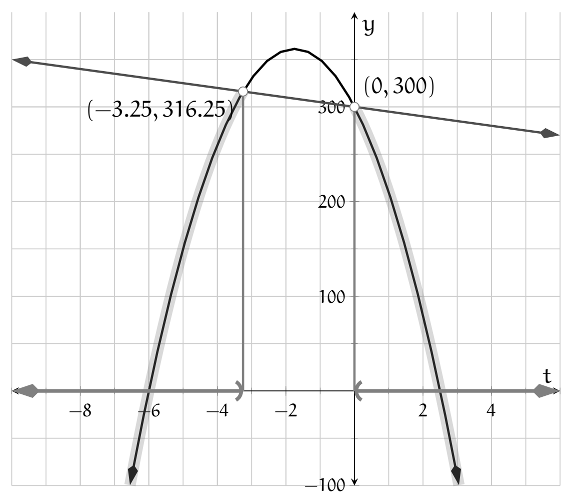


Figure 9.4.16

Occasionally, a curve abruptly "stops", and we need to recognize this in a solution to an inequality.

Example 9.4.17 Solve the inequality $1 - x > \sqrt{x+5}$ using a graph.
Explanation.

We plot $y = 1-x$ and $y = \sqrt{x+5}$, and then look for the intersection(s) of the graphs. The two curves intersect at $(-1, 2)$.

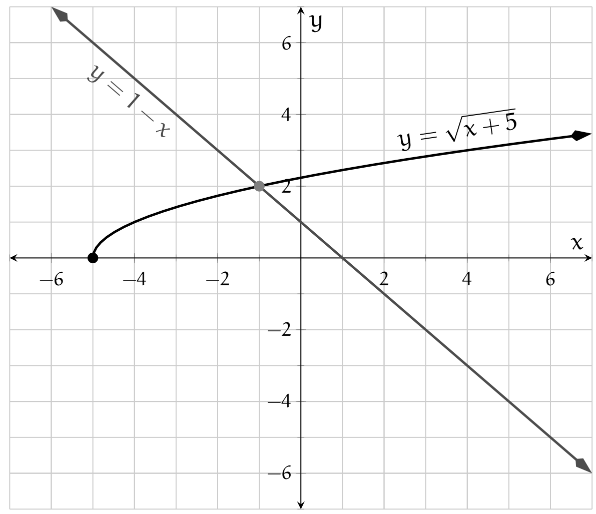


Figure 9.4.18: $y = 1 - x$ and $y = \sqrt{x+5}$

Since the inequality is $\overbrace{1-x}^{\text{line}} > \overbrace{\sqrt{x+5}}^{\text{half-parabola}}$, we want to identify the region where the line is higher than the half-parabola. While the line extends higher and higher off to the left, the half-parabola abruptly stops at $(-5, 0)$. So the solution set needs to stop at the corresponding place. As illustrated, the solution set is the interval $[-5, -1)$.

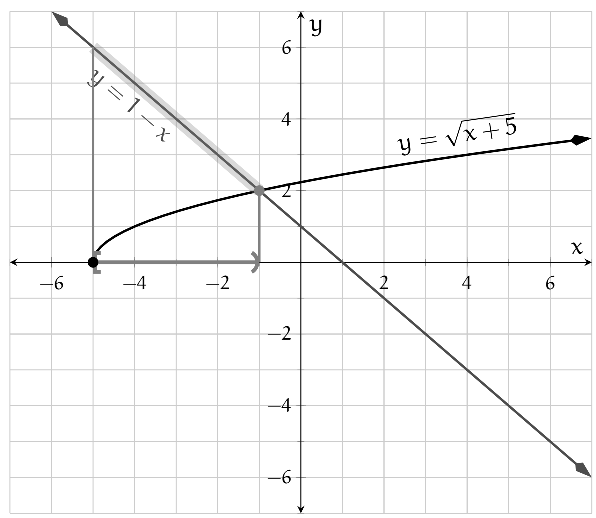


Figure 9.4.19: $y = 1 - x$ and $y = \sqrt{x+5}$

9.4.3 Reading Questions

1. Suppose you have an equation where x is the only variable. In order to solve that equation, explain how you could use a graph. Assume that some technology can provide you with any graph you would like to see.

2. The curves $y = x^4 - 3x^2 + x$ and $y = 1 - \sqrt{x-1}$ cross at three locations. How many solutions are there to $x^4 - 3x^2 + x = 1 - \sqrt{x-1}$?

3. The solution set to an inequality is generally not a single number or a small collection of numbers. In general, the solution set to an inequality is a $\boxed{}$.

9.4.4 Exercises

Points of Intersection Use technology to make some graphs and determine how many times the graphs of the following curves cross each other.

1. $y = (441 - 17x)(-67 - 16x)$ and $y = -8000$ intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

2. $y = (-143 - 12x)(-344 + 11x)$ and $y = -9000$ intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

3. $y = -4x^3 - x^2 + 5x$ and $y = 7x - 4$ intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

4. $y = -x^3 - 3x^2 - 6x$ and $y = 2x - 4$ intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

5. $y = -0.5\left(5x^2 + 2\right)$ and $y = 0.45(7x - 3)$ intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

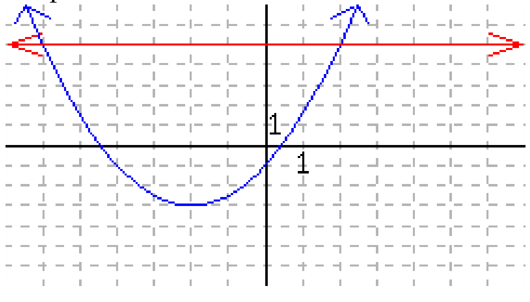
6. $y = -0.5\left(6x^2 - 9\right)$ and $y = -0.46(4x - 9)$ intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

7. $y = 1.05(x + 9)^2 - 1.05$ and $y = 1.1x - 1$ intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

8. $y = 1.5(x - 4)^2 + 6.45$ and $y = -0.05x - 1$ intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

Solving Equations and Inequalities Graphically

9. The equations $y = \frac{1}{2}x^2 + 2x - 1$ and $y = 5$ are plotted.

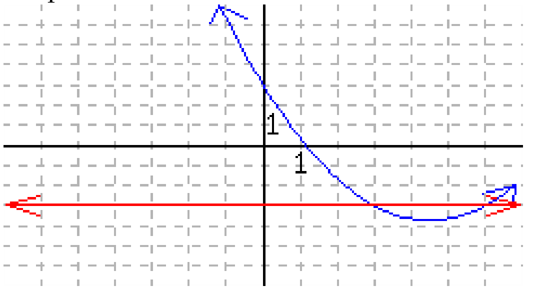


a. What are the points of intersection?

b. Solve $\frac{1}{2}x^2 + 2x - 1 = 5$.

c. Solve $\frac{1}{2}x^2 + 2x - 1 > 5$.

10. The equations $y = \frac{1}{3}x^2 - 3x + 3$ and $y = -3$ are plotted.

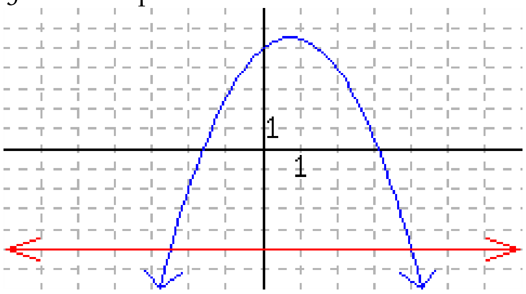


a. What are the points of intersection?

b. Solve $\frac{1}{3}x^2 - 3x + 3 = -3$.

c. Solve $\frac{1}{3}x^2 - 3x + 3 > -3$.

11. The equations $y = -x^2 + 1.5x + 5$ and $y = -5$ are plotted.

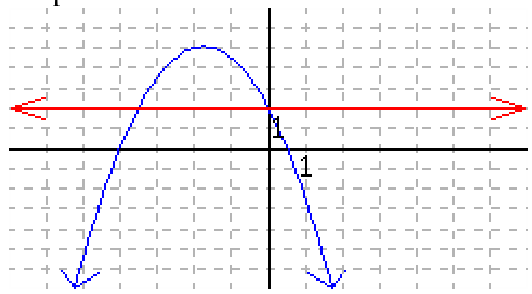


a. What are the points of intersection?

b. Solve $-x^2 + 1.5x + 5 = -5$.

c. Solve $-x^2 + 1.5x + 5 > -5$.

12. The equations $y = -x^2 - 3.5x + 2$ and $y = 2$ are plotted.

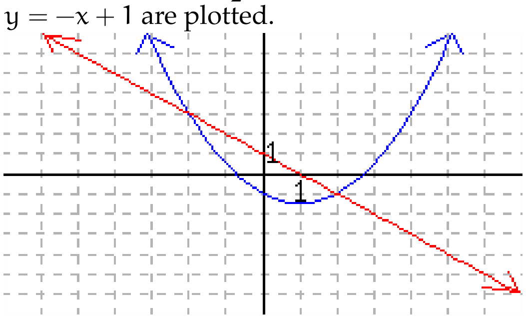


a. What are the points of intersection?

b. Solve $-x^2 - 3.5x + 2 = 2$.

c. Solve $-x^2 - 3.5x + 2 > 2$.

13. The equations $y = \frac{1}{2}x^2 - x - 1$ and $y = -x + 1$ are plotted.

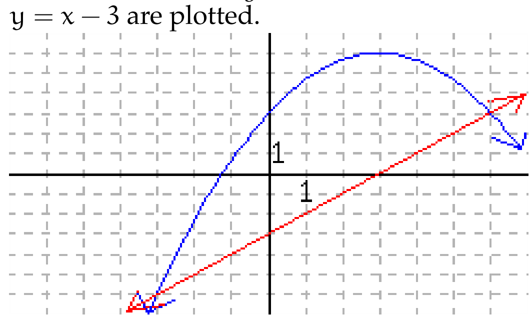


a. What are the points of intersection?

b. Solve $\frac{1}{2}x^2 - x - 1 = -x + 1$.

c. Solve $\frac{1}{2}x^2 - x - 1 > -x + 1$.

14. The equations $y = \frac{-1}{3}x^2 + 2x + 3$ and $y = x - 3$ are plotted.

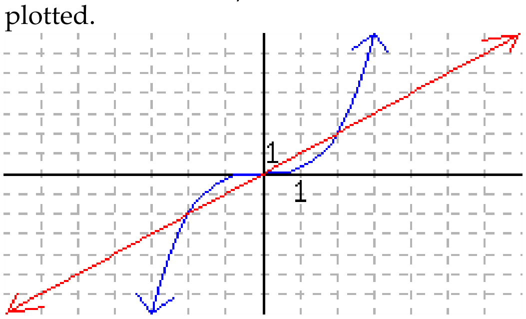


a. What are the points of intersection?

b. Solve $\frac{-1}{3}x^2 + 2x + 3 = x - 3$.

c. Solve $\frac{-1}{3}x^2 + 2x + 3 > x - 3$.

15. The equations $y = \frac{1}{4}x^3$ and $y = x$ are plotted.

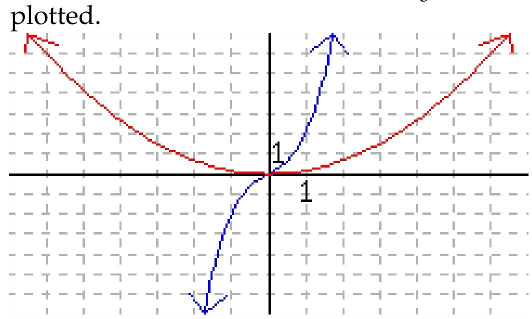


a. What are the points of intersection?

b. Solve $\frac{1}{4}x^3 = x$.

c. Solve $\frac{1}{4}x^3 > x$.

16. The equations $y = x^3 + x$ and $y = \frac{1}{6}x^2$ are plotted.

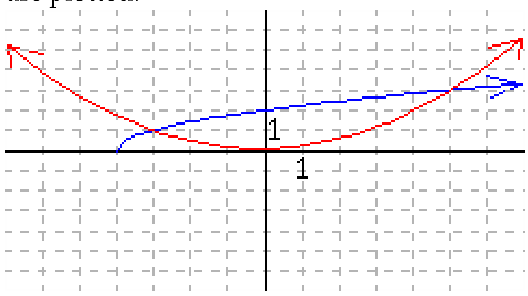


a. What are the points of intersection?

b. Solve $x^3 + x = \frac{1}{6}x^2$.

c. Solve $x^3 + x > \frac{1}{6}x^2$.

17. The equations $y = \sqrt{x+4}$ and $y = \frac{4x^2+x+3}{36}$ are plotted.

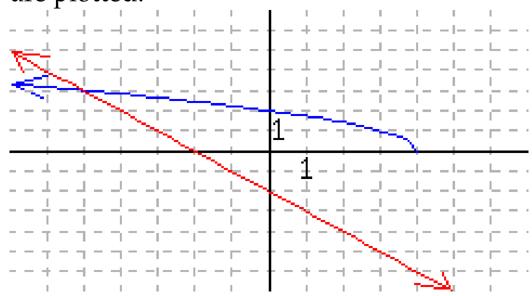


a. What are the points of intersection?

b. Solve $\sqrt{x+4} = \frac{4x^2+x+3}{36}$.

c. Solve $\sqrt{x+4} > \frac{4x^2+x+3}{36}$.

18. The equations $y = \sqrt{4-x}$ and $y = -2-x$ are plotted.

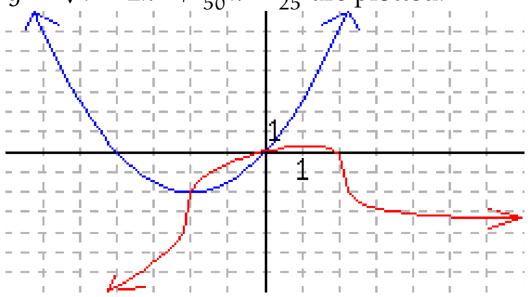


a. What are the points of intersection?

b. Solve $\sqrt{4-x} = -2-x$.

c. Solve $\sqrt{4-x} > -2-x$.

19. The equations $y = \frac{1}{2}x^2 + 2x$ and $y = \sqrt[3]{9-2x^2} + \frac{23}{50}x - \frac{52}{25}$ are plotted.

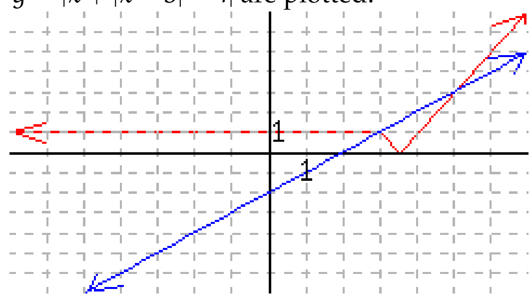


a. What are the points of intersection?

b. Solve $\frac{1}{2}x^2 + 2x = \sqrt[3]{9-2x^2} + \frac{23}{50}x - \frac{52}{25}$.

c. Solve $\frac{1}{2}x^2 + 2x > \sqrt[3]{9-2x^2} + \frac{23}{50}x - \frac{52}{25}$.

20. The equations $y = x - 2$ and $y = |x + |x-3| - 4|$ are plotted.



a. What are the points of intersection?

b. Solve $x - 2 = |x + |x-3| - 4|$.

c. Solve $x - 2 > |x + |x-3| - 4|$.

9.5 Topics in Graphing Chapter Review

9.5.1 Review of Graphing

In Section 9.1, we reviewed the fundamentals of graph-making. In particular, given an equation of the form $y =$ expression in x, the fundamental approach to making a graph is to make a table of points to plot.

We also looked back at the notions of "intercepts" on a graph. In the case of a linear equation in x and y, finding the x- and y-intercepts can be a way to create a graph.

9.5.2 Key Features of Quadratic Graphs

In Section 9.2, we identified the key features of a quadratic graph (which takes the shape of a parabola). The key features are the direction that it opens, the vertex, the axis of symmetry, the vertical intercept, and the horizontal intercepts (if there are any).

If the equation for a quadratic curve is $y = ax^2 + bx + c$, then the formula $h = -\frac{b}{2a}$ gives the first coordinate of the vertex. So you can find the location of the vertex with that coordinate and subbing that number into the equation to find the second coordinate.

If we know the location of a parabola's vertex and the direction that it opens, we can sketch the parabola. It helps to make a table finding a few points the the left and to the right of the vertex. The symmetry of a parabola means you only need to find points on one side to automatically get corresponding points on the other side.

9.5.3 Graphing Quadratic Equations

In Section 9.3, we practiced finding the exact locations of the vertical and horizontal intercepts for a quadratic equation curve. The vertical intercept can be found by lettting $x = 0$. The result is a number on the y-axis.

The horizontal intercepts can be found by setting y equal to 0. This leaves you with a quadratic equation in one variable, x, and the quadratic formula can be used to solve for x. There might be no solutions, as is the case when the parabola doesn't touch the x-axis. There might be one solution, when the vertex is on the x-axis. Or there might be two solutions, and therefore two horizontal intercepts.

When we know the exact locations of the intercepts (as well as the location of the vertex as found in Section 9.2) then we can plot accurate graphs of quadratic equations.

9.5.4 Graphically Solving Equations and Inequalities

In Section 9.4, we see how a graph can be used to solve an equation or inequality. Each side of an equation gives you a curve, and where the two curves cross tells you where there are solutions to the equation.

For example, to solve the equation $x^2 + x - 1 = 2x + 1$, we could plot two curves: $y = x^2 + x - 1$ and $y = 2x + 1$. We might use a computer to make the graphs for us, as in Figure 9.5.1.

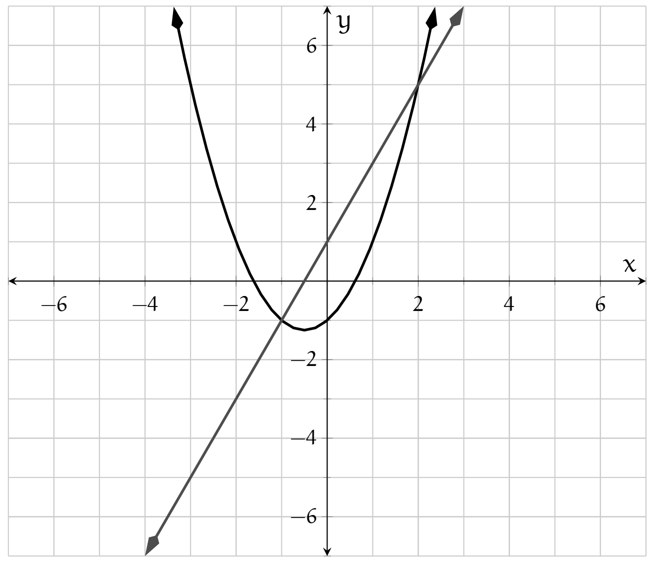


Figure 9.5.1: $y = x^2 + x - 1$ and $y = 2x + 1$

Since the curves cross at $(-1, -1)$ and $(2, 5)$, the solutions are $x = -1$ and $x = 2$. This means the solution set is $\{-1, 2\}$.

9.5.5 Exercises

Review of Graphing Make a table for the equation.

1.

x	$y = -9x$

2.

x	$y = -3x + 5$

3.

x	$y = \frac{10}{3}x + 1$

4.

x	$y = -\frac{5}{4}x + 3$

5. Create a table of ordered pairs and then make a plot of the equation $y = 2x + 3$.

6. Create a table of ordered pairs and then make a plot of the equation $y = -\frac{3}{4}x + 2$.

7. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$8x + 7y = -168$$

	x-value	y-value	Location (as an ordered pair)
y-intercept	______	______	____________
x-intercept	______	______	____________

8. Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$8x - 5y = -80$$

	x-value	y-value	Location (as an ordered pair)
y-intercept	______	______	____________
x-intercept	______	______	____________

9. Find the x- and y-intercepts of the line with equation $5x - 6y = -90$. Then find one other point on the line. Use your results to graph the line.

10. Find the x- and y-intercepts of the line with equation $x + 5y = -15$. Then find one other point on the line. Use your results to graph the line.

Key Features of Quadratic Graphs Find the axis of symmetry and vertex of the quadratic function.

11. $y = 5x^2 - 50x + 3$

12. $y = -3 - 30x - 5x^2$

13. $y = -1 - x^2 + 6x$

14. $y = -2x^2 + 20x$

15. $y = 2 - x^2$

16. $y = -2x^2 - 10x + 4$

17. $y = 2x^2 + 10x - 2$

18. $y = 3x^2$

19. $y = 0.4x^2 - 4$

20. $y = 5(x + 3)^2 + 4$

For the given quadratic equation, find the vertex. Then create a table of ordered pairs centered around the vertex and make a graph.

21. $y = x^2 + 2$

22. $y = x^2 - 5$

23. $y = (x - 2)^2$

24. $y = (x + 3)^2$

25. For $y = 4x^2 - 8x + 5$, determine the vertex, create a table of ordered pairs, and then make a graph.

26. For $y = -x^2 + 4x + 2$, determine the vertex, create a table of ordered pairs, and then make a graph.

27. For $y = x^2 - 5x + 3$, determine the vertex, create a table of ordered pairs, and then make a graph.

28. For $y = -2x^2 - 5x + 6$, determine the vertex, create a table of ordered pairs, and then make a graph.

29. Consider two numbers where one number is 4 less than a second number. Find a pair of such numbers that has the least product possible. One approach is to let x represent the smaller number, and write a formula for a function of x that outputs the product of the two numbers. Then find its vertex and interpret it.

These two numbers are [ ] and the least possible product is [ ].

30. You will build a rectangular sheep enclosure next to a river. There is no need to build a fence along the river, so you only need to build on three sides. You have a total of 420 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum possible area. One approach is to let x represent the length of fencing that runs perpendicular to the river, and write a formula for a function of x that outputs the area of the enclosure. Then find its vertex and interpret it.

The length of the pen (parallel to the river) should be [ ], the width (perpendicular to the river) should be [ ], and the maximum possible area is [ ].

Graphing Quadratic Equations Find the y-intercept and any x-intercept(s) of the quadratic curve.

31. $y = x^2 - 2x - 8$

32. $y = -x^2 + 1$

33. $y = x^2 + 6x + 9$

34. $y = x^2 + 4x + 7$

35. $y = x^2 + 8x + 5$

36. $y = 5x^2 - 8x - 4$

37. $y = -x + 18 - 5x^2$

38. $y = 5x - 2x^2$

Graph each curve by algebraically determining its key features.

39. $y = x^2 - 7x + 12$

40. $y = -x^2 - x + 20$

41. $y = x^2 - 8x + 16$

42. $y = x^2 - 4$

43. $y = x^2 + 6x$

44. $y = -x^2 + 5x$

45. $y = x^2 + 4x + 7$

46. $y = x^2 + 2x - 5$

47. $y = -x^2 + 4x - 1$

48. $y = 2x^2 - 4x - 30$

49. An object was shot up into the air with an initial vertical speed of 384 feet per second. Its height as time passes can be modeled by the quadratic equation $y = -16t^2 + 384t$. Here t represents the number of seconds since the object's release, and y represents the object's height in feet.

a. After [ ], this object reached its maximum height of [ ].

b. This object flew for [ ] before it landed on the ground.

c. This object was [ ] in the air 3 s after its release.

d. This object was 704 ft high at two times: once [ ] after its release, and again later [ ] after its release.

50. A remote control aircraft will perform a stunt by flying toward the ground and then up. Its height, in feet, can be modeled by the equation $h = t^2 - 10t + 28$, where t is in seconds. The plane (☐ will ☐ will not) hit the ground during this stunt.

51. An object is launched upward at the height of 280 meters. Its height can be modeled by

$$h = -4.9t^2 + 70t + 280,$$

where h stands for the object's height in meters, and t stands for time passed in seconds since its launch. The object's height will be 330 meters twice before it hits the ground. Find how many seconds since the launch would the object's height be 330 meters. Round your answers to two decimal places if needed.

The object's height would be 330 meters the first time at ☐ seconds, and then the second time at ☐ seconds.

52. Currently, an artist can sell 230 paintings every year at the price of \$70.00 per painting. Each time he raises the price per painting by \$10.00, he sells 10 fewer paintings every year.

Assume he will raise the price per painting x times, then he will sell $230 - 10x$ paintings every year at the price of $70 + 10x$ dollars. His yearly income can be modeled by the equation:

$$i = (70 + 10x)(230 - 10x)$$

where i stands for his yearly income in dollars. If the artist wants to earn \$21,600.00 per year from selling paintings, what new price should he set?

To earn \$21,600.00 per year, the artist could sell his paintings at two different prices. The lower price is ☐ per painting, and the higher price is ☐ per painting.

Graphically Solving Equations and Inequalities Use technology to make some graphs and determine how many times the graphs of the following curves cross each other.

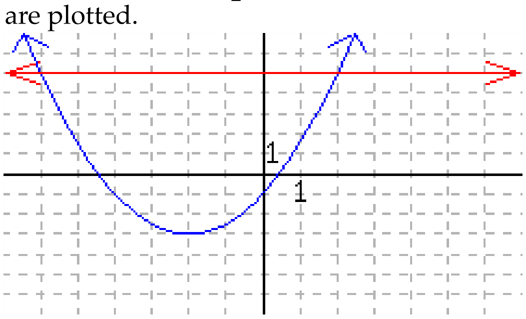
53. $y = (286 + 5x)(78 + 10x)$ and $y = 6000$ intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

54. $y = 5x^3 - x^2 - 6x$ and $y = -4x + 3$ intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

55. $y = 0.2(8x^2 + 2)$ and $y = 0.2(x - 9)$ intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

56. $y = 1.85(x - 4)^2 - 8.4$ and $y = x + 1$ intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

57. The equations $y = \frac{1}{2}x^2 + 2x - 1$ and $y = 5$ are plotted.

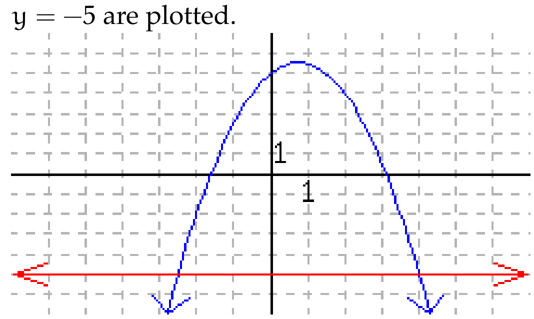


a. What are the points of intersection?

b. Solve $\frac{1}{2}x^2 + 2x - 1 = 5$.

c. Solve $\frac{1}{2}x^2 + 2x - 1 > 5$.

58. The equations $y = -x^2 + 1.5x + 5$ and $y = -5$ are plotted.

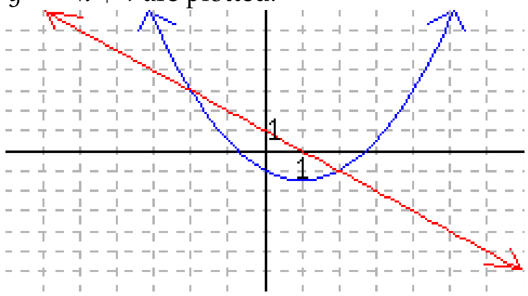


a. What are the points of intersection?

b. Solve $-x^2 + 1.5x + 5 = -5$.

c. Solve $-x^2 + 1.5x + 5 > -5$.

59. The equations $y = \frac{1}{2}x^2 - x - 1$ and $y = -x + 1$ are plotted.

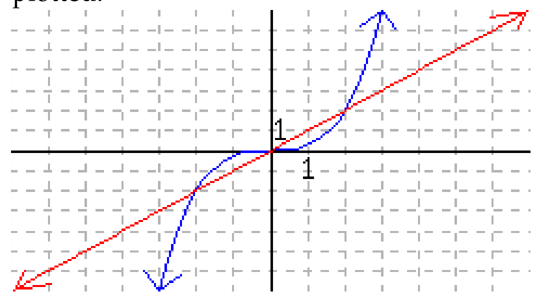


a. What are the points of intersection?

b. Solve $\frac{1}{2}x^2 - x - 1 = -x + 1$.

c. Solve $\frac{1}{2}x^2 - x - 1 > -x + 1$.

60. The equations $y = \frac{1}{4}x^3$ and $y = x$ are plotted.

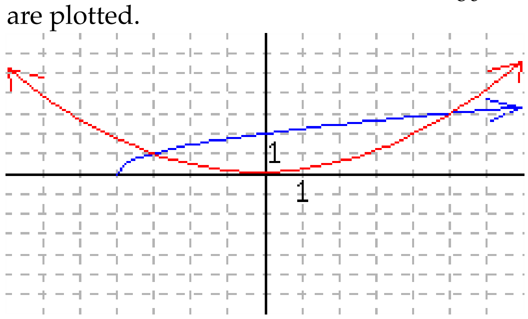


a. What are the points of intersection?

b. Solve $\frac{1}{4}x^3 = x$.

c. Solve $\frac{1}{4}x^3 > x$.

61. The equations $y = \sqrt{x+4}$ and $y = \frac{4x^2+x+3}{36}$ are plotted.

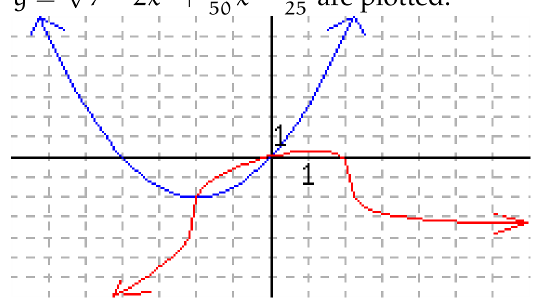


a. What are the points of intersection?

b. Solve $\sqrt{x+4} = \frac{4x^2+x+3}{36}$.

c. Solve $\sqrt{x+4} > \frac{4x^2+x+3}{36}$.

62. The equations $y = \frac{1}{2}x^2 + 2x$ and $y = \sqrt[3]{9 - 2x^2} + \frac{23}{50}x - \frac{52}{25}$ are plotted.



a. What are the points of intersection?

b. Solve
$\frac{1}{2}x^2 + 2x = \sqrt[3]{9 - 2x^2} + \frac{23}{50}x - \frac{52}{25}$.

c. Solve
$\frac{1}{2}x^2 + 2x > \sqrt[3]{9 - 2x^2} + \frac{23}{50}x - \frac{52}{25}$.

Part III

Preparation for College Algebra

Chapter 10

Factoring

10.1 Factoring Out the Common Factor

In Chapter 5, we learned how to multiply polynomials, such as when you start with $(x+2)(x+3)$ and obtain $x^2 + 5x + 6$. This chapter, starting with this section, is about the *opposite* process—factoring. For example, starting with $x^2 + 5x + 6$ and obtaining $(x + 2)(x + 3)$. We will start with the simplest kind of factoring: for example starting with $x^2 + 2x$ and obtaining $x(x + 2)$.

10.1.1 Motivation for Factoring

When you write $x^2 + 2x$, you have an algebraic expression built with two terms—two parts that are *added* together. When you write $x(x + 2)$, you have an algebraic expression built with two factors—two parts that are *multiplied* together. Factoring is useful, because sometimes (but not always) having your expression written as parts that are *multiplied* together makes it easy to simplify the expression.

You've seen this with fractions. To simplify $\frac{15}{35}$, breaking down the numerator and denominator into factors is useful: $\frac{3\cdot5}{7\cdot5}$. Now you can see that the factors of 5 cancel.

There are other reasons to appreciate the value in factoring. One reason is that there is a relationship between a factored polynomial and the horizontal intercepts of its graph. For example in the graph of $y = (x+2)(x-3)$, the horizontal intercepts are $(-2, 0)$ and $(3, 0)$. Note the x-values are -2 and 3, and think about what happens when you substitue those numbers in for x in $y = (x+2)(x-3)$. We will explore this more fully in Section 13.2.

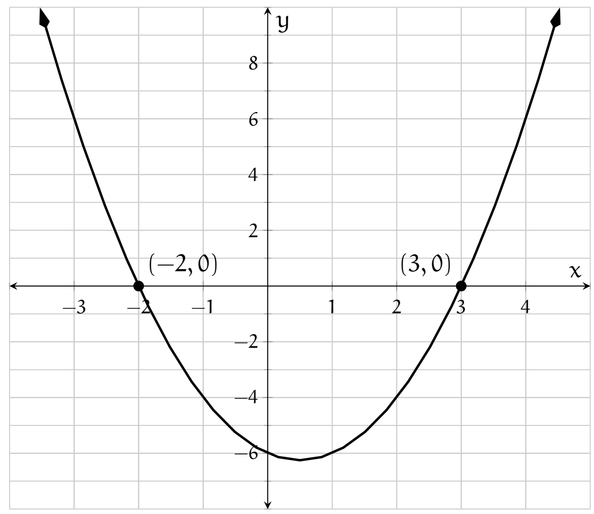


Figure 10.1.2: A graph of $y = (x+2)(x-3)$

10.1.2 Identifying the Greatest Common Factor

The most basic technique for factoring involves recognizing the **greatest common factor** between two expressions, which is the largest factor that goes in evenly to both expressions. For example, the greatest common factor between 6 and 8 is 2, since 2 divides nicely into both 6 and 8 and no larger number would divide nicely into both 6 and 8.

Similarly, the greatest common factor between $4x$ and $3x^2$ is x. If you write $4x$ as a product of its factors, you have $2 \cdot 2 \cdot x$. And if you fully factor $3x^2$, you have $3 \cdot x \cdot x$. The only factor they have in common is x, so that is the greatest common factor. No larger expression goes in nicely to both expressions.

Example 10.1.3 Finding the Greatest Common Factor. What is the common factor between $6x^2$ and $70x$? Break down each of these into its factors:

$$6x^2 = 2 \cdot 3 \cdot x \cdot x \qquad\qquad 70x = 2 \cdot 5 \cdot 7 \cdot x$$

And identify the common factors:

$$6x^2 = \overset{\downarrow}{2} \cdot 3 \cdot \overset{\downarrow}{x} \cdot x \qquad\qquad 70x = \overset{\downarrow}{2} \cdot 5 \cdot 7 \cdot \overset{\downarrow}{x}$$

With 2 and x in common, the greatest common factor is $2x$.

Checkpoint 10.1.4

a. The greatest common factor between $6x$ and $8x$ is ________.

b. The greatest common factor between $14x^2$ and $10x$ is ________.

c. The greatest common factor between $6y^2$ and $7y^2$ is ________.

d. The greatest common factor between $12xy^2$ and $9xy$ is ________.

e. The greatest common factor between $6x^3$, $2x^2$, and $8x$ is $\boxed{}$.

Explanation.

a. Since $6x$ completely factors as $\overset{\downarrow}{2} \cdot 3 \cdot \overset{\downarrow}{x}$ ... and $8x$ completely factors as $\overset{\downarrow}{2} \cdot 2 \cdot 2 \cdot \overset{\downarrow}{x}$, ...
... the greatest common factor is $2x$.

b. Since $14x^2$ completely factors as $\overset{\downarrow}{2} \cdot 7 \cdot \overset{\downarrow}{x} \cdot x$ ... and $10x$ completely factors as $\overset{\downarrow}{2} \cdot 5 \cdot \overset{\downarrow}{x}$, ...
... the greatest common factor is $2x$.

c. Since $6y^2$ completely factors as $2 \cdot 3 \cdot \overset{\downarrow}{y} \cdot \overset{\downarrow}{y}$ ... and $7y^2$ completely factors as $7 \cdot \overset{\downarrow}{y} \cdot \overset{\downarrow}{y}$, ...
... the greatest common factor is y^2.

d. Since $12xy^2$ completely factors as $2 \cdot 2 \cdot \overset{\downarrow}{3} \cdot \overset{\downarrow}{x} \cdot \overset{\downarrow}{y} \cdot y$ ... and $9xy$ completely factors as $\overset{\downarrow}{3} \cdot 3 \cdot \overset{\downarrow}{x} \cdot \overset{\downarrow}{y}$, ...
... the greatest common factor is $3xy$.

e. Since $6x^3$ completely factors as $\overset{\downarrow}{2} \cdot 3 \cdot \overset{\downarrow}{x} \cdot x \cdot x$ ... $2x^2$ completely factors as $\overset{\downarrow}{2} \cdot \overset{\downarrow}{x} \cdot x$, ...and $8x$ completely factors as $\overset{\downarrow}{2} \cdot 2 \cdot 2 \cdot \overset{\downarrow}{x}$, ...
... the greatest common factor is $2x$.

10.1.3 Factoring Out the Greatest Common Factor

We have learned the distributive property: $a(b+c) = ab + ac$. Perhaps you have thought of this as a way to "distribute" the number a to each of b and c. In this section, we will use the distributive property in the opposite way. If you have an expression $ab + ac$, it is equal to $a(b+c)$. In that example, we factored out a, which is the common factor between ab and ac.

The following steps use the distributive property to factor out the greatest common factor between two or more terms.

Process 10.1.5 Factoring Out the Greatest Common Factor.

1. *Identify the greatest common factor in all terms.*
2. *Write the greatest common factor outside a pair of parentheses with the appropriate addition or subtraction signs inside.*
3. *For each term from the original expression, what would you multiply the greatest common factor by to result in that term? Write your answer in the parentheses.*

Example 10.1.6 To factor $12x^2 + 15x$:

1. The greatest common factor between $12x^2$ and $15x$ is $3x$.
2. $3x(\quad + \quad)$
3. $3x(4x + 5)$

Example 10.1.7 Factor the polynomial $3x^3 + 3x^2 - 9$.

1. We identify the greatest common factor as 3, because 3 is the only common factor between $3x^3$, $3x^2$ and 9.

2. We write:
$$3x^3 + 3x^2 - 9 = 3(\quad + \quad - \quad).$$

3. We ask the question "3 times what gives $3x^3$?" The answer is x^3. Now we have:
$$3x^3 + 3x^2 - 9 = 3(x^3 + \quad - \quad).$$
We ask the question "3 times what gives $3x^2$?" The answer is x^2. Now we have:
$$3x^3 + 3x^2 - 9 = 3(x^3 + x^2 - \quad).$$
We ask the question "3 times what gives 9?" The answer is 3. Now we have:
$$3x^3 + 3x^2 - 9 = 3(x^3 + x^2 - 3).$$

To check that this is correct, multiplying through $3(x^3 + x^2 - 3)$ should give the original expression $3x^3 + 3x^2 - 9$. We check this, and it does.

Checkpoint 10.1.8 Factor the polynomial $4x^3 + 12x^2 - 12x$.

Explanation. In this exercise, $4x$ is the greatest common factor. We find

$$\begin{aligned}4x^3 + 12x^2 - 12x &= 4x(\quad + \quad - \quad)\\ &= 4x(x^2 + \quad - \quad)\\ &= 4x(x^2 + 3x - \quad)\\ &= 4x(x^2 + 3x - 3)\end{aligned}$$

Note that you might fail to recognize that $4x$ is the greatest common factor. At first you might only find that, say, 4 is a common factor. This is OK—you can factor out the 4 and continue from there:

$$\begin{aligned}4x^3 + 12x^2 - 12x &= 4(\quad + \quad - \quad)\\ &= 4(x^3 + \quad - \quad)\\ &= 4(x^3 + 3x^2 - \quad)\\ &= 4(x^3 + 3x^2 - 3x)\end{aligned}$$

Now examine the second factor here and there is *still* a common factor, x. Factoring this out too.

$$\begin{aligned}&= 4x(\quad + \quad - \quad)\\ &= 4x(x^2 + \quad - \quad)\\ &= 4x(x^2 + 3x - \quad)\\ &= 4x(x^2 + 3x - 3)\end{aligned}$$

So there is more than one way to find the answer.

10.1.4 Visualizing With Rectangles

In Section 5.4, we learned one way to multiply polynomials using rectangle diagrams. Similarly, we can factor a polynomial with a rectangle diagram.

Process 10.1.9 Factoring Out the Greatest Common Factor Using Rectangles.

1. *Put the terms into adjacent rectangles. Think of these as labeling the areas of each rectangle.*
2. *Identify the greatest common factor, and mark the height of the overall rectangle with it.*
3. *Mark the width of each rectangle based on each rectangle's area and height.*
4. *Since the overall rectangle's area equals its width times its height, the height is one factor, and the sum of the widths is another factor.*

Example 10.1.10 We will factor $12x^2 + 15x$, the same polynomial from the example in Algorithm 10.1.5, so that you may compare the two styles.

$$\begin{array}{|c|c|}\hline 12x^2 & 15x \\ \hline \end{array} \qquad 3x\ \begin{array}{|c|c|}\hline 12x^2 & 15x \\ \hline \end{array} \qquad \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{4x} & \multicolumn{1}{c}{5} \\ \cline{2-3} 3x & 12x^2 & 15x \\ \cline{2-3} \end{array}$$

So $12x^2 + 15x$ factors as $3x(4x + 5)$.

10.1.5 More Examples of Factoring out the Common Factor

Previous examples did not cover every nuance with factoring out the greatest common factor. Here are a few more factoring examples that attempt to do so.

Example 10.1.11 Factor $-35m^5 + 5m^4 - 10m^3$.

First, we identify the common factor. The number 5 is the greatest common factor of the three coefficients (which were -35, 5, and -10) and also m^3 is the largest expression that divides m^5, m^4, and m^3. Therefore the greatest common factor is $5m^3$.

In this example, the leading term is a negative number. When this happens, we will make it common practice to take that negative as part of the greatest common factor. So we will proceed by factoring out $-5m^3$. Note the signs change inside the parentheses.

$$\begin{aligned} -35m^5 + 5m^4 - 10m^3 &= -5m^3(\qquad - \qquad + \quad) \\ &= -5m^3(7m^2 - \qquad + \quad) \\ &= -5m^3(7m^2 - m + \quad) \\ &= -5m^3(7m^2 - m + 2) \end{aligned}$$

Example 10.1.12 Factor $14 - 7n^2 + 28n^4 - 21n$.

Notice that the terms are not in a standard order, with powers of n decreasing as you read left to right. It is usually a best practice to rearrange the terms into the standard order first.

$$14 - 7n^2 + 28n^4 - 21n = 28n^4 - 7n^2 - 21n + 14.$$

The number 7 divides all of the numerical coefficients. Separately, no power of n is part of the greatest common factor because the 14 term has no n factors. So the greatest common factor is just 7. We proceed by factoring that out:

$$\begin{aligned} 14 - 7n^2 + 28n^4 - 21n &= 28n^4 - 7n^2 - 21n + 14 \\ &= 7(4n^4 - n^2 - 3n + 2) \end{aligned}$$

Example 10.1.13 Factor $24ab^2 + 16a^2b^3 - 12a^3b^2$.

There are two variables in this polynomial, but that does not change the factoring strategy. The greatest numerical factor between the three terms is 4. The variable a divides all three terms, and b^2 divides all three terms. So we have:

$$24ab^2 + 16a^2b^3 - 12a^3b^2 = 4ab^2(6 + 4ab - 3a^2)$$

Example 10.1.14 Factor $4m^2n - 3xy$.

There are no common factors in those two terms (unless you want to count 1 or -1, but we do not count these for the purposes of identifying a greatest common factor). In this situation we can say the polynomial is **prime** or **irreducible**, and leave it as it is.

Example 10.1.15 Factor $-x^3 + 2x + 18$.

There are no common factors in those three terms, and it would be correct to state that this polynomial is prime or irreducible. However, since its leading coefficient is negative, it may be wise to factor out a negative sign. So, it could be factored as $-(x^3 - 2x - 18)$. Note that *every* term is negated as the leading negative sign is extracted.

10.1.6 Reading Questions

1. Given two terms, how would you describe their "greatest common factor?"
2. If a simplified polynomial has four terms, and you factor out its greatest common factor, how many terms will remain inside a set of parentheses?

10.1.7 Exercises

Review and Warmup Multiply the polynomials.

1. $-4x(x-2)$
2. $-x(x-7)$
3. $-6x(9x-9)$
4. $-7x(4x+9)$
5. $6x^2(x+8)$
6. $8x^2(x-4)$
7. $10t^2(8t^2-5t)$
8. $7x^2(5x^2-9x)$

Identifying Common Factors Find the greatest common factor of the following terms.

9. 4 and $20x$
10. 10 and $90y$
11. $7y$ and $28y^2$
12. $4r$ and $28r^2$
13. $10r^3$ and $-100r^4$
14. $6t^3$ and $-42t^4$
15. $3t^{19}$ and $-18t^{15}$
16. $9t^{12}$ and $-81t^{11}$
17. $6x^{17}, -12x^{14}, 30x^2$
18. $3x^{11}, -15x^{10}, 27x^3$
19. $5x^{16}y^7, -40x^{10}y^{12}, 10x^4y^{13}$
20. $2x^{16}y^{10}, -6x^{11}y^{11}, 10x^7y^{14}$

Factoring out the Common Factor Factor the given polynomial.

21. $3r+3$
22. $8r+8$

23. $5t - 5$

24. $2t - 2$

25. $-8t - 8$

26. $-5x - 5$

27. $2x - 18$

28. $8y + 32$

29. $12y^2 + 32$

30. $90r^2 - 20$

31. $18r^2 + 9r + 72$

32. $60t^2 + 70t + 60$

33. $32t^4 - 12t^3 + 24t^2$

34. $10t^4 + 12t^3 + 4t^2$

35. $20x^5 - 35x^4 + 45x^3$

36. $50x^5 + 10x^4 + 15x^3$

37. $28y - 20y^2 + 20y^3$

38. $72y + 48y^2 + 40y^3$

39. $5r^2 + 11$

40. $16r^2 + 9$

41. $8xy + 8y$

42. $9xy + 9y$

43. $10x^{11}y^5 + 60y^5$

44. $2x^7y^5 + 6y^5$

45. $6x^5y^9 - 18x^4y^9 + 21x^3y^9$

46. $63x^5y^{10} - 35x^4y^{10} + 35x^3y^{10}$

47. $40x^5y^6z^5 - 10x^4y^6z^4 + 25x^3y^6z^3$

48. $24x^5y^4z^9 + 20x^4y^4z^8 + 8x^3y^4z^7$

10.2 Factoring by Grouping

This section covers a technique for factoring polynomials like x^3+3x^2+2x+6, which factors as $\left(x^2+2\right)(x+3)$. If there are *four* terms, the technique in this section *might* help you to factor the polynomial. Additionally, this technique is a stepping stone to a factoring technique in Section 10.3 and Section 10.4.

10.2.1 Factoring out Common Polynomials

Recall that to factor $3x+6$, we factor out the common factor 3:

$$\begin{aligned} 3x+6 &= \overset{\downarrow}{3}x+\overset{\downarrow}{3}\cdot 2 \\ &= 3(x+2) \end{aligned}$$

The "3" here could have been something more abstract, and it still would be valid to factor it out:

$$\begin{aligned} xA+2A &= x\overset{\downarrow}{A}+2\overset{\downarrow}{A} \\ &= A(x+2) \end{aligned} \qquad\qquad \begin{aligned} x\text{🍎}+2\text{🍎} &= x\overset{\downarrow}{\text{🍎}}+2\overset{\downarrow}{\text{🍎}} \\ &= \text{🍎}(x+2) \end{aligned}$$

In fact, even "larger" things can be factored out, as in this example:

$$\begin{aligned} x(a+b)+2(a+b) &= x\overbrace{(a+b)}^{\downarrow}+2\overbrace{(a+b)}^{\downarrow} \\ &= (a+b)(x+2) \end{aligned}$$

In this last example, we factored out the binomial factor $(a+b)$. Factoring out binomials is the essence of this section, so let's see that a few more times:

$$\begin{aligned} x(x+2)+3(x+2) &= x\overbrace{(x+2)}^{\downarrow}+3\overbrace{(x+2)}^{\downarrow} \\ &= (x+2)(x+3) \end{aligned}$$

$$\begin{aligned} z^2(2y+5)+3(2y+5) &= z^2\overbrace{(2y+5)}^{\downarrow}+3\overbrace{(2y+5)}^{\downarrow} \\ &= (2y+5)(z^2+3) \end{aligned}$$

And even with an expression like $Q^2(Q-3)+Q-3$, if we re-write it in the right way using a 1 and some parentheses, then it too can be factored:

$$\begin{aligned} Q^2(Q-3)+Q-3 &= Q^2(Q-3)+1(Q-3) \\ &= Q^2\overbrace{(Q-3)}^{\downarrow}+1\overbrace{(Q-3)}^{\downarrow} \\ &= (Q-3)(Q^2+1) \end{aligned}$$

The truth is you are unlikely to come upon an expression like $x(x+2)+3(x+2)$, as in these examples. Why wouldn't someone have multiplied that out already? Or factored it all the way? So far in this section, we have only been looking at a stepping stone to a real factoring technique called **factoring by grouping**.

10.2.2 Factoring by Grouping

Factoring by grouping is a factoring technique that *sometimes* works on polynomials with four terms. Here is an example.

Example 10.2.2 Suppose we must factor $x^3 - 3x^2 + 5x - 15$. Note that there are four terms, and they are written in descending order of the powers of x. "Grouping" means to group the first two terms and the last two terms together:

$$x^3 - 3x^2 + 5x - 15 = \left(x^3 - 3x^2\right) + (5x - 15)$$

Now, each of these two groups has its own greatest common factor we can factor out:

$$= x^2(x-3) + 5(x-3)$$

In a sense, we are "lucky" because we now see matching binomials that can themselves be factored out:

$$\begin{aligned} &= x^2\overbrace{(x-3)}^{\downarrow} + 5\overbrace{(x-3)}^{\downarrow} \\ &= (x-3)\left(x^2+5\right) \end{aligned}$$

And so we have factored $x^3 - 3x^2 + 5x - 15$ as $(x-3)\left(x^2+5\right)$. But to be sure, if we multiply this back out, it should recover the original $x^3 - 3x^2 + 5x - 15$. To confirm your factoring is correct, you should always multiply out your factored result to check that it matches the original polynomial.

Checkpoint 10.2.3 Factor $x^3 + 4x^2 + 2x + 8$.

Explanation. We will break the polynomial into two groups: $x^3 + 4x^2$ and $2x + 8$.

$$\begin{aligned} x^3 + 4x^2 + 2x + 8 &= \left(x^3 + 4x^2\right) + (2x + 8) \\ &= x^2(x+4) + 2(x+4) \\ &= (x+4)\left(x^2+2\right) \end{aligned}$$

Example 10.2.4 Factor $t^3 - 5t^2 - 3t + 15$. This example has a complication with negative signs. If we try to break up this polynomial into two groups as $\left(t^3 - 5t^2\right) - (3t + 15)$, then we've made an error! In that last expression, we are *subtracting* a group with the term 15, so overall it subtracts 15. The original polynomial *added* 15, so we are off course.

One way to handle this is to treat subtraction as addition of a negative:

$$\begin{aligned} t^3 - 5t^2 - 3t + 15 &= t^3 - 5t^2 + (-3t) + 15 \\ &= \left(t^3 - 5t^2\right) + (-3t + 15) \end{aligned}$$

Now we can proceed to factor out common factors from each group. Since the second group leads with a negative coefficient, we'll factor out -3. This will result in the " $+15$" becoming " -5."

$$\begin{aligned} &= t^2(t-5) + (-3)(t-5) \\ &= t^2\overbrace{(t-5)}^{\downarrow} - 3\overbrace{(t-5)}^{\downarrow} \end{aligned}$$

$$= (t-5)(t^2-3)$$

And remember that we can confirm this is correct by multiplying it out. If we made no mistakes, it should result in the original $t^3 - 5t^2 - 3t + 15$.

Checkpoint 10.2.5 Factor $6q^3 - 9q^2 - 4q + 6$.

Explanation. We will break the polynomial into two groups: $6q^3 - 9q^2$ and $-4q + 6$.

$$\begin{aligned} 6q^3 - 9q^2 - 4q + 6 &= \left(6q^3 - 9q^2\right) + (-4q + 6) \\ &= 3q^2(2q-3) - 2(2q-3) \\ &= (2q-3)\left(3q^2-2\right) \end{aligned}$$

Example 10.2.6 Factor $x^3 - 3x^2 + x - 3$. To succeed with this example, we will need to "factor out" a trivial number 1 that isn't apparent until we make it so.

$$\begin{aligned} x^3 - 3x^2 + x - 3 &= \left(x^3 - 3x^2\right) + (x-3) \\ &= x^2(x-3) + 1(x-3) \\ &= x^2\overbrace{(x-3)}^{\downarrow} + 1\overbrace{(x-3)}^{\downarrow} \\ &= (x-3)\left(x^2+1\right) \end{aligned}$$

Notice how we changed $x - 3$ to $+1(x-3)$, so we wouldn't forget the $+1$ in the final factored form. As always, we should check this is correct by multiplying it out.

Checkpoint 10.2.7 Factor $6t^6 + 9t^4 + 2t^2 + 3$.

Explanation. We will break the polynomial into two groups: $6t^6 + 9t^4$ and $2t^2 + 3$.

$$\begin{aligned} 6t^6 + 9t^4 + 2t^2 + 3 &= \left(6t^6 + 9t^4\right) + \left(2t^2 + 3\right) \\ &= 3t^4\left(2t^2+3\right) + 1\left(2t^2+3\right) \\ &= \left(2t^2+3\right)\left(3t^4+1\right) \end{aligned}$$

Example 10.2.8 Factor $xy^2 - 10y^2 - 2x + 20$. The technique can work when there are multiple variables too.

$$\begin{aligned} xy^2 - 10y^2 - 2x + 20 &= \left(xy^2 - 10y^2\right) + (-2x + 20) \\ &= y^2(x-10) + (-2)(x-10) \\ &= y^2\overbrace{(x-10)}^{\downarrow} - 2\overbrace{(x-10)}^{\downarrow} \\ &= (x-10)\left(y^2-2\right). \end{aligned}$$

Unfortunately, this technique is not guaranteed to work on every polynomial with four terms. In fact, *most* randomly selected four-term polynomials will not factor using this method and those selected here should be considered "nice." Here is an example that will not factor with grouping:

$$\begin{aligned} x^3 + 6x^2 + 11x + 6 &= \left(x^3 + 6x^2\right) + (11x + 6) \\ &= x^2\underbrace{(x+6)}_{?} + 1\underbrace{(11x+6)}_{?} \end{aligned}$$

In this example, at the step where we hope to see the same binomial appearing twice, we see two different binomials. It doesn't mean that this kind of polynomial can't be factored, but it does mean that "factoring by grouping" is not going to help. This polynomial actually factors as $(x+1)(x+2)(x+3)$. So the fact that grouping fails to factor the polynomial doesn't tell us whether or not it is prime.

10.2.3 Reading Questions

1. Factoring by grouping is a factoring technique for when a polynomial has $\boxed{}$ terms.

10.2.4 Exercises

Review and Warmup Factor the given polynomial.

1. $-5t-5$
2. $-10x-10$
3. $7x+14$
4. $4y-24$
5. $30y^2-100$
6. $21r^2+35$

Factoring out Common Polynomials Factor the given polynomial.

7. $r(r+4)-6(r+4)$
8. $t(t-10)-4(t-10)$
9. $x(y+9)+7(y+9)$
10. $x(y-10)+2(y-10)$
11. $2x(x+y)-9(x+y)$
12. $3x(x+y)+7(x+y)$
13. $3y^4(4y+9)+4y+9$
14. $9y^3(8y-7)+8y-7$
15. $12r^4(r+10)+6r^3(r+10)+60r^2(r+10)$
16. $10r^4(r-15)-30r^3(r-15)+35r^2(r-15)$

Factoring by Grouping Factor the given polynomial.

17. $t^2+8t+9t+72$
18. $t^2-5t+6t-30$
19. $t^2+2t+4t+8$
20. $x^2-8x+10x-80$
21. $x^3+5x^2+7x+35$
22. $y^3-2y^2+5y-10$
23. $y^3-7y^2+2y-14$
24. $r^3+5r^2+8r+40$
25. $xy+7x+4y+28$
26. $xy+8x-9y-72$
27. $xy-9x-3y+27$
28. $xy+10x+6y+60$
29. $2x^2+4xy+9xy+18y^2$
30. $3x^2+15xy+8xy+40y^2$
31. $4x^2+36xy+9xy+81y^2$
32. $5x^2-5xy+7xy-7y^2$
33. x^3-6-7x^3y+42y
34. x^3+7-2x^3y-14y
35. x^3+6+8x^3y+48y
36. x^3-9-8x^3y+72y
37. $10t^5+20t^4-15t^4-30t^3+25t^3+50t^2$
38. $24x^5+48x^4+16x^4+32x^3+40x^3+80x^2$

10.3 Factoring Trinomials with Leading Coefficient One

In Chapter 5, we learned how to multiply binomials like $(x+2)(x+3)$ and obtain the trinomial x^2+5x+6. In this section, we will learn how to undo that. So we'll be starting with a trinomial like x^2+5x+6 and obtaining its factored form $(x+2)(x+3)$. The trinomials that we'll factor in this section all have leading coefficient 1, but Section 10.4 will cover some more general trinomials.

10.3.1 Factoring Trinomials by Listing Factor Pairs

Consider the example $x^2+5x+6=(x+2)(x+3)$. There are at least three things that are important to notice:

- The leading coefficient of x^2+5x+6 is 1.
- The two factors on the right use the numbers 2 and 3, and when you *multiply* these you get the 6.
- The two factors on the right use the numbers 2 and 3, and when you *add* these you get the 5.

So the idea is that if you need to factor x^2+5x+6 and you somehow discover that 2 and 3 are special numbers (because $2\cdot 3=6$ and $2+3=5$), then you can conclude that $(x+2)(x+3)$ is the factored form of the given polynomial.

Example 10.3.2 Factor $x^2+13x+40$. Since the leading coefficient is 1, we are looking to write this polynomial as $(x+?)(x+?)$ where the question marks are two possibly different, possibly negative, numbers. We need these two numbers to multiply to 40 and add to 13. How can you track these two numbers down? Since the numbers need to multiply to 40, one method is to list all **factor pairs** of 40 in a table just to see what your options are. We'll write every *pair of factors* that multiply to 40.

$1\cdot 40$	$-1\cdot(-40)$
$2\cdot 20$	$-2\cdot(-20)$
$4\cdot 10$	$-4\cdot(-10)$
$5\cdot 8$	$-5\cdot(-8)$

We wanted to find *all* factor pairs. To avoid missing any, we started using 1 as a factor, and then slowly increased that first factor. The table skips over using 3 as a factor, because 3 is not a factor of 40. Similarly the table skips using 6 and 7 as a factor. And there would be no need to continue with 8 and beyond, because we already found "large" factors like 8 as the partners of "small" factors like 5.

There is an entire second column where the signs are reversed, since these are also ways to multiply two numbers to get 40. In the end, there are eight factor pairs.

We need a pair of numbers that also *adds* to 13. So we check what each of our factor pairs add up to:

Factor Pair	Sum of the Pair
$1\cdot 40$	41
$2\cdot 20$	22
$4\cdot 10$	14
$5\cdot 8$	13 (what we wanted)

Factor Pair	Sum of the Pair
$-1\cdot(-40)$	(no need to go this far)
$-2\cdot(-20)$	(no need to go this far)
$-4\cdot(-10)$	(no need to go this far)
$-5\cdot(-8)$	(no need to go this far)

The winning pair of numbers is 5 and 8. Again, what matters is that $5\cdot 8=40$, and $5+8=13$. So we can conclude that $x^2+13x+40=(x+5)(x+8)$.

To ensure that we made no mistakes, here are some possible checks.

Multiply it Out. Multiplying out our answer $(x+5)(x+8)$ should give us $x^2+13x+40$.

$$\begin{aligned}(x+5)(x+8) &= (x+5)\cdot x + (x+5)\cdot 8\\ &= x^2+5x+8x+40\\ &\stackrel{\checkmark}{=} x^2+13x+40\end{aligned}$$

We could also use a rectangular area diagram to verify the factorization is correct:

	x	5
x	x^2	$5x$
8	$8x$	40

Evaluating. If the answer really is $(x+5)(x+8)$, then notice how evaluating at -5 would result in 0. So the original expression should also result in 0 if we evaluate at -5. And similarly, if we evaluate it at -8, $x^2+13x+40$ should be 0.

$$\begin{aligned}(-5)^2+13(-5)+40 &\stackrel{?}{=} 0\\ 25-65+40 &\stackrel{?}{=} 0\\ 0 &\stackrel{\checkmark}{=} 0\end{aligned}
\qquad
\begin{aligned}(-8)^2+13(-8)+40 &\stackrel{?}{=} 0\\ 64-104+40 &\stackrel{?}{=} 0\\ 0 &\stackrel{\checkmark}{=} 0.\end{aligned}$$

This also gives us evidence that the factoring was correct.

Example 10.3.3 Factor $y^2-11y+24$. The negative coefficient is a small complication from Example 10.3.2, but the process is actually still the same.

Explanation. We need a pair of numbers that multiply to 24 and add to -11. Note that we *do* care to keep track that they sum to a negative total.

Factor Pair	Sum of the Pair
$1\cdot 24$	25
$2\cdot 12$	14
$3\cdot 8$	11 (close; wrong sign)
$4\cdot 6$	10

Factor Pair	Sum of the Pair
$-1\cdot(-24)$	-25
$-2\cdot(-12)$	-14
$-3\cdot(-8)$	-11 (what we wanted)
$-4\cdot(-6)$	(no need to go this far)

So $y^2-11y+24=(y-3)(y-8)$. To confirm that this is correct, we should check. Either by multiplying out the factored form:

$$\begin{aligned}(y-3)(y-8) &= (y-3)\cdot y-(y-3)\cdot 8\\ &= y^2-3y-8y+24\\ &\stackrel{\checkmark}{=} y^2-11y+24\end{aligned}$$

	y	-3
y	y^2	$-3y$
-8	$-8y$	24

Or by evaluating the original expression at 3 and 8:

$$\begin{aligned}3^2-11(3)+24 &\stackrel{?}{=} 0\\ 9-33+24 &\stackrel{?}{=} 0\\ 0 &\stackrel{\checkmark}{=} 0\end{aligned}
\qquad
\begin{aligned}8^2-11(8)+24 &\stackrel{?}{=} 0\\ 64-88+24 &\stackrel{?}{=} 0\\ 0 &\stackrel{\checkmark}{=} 0.\end{aligned}$$

Our factorization passes the tests.

Example 10.3.4 Factor z^2+5z-6. The negative coefficient is again a small complication from Example 10.3.2, but the process is actually still the same.

Explanation. We need a pair of numbers that multiply to -6 and add to 5. Note that we *do* care to keep track that they multiply to a negative product.

Factor Pair	Sum of the Pair
$1 \cdot (-6)$	-5 (close; wrong sign)
$2 \cdot (-3)$	14

Factor Pair	Sum of the Pair
$-1 \cdot 6$	5 (what we wanted)
$-2 \cdot 3$	(no need to go this far)

So $z^2 + 5z - 6 = (z-1)(z+6)$. To confirm that this is correct, we should check. Either by multiplying out the factored form:

$$\begin{aligned}(z-1)(z+6) &= (z-1)\cdot z + (z-1)\cdot 6\\ &= z^2 - z + 6z - 6\\ &\stackrel{\checkmark}{=} z^2 + 5z - 6\end{aligned}$$

	z	-1
z	z^2	$-z$
6	$6z$	-6

Or by evaluating the original expression at 1 and -6:

$$\begin{aligned}1^2 + 5(1) - 6 &\stackrel{?}{=} 0\\ 1 + 5 - 6 &\stackrel{?}{=} 0\\ 0 &\stackrel{\checkmark}{=} 0\end{aligned}$$

$$\begin{aligned}(-6)^2 + 5(-6) - 6 &\stackrel{?}{=} 0\\ 36 - 30 - 6 &\stackrel{?}{=} 0\\ 0 &\stackrel{\checkmark}{=} 0.\end{aligned}$$

Our factorization passes the tests.

Checkpoint 10.3.5 Factor $m^2 - 6m - 40$.

Explanation. We need a pair of numbers that multiply to -40 and add to -6. Note that we *do* care to keep track that they multiply to a negative product and sum to a negative total.

Factor Pair	Sum of the Pair
$1 \cdot (-40)$	-39
$2 \cdot (-20)$	-18
$4 \cdot (-10)$	-6 (what we wanted)
(no need to continue)	...

So $m^2 - 6m - 40 = (m+4)(m-10)$.

10.3.2 Connection to Grouping

The factoring method we just learned is actually taking a shortcut compared to what we will learn in Section 10.4. To prepare yourself for that more complicated factoring technique, you may want to try taking the "scenic route" instead of that shortcut.

Example 10.3.6 Let's factor $x^2 + 13x + 40$ again (the polynomial from Example 10.3.2). As before, it is important to discover that 5 and 8 are important numbers, because they multiply to 40 and add to 13. As before, listing out all of the factor pairs is one way to discover the 5 and the 8.

Instead of jumping to the factored answer, we can show how $x^2 + 13x + 40$ factors in a more step-by-step fashion using 5 and 8. Since they add up to 13, we can write:

$$x^2 + \overset{\downarrow}{13}x + 40 = x^2 + \overset{\downarrow}{\overbrace{5x + 8x}} + 40$$

We have intentionally split up the trinomial into an unsimplified polynomial with four terms. In Section 10.2, we handled such four-term polynomials by grouping:

$$= \left(x^2 + 5x\right) + (8x + 40)$$

Now we can factor out each group's greatest common factor:

$$
\begin{aligned}
&= x(x+5) + 8(x+5) \\
&= x\overbrace{(x+5)}^{\downarrow} + 8\overbrace{(x+5)}^{\downarrow} \\
&= (x+5)(x+8)
\end{aligned}
$$

And we have found that $x^2 + 13x + 40$ factors as $(x+5)(x+8)$ without taking the shortcut.

This approach takes more time, and ultimately you may not use it much. However, if you try a few examples this way, it may make you more comfortable with the more complicated technique in Section 10.4.

10.3.3 Trinomials with Higher Powers

So far we have only factored examples of **quadratic** trinomials: trinomials whose highest power of the variable is 2. However, this technique can also be used to factor trinomials where there is a larger highest power of the variable. It only requires that the highest power is even, that the next highest power is half of the highest power, and that the third term is a constant term.

In the four examples below, check:

1. if the highest power is even
2. if the next highest power is half of the highest power
3. if the last term is constant

Factor pairs *will* help with…

- $y^6 - 23y^3 - 50$
- $h^{16} + 22h^8 + 105$

Factor pairs *won't* help with…

- $y^5 - 23y^3 - 50$
- $h^{16} + 22h^8 + 105h^2$

Example 10.3.7 Factor $h^{16} + 22h^8 + 105$. This polynomial is one of the examples above where using factor pairs will help. We find that $7 \cdot 15 = 105$, and $7 + 15 = 22$, so the numbers 7 and 15 can be used:

$$
\begin{aligned}
h^{16} + 22h^8 + 105 &= h^{16} + \overbrace{7h^8 + 15h^8} + 105 \\
&= \left(h^{16} + 7h^8\right) + \left(15h^8 + 105\right) \\
&= h^8\left(h^8 + 7\right) + 15\left(h^8 + 7\right) \\
&= \left(h^8 + 7\right)\left(h^8 + 15\right)
\end{aligned}
$$

Actually, once we settled on using 7 and 15, we could have concluded that $h^{16} + 22h^8 + 105$ factors as $\left(h^8 + 7\right)\left(h^8 + 15\right)$, if we know which power of h to use. We'll always use half the highest power in these factorizations.

In any case, to confirm that this is correct, we should check by multiplying out the factored form:

$$
\begin{aligned}
(h^8+7)(h^8+15) &= (h^8+7)\cdot h^8 + (h^8+7)\cdot 15 \\
&= h^{16} + 7h^8 + 15h^8 + 105 \\
&\stackrel{\checkmark}{=} h^{16} + 22h^8 + 15
\end{aligned}
$$

	h^8	7
h^8	h^{16}	$7h^8$
15	$15h^8$	105

Our factorization passes the tests.

Checkpoint 10.3.8 Factor $y^6 - 23y^3 - 50$.

Explanation. We need a pair of numbers that multiply to -50 and add to -23. Note that we *do* care to keep track that they multiply to a negative product and sum to a negative total.

Factor Pair	Sum of the Pair
$1 \cdot (-50)$	-49
$2 \cdot (-25)$	-23 (what we wanted)
(no need to continue)	...

So $y^6 - 23y^3 - 50 = \left(y^3 - 25\right)\left(y^3 + 2\right)$.

10.3.4 Factoring in Stages

Sometimes factoring a polynomial will take two or more "stages." Always begin factoring a polynomial by factoring out its greatest common factor, and *then* apply a second stage where you use a technique from this section. The process of factoring a polynomial is not complete until each of the factors cannot be factored further.

Example 10.3.9 Factor $2z^2 - 6z - 80$.

Explanation. We will first factor out the common factor, 2:

$$2z^2 - 6z - 80 = 2\left(z^2 - 3z - 40\right)$$

Now we are left with a factored expression that might factor more. Looking inside the parentheses, we ask ourselves, "what two numbers multiply to be -40 and add to be -3?" Since 5 and -8 do the job the full factorization is:

$$\begin{aligned} 2z^2 - 6z - 80 &= 2\left(z^2 - 3z - 40\right) \\ &= 2(z+5)(z-8) \end{aligned}$$

Example 10.3.10 Factor $-r^2 + 2r + 24$.

Explanation. The three terms don't exactly have a common factor, but as discussed in Section 10.1, when the leading term has a negative sign, it is often helpful to factor out that negative sign:

$$-r^2 + 2r + 24 = -\left(r^2 - 2r - 24\right).$$

Looking inside the parentheses, we ask ourselves, "what two numbers multiply to be -24 and add to be -2?" Since -6 and 4 work here and the full factorization is shown:

$$\begin{aligned} -r^2 + 2r + 24 &= -\left(r^2 - 2r - 24\right) \\ &= -(r-6)(r+4) \end{aligned}$$

Example 10.3.11 Factor $p^2q^3 + 4p^2q^2 - 60p^2q$.

Explanation. First, always look for the greatest common factor: in this trinomial it is p^2q. After factoring this out, we have

$$p^2q^3 + 4p^2q^2 - 60p^2q = p^2q\left(q^2 + 4q - 60\right).$$

Looking inside the parentheses, we ask ourselves, "what two numbers multiply to be -60 and add to be 4?"

Since 10 and -6 fit the bill, the full factorization can be shown below:

$$\begin{aligned} p^2q^3 + 4p^2q^2 - 60p^2q &= p^2q\left(q^2 + 4q - 60\right) \\ &= p^2q(q+10)(q-6) \end{aligned}$$

10.3.5 More Trinomials with Two Variables

You might encounter a trinomial with two variables that can be factored using the methods we've discussed in this section. It can be tricky though: $x^2 + 5xy + 6y^2$ has two variables and it *can* factor using the methods from this section, but $x^2 + 5x + 6y^2$ also has two variables and it *cannot* be factored. So in examples of this nature, it is even more important to check that factorizations you find actually work.

Example 10.3.12 Factor $x^2 + 5xy + 6y^2$. This is a trinomial, and the coefficient of x is 1, so maybe we can factor it. We want to write $(x+?)(x+?)$ where the question marks will be *something* that makes it all multiply out to $x^2 + 5xy + 6y^2$.

Since the last term in the polynomial has a factor of y^2, it is natural to wonder if there is a factor of y in each of the two question marks. If there were, these two factors of y would multiply to y^2. So it is natural to wonder if we are looking for $(x + ?y)(x + ?y)$ where now the question marks are just numbers.

At this point we can think like we have throughout this section. Are there some numbers that multiply to 6 and add to 5? Yes, specifically 2 and 3. So we suspect that $(x + 2y)(x + 3y)$ might be the factorization.

To confirm that this is correct, we should check by multiplying out the factored form:

$$\begin{aligned} (x+2y)(x+3y) &= (x+2y)\cdot x + (x+2y)\cdot 3y \\ &= x^2 + 2xy + 3xy + 6y^2 \\ &\stackrel{\checkmark}{=} x^2 + 5xy + 6y^2 \end{aligned}$$

	x	$2y$
x	x^2	$2xy$
$3y$	$3xy$	$6y^2$

Our factorization passes the tests.

In Section 10.4, there is a more definitive method for factoring polynomials of this form.

10.3.6 Reading Questions

1. To factor $x^2 + bx + c$, you look for two numbers that do what?

2. How many factor pairs are there for the number 6?

10.3.7 Exercises

Review and Warmup Multiply the polynomials.

1. $(y+5)(y+10)$
2. $(r+1)(r+4)$
3. $(r+8)(r-3)$
4. $(r+4)(r-8)$
5. $(t-10)(t-4)$
6. $(t-4)(t-10)$
7. $3(x+2)(x+3)$
8. $-4(y-1)(y-9)$
9. $2(y-10)(y-3)$
10. $-2(r+7)(r+6)$

Factoring Trinomials with Leading Coefficient One Factor the given polynomial.

11. $r^2 + 12r + 20$
12. $r^2 + 13r + 42$
13. $r^2 + 13r + 30$
14. $t^2 + 14t + 40$
15. $t^2 + 3t - 18$
16. $x^2 + 7x - 30$

17. $x^2 + 2x - 63$

18. $y^2 - y - 30$

19. $y^2 - 7y + 10$

20. $r^2 - 14r + 45$

21. $r^2 - 8r + 15$

22. $r^2 - 10r + 16$

23. $t^2 + 10t + 16$

24. $t^2 + 11t + 30$

25. $x^2 + 11x + 10$

26. $x^2 + 12x + 32$

27. $y^2 - 6y - 40$

28. $y^2 + 6y - 7$

29. $r^2 + 3r - 28$

30. $r^2 - 3r - 4$

31. $r^2 - 6r + 5$

32. $t^2 - 16t + 63$

33. $t^2 - 7t + 12$

34. $x^2 - 18x + 80$

35. $x^2 + x + 6$

36. $y^2 + 3y + 10$

37. $y^2 + 9$

38. $r^2 - 4r + 6$

39. $r^2 + 6r + 9$

40. $r^2 + 22r + 121$

41. $t^2 + 14t + 49$

42. $t^2 + 6t + 9$

43. $x^2 - 20x + 100$

44. $x^2 - 12x + 36$

45. $y^2 - 4y + 4$

46. $y^2 - 20y + 100$

47. $2r^2 - 2r - 40$

48. $2r^2 - 2r - 4$

49. $10r^2 - 10$

50. $2t^2 + 6t - 20$

51. $10t^2 - 30t + 20$

52. $2x^2 - 18x + 16$

53. $3x^2 - 15x + 12$

54. $3y^2 - 21y + 18$

55. $2y^6 + 20y^5 + 18y^4$

56. $2y^7 + 14y^6 + 12y^5$

57. $3r^6 + 24r^5 + 21r^4$

58. $5r^7 + 20r^6 + 15r^5$

59. $10t^7 - 20t^6 - 30t^5$

60. $9t^4 + 9t^3 - 18t^2$

61. $3x^{10} - 12x^8$

62. $5x^6 - 20x^5 - 25x^4$

63. $2y^7 - 8y^6 + 6y^5$

64. $5y^5 - 15y^4 + 10y^3$

65. $3y^7 - 12y^6 + 9y^5$

66. $6r^4 - 18r^3 + 12r^2$

67. $-r^2 - 4r + 45$

68. $-t^2 + 4t + 12$

69. $-t^2 + 5t + 24$

70. $-x^2 - 4x + 5$

71. $x^2 + 12xr + 20r^2$

72. $y^2 + 7yr + 6r^2$

73. $y^2 - 3yt - 10t^2$

74. $y^2 - 4yx - 32x^2$

75. $r^2 - 7rt + 12t^2$

76. $r^2 - 5ry + 4y^2$

77. $t^2 + 16tx + 64x^2$

78. $t^2 + 2tr + r^2$

79. $x^2 - 12xy + 36y^2$

80. $x^2 - 22xt + 121t^2$

81. $4y^2 + 20y + 16$

82. $2y^2 + 18y + 16$

83. $3x^2y + 15xy + 18y$

84. $2x^2y + 6xy + 4y$

85. $7a^2b - 7ab - 14b$

86. $10a^2b + 10ab - 20b$

87. $2x^2y - 22xy + 20y$

88. $9x^2y - 27xy + 18y$

89. $3x^3y + 24x^2y + 21xy$

90. $2x^3y + 16x^2y + 30xy$

91. $x^2y^2 + x^2yz - 20x^2z^2$

92. $x^2y^2 + 8x^2yz - 9x^2z^2$

93. $r^2 + 0.9r + 0.2$

94. $r^2 + 0.9r + 0.14$

95. $t^2x^2 + 6tx + 5$

96. $t^2y^2 + 5ty + 6$

97. $x^2t^2 - 2xt - 24$

98. $x^2r^2 + xr - 20$

99. $y^2t^2 - 8yt + 7$

100. $y^2x^2 - 9yx + 14$

101. $6y^2r^2 + 18yr + 12$

102. $5r^2y^2 + 15ry + 10$

103. $7r^2t^2 - 7$

104. $3t^2r^2 + 6tr - 24$

105. $6x^2y^3 - 18xy^2 + 12y$

106. $6x^2y^3 - 24xy^2 + 18y$

Factor the given polynomial.

107. $(a+b)\,x^2 + 8(a+b)\,x + 12(a+b)$

108. $(a+b)\,y^2 + 9(a+b)\,y + 18(a+b)$

Challenge

109. What integers can go in the place of b so that the quadratic expression $x^2 + bx + 10$ is factorable?

10.4 Factoring Trinomials with a Nontrivial Leading Coefficient

In Section 10.3, we learned how to factor $ax^2 + bx + c$ when $a = 1$. In this section, we will examine the situation when $a \neq 1$. The techniques are similar to those in the last section, but there are a few important differences that will make-or-break your success in factoring these.

10.4.1 The AC Method

The AC Method is a technique for factoring trinomials like $4x^2+5x-6$, where there is no greatest common factor, and the leading coefficient is not 1.

Please note at this point that if we try the method in the previous section and ask ourselves the question "what two numbers multiply to be -6 and add to be 5?," we might come to the *erroneous* conclusion that $4x^2 + 5x - 6$ factors as $(x + 6)(x - 1)$. If we expand $(x + 6)(x - 1)$, we get

$$(x+6)(x-1) = x^2+5x-6$$

This expression is *almost* correct, except for the missing leading coefficient, 4. Dealing with this missing coefficient requires starting over with the AC method. If you are only interested in the steps for using the technique, skip ahead to Algorithm 10.4.3.

The example below explains *why* the "AC Method" works, which will be more carefully outlined a bit later. Understanding all of the details might take a few rereads, and coming back to this example after mastering the algorithm may be the best course of action.

Example 10.4.2 Expand the expression $(px + q)(rx + s)$ and analyze the result to gain insight into factoring $4x^2 + 5x - 6$.

Explanation. Factoring is the opposite process from multiplying polynomials together. We can gain some insight into how to factor complicated polynomials by taking a closer look at what happens when two generic polynomials are multiplied together:

$$\begin{aligned}(px+q)(rx+s) &= (px+q)(rx) + (px+q)s \\ &= (px)(rx) + q(rx) + (px)s + qs \\ &= (pr)x^2 + qrx + psx + qs \\ &= (pr)x^2 + (qr+ps)x + qs \qquad (10.4.1)\end{aligned}$$

When you encounter a trinomial like $4x^2 + 5x - 6$ and you wish to factor it, the leading coefficient, 4, is the (pr) from Equation (10.4.1). Similarly, the -6 is the qs, and the 5 is the $(qr + ps)$.

Now, if you multiply the leading coefficient and constant term from Equation (10.4.1), you have $(pr)(qs)$, which equals $pqrs$. Notice that if we factor this number in just the right way, $(qr)(ps)$, then we have two factors that add to the middle coefficient from Equation (10.4.1), $(qr + ps)$.

Can we do all this with the example $4x^2 + 5x - 6$? Multiplying 4 and -6 makes -24. Is there some way to factor -24 into two factors which add to 5? We make a table of factor pairs for -24 to see:

Factor Pair	Sum of the Pair
$-1 \cdot 24$	23
$-2 \cdot 12$	10
$-3 \cdot 8$	5 (what we wanted)
$-4 \cdot 6$	(no need to go this far)

Factor Pair	Sum of the Pair
$1 \cdot (-24)$	(no need to go this far)
$2 \cdot (-12)$	(no need to go this far)
$3 \cdot (-8)$	(no need to go this far)
$4 \cdot (-6)$	(no need to go this far)

So that 5 in $4x^2+5x-6$, which is equal to the abstract $(qr+ps)$ from Equation (10.4.1), breaks down as $-3+8$.

We can take -3 to be the qr and 8 to be the ps. Once we intentionally break up the 5 this way, factoring by grouping (see Section 10.2) can take over and is guaranteed to give us a factorization.

$$4x^2 \overbrace{+5x} -6 = 4x^2 \overbrace{-3x+8x} -6$$

Now that there are four terms, group them and factor out each group's greatest common factor.

$$\begin{aligned} &= \left(4x^2-3x\right)+(8x-6)\\ &= x(4x-3)+2(4x-3)\\ &= (4x-3)(x+2) \end{aligned}$$

And this is the factorization of $4x^2+5x-6$. This whole process is known as the "AC method," since it begins by multiplying a and c from the generic ax^2+bx+c.

Here is a summary of the algorithm:

Process 10.4.3 The AC Method. *To factor* ax^2+bx+c*:*

1. *Multiply* $a \cdot c$.
2. *Make a table of factor pairs for* ac. *Look for a pair that adds to* b. *If you cannot find one, the polynomial is irreducible.*
3. *If you did find a factor pair summing to* b, *replace* b *with an explicit sum, and distribute* x. *With the four terms you have at this point, use factoring by grouping to continue. You are guaranteed to find a factorization.*

Example 10.4.4 Factor $10x^2+23x+6$.

1. $10 \cdot 6 = 60$
2. Use a list of factor pairs for 60 to find that 3 and 20 are a pair that sums to 23.
3. Intentionally break up the 23 as $3+20$:

$$\begin{aligned} &10x^2 \overbrace{+23x} +6\\ &= 10x^2 \overbrace{+3x+20x} +6\\ &= \left(10x^2+3x\right)+(20x+6)\\ &= x(10x+3)+2(10x+3)\\ &= (10x+3)(x+2) \end{aligned}$$

Example 10.4.5 Factor $2x^2-5x-3$.

Explanation. Always start the factoring process by examining if there is a greatest common factor. Here there is not one. Next, note that this is a trinomial with a leading coefficient that is not 1. So the AC Method may be of help.

1. Multiply $2 \cdot (-3) = -6$.
2. Examine factor pairs that multiply to -6, looking for a pair that sums to -5:

Factor Pair	Sum of the Pair
$1 \cdot -6$	-5 (what we wanted)
$2 \cdot -3$	(no need to go this far)

Factor Pair	Sum of the Pair
$-1 \cdot 6$	(no need to go this far)
$-2 \cdot 3$	(no need to go this far)

3. Intentionally break up the -5 as $1 + (-6)$:

$$\begin{aligned} 2x^2 \overbrace{-5x}-3 &= 2x^2 \overbrace{+x-6x}-3 \\ &= \left(2x^2 + x\right) + (-6x - 3) \\ &= x(2x+1) - 3(2x+1) \\ &= (2x+1)(x-3) \end{aligned}$$

So we believe that $2x^2 - 5x - 3$ factors as $(2x+1)(x-3)$, and we should check by multiplying out the factored form:

$$\begin{aligned} (2x+1)(x-3) &= (2x+1) \cdot x + (2x+1) \cdot (-3) \\ &= 2x^2 + x - 6x - 3 \\ &\stackrel{\checkmark}{=} 2x^2 - 5x - 3 \end{aligned}$$

	$2x$	1
x	$2x^2$	x
-3	$-6x$	-3

Our factorization passes the tests.

Example 10.4.6 Factor $6p^2 + 5pq - 6q^2$. Note that this example has two variables, but that does not really change our approach.

Explanation. There is no greatest common factor. Since this is a trinomial, we try the AC Method.

1. Multiply $6 \cdot (-6) = -36$.

2. Examine factor pairs that multiply to -36, looking for a pair that sums to 5:

Factor Pair	Sum of the Pair
$1 \cdot -36$	-35
$2 \cdot -18$	-16
$3 \cdot -12$	-9
$4 \cdot -9$	-5 (close; wrong sign)
$6 \cdot -6$	0

Factor Pair	Sum of the Pair
$-1 \cdot 36$	35
$-2 \cdot 18$	16
$-3 \cdot 12$	9
$-4 \cdot 9$	5 (what we wanted)

3. Intentionally break up the 5 as $-4 + 9$:

$$\begin{aligned} 6p^2 \overbrace{+5pq}-6q^2 &= 6p^2 \overbrace{-4pq+9pq}-6q^2 \\ &= \left(6p^2 - 4pq\right) + \left(9pq - 6q^2\right) \\ &= 2p(3p-2q) + 3q(3p-2q) \\ &= (3p-2q)(2p+3q) \end{aligned}$$

So we believe that $6p^2 + 5pq - 6q^2$ factors as $(3p-2q)(2p+3q)$, and we should check by multiplying out the factored form:

$$\begin{aligned} (3p-2q)(2p+3q) &= (3p-2q) \cdot 2p + (3p-2q) \cdot 3q \\ &= 6p^2 - 4pq + 9pq - 6q^2 \\ &\stackrel{\checkmark}{=} 6p^2 + 5pq - 6q^2 \end{aligned}$$

	$3p$	$-2q$
$2p$	$6p^2$	$-4pq$
$3q$	$9pq$	$-6q^2$

Our factorization passes the tests.

10.4.2 Factoring in Stages

Sometimes factoring a polynomial will take two or more "stages." For instance you may need to begin factoring a polynomial by factoring out its greatest common factor, and *then* apply a second stage where you use a technique from this section. The process of factoring a polynomial is not complete until each of the factors cannot be factored further.

Example 10.4.7 Factor $18n^2 - 21n - 60$.

Explanation. Notice that 3 is a common factor in this trinomial. We should factor it out first:

$$18n^2 - 21n - 60 = 3\left(6n^2 - 7n - 20\right)$$

Now we are left with two factors, one of which is $6n^2 - 7n - 20$, which might factor further. Using the AC Method:

1. $6 \cdot -20 = -120$
2. Examine factor pairs that multiply to -120, looking for a pair that sums to -7:

Factor Pair	Sum of the Pair
$1 \cdot -120$	-119
$2 \cdot -60$	-58
$3 \cdot -40$	-37
$4 \cdot -30$	-26
$5 \cdot -24$	-19
$6 \cdot -20$	-14
$8 \cdot -15$	-7 (what we wanted)
$10 \cdot -12$	(no need to go this far)

Factor Pair	Sum of the Pair
$-1 \cdot 120$	(no need to go this far)
$-2 \cdot 60$	(no need to go this far)
$-3 \cdot 40$	(no need to go this far)
$-4 \cdot 30$	(no need to go this far)
$-5 \cdot 24$	(no need to go this far)
$-6 \cdot 20$	(no need to go this far)
$-8 \cdot 15$	(no need to go this far)
$-10 \cdot 12$	(no need to go this far)

3. Intentionally break up the -7 as $8 + (-15)$:

$$\begin{aligned}
18n^2 - 21n - 60 &= 3\left(6n^2 \overbrace{-7n} - 20\right)\\
&= 3\left(6n^2 \overbrace{+8n - 15n} - 20\right)\\
&= 3\left(\left(6n^2 + 8n\right) + \left(-15n - 20\right)\right)\\
&= 3\left(2n(3n+4) - 5(3n+4)\right)\\
&= 3(3n+4)(2n-5)
\end{aligned}$$

So we believe that $18n^2 - 21n - 60$ factors as $3(3n+4)(2n-5)$, and you should check by multiplying out the factored form.

Example 10.4.8 Factor $-16x^3y - 12x^2y + 18xy$.

Explanation. Notice that $2xy$ is a common factor in this trinomial. Also the leading coefficient is negative, and as discussed in Section 10.1, it is wise to factor that out as well. So we find:

$$-16x^3y - 12x^2y + 18xy = -2xy\left(8x^2 + 6x - 9\right)$$

Now we are left with one factor being $8x^2 + 6x - 9$, which might factor further. Using the AC Method:

1. $8 \cdot -9 = -72$
2. Examine factor pairs that multiply to -72, looking for a pair that sums to 6:

Factor Pair	Sum of the Pair
$1 \cdot -72$	-71
$2 \cdot -36$	-34
$3 \cdot -24$	-21
$4 \cdot -18$	-14
$6 \cdot -12$	-6 (close; wrong sign)
$8 \cdot -9$	-1

Factor Pair	Sum of the Pair
$-1 \cdot 72$	71
$-2 \cdot 36$	34
$-3 \cdot 24$	21
$-4 \cdot 18$	14
$-6 \cdot 12$	6 (what we wanted)
$-8 \cdot 9$	(no need to go this far)

3. Intentionally break up the 6 as $-6 + 12$:

$$\begin{aligned}
-16x^3y - 12x^2y + 18xy &= -2xy\left(8x^2 \overbrace{+\,6x} - 9\right)\\
&= -2xy\left(8x^2 \overbrace{-\,6x + 12x} - 9\right)\\
&= -2xy\left(\left(8x^2 - 6x\right) + (12x - 9)\right)\\
&= -2xy\left(2x(4x-3) + 3(4x-3)\right)\\
&= -2xy(4x-3)(2x+3)
\end{aligned}$$

So we believe that $-16x^3y - 12x^2y + 18xy$ factors as $-2xy(4x-3)(2x+3)$, and you should check by multiplying out the factored form.

10.4.3 Reading Questions

1. The AC Method is really trying to turn a trinomial into a polynomial with $\boxed{}$ terms so that factoring by grouping may be used.

2. When you are tyring to factor a polynomial and the leading coefficient is not 1, what should you try to do *before* you try the AC Method?

10.4.4 Exercises

Review and Warmup Multiply the polynomials.

1. $(4y+10)(2y+9)$ **2.** $(2r+9)(4r+3)$ **3.** $(6r-8)(3r-8)$

4. $(4t-10)(2t-7)$ **5.** $(3t-10)(t+9)$ **6.** $(9x-6)(x+2)$

7. $\left(6x^3+9\right)\left(x^2+3\right)$ **8.** $\left(2y^3+4\right)\left(y^2+10\right)$

Factoring Trinomials with a Nontrivial Leading Coefficient Factor the given polynomial.

9. $5y^2 + 6y + 1$

10. $3y^2 + 17y + 10$

11. $2r^2 + 11r - 21$

12. $5r^2 + 23r - 10$

13. $2t^2 - 13t + 15$

14. $3t^2 - 16t + 20$

15. $5x^2 + 6x + 10$

16. $2x^2 + 3x + 6$

17. $4x^2 + 13x + 9$

18. $6y^2 + 23y + 15$

19. $4y^2 - 11y - 3$

20. $8r^2 - r - 7$

21. $6r^2 - 13r + 7$

22. $4t^2 - 17t + 18$

23. $6t^2 + 19t + 10$

24. $8x^2 + 18x + 9$

25. $10x^2 - x - 24$

26. $10x^2 + 3x - 27$

27. $6y^2 - 29y + 9$

28. $10y^2 - 17y + 6$

29. $18r^2 + 27r + 9$

30. $4r^2 + 34r + 16$

31. $35t^2 + 28t - 7$

32. $14t^2 - 7t - 21$

33. $4x^2 - 26x + 12$

34. $15x^2 - 24x + 9$

35. $4x^9 + 18x^8 + 14x^7$

36. $14y^9 + 21y^8 + 7y^7$

37. $16y^6 - 8y^5 - 24y^4$

38. $8r^9 - 12r^8 - 20r^7$

39. $6r^7 - 8r^6 + 2r^5$

40. $10t^9 - 35t^8 + 15t^7$

41. $3t^2r^2 + 13tr + 12$

42. $3x^2y^2 + 14xy + 16$

43. $2x^2t^2 + 3xt - 2$

44. $2x^2r^2 - 3xr - 2$

45. $5y^2x^2 - 23yx + 12$

46. $5y^2t^2 - 14yt + 8$

47. $3r^2 + 13rx + 4x^2$

48. $5r^2 + 17rx + 14x^2$

49. $3t^2 - 26ty - 9y^2$

50. $5t^2 - 6tr - 8r^2$

51. $3x^2 - 11xr + 8r^2$

52. $5x^2 - 9xt + 4t^2$

53. $4x^2 + 11xy + 7y^2$

54. $4y^2 + 25yx + 6x^2$

55. $8y^2 - 17yt - 21t^2$

56. $4r^2 - 9ry - 9y^2$

57. $6r^2 - 31rx + 5x^2$

58. $8t^2 - 17tr + 2r^2$

59. $12t^2 + 25ty + 12y^2$

60. $12x^2 + 17xt + 6t^2$

61. $9x^2 - 9xr - 28r^2$

62. $15x^2 - 7xy - 2y^2$

63. $4y^2 - 16yt + 7t^2$

64. $12y^2 - 23yr + 10r^2$

65. $18r^2y^2 + 27ry + 9$

66. $15r^2t^2 + 20rt + 5$

67. $21t^2r^2 - 7tr - 28$

68. $15t^2x^2 + 12tx - 36$

69. $6x^9t^2 - 26x^8t + 24x^7$

70. $10x^7r^2 - 18x^6r + 8x^5$

71. $10x^2 + 22xy + 12y^2$

72. $8x^2 + 28xy + 20y^2$

73. $6a^2 + 10ab - 4b^2$

74. $10a^2 + 8ab - 2b^2$

75. $10x^2 - 24xy + 8y^2$

76. $9x^2 - 12xy + 3y^2$

77. $10x^2y + 32xy^2 + 6y^3$

78. $9x^2y + 21xy^2 + 6y^3$

Factor the given polynomial.

79. $12x^2(y+6) + 28x(y+6) + 16(y+6)$

80. $4x^2(y-6) + 18x(y-6) + 8(y-6)$

81. $9x^2(y+9) + 21x(y+9) + 6(y+9)$

82. $25x^2(y+2) + 30x(y+2) + 5(y+2)$

83. a. Factor the given polynomial.$3x^2 + 19x + 6$

b. Use your previous answer to factor$3(y-7)^2 + 19(y-7) + 6$

84. a. Factor the given polynomial.$2x^2 + 19x + 9$

b. Use your previous answer to factor$2(y+3)^2 + 19(y+3) + 9$

Challenge

85. What integers can go in the place of b so that the quadratic expression $3x^2 + bx + 8$ is factorable?

10.5 Factoring Special Polynomials

Certain polynomials have patterns that you can train yourself to recognize. And when they have these patterns, there are formulas you can use to factor them, much more quickly than using the techniques from Section 10.3 and Section 10.4.

10.5.1 Difference of Squares

If b is some positive integer, then when you multiply $(x-b)(x+b)$:

$$\begin{aligned}(x-b)(x+b) &= x^2 - bx + bx - b^2\\ &= x^2 - b^2.\end{aligned}$$

The $-bx$ and the $+bx$ cancel each other out. So this is telling us that

$$x^2 - b^2 = (x-b)(x+b).$$

And so if we ever encounter a polynomial of the form $x^2 - b^2$ (a "difference of squares") then we have a quick formula for factoring it. Just identify what "b" is, and use that in $(x-b)(x+b)$.

To use this formula, it's important to recognize which numbers are perfect squares, as in Figure 6.1.4.

Example 10.5.2 Factor $x^2 - 16$.

Explanation. The "16" being subtracted here is a perfect square. It is the same as 4^2. So we can take $b = 4$ and write:

$$\begin{aligned}x^2 - 16 &= (x-b)(x+b)\\ &= (x-4)(x+4)\end{aligned}$$

Checkpoint 10.5.3 Try to factor one yourself:

Factor $x^2 - 49$.

Explanation. The "49" being subtracted here is a perfect square. It is the same as 7^2. So we can take $b = 7$ and write:

$$\begin{aligned}x^2 - 49 &= (x-b)(x+b)\\ &= (x-7)(x+7)\end{aligned}$$

We can do a little better. There is nothing special about starting with "x^2" in these examples. In full generality:

Fact 10.5.4 The Difference of Squares Formula. *If A and B are any algebraic expressions, then:*

$$A^2 - B^2 = (A-B)(A+B).$$

Example 10.5.5 Factor $1 - p^2$.

Explanation. The "1" at the beginning of this expression is a perfect square; it's the same as 1^2. The "p^2" being subtracted here is also perfect square. We can take $A = 1$ and $B = p$, and use The Difference of Squares Formula:

$$\begin{aligned}1 - p^2 &= (A-B)(A+B)\\ &= (1-p)(1+p)\end{aligned}$$

Example 10.5.6 Factor $m^2n^2 - 4$.

Explanation. Is the "m^2n^2" at the beginning of this expression a perfect square? By the rules for exponents, it is the same as $(mn)^2$, so yes, it is a perfect square and we may take $A = mn$. The "4" being subtracted here is also perfect square. We can take $B = 2$. The Difference of Squares Formula tells us:

$$\begin{aligned} m^2n^2 - 4 &= (A - B)(A + B) \\ &= (mn - 2)(mn + 2) \end{aligned}$$

Checkpoint 10.5.7 Try to factor one yourself:

Factor $4z^2 - 9$.

Explanation. The "$4z^2$" at the beginning here is a perfect square. It is the same as $(2z)^2$. So we can take $A = 2z$. The "9" being subtracted is also a perfect square, so we can take $B = 3$:

$$\begin{aligned} 4z^2 - 9 &= (A - B)(A + B) \\ &= (2z - 3)(2z + 3) \end{aligned}$$

Example 10.5.8 Factor $x^6 - 9$.

Explanation. Is the "x^6" at the beginning of this expression is a perfect square? It may appear to be a *sixth* power, but it is *also* a perfect square because we can write $x^6 = \left(x^3\right)^2$. So we may take $A = x^3$. The "9" being subtracted here is also perfect square. We can take $B = 3$. The Difference of Squares Formula tells us:

$$\begin{aligned} x^6 - 9 &= (A - B)(A + B) \\ &= (x^3 - 3)(x^3 + 3) \end{aligned}$$

Warning 10.5.9 It's a common mistake to write something like $x^2 + 16 = (x + 4)(x - 4)$. This is not what The Difference of Squares Formula allows you to do, and this is in fact incorrect. The issue is that $x^2 + 16$ is a *sum* of squares, not a *difference*. And it happens that $x^2 + 16$ is actually prime. In fact, any sum of squares without a common factor will always be prime.

10.5.2 Perfect Square Trinomials

If we expand $(A + B)^2$:

$$\begin{aligned} (A + B)^2 &= (A + B)(A + B) \\ &= A^2 + BA + AB + B^2 \\ &= A^2 + 2AB + B^2. \end{aligned}$$

The BA and the AB equal each other and double up when added together. So this is telling us that

$$A^2 + 2AB + B^2 = (A + B)^2.$$

And so if we ever encounter a polynomial of the form $A^2 + 2AB + B^2$ (a "perfect square trinomial") then we have a quick formula for factoring it.

The tricky part is recognizing when a trinomial you have encountered is in this special form. Ask yourself:

1. Are the first and last terms perfect square? If so, jot down what A and B would be.

2. When you multiply 2 with what you wrote down for A and B, i.e. 2AB, do you have the middle term? If you have this middle term exactly, then your polynomial factors as $(A+B)^2$. If the middle term is the negative of 2AB, then the sign on your B can be reversed, and your polynomial factors as $(A-B)^2$.

Fact 10.5.10 The Perfect Square Trinomial Formula. *If* A *and* B *are any algebraic expressions, then:*

$$A^2 + 2AB + B^2 = (A+B)^2 \qquad \textit{and} \qquad A^2 - 2AB + B^2 = (A-B)^2$$

Example 10.5.11 Factor $x^2 + 6x + 9$.

Explanation. The first term, x^2, is clearly a perfect square. So we could take $A = x$. The last term, 9, is also a perfect square since it is equal to 3^2. So we could take $B = 3$. Now we multiply $2AB = 2 \cdot x \cdot 3$, and the result is 6x. This is the middle term, which is what we hope to see.

So we can use The Perfect Square Trinomial Formula:

$$\begin{aligned} x^2 + 6x + 9 &= (A+B)^2 \\ &= (x+3)^2 \end{aligned}$$

Example 10.5.12 Factor $4x^2 - 20xy + 25y^2$.

Explanation. The first term, $4x^2$, is a perfect square because it equals $(2x)^2$. So we could take $A = 2x$. The last term, $25y^2$, is also a perfect square since it is equal to $(5y)^2$. So we could take $B = 5y$. Now we multiply $2AB = 2 \cdot (2x) \cdot (5y)$, and the result is 20xy. This is the *negative* of the middle term, which we can work with. The factored form will be $(A-B)^2$ instead of $(A+B)^2$.

So we can use The Perfect Square Trinomial Formula:

$$\begin{aligned} 4x^2 - 20xy + 25y^2 &= (A-B)^2 \\ &= (2x-5y)^2 \end{aligned}$$

Checkpoint 10.5.13 Try to factor one yourself:

Factor $16q^2 + 56q + 49$.

Explanation. The first term, $16q^2$, is a perfect square because it equals $(4q)^2$. So we could take $A = 4q$. The last term, 49, is also a perfect square since it is equal to 7^2. So we could take $B = 7$. Now we multiply $2AB = 2 \cdot (4q) \cdot 7$, and the result is 56q. This is the middle term, which is what we hope to see.

So we can use The Perfect Square Trinomial Formula:

$$\begin{aligned} 16q^2 + 56q + 49 &= (A+B)^2 \\ &= (4q+7)^2 \end{aligned}$$

Warning 10.5.14 It is not enough to just see that the first and last terms are perfect squares. For example, $9x^2 + 10x + 25$ has its first term equal to $(3x)^2$ and its last term equal to 5^2. But when you examine $2 \cdot (3x) \cdot 5$ the result is 30x, *not* equal to the middle term. So The Perfect Square Trinomial Formula doesn't apply here. In fact, this polynomial doesn't factor at all.

Remark 10.5.15 To factor these perfect square trinomials, we *could* use methods from Section 10.3 and Section 10.4. As an exercise for yourself, try to factor each of the three previous examples using those methods. The advantage to using The Perfect Square Trinomial Formula is that it is much faster. With some practice, all of the work for using it can be done mentally.

10.5.3 Factoring in Stages

Sometimes factoring a polynomial will take two or more "stages." You might use one of the special patters to factor something into two factors, and then those factors might factor even more. When the task is to *factor* a polynomial, the intention is that you *fully* factor it, breaking down the pieces into even smaller pieces when that is possible.

Example 10.5.16 Factor out any greatest common factor. Factor $12z^3 - 27z$.

Explanation. The two terms of this polynomial have greatest common factor $3z$, so the first step in factoring should be to factor this out:

$$3z\left(4z^2 - 9\right).$$

Now we have two factors. There is nothing for us to do with $3z$, but we should ask if $\left(4z^2 - 9\right)$ can factor further. And in fact, that is a difference of squares. So we can apply The Difference of Squares Formula. The full process would be:

$$\begin{aligned} 12z^3 - 27z &= 3z\left(4z^2 - 9\right) \\ &= 3z(2z-3)(2z+3) \end{aligned}$$

Example 10.5.17 Recognize a *second* special pattern. Factor $p^4 - 1$.

Explanation. Since p^4 is the same as $\left(p^2\right)^2$, we have a difference of squares here. We can apply The Difference of Squares Formula:

$$p^4 - 1 = \left(p^2 - 1\right)\left(p^2 + 1\right)$$

It doesn't end here. Of the two factors we found, $\left(p^2 + 1\right)$ cannot be factored further. But the other one, $\left(p^2 - 1\right)$ is *also* a difference of squares. So we should apply The Difference of Squares Formula again:

$$= (p-1)(p+1)\left(p^2 + 1\right)$$

Example 10.5.18 Factor $32x^6y^2 - 48x^5y + 18x^4$.

Explanation. The first step of factoring any polynomial is to factor out the common factor if possible. For this trinomial, the common factor is $2x^4$, so we write

$$32x^6y^2 - 48x^5y + 18x^4 = 2x^4(16x^2y^2 - 24xy + 9).$$

The square numbers 16 and 9 in $16x^2y^2 - 24xy + 9$ hint that maybe we could use The Perfect Square Trinomial Formula. Taking $A = 4xy$ and $B = 3$, we multiply $2AB = 2 \cdot (4xy) \cdot 3$. The result is $24xy$, which is the negative of our middle term. So the whole process is:

$$\begin{aligned} 32x^6y^2 - 48x^5y + 18x^4 &= 2x^4(16x^2y^2 - 24xy + 9) \\ &= 2x^4(4xy - 3)^2 \end{aligned}$$

10.5.4 Reading Questions

1. Describe two special patterns where it is possible to memorize a quick factoring shortcut as discussed in this section.

10.5.5 Exercises

Review and Warmup Expand the square of a ***bi***nomial.

1. $(10x+1)^2$ **2.** $(7y+5)^2$ **3.** $(y-8)^2$

4. $(r-2)^2$ **5.** $\left(r^8+1\right)^2$ **6.** $\left(t^4-8\right)^2$

Multiply the polynomials.

7. $(t+10)(t-10)$ **8.** $(t+13)(t-13)$ **9.** $(2x-2)(2x+2)$

10. $(6x+7)(6x-7)$ **11.** $\left(4y^9-7\right)\left(4y^9+7\right)$ **12.** $\left(2y^6+4\right)\left(2y^6-4\right)$

Factoring Factor the given polynomial.

13. r^2-100 **14.** r^2-36 **15.** t^2-144

16. $81t^2-25$ **17.** t^2y^2-100 **18.** x^2t^2-121

19. $64x^2r^2-49$ **20.** $16y^2x^2-121$ **21.** $36-y^2$

22. $1-r^2$ **23.** $49-16r^2$ **24.** $1-144t^2$

25. t^4-64 **26.** t^4-49 **27.** $121x^4-144$

28. $64x^4-25$ **29.** $y^{14}-100$ **30.** y^6-121

31. $49x^4-64y^4$ **32.** $81x^4-4y^4$ **33.** $x^{14}-100y^{12}$

34. $x^{10}-121y^{14}$ **35.** t^2+4t+4 **36.** $x^2+20x+100$

37. $x^2-12x+36$ **38.** y^2-2y+1 **39.** $100y^2+20y+1$

40. $36r^2+12r+1$ **41.** $4r^2-4r+1$ **42.** $81t^2-18t+1$

43. $64t^2y^2-16ty+1$ **44.** $64t^2x^2-16tx+1$ **45.** $x^2+12xr+36r^2$

46. $x^2+22xy+121y^2$ **47.** $y^2-8yt+16t^2$ **48.** $y^2-18yr+81r^2$

49. $9r^2+24ry+16y^2$ **50.** $64r^2+48rt+9t^2$ **51.** $36t^2-60tr+25r^2$

52. $25t^2-60tx+36x^2$ **53.** $81t^4-16$ **54.** $16x^4-1$

55. $8x^2-72$ **56.** $6y^2-96$ **57.** $6y^3-6y$

58. $11r^3-44r$ **59.** $5r^3t^3-125rt$ **60.** $5t^4y^4-20t^2y^2$

61. $3-3t^2$ **62.** $125-5t^2$ **63.** $27x^2+18x+3$

64. $20x^2+20x+5$ **65.** $32y^2r^2+32yr+8$ **66.** $90y^2x^2+60yx+10$

67. $64r^2-32r+4$ **68.** $45r^2-30r+5$ **69.** $25t^7+10t^6+t^5$

70. $144t^9+24t^8+t^7$ **71.** $64t^5-16t^4+t^3$ **72.** $16x^8-8x^7+x^6$

73. $12x^4+12x^3+3x^2$ **74.** $75y^{10}+30y^9+3y^8$ **75.** $12y^6-12y^5+3y^4$

76. $90r^{10}-60r^9+10r^8$ **77.** $5r^4-80$ **78.** $7r^4-7$

79. t^2+100 **80.** t^2+36 **81.** $6x^3+24x$

82. $2x^3 + 8x$

83. $0.25y - y^3$

84. $0.01y - y^3$

Challenge

85. Select the expression which is equivalent to $555^2 - 666^2$.

- ⊙ $1221(1221)$
- ⊙ $1221(-111)$
- ⊙ $-111(-111)$
- ⊙ none of the above

10.6 Factoring Strategies

10.6.1 Factoring Strategies

Deciding which method to use when factoring a random polynomial can seem like a daunting task. Understanding all of the techniques that we have learned and how they fit together can be done using a decision tree.

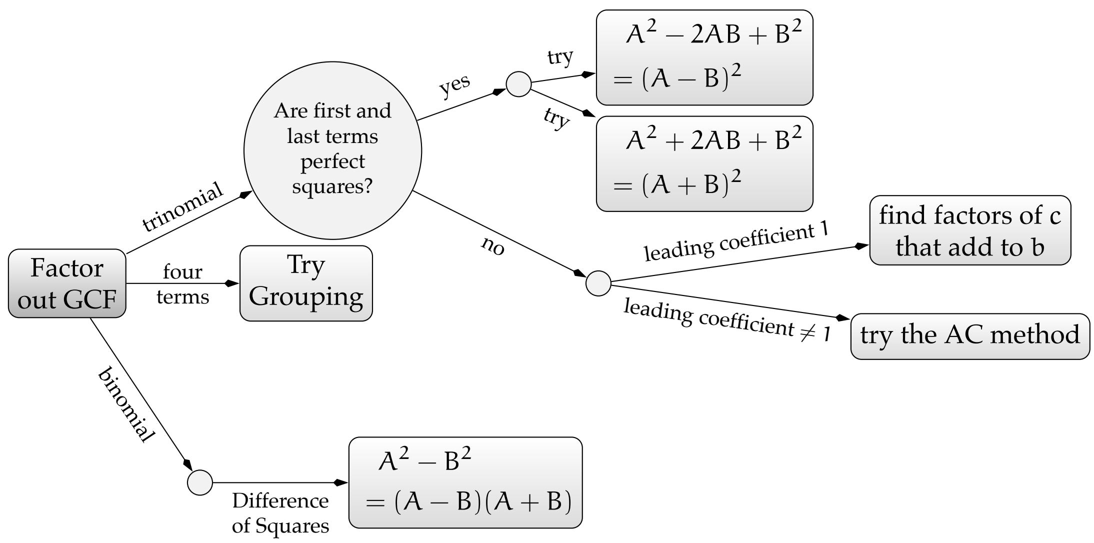


Figure 10.6.2: Factoring Decision Tree

Using the decision tree can guide us when we are given an expression to factor.

Example 10.6.3 Factor the expression $4k^2 + 12k - 40$ completely.

Explanation. Start by noting that the GCF is 4. Factoring this out, we get

$$4k^2 + 12k - 40 = 4\left(k^2 + 3k - 10\right).$$

Following the decision tree, we now have a trinomial where the leading coefficient is 1 and we need to look for factors of -10 that add to 3. We find that -2 and 5 work. So, the full factorization is:

$$\begin{aligned} 4k^2 + 12k - 40 &= 4\left(k^2 + 3k - 10\right) \\ &= 4(k-2)(k+5) \end{aligned}$$

Example 10.6.4 Factor the expression $64d^2 + 144d + 81$ completely.

Explanation. Start by noting that the GCF is 1, and there is no GCF to factor out. We continue along the decision tree for a trinomial. Notice that both 64 and 81 are perfect squares and that this expression might factor using the pattern $A^2+2AB+B^2 = (A+B)^2$. To find A and B, take the square roots of the first and last terms, so $A = 8d$ and $B = 9$. We have to check that the middle term is correct: since $2AB = 2(8d)(9) = 144d$ matches our middle term, the expression must factor as

$$64d^2 + 144d + 81 = (8d+9)^2.$$

Example 10.6.5 Factor the expression $10x^2y - 12xy^2$ completely.

Explanation. Start by noting that the GCF is $2xy$. Factoring this out, we get

$$10x^2y - 12xy^2 = 2xy(5x - 6y).$$

Since we have a binomial inside the parentheses, the only options on the decision tree for a binomial involve squares or cubes. Since there are none, we conclude that $2xy(5x - 6y)$ is the complete factorization.

Example 10.6.6 Factor the expression $9b^2 - 25y^2$ completely.

Explanation. Start by noting that the GCF is 1, and there is no GCF to factor out. We continue along the decision tree for a binomial and notice that we now have a difference of squares, $A^2 - B^2 = (A - B)(A + B)$. To find the values for A and B that fit the patterns, just take the square roots. So $A = 3b$ since $(3b)^2 = 9b^2$ and $B = 5y$ since $(5y)^2 = 25y^2$. So, the expression must factor as

$$9b^2 - 25y^2 = (3b - 5y)(3b + 5y).$$

Example 10.6.7 Factor the expression $24w^3 + 6w^2 - 9w$ completely.

Explanation. Start by noting that the GCF is $3w$. Factoring this out, we get

$$24w^3 + 6w^2 - 9w = 3w\left(8w^2 + 2w - 3\right).$$

Following the decision tree, we now have a trinomial inside the parentheses where $a \neq 1$. We should try the AC method because neither 8 nor -3 are perfect squares. In this case, $ac = -24$ and we must find two factors of -24 that add to be 2. The numbers 6 and -4 work in this case. The rest of the factoring process is:

$$\begin{aligned}
24w^3 + 6w^2 - 9w &= 3w\left(8w^2 \overbrace{+\,2w} - 3\right)\\
&= 3w\left(8w^2 \overbrace{+\,6w - 4w} - 3\right)\\
&= 3w\left(\left(8w^2 + 6w\right) + (-4w - 3)\right)\\
&= 3w\left(2w(4w + 3) - 1(4w + 3)\right)\\
&= 3w(4w + 3)(2w - 1)
\end{aligned}$$

Example 10.6.8 Factor the expression $-6xy + 9y + 2x - 3$ completely.

Explanation. Start by noting that the GCF is 1, and there is no GCF to factor out. We continue along the decision tree. Since we have a four-term polynomial, we should try to factor by grouping. The full process is:

$$\begin{aligned}
-6xy + 9y + 2x - 3 &= (-6xy + 9y) + (2x - 3)\\
&= -3y(2x - 3) + 1(2x - 3)\\
&= (2x - 3)(-3y + 1)
\end{aligned}$$

Note that the negative sign in front of the $3y$ can be factored out if you wish. That would look like:

$$= -(2x - 3)(3y - 1)$$

Example 10.6.9 Factor the expression $4w^3 - 20w^2 + 24w$ completely.

Explanation. Start by noting that the GCF is $4w$. Factoring this out, we get

$$4w^3 - 20w^2 + 24w = 4w\left(w^2 - 5w + 6\right).$$

Following the decision tree, we now have a trinomial with $a = 1$ inside the parentheses. So, we can look for factors of 6 that add up to -5. Since -3 and -2 fit the requirements, the full factorization is:

$$\begin{aligned} 4w^3 - 20w^2 + 24w &= 4w\left(w^2 - 5w + 6\right) \\ &= 4w(w-3)(w-2) \end{aligned}$$

Example 10.6.10 Factor the expression $9 - 24y + 16y^2$ completely.

Explanation. Start by noting that the GCF is 1, and there is no GCF to factor out. Continue along the decision tree. We now have a trinomial where both the first term, 9, and last term, $16y^2$, look like perfect squares. To use the perfect squares difference pattern, $A^2 - 2AB + B^2 = (A - B)^2$, recall that we need to mentally take the square roots of these two terms to find A and B. So, $A = 3$ since $3^2 = 9$, and $B = 4y$ since $(4y)^2 = 16y^2$. Now we have to check that $2AB$ matches $24y$:

$$2AB = 2(3)(4y) = 24y.$$

So the full factorization is:

$$9 - 24y + 16y^2 = (3 - 4y)^2.$$

Example 10.6.11 Factor the expression $9 - 25y + 16y^2$ completely.

Explanation. Start by noting that the GCF is 1, and there is no GCF to factor out. Since we now have a trinomial where both the first term and last term are perfect squares in exactly the same way as in Example 10. However, we cannot apply the perfect squares method to this problem because it worked when $2AB = 24y$. Since our middle term is $25y$, we can be certain that it won't be a perfect square.

Continuing on with the decision tree, our next option is to use the AC method. You might be tempted to rearrange the order of the terms, but that is unnecessary. In this case, $ac = 144$ and we need to come up with two factors of 144 that add to be -25. After a brief search, we conclude that those values are -16 and -9. The remainder of the factorization is:

$$\begin{aligned} 9\overbrace{-25y}+16y^2 &= 9\overbrace{-16y-9y}+16y^2 \\ &= (9-16y) + \left(-9y + 16y^2\right) \\ &= 1(9-16y) - y(9+16y) \\ &= (9-16y)(1-y) \end{aligned}$$

Example 10.6.12 Factor the expression $20x^4 + 13x^3 - 21x^2$ completely.

Explanation. Start by noting that the GCF is x^2. Factoring this out, we get

$$20x^4 + 13x^3 - 21x^2 = x^2\left(20x^2 + 13x - 21\right).$$

Following the decision tree, we now have a trinomial inside the parentheses where $a \neq 1$ and we should try the AC method. In this case, $ac = -420$ and we need factors of -420 that add to 13.

Factor Pair	Sum
$1 \cdot -420$	-419
$2 \cdot -210$	-208
$3 \cdot -140$	-137
$4 \cdot -105$	-101

Factor Pair	Sum
$5 \cdot -84$	-79
$6 \cdot -70$	-64
$7 \cdot -60$	-53
$10 \cdot -42$	-32

Factor Pair	Sum
$12 \cdot -35$	-23
$14 \cdot -30$	-16
$15 \cdot -28$	-13
$20 \cdot -21$	-1

In the table of the factor pairs of -420 we find $15+(-28)=-13$, the opposite of what we want, so we want the opposite numbers: -15 and 28. The rest of the factoring process is shown:

$$\begin{aligned}
20x^4+13x^3-21x^2 &= x^2\left(20x^2\overbrace{+13x}-21\right)\\
&= x^2\left(20x^2\overbrace{-15x+28x}-21\right)\\
&= x^2\left(\left(20x^2-15x\right)+(28x-21)\right)\\
&= x^2\left(5x(4x-3)+7(4x-3)\right)\\
&= x^2(4x-3)(5x+7)
\end{aligned}$$

10.6.2 Reading Questions

1. Do you find a factoring flowchart helpful?

10.6.3 Exercises

Strategies Which factoring techniques/tools will be useful for factoring the polynomial below? Check all that apply.
☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to the constant term and sum to the linear coefficient ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

1. $64B^2-216Bb+140b^2$

2. $6m^6+384m^3x^3$

3. $49n^3-35n^2-28n+20$

4. $4q^4-2916q$

5. $9y^2-24y+16$

6. $3r^4-39r^3+120r^2$

7. $40x^2-408xa+432a^2$

8. b^3-b^2-2b+2

9. $54A^6-686A^3B^3$

10. $63n-81$

11. $32m^3-50mp^2$

12. n^3+64A^3

13. $210q^3t-168q^3-270q^2t+216q^2$

14. $4y^5-2916y^2p^3$

Factoring Factor the given polynomial.

15. $6x+6$

16. $-3y-3$

17. $36y^2+27$

18. $30r^4-12r^3+42r^2$

19. $6r+12r^2+15r^3$

20. $8xy+8y$

21. $54x^5y^8-6x^4y^8+42x^3y^8$

22. $t^2-2t+3t-6$

23. $xy+2x+8y+16$

24. x^3-3+2x^3y-6y

25. $y^2 - 3y - 4$

26. $5y^2 - 2y - 7$

27. $2r^2y^2 + 3ry - 9$

28. $2r^2 - 6r + 5$

29. $4r^2 - 7r - 2$

30. $8t^2 + 22t + 15$

31. $12t^2 - 23t + 10$

32. $3x^2 + 10xr + 3r^2$

33. $3x^2 - 13xy + 12y^2$

34. $4y^2 + 3yt - 7t^2$

35. $8y^2 + 22yr + 9r^2$

36. $8r^2 - 18rx + 9x^2$

37. $12r^2 - 8r - 4$

38. $15r^2t^2 - 3rt - 12$

39. $10t^9 + 25t^8 + 15t^7$

40. $6t^{10} - 9t^9 + 3t^8$

41. $6x^2 + 20xy + 14y^2$

42. $10x^2 - 34xy + 12y^2$

43. $y^2 + 9y + 8$

44. $y^2 - 6y + 5$

45. $r^2 + 10rx + 16x^2$

46. $r^2y^2 - 4ry - 12$

47. $r^2 - 10ry + 24y^2$

48. $4t^2x^2 + 12tx + 8$

49. $2t^2 + 8t - 10$

50. $3x^4 + 18x^3 + 24x^2$

51. $7x^9 - 28x^8 + 21x^7$

52. $2x^2y + 6xy + 4y$

53. $2x^2y - 18xy + 28y$

54. $2x^2y^3 - 10xy^2 + 8y$

55. $x^2y^2 + 6x^2yz - 7x^2z^2$

56. $r^2 + 0.9r + 0.2$

57. $t^2 - 144$

58. $t^2r^2 - 16$

59. $9 - x^2$

60. $x^4 - 121$

61. $y^{12} - 49$

62. $x^6 - 36y^{14}$

63. $81y^4 - 16$

64. $3r^3 - 147r$

65. $r^2 + 4$

66. $32 - 8t^2$

67. $t^2 + 12t + 36$

68. $x^2 - 12xy + 36y^2$

69. $x^2 - 18x + 81$

70. $36y^2 - 12y + 1$

71. $y^2 + 18yx + 81x^2$

72. $9y^2 + 30yr + 25r^2$

73. $98r^2y^2 + 28ry + 2$

74. $4r^{10} + 4r^9 + r^8$

75. $98t^8 + 28t^7 + 2t^6$

76. $0.16t - t^3$

77. $2x^4 - 162$

78. $x^2 - 14x + 49 - 64y^2$

10.7 Solving Quadratic Equations by Factoring

10.7.1 Introduction

We have learned how to factor trinomials like $x^2 + 5x + 6$ into $(x + 2)(x + 3)$. This skill can be used to solve an equation like $x^2 + 5x + 6 = 0$, which is a quadratic equation. Note that we solved equations like this in Chapter 7, but here we will use factoring, a new method.

Definition 10.7.1 Quadratic Equation. A **quadratic equation** is is an equation in the form $ax^2 + bx + c = 0$ with $a \neq 0$. We also consider equations such as $x^2 = x + 3$ and $5x^2 + 3 = (x + 1)^2 + (x + 1)(x - 3)$ to be quadratic equations, because we can expand any multiplication, add or subtract terms from both sides, and combine like terms to get the form $ax^2 + bx + c = 0$. The form $ax^2 + bx + c = 0$ is called the **standard form** of a quadratic equation. ◊

Before we begin exploring the method of solving quadratic equations by factoring, we'll identify what types of equations are quadratic and which are not.

Checkpoint 10.7.3 Identify which of the items are quadratic equations.

a. The equation $2x^2 + 5x = 7$ (☐ is ☐ is not) a quadratic equation.

b. The equation $5 - 2x = 3$ (☐ is ☐ is not) a quadratic equation.

c. The equation $15 - x^3 = 3x^2 + 9x$ (☐ is ☐ is not) a quadratic equation.

d. The equation $(x + 3)(x - 4) = 0$ (☐ is ☐ is not) a quadratic equation.

e. The equation $x(x + 1)(x - 1) = 0$ (☐ is ☐ is not) a quadratic equation.

f. The expression $x^2 - 5x + 6$ (☐ is ☐ is not) a quadratic equation.

g. The equation $(2x - 3)(x + 5) = 12$ (☐ is ☐ is not) a quadratic equation.

Explanation.

a. The equation $2x^2 + 5x = 7$ *is* a quadratic equation. To write it in standard form, simply subtract 7 from both sides.

b. The equation $5 - 2x = 3$ *is not* quadratic. It is a linear equation.

c. The equation $15 + x^3 = 3x^2 + 9x$ *is not* a quadratic equation because of the x^3 term.

d. The equation $(x + 3)(x - 4) = 0$ *is* a quadratic equation. If we expand the left-hand side of the equation, we would get something in standard form.

e. The equation $x(x + 1)(x - 1) = 0$ *is not* a quadratic equation. If we expanded the left-hand side of the equation, we would have an expression with an x^3 term, which automatically makes it not quadratic.

f. The expression $x^2 - 5x + 6$ *is not* a quadratic equation; it's not an *equation* at all. Instead, this is a quadratic *expression*.

g. The equation $(2x - 3)(x + 5) = 12$ *is* a quadratic equation. Multiplying out the left-hand side, and subtracting 12 form both sides, we would have a quadratic equation in standard form.

Now we'll look at an application that demonstrates the need and method for solving a quadratic equation by factoring.

Nita is in a physics class that launches a tennis ball from a rooftop 80 feet above the ground. They fire it directly upward at a speed of 64 feet per second and measure the time it takes for the ball to hit the ground below. We can model the height of the tennis ball, h, in feet, with the quadratic equation

$$h = -16t^2 + 64t + 80,$$

where t represents the time in seconds after the launch. Using the model we can predict when the ball will hit the ground.

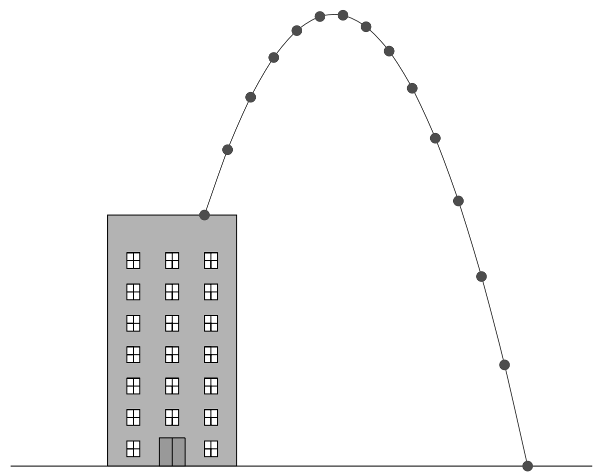


Figure 10.7.4: A Diagram of the Ball Thrown from the Roof

The ground has a height of 0, or $h = 0$. We will substitute 0 for h in the equation and we have

$$-16t^2 + 64t + 80 = 0$$

We need to solve this quadratic equation for t to find when the ball will hit the ground.

The key strategy for solving a *linear* equation is to separate the variable terms from the constant terms on either side of the equal sign. It turns out that this same method *will not work* for quadratic equations. Fortunately, we already have spent a good amount of time discussing a method that *will* work: factoring. If we can factor the polynomial on the left-hand side, we will be on the home stretch to solving the whole equation.

We will look for a common factor first, and see that we can factor out -16. Then we can finish factoring the trinomial:

$$\begin{aligned} -16t^2 + 64t + 80 &= 0 \\ -16(t^2 - 4t - 5) &= 0 \\ -16(t + 1)(t - 5) &= 0 \end{aligned}$$

In order to finish solving the equation, we need to understand the following property. This property explains why it was *incredibly important* to not move the 80 in our example over to the other side of the equation before trying to factor.

Fact 10.7.5 Zero Product Property. *If the product of two or more numbers is equal to zero, then at least one of the numbers must be zero.*

One way to understand this property is to think about the equation $a \cdot b = 0$. Maybe $b = 0$, because that would certainly cause the equation to be true. But suppose that $b \neq 0$. Then it is safe to divide both sides by b, and the resulting equation says that $a = 0$. So no matter what, either $a = 0$ or $b = 0$.

To understand this property more, let's look at a few products:

$$4 \cdot 7 = 28 \qquad 4 \cdot 0 = 0 \qquad 4 \cdot 7 \cdot 3 = 84$$

$$0 \cdot 7 = 0 \qquad -4 \cdot 0 = 0 \qquad 4 \cdot 0 \cdot 3 = 0$$

When none of the factors are 0, the result is never 0. The only way to get a product of 0 is when one of the factors is 0. This property is unique to the number 0 and can be used no matter how many numbers are multiplied together.

Now we can see the value of factoring. We have three factors in our equation

$$-16(t+1)(t-5) = 0.$$

The first factor is the number -16. The second and third factors, $t+1$ and $t-5$, are expressions that represent numbers. Since the product of the three factors is equal to 0, one of the factors must be zero.

Since -16 is not 0, either $t+1$ or $t-5$ must be 0. This gives us two equations to solve:

$$\begin{aligned} t+1 &= 0 & \text{or} && t-5 &= 0 \\ t+1-1 &= 0-1 & \text{or} && t-5+5 &= 0+5 \\ t &= -1 & \text{or} && t &= 5 \end{aligned}$$

We have found two solutions, -1 and 5. A quadratic expression will have at most two linear factors, not including any constants, so it can have up to two solutions.

Let's check each of our two solutions -1 and 5:

$$\begin{aligned} -16t^2+64t+80 &= 0 & -16t^2+64t+80 &= 0 \\ -16(-1)^2+64(-1)+80 &\stackrel{?}{=} 0 & -16(5)^2+64(5)+80 &\stackrel{?}{=} 0 \\ -16(1)-64+80 &\stackrel{?}{=} 0 & -16(25)+320+80 &\stackrel{?}{=} 0 \\ -16-64+80 &\stackrel{?}{=} 0 & -400+320+80 &\stackrel{?}{=} 0 \\ 0 &\stackrel{\checkmark}{=} 0 & 0 &\stackrel{\checkmark}{=} 0 \end{aligned}$$

We have verified our solutions. While there are two solutions to the equation, the solution -1 is not relevant to this physics model because it is a negative time which would tell us something about the ball's height *before* it was launched. The solution 5 does make sense. According to the model, the tennis ball will hit the ground 5 seconds after it is launched.

10.7.2 Further Examples

We'll now look at further examples of solving quadratic equations by factoring. The general process is outlined here:

Process 10.7.6 Solving Quadratic Equations by Factoring.

Simplify *Simplify the equation using distribution and by combining like terms.*

Isolate *Move all terms onto one side of the equation so that the other side has 0.*

Factor *Factor the quadratic expression.*

Apply the Zero Product Property *Apply the Zero Product Property.*

Solve *Solve the equation(s) that result after the zero product property was applied.*

Example 10.7.7 Solve $x^2 - 5x - 14 = 0$ by factoring.

Explanation.

$$x^2 - 5x - 14 = 0$$
$$(x-7)(x+2) = 0$$

$$\begin{aligned} x - 7 &= 0 & \text{or} && x + 2 &= 0 \\ x - 7 + 7 &= 0 + 7 & \text{or} && x + 2 - 2 &= 0 - 2 \\ x &= 7 & \text{or} && x &= -2 \end{aligned}$$

The solutions are -2 and 7, so the solution set is written as $\{-2, 7\}$.

Example 10.7.8 Solve $x^2 - 5x - 14 = 0$ by using graphing technology.

Explanation. We have already solved the equation $x^2 - 5x - 14 = 0$ by factoring in Example 10.7.7, and now we can analyze the significance of the solutions graphically. What will -2 and 7 mean on the graph?

To solve this equation graphically, we first make a graph of $y = x^2 - 5x - 14$ and of $y = 0$. Both of these graphs are shown in Figure 10.7.9 in an appropriate window.

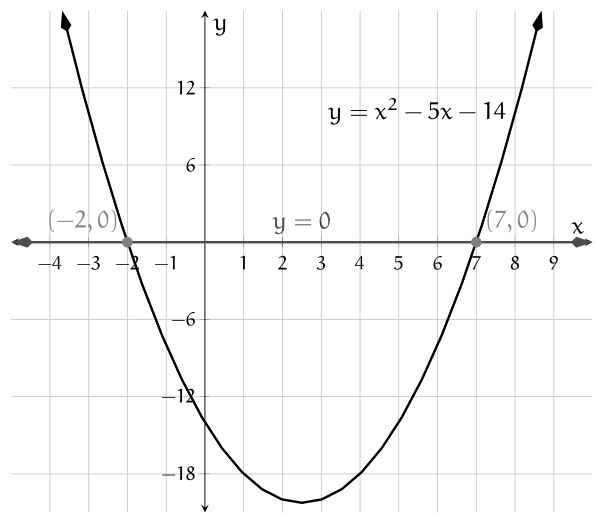


Figure 10.7.9: A graph of $y = x^2 - 5x - 14$ and $y = 0$.

From the graph provided we can see that the solutions (the x-values where the graphs intersect) are -2 and 7.

If the two factors of a polynomial happen to be the same, the equation will only have one solution. Let's look at an example of that.

Example 10.7.10 A Quadratic Equation with Only One Solution. Solve $x^2 - 10x + 25 = 0$ by factoring.

Explanation.

$$\begin{aligned} x^2 - 10x + 25 &= 0 \\ (x-5)(x-5) &= 0 \\ (x-5)^2 &= 0 \\ x - 5 &= 0 \end{aligned}$$

$$x - 5 + 5 = 0 + 5$$
$$x = 5$$

The solution is 5, so the solution set is written as $\{5\}$.

While we are examining this problem, let's compare the algebraic solution to a graphical solution.

To solve this equation graphically, we first make a graph of $y = x^2 - 10x + 25$ and of $y = 0$. Both of these graphs are shown in Figure 10.7.11 in an appropriate window.

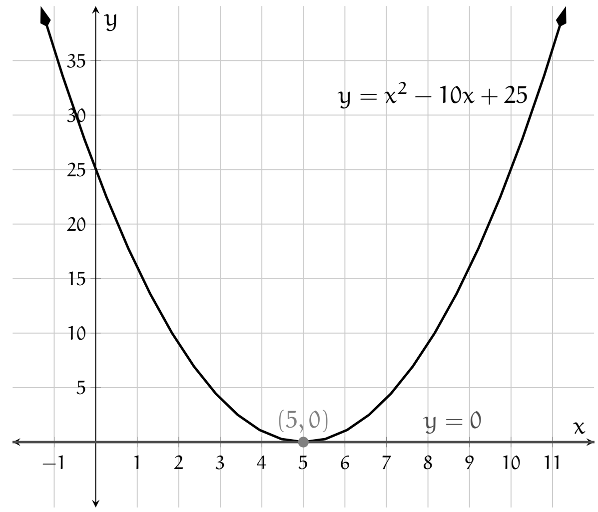


Figure 10.7.11: A graph of $y = x^2 - 10x + 25$ and $y = 0$.

From the graph provided, we can see that the reason that the equation $x^2 - 10x + 25 = 0$ has only one solution is that the parabola $y = x^2 - 10x + 25$ touches the line $y = 0$ only once. So again, the solution is 5.

Example 10.7.12 Factor Out a Common Factor. Solve $5x^2 + 55x + 120 = 0$ by factoring.

Explanation. Note that the terms are all divisible by 5, so we can factor that out to start.

$$5x^2 + 55x + 120 = 0$$
$$5(x^2 + 11x + 24) = 0$$
$$5(x + 8)(x + 3) = 0$$

$$x + 8 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = -8 \quad \text{or} \quad x = -3$$

The solution set is $\{-8, -3\}$.

Example 10.7.13 Factoring Using the AC Method. Solve $3x^2 - 7x + 2 = 0$ by factoring.

Explanation. Recall that we multiply $3 \cdot 2 = 6$ and find a factor pair that multiplies to 6 and adds to -7. The factors are -6 and -1. We use the two factors to replace the middle term with $-6x$ and $-x$.

$$3x^2 - 7x + 2 = 0$$
$$3x^2 - 6x - x + 2 = 0$$
$$(3x^2 - 6x) + (-x + 2) = 0$$

$$\begin{aligned}3x(x-2)-1(x-2)&=0\\(3x-1)(x-2)&=0\end{aligned}$$

$$\begin{aligned}3x-1&=0 &\quad\text{or}\quad && x-2&=0\\3x&=1 &\quad\text{or}\quad && x&=2\\x&=\frac{1}{3} &\quad\text{or}\quad && x&=2\end{aligned}$$

The solution set is $\left\{\frac{1}{3},2\right\}$.

So far the equations have been written in standard form, which is

$$ax^2+bx+c=0$$

If an equation is not given in standard form then we must rearrange it in order to use the Zero Product Property.

Example 10.7.14 Writing in Standard Form. Solve $x^2-10x=24$ by factoring.

Explanation. There is nothing like the Zero Product Property for the number 24. We must have a 0 on one side of the equation to solve quadratic equations using factoring.

$$\begin{aligned}x^2-10x&=24\\x^2-10x-24&=24-24\\x^2-10x-24&=0\\(x-12)(x+2)&=0\end{aligned}$$

$$\begin{aligned}x-12&=0 &\quad\text{or}\quad && x+2&=0\\x&=12 &\quad\text{or}\quad && x&=-2\end{aligned}$$

The solution set is $\{-2,12\}$.

Example 10.7.15 Writing in Standard Form. Solve $(x+4)(x-3)=18$ by factoring.

Explanation. Again, there is nothing like the Zero Product Property for a number like 18. We must expand the left side and subtract 18 from both sides.

$$\begin{aligned}(x+4)(x-3)&=18\\x^2+x-12&=18\\x^2+x-12-18&=18-18\\x^2+x-30&=0\\(x+6)(x-5)&=0\end{aligned}$$

$$\begin{aligned}x+6&=0 &\quad\text{or}\quad && x-5&=0\\x&=-6 &\quad\text{or}\quad && x&=5\end{aligned}$$

The solution set is $\{-6,5\}$.

Example 10.7.16 A Quadratic Equation with No Constant Term. Solve $2x^2 = 5x$ by factoring.

Explanation. It may be tempting to divide both sides of the equation by x. But x is a variable, and for all we know, maybe $x = 0$. So it is not safe to divide by x. As a general rule, never divide an equation by a variable in the solving process. Instead, we will put the equation in standard form.

$$\begin{aligned} 2x^2 &= 5x \\ 2x^2 - 5x &= 5x - 5x \\ 2x^2 - 5x &= 0 \end{aligned}$$

We can factor out x.

$$x(2x-5) = 0$$

$$\begin{aligned} x &= 0 & \text{or} && 2x-5 &= 0 \\ x &= 0 & \text{or} && 2x &= 5 \\ x &= 0 & \text{or} && x &= \frac{5}{2} \end{aligned}$$

The solution set is $\left\{0, \frac{5}{2}\right\}$. In general, if a quadratic equation does not have a constant term, then 0 will be one of the solutions.

While we are examining this problem, let's compare the algebraic solution to a graphical solution.

To solve this equation graphically, we first make a graph of $y = 2x^2$ and of $y = 5x$. Both of these graphs are shown in Figure 10.7.17 in an appropriate window.

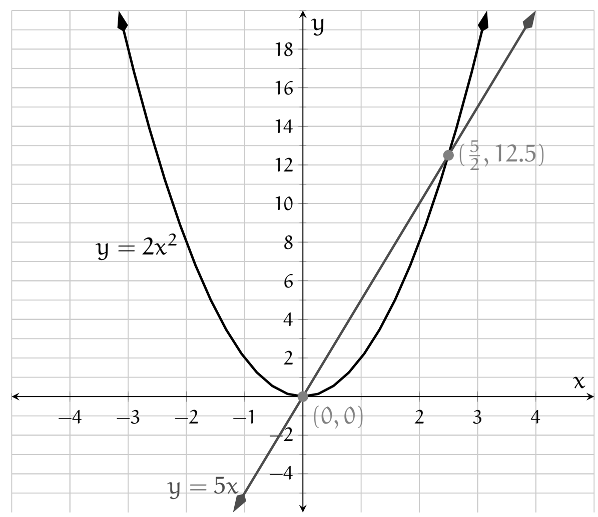


Figure 10.7.17: A graph of $y = 2x^2$ and $y = 5x$.

From the graph provided, we can see that the equation $2x^2 = 5x$ has two solutions because the graph of $y = 2x^2$ crosses the graph of $y = 5x$ twice. Those solutions, the x-values where the graphs cross, appear to be 0 and $\frac{5}{2} = 2.5$.

Example 10.7.18 Factoring a Special Polynomial. Solve $x^2 = 9$ by factoring.

Explanation. We can put the equation in standard form and use factoring. In this case, we find a difference of squares.

$$\begin{aligned} x^2 &= 9 \\ x^2 - 9 &= 0 \\ (x+3)(x-3) &= 0 \end{aligned}$$

$$\begin{aligned} x+3 &= 0 && \text{or} && x-3 &= 0 \\ x &= -3 && \text{or} && x &= 3 \end{aligned}$$

The solution set is $\{-3, 3\}$.

Example 10.7.19 Solving an Equation with a Higher Degree. Solve $2x^3 - 10x^2 - 28x = 0$ by factoring.

Explanation. Although this equation is not quadratic, it does factor so we can solve it by factoring.

$$\begin{aligned} 2x^3 - 10x^2 - 28x &= 0 \\ 2x(x^2 - 5x - 14) &= 0 \\ 2x(x-7)(x+2) &= 0 \end{aligned}$$

$$\begin{aligned} 2x &= 0 && \text{or} && x-7 &= 0 && \text{or} && x+2 &= 0 \\ x &= 0 && \text{or} && x &= 7 && \text{or} && x &= -2 \end{aligned}$$

The solution set is $\{-2, 0, 7\}$.

10.7.3 Applications

Example 10.7.20 Kicking it on Mars.
Some time in the recent past, Filip traveled to Mars for a vacation with his kids, Henrik and Karina, who wanted to kick a soccer ball around in the comparatively reduced gravity. Karina stood at point K and kicked the ball over her dad standing at point F to Henrik standing at point H. The height of the ball off the ground, h in feet, can be modeled by the equation $h = -0.01\left(x^2 - 70x - 1800\right)$, where x is how far to the right the ball is from Filip. Note that distances to the left of Filip will be negative.

a. Find out how high the ball was above the ground when it passed over Filip's head.

b. Find the distance from Karina to Henrik.

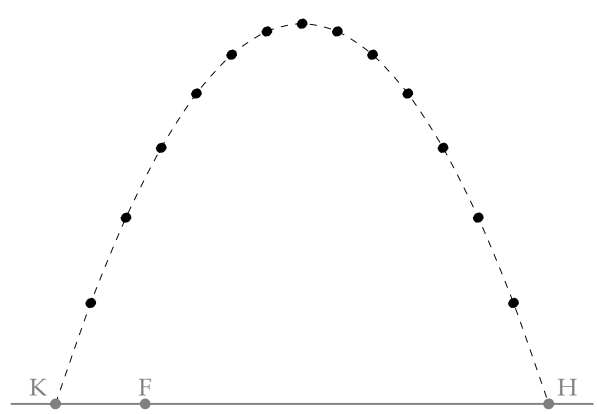


Figure 10.7.21: A Soccer Kick on Mars

Explanation.

a. The ball was neither left nor right of Filip when it went over him, so $x = 0$. Plugging that value into our equation for x,

$$\begin{aligned} h &= -0.01\left(0^2 - 70(0) - 1800\right) \\ &= -0.01(-1800) \\ &= 18 \end{aligned}$$

It seems that the soccer ball was 18 feet above the ground when it flew over Filip.

b. The distance from Karina to Henrik is the same as the distance from point K to point H. These are the horizontal intercepts of the graph of the given formula: $h = -0.01\left(x^2 - 70x - 1800\right)$. To find the horizontal intercepts, set $h = 0$ and solve for x.

$$0 = -0.01\left(x^2 - 70x - 1800\right)$$

Note that we can divide by -0.01 on both sides of the equation to simplify.

$$\begin{aligned} 0 &= x^2 - 70x - 1800 \\ 0 &= (x - 90)(x + 20) \end{aligned}$$

So, either:

$$\begin{aligned} x - 90 &= 0 &\quad\text{or}\quad&& x + 20 &= 0 \\ x &= 90 &\quad\text{or}\quad&& x &= -20 \end{aligned}$$

Since the x-values are how far right or left the points are from Filip, Karina is standing 20 feet left of Filip and Henrik is standing 90 feet right of Filip. Thus, the two kids are 110 feet apart.

It is worth noting that if this same kick, with same initial force at the same angle, took place on Earth, the ball would have traveled less than 30 feet from Karina before landing!

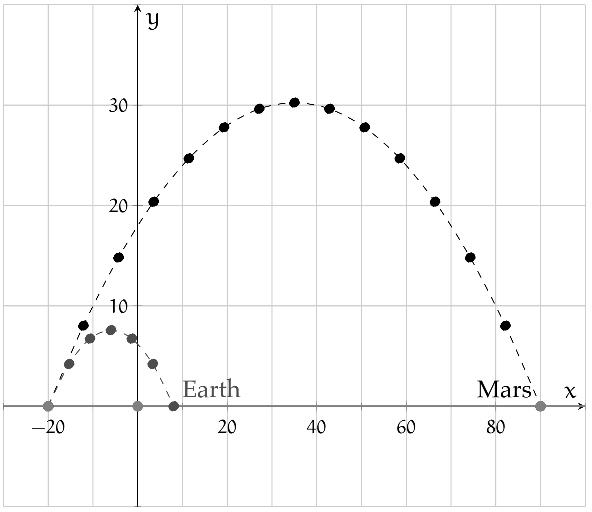


Figure 10.7.22: A Soccer Kick on Mars and the Same Kick on Earth

Example 10.7.23 An Area Application. Rajesh has a hot tub and he wants to build a deck around it. The hot tub is 7 ft by 5 ft and it is covered by a roof that is 99 ft^2. How wide can he make the deck so that it will be covered by the roof?

Explanation. We will define x to represent the width of the deck (in feet). Here is a diagram to help us understand the scenario.

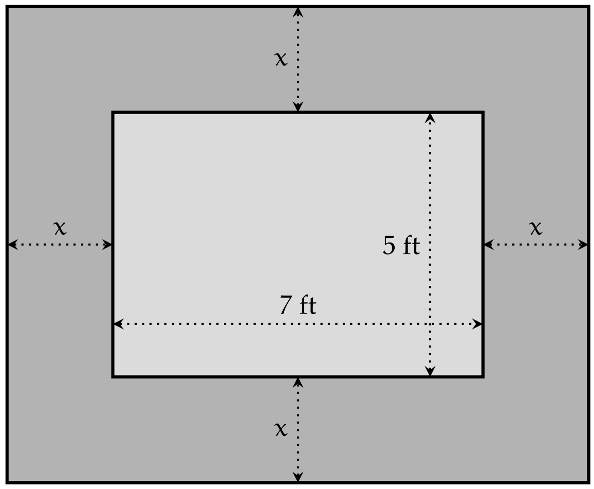


Figure 10.7.24: Diagram for the Deck

The overall length is $7 + 2x$ feet, because Rajesh is adding x feet on each side. Similarly, the overall width is $5 + 2x$ feet.

The formula for the area of a rectangle is area = length · width. Since the total area of the roof is 99 ft^2, we can write and solve the equation:

$$\begin{aligned}
(7 + 2x)(5 + 2x) &= 99 \\
4x^2 + 24x + 35 &= 99 \\
4x^2 + 24x + 35 - 99 &= 99 - 99 \\
4x^2 + 24x - 64 &= 0 \\
4(x^2 + 6x - 16) &= 0 \\
4(x + 8)(x - 2) &= 0
\end{aligned}$$

$$\begin{aligned}
x + 8 &= 0 & \text{or} && x - 2 &= 0 \\
x &= -8 & \text{or} && x &= 2
\end{aligned}$$

Since a length cannot be negative, we take $x = 2$ as the only applicable solution. Rajesh should make the deck 2 ft wide on each side to fit under the roof.

10.7.4 Reading Questions

1. If you use factoring to solve a polynomial equation, what number should be the only thing on one side of the equation?

2. When you are tyring to solve a quadratic equation where the leading coefficient is not 1m and you want to use factoring, and you have moved all terms to one side so that the other side is 0, what should you look for before trying anything else?

10.7.5 Exercises

Warmup and Review Factor the given polynomial.

1. $10y - 60$
2. $7y + 63$
3. $r^2 + 3r - 18$
4. $r^2 + 6r - 27$
5. $3t^2 - 4t + 1$
6. $2t^2 - 11t + 15$
7. $30x^2 + 12x + 42$
8. $27x^2 - 72x + 72$
9. $9y^4 - 100$
10. $100y^4 - 9$

Solve Quadratic Equations by Factoring Solve the equation.

11. $(x - 1)(x + 3) = 0$
12. $(x + 2)(x - 10) = 0$
13. $89(x + 4)(16x - 5) = 0$
14. $90(x + 7)(9x - 5) = 0$
15. $x^2 + 11x + 10 = 0$
16. $x^2 + 9x + 8 = 0$
17. $x^2 - 3x - 28 = 0$
18. $x^2 - 4x - 5 = 0$
19. $x^2 - 11x + 28 = 0$
20. $x^2 - 13x + 42 = 0$
21. $x^2 + 16x = -60$
22. $x^2 + 15x = -56$
23. $x^2 + 4x = 21$
24. $x^2 - x = 90$
25. $x^2 - 15x = -50$
26. $x^2 - 17x = -72$
27. $x^2 = 5x$
28. $x^2 = 3x$
29. $6x^2 = 36x$
30. $7x^2 = -49x$
31. $8x^2 = 3x$
32. $9x^2 = 7x$
33. $x^2 - 24x + 144 = 0$
34. $x^2 - 2x + 1 = 0$
35. $x^2 = 4x - 4$
36. $x^2 = 8x - 16$
37. $36x^2 = -60x - 25$
38. $49x^2 = -14x - 1$
39. $4x^2 = -25x - 36$
40. $4x^2 = -37x - 40$
41. $x^2 - 36 = 0$
42. $x^2 - 81 = 0$
43. $25x^2 - 144 = 0$
44. $25x^2 - 9 = 0$
45. $81x^2 = 121$
46. $25x^2 = 16$
47. $x(x + 11) = 12$
48. $x(x - 6) = 16$
49. $x(4x + 33) = 70$
50. $x(3x + 20) = 63$
51. $(x - 5)(x + 2) = -6$
52. $(x - 1)(x + 4) = -6$
53. $(x - 1)(3x + 4) = 2x^2 - 2$
54. $(x - 1)(4x + 7) = 3x^2 - 9$
55. $x(x - 12) = -3(2x + 3)$
56. $x(x - 4) = -(2x + 1)$
57. $49x^2 + 84x + 36 = 0$
58. $9x^2 + 24x + 16 = 0$
59. $(x + 3)\left(x^2 + 17x + 72\right) = 0$
60. $(x - 9)\left(x^2 + 15x + 50\right) = 0$
61. $x\left(x^2 - 1\right) = 0$
62. $x\left(x^2 - 4\right) = 0$
63. $x^3 - 8x^2 + 15x = 0$
64. $x^3 - 13x^2 + 30x = 0$

Quadratic Equation Application Problems

65. Two numbers' sum is 2, and their product is -35. Find these two numbers.
These two numbers are $\boxed{}$.

66. Two numbers' sum is -2, and their product is -35. Find these two numbers.
These two numbers are $\boxed{}$.

67. A rectangle's base is 9 cm longer than its height. The rectangle's area is 136 cm^2. Find this rectangle's dimensions.

The rectangle's height is ______.

The rectangle's base is ______.

68. A rectangle's base is 8 cm longer than its height. The rectangle's area is 128 cm^2. Find this rectangle's dimensions.

The rectangle's height is ______.

The rectangle's base is ______.

69. A rectangle's base is 8 in shorter than three times its height. The rectangle's area is 35 in^2. Find this rectangle's dimensions.

The rectangle's height is ______.

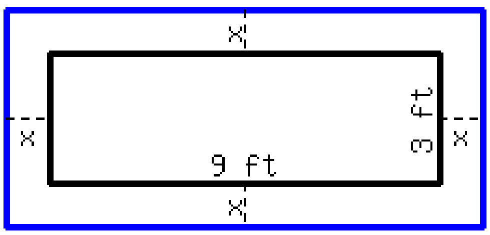
The rectangle's base is ______.

70. A rectangle's base is 2 in shorter than five times its height. The rectangle's area is 16 in^2. Find this rectangle's dimensions.

The rectangle's height is ______.

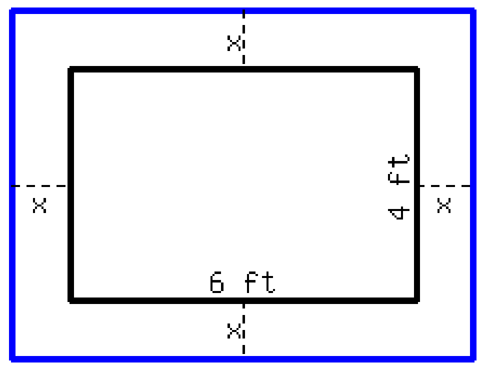
The rectangle's base is ______.

71. There is a rectangular lot in the garden, with 9 ft in length and 3 ft in width. You plan to expand the lot by an equal length around its four sides, and make the area of the expanded rectangle 55 ft^2. How long should you expand the original lot in four directions?



You should expand the original lot by ______ in four directions.

72. There is a rectangular lot in the garden, with 6 ft in length and 4 ft in width. You plan to expand the lot by an equal length around its four sides, and make the area of the expanded rectangle 48 ft^2. How long should you expand the original lot in four directions?



You should expand the original lot by ______ in four directions.

Challenge

73. Give an example of a cubic equation that has three solutions: one solution is $x = 4$, the second solution is $x = -2$, and the third solution is $x = \frac{2}{3}$.

74. Solve for x in the equation $48x^{46} - 3x^{44} = 0$.

10.8 Factoring Chapter Review

10.8.1 Factoring out the GCF

In Section 10.1 we covered how to factor out the greatest common factor. Recall that the **greatest common factor** between two expressions is the largest factor that goes in evenly to both expressions.

Example 10.8.1 Finding the Greatest Common Factor. What is the greatest common factor between $12x^3y$ and $42x^2y^2$?

Explanation. Break down each of these into its factors:

$$12x^3y = (2 \cdot 2) \cdot 3 \cdot (x \cdot x \cdot x) \cdot y \qquad 42x^2y^2 = 2 \cdot 3 \cdot 7 \cdot (x \cdot x) \cdot (y \cdot y)$$

Identify the common factors:

$$12x^3y = \overset{\downarrow}{2} \cdot 2 \cdot \overset{\downarrow}{3} \cdot \overset{\downarrow}{x} \cdot \overset{\downarrow}{x} \cdot x \cdot \overset{\downarrow}{y} \qquad 42x^2y^2 = \overset{\downarrow}{2} \cdot \overset{\downarrow}{3} \cdot 7 \cdot \overset{\downarrow}{x} \cdot \overset{\downarrow}{x} \cdot \overset{\downarrow}{y} \cdot y$$

With 2, 3, two x's and a y in common, the greatest common factor is $6x^2y$.

Example 10.8.2 What is the greatest common factor between $18c^3y^2$ and $27y^3c$?

Explanation. Break down each into factors. You can definitely do this mentally with practice.

$$18c^3y^2 = 2 \cdot 3 \cdot 3 \cdot c \cdot c \cdot c \cdot y \cdot y \qquad 27y^3c = 3 \cdot 3 \cdot 3 \cdot y \cdot y \cdot y \cdot c$$

And take note of the common factors.

$$18c^3y^2 = 2 \cdot \overset{\downarrow}{3} \cdot \overset{\downarrow}{3} \cdot \overset{\downarrow}{c} \cdot c \cdot c \cdot \overset{\downarrow}{y} \cdot \overset{\downarrow}{y} \qquad 27y^3c = \overset{\downarrow}{3} \cdot \overset{\downarrow}{3} \cdot 3 \cdot \overset{\downarrow}{y} \cdot \overset{\downarrow}{y} \cdot y \cdot c$$

And so the GCF is $9y^2c$

Example 10.8.3 Factoring out the Greatest Common Factor. Factor out the GCF from the expression $32mn^2 - 24m^2n - 12mn$.

Explanation. To factor out the GCF from the expression $32mn^2 - 24m^2n - 12mn$, first note that the GCF to all three terms is $4mn$. Begin by writing that in front of a blank pair of parentheses and fill in the missing pieces.

$$\begin{aligned} 32mn^2 - 24m^2n - 12mn &= 4mn(\quad - \quad - \quad) \\ &= 4mn(8n - 6m - 3) \end{aligned}$$

Example 10.8.4 Factor out the GCF from the expression $14x^3 - 35x^2$.

Explanation. First note that the GCF of the terms in $14x^3 - 35x^2$ is $7x^2$. Factoring this out, we have:

$$\begin{aligned} 14x^3 - 35x^2 &= 7x^2\,(\quad - \quad) \\ &= 7x^2\,(2x - 5) \end{aligned}$$

Example 10.8.5 Factor out the GCF from the expression $36m^3n^2 - 18m^2n^5 + 24mn^3$.

Explanation. First note that the GCF of the terms in $36m^3n^2 - 18m^2n^5 + 24mn^3$ is $6mn^2$. Factoring this out, we have:

$$\begin{aligned} 36m^3n^2 - 18m^2n^5 + 24mn^3 &= 6mn^2\left(\quad - \quad + \quad \right) \\ &= 6mn^2\left(6m^2 - 3mn^3 + 4n\right) \end{aligned}$$

Example 10.8.6 Factor out the GCF from the expression $42f^3w^2 - 8w^2 + 9f^3$.

Explanation. First note that the GCF of the terms in $42f^3w^2 - 8w^2 + 9f^3$ is 1, so we call the expression prime. The only way to factor the GCF out of this expression is:

$$42f^3w^2 - 8w^2 + 9f^3 = 1\left(42f^3w^2 - 8w^2 + 9f^3\right)$$

10.8.2 Factoring by Grouping

In Section 10.2 we covered how to factor by grouping. Recall that factoring using grouping is used on four-term polynomials, and also later in the AC method in Section 10.4. Begin by grouping two pairs of terms and factoring out their respective GCF; if all is well, we should be left with two matching pieces in parentheses that can be factored out in their own right.

Example 10.8.7 Factor the expression $2x^3 + 5x^2 + 6x + 15$ using grouping.

Explanation.

$$\begin{aligned} 2x^3 + 5x^2 + 6x + 15 &= \left(2x^3 + 5x^2\right) + (6x + 15) \\ &= x^2(2x+5) + 3(2x+5) \\ &= \left(x^2+3\right)(2x+5) \end{aligned}$$

Example 10.8.8 Factor the expression $2xy - 3x - 8y + 12$ using grouping.

Explanation.

$$\begin{aligned} 2xy - 3x + 8y - 12 &= (2xy - 3x) + (-8y + 12) \\ &= x(2y-3) - 4(2y-3) \\ &= (x-4)(2y-3) \end{aligned}$$

Example 10.8.9 Factor the expression $xy - 2 - 2x + y$ using grouping.

Explanation. This is a special example because if we try to follow the algorithm without considering the bigger context, we will fail:

$$xy - 2 - 2x + y = (xy - 2) + (-2x + y)$$

Note that there is no common factor in either grouping, besides 1, but the groupings themselves don't match. We should now recognize that whatever we are doing isn't working and try something else. It turns out that this polynomial *isn't* prime; all we need to do is rearrange the polynomial into standard form where the degrees decrease from left to right before grouping.

$$\begin{aligned} xy - 2 - 2x + y &= xy - 2x + y - 2 \\ &= (xy - 2x) + (y - 2) \\ &= x(y-2) + 1(y-2) \end{aligned}$$

$$= (x+1)(y-2)$$

Example 10.8.10 Factor the expression $15m^2 - 3m - 10mn + 2n$ using grouping.
Explanation.

$$\begin{aligned} 15m^2 - 3m - 10mn + 2n &= \left(15m^2 - 3m\right) + (-10mn + 2n) \\ &= 3m\,(5m-1) - 2n\,(5m-1) \\ &= (3m-2n)(5m-1) \end{aligned}$$

10.8.3 Factoring Trinomials with Leading Coefficient 1

In Section 10.3 we covered factoring expressions that look like $x^2 + bx + c$. The trick was to look for two numbers whose product was c and whose sum was b. Always remember to look for a greatest common factor first, before looking for factor pairs.

Example 10.8.11 Answer the questions to practice for the factor pairs method.

a. What two numbers multiply to be 6 and add to be 5?

b. What two numbers multiply to be −6 and add to be 5?

c. What two numbers multiply to be −6 and add to be −1?

d. What two numbers multiply to be 24 and add to be −10?

e. What two numbers multiply to be −24 and add to be 2?

f. What two numbers multiply to be −24 and add to be −5?

g. What two numbers multiply to be 420 and add to be 44?

h. What two numbers multiply to be −420 and add to be −23?

i. What two numbers multiply to be 420 and add to be −41?

Explanation.

a. What two numbers multiply to be 6 and add to be 5? The numbers are 2 and 3.

b. What two numbers multiply to be −6 and add to be 5? The numbers are 6 and −1.

c. What two numbers multiply to be −6 and add to be −1? The numbers are −3 and 2.

d. What two numbers multiply to be 24 and add to be −10? The numbers are −6 and −4.

e. What two numbers multiply to be −24 and add to be 2? The numbers are 6 and −4.

f. What two numbers multiply to be −24 and add to be −5? The numbers are −8 and 3.

g. What two numbers multiply to be 420 and add to be 44? The numbers are 30 and 14.

h. What two numbers multiply to be −420 and add to be −23? The numbers are −35 and 12.

i. What two numbers multiply to be 420 and add to be −41? The numbers are −20 and −21.

Note that for parts g–i, the factors of 420 are important. Below is a table of factors of 420 which will make it much clearer how the answers were found. To generate a table like this, we start with 1, and we work our way up the factors of 420.

Factor Pair
$1 \cdot 420$
$2 \cdot 210$
$3 \cdot 140$
$4 \cdot 105$

Factor Pair
$5 \cdot 84$
$6 \cdot 70$
$7 \cdot 60$
$10 \cdot 42$

Factor Pair
$12 \cdot 35$
$14 \cdot 30$
$15 \cdot 28$
$20 \cdot 21$

It is now much easier to see how to find the numbers in question. For example, to find two numbers that multiply to be -420 and add to be -23, look in the table for two factors that are 23 apart and assign a negative sign appropriately. As we found earlier, the numbers that are 23 apart are 12 and 35, and making the larger one negative, we have our answer: 12 and -35.

Example 10.8.12 Factor the expression $x^2 - 3x - 28$

Explanation. To factor the expression $x^2 - 3x - 28$, think of two numbers that multiply to be -28 and add to be -3. In the Section 10.3, we created a table of all possibilities of factors, like the one shown, to be sure that we never missed the right numbers; however, we encourage you to try this mentally for most problems.

Factor Pair	Sum of the Pair
$-1 \cdot 28$	27
$-2 \cdot 14$	12
$-4 \cdot 7$	3 (close; wrong sign)

Factor Pair	Sum of the Pair
$1 \cdot (-28)$	-27
$2 \cdot (-14)$	-12
$4 \cdot (-7)$	-3 (what we wanted)

Since the two numbers in question are 4 and -7 that means that

$$x^2 - 3x - 28 = (x+4)(x-7)$$

Remember that you can always multiply out your factored expression to verify that you have the correct answer. We will use the FOIL expansion.

$$\begin{aligned}(x+4)(x-7) &= x^2 - 7x + 4x - 28 \\ &\stackrel{\checkmark}{=} x^2 - 3x - 28\end{aligned}$$

Example 10.8.13 Factoring in Stages. Completely factor the expression $4x^3 - 4x^2 - 120x$.

Explanation. Remember that some expressions require more than one step to completely factor. To factor $4x^3 - 4x^2 - 120x$, first, always look for any GCF; after that is done, consider other options. Since the GCF is $4x$, we have that

$$4x^3 - 4x^2 - 120x = 4x\left(x^2 - x - 30\right).$$

Now the factor inside parentheses might factor further. The key here is to consider what two numbers multiply to be -30 and add to be -1. In this case, the answer is -6 and 5. So, to completely write the factorization, we have:

$$\begin{aligned}4x^3 - 4x^2 - 120x &= 4x\left(x^2 - x - 30\right) \\ &= 4x(x-6)(x+5)\end{aligned}$$

Example 10.8.14 Factoring Expressions with Higher Powers. Completely factor the expression $p^{10} - 6p^5 - 72$.

Explanation. If we have a trinomial with an even exponent on the leading term, and the middle term has an exponent that is half the leading term exponent, we can still use the factor pairs method. To factor $p^{10} - 6p^5 - 72$, we note that the middle term exponent 5 is half of the leading term exponent 10, and that two numbers that multiply to be -72 and add to be -6 are -12 and 6. So the factorization of the expression is

$$p^{10} - 6p^5 - 72 = \left(p^5 - 12\right)\left(p^5 + 6\right)$$

Example 10.8.15 Factoring Expressions with Two Variables. Completely factor the expression $x^2 - 3xy - 70y^2$.

Explanation. If an expression has two variables, like $x^2 - 3xy - 70y^2$, we pretend for a moment that the expression is $x^2 - 3x - 70$. To factor this expression we ask ourselves "what two numbers multiply to be -70 and add to be -3?" The two numbers in question are 7 and -10. So $x^2 - 3x - 70$ factors as $(x + 7)(x - 10)$.

To go back to the original problem now, make the two numbers $7y$ and $-10y$. So, the full factorization is

$$x^2 - 3xy - 70y^2 = (x + 7y)(x - 10y)$$

With problems like this, it is important to verify the your answer to be sure that all of the variables ended up where they were supposed to. So, to verify, FOIL your answer.

$$\begin{aligned}(x + 7y)(x - 10y) &= x^2 - 10xy + 7yx - 70y^2 \\ &= x^2 - 10xy + 7xy - 70y^2 \\ &\stackrel{\checkmark}{=} x^2 - 3xy - 70y^2\end{aligned}$$

Example 10.8.16 Completely factor the expressions.

a. $x^2 - 11x + 30$

b. $-s^2 + 3s + 28$

c. $g^2 - 3g - 24$

d. $w^2 - wr - 30r^2$

e. $z^8 + 2z^4 - 63$

Explanation.

a. $x^2 - 11x + 30 = (x - 6)(x - 5)$

b. $\begin{aligned}-s^2 + 3s + 28 &= -\left(s^2 - 3s - 28\right) \\ &= -(s - 7)(s + 4)\end{aligned}$

c. $g^2 - 3g - 24$ is prime. No two integers multiply to be -24 and add to be -3.

d. $w^2 - wr - 30r^2 = (w - 6r)(w + 5r)$

e. $z^8 + 2z^4 - 63 = \left(z^4 - 7\right)\left(z^4 + 9\right)$

10.8.4 Factoring Trinomials with Non-Trivial Leading Coefficient

In Section 10.4 we covered factoring trinomials of the form $ax^2 + bx + c$ when $a \neq 1$ using the AC method.

Example 10.8.17 Using the AC Method. Completely factor the expression $9x^2 - 6x - 8$.

Explanation. To factor the expression $9x^2 - 6x - 8$, we first find ac:

1. $9 \cdot (-8) = -72$.

2. Examine factor pairs that multiply to -72, looking for a pair that sums to -6:

Factor Pair	Sum of the Pair
$1 \cdot -72$	-71
$2 \cdot -36$	-34
$3 \cdot -24$	-21
$4 \cdot -18$	-14
$6 \cdot -12$	-6
$8 \cdot -9$	(no need to go this far)

Factor Pair	Sum of the Pair
$-1 \cdot 72$	(no need to go this far)
$-2 \cdot 36$	(no need to go this far)
$-3 \cdot 24$	(no need to go this far)
$-4 \cdot 18$	(no need to go this far)
$-6 \cdot 12$	(no need to go this far)
$-8 \cdot 9$	(no need to go this far)

3. Intentionally break up the -6 as $6 + (-12)$ and then factor using grouping:

$$\begin{aligned} 9x^2 \overbrace{-6x} -8 &= 9x^2 \overbrace{+6x - 12x} -8 \\ &= \left(9x^2 + 6x\right) + (-12x - 8) \\ &= 3x(3x+2) - 4(3x+2) \\ &= (3x+2)(3x-4) \end{aligned}$$

Example 10.8.18 Completely factor the expression $3x^2 + 5x - 6$.

Explanation. First note that there is no GCF besides 1 and that $ac = -18$. To look for two factors of -18 that add up to 5, we will make a factor pair table.

Factor Pair	Sum of the Pair
$1 \cdot -18$	-17
$2 \cdot -9$	-7
$3 \cdot -6$	-3

Factor Pair	Sum of the Pair
$-1 \cdot 18$	17
$-2 \cdot 9$	7
$-3 \cdot 6$	3

Since none of the factor pairs of -18 sum to 5, we must conclude that this trinomial is prime. The only way to factor it is $3x^2 + 5x - 6 = 1\left(3x^2 + 5x - 6\right)$.

Example 10.8.19 Completely factor the expression $3y^2 + 20y - 63$.

Explanation. First note that $ac = -189$. Looking for two factors of -189 that add up to 20, we find 27 and -7. Breaking up the $+20$ into $+27 - 7$, we can factor using grouping.

$$\begin{aligned} 3y^2 \overbrace{+20y} -63 &= 3y^2 \overbrace{+27y - 7y} -63 \\ &= \left(3y^2 + 27y\right) + (-7y - 63) \\ &= 3y(y+9) - 7(y+9) \\ &= (y+9)(3y-7) \end{aligned}$$

Example 10.8.20 Factoring in Stages with the AC Method. Completely factor the expression $8y^3 + 54y^2 + 36y$.

Explanation. Recall that some trinomials need to be factored in stages: the first stage is always to factor out the GCF. To factor $8y^3 + 54y^2 + 36y$, first note that the GCF of the three terms in the expression is $2y$. Then apply the AC method:

$$8y^3 + 54y^2 + 36y = 2y\left(4y^2 + 27y + 18\right)$$

Now we find $ac = 4 \cdot 18 = 72$. What two factors of 72 add up to 27? After checking a few numbers, we find that 3 and 24 fit the requirements. So:

$$\begin{aligned}
&= 2y\left(4y^2 \overbrace{+27y} +18\right)\\
&= 2y\left(4y^2 \overbrace{+3y+24y} +18\right)\\
&= 2y\left(\left(4y^2+3y\right)+(24y+18)\right)\\
&= 2y\left(y(4y+3)+6(4y+3)\right)\\
&= 2y(4y+3)(y+6)
\end{aligned}$$

Example 10.8.21 Completely factor the expression $18x^3 + 26x^2 + 4x$.

Explanation. First note that there is a GCF of $2x$ which should be factored out first. Doing this leaves us with $18x^3 + 26x^2 + 8x = 2x\left(9x^2 + 13x + 4\right)$. Now we apply the AC method on the factor in the parentheses. So, $ac = 36$, and we must find two factors of 36 that sum to be 13. These two factors are 9 and 4. Now we can use grouping.

$$\begin{aligned}
18x^3 + 26x^2 + 8x &= 2x\left(9x^2 \overbrace{+13x} +4\right)\\
&= 2x\left(9x^2 \overbrace{+9x+4x} +4\right)\\
&= 2x\left(\left(9x^2+9x\right)+(4x+4)\right)\\
&= 2x\left(9x(x+1)+4(x+1)\right)\\
&= 2x(x+1)(9x+4)
\end{aligned}$$

10.8.5 Factoring Special Forms

In Section 10.5 we covered how to factor binomials and trinomials using formulas. Using these formulas, when appropriate, often drastically increased the speed of factoring. Below is a summary of the formulas covered. For each, consider that A and B could be any algebraic expressions.

Difference of Squares $A^2 - B^2 = (A+B)(A-B)$

Perfect Square Sum $A^2 + 2AB + B^2 = (A+B)^2$

Perfect Square Difference $A^2 - 2AB + B^2 = (A+B)^2$

Example 10.8.22 Factoring the Form $A^2 - 2AB + B^2$. Completely factor the expression $16y^2 - 24y + 9$.

Explanation. To factor $16y^2 - 24y + 9$ we notice that the expression might be of the form $A^2 - 2AB + B^2$. To find A and B, we mentally take the square root of both the first and last terms of the original expression. The square root of $16y^2$ is $4y$ since $(4y)^2 = 4^2y^2 = 16y^2$. The square root of 9 is 3. So, we conclude that $A = 4y$ and $B = 3$. Recall that we now need to check that the $24y$ matches our $2AB$. Using our values for A and B, we indeed see that $2AB = -2(4y)(3) = 24y$. So, we conclude that

$$16y^2 - 24y + 9 = (4y-3)^2.$$

Example 10.8.23 Mixed Special Forms Factoring.

a. Completely factor the expression $9w^2 + 12w + 4$.

b. Completely factor the expression $4q^2 - 81$.

c. Completely factor the expression $9p^2 + 25$.

d. Completely factor the expression $121b^2 - 36$.

e. Completely factor the expression $25u^2 - 70u + 49$.

Explanation. The first step for each problem is to try to fit the expression to one of the special factoring forms.

a. To factor $9w^2 + 12w + 4$ we notice that the expression might be of the form $A^2 + 2AB + B^2$ where $A = 3w$ and $B = 2$. With this formula we need to check the value of $2AB$ which in this case is $2AB = 2(3w)(2) = 12w$. Since the value of $2AB$ is correct, the expression must factor as

$$9w^2 + 12w + 4 = (3w + 2)^2$$

b. To factor $4q^2 - 81$ we notice that the expression is of the form $A^2 - B^2$ where $A = 2q$ and $B = 9$. Thus, the expression must factor as

$$4q^2 - 81 = (2q - 9)(2q + 9)$$

c. To factor $9p^2 + 25$ we notice that the expression is of the form $A^2 + B^2$. This is called a sum of squares. If you recall from the section, the sum of squares is always prime. So $9p^2 + 25$ is prime.

d. To completely factor the expression $121b^2 - 36$ first note that the expression is of the form $A^2 - B^2$ where $A = 11b$ and $B = 6$. So, the expression factors as

$$121b^2 - 36 = (11b + 6)(11b - 6).$$

e. To completely factor the expression $25u^2 - 70u + 49$ first note that the expression might be of the form $A^2 - 2AB + B^2$ where $A = 5u$ and $B = 7$. Now, we check that $2AB$ matches the middle term: $2AB = 2(5u)(7) = 70u$. So, the expression factors as

$$25u^2 - 70u + 49 = (5u - 7)^2.$$

10.8.6 Factoring Strategies

In Section 10.6 we covered a factoring decision tree to help us decide what methods to try when factoring a given expression. Remember to always factor out the GCF first.

Example 10.8.24 Factor the expressions using an effective method.

a. $24xy - 20x - 18y + 15$.

b. $12t^2 + 36t + 27$.

c. $8u^2 + 14u - 9$.

d. $18c^2 - 98p^2$.

Explanation.

a. To factor the expression $24xy - 20x - 18y + 15$, we first look for a GCF. Since the GCF is 1, we can move further on the flowchart. Since this is a four-term polynomial, we will try grouping.

$$\begin{aligned} 24xy - 20x - 18y + 15 &= 24xy + (-20x) + (-18y) + 15 \\ &= (24xy - 20x) + (-18y + 15) \\ &= 4x(6y - 5) + (-3)(6y - 5) \\ &= 4x\overbrace{(6x - 5)} - 3\overbrace{(6x - 5)} \\ &= (6x - 5)(4x - 3) \end{aligned}$$

b. To factor the expression $12t^2 + 36t + 27$, we first look for a GCF. Since the GCF is 3, first we will factor that out.

$$12t^2 + 36t + 27 = 3\left(4t^2 + 12t + 9\right)$$

Next, we can note that the first and last terms are perfect squares where $A^2 = 4t^2$ and $B = 9$; so $A = 2t$ and $B = 3$. To check the middle term, $2AB = 12t$. So the expression factors as a perfect square.

$$\begin{aligned} 12t^2 + 36t + 27 &= 3\left(4t^2 + 12t + 9\right) \\ &= 3(2t + 3)^2 \end{aligned}$$

c. To factor the expression $8u^2 + 14u - 9$, we first look for a GCF. Since the GCF is 1, we can move further on the flowchart. Since the expression is a trinomial with leading coefficient other than 1, we should try the AC method. Note that $AC = -72$ and factor pairs of -72 that add up to 14 are 18 and -4.

$$\begin{aligned} 8u^2 + 14u - 9 &= 8u^2 + 18u - 4u - 9 \\ &= \left(8u^2 + 18\right) + (-4u - 9) \\ &= 2u(4u + 9) - 1(4u + 9) \\ &= (2u - 1)(4u + 9) \end{aligned}$$

d. To factor the expression $18c^2 - 98p^2$, we first look for a GCF. Since the GCF is 2, first we will factor that out.

$$18c^2 - 98p^2 = 2\left(9c^2 - 49p^2\right)$$

Now we notice that we have a binomial where both the first and second terms can be written as squares: $9c^2 = (3c)^2$ and $49p^2 = (7p)^2$.

$$\begin{aligned} 18c^2 - 98p^2 &= 2\left(9c^2 - 49p^2\right) \\ &= 2(3c - 7p)(3c + 7p) \end{aligned}$$

10.8.7 Solving Quadratic Equations by Factoring

In Section 10.7 we covered the zero product property and learned an algorithm for solving quadratic equations by factoring.

Example 10.8.25 Solving Using Factoring. Solve the quadratic equations using factoring.

a. $x^2 - 2x - 15 = 0$

b. $4x^2 - 40x = -96$

c. $6x^2 + x - 12 = 0$

d. $(x-3)(x+2) = 14$

e. $x^3 - 64x = 0$

Explanation.

a. Use factor pairs.

$$\begin{aligned} x^2 - 2x - 15 &= 0 \\ (x-5)(x+3) &= 0 \end{aligned}$$

$$\begin{aligned} x - 5 &= 0 & &\text{or} & x + 3 &= 0 \\ x &= 5 & &\text{or} & x &= -3 \end{aligned}$$

So the solution set is $\{5, -3\}$.

b. Start by putting the equation in standard form and factoring out the greatest common factor.

$$\begin{aligned} 4x^2 - 40x &= -96 \\ 4x^2 - 40x + 96 &= 0 \\ 4\left(x^2 - 10x + 24\right) &= 0 \\ 4(x-6)(x-4) &= 0 \end{aligned}$$

$$\begin{aligned} x - 6 &= 0 & &\text{or} & x - 4 &= 0 \\ x &= 6 & &\text{or} & x &= 4 \end{aligned}$$

So the solution set is $\{4, 6\}$.

c. Use the AC method.

$$6x^2 + x - 12 = 0$$

Note that $a \cdot c = -72$ and that $9 \cdot -8 = -72$ and $9 - 8 = 1$

$$\begin{aligned} 6x^2 + 9x - 8x - 12 &= 0 \\ \left(6x^2 + 9x\right) + (-8x - 12) &= 0 \\ 3x\,(2x+3) - 4\,(2x+3) &= 0 \\ (2x+3)(3x-4) &= 0 \end{aligned}$$

$$\begin{aligned} 2x + 3 &= 0 & &\text{or} & 3x - 4 &= 0 \\ x &= -\frac{3}{2} & &\text{or} & x &= \frac{4}{3} \end{aligned}$$

So the solution set is $\left\{-\frac{3}{2}, \frac{4}{3}\right\}$.

d. Start by putting the equation in standard form.

$$\begin{aligned}(x-3)(x+2) &= 14\\ x^2-x-6 &= 14\\ x^2-x-20 &= 0\\ (x-5)(x+4) &= 0\end{aligned}$$

$$\begin{array}{ccc} x-5=0 & \text{or} & x+4=0\\ x=5 & \text{or} & x=-4\end{array}$$

So the solution set is $\{5,-4\}$.

e. Even though this equation has a power higher than 2, we can still find all of its solutions by following the algorithm. Start by factoring out the greatest common factor.

$$\begin{aligned}x^3-64x &= 0\\ x\left(x^2-64\right) &= 0\\ x(x-8)(x+8) &= 0\end{aligned}$$

$$\begin{array}{ccccc} x=0 & \text{or} & x-8=0 & \text{or} & x+8=0\\ x=0 & \text{or} & x=8 & \text{or} & x=-8\end{array}$$

So the solution set is $\{0,8,-8\}$.

10.8.8 Exercises

Factoring out the Common Factor

1. Find the greatest common factor of the following terms.
$3r$ and $15r^2$

2. Find the greatest common factor of the following terms.
$9r$ and $72r^2$

3. Find the greatest common factor of the following terms.
$6t^{11}$ and $-60t^{10}$

4. Find the greatest common factor of the following terms.
$3t^{16}$ and $-12t^{15}$

5. Find the greatest common factor of the following terms.
$6x^{20}y^8, -18x^{17}y^9, 12x^{12}y^{10}$

6. Find the greatest common factor of the following terms.
$3x^{20}y^4, -21x^{18}y^9, 15x^{17}y^{11}$

7. Factor the given polynomial.
$90x^2-40x+10$

8. Factor the given polynomial.
$80y^2-60y+30$

9. Factor the given polynomial.
$28y^2-27$

10. Factor the given polynomial.
$4r^2-3$

11. Factor the given polynomial.
$r(r-8)+9(r-8)$

12. Factor the given polynomial.
$t(t+5)+6(t+5)$

Factoring by Grouping Factor the given polynomial.

13. $t^2 - 2t + 4t - 8$
14. $x^2 + 8x + 10x + 80$
15. $x^3 - 5x^2 + 8x - 40$
16. $x^3 + 10x^2 + 6x + 60$
17. $xy + 5x - 10y - 50$
18. $xy - 6x + 5y - 30$
19. $7x^2 + 63xy + 8xy + 72y^2$
20. $8x^2 - 16xy + 5xy - 10y^2$

Factoring Trinomials with Leading Coefficient One Factor the given polynomial.

21. $t^2 + 10t + 21$
22. $t^2 + 10t + 9$
23. $x^2 - x - 30$
24. $x^2 - 5x - 24$
25. $x^2 + x + 4$
26. $y^2 + 5$
27. $y^2 - 4y + 4$
28. $r^2 - 20r + 100$
29. $6r^2 - 24r + 18$
30. $6t^2 - 18t + 12$
31. $-t^2 - 5t + 24$
32. $-x^2 + 4x + 5$
33. $x^2 - xr - 6r^2$
34. $x^2 + 5xr - 6r^2$
35. $y^2 - 8yx + 15x^2$
36. $y^2 - 7yx + 10x^2$

Factoring Trinomials with a Nontrivial Leading Coefficient Factor the given polynomial.

37. $3r^2 + 23r - 8$
38. $3r^2 + 5r - 28$
39. $3t^2 + 6t + 7$
40. $3t^2 + t + 6$
41. $4x^2 - 8x + 3$
42. $6x^2 - 17x + 5$
43. $18x^2 - 30x + 12$
44. $8y^2 - 12y + 4$
45. $4y^9 - 22y^8 + 24y^7$
46. $4r^{10} - 22r^9 + 18r^8$
47. $6r^2x^2 - 21rx - 27$
48. $20t^2r^2 + 10tr - 30$
49. $6x^2 + 20xy + 16y^2$
50. $10x^2 + 35xy + 15y^2$

Factoring Special Polynomials Factor the given polynomial.

51. $x^2 - 100$
52. $x^2 - 36$
53. $4y^2 - 9$
54. $100y^2 - 121$
55. $r^{14} - 36$
56. $r^6 - 4$
57. $100t^2 - 20t + 1$
58. $36t^2 - 12t + 1$
59. $49t^2 - 56tx + 16x^2$
60. $121x^2 - 88xr + 16r^2$
61. $16x^4 - 1$
62. $81y^4 - 16$
63. $4y^3 - 100y$
64. $8r^3 - 128r$
65. $6r^3t^4 - 150rt^2$
66. $3t^3r^3 - 108tr$
67. $48 - 3t^2$
68. $3 - 3t^2$
69. $x^2 + 49$
70. $x^2 + 9$

Factoring Strategies Which factoring techniques/tools will be useful for factoring the polynomial below? Check all that apply.
☐ Factoring out a GCF ☐ Factoring by grouping ☐ Finding two numbers that multiply to the constant term and sum to the linear coefficient ☐ The AC Method ☐ Difference of Squares ☐ Difference of Cubes ☐ Sum of Cubes ☐ Perfect Square Trinomial ☐ None of the above

71. $49a^2 - 70ap + 25p^2$
72. $9c^2 - 81cA + 72A^2$

Factor the given polynomial.

73. $6r - 8r^2 + 6r^3$
74. $7xy + 7y$
75. $4t^2 - 16ty + 7y^2$
76. $t^2 - 7t - 18$
77. $4t^2 + 36tr + 81r^2$
78. $x^2y^2 + 4x^2yz - 5x^2z^2$

79. $15x^2 + 26xt + 7t^2$

80. $y^2 - 11y + 24$

81. $16y^4 - 81$

82. $2r^3 - 2r$

Solving Quadratic Equations by Factoring Solve the equation.

83. $x^2 - 4x - 21 = 0$

84. $x^2 + 2x - 24 = 0$

85. $x^2 + 19x = -90$

86. $x^2 + 17x = -70$

87. $x^2 = 8x$

88. $x^2 = 6x$

89. $x^2 - 8x + 16 = 0$

90. $x^2 - 12x + 36 = 0$

91. $4x^2 = -27x - 44$

92. $4x^2 = -23x - 28$

93. A rectangle's base is 1 in shorter than twice its height. The rectangle's area is 45 in^2. Find this rectangle's dimensions.

The rectangle's height is $\boxed{}$.

The rectangle's base is $\boxed{}$.

94. A rectangle's base is 1 in shorter than five times its height. The rectangle's area is 120 in^2. Find this rectangle's dimensions.

The rectangle's height is $\boxed{}$.

The rectangle's base is $\boxed{}$.

Chapter 11

Functions

11.1 Function Basics

In this section, we will introduce a topic that will be essential for continued mathematical learning: functions. Functions should be thought of as machines that turn one number into another number, much like a cash register can turn a number of pounds of fruit into a price.

11.1.1 Informal Definition of a Function

We are familiar with the $\sqrt{}$ symbol. This symbol is used to turn numbers into their square roots. Sometimes it's simple to do this on paper or in our heads, and sometimes it helps to have a calculator. We can see some calculations in Figure 11.1.2.

$$\begin{aligned} \sqrt{9} &= 3 \\ \sqrt{1/4} &= 1/2 \\ \sqrt{2} &\approx 1.41 \end{aligned}$$

Figure 11.1.2: Values of $\sqrt{x}$

The $\sqrt{}$ symbol represents a *process*; it's a way for us to turn numbers into other numbers. This idea of having a process for turning numbers into other numbers is the fundamental topic of this chapter.

Definition 11.1.3 Function (Informal Definition). A **function** is a process for turning numbers into (potentially) different numbers. The process must be *consistent*, in that whenever you apply it to some particular number, you always get the same result. ♢

Section 11.5 covers a more technical definition for functions, and covers topics that are more appropriate when using that definition. Definition 11.1.3 is so broad that you probably use functions all the time.

Example 11.1.4 In each of these examples, some process is used for turning one number into another.

- If you convert a person's birth year into their age, you are using a function.
- If you look up the Kelly Blue Book value of a Honda Odyssey based on how old it is, you are using a function.
- If you use the expected guest count for a party to determine how many pizzas you should order, you are using a function.

The $\sqrt{}$ function is consistent; for example, every time you evaluate $\sqrt{9}$, you always get 3. One interesting fact is that $\sqrt{}$ is not found on most keyboards, and yet computers can still find square roots. Computer technicians write sqrt( ) when they want to compute a square root, as we see in Figure 11.1.5.

$$\begin{aligned} \text{sqrt}(9) &= 3 \\ \text{sqrt}(1/4) &= 1/2 \\ \text{sqrt}(2) &\approx 1.41 \end{aligned}$$

Figure 11.1.5: Values of sqrt(x)

The parentheses in sqrt ( ) are very important. To see why, try to put yourself in the "mind" of a computer. The computer will recognize `sqrt` and know that it needs to compute a square root but without parentheses it will think that it needs to compute `sqrt4` and then put a `9` on the end, which would produce a final result of 29. This is probably not what was intended. And so the purpose of the parentheses in `sqrt(49)` is to be deliberately clear.

Functions have their own names. We've seen a function named sqrt, but any name you can imagine is allowable. In the sciences, it is common to name functions with whole words, like weight or health_index. In math, we often abbreviate such function names to w or h. And of course, since the word "function" itself starts with "f," we will often name a function f.

Warning 11.1.6 Notation Ambiguity. In some contexts, the symbol t might represent a variable (a number that is represented by a letter) and in other contexts, t might represent a function (a process for changing numbers into other numbers). By staying conscious of the *context* of an investigation, we avoid confusion.

Next we need to discuss how we go about using a function's name.

Definition 11.1.7 Function Notation. The standard notation for referring to functions involves giving the function itself a name, and then writing:

$$\begin{array}{c}\text{name}\\ \text{of}\\ \text{function}\end{array}\left(\text{input}\right)$$

◊

Example 11.1.8 $f(13)$ is pronounced "f of 13." The word "of" is very important, because it reminds us that f is a process and we are about to apply that process to the input value 13. So f is the function, 13 is the input, and $f(13)$ is the output we'd get from using 13 as input.

$f(x)$ is pronounced "f of x." This is just like the previous example, except that the input is not any specific number. The value of x could be 13 or any other number. Whatever x's value, $f(x)$ means the corresponding output from the function f.

BudgetDeficit(2017) is pronounced "BudgetDeficit of 2017." This is probably about a function that takes a year as input, and gives that year's federal budget deficit as output. The process here of changing a year into a dollar amount might not involve any mathematical formula, but rather looking up information from the Congressional Budget Office's website.

Note 11.1.9 While a function has a name like f, and the input to that function often has a variable name like x, the expression $f(x)$ represents the output of the function. To be clear, $f(x)$ is *not* a function. Rather, f is a function, and $f(x)$ its output when the number x was used as input.

Checkpoint 11.1.10 Suppose you see the sentence, "If x is the number of software licenses you buy for your office staff, then $c(x)$ is the total cost of the licenses."

a. In the function notation, what represents input? ______.

b. What is the function here? $\boxed{}$.

c. What represents output? $\boxed{}$.

Explanation. The input is x, the function is c, and $c(x)$ is the output from c when the input is x.

Warning 11.1.11 More Notation Ambiguity. As mentioned in Warning 11.1.6, we need to remain conscious of the context of any symbol we are using. Consider the expression $a(b)$. This could easily mean the output of a function a with input b. It could also mean that two numbers a and b need to be multiplied. It all depends on the context in which these symbols are being used.

Sometimes it's helpful to think of a function as a machine, as in Figure 11.1.12. A *function* has the capacity to take in all kinds of different numbers into it's hopper (feeding tray) as inputs and transform them into their outputs.

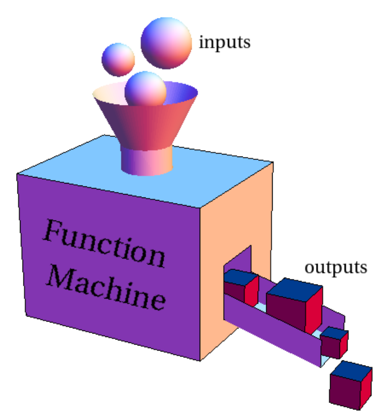


Figure 11.1.12: Imagining a function as a machine. (Image by Duane Nykamp using Mathematica.)

11.1.2 Tables and Graphs

Since functions are potentially complicated, we want ways to understand them more easily. Two basic tools for understanding a function better are tables and graphs.

Example 11.1.13 A Table for the Budget Deficit Function. Consider the function BudgetDeficit, that takes a year as its input and outputs the US federal budget deficit for that year. For example, the Congressional Budget Office's website tells us that BudgetDeficit(2009) is \$1.41 trillion. If we'd like to understand this function better, we might make a table of all the inputs and outputs we can find. Using the CBO's website (www.cbo.gov/topics/budget), we can put together Table 11.1.14.

input x (year)	output BudgetDeficit(x) ($trillion)
2007	0.16
2008	0.46
2009	1.4
2010	1.3
2011	1.3
2012	1.1
2013	0.68
2014	0.48
2015	0.44
2016	0.59
2017	0.67
2018	0.78

Table 11.1.14: The Federal Budget Deficit

How is this table helpful? There are things about the function that we can see now by looking at the numbers in this table.

- We can see that the budget deficit had a spike between 2008 and 2009.
- And it fell again between 2012 and 2013.
- It appears to stay roughly steady for several years at a time, with occasional big jumps or drops.

These observations help us understand the function BudgetDeficit a little better.

Checkpoint 11.1.15 According to Table 11.1.14, what is the value of BudgetDeficit(2015)?

Explanation. Table 11.1.14 shows that when the input is 2015, the output is 0.44. So BudgetDeficit(2015) $=$ 0.44. In context, that means that in 2015 the budget deficit was $0.44 trillion.

Example 11.1.16 A Table for the Square Root Function. Let's return to our example of the function sqrt. Tabulating some inputs and outputs reveals Figure 11.1.17.

input, x	output, sqrt(x)
0	0
1	1
2	≈ 1.41
3	≈ 1.73
4	2
5	≈ 2.24
6	≈ 2.45
7	≈ 2.65
8	≈ 2.83
9	3

Figure 11.1.17

How is this table helpful? Here are some observations that we can make now.

- We can see that when input numbers increase, so do output numbers.
- We can see even though outputs are increasing, they increase by less and less with each step forward in x.

These observations help us understand sqrt a little better. For instance, based on these observations which do you think is larger: the difference between sqrt(23) and sqrt(24), or the difference between sqrt(85) and sqrt(86)?

Checkpoint 11.1.18 According to Figure 11.1.17, what is the value of sqrt(6)?

Explanation. Figure 11.1.17 shows that when the input is 6, the output is about 2.45. So sqrt(6) $\approx$ 2.45.

Another powerful tool for understanding a function better is a graph. Given a function f, one way to make its graph is to take a table of input and output values, and read each row as the coordinates of a point in the xy-plane.

Example 11.1.19 A Graph for the Budget Deficit Function. Returning to the function BudgetDeficit that we studied in Example 11.1.13, in order to make a graph of this function we view Table 11.1.14 as a list of points with x and y coordinates, as in Figure 11.1.20. We then plot these points on a set of coordinate axes, as in Figure 11.1.21. The points have been connected with a curve so that we can see the overall pattern given by the progression of points. Since there was not any actual data for inputs in between any two years, the curve is dashed. That is, this curve is dashed because it just represents someone's best guess as to how to connect the plotted points. Only the plotted points themselves are precise.

(input, output) $(x, \text{BudgetDeficit}(x))$
$(2007, 0.16)$
$(2008, 0.46)$
$(2009, 1.4)$
$(2010, 1.3)$
$(2011, 1.3)$
$(2012, 1.1)$
$(2013, 0.68)$
$(2014, 0.48)$
$(2015, 0.44)$
$(2016, 0.59)$
$(2017, 0.67)$
$(2018, 0.78)$

Figure 11.1.20

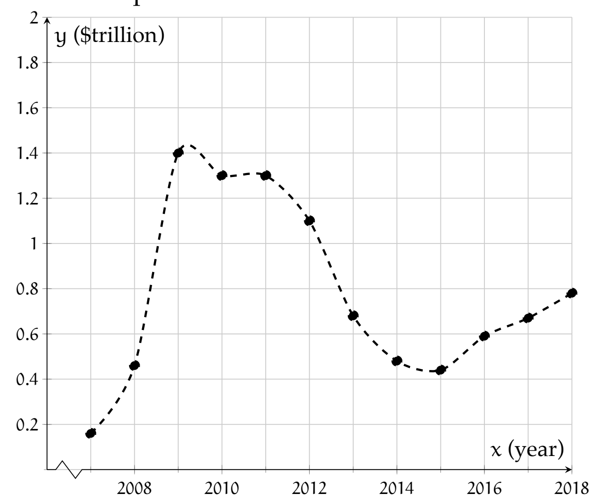


Figure 11.1.21: $y = \text{BudgetDeficit}(x)$

How has this graph helped us to understand the function better? All of the observations that we made in Example 11.1.13 are perhaps even more clear now. For instance, the spike in the deficit between 2008 and 2009 is now visually apparent. Seeking an explanation for this spike, we recall that there was a financial crisis in late 2008. Revenue from income taxes dropped at the same time that federal money was spent to prevent further losses.

Example 11.1.22 A Graph for the Square Root Function. Let's now construct a graph for sqrt. Tabulating inputs and outputs gives the points in Figure 11.1.23, which in turn gives us the graph in Figure 11.1.24.

(input, output) $(x, \text{sqrt}(x))$
$(0, 0)$
$(1, 1)$
$\approx (2, 1.41)$
$\approx (3, 1.73)$
$(4, 2)$
$\approx (5, 2.24)$
$\approx (6, 2.45)$
$\approx (7, 2.65)$
$\approx (8, 2.83)$
$(9, 3)$

Figure 11.1.23

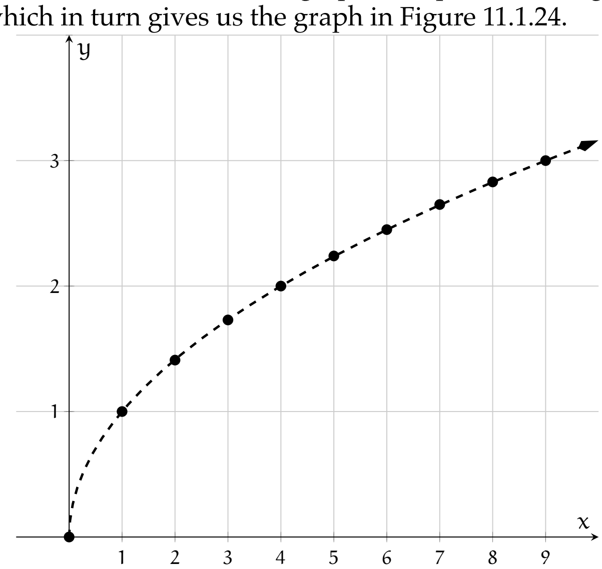


Figure 11.1.24: $y = \text{sqrt}(x)$

Just as in the previous example, we've plotted points where we have concrete coordinates, and then we have made our best attempt to connect those points with a curve. Unlike the previous example, here we believe that points will continue to follow the same pattern indefinitely to the right, and so we have added an arrowhead to the graph.

What has this graph done to improve our understanding of sqrt? As inputs (x-values) increase, the outputs (y-values) increase too, although not at the same rate. In fact we can see that our graph is steep on its left, and less steep as we move to the right. This confirms our earlier observation in Example 11.1.16 that outputs increase by smaller and smaller amounts as the input increases.

Remark 11.1.25 Graph of a Function. Given a function f, when we refer to a **graph of** f we are *not* referring to an entire picture, like Figure 11.1.24. A graph of f is only *part* of that picture—the curve and the points that it connects. Everything else (axes, tick marks, the grid, labels, and the surrounding white space) is just useful decoration so that we can read the graph more easily.

Remark 11.1.26 A Common Wording Misunderstanding. It is common to refer to the graph of f as the **graph of the equation** $y = f(x)$. However, we should avoid saying "the graph of $f(x)$." That would indicate a misunderstanding of our notation. Since $f(x)$ is the output for a certain input x. That means that $f(x)$ is just a number and not worthy of a two-dimensional picture.

While it is important to be able to make a graph of a function f, we also need to be capable of looking at a graph and reading it well. A graph of f provides us with helpful specific information about f; it tells us what f does to its input values. When we were making graphs, we plotted points of the form

$$(\text{input}, \text{output})$$

Now given a graph of f, we interpret coordinates in the same way.

Example 11.1.27 In Figure 11.1.28 we have a graph of a function f. If we wish to find $f(1)$, we recognize that 1 is being used as an input. So we would want to find a point of the form $(1, \quad)$. Seeking out x-coordinate 1 in Figure 11.1.28, we find that the only such point is $(1, 2)$. Therefore the output for 1 is 2; in other words $f(1) = 2$.

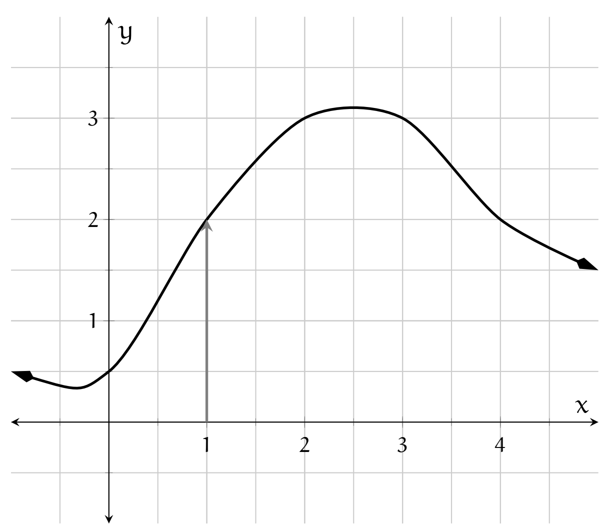


Figure 11.1.28: $y = f(x)$

Checkpoint 11.1.29 Use the graph of f in Figure 11.1.28 to find $f(0)$, $f(3)$, and $f(4)$.

a. $f(0)$

b. $f(3)$

c. $f(4)$

Explanation.

a. $f(0) = 0.5$, since $(0, 0.5)$ is on the graph.

b. $f(3) = 3$, since $(3, 3)$ is on the graph.

c. $f(4) = 2$, since $(4, 2)$ is on the graph.

Example 11.1.30 Unemployment Rates.
Suppose that u is the unemployment function of time. That is, $u(t)$ is the unemployment rate in the United States in year t. The graph of the equation $y = u(t)$ is given in Figure 11.1.31 (data.bls.gov/timeseries/LNS14000000).

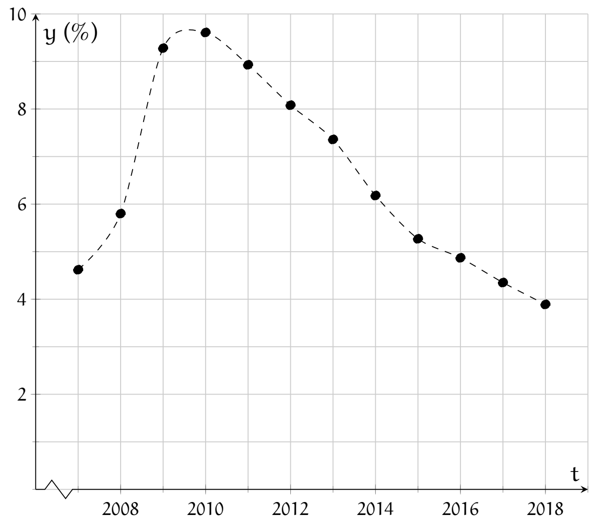


Figure 11.1.31: Unemployment in the United States

What was the unemployment in 2008? It is a straightforward matter to use Figure 11.1.31 to find that unemployment was almost 6% in 2008. Asking this question is exactly the same thing as asking to find $u(2008)$. That is, we have one question that can either be asked in an everyday-English way or which can be asked in a terse, mathematical notation-heavy way:

"What was unemployment in 2008?" "Find $u(2008)$."

If we use the table to establish that $u(2009) \approx 9.25$, then we should be prepared to translate that into everyday-English using the context of the function: In 2009, unemployment in the U.S. was about 9.25%.

If we ask the question "when was unemployment at 5%," we can read the graph and see that there were two such times: mid-2007 and about 2016. But there is again a more mathematical notation-heavy way to ask this question. Namely, since we are being told that the output of u is 5, we are being asked to solve the equation $u(t) = 5$. So the following communicate the same thing:

"When was unemployment at 5%?" "Solve the equation $u(t) = 5$."

And our answer to this question is:

"Unemployment was at 5% in mid-2007 and about 2016." "$t \approx 2007.5$ or $t \approx 2016$."

Checkpoint 11.1.32 Use the graph of u in Figure 11.1.31 to answer the following.

a. Find $u(2011)$ and interpret it.

Interpretation:

b. Solve the equation $u(t) = 6$ and interpret your solution(s).

$t \approx$ ☐ or $t \approx$ ☐

Interpretation:

Explanation.

a. $u(2011) \approx 9$; In 2011 the US unemployment rate was about 9%.

b. $t \approx 2008$ or $t \approx 2014$; The points at which unemployment was 6% were in early 2008 and early 2014.

11.1.3 Translating Between Four Descriptions of the Same Function

We have noted that functions are complicated, and we want ways to make them easier to understand. It's common to find a problem involving a function and not know how to find a solution to that problem. Most functions have at least four standard ways to think about them, and if we learn how to translate between these four perspectives, we often find that one of them makes a given problem easier to solve.

The four modes for working with a given function are

- a verbal description
- a table of inputs and outputs
- a graph of the function
- a formula for the function

This has been visualized in Figure 11.1.33.

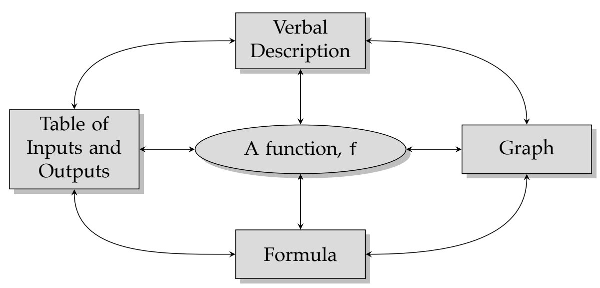


Figure 11.1.33: Function Perspectives

Example 11.1.34 Consider a function f that squares its input and then adds 1. Translate this verbal description of f into a table, a graph, and a formula.

Explanation.

To make a table for f, we'll have to select some input x-values. These choices are left entirely up to us, so we might as well choose small, easy-to-work-with values. However we shouldn't shy away from negative input values. Given the verbal description, we should be able to compute a column of output values. Figure 11.1.35 is one possible table that we might end up with.

x	$f(x)$
-2	$(-2)^2 + 1 = 5$
-1	$(-1)^2 + 1 = 2$
0	$0^2 + 1 = 1$
1	$1^2 + 1 = 2$
2	5
3	10
4	17

Figure 11.1.35

Once we have a table for f, we can make a graph for f as in Figure 11.1.36, using the table to plot points.

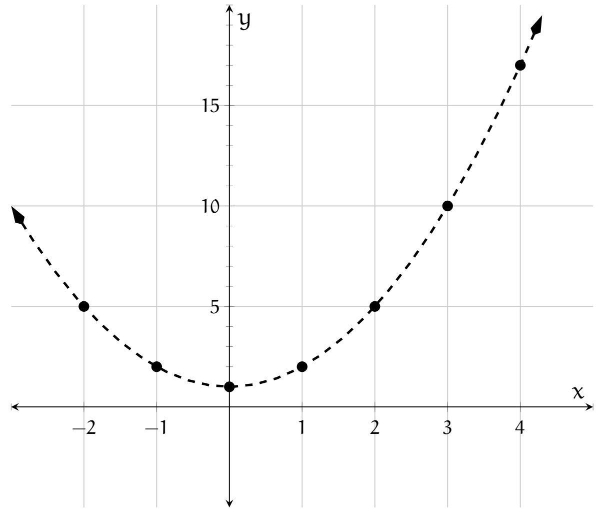


Figure 11.1.36: $y = f(x)$

Lastly, we must find a formula for f. This means we need to write an algebraic expression that says the same thing about f as the verbal description, the table, and the graph. For this example, we can focus on the verbal description. Since f takes its input, squares it, and adds 1, we have that

$$f(x) = x^2 + 1.$$

Example 11.1.37 Let F be the function that takes a Celsius temperature as input and outputs the corresponding Fahrenheit temperature. Translate this verbal description of F into a table, a graph, and a formula.

Explanation. To make a table for F, we will need to rely on what we know about Celsius and Fahrenheit temperatures. It is a fact that the freezing temperature of water at sea level is 0 °C, which equals 32 °F. Also, the boiling temperature of water at sea level is 100 °C, which is the same as 212 °F. One more piece of information we might have is that standard human body temperature is 37 °C, or 98.6 °F. All of this is compiled in Figure 11.1.38. Note that we tabulated inputs and outputs by working with the context of the function, not with any computations.

C	$F(C)$
0	32
37	98.6
100	212

Figure 11.1.38

Once a table is established, making a graph by plotting points is a simple matter, as in Figure 11.1.39. The three plotted points seem to be in a straight line, so we think it is reasonable to connect them in that way.

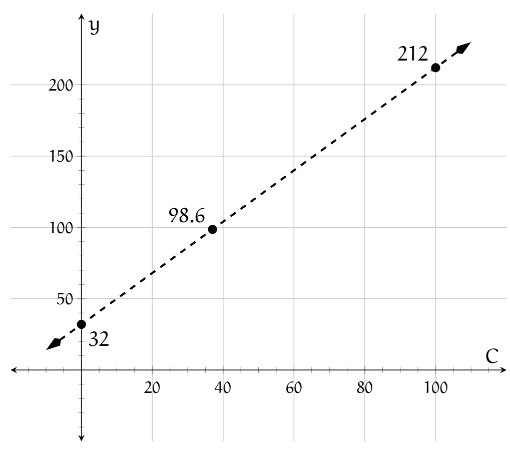


Figure 11.1.39: $y = F(C)$

To find a formula for F, the verbal definition is not of much direct help. But F's graph does seem to be a straight line. And linear equations are familiar to us. This line has a y-intercept at $(0, 32)$ and a slope we can calculate: $\frac{212-32}{100-0} = \frac{180}{100} = \frac{9}{5}$. So the equation of this line is $y = \frac{9}{5}C + 32$. On the other hand, the equation of this graph is $y = F(C)$, since it is a graph of the function F. So evidently,

$$F(C) = \frac{9}{5}C + 32.$$

11.1.4 Reading Questions

1. When g is a function, how should you say out loud "$g(x)$?"

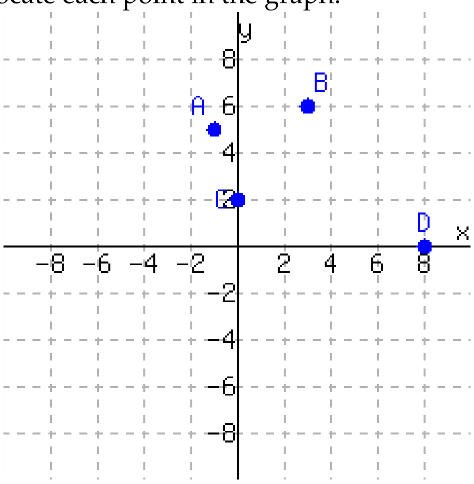
2. There are four main ways to communicate how a function turns its inputs into its outputs. What are they?

3. What is usually an acceptable way to type "the square root of x" if you have to type it using a regular keyboard?

11.1.5 Exercises

Review and Warmup

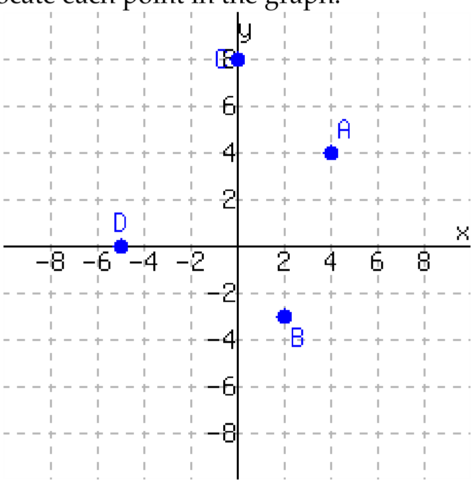
1. Locate each point in the graph:



Write each point's position as an ordered pair, like $(1, 2)$.

$A =$ ______ $B =$ ______
$C =$ ______ $D =$ ______

2. Locate each point in the graph:



Write each point's position as an ordered pair, like $(1, 2)$.

$A =$ ______ $B =$ ______
$C =$ ______ $D =$ ______

3. Evaluate $\dfrac{2t-1}{8t}$ for $t = -4$.

4. Evaluate $\dfrac{6t-4}{8t}$ for $t = 10$.

5. a. Evaluate $2x^2$ when $x = 4$.

b. Evaluate $(2x)^2$ when $x = 4$.

6. a. Evaluate $3x^2$ when $x = 2$.

b. Evaluate $(3x)^2$ when $x = 2$.

Function Formulas and Evaluation Evaluate the function at the given values.

7. $H(x) = x - 4$
a. $H(5)$
b. $H(-2)$
c. $H(0)$

8. $G(x) = x - 1$
a. $G(3)$
b. $G(-5)$
c. $G(0)$

9. $F(x) = 4x$
a. $F(3)$
b. $F(-4)$
c. $F(0)$

10. $G(x) = 10x$
a. $G(1)$
b. $G(-5)$
c. $G(0)$

11. $H(x) = -3x + 5$
a. $H(4)$
b. $H(-2)$
c. $H(0)$

12. $K(x) = -5x + 9$
a. $K(3)$
b. $K(-2)$
c. $K(0)$

13. $K(x) = -x + 9$
a. $K(2)$
b. $K(-5)$
c. $K(0)$

14. $f(x) = -x + 6$
a. $f(4)$
b. $f(-1)$
c. $f(0)$

15. $g(x) = x^2 - 8$
a. $g(4)$
b. $g(-2)$
c. $g(0)$

16. $h(t) = t^2 - 3$
a. $h(4)$
b. $h(-4)$
c. $h(0)$

17. $F(y) = -y^2 + 9$
a. $F(1)$
b. $F(-2)$
c. $F(0)$

18. $F(x) = -x^2 - 3$
a. $F(5)$
b. $F(-2)$
c. $F(0)$

19. $G(t) = 6$
a. $G(3)$
b. $G(6)$
c. $G(0)$

20. $H(y) = -7$
a. $H(2)$
b. $H(-7)$
c. $H(0)$

21. $K(x) = \dfrac{7x}{2x + 10}$
a. $K(5)$
b. $K(-1)$

22. $K(x) = \dfrac{7x}{-10x + 4}$
a. $K(5)$
b. $K(-6)$

23. $f(x) = \dfrac{3}{x - 2}$.
a. $f(1)$
b. $f(2)$

24. $g(x) = -\dfrac{70}{x - 7}$.
a. $g(14)$
b. $g(7)$

25. $h(x) = -3x - 4$
a. $h(7)$
b. $h(-4)$

26. $F(x) = -6x + 3$
a. $F(5)$
b. $F(-5)$

27. $F(x) = x^2 + 5x - 5$
a. $F(1)$
b. $F(-3)$

28. $G(x) = x^2 + 2x$
a. $G(0)$
b. $G(-3)$

29. $H(x) = -3x^2 + 5x + 3$
a. $H(2)$
b. $H(-2)$

30. $K(x) = -2x^2 - 2x - 1$
a. $K(1)$
b. $K(-3)$

31. $K(x) = \sqrt{x}$.
a. $K(49)$
b. $K\left(\frac{64}{9}\right)$
c. $K(-6)$

32. $f(x) = \sqrt{x}$.
a. $f(16)$
b. $f\left(\frac{4}{9}\right)$
c. $f(-6)$

33. $g(x) = \sqrt[3]{x}$
a. $g(-1)$
b. $g\left(\frac{64}{27}\right)$

34. $h(x) = \sqrt[3]{x}$
a. $h(-27)$
b. $h\left(\frac{1}{27}\right)$

35. $F(x) = -12$
a. $F(4)$
b. $F(-8)$

36. $F(x) = 15$
a. $F(7)$
b. $F(-2)$

Function Formulas and Solving Equations

37. Solve for x, where $G(x) = 12x + 6$.

a. $G(x) = -42$

b. $G(x) = -3$

38. Solve for x, where $H(x) = -8x - 5$.

a. $H(x) = 19$

b. $H(x) = 7$

39. Solve for x, where $K(x) = x^2 + 7$.

a. $K(x) = 8$

b. $K(x) = 6$

40. Solve for x, where $K(x) = x^2 - 1$.

a. $K(x) = 8$

b. $K(x) = -4$

41. Solve for x, where $f(x) = x^2 + x - 73$.
$f(x) = -1$

42. Solve for x, where $g(x) = x^2 + 3x - 25$.
$g(x) = -7$

43. If h is a function defined by $h(y) = 4y + 9$,

a. Find $h(0)$.

b. Solve $h(y) = 0$.

44. If f is a function defined by $f(y) = -4y + 2$,

a. Find $f(0)$.

b. Solve $f(y) = 0$.

45. If H is a function defined by $H(r) = 4r^2 - 4$,

a. Find $H(0)$.

b. Solve $H(r) = 0$.

46. If h is a function defined by $h(r) = r^2 - 1$,

a. Find $h(0)$.

b. Solve $h(r) = 0$.

47. If f is a function defined by
$f(t) = t^2 - 9t + 18$,

a. Find $f(0)$.

b. Solve $f(t) = 0$.

48. If G is a function defined by
$G(t) = t^2 + 2t - 35$,

a. Find $G(0)$.

b. Solve $G(t) = 0$.

Functions and Points on a Graph

49. a. If $K(4) = 2$, then the point $\boxed{\quad}$ is on the graph of K.

b. If $(3, 0)$ is on the graph of K, then $K(3) = \boxed{\quad}$.

50. a. If $f(10) = 12$, then the point $\boxed{\quad}$ is on the graph of f.

b. If $(10, 7)$ is on the graph of f, then $f(10) = \boxed{\quad}$.

51. If $g(r) = x$, then the point $\boxed{\quad}$ is on the graph of g.

52. If $h(y) = r$, then the point $\boxed{\quad}$ is on the graph of h.

53. If (t, x) is on the graph of h, then $h(t) = \boxed{\quad}$.

54. If (r, y) is on the graph of F, then $F(r) = \boxed{\quad}$.

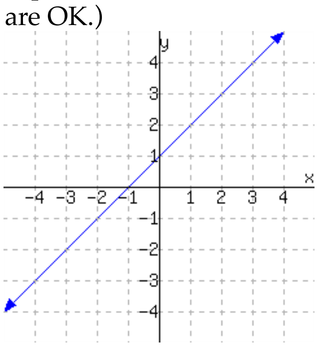
55. For the function G, when $x = 1$, the output is 0.
Choose all true statements.

☐ $G(1) = 0$ ☐ $G(0) = 1$ ☐ The function's value is 1 at 0. ☐ The point $(1, 0)$ is on the graph of the function. ☐ The function's value is 0 at 1. ☐ The point $(0, 1)$ is on the graph of the function.

56. For the function H, when $x = -2$, the output is -11.
Choose all true statements.
☐ The function's value is -2 at -11. ☐ The point $(-11, -2)$ is on the graph of the function. ☐ $H(-11) = -2$ ☐ The point $(-2, -11)$ is on the graph of the function. ☐ The function's value is -11 at -2. ☐ $H(-2) = -11$

Function Graphs

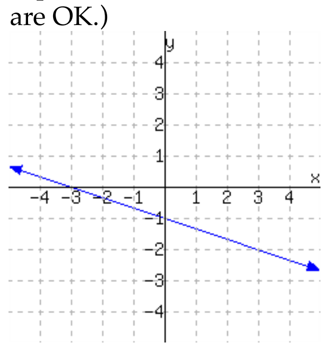
57. Use the graph of K below to evaluate the given expressions. (Estimates are OK.)



a. $K(-1) =$

b. $K(3) =$

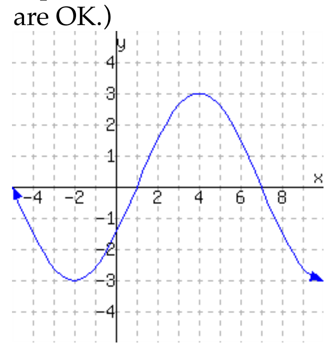
58. Use the graph of K below to evaluate the given expressions. (Estimates are OK.)



a. $K(-3) =$

b. $K(1) =$

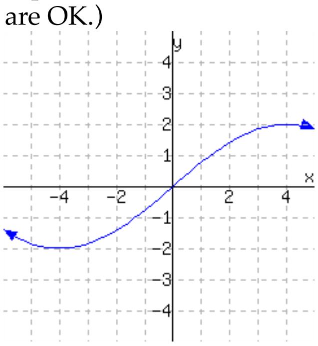
59. Use the graph of f below to evaluate the given expressions. (Estimates are OK.)



a. $f(-2) =$

b. $f(7) =$

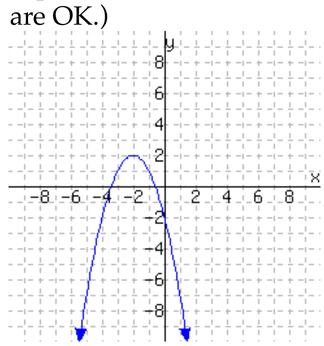
60. Use the graph of g below to evaluate the given expressions. (Estimates are OK.)



a. $g(-4) =$

b. $g(0) =$

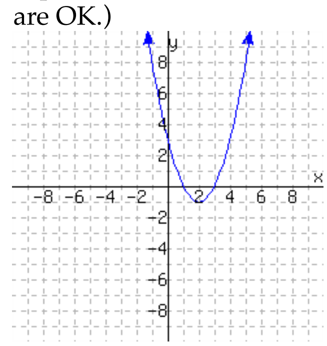
61. Use the graph of h below to evaluate the given expressions. (Estimates are OK.)



a. $h(-2) =$

b. $h(1) =$

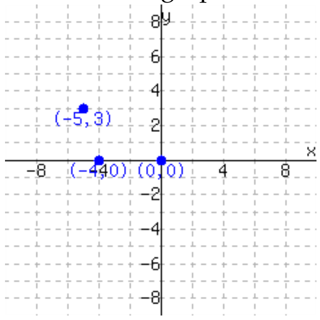
62. Use the graph of h below to evaluate the given expressions. (Estimates are OK.)



a. $h(2) =$

b. $h(3) =$

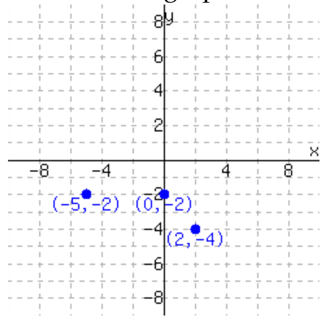
63. Function f is graphed.



a. Find $f(-5)$.

b. Solve $f(x) = 0$.

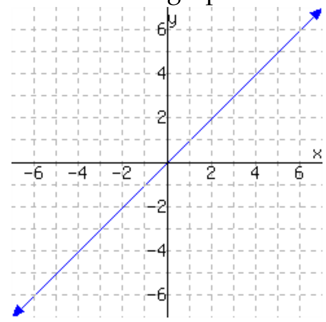
64. Function f is graphed.



a. Find $f(2)$.

b. Solve $f(x) = -2$.

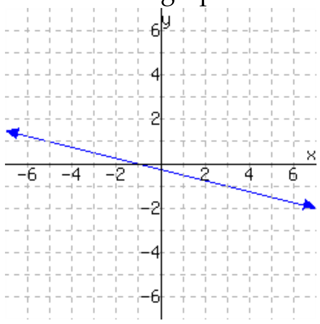
65. Function f is graphed.



a. Find $f(2) =$.

b. Solve $f(x) = 3$.

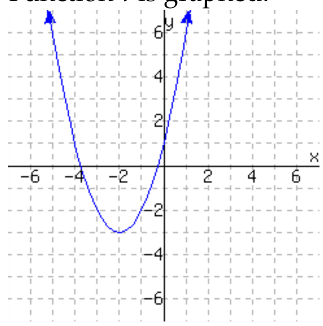
66. Function f is graphed.



a. Find $f(3) =$.

b. Solve $f(x) = 0$.

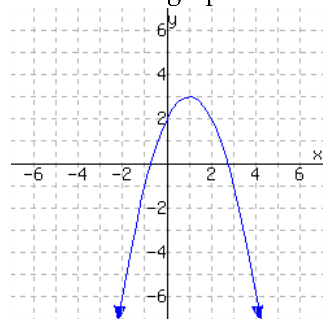
67. Function f is graphed.



a. Find $f(-1) =$.

b. Solve $f(x) = 1$.

68. Function f is graphed.



a. Find $f(-1) =$.

b. Solve $f(x) = 2$.

Function Tables

69. Use the table of values for g below to evaluate the given expressions.

x	0	2	4	6	8
$g(x)$	6.1	9.3	6.2	9.7	4.5

a. $g(2) =$

b. $g(8) =$

70. Use the table of values for h below to evaluate the given expressions.

x	-1	0	1	2	3
$h(x)$	2.8	9.1	0.1	-1.4	9.4

a. $h(0) =$

b. $h(2) =$

71. Make a table of values for the function G, defined by $G(x) = -2x^2$. Based on values in the table, sketch a graph of G.

x	$G(x)$
___	___
___	___
___	___
___	___
___	___

72. Make a table of values for the function H, defined by $H(x) = \dfrac{2^x + 2}{x^2 + 3}$. Based on values in the table, sketch a graph of H.

x	$H(x)$
___	___
___	___
___	___
___	___
___	___

Translating Between Different Representations of a Function

73. Here is a verbal description of a function G. "Cube the input x to obtain the output y."

a. Give a numeric representation of G.

x	0	1	2	3	4
$G(x)$	__	__	__	__	__

b. Give a formula for G.

74. Here is a verbal description of a function H. "Cube the input x to obtain the output y."

a. Give a numeric representation of H.

x	0	1	2	3	4
$H(x)$	__	__	__	__	__

b. Give a formula for H.

75. Here is a verbal description of a function K. "Double the input x and then subtract three to obtain the output y."

a. Give a numeric representation of K:

x	0	1	2	3	4
$K(x)$	__	__	__	__	__

b. Give a formula for K.

76. Here is a verbal description of a function K. "Quadruple the input x and then subtract seven to obtain the output y."

a. Give a numeric representation of K:

x	0	1	2	3	4
$K(x)$	__	__	__	__	__

b. Give a formula for K.

77. Express the function f numerically with the table.

$$f(x) = 2x^3 - \frac{1}{2}x^2$$

x	−3	−2	−1	0	1	2	3
$f(x)$	__	__	__	__	__	__	__

On graphing paper, you should be able to give a graphical representation of f too.

78. Express the function g numerically with the table.

$$g(x) = x^2 - \frac{1}{2}x$$

x	−3	−2	−1	0	1	2	3
$g(x)$	__	__	__	__	__	__	__

On graphing paper, you should be able to give a graphical representation of g too.

79. Express the function h numerically with the table.

$$h(x) = \frac{8 - x}{7 + x}$$

x	−3	−2	−1	0	1	2	3
$h(x)$	__	__	__	__	__	__	__

On graphing paper, you should be able to give a graphical representation of h too.

80. Express the function h numerically with the table.

$$h(x) = \frac{5 - x}{4 + x}$$

x	−3	−2	−1	0	1	2	3
$h(x)$	__	__	__	__	__	__	__

On graphing paper, you should be able to give a graphical representation of h too.

Functions in Context

81. Phil started saving in a piggy bank on his birthday. The function $f(x) = 2x + 2$ models the amount of money, in dollars, in Phil's piggy bank. The independent variable represents the number of days passed since his birthday.

Interpret the meaning of $f(4) = 10$.

- ⊙ ***A***. Four days after Phil started his piggy bank, there were $10 in it.
- ⊙ ***B***. The piggy bank started with $10 in it, and Phil saves $4 each day.
- ⊙ ***C***. The piggy bank started with $4 in it, and Phil saves $10 each day.
- ⊙ ***D***. Ten days after Phil started his piggy bank, there were $4 in it.

82. An arcade sells multi-day passes. The function $g(x) = \frac{1}{3}x$ models the number of days a pass will work, where x is the amount of money paid, in dollars.

Interpret the meaning of $g(12) = 4$.

- ⊙ ***A***. Each pass costs $12, and it works for 4 days.
- ⊙ ***B***. If a pass costs $4, it will work for 12 days.
- ⊙ ***C***. If a pass costs $12, it will work for 4 days.
- ⊙ ***D***. Each pass costs $4, and it works for 12 days.

83. Maygen will spend $175 to purchase some bowls and some plates. Each bowl costs $3, and each plate costs $5. The function $p(b) = -\frac{3}{5}b + 35$ models the number of plates Maygen will purchase, where b represents the number of bowls Maygen will purchase.

Interpret the meaning of $p(45) = 8$.

- ⊙ ***A***. If 45 bowls are purchased, then 8 plates will be purchased.
- ⊙ ***B***. $8 will be used to purchase bowls, and $45 will be used to purchase plates.
- ⊙ ***C***. If 8 bowls are purchased, then 45 plates will be purchased.
- ⊙ ***D***. $45 will be used to purchase bowls, and $8 will be used to purchase plates.

84. Carly will spend $450 to purchase some bowls and some plates. Each plate costs $2, and each bowl costs $9. The function $q(x) = -\frac{2}{9}x + 50$ models the number of bowls Carly will purchase, where x represents the number of plates to be purchased.

Interpret the meaning of $q(27) = 44$.

- ⊙ ***A***. 44 plates and 27 bowls can be purchased.
- ⊙ ***B***. $44 will be used to purchase bowls, and $27 will be used to purchase plates.
- ⊙ ***C***. $27 will be used to purchase bowls, and $44 will be used to purchase plates.
- ⊙ ***D***. 27 plates and 44 bowls can be purchased.

85. Find a formula for the function f that gives the number of hours in x years.

86. Find a formula for the function f that gives the number of minutes in x days.

87. Suppose that M is the function that computes how many miles are in x feet. Find the formula for M. If you do not know how many feet are in one mile, you can look it up on Google.

Evaluate $M(13000)$ and interpret the result.

There are about [ ] miles in [ ] feet.

88. Suppose that K is the function that computes how many kilograms are in x pounds. Find the formula for K. If you do not know how many pounds are in one kilogram, you can look it up on Google.

Evaluate $K(159)$ and interpret the result.

Something that weighs [ ] pounds would weigh about [ ] kilograms.

89. Suppose that f is the function that the phone company uses to determine what your bill will be (in dollars) for a long-distance phone call that lasts t minutes. Each call costs a fixed price of \$2.65 plus 11 cents per minute. Write a formula for this linear function f.

90. Suppose that f is the function that gives the total cost (in dollars) of downhill skiing x times during a season with a \$500 season pass. Write a formula for f.

91. Suppose that f is the function that tells you how many dimes are in x dollars. Write a formula for f.

92. The function C models the the number of customers in a store t hours since the store opened.

t	0	1	2	3	4	5	6	7
$C(t)$	0	40	78	95	99	78	39	0

a. Find $C(6)$.

b. Interpret the meaning of $C(6)$.

- ⊙ ***A***. There were 39 customers in the store 6 hours after the store opened.
- ⊙ ***B***. In 6 hours since the store opened, the store had an average of 39 customers per hour.
- ⊙ ***C***. There were 6 customers in the store 39 hours after the store opened.
- ⊙ ***D***. In 6 hours since the store opened, there were a total of 39 customers.

c. Solve $C(t) = 78$ for t. $t =$ [ ]

d. Interpret the meaning of Part c's solution(s).

- ⊙ ***A***. There were 78 customers in the store 2 hours after the store opened.
- ⊙ ***B***. There were 78 customers in the store 2 hours after the store opened, and again 5 hours after the store opened.
- ⊙ ***C***. There were 78 customers in the store 5 hours after the store opened.
- ⊙ ***D***. There were 78 customers in the store either 2 hours after the store opened, or 5 hours after the store opened.

93. Let $s(t) = 13t^2 - 3t + 200$, where s is the position (in mi) of a car driving on a straight road at time t (in hr). The car's velocity (in mi/hr) at time t is given by $v(t) = 26t - 3$.

a. *Using function notation*, express the car's position after 1.5 hours. The answer here is not a formula, it's just something using function notation like `f(8)`.

b. Where is the car then? The answer here is a number with units.

c. *Use function notation* to express the question, "When is the car going 59 $\frac{\text{mi}}{\text{hr}}$?" The answer is an equation that uses function notation; something like `f(x)=23`. You are not being asked to actually solve the equation, just to write down the equation.

d. Where is the car when it is going 75 $\frac{\text{mi}}{\text{hr}}$? The answer here is a number with units. You are being asked a question about its position, but have been given information about its speed.

94. Let $s(t) = 13t^2 + t + 100$, where s is the position (in mi) of a car driving on a straight road at time t (in hr). The car's velocity (in mi/hr) at time t is given by $v(t) = 26t + 1$.

a. *Using function notation*, express the car's position after 3.4 hours. The answer here is not a formula, it's just something using function notation like `f(8)`.

b. Where is the car then? The answer here is a number with units.

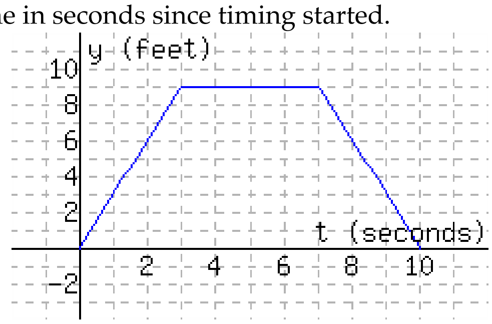
c. *Use function notation* to express the question, "When is the car going 58 $\frac{\text{mi}}{\text{hr}}$?" The answer is an equation that uses function notation; something like `f(x)=23`. You are not being asked to actually solve the equation, just to write down the equation.

d. Where is the car when it is going 27 $\frac{\text{mi}}{\text{hr}}$? The answer here is a number with units. You are being asked a question about its position, but have been given information about its speed.

95. Describe your own example of a function that has real context to it. You will need some kind of input variable, like "number of years since 2000" or "weight of the passengers in my car." You will need a process for using that number to bring about a different kind of number. The process does not need to involve a formula; a verbal description would be great, as would a formula.

Give your function a name. Write the symbol(s) that you would use to represent input. Write the symbol(s) that you would use to represent output.

96. The following figure has the graph $y = d(t)$, which models a particle's distance from the starting line in feet, where t stands for time in seconds since timing started.



a. Find $d(7)$.

b. Interpret the meaning of $d(7)$.

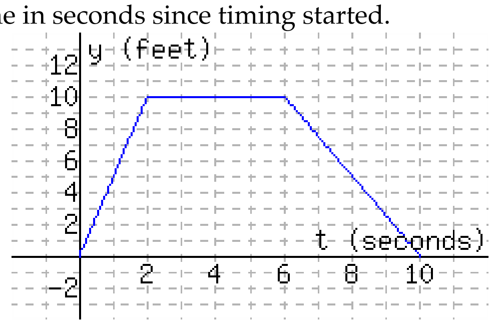
- ⊙ ***A***. In the first 7 seconds, the particle moved a total of 9 feet.
- ⊙ ***B***. The particle was 9 feet away from the starting line 7 seconds since timing started.
- ⊙ ***C***. In the first 9 seconds, the particle moved a total of 7 feet.
- ⊙ ***D***. The particle was 7 feet away from the starting line 9 seconds since timing started.

c. Solve $d(t) = 6$ for t. $t =$ ☐

d. Interpret the meaning of part c's solution(s).

- ⊙ *A*. The particle was 6 feet from the starting line 2 seconds since timing started, or 8 seconds since timing started.
- ⊙ *B*. The particle was 6 feet from the starting line 2 seconds since timing started, and again 8 seconds since timing started.
- ⊙ *C*. The particle was 6 feet from the starting line 2 seconds since timing started.
- ⊙ *D*. The particle was 6 feet from the starting line 8 seconds since timing started.

97. The following figure has the graph $y = d(t)$, which models a particle's distance from the starting line in feet, where t stands for time in seconds since timing started.



a. Find $d(9)$.

b. Interpret the meaning of $d(9)$.

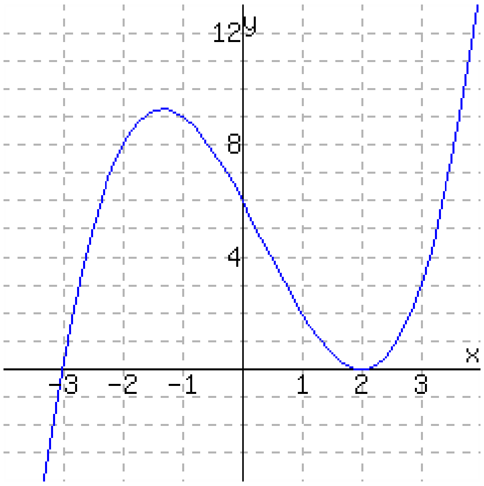
- ⊙ *A*. The particle was 2.5 feet away from the starting line 9 seconds since timing started.
- ⊙ *B*. The particle was 9 feet away from the starting line 2.5 seconds since timing started.
- ⊙ *C*. In the first 9 seconds, the particle moved a total of 2.5 feet.
- ⊙ *D*. In the first 2.5 seconds, the particle moved a total of 9 feet.

c. Solve $d(t) = 5$ for t. $t =$ ☐

d. Interpret the meaning of part c's solution(s).

- ⊙ *A*. The particle was 5 feet from the starting line 1 seconds since timing started.
- ⊙ *B*. The particle was 5 feet from the starting line 1 seconds since timing started, and again 8 seconds since timing started.
- ⊙ *C*. The particle was 5 feet from the starting line 1 seconds since timing started, or 8 seconds since timing started.
- ⊙ *D*. The particle was 5 feet from the starting line 8 seconds since timing started.

98. Use the graph of h in the figure to fill in the table.



x	-2	-1	0	1	2
$h(x)$	________	________	________	________	________

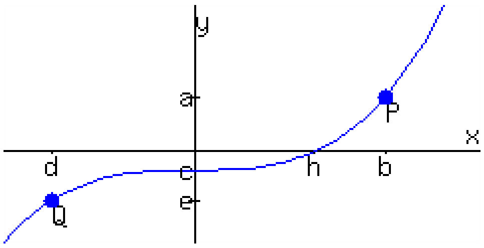
a. Evaluate $h(3) - h(0)$.

b. Evaluate $h(2) - h(-1)$.

c. Evaluate $2h(-1)$.

d. Evaluate $h(0) + 3$.

99. Use the given graph of a function f, along with a, b, c, d, e, and h to answer the following questions. Some answers are points, and should be entered as ordered pairs. Some answers ask you to solve for x, so the answer should be in the form ***x*=...**



a. What are the coordinates of the point P?

b. What are the coordinates of the point Q?

c. Evaluate $f(b)$. (The answer is symbolic, not a specific number.)

d. Solve $f(x) = e$ for x. (The answer is symbolic, not a specific number.)

e. Suppose $c = f(z)$. Solve the equation $z = f(x)$ for x.

11.2 Domain and Range

A function is a process for turning input values into output values. Occasionally a function f will have input values for which the process breaks down.

11.2.1 Domain

Example 11.2.2 Let P be the population of Portland as a function of the year. According to Google[1] we can say that:

$$P(2016) = 639863 \qquad P(1990) = 487849$$

But what if we asked to find $P(1600)$? The question doesn't really make sense anymore. The Multnomah tribe lived in villages in the area, but the city of Portland was not incorporated until 1851. We say that $P(1600)$ is *undefined*.

Example 11.2.3 If m is a person's mass in kg, let $w(m)$ be their weight in lb. There is an approximate formula for w:

$$w(m) \approx 2.2m$$

From this formula we can find:

$$w(50) \approx 110 \qquad w(80) \approx 176$$

which tells us that a 50- kg person weighs 110 lb, and an 80- kg person weighs 176 lb.

What if we asked for $w(-100)$? In the context of this example, we would be asking for the weight of a person whose mass is -100 kg. This is clearly nonsense. That means that $w(-100)$ is *undefined*. Note that the *context* of the example is telling us that $w(-100)$ is undefined even though the formula alone might suggest that $w(-100) = -220$.

Example 11.2.4 Let g have the formula

$$g(x) = \frac{x}{x-7}.$$

For most x-values, $g(x)$ is perfectly computable:

$$g(2) = -\frac{2}{5} \qquad g(14) = 2.$$

But if we try to compute $g(7)$, we run into an issue of arithmetic.

$$\begin{aligned} g(7) &= \frac{7}{7-7} \\ &= \frac{7}{0} \end{aligned}$$

The expression $\frac{7}{0}$ is *undefined*. There is no number that this could equal.

[1] https://www.google.com/publicdata/explore?ds=kf7tgg1uo9ude_&met_y=population&hl=en&dl=en#!ctype=l&strail=false &bcs=d&nselm=h&met_y=population&scale_y=lin&ind_y=false&rdim=country&idim=place:4159000&ifdim=country &hl=en_US&dl=en&ind=false

Checkpoint 11.2.5 If $f(x) = \frac{x+2}{x+8}$, find an input for f that would cause an undefined output.

The number [____] would cause an undefined output.

Explanation. Trying -8 as an input value would not work out; it would lead to division by 0.

These examples should motivate the following definition.

Definition 11.2.6 Domain. The **domain** of a function f is the collection of all of its valid input values. ◊

Example 11.2.7 Referring to the functions from Examples 11.2.2–11.2.4

- The domain of P is all years starting from 1851 and later. It would also be reasonable to say that the domain is actually all years from 1851 up to the current year, since we cannot guarantee that Portland will exist forever.
- The domain of w is all positive real numbers. It is nonsensical to have a person with negative mass or even one with zero mass. While there is some lower bound for the smallest mass a person could have, and also an upper bound for the largest mass a person could have, these boundaries are gray. We can say for sure that non-positive numbers should never be used as inputs for w.
- The domain of g is all real numbers except 7. This is the only number that causes a breakdown in g's formula.

11.2.2 Interval, Set, and Set-Builder Notation

Communicating the domain of a function can be wordy. In mathematics, we can communicate the same information using concise notation that is accepted for use almost everywhere. Table 11.2.8 contains example functions from this section and their domains, and demonstrates *interval notation* for these domains. Basic interval notation is covered in Section 1.3, but some of our examples here go beyond what that section covers.

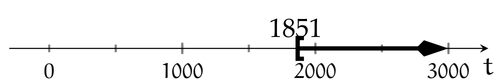
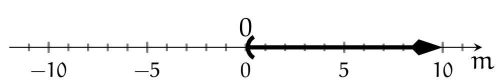
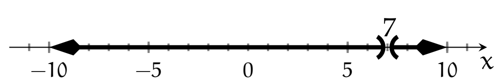
Function	Verbal Domain	Number Line Illustration	Interval Notation
P from Example 11.2.2	all years 1851 and greater		$[1851, \infty)$
w from Example 11.2.3	all real numbers greater than 0		$(0, \infty)$
g from Example 11.2.4	all real numbers except 7		$(-\infty, 7) \cup (7, \infty)$

Figure 11.2.8: Domains from Earlier Examples

Again, basic interval notation is covered in Section 1.3, but one thing appears in Table 11.2.8 that is not explained in that earlier section: the $\cup$ symbol, which we see in the domain of g.

Occasionally there is a need to consider number line pictures such as Figure 11.2.9, where two or more intervals appear.

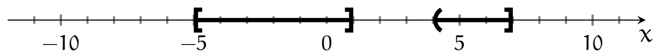


Figure 11.2.9: A number line with a union of two intervals

This picture is trying to tell you to consider numbers that are between -5 and 1, together with numbers that are between 4 and 7. That word "together" is related to the word "union," and in math the **union symbol**, $\cup$, captures this idea. It means to combine two ideas together, even if they are separate ideas. Think of it as putting everything from two baskets into one basket: a basket of oranges and a basket of apples combined into one big basket still contains oranges and apples, but now it can be thought of as a single idea. So we can write the numbers in this picture as

$$[-5, 1] \cup (4, 7]$$

(which uses interval notation).

With the domain of g in Table 11.2.8, the number line picture shows us another "union" of two intervals. They are very close together, but there are still two separated intervals in that picture: $(-\infty, 7)$ and $(7, \infty)$. Their union is represented by $(-\infty, 7) \cup (7, \infty)$.

Checkpoint 11.2.10 What is the domain of the function sqrt, where $\operatorname{sqrt}(x) = \sqrt{x}$, using interval notation?

Explanation. The function sqrt cannot take a negative number as an input. It can however take any positive number as input, or the number 0 as input. Representing this on a number line, we find the domain is $[0, \infty)$ in interval notation.

Checkpoint 11.2.11 What is the domain of the function ℓ where $\ell(x) = \frac{2}{x-3}$, using interval notation?

Explanation. The function ℓ cannot take a 3 as an input. It can however take any other number as input. Representing this on a number line, we have an interval $(-\infty, 3)$ to the left of 3, and $(3, \infty)$ to the right of 3. So we find the domain is $(-\infty, 3) \cup (3, \infty)$.

Sometimes we will consider collections of only a short list of numbers. In those cases, we use **set notation** (first introduced in Section A.6). With set notation, we have a list of numbers in mind, and we simply list all of those numbers. Curly braces are standard for surrounding the list. Table 11.2.12 illustrates set notation in use.

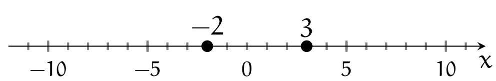
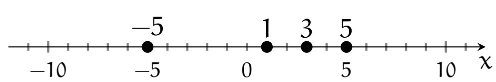
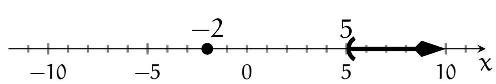
Picture of Set	Set Notation
	$\{-2, 3\}$
	$\{-5, 1, 3, 5\}$
	$\{-2\} \cup (5, \infty)$

Figure 11.2.12: Set Notation

Checkpoint 11.2.13 A change machine lets you put in an x-dollar bill, and gives you $f(x)$ nickels in return equal in value to x dollars. Any current, legal denomination of US paper money can be fed to the change machine. What is the domain of f?

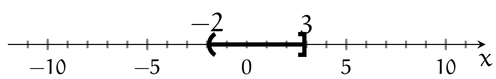
Explanation. The current, legal denominations of US paper money are \$1, \$2, \$5, \$10, \$20, \$50, and \$100. So the domain of f is the set $\{1, 2, 5, 10, 20, 50, 100\}$.

While most collections of numbers that we will encounter can be described using a combination of interval notation and set notation, there is another commonly used notation that is very useful in algebra: **set-builder notation**, which was introduced in Section 1.3. Set-builder notation also uses curly braces. Set-builder notation provides a template for what a number that is under consideration might look like, and then it gives you restrictions on how to use that template. A very basic example of set-builder notation is

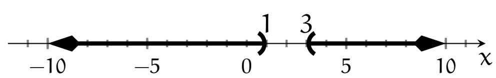
$$\{x \mid x \geq 3\}.$$

Verbally, this is "the set of all x such that x is greater than or equal to 3." Table 11.2.14 gives more examples of set-builder notation in use.

Picture of Set | Set Notation



$\{x \mid -2 < x \leq 3\}$



$\{x \mid x < 1 \text{ or } x > 3\}$

Figure 11.2.14: Set-Builder Notation

Checkpoint 11.2.15 What is the domain of the function sqrt, where $\operatorname{sqrt}(x) = \sqrt{x}$, using set-builder notation?

Explanation. The function sqrt cannot take a negative number as an input. It can however take any positive number as input, or the number 0 as input. Representing this on a number line, we find the domain is $\{x \mid x \geq 0\}$ in set-builder notation.

Example 11.2.16 What is the domain of the function A, where $A(x) = \frac{2x+1}{x^2-2x-8}$?

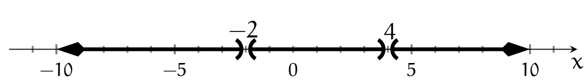
Note that if you plugged in some value for x, the only thing that might go wrong is if the denominator equals 0. So a *bad* value for x would be when

$$x^2 - 2x - 8 = 0$$
$$(x + 2)(x - 4) = 0$$

Here, we used a basic factoring technique from Section 10.3. To continue, either

$$\begin{aligned} x + 2 &= 0 &\quad \text{or} \quad && x - 4 &= 0 \\ x &= -2 &\quad \text{or} \quad && x &= 4. \end{aligned}$$

These are the *bad* x-values because they lead to division by 0 in the formula for A. So on a number line, if we wanted to picture the domain of A, we would make a sketch like:



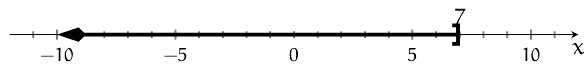
So the domain is the union of three intervals: $(-\infty, -2) \cup (-2, 4) \cup (4, \infty)$.

Example 11.2.17 What is the domain of the function B, where $B(x) = \sqrt{7-x} + 3$?

Note that if you plugged in some value for x, the only thing that might go wrong is if the value in the radical is negative. So the *good* values for x would be when

$$\begin{aligned} 7 - x &\geq 0 \\ 7 &\geq x \\ x &\leq 7 \end{aligned}$$

So on a number line, if we wanted to picture the domain of B, we would make a sketch like:



So the domain is the interval $(-\infty, 7]$.

There are three main properties of functions that cause numbers to be excluded from a domain, which are summarized here.

List 11.2.18: Summary of Domain Restrictions

Denominators Division by zero is undefined. So if a function contains an expression in a denominator, it will only be defined where that expression is not equal to zero.

Example 11.2.16 demonstrates this.

Square Roots The square root of a negative number is undefined. So if a function contains a square root, it will only be defined when the expression inside that radical is greater than or equal to zero. (This is actually true for any even nth radical.)

Example 11.2.17 demonstrates this.

Context Some numbers are nonsensical in context. If a function has real-world context, then this may add additional restrictions on the input values.

Example 11.2.3 demonstrates this.

11.2.3 Range

The domain of a function is the collection of its valid inputs; there is a similar notion for *output*.

Definition 11.2.19 Range. The **range** of a function f is the collection of all of its possible output values. ◊

Example 11.2.20 Let f be the function defined by the formula $f(x) = x^2$. Finding f's *domain* is straightforward. Any number anywhere can be squared to produce an output, so f has domain $(-\infty, \infty)$. What is the *range* of f?

Explanation. We would like to describe the collection of possible numbers that f can give as output. We will use a graphical approach. Figure 11.2.21 displays a graph of f, and the visualization that reveals f's range.

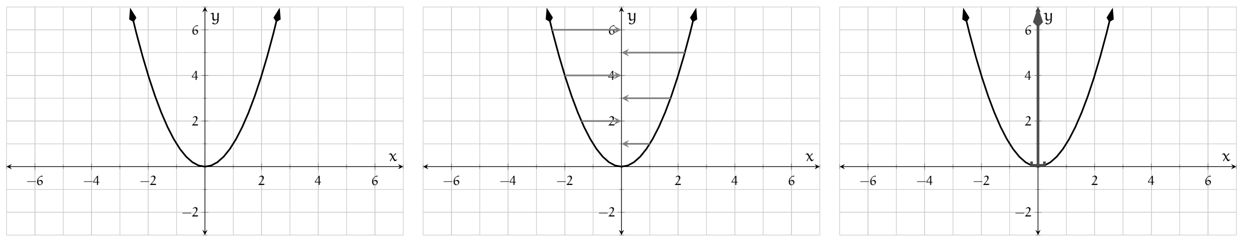


Figure 11.2.21: $y = f(x)$ where $f(x) = x^2$. The second graph illustrates how to visualize the range. In the third graph, the range is marked as an interval on the y-axis.

Output values are the y-coordinates in a graph. If we "slide the ink" left and right over to the y-axis to emphasize what the y-values in the graph are, we have y-values that start from 0 and continue upward forever. Therefore the range is $[0, \infty)$.

Example 11.2.22 Given the function g graphed in Figure 11.2.23, find the domain and range of g.

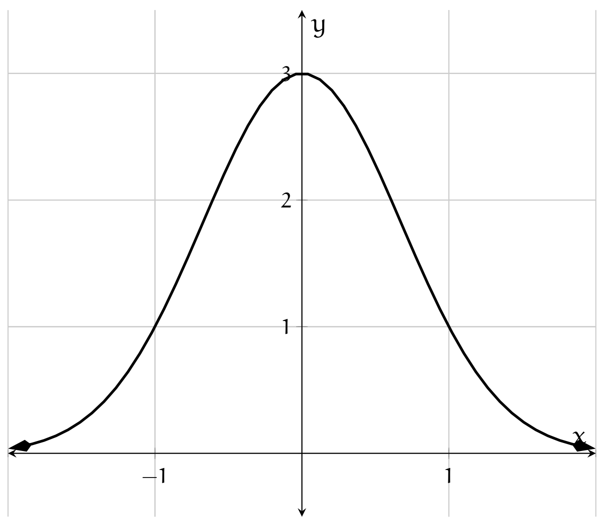


Figure 11.2.23: $y = g(x)$

Explanation. To find the domain, we can visualize all of the x-values that are valid inputs for this function by "sliding the ink" down onto the x-axis. The arrows at the far left and far right of the curve indicate that whatever pattern we see in the graph continues off to the left and right. Here, we see that the arms of the graph appear to be tapering down to the x-axis and extending left and right forever. Every x-value can be used to get an output for the function, so the domain is $(-\infty, \infty)$.

If we visualize the possible *outputs* by "sliding the ink" sideways onto the y-axis, we find that outputs as high as 3 are possible (including 3 itself). The outputs appear to get very close to 0 when x is large, but they aren't quite equal to 0. So the range is $(0, 3]$.

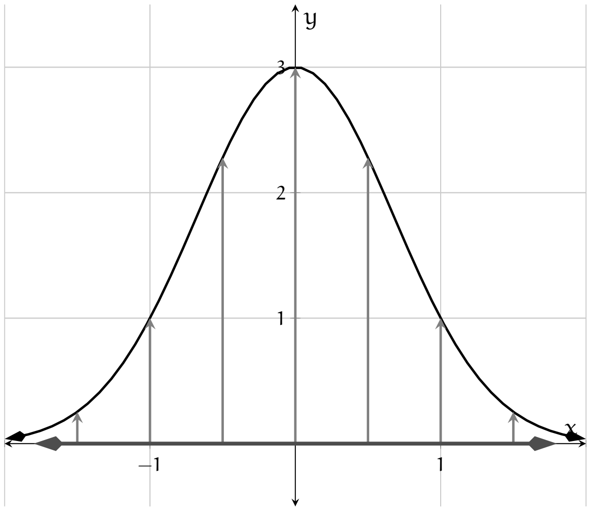


Figure 11.2.24: Domain of g

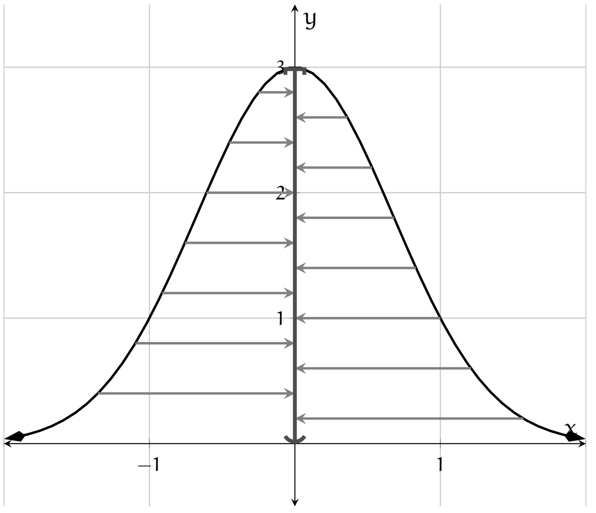
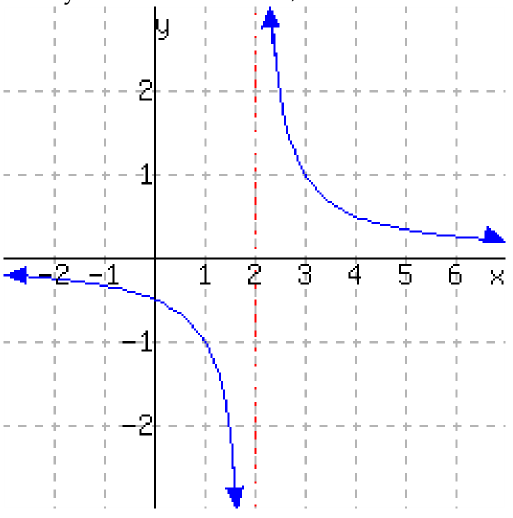


Figure 11.2.25: Range of g

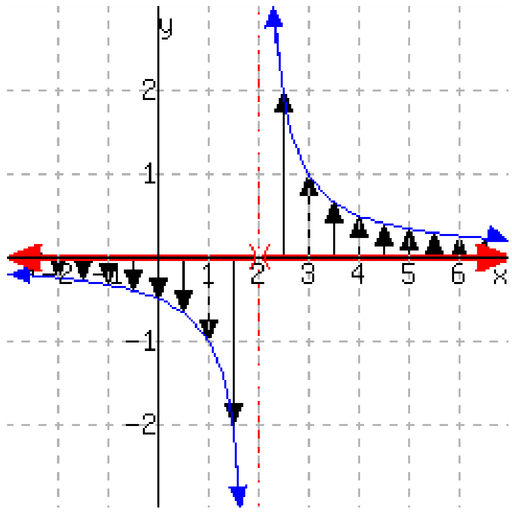
Checkpoint 11.2.26 Given the function h graphed below, find the domain and range of h. Note there is an invisible vertical line at $x = 2$, and the two arms of the graph are extending downward (and upward) forever, getting arbitrarily close to that vertical line, but never touching it. Also note that the two arms extend forever to the left and right, getting arbitrarily close to the x-axis, but never touching it.



The domain of h is ______ and the range of h is ______.

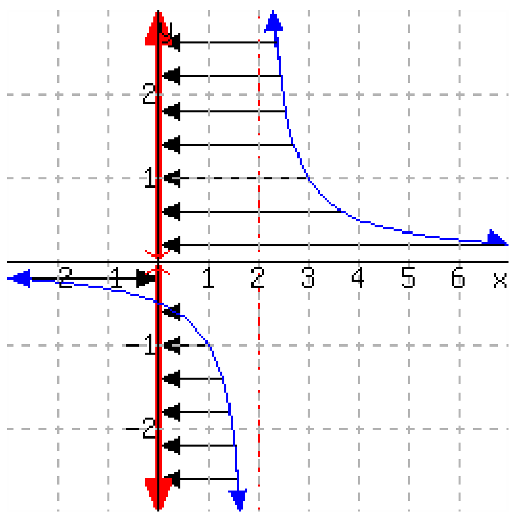
Explanation. To find the domain, we try to visualize all of the x-values that are valid inputs for this function. The arrows pointing left and right on the curve indicate that whatever pattern we see in the graph continues off to the left and right. So for x-values far to the right or left, we will be able to get an output for h.

The arrows pointing up and down are supposed to indicate that the curve will get closer and closer to the vertical line $x = 2$ after the curve leaves the viewing window we are using. So even when x is some number very close to 2, we will be able to get an output for h.



The one x-value that doesn't behave is $x = 2$. If we tried to use that as an input, there is no point on the graph directly above or below that on the x-axis. So the domain is $(-\infty, 2) \cup (2, \infty)$.

To find the range, we try to visualize all of the y-values that are possible outputs for this function. Sliding the ink of the curve left/right onto the y-axis reveals that $y = 0$ is the only y-value that we could never obtain as an output. So the range is $(-\infty, 0) \cup (0, \infty)$.



The examples of finding domain and range so far have all involved either a verbal description of a function, a formula for that function, or a graph of that function. Recall that there is a fourth perspective on functions: a table. In the case of a table, we have very limited information about the function's inputs and outputs. If the table is all that we have, then there are a handful of input values listed in the table for which we know outputs. For any other input, the output is undefined.

Example 11.2.27 Consider the function k given in Figure 11.2.28. What is the domain and range of k?

x	$k(x)$
3	4
8	5
10	5

Figure 11.2.28

Explanation. All that we know about k is that $k(3) = 4$, $k(8) = 5$, and $k(10) = 5$. Without any other information such as a formula for k or a context for k that tells us its verbal description, we must assume that its domain is $\{3, 8, 10\}$; these are the only valid input for k. Similarly, k's range is $\{4, 5\}$.

Note that we have used set notation, not interval notation, since the answers here were *lists* of x-values (for the domain) and y-values (for the range). Also note that we could graph the information that we have about k in Figure 11.2.29, and the visualization of "sliding ink" to determine domain and range still works.

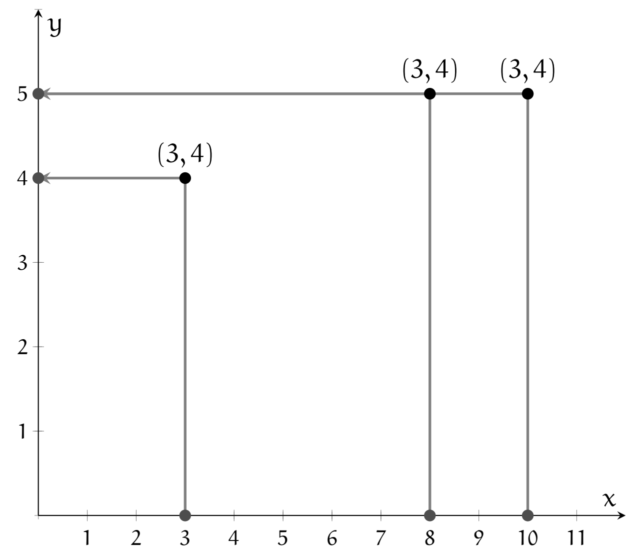


Figure 11.2.29

Warning 11.2.30 Finding Range from a Formula. Sometimes it is possible to find the range of a function using its formula without seeing its graph or a table. However, this often requires advanced techniques learned in calculus. Therefore when you are asked to find the range of a function based on its formula alone, your approach should be to examine a graph.

11.2.4 Reading Questions

1. Use a complete sentence to describe what is the domain of a function.

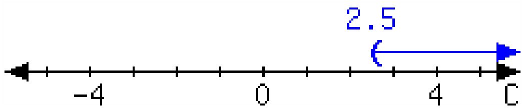
2. When you have a formula for a function, what is one thing that might tell you a number that is excluded from the domain?

3. To find the range of a function, it's more helpful to have its $\boxed{}$ than its formula.

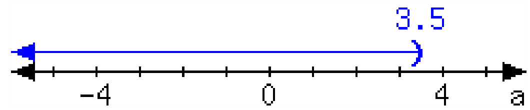
11.2.5 Exercises

Review and Warmup

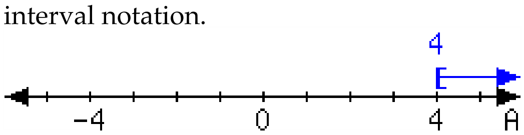
1. For the interval expressed in the number line, write it using set-builder notation and interval notation.



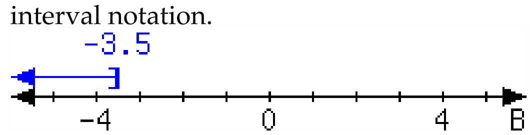
2. For the interval expressed in the number line, write it using set-builder notation and interval notation.



3. For the interval expressed in the number line, write it using set-builder notation and interval notation.



4. For the interval expressed in the number line, write it using set-builder notation and interval notation.



5. Solve this compound inequality, and write your answer in *interval notation*.
$x \geq -2$ and $x \leq -1$

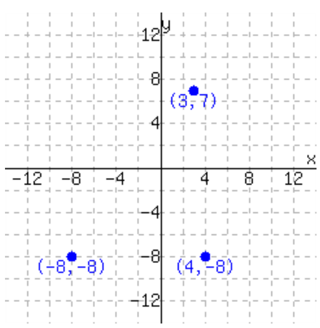
6. Solve this compound inequality, and write your answer in *interval notation*.
$x \geq 2$ and $x < 4$

7. Solve this compound inequality, and write your answer in *interval notation*.
$x \geq 2$ or $x \leq -2$

8. Solve this compound inequality, and write your answer in *interval notation*.
$x > -1$ or $x < -5$

Domain and Range From a Graph A function is graphed. Find its domain and range.

9.



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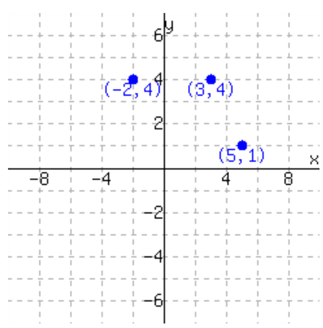

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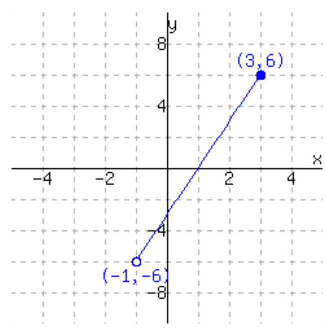

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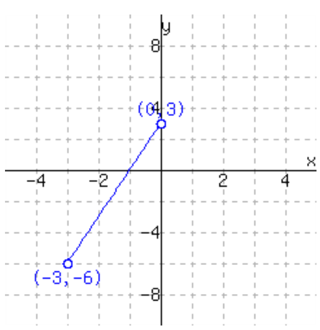

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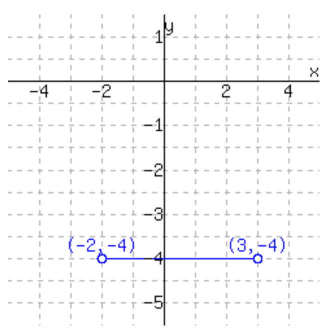

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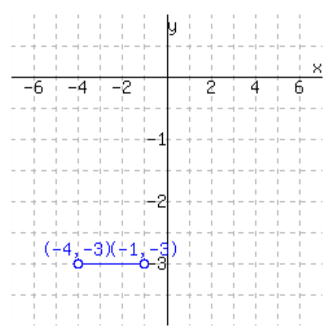

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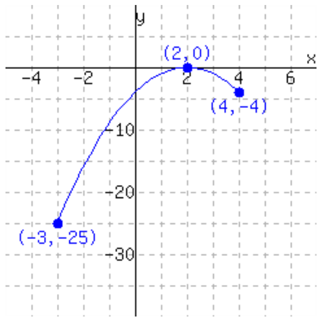

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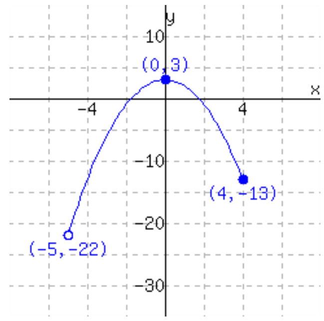

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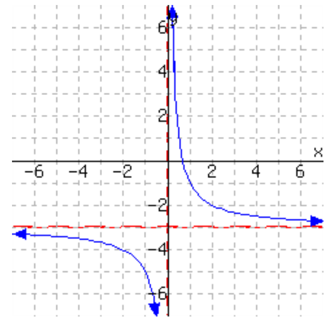

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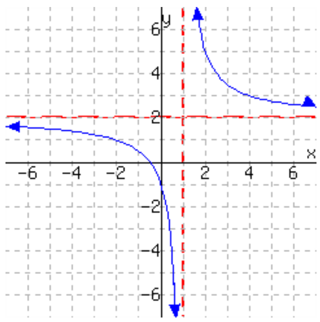

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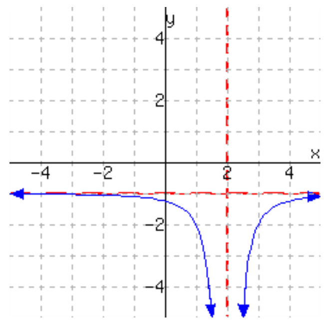

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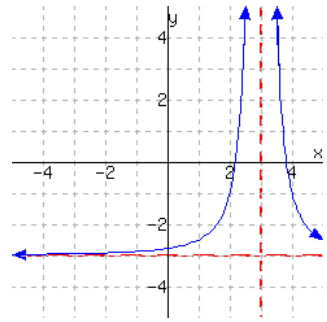

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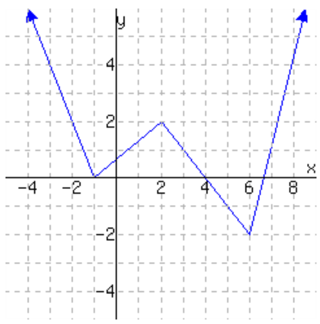

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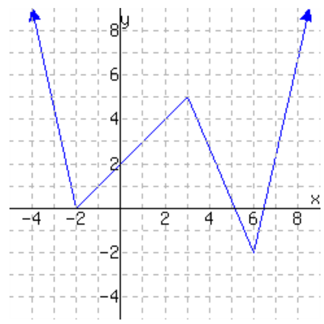

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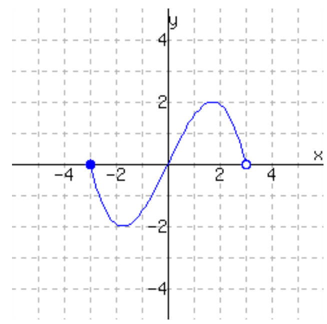

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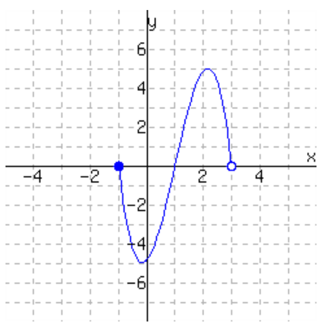

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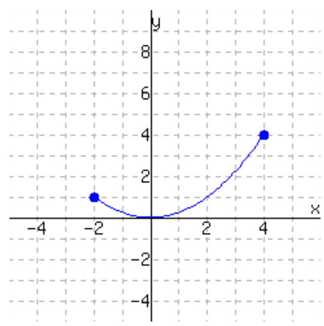

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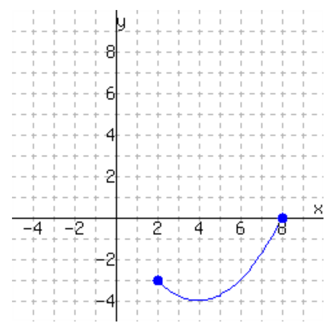

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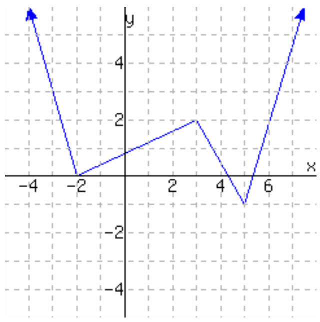

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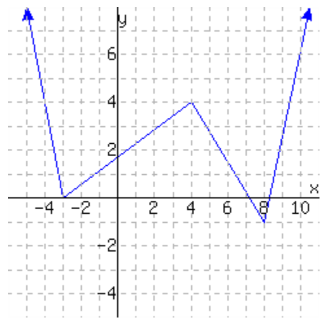

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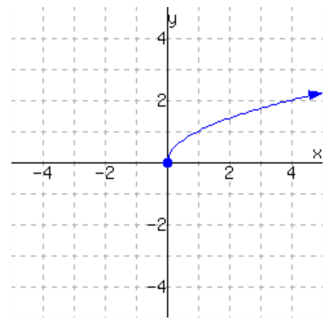

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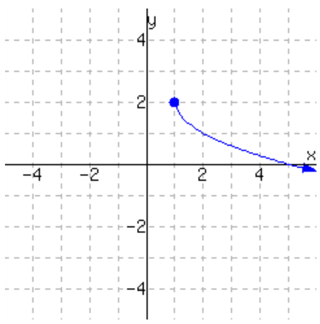

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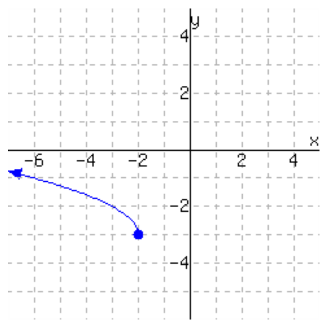

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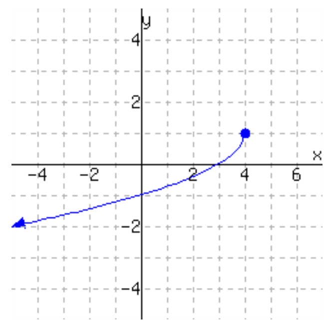

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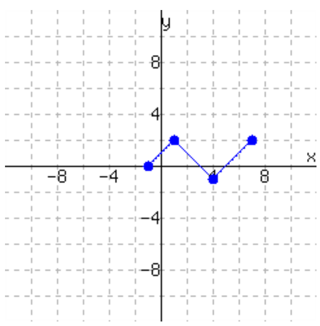

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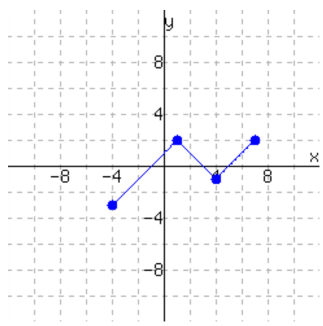

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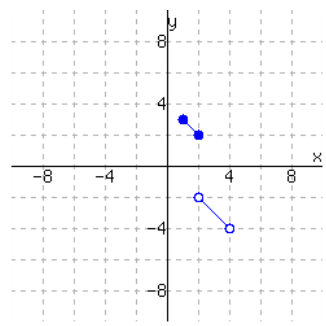

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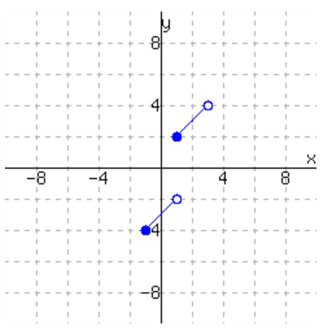


Domain From a Formula

37. Find the domain of H where $H(x) = -7x + 4$.

38. Find the domain of K where $K(x) = 7x - 8$.

39. Find the domain of K where $K(x) = \frac{5}{6}x^4$.

40. Find the domain of f where $f(x) = \frac{2}{5}x^3$.

41. Find the domain of g where $g(x) = |6x - 3|$.

42. Find the domain of h where $h(x) = |-2x + 6|$.

43. Find the domain of h where $h(x) = \frac{2x}{x+9}$.

44. Find the domain of F where $F(x) = \frac{5x}{x-5}$.

45. Find the domain of G where $G(x) = \frac{x}{2x+3}$.

46. Find the domain of H where $H(x) = \frac{4x}{7x+10}$.

47. Find the domain of K where $K(x) = \frac{4x+8}{x^2 - 11x + 30}$.

48. Find the domain of K where $K(x) = -\frac{3x+5}{x^2 - x - 30}$.

49. Find the domain of f where $f(x) = \frac{10x+4}{x^2+9x}$.

50. Find the domain of g where $g(x) = \frac{3x-8}{x^2+3x}$.

51. Find the domain of h where $h(x) = \frac{1-4x}{x^2-49}$.

52. Find the domain of h where $h(x) = \frac{9x+9}{x^2-81}$.

53. Find the domain of F where $F(x) = \frac{2x-3}{16x^2-49}$.

54. Find the domain of G where $G(x) = \frac{6-5x}{81x^2-4}$.

55. Find the domain of H where $H(x) = \frac{8x-6}{x^2+3}$.

56. Find the domain of K where $K(x) = \frac{x+2}{x^2+10}$.

57. Find the domain of the function.
$K(x) = -\frac{6}{\sqrt{x-10}}$

58. Find the domain of the function.
$f(x) = \frac{8}{\sqrt{x-1}}$

59. Find the domain of the function.
$g(x) = \sqrt{6-x}$

60. Find the domain of the function.
$h(x) = \sqrt{3-x}$

61. Find the domain of the function.
$h(x) = \sqrt{9+13x}$

62. Find the domain of the function.
$F(x) = \sqrt{6+11x}$

63. Find the domain of A where $A(x) = \frac{x+13}{x^2-9}$.

64. Find the domain of m where $m(x) = \frac{x+16}{x^2-361}$.

65. Find the domain of b where $b(x) = \frac{16x-2}{x^2+7x-98}$.

66. Find the domain of m where $m(x) = \frac{16x-11}{x^2+7x-98}$.

67. Find the domain of r where $r(x) = \frac{\sqrt{2+x}}{8-x}$.

68. Find the domain of B where $B(x) = \frac{\sqrt{4+x}}{4-x}$.

Domain and Range Using Context

69. Thanh bought a used car for \$7,800. The car's value decreases at a constant rate each year. After 5 years, the value decreased to \$6,300.

Use a function to model the car's value as the number of years increases. Find this function's domain and range in this context.

The function's domain in this context is ______.

The function's range in this context is ______.

70. Carmen bought a used car for $8,400. The car's value decreases at a constant rate each year. After 10 years, the value decreased to $5,400.

Use a function to model the car's value as the number of years increases. Find this function's domain and range in this context.

The function's domain in this context is ______.

The function's range in this context is ______.

71. Assume a car uses gas at a constant rate. After driving 25 miles since a full tank of gas was purchased, there was 9 gallons of gas left; after driving 65 miles since a full tank of gas was purchased, there was 7.4 gallons of gas left.

Use a function to model the amount of gas in the tank (in gallons). Let the independent variable be the number of miles driven since a full tank of gas was purchased. Find this function's domain and range in this context.

The function's domain in this context is ______.

The function's range in this context is ______.

72. Assume a car uses gas at a constant rate. After driving 30 miles since a full tank of gas was purchased, there was 13.2 gallons of gas left; after driving 60 miles since a full tank of gas was purchased, there was 11.4 gallons of gas left.

Use a function to model the amount of gas in the tank (in gallons). Let the independent variable be the number of miles driven since a full tank of gas was purchased. Find this function's domain and range in this context.

The function's domain in this context is ______.

The function's range in this context is ______.

73. Joseph inherited a collection of coins when he was 14 years old. Ever since, he has been adding into the collection the same number of coins each year. When he was 20 years old, there were 510 coins in the collection. When he was 30 years old, there were 910 coins in the collection. At the age of 51, Joseph donated all his coins to a museum.

Use a function to model the number of coins in Joseph's collection, starting in the year he inherited the collection, and ending in the year the collection was donated. Find this function's domain and range in this context.

The function's domain in this context is ______.

The function's range in this context is ______.

74. Virginia inherited a collection of coins when she was 15 years old. Ever since, she has been adding into the collection the same number of coins each year. When she was 20 years old, there were 330 coins in the collection. When she was 30 years old, there were 530 coins in the collection. At the age of 57, Virginia donated all her coins to a museum.

Use a function to model the number of coins in Virginia's collection, starting in the year she inherited the collection, and ending in the year the collection was donated. Find this function's domain and range in this context.

The function's domain in this context is ______.

The function's range in this context is ______.

75. Assume a tree grows at a constant rate. When the tree was planted, it was 4 feet tall. After 8 years, the tree grew to 10.4 feet tall.

Use a function to model the tree's height as years go by. Assume the tree can live 190 years, find this function's domain and range in this context.

The function's domain in this context is ______.

The function's range in this context is ______.

76. Assume a tree grows at a constant rate. When the tree was planted, it was 2.1 feet tall. After 10 years, the tree grew to 8.1 feet tall.

Use a function to model the tree's height as years go by. Assume the tree can live 170 years, find this function's domain and range in this context.

The function's domain in this context is ______.

The function's range in this context is ______.

77. An object was shot up into the air at an initial vertical speed of 384 feet per second. Its height as time passes can be modeled by the quadratic function f, where $f(t) = -16t^2 + 384t$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height in feet.

Find the function's domain and range in this context.

The function's domain in this context is ______.

The function's range in this context is ______.

78. An object was shot up into the air at an initial vertical speed of 416 feet per second. Its height as time passes can be modeled by the quadratic function f, where $f(t) = -16t^2 + 416t$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height in feet.

Find the function's domain and range in this context.

The function's domain in this context is ______.

The function's range in this context is ______.

79. From a clifftop over the ocean 376.32 m above sea level, an object was shot straight up into the air with an initial vertical speed of 110.74 $\frac{\text{m}}{\text{s}}$. On its way down it missed the cliff and fell into the ocean, where it floats on the surface. Its height (above sea level) as time passes can be modeled by the quadratic function f, where $f(t) = -4.9t^2 + 110.74t + 376.32$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height (above sea level) in meters.

Find the function's domain and range in this context.

The function's domain in this context is ______.

The function's range in this context is ______.

80. From a clifftop over the ocean 324.87 m above sea level, an object was shot straight up into the air with an initial vertical speed of 93.59 $\frac{\text{m}}{\text{s}}$. On its way down it missed the cliff and fell into the ocean, where it floats on the surface. Its height (above sea level) as time passes can be modeled by the quadratic function f, where $f(t) = -4.9t^2 + 93.59t + 324.87$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height (above sea level) in meters.

Find the function's domain and range in this context.

The function's domain in this context is $\boxed{}$.

The function's range in this context is $\boxed{}$.

81. You will build a rectangular sheep pen next to a river. There is no need to build a fence along the river, so you only need to build three sides. You have a total of 460 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum area.

Use a function to model the area of the rectangular pen, with respect to the length of the width (the two sides perpendicular to the river). Find the function's domain and range in this context.

The function's domain is $\boxed{}$.

The function's range is $\boxed{}$.

82. You will build a rectangular sheep pen next to a river. There is no need to build a fence along the river, so you only need to build three sides. You have a total of 480 feet of fence to use. Find the dimensions of the pen such that you can enclose the maximum area.

Use a function to model the area of the rectangular pen, with respect to the length of the width (the two sides perpendicular to the river). Find the function's domain and range in this context.

The function's domain is $\boxed{}$.

The function's range is $\boxed{}$.

83. A student's first name is a function of their student identification number.

(a) Describe the domain for this function in a sentence. Specifics are not needed.

(b) Describe the range for this function in a sentence. Specifics are not needed.

84. The year a car was made is a function of its VIN (Vehicle Identification Number).

(a) Describe the domain for this function in a sentence. Specifics are not needed.

(b) Describe the range for this function in a sentence. Specifics are not needed.

Challenge

85. For each part, sketch the graph of a function with the given domain and range.

a. The domain is $(0, \infty)$ and the range is $(-\infty, 0)$.

b. The domain is $(1, 2)$ and the range is $(3, 4)$.

c. The domain is $(0, \infty)$ and the range is $[2, 3]$.

d. The domain is $(1, 2)$ and the range is $(-\infty, \infty)$.

e. The domain is $(-\infty, \infty)$ and the range is $(-1, 1)$.

f. The domain is $(0, \infty)$ and the range is $[0, \infty)$.

11.3 Using Technology to Explore Functions

Graphing technology allows us to explore the properties of functions more deeply than we can with only pencil and paper. It can quickly create a table of values, and quickly plot the graph of a function. Such technology can also evaluate functions, solve equations with functions, find maximum and minimum values, and explore other key features.

There are many graphing technologies currently available, including (but not limited to) physical (handheld) graphing calculators, *Desmos*, *GeoGebra*, *Sage*, and *WolframAlpha*.

This section will focus on *how* technology can be used to explore functions and their key features. Although the choice of particular graphing technology varies by each school and curriculum, the main ways in which technology is used to explore functions is the same and can be done with each of the technologies above.

11.3.1 Finding an Appropriate Window

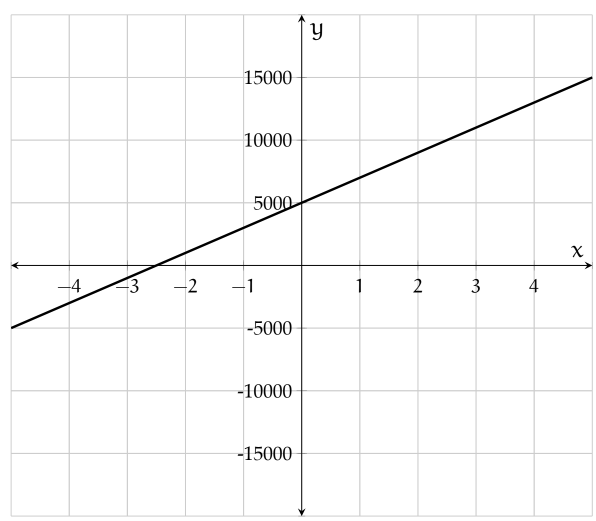
With a simple linear equation like $y = 2x + 5$, most graphing technologies will show this graph in a good window by default. A common default window goes from $x = -10$ to $x = 10$ and $y = -10$ to $y = 10$.

What if we wanted to graph something with a much larger magnitude though, such as $y = 2000x+5000$? If we tried to view this for $x = -10$ to $x = 10$ and $y = -10$ to $y = 10$, the function would appear as an almost vertical line since it has such a steep slope.

Using technology, we will create a table of values for this function as shown in Figure 11.3.2(a). Then we will set the x-values for which we view the function to go from $x = -5$ to $x = 5$ and the y-values from $y = -20{,}000$ to $y = 20{,}000$. The graph is shown in Figure 11.3.2(b).

x	$y = 2000x + 5000$
-5	-5000
-4	-3000
-3	-1000
-2	1000
-1	3000
0	5000
1	7000
2	9000
3	11000
4	13000
5	15000

(a) A table of values



(b) Graphed with an appropriate window

Figure 11.3.2: Creating a table of values to determine an appropriate graphing window

Now let's practice finding an appropriate viewing window with a less familiar function.

Example 11.3.3 Find an appropriate window for $q(x) = \frac{x^3}{100} - 2x + 1$.

Entering this function into graphing technology, we input `q(x)=(x^3)/100-2x+1`. A default window will generally give us something like this:

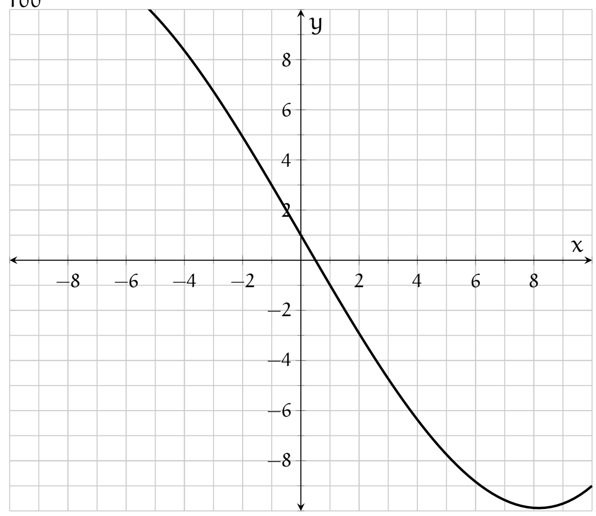


Figure 11.3.4: Function q graphed in the default window.

We can tell from the lower right corner of Figure 11.3.4 that we're not quite viewing all of the important details of this function. To determine a better window, we could use technology to make a table of values. Another more rudimentary option is to double the viewing constraints for x and y, as shown in Figure 11.3.5. Many graphing technologies have the ability to zoom in and out quickly.

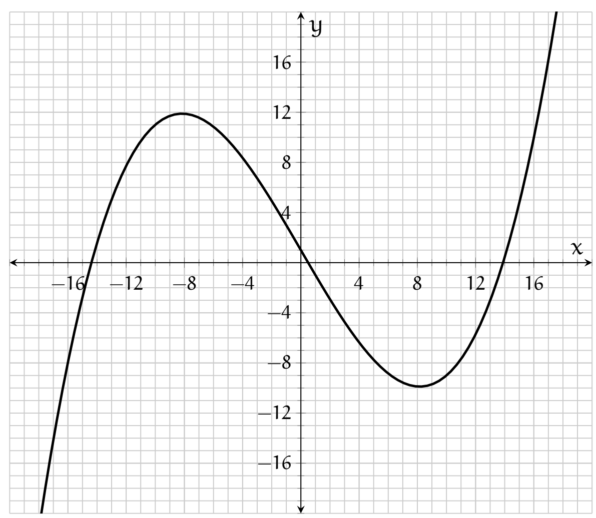


Figure 11.3.5: Function q graphed in an expanded window.

11.3.2 Using Technology to Determine Key Features of a Graph

The key features of a graph can be determined using graphing technology. Here, we'll show how to determine the x-intercepts, y-intercepts, and maximum/minimum values using technology.

Example 11.3.6 Graph the function given by $p(x) = -1000x^2 - 100x + 40$. Determine an appropriate viewing window, and then use graphing technology to determine the following:

a. Determine the x-intercepts of the function.

b. Determine the y-intercept of the function.

c. Determine the maximum function value and where it occurs.

Explanation.

To start, we'll take a quick view of this function in a default window. We can see that we need to zoom in on the x-values, but we need to zoom out on the y-values.
From the graph we see that the x-values might as well run from about -0.5 to 0.5, so we will look at x-values in that window in increments of 0.1, as shown in Table 11.3.8(a). This table allows us to determine an appropriate viewing window for $y = p(x)$ which is shown in Figure 11.3.8(b). The table suggests we should go a little higher than 40 on the y-axis, and it would be OK to go the same distance in the negative direction to keep the x-axis centered.

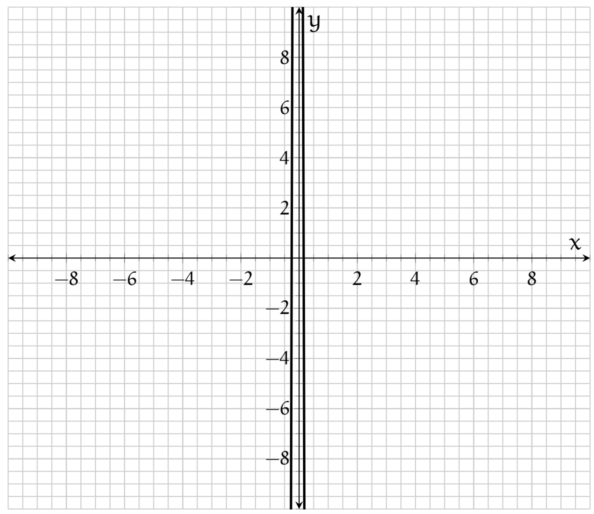
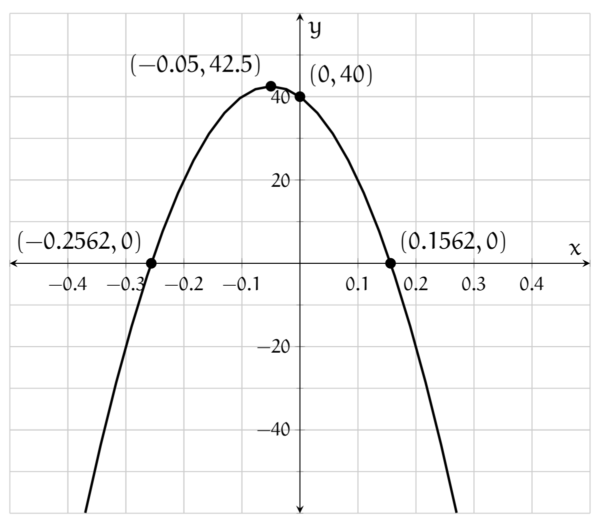


Figure 11.3.7: Graph of $y = p(x)$ in an inappropriate window

x	$p(x)$
-0.5	-160
-0.4	-80
-0.3	-20
-0.2	20
-0.1	40
0	40
0.1	20
0.2	-20
0.3	-80
0.4	-160
0.5	-260

(a) Function values for $y = p(x)$



(b) Graph of $y = p(x)$ in an appropriate window showing key features

Figure 11.3.8: Creating a table of values to determine an appropriate graphing window

We can now use Figure 11.3.8(b) to determine the x-intercepts, the y-intercept, and the maximum function value.

a. To determine the x-intercepts, we will find the points where y is zero. These are about $(-0.2562, 0)$ and $(0.1562, 0)$.

b. To determine the y-intercept, we need the point where x is zero. This point is $(0, 40)$.

c. The highest point on the graph is the vertex, which is about $(-0.05, 42.5)$. So the maximum function value is 42.5 and occurs at -0.05.

d. We can see that the function is defined for all x-values, so the domain is $(-\infty, \infty)$. The maximum function value is 42.5, and there is no minimum function value. Thus the range is $(-\infty, 42.5]$.

Example 11.3.9 Graphing Technology Limitations. If we use graphing technology to graph the function g where $g(x) = 0.0002x^2 + 0.00146x + 0.00266$, we may be mislead by the way values are rounded. Without technology, we know that this function is a quadratic function and therefore has at most two x-intercepts and has a vertex that will determine the minimum function value. However, using technology we could obtain a graph with the following key points:

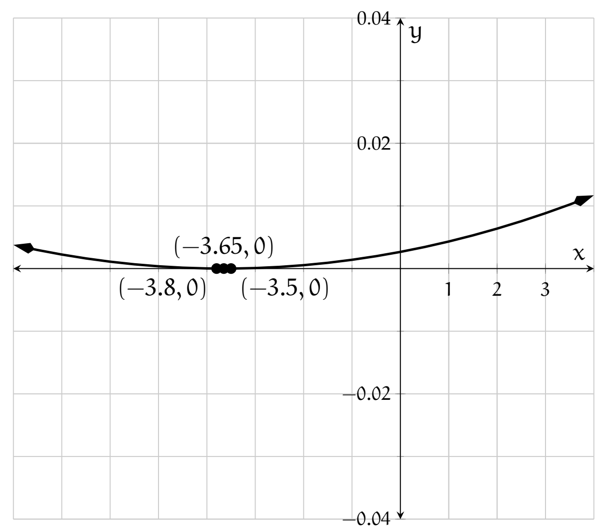


Figure 11.3.10: Misleading graph

This *looks* like there are three x-intercepts, which we know is not possible for a quadratic function. We can evaluate g at $x = -3.65$ and determine that $g(-3.65) = -0.0000045$, which is *approximately* zero when rounded. So the true vertex of this function is $(-3.65, -0.0000045)$, and the minimum value of this function is -0.0000045 (not zero).

Every graphing tool generally has some type of limitation like this one, and it's good to be aware that these limitations exist.

11.3.3 Solving Equations and Inequalities Graphically Using Technology

To *algebraically* solve an equation like $h(x) = v(x)$ for

$$h(x) = -0.01(x - 90)(x + 20) \qquad \text{and} \qquad v(x) = -0.04(x - 10)(x - 80),$$

we'd start by setting up

$$-0.01(x - 90)(x + 20) = -0.04(x - 10)(x - 80)$$

To solve this, we'd then simplify each side of the equation, set it equal to zero, and finally use the quadratic formula like we did in Section 7.2.

An alternative is to *graphically* solve this equation, which is done by graphing

$$y = -0.01(x-90)(x+20) \qquad \text{and} \qquad y = -0.04(x-10)(x-80).$$

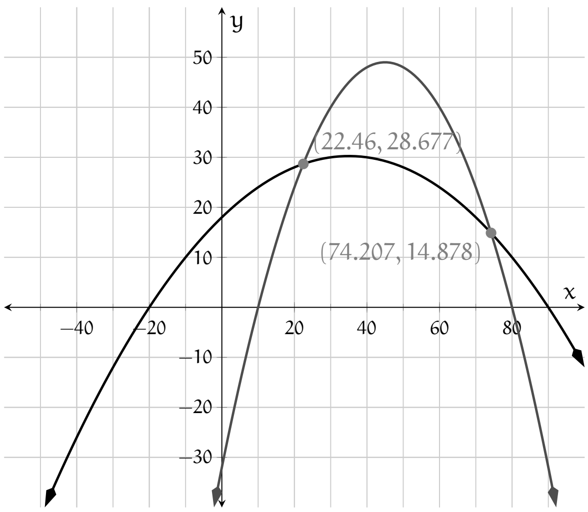


Figure 11.3.11: Points of intersection for $h(x) = v(x)$

The points of intersection, $(22.46, 28.677)$ and $(74.207, 14.878)$, show where these functions are equal. This means that the x-values give the solutions to the equation $-0.01(x-90)(x+20) = -0.04(x-10)(x-80)$. So the solutions are approximately 22.46 and 74.207, and the solution set is approximately $\{22.46, 74.207\}$.

Similarly, to *graphically* solve an equation like $h(x) = 25$ for

$$h(x) = -0.01(x-90)(x+20),$$

we can graph

$$y = -0.01(x-90)(x+20) \qquad \text{and} \qquad y = 25$$

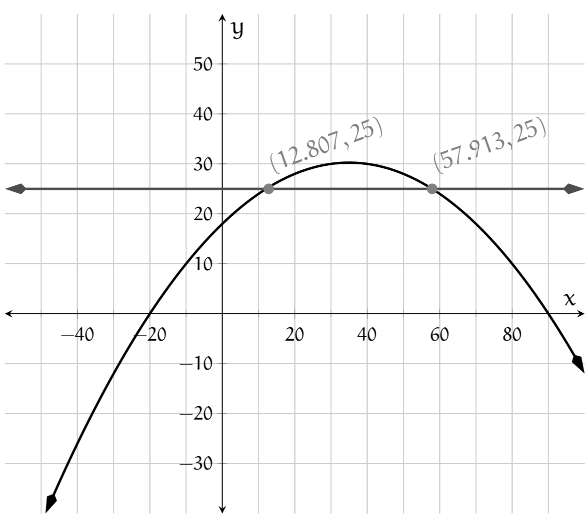


Figure 11.3.12: Points of intersection for $h(x) = 25$

The points of intersection are $(12.807, 25)$ and $(57.913, 25)$, which tells us that the solutions to $h(x) = 25$ are approximately 12.807 and 57.913. The solution set is approximately $\{12.807, 57.913\}$.

Example 11.3.13 Use graphing technology to solve the following inequalities:

a. $-20t^2 - 70t + 300 \geq -5t + 300$

b. $-20t^2 - 70t + 300 < -5t + 300$

Explanation. To solve these inequalities graphically, we will start by graphing the equations $y = -20t^2 - 70t + 300$ and $y = -5t + 300$ and determining the points of intersection:

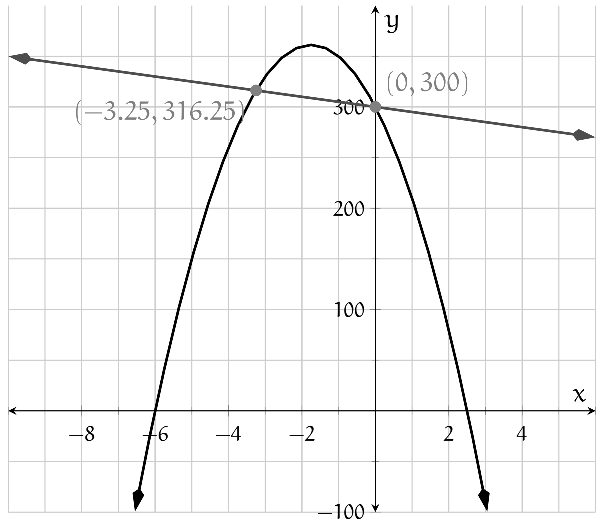


Figure 11.3.14: Points of intersection for $y = -20t^2 - 70t + 300$ and $y = -5t + 300$

a. To solve $-20t^2 - 70t + 300 \geq -5t + 300$, we need to determine where the y-values of the graph of $y = -20t^2 - 70t + 300$ are *greater* than the y-values of the graph of $y = -5t + 300$ in addition to the values where the y-values are equal. This region is highlighted in Figure 11.3.15.

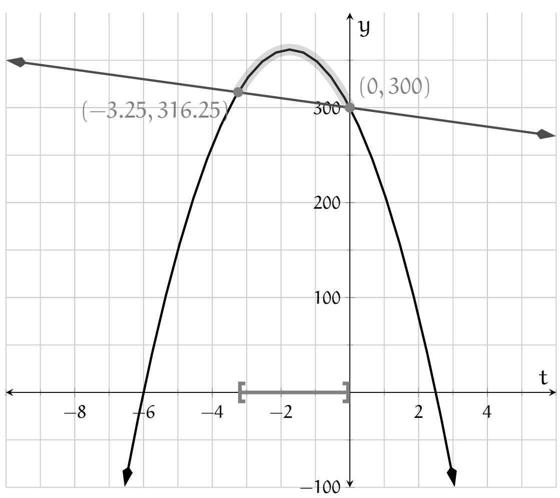


Figure 11.3.15

We can see that this region includes all values of t between, and including, $t = -3.25$ and $t = 0$. So the solutions to this inequality include all values of t for which $-3.25 \leq t \leq 0$. We can write this solution set in interval notation as $[-3.25, 0]$ or in set-builder notation as $\{t \mid -3.25 \leq t \leq 0\}$.

b. To now solve $-20t^2 - 70t + 300 < -5t + 300$, we will need to determine where the y-values of the graph of $y = -20t^2 - 70t + 300$ are *less than* the y-values of the graph of $y = -5t + 300$. This region is highlighted in Figure 11.3.16.

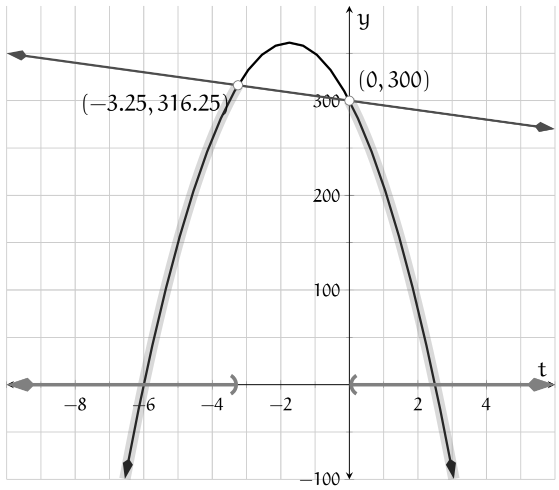


Figure 11.3.16

We can see that $-20t^2 - 70t + 300 < -5t + 300$ for all values of t where $t < -3.25$ or $t > 0$. We can write this solution set in interval notation as $(-\infty, -3.25) \cup (0, \infty)$ or in set-builder notation as $\{t \mid t < -3.25 \text{ or } t > 0\}$.

11.3.4 Reading Questions

1. If you use technology to create a graph of a function, you will have to choose a good $\boxed{}$ to capture all of its features.

2. Describe the process to graphically solve an equation in the form $f(x) = g(x)$.

11.3.5 Exercises

Using Technology to Create a Table of Function Values Use technology to make a table of values for the function.

1. $H(x) = -3x^2 + 14x + 3$

x	$H(x)$

2. $K(x) = 3x^2 - 8x - 3$

x	$K(x)$

3. $f(x) = -0.2x^2 + 80x + 93$

x	$f(x)$

4. $g(x) = -3x^2 - 160x + 38$

x	$g(x)$

5. $h(x) = 7x^3 + 10x - 18$

x	$h(x)$

6. $h(x) = -2x^3 + 190x - 74$

x	$h(x)$

Determining Appropriate Windows

7. Let $f(x) = -5670x + 4316$. Choose an appropriate window for graphing f that shows its key features.

The x-interval could be ______ and the y-interval could be ______.

8. Let $f(x) = 633x + 787$. Choose an appropriate window for graphing f that shows its key features.

The x-interval could be ______ and the y-interval could be ______.

9. Let $f(x) = -742x^2 - 210x - 6418$. Choose an appropriate window for graphing f that shows its key features.

The x-interval could be ______ and the y-interval could be ______.

10. Let $f(x) = -852x^2 - 622x + 6289$. Choose an appropriate window for graphing f that shows its key features.

The x-interval could be ______ and the y-interval could be ______.

11. Let $f(x) = 0.00049x^2 + 0.0011x - 0.59$. Choose an appropriate window for graphing f that shows its key features.

The x-interval could be ______ and the y-interval could be ______.

12. Let $f(x) = -0.00012x^2 - 0.0028x + 0.58$. Choose an appropriate window for graphing f that shows its key features.

The x-interval could be ______ and the y-interval could be ______.

Finding Points of Intersection

13. Use technology to determine how many times the equations $y = (370 - 13x)(-307 + 20x)$ and $y = 4000$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times)
.

14. Use technology to determine how many times the equations $y = (-214 - 8x)(416 + 6x)$ and $y = 3000$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times)
.

15. Use technology to determine how many times the equations $y = -2x^3 - x^2 + 4x$ and $y = 3x + 1$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

16. Use technology to determine how many times the equations $y = 9x^3 - 3x^2 - 3x$ and $y = x + 2$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

17. Use technology to determine how many times the equations $y = 0.1\left(6x^2 + 1\right)$ and $y = -0.74(6x - 8)$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

18. Use technology to determine how many times the equations $y = 0.2\left(7x^2 + 9\right)$ and $y = 0.57(3x + 6)$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

19. Use technology to determine how many times the equations $y = 1.4(x - 8)^2 - 2.35$ and $y = 0.1x$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

20. Use technology to determine how many times the equations $y = 1.85(x - 1)^2 + 5.2$ and $y = -x$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

Using Technology to Find Key Features of a Graph

21. For the function j defined by

$$j(x) = -\frac{2}{5}(x - 3)^2 + 6,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.

b. The vertex.

c. The domain.

d. The range.

22. For the function k defined by

$$k(x) = 2(x + 1)^2 + 10,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.

b. The vertex.

c. The domain.

d. The range.

23. For the function L defined by

$$L(x) = 3000x^2 + 10x + 4,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.

b. The vertex.

c. The domain.

d. The range.

24. For the function M defined by

$$M(x) = -(300x - 2950)^2,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.

b. The vertex.

c. The domain.

d. The range.

25. For the function N defined by

$$N(x) = (300x - 1.05)^2,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.

b. The vertex.

c. The domain.

d. The range.

26. For the function B defined by

$$B(x) = x^2 - 0.05x + 0.0006,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.

b. The vertex.

c. The domain.

d. The range.

Solving Equations and Inequalities Graphically Using Technology

27. Let $s(x) = \frac{1}{5}x^2 - 2x + 10$ and $t(x) = -x + 40$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $s(x) = t(x)$.

c. Solve $s(x) > t(x)$.

d. Solve $s(x) \leq t(x)$.

28. Let $w(x) = \frac{1}{4}x^2 - 3x - 8$ and $m(x) = x + 12$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $w(x) = m(x)$.

c. Solve $w(x) > m(x)$.

d. Solve $w(x) \leq m(x)$.

29. Let $f(x) = 4x^2 + 5x - 1$ and $g(x) = 5$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $f(x) = g(x)$.

c. Solve $f(x) < g(x)$.

d. Solve $f(x) \geq g(x)$.

30. Let $p(x) = 6x^2 - 3x + 4$ and $k(x) = 7$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $p(x) = k(x)$.

c. Solve $p(x) < k(x)$.

d. Solve $p(x) \geq k(x)$.

31. Let $q(x) = -4x^2 - 24x + 10$ and $r(x) = 2x + 22$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $q(x) = r(x)$.

c. Solve $q(x) > r(x)$.

d. Solve $q(x) \leq r(x)$.

32. Let $h(x) = -10x^2 - 5x + 3$ and $j(x) = -3x - 9$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $h(x) = j(x)$.

c. Solve $h(x) > j(x)$.

d. Solve $h(x) \leq j(x)$.

33. Use graphing technology to solve the equation $0.4x^2 + 0.5x - 0.2 = 2.4$. Approximate the solution(s) if necessary.

34. Use graphing technology to solve the equation $-0.25x^2 - 2x + 1.75 = 4.75$. Approximate the solution(s) if necessary.

35. Use graphing technology to solve the equation $(200 + 5x)(100 - 2x) = 15000$. Approximate the solution(s) if necessary.

36. Use graphing technology to solve the equation $(200 - 5x)(100 + 10x) = 20000$. Approximate the solution(s) if necessary.

37. Use graphing technology to solve the equation $2x^3 - 5x + 1 = -\frac{1}{2}x + 1$. Approximate the solution(s) if necessary.

38. Use graphing technology to solve the equation $-x^3 + 8x = -4x + 16$. Approximate the solution(s) if necessary.

39. Use graphing technology to solve the equation $-0.05x^2 - 2.03x - 19.6 = 0.05x^2 + 1.97x + 19.4$. Approximate the solution(s) if necessary.

40. Use graphing technology to solve the equation $-0.02x^2 + 1.97x - 51.5 = 0.05(x - 50)^2 - 0.03(x - 50)$. Approximate the solution(s) if necessary.

41. Use graphing technology to solve the equation $-200x^2 + 60x - 55 = -20x - 40$. Approximate the solution(s) if necessary.

42. Use graphing technology to solve the equation $150x^2 - 20x + 50 = 100x + 40$. Approximate the solution(s) if necessary.

43. Use graphing technology to solve the inequality $2x^2 + 5x - 3 > -5$. State the solution set using interval notation, and approximate if necessary.

44. Use graphing technology to solve the inequality $-x^2 + 4x - 7 > -12$. State the solution set using interval notation, and approximate if necessary.

45. Use graphing technology to solve the inequality $10x^2 - 11x + 7 \leq 7$. State the solution set using interval notation, and approximate if necessary.

46. Use graphing technology to solve the inequality $-10x^2 - 15x + 4 \leq 9$. State the solution set using interval notation, and approximate if necessary.

47. Use graphing technology to solve the inequality $-x^2 - 6x + 1 > x + 5$. State the solution set using interval notation, and approximate if necessary.

48. Use graphing technology to solve the inequality $3x^2 + 5x - 4 > -2x + 1$. State the solution set using interval notation, and approximate if necessary.

49. Use graphing technology to solve the inequality $-10x + 4 \leq 20x^2 - 34x + 6$. State the solution set using interval notation, and approximate if necessary.

50. Use graphing technology to solve the inequality $-15x^2 - 6 \leq 10x - 4$. State the solution set using interval notation, and approximate if necessary.

51. Use graphing technology to solve the inequality $\frac{1}{2}x^2 + \frac{3}{2}x \geq \frac{1}{2}x - \frac{3}{2}$. State the solution set using interval notation, and approximate if necessary.

52. Use graphing technology to solve the inequality $\frac{3}{4}x \geq \frac{1}{4}x^2 - 3x$. State the solution set using interval notation, and approximate if necessary.

11.4 Simplifying Expressions with Function Notation

In this section, we will discuss algebra simplification that will appear in many facets of education. Simplification is a skill, like cooking noodles or painting a wall. It may not always be exciting, but it does serve a purpose. Also like cooking noodles or painting a wall, it isn't usually difficult, and yet there are common avoidable mistakes that people make. With practice from this section, you'll have experience to prevent yourself from overcooking the noodles or ruining your paintbrush.

11.4.1 Negative Signs in and out of Function Notation

Let's start by reminding ourselves about the meaning of function notation. When we write $f(x)$, we have a process f that is doing something to an input value x. Whatever is inside those parentheses is the input to the function. What if we use something for input that is not quite as simple as "x?"

Example 11.4.2 Find and simplify a formula for $f(-x)$, where $f(x) = x^2 + 3x - 4$.

Explanation. Those parentheses encase "$-x$," so we are meant to treat "$-x$" as the input. The rule that we have been given for f is

$$f(x) = x^2 + 3x - 4.$$

But the x's that are in this formula are just place-holders. What f does to a number can just as easily be communicated with

$$f(\ ) = (\ )^2 + 3(\ ) - 4.$$

So now that we are meant to treat "$-x$" as the input, we will insert "$-x$" into those slots, after which we can do more familiar algebraic simplification:

$$\begin{aligned} f(\quad) &= (\quad)^2 + 3(\quad) - 4 \\ f(-x) &= (-x)^2 + 3(-x) - 4 \\ &= x^2 - 3x - 4 \end{aligned}$$

The previous example contrasts nicely with this one:

Example 11.4.3 Find and simplify a formula for $-f(x)$, where $f(x) = x^2 + 3x - 4$.

Explanation. Here, the parentheses only encase "x." The negative sign is on the outside. So the way to see this expression is that first f will do what it does to x, and then that result will be negated:

$$\begin{aligned} -f(x) &= -(x^2 + 3x - 4) \\ &= -x^2 - 3x + 4 \end{aligned}$$

Note that the answer to this exercise, which was to simplify $-f(x)$, is different from the answer to Example 11.4.2, which was to simplify $f(-x)$. In general you cannot pass a negative sign in and out of function notation and still have the same quantity.

In Example 11.4.2 and Example 11.4.3, we are working with the expressions $f(-x)$ and $-f(x)$, and trying to find "simplified" formulas. If it seems strange to be doing these things, perhaps this applied example will help.

Checkpoint 11.4.4 The NASDAQ Composite Index measures how well a portion of the stock market is doing. Suppose $N(t)$ is the value of the index t days after January 1, 2018. A formula for N is $N(t) =$

$3.34t^2 + 26.2t + 6980$.

What if you wanted a new function, B, that gives the value of the NASDAQ index t days *before* January 1, 2018? Technically, t days *before* is the same as *negative* t days after. So $B(t)$ is the same as $N(-t)$, and now the expression $N(-t)$ means something. Find a simplified formula for $N(-t)$.

$N(-t)$

Explanation.

$$\begin{aligned} N(\quad) &= 3.34(\quad)^2 + 26.2(\quad) + 6980 \\ N(-t) &= 3.34(-t)^2 + 26.2(-t) + 6980 \\ &= 3.34t^2 - 26.2t + 6980 \end{aligned}$$

11.4.2 Other Nontrivial Simplifications

Example 11.4.5 Find and simplify a formula for $h(5x)$, where $h(x) = \frac{x}{x-2}$.

Explanation. The parentheses encase "5x," so we are meant to treat "5x" as the input.

$$\begin{aligned} h(\quad) &= \frac{(\quad)}{(\quad)-2} \\ h(5x) &= \frac{5x}{5x-2} \\ &= \frac{5x}{5x-2} \end{aligned}$$

Example 11.4.6 Find and simplify a formula for $\frac{1}{3}g(3x)$, where $g(x) = 2x^2 + 8$.

Explanation. Do the $\frac{1}{3}$ and the 3 cancel each other? No. The 3 is part of the input, affecting x right away. Then g does whatever it does to 3x, and *then* we multiply the result by $\frac{1}{3}$. Since the function g acts "in between," we don't have the chance to cancel the 3 with the $\frac{1}{3}$. Let's see what actually happens:

Those parentheses encase "3x," so we are meant to treat "3x" as the input. We will keep the $\frac{1}{3}$ where it is until it is possible to simplify:

$$\begin{aligned} \frac{1}{3}g(\quad) &= \frac{1}{3}\left(2(\quad)^2 + 8\right) \\ \frac{1}{3}g(3x) &= \frac{1}{3}\left(2(3x)^2 + 8\right) \\ &= \frac{1}{3}\left(2\left(9x^2\right) + 8\right) \\ &= \frac{1}{3}\left(18x^2 + 8\right) \\ &= 6x^2 + \frac{8}{3} \end{aligned}$$

Example 11.4.7 If $k(x) = x^2 - 3x$, find and simplify a formula for $k(x-4)$.

Explanation. This type of exercise is often challenging for algebra students. But let's focus on those parentheses one more time. They encase "$x - 4$," so we are meant to treat "$x - 4$" as the input.

$$\begin{aligned} k(\qquad) &= (\qquad)^2 - 3(\qquad) \\ k(x-4) &= (x-4)^2 - 3(x-4) \end{aligned}$$

$$= x^2 - 8x + 16 - 3x + 12$$
$$= x^2 - 11x + 28$$

Checkpoint 11.4.8 If $q(x) = x + \sqrt{x + 8}$, find and simplify a formula for $q(x + 5)$.

$q(x + 5)$

Explanation. Starting with the generic formula for q:

$$\begin{aligned} q(\qquad) &= (\qquad) + \sqrt{(\qquad) + 8} \\ q(x+5) &= x + 5 + \sqrt{x + 5 + 8} \\ &= x + 5 + \sqrt{x + 13} \end{aligned}$$

Example 11.4.9 If $f(x) = \frac{1}{x}$, find and simplify a formula for $f(x + 3) + 2$.

Explanation. Do not be tempted to add the 3 and the 2. The 3 is added to input *before* the function f does its work. The 2 is added to the result *after* f has done its work.

$$\begin{aligned} f(\qquad) + 2 &= \frac{1}{(\qquad)} + 2 \\ f(x+3) + 2 &= \frac{1}{x+3} + 2 \end{aligned}$$

This last expression is considered fully simplified. However you might combine the two terms using a technique from Section 12.3.

The tasks we have practiced in this section are the kind of tasks that will make it easier to understand interesting and useful material in college algebra and calculus.

11.4.3 Reading Questions

1. Explain how $f(x + 2)$ probably does *not* mean that f is being multiplied by $x + 2$.

11.4.4 Exercises

Review and Warmup

1. Use the distributive property to write an equivalent expression to $5(n + 5)$ that has no grouping symbols.

2. Use the distributive property to write an equivalent expression to $2(q + 9)$ that has no grouping symbols.

3. Use the distributive property to write an equivalent expression to $-4(x - 6)$ that has no grouping symbols.

4. Use the distributive property to write an equivalent expression to $-7(r + 3)$ that has no grouping symbols.

5. Multiply the polynomials.
$2\,(y + 1)^2$

6. Multiply the polynomials.
$4\,(y + 7)^2$

7. Expand the square of a ***bi***nomial.
$(4y + 9)^2$

8. Expand the square of a ***bi***nomial.
$(10r + 3)^2$

Simplifying Function Expressions

9. Simplify $G(r+7)$, where $G(r) = 7 - 3r$.

10. Simplify $g(t+2)$, where $g(t) = 7 - 7t$.

11. Simplify $K(-t)$, where $K(t) = 5 + t$.

12. Simplify $F(-x)$, where $F(x) = 7 + 8x$.

13. Simplify $g(x+4)$, where $g(x) = 6 - 4.4x$.

14. Simplify $K(y+8)$, where $K(y) = 6 + 7.3y$.

15. Simplify $F(y - \frac{5}{7})$, where $F(y) = -\frac{3}{4} + \frac{7}{9}y$.

16. Simplify $g(y + \frac{3}{5})$, where $g(y) = -\frac{1}{7} + \frac{7}{5}y$.

17. Simplify $H(r) + 1$, where $H(r) = -6r + 6$.

18. Simplify $F(r) + 5$, where $F(r) = 6r + 5$.

19. Simplify $f(t) + 8$, where $f(t) = 5 + 1.1t$.

20. Simplify $H(t) + 3$, where $H(t) = 5 - 3.4t$.

21. Simplify $F(7x)$, where $F(x) = -8x^2 + 5x + 7$.

22. Simplify $f(3x)$, where $f(x) = 4x^2 + 4x - 1$.

23. Simplify $H(-y)$, where $H(y) = y^2 - y + 4$.

24. Simplify $h(-y)$, where $h(y) = 8y^2 - 5y + 4$.

25. Simplify $5f(y)$, where $f(y) = 7y^2 + 4y + 7$.

26. Simplify $8G(r)$, where $G(r) = 2r^2 + 4r - 1$.

27. Simplify $h(r-5)$, where $h(r) = -2.4r^2 + 3r + 7$.

28. Simplify $f(t+3)$, where $f(t) = -6.9t^2 + 3t - 2$.

29. Simplify $G(t) + 2$, where $G(t) = 5t^2 + 3t + 7$.

30. Simplify $h(x) + 5$, where $h(x) = 3x^2 - 2x + 3$.

31. Simplify $g(x+9)$, where $g(x) = \sqrt{3 - 7x}$.

32. Simplify $g(x+6)$, where $g(x) = \sqrt{2 + 2x}$.

33. Simplify $h(x) + 3$, where $h(x) = \sqrt{2 + 6x}$.

34. Simplify $F(x) + 9$, where $F(x) = \sqrt{2 - x}$.

35. Simplify $G(x+6)$, where $G(x) = 7x + \sqrt{2 - 5x}$.

36. Simplify $G(x+3)$, where $G(x) = -2x + \sqrt{1 + 7x}$.

37. Simplify $K(t+4)$, where $K(t) = \frac{7}{t+1}$.

38. Simplify $G(t+8)$, where $G(t) = -\frac{2}{-3t+1}$.

39. Simplify $h(-3x)$, where $h(x) = \frac{5x}{x^2+6}$.

40. Simplify $K(7x)$, where $K(x) = \frac{6x}{x^2-2}$.

41. Let f be a function given by $f(x) = -4x - 8$. Find and simplify the following:

a. $f(x) + 5 =$ ☐

b. $f(x+5) =$ ☐

c. $5f(x) =$ ☐

d. $f(5x) =$ ☐

42. Let f be a function given by $f(x) = -4x - 1$. Find and simplify the following:

a. $f(x) + 3 =$ ☐

b. $f(x+3) =$ ☐

c. $3f(x) =$ ☐

d. $f(3x) =$ ☐

43. Let f be a function given by $f(x) = 3x^2 + 4x$. Find and simplify the following:

a. $f(x) - 2 =$

b. $f(x - 2) =$

c. $-2f(x) =$

d. $f(-2x) =$

44. Let f be a function given by $f(x) = -4x^2 - 2x$. Find and simplify the following:

a. $f(x) - 3 =$

b. $f(x - 3) =$

c. $-3f(x) =$

d. $f(-3x) =$

Applications

45. A circular oil slick is expanding with radius, r in feet, at time t in hours given by $r = 18t - 0.3t^2$, for t in hours, $0 \leq t \leq 10$.

Find a formula for $A = f(t)$, the area of the oil slick as a function of time.

$A = f(t)$

with

(Be sure to include units!)

46. Suppose $T(t)$ represents the temperature outside, in Fahrenheit, at t hours past noon, and a formula for T is $T(t) = \frac{28t}{t^2+1} + 56$.

If we introduce $F(t)$ as the temperature outside, in Fahrenheit, at t hours past 1:00pm, then $F(t) = T(t + 1)$. Find a simplified formula for $T(t + 1)$.

$T(t + 1)$

47. Suppose $G(t)$ represents how many gigabytes of data has been downloaded t minutes after you started a download, and a formula for G is $G(t) = 20 - \frac{80}{t+4}$.

If we introduce $M(t)$ as how many megabytes of data has been downloaded t minutes after you started a download, then $M(t) = 1024G(t)$. Find a simplified formula for $1024G(t)$.

$1024G(t)$

11.5 Technical Definition of a Function

In Section 11.1, we discussed a conceptual understanding of functions and Definition 11.1.3. In this section we'll start with a more technical definition of what is a function, consistent with the ideas from Section 11.1.

11.5.1 Formally Defining a Function

Definition 11.5.2 Function (Technical Definition). A **function** is a collection of ordered pairs (x, y) such that any particular value of x is paired with at most one value for y. ◊

How is this definition consistent with the informal Definition 11.1.3, which describes a function as a *process*? Well, if you have a collection of ordered pairs (x, y), you can choose to view the left number as an input, and the right value as the output. If the function's name is f and you want to find $f(x)$ for a particular number x, look in the collection of ordered pairs to see if x appears among the first coordinates. If it does, then $f(x)$ is the (unique) y-value it was paired with. If it does not, then that x is just not in the domain of f, because you have no way to determine what $f(x)$ would be.

Example 11.5.3 Using Definition 11.5.1, a function f could be given by $\{(1, 4), (2, 3), (5, 3), (6, 1)\}$.

a. What is $f(1)$? Since the ordered pair $(1, 4)$ appears in the collection of ordered pairs, $f(1) = 4$.

b. What is $f(2)$? Since the ordered pair $(2, 3)$ appears in the collection of ordered pairs, $f(2) = 3$.

c. What is $f(3)$? None of the ordered pairs in the collection start with 3, so $f(3)$ is undefined, and we would say that 3 is not in the domain of f.

Example 11.5.4 A Function Given as a Table. Consider the function g expressed by Figure 11.5.5. How is this "a collection of ordered pairs?" With tables the connection is most easily apparent. Pair off each x-value with its y-value.

x	$g(x)$
12	0.16
15	3.2
18	1.4
21	1.4
24	0.98

Figure 11.5.5

In this case, we can view this function as:

$$\{(12, 0.16), (15, 3.2), (18, 3.2), (21, 1.4), (24, 0.98)\}.$$

Example 11.5.6 A Function Given as a Formula. Consider the function h expressed by the formula $h(x) = x^2$. How is this "a collection of ordered pairs?"

This time, the collection is *really big*. Imagine an x-value, like $x = 2$. We can calculate that $f(2) = 2^2 = 4$. So the input 2 pairs with the output 4 and the ordered pair $(2, 4)$ is part of the collection.

You could move on to *any* x-value, like say $x = 2.1$. We can calculate that $f(2.1) = 2.1^2 = 4.41$. So the input 2.1 pairs with the output 4.41 and the ordered pair $(2.1, 4.41)$ is part of the collection.

The collection is so large that we cannot literally list all the ordered pairs as was done in Example 11.5.3 and Example 11.5.4. We just have to imagine this giant collection of ordered pairs. And if it helps to conceptualize it, we know that the ordered pairs $(2, 4)$ and $(2.1, 4.41)$ are included.

Example 11.5.7 A Function Given as a Graph. Consider the functions p and q expressed in Figure 11.5.8 and Figure 11.5.9. How is each of these "a collection of ordered pairs?"

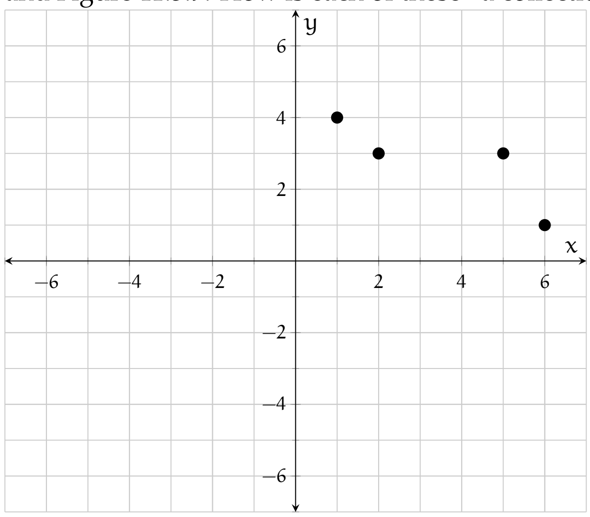


Figure 11.5.8: $y = p(x)$

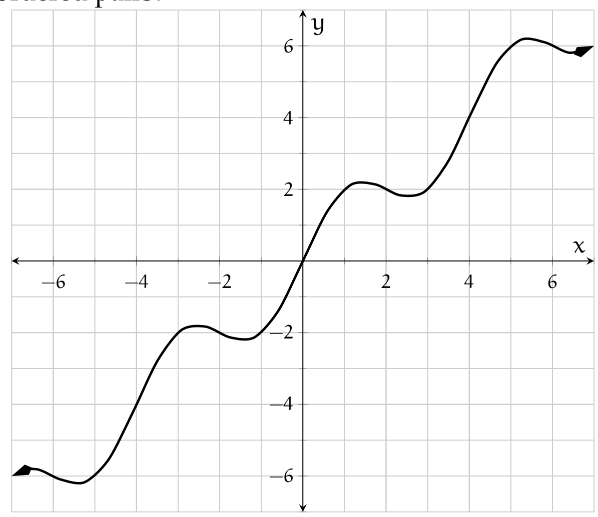


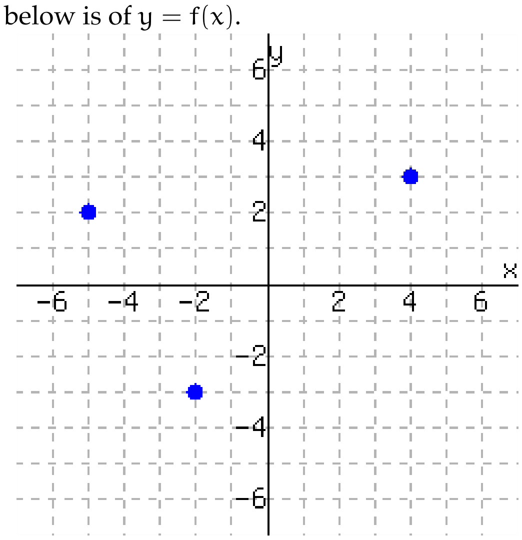
Figure 11.5.9: $y = q(x)$

In Figure 11.5.8, we see that $p(1) = 4$, $p(2) = 3$, $p(5) = 3$, and $p(6) = 1$. The graph *literally is* the collection

$$\{(1, 4), (2, 3), (5, 3), (6, 1)\}.$$

In Figure 11.5.9, we can see a few whole number function values, like $q(0) = 0$ and $q(1) = 2$. But the entire curve has infinitely many points on it and we'd never be able to list them all. We just have to imagine the giant collection of ordered pairs. And if it helps to conceptualize it, we know that the ordered pairs $(0, 0)$ and $(1, 2)$ are included.

Checkpoint 11.5.10 The graph below is of $y = f(x)$.



Write the function f as a set of ordered pairs.

Explanation. The function can be expressed as the set $\{(-5, 2), (-2, -3), (4, 3)\}$.

11.5.2 Identifying What is *Not* a Function

Just because you have a set of order pairs, a table, a graph, or an equation, it does not necessarily mean that you have a function. Conceptually, whatever you have needs to give consistent outputs if you feed it the same input. More technically, the set of ordered pairs is not allowed to have two ordered pairs that have the same x-value but different y-values.

Example 11.5.11 Consider each set of ordered pairs. Does it make a function?

a. $\left\{(5,9),(3,2),\left(\frac{1}{2},0.6\right),(5,1)\right\}$

b. $\left\{(-5,12),(3,7),\left(\sqrt{2},1\right),(-0.9,4)\right\}$

c. $\left\{(5,9),(3,9),\left(4.2,\sqrt{2}\right),\left(\frac{4}{3},\frac{1}{2}\right)\right\}$

d. $\left\{(5,9),(0.7,2),\left(\sqrt{25},3\right),\left(\frac{2}{3},\frac{3}{2}\right)\right\}$

Explanation.

a. This set of ordered pairs is *not* a function. The problem is that it has both $(5,9)$ and $(5,1)$. It uses the same x-value paired with two different y-values. We have no clear way to turn the input 5 into an output.

b. This set of ordered pairs *is* a function. It is a collection of ordered pairs, and the x-values are never reused.

c. This set of ordered pairs *is* a function. It is a collection of ordered pairs, and the x-values are never reused. You might note that the *output* value 9 appears twice, but that doesn't matter. That just tells us that the function turns 5 into 9 and it also turns 3 into 9.

d. This set of ordered pairs is *not* a function, but it's a little tricky. One of the ordered pairs uses $\sqrt{25}$ as an input value. But that is the same as 5, which is also used as an input value.

Now that we understand how some sets of ordered pairs might not be functions, what about tables, graphs, and equations? If we are handed one of these things, can we tell whether or not it is giving us a function?

Checkpoint 11.5.12 Does This Table Make a Function? Which of these tables make y a function of x?

a.

x	y
2	1
3	1
4	2
5	2
6	2

This table (☐ does ☐ does not) make y a function of x.

b.

x	y
8	3
9	2
5	1
2	0
8	1

This table (☐ does ☐ does not) make y a function of x.

c.

x	y
5	9
5	9
6	2
6	2
6	2

This table (☐ does ☐ does not) make y a function of x.

Explanation.

a. This table does make y a function of x. In the table, no x-value is repeated.

b. This table does not make y a function of x. In the table, the x-value 8 is repeated, and it is paired with two different y-values, 3 and 1.

c. This table does make y a function of x, but you have to think carefully. It's true that the x-value 5 is used more than once in the table. But in both places, the y-value is the same, 9. So there is no conceptual issue with asking for $f(5)$; it would definitely be 9. Similarly, the repeated use of 6 as an x-value is not a problem since it is always paired with output 2.

Example 11.5.13 Does This Graph Make a Function? Which of these graphs make y a function of x?

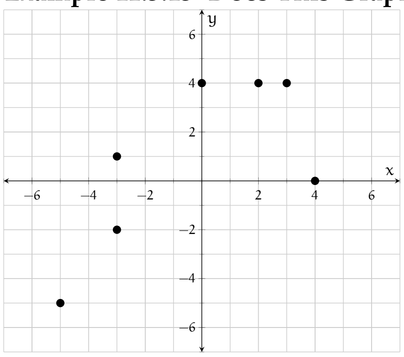


Figure 11.5.14

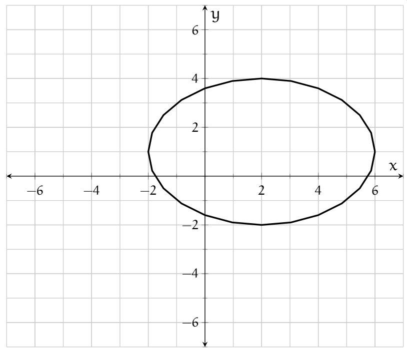


Figure 11.5.15

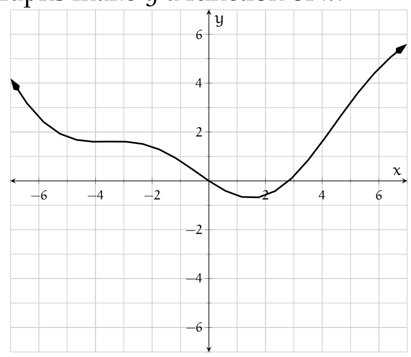


Figure 11.5.16

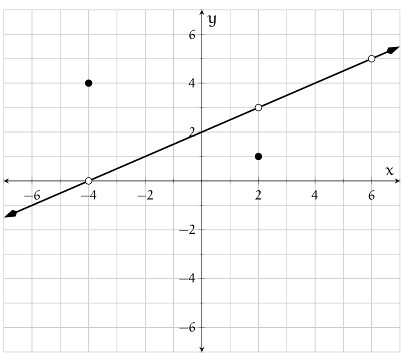


Figure 11.5.17

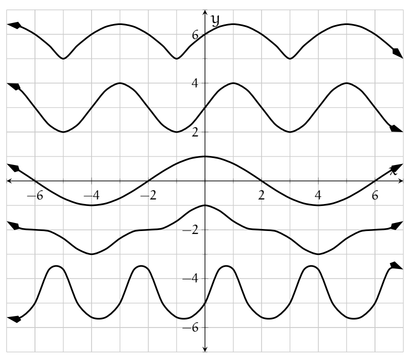


Figure 11.5.18

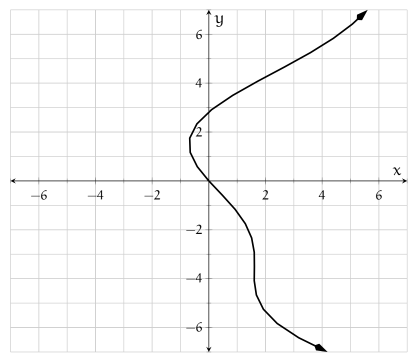


Figure 11.5.19

Explanation. The graph in Figure 11.5.14 does *not* make y a function of x. Two ordered pairs on that graph are $(-3, 1)$ and $(-3, -2)$, so an input value is used twice with different output values.

The graph in Figure 11.5.15 does *not* make y a function of x. There are many ordered pairs with the same input value but different output values. For example, $(2, -2)$ and $(2, 4)$.

The graph in Figure 11.5.16 *does* make y a function of x. It appears that no matter what x-value you choose on the x-axis, there is exactly one y-value paired up with it on the graph.

The graph in Figure 11.5.17 *does* make y a function of x, but we should discuss. The hollow dots on the line indicate that the line goes right up to that point, but never reaches it. We say there is a "hole" in the graph at these places. For two of these holes, there is a separate ordered pair immediately above or below the hole. The graph has the ordered pair $(-4, 4)$. It *also* has ordered pairs like (very close to -4, very close to 0), but it does not have $(-4, 0)$. Overall, there is no x-value that is used twice with different y-values, so this graph does make y a function of x

The graph in Figure 11.5.15 does *not* make y a function of x. There are many ordered pairs with the same input value but different output values. For example, $(0, 1)$, $(0, 3)$, $(0, -1)$, $(0, 5)$, and $(0, -6)$ all use $x = 0$.

The graph in Figure 11.5.15 does *not* make y a function of x. There are many ordered pairs with the same input value but different output values. For example at $x = 2$, there is both a positive and a negative associated y-value. It's hard to say exactly what these y-values are, but we don't have to.

This last set of examples might reveal something to you. For instance in Figure 11.5.15, the issue is that there are places on the graph with the same x-value, but different y-values. Visually, what that means is there are places on the graph that are directly above/below each other. Thinking about this leads to a quick visual "test" to determine if a graph gives y as a function of x.

Fact 11.5.20 Vertical Line Test. *Given a graph in the xy-plane, if a vertical line ever touches it in more than one place, the graph does* not *give y as a function of x. If vertical lines only ever touch the graph once or never at all, then the graph* does *give y as a function of x.*

Example 11.5.21 In each graph from Example 11.5.13, we can apply the Vertical Line Test.

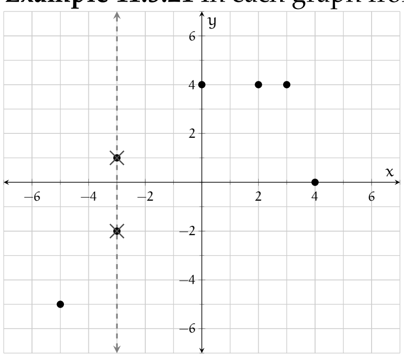


Figure 11.5.22: A vertical line touching the graph twice makes this graph not give y as a function of x.

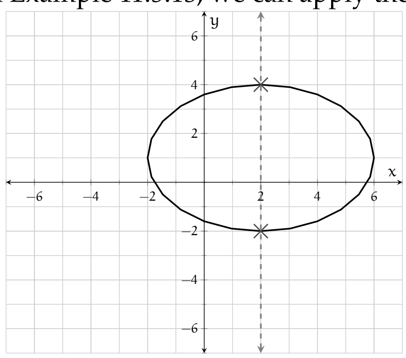


Figure 11.5.23: A vertical line touching the graph twice makes this graph not give y as a function of x.

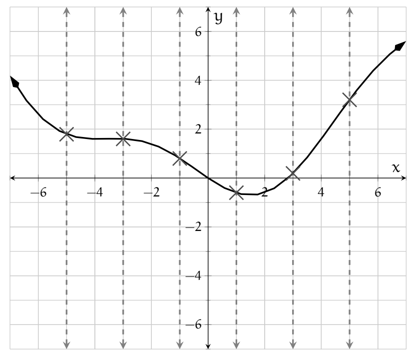


Figure 11.5.24: All vertical lines only touch the graph once, so this graph does give y as a function of x.

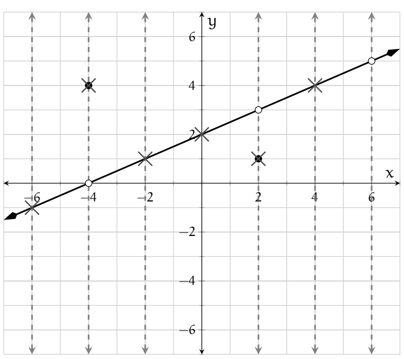


Figure 11.5.25: All vertical lines only touch the graph once, or not at all, so this graph does give y as a function of x.

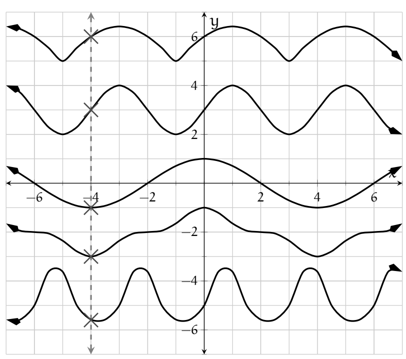


Figure 11.5.26: A vertical line touching the graph more than once makes this graph not give y as a function of x.

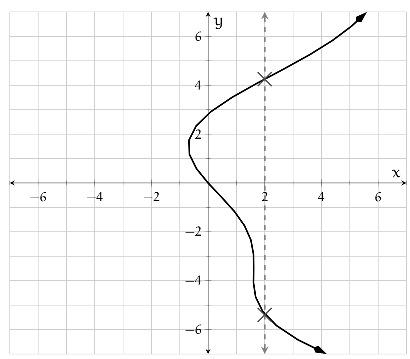


Figure 11.5.27: A vertical line touching the graph more than once makes this graph not give y as a function of x.

Lastly, we come to equations. Certain equations with variables x and y clearly make y a function of x. For example, $y = x^2 + 1$ says that if you have an x-value, all you have to do is substitute it into that equation and you will have determined an output y-value. You could then name the function f and give a formula for it: $f(x) = x^2 + 1$.

With other equations, it may not be immediately clear whether or not they make y a function of x.

Example 11.5.28 Do each of these equations make y a function of x?

a. $2x + 3y = 5$ b. $y = \pm\sqrt{x+4}$ c. $x^2 + y^2 = 9$

Explanation.

a. The equation $2x + 3y = 5$ *does* make y a function of x. Here are three possible explanations.

   i. You recognize that the graph of this equation would be a non-vertical line, and so it would pass the Vertical Line Test.

   ii. Imagine that you have a specific value for x and you substitute it in to $2x + 3y = 5$. Will you be able to use algebra to solve for y? All you will need is to simplify, subtract from both sides, and divide on both sides, so you will be able to determine y.

   iii. Can you just isolate y in terms of x? Yes, a few steps of algebra can turn $2x+3y = 5$ into $y = \frac{5-2x}{3}$. Now you have an explicit formula for y in terms of x, so y is a function of x.

b. The equation $y = \pm\sqrt{x+4}$ does *not* make y a function of x. Just having the $\pm$ (plus *or* minus) in the equation immediately tells you that for almost any valid x-value, there would be *two* associated y-values.

c. The equation $x^2 + y^2 = 9$ does *not* make y a function of x. Here are three possible explanations.

   i. Imagine that you have a specific value for x and you substitute it in to $x^2 + y^2 = 9$. Will you be able to use algebra to solve for y? For example, if you substitute in $x = 1$, then you have $1 + y^2 = 9$, which simplifies to $y^2 = 8$. Can you really determine what y is? No, because it could be $\sqrt{8}$ or it could be $-\sqrt{8}$. So this equation does not provide you with a way to turn x-values into y-values.

   ii. Can you just isolate y in terms of x? You might get started and use algebra to convert $x^2 + y^2 = 9$ into $y^2 = 9 - x^2$. But what now? The best you can do is acknowledge that y is either the positive or the negative square root of $9 - x^2$. You might write $y = \pm\sqrt{9 - x^2}$. But now for almost any valid x-value, there are *two* associated y-values.

   iii. You recognize that the graph of this equation would be a circle with radius 3, and so it would not pass the Vertical Line Test.

Checkpoint 11.5.29 Do each of these equations make y a function of x?

a. $5x^2 - 4y = 12$

This equation (☐ does ☐ does not) make y a function of x.

b. $5x - 4y^2 = 12$

This equation (☐ does ☐ does not) make y a function of x.

c. $x = \sqrt{y}$

This equation (☐ does ☐ does not) make y a function of x.

Explanation.

a. The equation $5x^2 - 4y = 12$ *does* make y a function of x. You can isolate y in terms of x. A few steps of algebra can turn $5x^2 - 4y = 12$ into $y = \frac{5x^2-12}{4}$. Now you have an explicit formula for y in terms of x, so y is a function of x.

b. The equation $5x - 4y^2 = 12$ does *not* make y a function of x. You cannot isolate y in terms of x. You might get started and use algebra to convert $5x - 4y^2 = 12$ into $y^2 = \frac{5x-12}{4}$. But what now? The best you can do is acknowledge that y is either the positive or the negative square root of $\frac{5x-12}{4}$. You might write $y = \pm\sqrt{\frac{5x-12}{4}}$. But now for almost any valid x-value, there are *two* associated y-values.

c. The equation $x = \sqrt{y}$ *does* make y a function of x. If you try substituting a non-negative x-value, then you can square both sides and you know exactly what the value of y is.

If you try substituting a negative x-value, then you are saying that $\sqrt{y}$ is negative which is impossible. So for negative x, there are no y-values. This is not a problem for the equation giving you a function. This just means that the domain of that function does not include negative numbers. Its domain would be $[0, \infty)$.

11.5.3 Reading Questions

1. Suppose you have a "relation". That is, a set of order pairs, a table of x- and y-values, a graph, or an equation in x and y. What is the one thing that could happen that would make the relation *not* be a function?

2. Explain how to use the vertical line test.

11.5.4 Exercises

Determining If Sets of Ordered Pairs Are Functions

1. Do these sets of ordered pairs make functions of x? What are their domains and ranges?

a. $\{(-10, 10), (2, 0)\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ______ and range ______.

b. $\{(-9, 3), (-6, 2), (-4, 6)\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ______ and range ______.

c. $\{(3, 9), (10, 0), (3, 0), (3, 4)\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ______ and range ______.

d. $\{(-8, 6), (-10, 10), (-8, 7), (3, 10), (8, 3)\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ______ and range ______.

2. Do these sets of ordered pairs make functions of x? What are their domains and ranges?

a. $\{(0, 7), (3, 4)\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ______ and range ______.

b. $\{(-3, 3), (3, 3), (6, 0)\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ______ and range ______.

c. $\{(9, 9), (6, 2), (8, 6), (9, 10)\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ______ and range ______.

d. $\{(-8, 0), (0, 5), (1, 7), (4, 9), (0, 8)\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ______ and range ______.

3. Does the following set of ordered pairs make for a function of x?

$\{(-1, 2), (-1, 5), (-7, 6), (-6, 2), (8, 9)\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ______ and range ______.

4. Does the following set of ordered pairs make for a function of x?

$\{(-10, 9), (6, 4), (-8, 5), (1, 2), (-9, 9)\}$

This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ______ and range ______.

Domain and Range

5. Below is all of the information that exists about a function H.

$H(1) = 3 \quad H(3) = 1 \quad H(6) = 4$

Write H as a set of ordered pairs.

H has domain ______ and range ______.

6. Below is all of the information about a function K.

$K(a) = 2 \quad K(b) = 1$

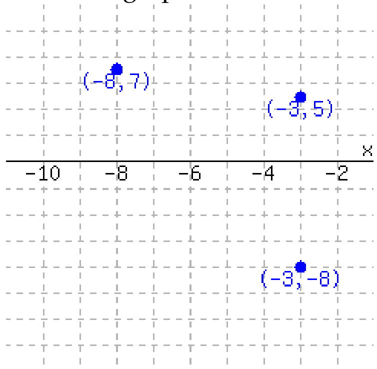
$K(c) = -2 \quad K(d) = 2$

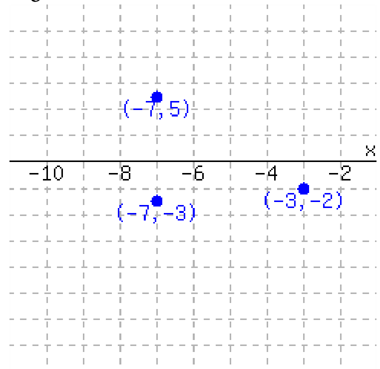
Write K as a set of ordered pairs.

K has domain ______ and range ______.

Determining If Graphs Are Functions

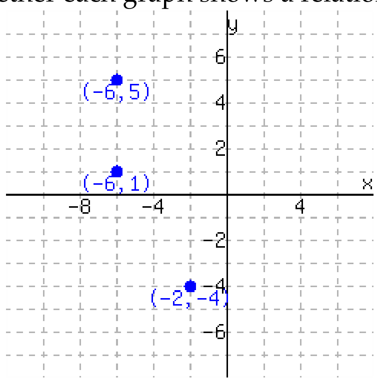
7. Decide whether each graph shows a relationship where y is a function of x.

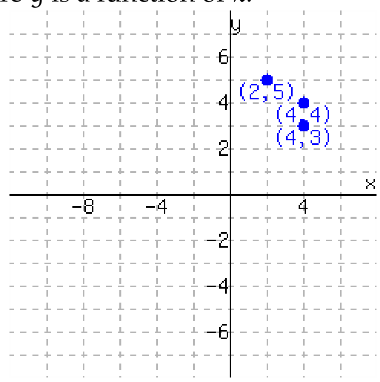




The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

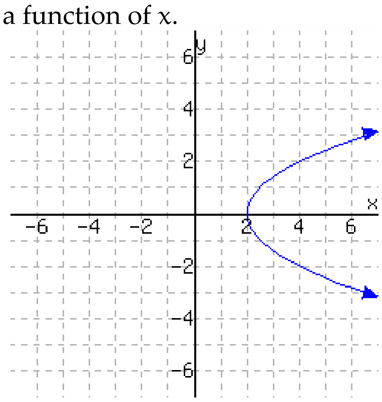
8. Decide whether each graph shows a relationship where y is a function of x.

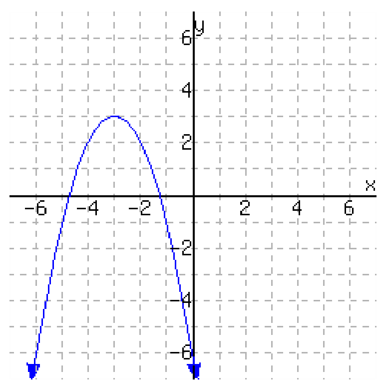




The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

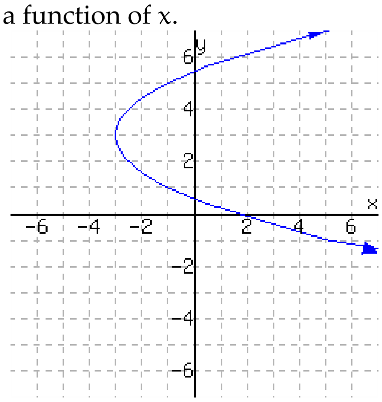
9. The following graphs show two relationships. Decide whether each graph shows a relationship where y is a function of x.

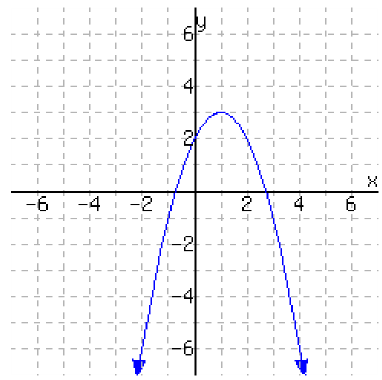




The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

10. The following graphs show two relationships. Decide whether each graph shows a relationship where y is a function of x.





The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

Determining If Tables Are Functions Determine whether or not the following table could be the table of values of a function. If the table can not be the table of values of a function, give an input that has more than one possible output.

11.

Input	Output
2	−1
4	1
6	−4
8	18
−2	−10

Could this be the table of values for a function? (☐ yes ☐ no)
If not, which input has more than one possible output? (☐ -2 ☐ 2 ☐ 4 ☐ 6 ☐ 8 ☐ None, the table represents a function.)

12.

Input	Output
2	4
4	−13
6	13
8	6
−2	−11

Could this be the table of values for a function? (☐ yes ☐ no)
If not, which input has more than one possible output? (☐ -2 ☐ 2 ☐ 4 ☐ 6 ☐ 8 ☐ None, the table represents a function.)

13.

Input	Output
−4	13
−3	2
−2	11
−3	13
−1	−18

Could this be the table of values for a function? (☐ yes ☐ no)
If not, which input has more than one possible output? (☐ -4 ☐ -3 ☐ -2 ☐ -1 ☐ None, the table represents a function.)

14.

Input	Output
−4	−8
−3	4
−2	−14
−3	19
−1	0

Could this be the table of values for a function? (☐ yes ☐ no)
If not, which input has more than one possible output? (☐ -4 ☐ -3 ☐ -2 ☐ -1 ☐ None, the table represents a function.)

Determining If Equations Are Functions

15. Select all of the following relations that make y a function of x. There are several correct answers.
☐ $y = \pm\sqrt{81 - x^2}$ ☐ $y = x^2$ ☐ $x = y^3$ ☐ $y = \sqrt[3]{x}$ ☐ $y = \frac{1}{x^2}$ ☐ $y = |x|$
☐ $x^2 + y^2 = 81$ ☐ $5x + 4y = 1$ ☐ $y = \frac{x+7}{8-x}$ ☐ $|y| = x$ ☐ $y = \sqrt{81 - x^2}$
☐ $x = y^2$

16. Select all of the following relations that make y a function of x. There are several correct answers.
☐ $x = y^2$ ☐ $5x + 5y = 1$ ☐ $x^2 + y^2 = 16$ ☐ $y = \frac{1}{x^3}$ ☐ $|y| = x$
☐ $x = y^3$ ☐ $y = \sqrt{36 - x^2}$ ☐ $y = x^5$ ☐ $y = \pm\sqrt{36 - x^2}$ ☐ $y = |x|$
☐ $y = \frac{x+2}{4-x}$ ☐ $y = \sqrt[8]{x}$

17. Some equations involving x and y define y as a function of x, and others do not. For example, if $x + y = 1$, we can solve for y and obtain $y = 1 - x$. And we can then think of $y = f(x) = 1 - x$. On the other hand, if we have the equation $x = y^2$ then y is not a function of x, since for a given positive value of x, the value of y could equal $\sqrt{x}$ or it could equal $-\sqrt{x}$. Select all of the following relations that make y a function of x. There are several correct answers.
☐ $|y| - x = 0$ ☐ $y + x^2 = 1$ ☐ $y^2 + x^2 = 1$ ☐ $3x + 5y + 9 = 0$ ☐ $y - |x| = 0$
☐ $x + y = 1$ ☐ $y^6 + x = 1$ ☐ $y^3 + x^4 = 1$

On the other hand, some equations involving x and y define x as a function of y (the other way round).

Select all of the following relations that make x a function of y. There are several correct answers.
☐ $y^2 + x^2 = 1$ ☐ $3x + 5y + 9 = 0$ ☐ $|y| - x = 0$ ☐ $y - |x| = 0$ ☐ $y^4 + x^5 = 1$

18. Some equations involving x and y define y as a function of x, and others do not. For example, if $x + y = 1$, we can solve for y and obtain $y = 1 - x$. And we can then think of $y = f(x) = 1 - x$. On the other hand, if we have the equation $x = y^2$ then y is not a function of x, since for a given positive value of x, the value of y could equal $\sqrt{x}$ or it could equal $-\sqrt{x}$. Select all of the following relations that make y a function of x. There are several correct answers.
☐ $y^6 + x = 1$ ☐ $y - |x| = 0$ ☐ $|y| - x = 0$ ☐ $y + x^2 = 1$ ☐ $4x + 2y + 4 = 0$
☐ $y^3 + x^4 = 1$ ☐ $x + y = 1$ ☐ $y^2 + x^2 = 1$

On the other hand, some equations involving x and y define x as a function of y (the other way round).

Select all of the following relations that make x a function of y. There are several correct answers.
☐ $y^4 + x^5 = 1$ ☐ $y - |x| = 0$ ☐ $y^2 + x^2 = 1$ ☐ $|y| - x = 0$ ☐ $4x + 2y + 4 = 0$

11.6 Functions Chapter Review

11.6.1 Function Basics

In Section 11.1 we defined functions informally, as well as function notation. We saw functions in four forms: verbal descriptions, formulas, graphs and tables.

Example 11.6.1 Informal Definition of a Function. Determine whether each example below describes a function.

a. The area of a circle given its radius.

b. The number you square to get 9.

Explanation.

a. The area of a circle given its radius is a function because there is a set of steps or a formula that changes the radius into the area of the circle. We could write $A(r) = \pi r^2$.

b. The number you square to get 9 is not a function because the process we would apply to get the result does not give a single answer. There are two different answers, -3 and 3. A function must give a single output for a given input.

Example 11.6.2 Tables and Graphs. Make a table and a graph of the function f, where $f(x) = x^2$.

Explanation.

First we will set up a table with negative and positive inputs and calculate the function values. The values are shown in Figure 11.6.3, which in turn gives us the graph in Figure 11.6.4.

input, x	output, $f(x)$
-3	9
-2	4
-1	1
0	0
1	2
2	4
3	9

Figure 11.6.3

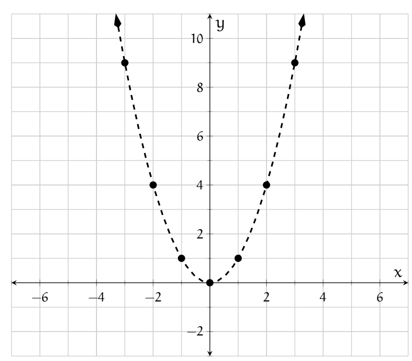


Figure 11.6.4: $y = f(x) = x^2$

Example 11.6.5 Translating between Four Descriptions of the Same Function. Consider a function f that triples its input and then adds 4. Translate this verbal description of f into a table, a graph, and a formula.

Explanation.

To make a table for f, we'll have to select some input x-values so we will choose some small negative and positive values that are easy to work with. Given the verbal description, we should be able to compute a column of output values. Table 11.6.6 is one possible table that we might end up with.

x	$f(x)$
-2	$3(-2) + 4 = -2$
-1	$3(-1) + 4 = 1$
0	$3(0) + 4 = 4$
1	$3(1) + 4 = 7$
2	$3(2) + 4 = 10$

Figure 11.6.6

Once we have a table for f, we can make a graph for f as in Figure 11.6.7, using the table to plot points. Lastly, we must find a formula for f. This means we need to write an algebraic expression that says the same thing about f as the verbal description, the table, and the graph. For this example, we can focus on the verbal description. Since f takes its input, triples it, and adds 4, we have the formula

$$f(x) = 3x + 4.$$

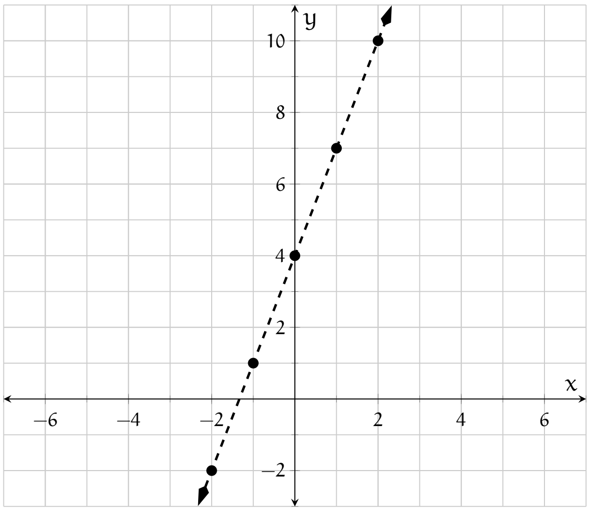


Figure 11.6.7: $y = f(x)$

11.6.2 Domain and Range

In Section 11.2 we saw the definition of domain and range, and three types of domain restrictions. We also learned how to write the domain and range in interval and set-builder notation.

Example 11.6.8 Determine the domain of p, where $p(x) = \dfrac{x}{2x-1}$.

Explanation. This is an example of the first type of domain restriction, when you have a variable in the denominator. The denominator cannot equal 0 so a *bad* value for x would be when

$$\begin{aligned} 2x - 1 &= 0 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

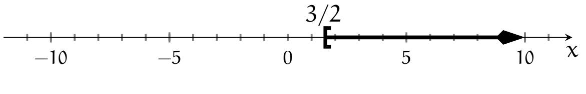
The domain is all real numbers except $\frac{1}{2}$.

Example 11.6.9 What is the domain of the function C, where $C(x) = \sqrt{2x-3} - 5$?

Explanation. This is an example of the second type of domain restriction where the value inside the radical cannot be negative. So the *good* values for x would be when

$$\begin{aligned} 2x - 3 &\geq 0 \\ 2x &\geq 3 \\ x &\geq \frac{3}{2} \end{aligned}$$

So on a number line, if we wanted to picture the domain of C, we would make a sketch like:



The domain is the interval $\left[\frac{3}{2}, \infty\right)$.

Example 11.6.10 Range.
Find the range of the function q using its graph shown in Figure 11.6.11.

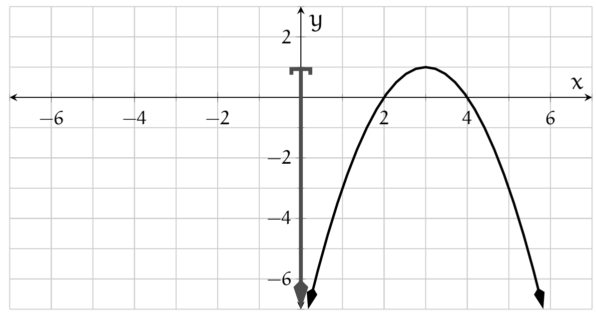


Figure 11.6.11: $y = q(x)$. The range is marked as an interval on the y-axis.

Explanation. The range is the collection of possible numbers that q can give for output. Figure 11.6.11 displays a graph of q, with the range shown as an interval on the y-axis.

The output values are the y-coordinates so we can see that the y-values start from 1 and continue downward forever. Therefore the range is $(-\infty, 1]$.

11.6.3 Using Technology to Explore Functions

In Section 11.3 we covered how to find a good graphing window and use it to identify all of the key features of a function. We also learned how to solve equations and inequalities using a graph. Here are some examples for review.

Example 11.6.12 Finding an Appropriate Window. Graph the function t, where $t(x) = (x+10)^2 - 15$, using technology and find a good viewing window.

Explanation.

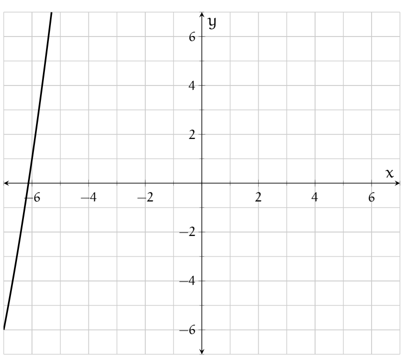


Figure 11.6.13: $y = t(x)$ in the viewing window of -7 to 7 on the x and y axes. We need to zoom out and move our window to the left.

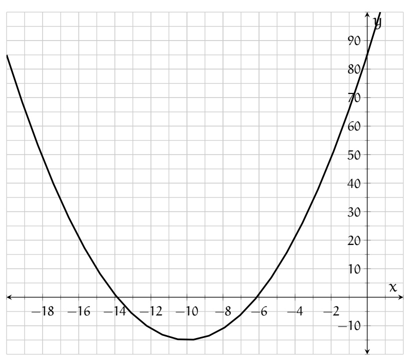


Figure 11.6.14: $y = t(x)$ in a good viewing window.

After some trial and error we found this window that goes from -20 to 2 on the x-axis and -20 to 100 on the y-axis.

Now we can see the vertex and all of the intercepts and we will identify them in the next example.

Example 11.6.15 Using Technology to Determine Key Features of a Graph. Use the previous graph in figure 11.6.14 to identify the intercepts, minimum or maximum function value, and the domain and range of the function t, where $t(x) = (x+10)^2 - 15$.

Explanation.

From our graph we can now identify the vertex at $(-10, -15)$, the y-intercept at $(0, 85)$, and the x-intercepts at approximately $(-13.9, 0)$ and $(-6.13, 0)$.

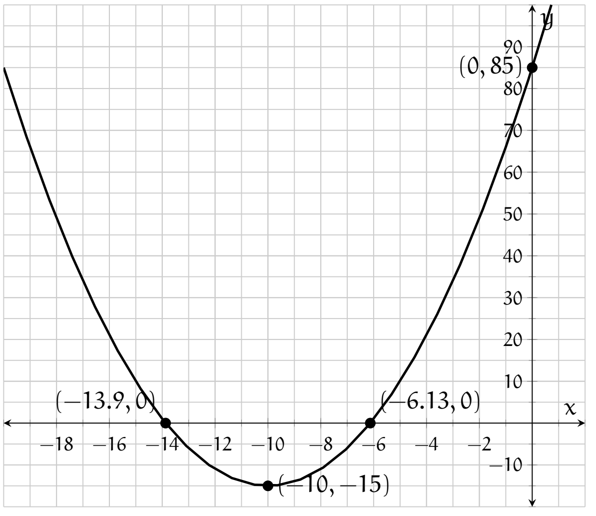


Figure 11.6.16: $y = t(x) = (x+10)^2 - 15$.

Example 11.6.17 Solving Equations and Inequalities Graphically Using Technology. Use graphing technology to solve the equation $t(x) = 40$, where $t(x) = (x+10)^2 - 15$.

Explanation.

To solve the equation $t(x) = 40$, we need to graph $y = t(x)$ and $y = 40$ on the same axes and find the x-values where they intersect.
From the graph we can see that the intersection points are approximately $(-17.4, 40)$ and $(-2.58, 40)$. The solution set is $\{-17.4, -2.58\}$.

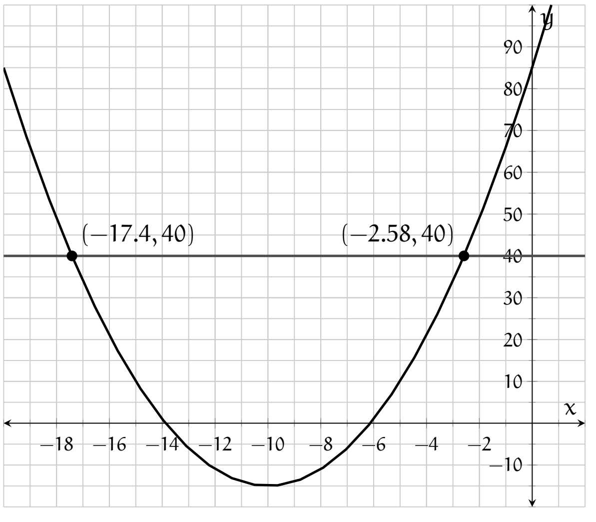


Figure 11.6.18: $y = t(x)$ where $t(x) = (x+10)^2 - 15$ and $y = 40$.

11.6.4 Simplifying Expressions with Function Notation

In Section 11.4 we learned about the difference between $f(-x)$ and $-f(x)$ and how to simplify them. We also learned how to simplify other changes to the input and output like $f(3x)$ and $\frac{1}{3}f(x)$. Here are some examples.

Example 11.6.19 Negative Signs in and out of Function Notation. Find and simplify a formula for $f(-x)$ and $-f(x)$, where $f(x) = -3x^2 - 7x + 1$.

Explanation. To find $f(-x)$, we use an input of $-x$ in our function f and simplify to get:

$$\begin{aligned} f(-x) &= -3(-x)^2 - 7(-x) + 1 \\ &= -3x^2 + 7x + 1 \end{aligned}$$

To find $-f(x)$, we take the opposite of the function f and simplify to get:

$$\begin{aligned} -f(x) &= -(-3x^2 - 7x + 1) \\ &= 3x^2 + 7x - 1 \end{aligned}$$

Example 11.6.20 Other Nontrivial Simplifications. If $g(x) = 2x^2 - 3x - 5$, find and simplify a formula for $g(x - 1)$.

Explanation. To find $g(x - 1)$, we put in $(x - 1)$ for the input. It is important to keep the parentheses because we have exponents and negative signs in the function.

$$\begin{aligned} g(x-1) &= 2(x-1)^2 - 3(x-1) - 5 \\ &= 2(x^2 - 2x + 1) - 3x + 3 - 5 \\ &= 2x^2 - 4x + 2 - 3x - 2 \\ &= 2x^2 - 7x \end{aligned}$$

11.6.5 Technical Definition of a Function

In Section 11.5 we gave a formal definition of a function 11.5.2 and learned to identify what is and is not a function with sets or ordered pairs, tables and graphs. We also used the vertical line test 11.5.20.

Example 11.6.21 Formally Defining a Function. We learned that sets of ordered pairs, tables and graphs can meet the formal definition of a function. Here is an example that shows a function in all three forms. We can verify that each input has at most one output.

$\{(1,4),(2,4),(3,3),(4,6),(5,-2)\}$

Figure 11.6.22: The function f represented as a collection of ordered pairs.

x	$f(x)$
1	4
2	4
3	3
4	6
5	-2

Figure 11.6.23: The function f represented as a table.

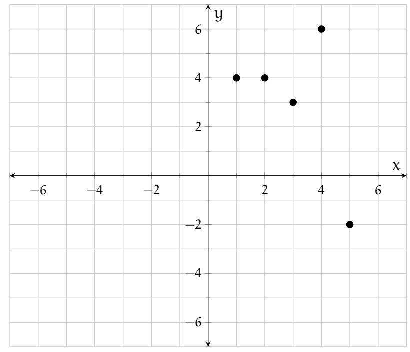


Figure 11.6.24: The function f represented as a graph.

Example 11.6.25 Identifying What is *Not* a Function. Identify whether each graph represents a function using the vertical line test 11.5.20.

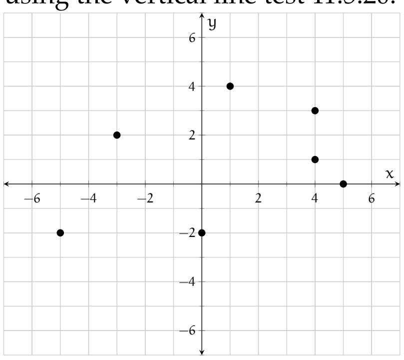


Figure 11.6.26

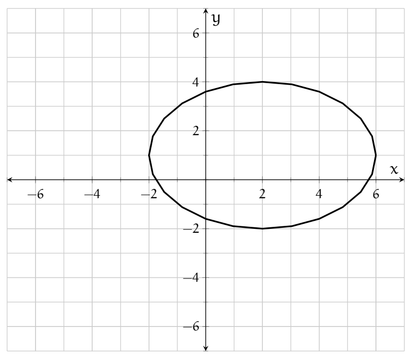


Figure 11.6.27

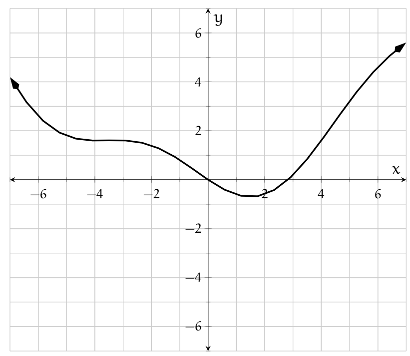


Figure 11.6.28

Explanation.

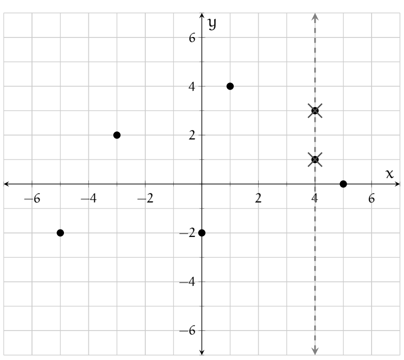


Figure 11.6.29: A vertical line touching the graph twice makes this graph not give y as a function of x.

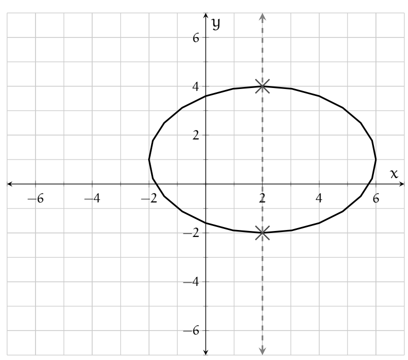


Figure 11.6.30: A vertical line touching the graph twice makes this graph not give y as a function of x.

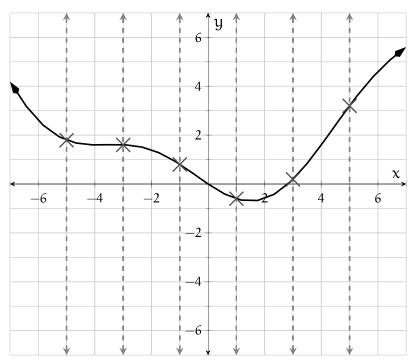


Figure 11.6.31: All vertical lines only touch the graph once, so this graph does give y as a function of x.

11.6.6 Exercises

Function Basics

1. Randi will spend \$225 to purchase some bowls and some plates. Each plate costs \$8, and each bowl costs \$5. The function $q(x) = -\frac{8}{5}x + 45$ models the number of bowls Randi will purchase, where x represents the number of plates to be purchased.
Interpret the meaning of $q(35) = -11$.

- ⊙ ***A***. −\$11 will be used to purchase bowls, and \$35 will be used to purchase plates.
- ⊙ ***B***. −11 plates and 35 bowls can be purchased.
- ⊙ ***C***. 35 plates and −11 bowls can be purchased.
- ⊙ ***D***. \$35 will be used to purchase bowls, and −\$11 will be used to purchase plates.

2. Douglas will spend \$150 to purchase some bowls and some plates. Each plate costs \$5, and each bowl costs \$6. The function $q(x) = -\frac{5}{6}x + 25$ models the number of bowls Douglas will purchase, where x represents the number of plates to be purchased.
Interpret the meaning of $q(12) = 15$.

- ⊙ ***A***. \$15 will be used to purchase bowls, and \$12 will be used to purchase plates.
- ⊙ ***B***. 12 plates and 15 bowls can be purchased.
- ⊙ ***C***. 15 plates and 12 bowls can be purchased.
- ⊙ ***D***. \$12 will be used to purchase bowls, and \$15 will be used to purchase plates.

3. Evaluate the function at the given values.
$G(x) = -\dfrac{9}{x-7}$.

a. $G(8)$

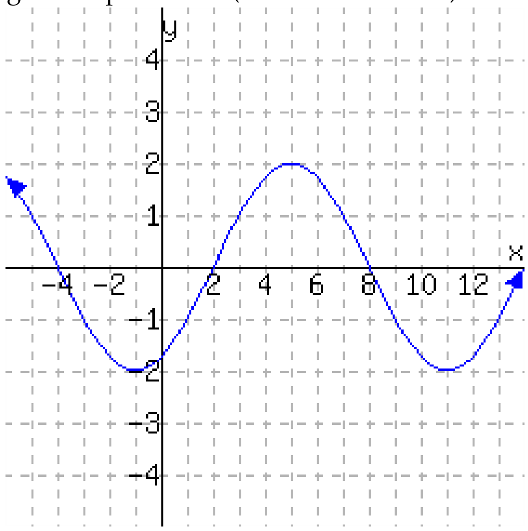
b. $G(7)$

4. Evaluate the function at the given values.
$G(x) = \dfrac{40}{x+8}$.

a. $G(2)$

b. $G(-8)$

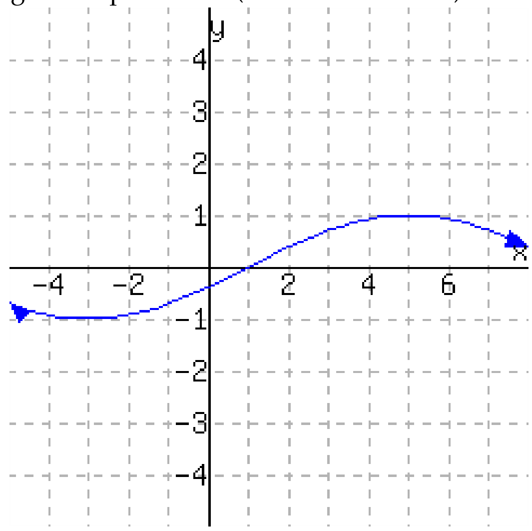
5. Use the graph of H below to evaluate the given expressions. (Estimates are OK.)



a. $H(-4) =$

b. $H(11) =$

6. Use the graph of K below to evaluate the given expressions. (Estimates are OK.)



a. $K(-3) =$

b. $K(5) =$

7. Use the table of values for f below to evaluate the given expressions.

x	-5	-1	3	7	11
$f(x)$	-1.5	-1.2	7.3	6.2	5.6

a. $f(3) =$

b. $f(7) =$

8. Use the table of values for f below to evaluate the given expressions.

x	0	2	4	6	8
$f(x)$	7.3	-1.3	1.3	7.3	-1.6

a. $f(0) =$

b. $f(8) =$

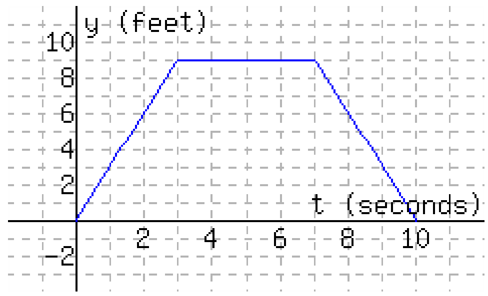
9. Make a table of values for the function h, defined by $h(x) = -4x^2$. Based on values in the table, sketch a graph of h.

x	$h(x)$

10. Make a table of values for the function H, defined by $H(x) = \dfrac{2^x - 3}{x^2 + 3}$. Based on values in the table, sketch a graph of H.

x	$H(x)$

11. The following figure has the graph $y = d(t)$, which models a particle's distance from the starting line in feet, where t stands for time in seconds since timing started.



a. Find $d(7)$.

b. Interpret the meaning of $d(7)$.

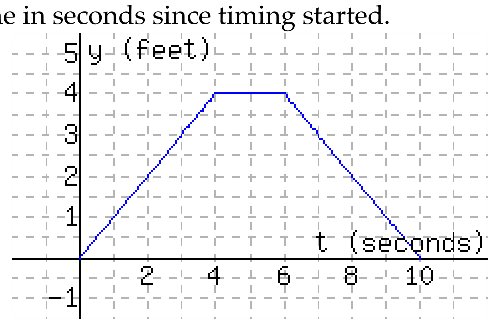
- ⊙ ***A***. The particle was 9 feet away from the starting line 7 seconds since timing started.
- ⊙ ***B***. The particle was 7 feet away from the starting line 9 seconds since timing started.
- ⊙ ***C***. In the first 9 seconds, the particle moved a total of 7 feet.
- ⊙ ***D***. In the first 7 seconds, the particle moved a total of 9 feet.

c. Solve $d(t) = 3$ for t. $t =$ ______

d. Interpret the meaning of part c's solution(s).

- ⊙ ***A***. The particle was 3 feet from the starting line 1 seconds since timing started, and again 9 seconds since timing started.
- ⊙ ***B***. The particle was 3 feet from the starting line 9 seconds since timing started.
- ⊙ ***C***. The particle was 3 feet from the starting line 1 seconds since timing started.
- ⊙ ***D***. The particle was 3 feet from the starting line 1 seconds since timing started, or 9 seconds since timing started.

12. The following figure has the graph $y = d(t)$, which models a particle's distance from the starting line in feet, where t stands for time in seconds since timing started.



a. Find $d(8)$.

b. Interpret the meaning of $d(8)$.

- ⊙ ***A***. In the first 2 seconds, the particle moved a total of 8 feet.
- ⊙ ***B***. The particle was 8 feet away from the starting line 2 seconds since timing started.

⊙ ***C***. In the first 8 seconds, the particle moved a total of 2 feet.

⊙ ***D***. The particle was 2 feet away from the starting line 8 seconds since timing started.

c. Solve $d(t) = 3$ for t. $t =$ ______

d. Interpret the meaning of part c's solution(s).

⊙ ***A***. The particle was 3 feet from the starting line 3 seconds since timing started, or 7 seconds since timing started.

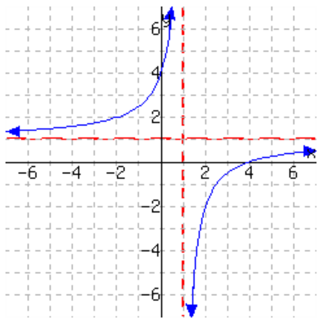
⊙ ***B***. The particle was 3 feet from the starting line 3 seconds since timing started.

⊙ ***C***. The particle was 3 feet from the starting line 7 seconds since timing started.

⊙ ***D***. The particle was 3 feet from the starting line 3 seconds since timing started, and again 7 seconds since timing started.

Domain and Range A function is graphed.

13.



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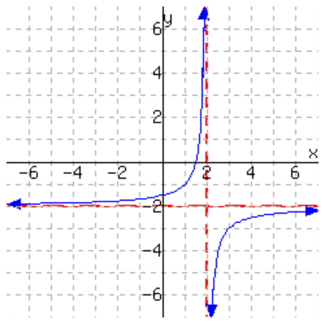

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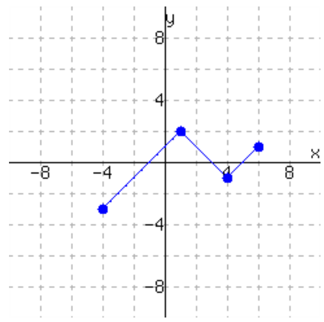

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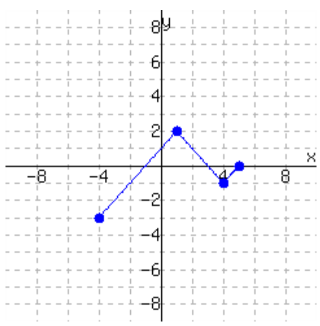

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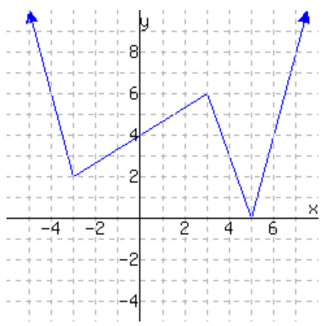

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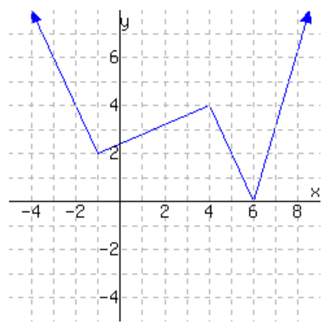


19. Find the domain of t where $t(x) = \dfrac{\sqrt{8+x}}{5-x}$.

20. Find the domain of C where $C(x) = \dfrac{\sqrt{10+x}}{2-x}$.

21. An object was shot up into the air at an initial vertical speed of 512 feet per second. Its height as time passes can be modeled by the quadratic function f, where $f(t) = -16t^2 + 512t$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height in feet.

Find the function's domain and range in this context.

The function's domain in this context is ______.

The function's range in this context is ______.

22. An object was shot up into the air at an initial vertical speed of 544 feet per second. Its height as time passes can be modeled by the quadratic function f, where $f(t) = -16t^2 + 544t$. Here t represents the number of seconds since the object's release, and $f(t)$ represents the object's height in feet.
Find the function's domain and range in this context.
The function's domain in this context is ______.
The function's range in this context is ______.

Using Technology to Explore Functions

23. Use technology to make a table of values for the function H defined by $H(x) = -4x^2 + 16x + 1$.

x	$H(x)$

24. Use technology to make a table of values for the function K defined by $K(x) = 2x^2 - 7x - 2$.

x	$K(x)$

25. Choose an appropriate window for graphing the function f defined by $f(x) = -1184x - 7607$ that shows its key features.
The x-interval could be ______ and the y-interval could be ______.

26. Choose an appropriate window for graphing the function f defined by $f(x) = -139x + 159$ that shows its key features.
The x-interval could be ______ and the y-interval could be ______.

27. Use technology to determine how many times the equations $y = -5x^3 + 2x^2 + x$ and $y = 6x + 4$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

28. Use technology to determine how many times the equations $y = -3x^3 - x^2 + 9x$ and $y = -5x + 6$ intersect. They intersect (☐ zero times ☐ one time ☐ two times ☐ three times) .

29. For the function L defined by

$$L(x) = 3000x^2 + 10x + 4,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.

b. The vertex.

c. The domain.

d. The range.

30. For the function M defined by

$$M(x) = -(300x - 2950)^2,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.

b. The vertex.

c. The domain.

d. The range.

31. Let $f(x) = 4x^2 + 5x - 1$ and $g(x) = 5$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $f(x) = g(x)$.

c. Solve $f(x) < g(x)$.

d. Solve $f(x) \geq g(x)$.

32. Let $p(x) = 6x^2 - 3x + 4$ and $k(x) = 7$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $p(x) = k(x)$.

c. Solve $p(x) < k(x)$.

d. Solve $p(x) \geq k(x)$.

33. Use graphing technology to solve the equation $-0.02x^2 + 1.97x - 51.5 = 0.05\,(x - 50)^2 - 0.03\,(x - 50)$. Approximate the solution(s) if necessary.

34. Use graphing technology to solve the equation $-200x^2 + 60x - 55 = -20x - 40$. Approximate the solution(s) if necessary.

35. Use graphing technology to solve the inequality $-15x^2 - 6 \leq 10x - 4$. State the solution set using interval notation, and approximate if necessary.

36. Use graphing technology to solve the inequality $\frac{1}{2}x^2 + \frac{3}{2}x \geq \frac{1}{2}x - \frac{3}{2}$. State the solution set using interval notation, and approximate if necessary.

Simplifying Expressions with Function Notation

37. Let f be a function given by $f(x) = -3x^2 + 3x$. Find and simplify the following:

a. $f(x) - 3 =$ ☐

b. $f(x-3) =$ ☐

c. $-3f(x) =$ ☐

d. $f(-3x) =$ ☐

38. Let f be a function given by $f(x) = 3x^2 - 4x$. Find and simplify the following:

a. $f(x) - 4 =$ ☐

b. $f(x-4) =$ ☐

c. $-4f(x) =$ ☐

d. $f(-4x) =$ ☐

39. Simplify $F(r) + 6$, where $F(r) = 3 - 5.1r$.

40. Simplify $g(r) + 9$, where $g(r) = 2 + 6.5r$.

Technical Definition of a Function

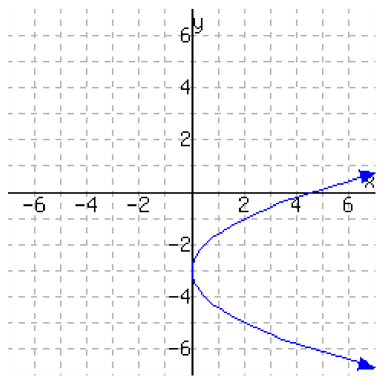
41. Does the following set of ordered pairs make for a function of x?
$\left\{(-3,3), (-5,9), (-5,0), (1,7), (-6,3)\right\}$
This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ☐ and range ☐.

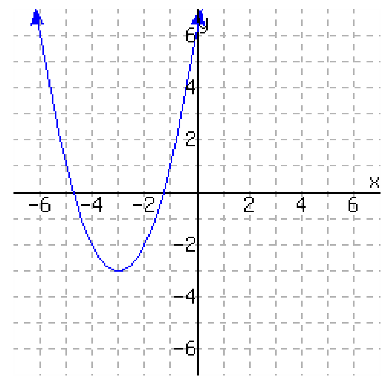
42. Does the following set of ordered pairs make for a function of x?
$\left\{(-8,9), (5,5), (-3,9), (-1,2), (-8,10)\right\}$
This set of ordered pairs (☐ describes ☐ does not describe) a function of x. This set of ordered pairs has domain ☐ and range ☐.

43. Below is all of the information that exists about a function f.
$f(-2) = 1 \quad f(0) = 5 \quad f(1) = 2$
Write f as a set of ordered pairs.
f has domain ☐ and range ☐.

44. Below is all of the information about a function f.
$f(a) = 5 \quad f(b) = 6$
$f(c) = 3 \quad f(d) = 6$
Write f as a set of ordered pairs.
f has domain ☐ and range ☐.

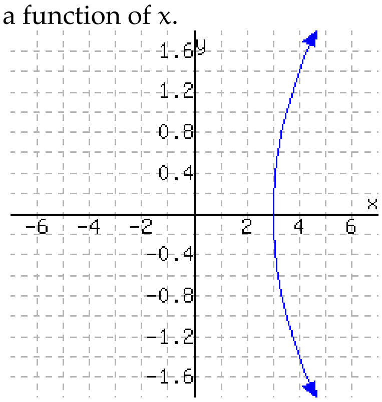
45. The following graphs show two relationships. Decide whether each graph shows a relationship where y is a function of x.

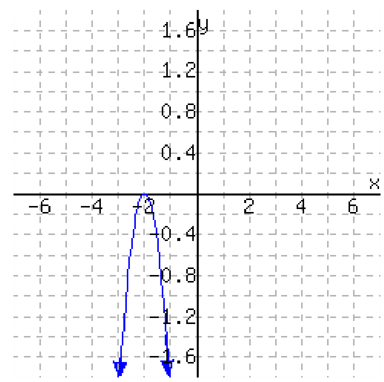




The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

46. The following graphs show two relationships. Decide whether each graph shows a relationship where y is a function of x.





The first graph (☐ does ☐ does not) give a function of x. The second graph (☐ does ☐ does not) give a function of x.

47. Some equations involving x and y define y as a function of x, and others do not. For example, if $x + y = 1$, we can solve for y and obtain $y = 1 - x$. And we can then think of $y = f(x) = 1 - x$. On the other hand, if we have the equation $x = y^2$ then y is not a function of x, since for a given positive value of x, the value of y could equal $\sqrt{x}$ or it could equal $-\sqrt{x}$. Select all of the following relations that make y a function of x. There are several correct answers.

☐ $y^3 + x^4 = 1$ ☐ $y + x^2 = 1$ ☐ $y^2 + x^2 = 1$ ☐ $y^6 + x = 1$ ☐ $y - |x| = 0$
☐ $5x + 8y + 9 = 0$ ☐ $x + y = 1$ ☐ $|y| - x = 0$

On the other hand, some equations involving x and y define x as a function of y (the other way round).

Select all of the following relations that make x a function of y. There are several correct answers.

☐ $|y| - x = 0$ ☐ $5x + 8y + 9 = 0$ ☐ $y^2 + x^2 = 1$ ☐ $y - |x| = 0$ ☐ $y^4 + x^5 = 1$

48. Some equations involving x and y define y as a function of x, and others do not. For example, if $x + y = 1$, we can solve for y and obtain $y = 1 - x$. And we can then think of $y = f(x) = 1 - x$. On the other hand, if we have the equation $x = y^2$ then y is not a function of x, since for a given positive value of x, the value of y could equal $\sqrt{x}$ or it could equal $-\sqrt{x}$. Select all of the following relations that make y a function of x. There are several correct answers.

☐ $|y| - x = 0$ ☐ $y + x^2 = 1$ ☐ $x + y = 1$ ☐ $y^2 + x^2 = 1$ ☐ $6x + 5y + 4 = 0$
☐ $y^3 + x^4 = 1$ ☐ $y^6 + x = 1$ ☐ $y - |x| = 0$

On the other hand, some equations involving x and y define x as a function of y (the other way round).

Select all of the following relations that make x a function of y. There are several correct answers.

☐ $y^2 + x^2 = 1$ ☐ $|y| - x = 0$ ☐ $y - |x| = 0$ ☐ $y^4 + x^5 = 1$ ☐ $6x + 5y + 4 = 0$

Determine whether or not the following table could be the table of values of a function. If the table can not be the table of values of a function, give an input that has more than one possible output.

49.

Input	Output
2	7
4	-18
6	10
8	-4
-2	1

Could this be the table of values for a function? (☐ yes ☐ no)
If not, which input has more than one possible output? (☐ -2 ☐ 2 ☐ 4 ☐ 6 ☐ 8 ☐ None, the table represents a function.)

50.

Input	Output
2	12
4	9
6	-14
8	-15
-2	1

Could this be the table of values for a function? (☐ yes ☐ no)
If not, which input has more than one possible output? (☐ -2 ☐ 2 ☐ 4 ☐ 6 ☐ 8 ☐ None, the table represents a function.)

51.

Input	Output
-4	6
-3	2
-2	0
-3	13
-1	-3

Could this be the table of values for a function? (☐ yes ☐ no)
If not, which input has more than one possible output? (☐ -4 ☐ -3 ☐ -2 ☐ -1 ☐ None, the table represents a function.)

52.

Input	Output
-4	-14
-3	4
-2	17
-3	19
-1	15

Could this be the table of values for a function? (☐ yes ☐ no)
If not, which input has more than one possible output? (☐ -4 ☐ -3 ☐ -2 ☐ -1 ☐ None, the table represents a function.)

Chapter 12

Rational Functions and Equations

12.1 Introduction to Rational Functions

In this chapter we will learn about rational functions, which are ratios of two polynomial functions. Creating this ratio inherently requires division, and we'll explore the effect this has on the graphs of rational functions and their domain and range.

12.1.1 Graphs of Rational Functions

Example 12.1.2
When a drug is injected into a patient, the drug's concentration in the patient's bloodstream can be modeled by the function C, with formula

$$C(t) = \frac{3t}{t^2 + 8}$$

where $C(t)$ gives the drug's concentration, in milligrams per liter, t hours since the injection. A new injection is needed when the concentration falls to 0.35 milligrams per liter. Using graphing technology, we will graph $y = \frac{3t}{t^2+8}$ and $y = 0.35$ to examine the situation and answer some important questions.

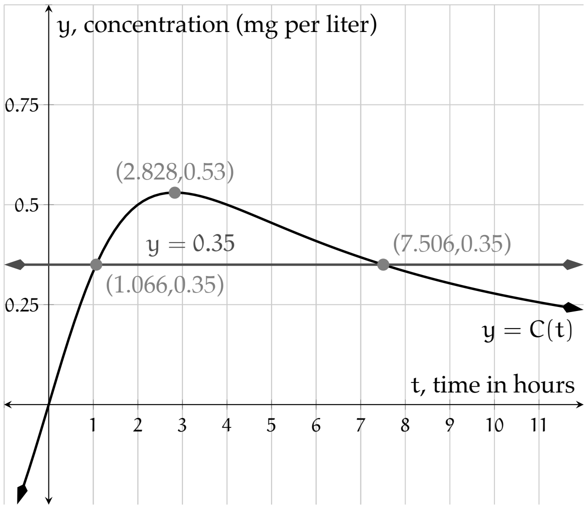


Figure 12.1.3: Graph of $C(t) = \frac{3t}{t^2+8}$

a. What is the concentration after 10 hours?

b. After how many hours since the first injection is the drug concentration greatest?

c. After how many hours since the first injection should the next injection be given?

d. What happens to the drug concentration if no further injections are given?

Explanation.

a. To determine the concentration after 10 hours, we will evaluate C at $t = 10$. After 10 hours, the concentration will be about $0.2777\ \frac{\text{mg}}{\text{L}}$.

$$\begin{aligned} C(10) &= \frac{3(10)}{10^2+8} \\ &= \frac{30}{108} \\ &= \frac{5}{18} \\ &\approx 0.2777 \end{aligned}$$

b. Using the graph, we can see that the maximum concentration of the drug will be $0.53\ \frac{\text{mg}}{\text{L}}$ and will occur after about 2.828 hours.

c. The approximate points of intersection $(1.066, 0.35)$ and $(7.506, 0.35)$ tell us that the concentration of the drug will reach $0.35\ \frac{\text{mg}}{\text{L}}$ after about 1.066 hours and again after about 7.506 hours. Given the rising, then falling shape of the graph, this means that another dose will need to be administered after about 7.506 hours.

d. From the initial graph, it appears that the concentration of the drug will diminish to zero with enough time passing. Exploring further, we can see both numerically and graphically that for larger and larger values of t, the function values get closer and closer to zero. This is shown in Figure 12.1.4 and Figure 12.1.5.

t	$C(t)$
24	$0.123\ldots$
48	$0.062\ldots$
72	$0.041\ldots$
96	$0.031\ldots$
120	$0.020\ldots$

Figure 12.1.4: Numerical Values for $C(t) = \frac{3t}{t^2+8}$

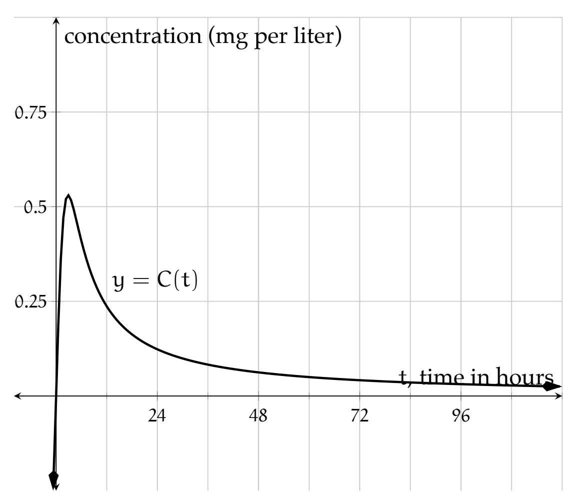


Figure 12.1.5: Graph of $C(t) = \frac{3t}{t^2+8}$

In Section 12.5, we'll explore how to algebraically solve $C(t) = 0.35$. For now, we will rely on technology to make the graph and determine intersection points.

The function C, where $C(t) = \frac{3t}{t^2+8}$, is a *rational function*, which is a type of function defined as follows.

Definition 12.1.6 Rational Function. A **rational function** f is a function in the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomial functions, but Q is not the constant zero function. ◊

Checkpoint 12.1.7 Identify which of the following are rational functions and which are not.

a. f defined by $f(x) = \frac{25x^2+3}{25x^2+3}$ (☐ is ☐ is not) a rational function.

b. Q defined by $Q(x) = \frac{5x^2+3\sqrt{x}}{2x}$ (☐ is ☐ is not) a rational function.

c. g defined by $g(t) = \frac{t\sqrt{5}-t^3}{2t+1}$ (☐ is ☐ is not) a rational function.

d. P defined by $P(x) = \frac{5x+3}{|2x+1|}$ (☐ is ☐ is not) a rational function.

e. h defined by $h(x) = \frac{3^x+1}{x^2+1}$ (☐ is ☐ is not) a rational function.

Explanation.

a. f defined by $f(x) = \frac{25x^2+3}{25x^2+3}$ is a rational function as its formula is a polynomial divided by another polynomial.

b. Q defined by $Q(x) = \frac{5x^2+3\sqrt{x}}{2x}$ is not a rational function because the numerator contains $\sqrt{x}$ and is therefore not a polynomial.

c. g defined by $g(t) = \frac{t\sqrt{5}-t^3}{2t+1}$ is a rational function as its formula is a polynomial divided by another polynomial.

d. P defined by $P(x) = \frac{5x+3}{|2x+1|}$ is not a rational function because the denominator contains the absolute value of an expression with variables in it.

e. h defined by $h(x) = \frac{3^x+1}{x^2+1}$ is not a rational function because the numerator contains 3^x, which has a variable in the exponent.

A rational function's graph is not always smooth like the one shown in Example 12.1.3. It could have breaks, as we'll see now.

Example 12.1.8 Build a table and sketch the graph of the function f where $f(x) = \frac{1}{x-2}$. Find the function's domain and range.

Since $x = 2$ makes the denominator 0, the function will be undefined for $x = 2$. We'll start by choosing various x-values and plotting the associated points.

x		f(x)	Point
-6	$\frac{1}{-6-2}$	-0.125	$(-6,-0.125)$
-4	$\frac{1}{-4-2}$	≈ -0.167	$\left(-4,-\frac{1}{6}\right)$
-2	$\frac{1}{-2-2}$	-0.25	$(-2,-0.25)$
0	$\frac{1}{0-2}$	-0.5	$(0,-0.5)$
1	$\frac{1}{1-2}$	-1	$(1,-1)$
2	$\frac{1}{2-2}$	undefined	
3	$\frac{1}{3-2}$	1	$(3,1)$
4	$\frac{1}{4-2}$	0.5	$(4,0.5)$

Figure 12.1.9: Initial Values of $f(x)=\frac{1}{x-2}$

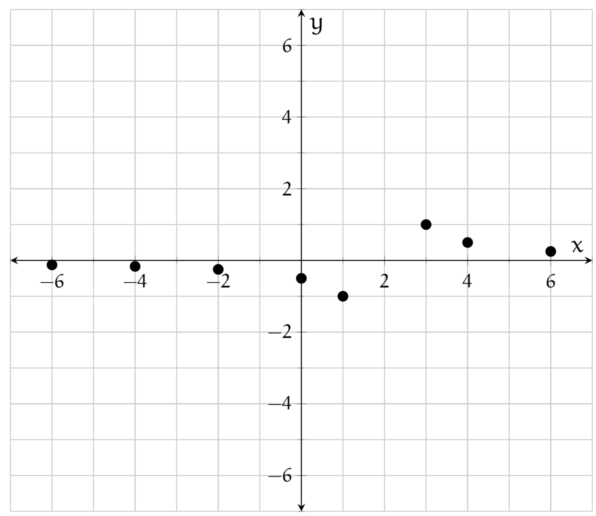


Figure 12.1.10: Initial Points for $f(x)=\frac{1}{x-2}$

Note that extra points were chosen near $x=2$ in the Figure 12.1.9, but it's still not clear on the graph what happens really close to $x=2$. It will be essential that we include at least one x-value between 1 and 2 and also between 2 and 3.

Further, we'll note that dividing one number by a number that is close to 0 yields a large number. For example, $\frac{1}{0.0005}=2000$. In fact, the smaller the number is that we divide by, the larger our result becomes. So when x gets closer and closer to 2, then $x-2$ gets closer and closer to 0. And then $\frac{1}{x-2}$ takes very large values.

When we plot additional points closer and closer to 2, we get larger and larger results. To the left of 2, the results are negative, so the connected curve has an arrow pointing downward there. The opposite happens to the right of $x=2$, and an arrow points upward. We'll also draw the vertical line $x=2$ as a dashed line to indicate that the graph never actually touches it.

x		f(x)	Point
-6	$\frac{1}{-6-2}$	-0.125	$(-6,-0.125)$
-4	$\frac{1}{-4-2}$	≈ -0.167	$\left(-4,-\frac{1}{6}\right)$
-2	$\frac{1}{-2-2}$	-0.25	$(-2,-0.25)$
0	$\frac{1}{0-2}$	-0.5	$(0,-0.5)$
1	$\frac{1}{1-2}$	-1	$(1,-1)$
1.5	$\frac{1}{1.5-2}$	-2	$(1.5,-2)$
1.8	$\frac{1}{1.8-2}$	-5	$(1.8,-5)$
2	$\frac{1}{2-2}$	undefined	
2.1	$\frac{1}{2.2-2}$	5	$(2.2,5)$
2.5	$\frac{1}{2.5-2}$	2	$(2.5,2)$
3	$\frac{1}{3-2}$	1	$(3,1)$
4	$\frac{1}{4-2}$	0.5	$(4,0.5)$

Figure 12.1.11: Values of $f(x)=\frac{1}{x-2}$

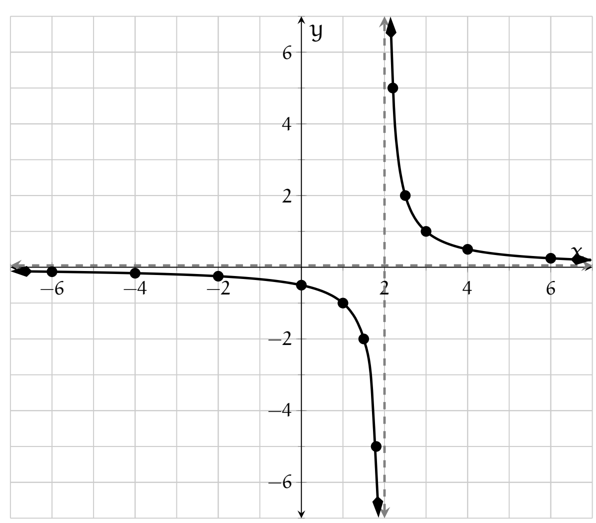


Figure 12.1.12: Full Graph of $f(x)=\frac{1}{x-2}$

Note that in Figure 12.1.12, the line $y = 0$ was also drawn as a dashed line. This is because the values of $y = f(x)$ will get closer and closer to zero as the inputs become more and more positive (or negative).

We know that the domain of this function is $(-\infty, 2) \cup (2, \infty)$ as the function is undefined at 2. We can determine this algebraically, and it is also evident in the graph.

We can see from the graph that the range of the function is $(-\infty, 0) \cup (0, \infty)$. See Checkpoint 11.2.26 for a discussion of how to see the range using a graph like this one.

Remark 12.1.13 The line $x = 2$ in Example 12.1.8 is referred to as a **vertical asymptote**. The line $y = 0$ is referred to as a **horizontal asymptote**. We'll use this vocabulary when referencing such lines, but the classification of vertical asymptotes and horizontal asymptotes is beyond the scope of this book.

Example 12.1.14 Algebraically find the domain of $g(x) = \frac{3x^2}{x^2-2x-24}$. Use technology to sketch a graph of this function.

Explanation. To find a rational function's domain, we set the denominator equal to 0 and solve:

$$x^2 - 2x - 24 = 0$$
$$(x-6)(x+4) = 0$$

$$\begin{aligned} x - 6 &= 0 \quad &\text{or} \quad && x + 4 &= 0 \\ x &= 6 \quad &\text{or} \quad && x &= -4 \end{aligned}$$

Since $x = 6$ and $x = -4$ will cause the denominator to be 0, they are excluded from the domain. The function's domain is $\{x \mid x \neq 6, x \neq -4\}$. In interval notation, the domain is $(-\infty, -4) \cup (-4, 6) \cup (6, \infty)$.

To begin creating this graph, we'll use technology to create a table of function values, making sure to include values near both -4 and 6. We'll sketch an initial plot of these.

x	$\frac{3x^2}{x^2-2x-24}$	x	$\frac{3x^2}{x^2-2x-24}$
-10	3.125	1	-0.12
-9	3.24	2	-0.5
-8	$3.428\ldots$	3	-1.285
-7	$3.769\ldots$	4	-3
-6	4.5	5	$-8.333\ldots$
-5	$6.818\ldots$	6	undefined
-4	undefined	7	$13.363\ldots$
-3	-3	8	8
-2	-0.75	9	$6.230\ldots$
-1	$-0.142\ldots$	10	$5.357\ldots$
0	$0\ldots$		

Figure 12.1.15: Numerical Values for g

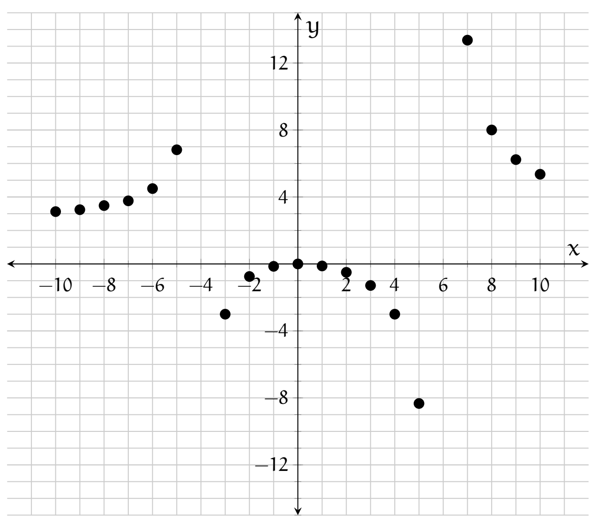


Figure 12.1.16: Initial Set-Up to Graph g

We can now begin to see what happens near $x = -4$ and $x = 6$. These are referred to as vertical asymptotes and will be graphed as dashed vertical lines as they are features of the graph but do not include function values.

The last thing we need to consider is what happens for large positive values of x and large negative values of x. Choosing a few values, we find:

x	$g(x)$
1000	3.0060...
2000	3.0030...
3000	3.0020...
4000	3.0015...

Figure 12.1.17: Values for Large Positive x

x	$g(x)$
−1000	2.9940...
−2000	2.9970...
−3000	2.9980...
−4000	2.9985...

Figure 12.1.18: Values for Large Negative x

Thus for really large positive x and for really large negative x, we see that the function values get closer and closer to $y = 3$. This is referred to as the horizontal asymptote, and will be graphed as a dashed horizontal line on the graph.

Putting all of this together, we can sketch a graph of this function.

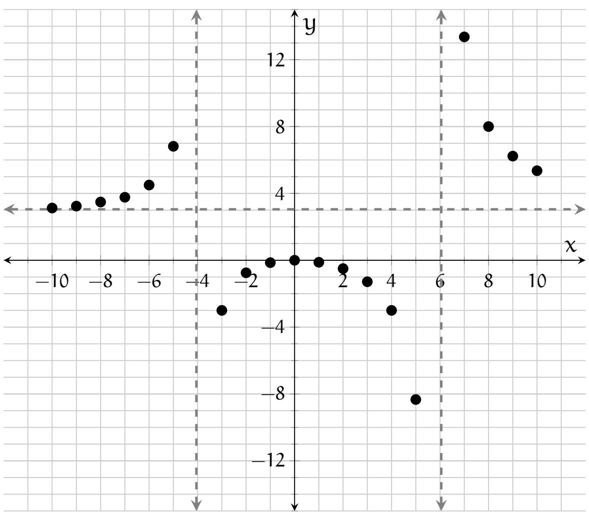


Figure 12.1.19: Asymptotes Added for Graphing $g(x) = \frac{3x^2}{x^2-2x-24}$

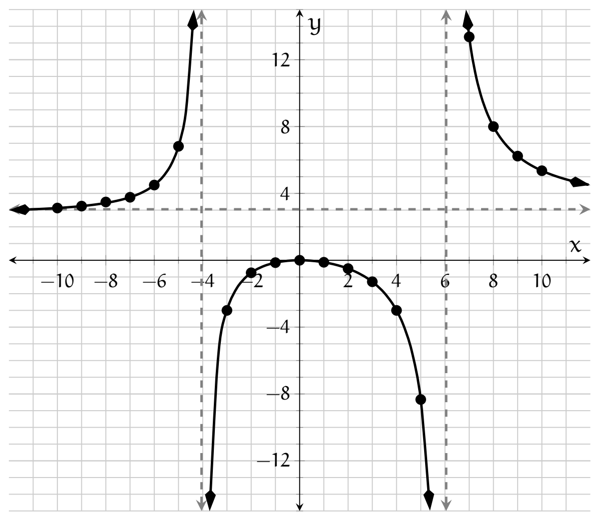


Figure 12.1.20: Full Graph of $g(x) = \frac{3x^2}{x^2-2x-24}$

Let's look at another example where a rational function is used to model real life data.

Example 12.1.21 The monthly operation cost of Saqui's shoe company is approximately $300,000.00. The cost of producing each pair of shoes is $30.00. As a result, the cost of producing x pairs of shoes is $30x + 300000$ dollars, and the average cost of producing each pair of shoes can be modeled by

$$\bar{C}(x) = \frac{30x + 300000}{x}.$$

Answer the following questions with technology.

a. What's the average cost of producing 100 pairs of shoes? Of producing 1000 pairs? Of producing 10,000 pairs? What's the pattern?

b. To make the average cost of producing each pair of shoes cheaper than $50.00, at least how many pairs of shoes must Saqui's company produce?

c. Assume that her company's shoes are very popular. What happens to the average cost of producing shoes if more and more people keep buying them?

Explanation. We will graph the function with technology. After adjusting window settings, we have:

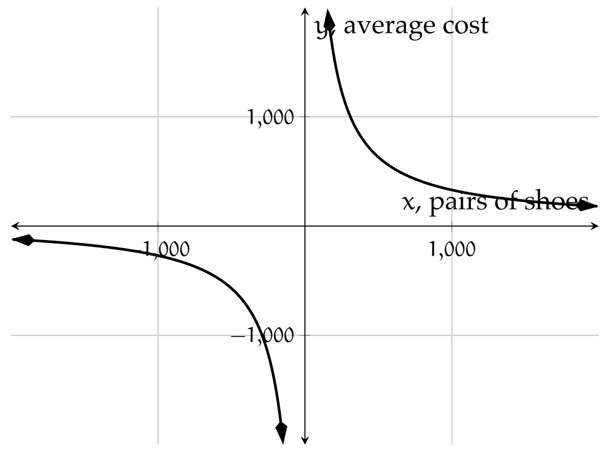


Figure 12.1.22: Graph of $\bar{C}(x) = \frac{30x+300000}{x}$

a. To answer this question, we locate the points where x values are 100, 1000 and 10,000. They are $(100, 3030)$, $(1000, 330)$ and $(10000, 60)$. They imply:

   - If the company produces 100 pairs of shoes, the average cost of producing one pair is \$3030.00.
   - If the company produces 1,000 pairs of shoes, the average cost of producing one pair is \$330.00.
   - If the company produces 10,000 pairs of shoes, the average cost of producing one pair is \$60.00.

   We can see the more shoes her company produces, the lower the average cost.

b. To answer this question, we locate the point where its y-value is 50. With technology, we graph both $y = \bar{C}(x)$ and $y = 50$, and locate their intersection.

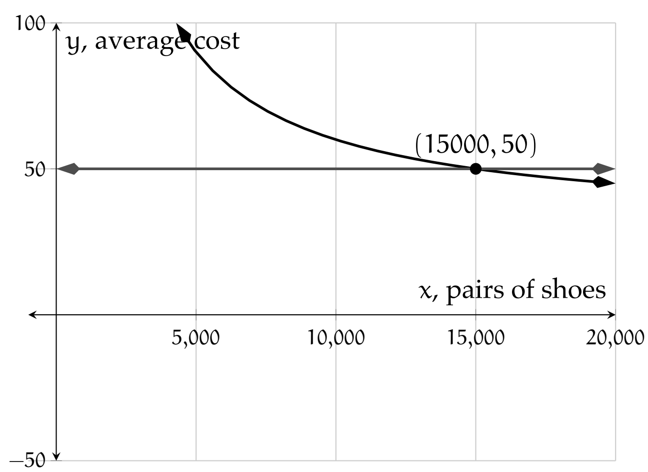


Figure 12.1.23: Intersection of $\bar{C}(x) = \frac{30x+300000}{x}$ and $y = 50$

The intersection $(15000, 50)$ implies the average cost of producing one pair is $50.00 if her company produces 15,000 pairs of shoes.

c. To answer this question, we substitute x with some large numbers, and use technology to create a table of values:

x	$g(x)$
100000	33
1000000	31
10000000	30.03
100000000	30.003

Figure 12.1.24: Values for Large Positive x

We can estimate that the average cost of producing one pair is getting closer and closer to $30.00 as her company produces more and more pairs of shoes.

Note that the cost of producing each pair is $30.00. This implies, for big companies whose products are very popular, the cost of operations can be ignored when calculating the average cost of producing each unit of product.

12.1.2 Reading Questions

1. What makes a function be a "rational" function?

2. Describe what an asymptote is.

3. If there is a rational function with a vertical asymptote at $x = 7$, what does that mean about the denominator of the rational function?

12.1.3 Exercises

Rational Functions in Context

1. The population of deer in a forest can be modeled by

$$P(x) = \frac{720x + 2310}{3x + 7}$$

where x is the number of years in the future. Answer the following questions.

a. How many deer live in this forest this year?

b. How many deer will live in this forest 27 years later? Round your answer to an integer.

c. After how many years, the deer population will be 247? Round your answer to an integer.

d. Use a calculator to answer this question: As time goes on, the population levels off at about how many deer?

2. The population of deer in a forest can be modeled by

$$P(x) = \frac{320x + 2200}{4x + 5}$$

where x is the number of years in the future. Answer the following questions.

a. How many deer live in this forest this year?

b. How many deer will live in this forest 14 years later? Round your answer to an integer.

c. After how many years, the deer population will be 100? Round your answer to an integer.

d. Use a calculator to answer this question: As time goes on, the population levels off at about how many deer?

3. In a certain store, cashiers can serve 50 customers per hour on average. If x customers arrive at the store in a given hour, then the average number of customers C waiting in line can be modeled by the function

$$C(x) = \frac{x^2}{2500 - 50x}$$

where $x < 50$.
Answer the following questions with a graphing calculator. Round your answers to integers.

a. If 38 customers arrived in the store in the past hour, there are approximately ______ customers waiting in line.

b. If there are 8 customers waiting in line, approximately ______ customers arrived in the past hour.

4. In a certain store, cashiers can serve 55 customers per hour on average. If x customers arrive at the store in a given hour, then the average number of customers C waiting in line can be modeled by the function

$$C(x) = \frac{x^2}{3025 - 55x}$$

where $x < 55$.
Answer the following questions with a graphing calculator. Round your answers to integers.

a. If 48 customers arrived in the store in the past hour, there are approximately ______ customers waiting in line.

b. If there are 2 customers waiting in line, approximately ______ customers arrived in the past hour.

Identify Rational Functions Select all rational functions. There are several correct answers.

5.

☐ $t(x) = \frac{5-7x^3}{6x^{0.7}+2x-3}$ ☐ $r(x) = \frac{6x^2+2x-3}{5-7x^{-6}}$ ☐ $c(x) = \frac{6x^2+2x-3}{5+|x|}$ ☐ $b(x) = \frac{6x^2+2x-3}{5}$

☐ $s(x) = \frac{\sqrt{6}x^2+2x-3}{5-7x^6}$ ☐ $n(x) = \frac{6x^2+2\sqrt{x}-3}{5-7x^6}$ ☐ $a(x) = \frac{6x^2+2x-3}{5-7x^6}$ ☐ $m(x) = \frac{6x+2}{6x+2}$

☐ $h(x) = \frac{5}{6x^2+2x-3}$

6.

☐ $n(x) = \frac{7x^2+6\sqrt{x}-6}{3-7x^7}$ ☐ $t(x) = \frac{3-7x^3}{7x^{0.7}+6x-6}$ ☐ $m(x) = \frac{7x+6}{7x+6}$ ☐ $r(x) = \frac{7x^2+6x-6}{3-7x^{-7}}$

☐ $c(x) = \frac{7x^2+6x-6}{3+|x|}$ ☐ $h(x) = \frac{3}{7x^2+6x-6}$ ☐ $b(x) = \frac{7x^2+6x-6}{3}$ ☐ $s(x) = \frac{\sqrt{7x^2}+6x-6}{3-7x^7}$

☐ $a(x) = \frac{7x^2+6x-6}{3-7x^7}$

Domain

7. Find the domain of K where $K(x) = \frac{x}{x+4}$.

8. Find the domain of K where $K(x) = \frac{5x}{x-10}$.

9. Find the domain of f where $f(x) = \frac{3x+9}{x^2+4x-12}$.

10. Find the domain of g where $g(x) = -\frac{4x+3}{x^2-3x-40}$.

11. Find the domain of h where $h(x) = \frac{9x+6}{x^2+7x}$.

12. Find the domain of F where $F(x) = \frac{2x-7}{x^2+2x}$.

13. Find the domain of F where $F(x) = \frac{2-5x}{x^2-49}$.

14. Find the domain of G where $G(x) = \frac{8x-10}{x^2-100}$.

15. Find the domain of the function c defined by $c(x) = \frac{x+6}{x^4}$

16. Find the domain of the function n defined by $n(x) = \frac{x+8}{x^2}$

17. Find the domain of the function t defined by $t(x) = \frac{x+10}{x^2+49}$

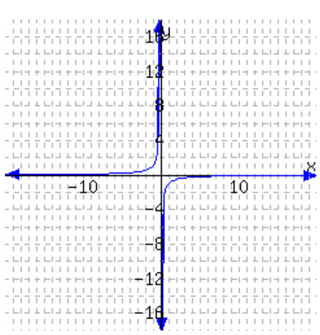
18. Find the domain of the function p defined by $p(x) = \frac{x-8}{x^2+16}$

19. Find the domain of the function r defined by $r(x) = \frac{x-6}{x-6}$

20. Find the domain of the function n defined by $n(x) = \frac{x-4}{x-4}$

A function is graphed. Find its domain.

21.



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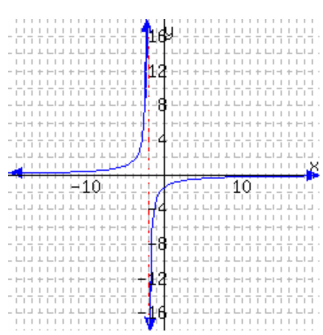

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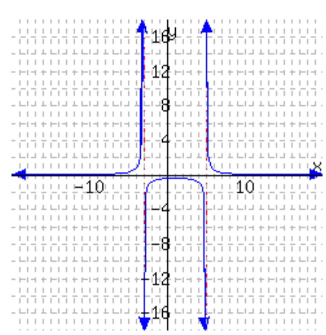

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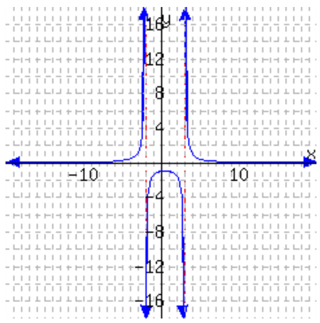

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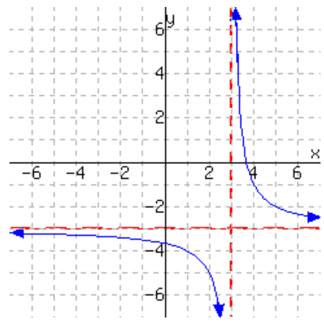

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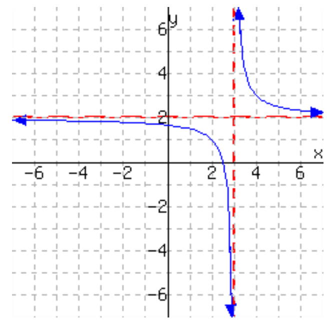

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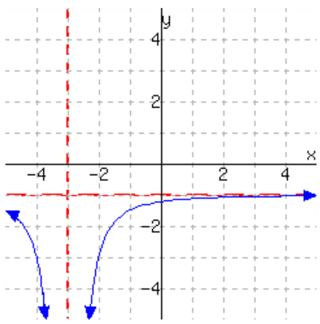

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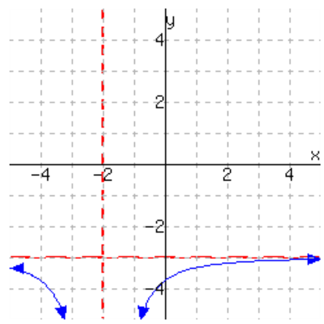


Graphing Technology

29. In a forest, the number of deer can be modeled by the function $f(t) = \frac{200t+350}{0.8t+7}$, where t stands for the number of years from now. Answer the question with technology.

After 20 years, there would be approximately deer living in the forest.

30. In a forest, the number of deer can be modeled by the function $f(t) = \frac{240t+240}{0.6t+4}$, where t stands for the number of years from now. Answer the question with technology.

After years, there would be approximately 230 deer living in the forest.

31. In a forest, the number of deer can be modeled by the function $f(t) = \frac{60t+630}{0.3t+9}$, where t stands for the number of years from now. Answer the question with technology.

As time goes on, the population levels off at approximately deer living in the forest.

32. The concentration of a drug in a patient's blood stream, in milligrams per liter, can be modeled by the function $C(t) = \frac{7t}{t^2+6}$, where t is the number of hours since the drug is injected. Answer the following question with technology.

The drug's concentration after 1 hours is milligrams per liter.

33. The concentration of a drug in a patient's blood stream, in milligrams per liter, can be modeled by the function $C(t) = \frac{8t}{t^2+5}$, where t is the number of hours since the drug is injected. Answer the following question with technology.

hours since injection, the drug's concentration is 1.72 milligrams per liter.

34. The concentration of a drug in a patient's blood stream, in milligrams per liter, can be modeled by the function $C(t) = \frac{9t}{t^2+6}$, where t is the number of hours since the drug is injected. Answer the following question with technology.

$\boxed{}$ hours since injection, the drug's concentration is at the maximum value of $\boxed{}$ milligrams per liter.

35. The concentration of a drug in a patient's blood stream, in milligrams per liter, can be modeled by the function $C(t) = \frac{9t}{t^2+7}$, where t is the number of hours since the drug is injected. Answer the following question with technology.

As time goes on, the drug's concentration in the patient's blood stream levels off at approximately $\boxed{}$ milligrams per liter.

12.2 Multiplication and Division of Rational Expressions

In the last section, we learned some rational function applications. In this section, we will learn how to simplify rational expressions, and how to multiply and divide them.

12.2.1 Simplifying Rational Expressions

Consider the two rational functions below. At first glance, which function *looks* simpler?

$$f(x) = \frac{8x^3 - 12x^2 + 8x - 12}{2x^3 - 3x^2 + 10x - 15} \qquad g(x) = \frac{4(x^2+1)}{x^2+5}, \text{ for } x \neq \frac{3}{2}$$

It can be argued that the function g is simpler, at least with regard to the ease with which we can determine its domain, quickly evaluate it, and also determine where its function value is zero. All of these things are considerably more difficult with the function f.

These two functions are actually the *same* function. Using factoring and the same process of canceling that's used with numerical ratios, we will learn how to simplify the function f into the function g. (The full process for simplifying $f(x) = \frac{8x^3-12x^2+8x-12}{2x^3-3x^2+10x-15}$ will be shown in Example 12.2.8.)

To see a simple example of the process for simplifying a rational function or expression, let's look at simplifying $\frac{14}{21}$ and $\frac{(x+2)(x+7)}{(x+3)(x+7)}$ by canceling common factors:

$$\begin{aligned} \frac{14}{21} &= \frac{2 \cdot \cancel{7}}{3 \cdot \cancel{7}} \\ &= \frac{2}{3} \end{aligned} \qquad \begin{aligned} \frac{(x+2)(x+7)}{(x+3)(x+7)} &= \frac{(x+2)\cancel{(x+7)}}{(x+3)\cancel{(x+7)}} \\ &= \frac{x+2}{x+3}, \text{ for } x \neq -7 \end{aligned}$$

The statement "for $x \neq -7$" was added when the factors of $x+7$ were canceled. This is because $\frac{(x+2)(x+7)}{(x+3)(x+7)}$ was undefined for $x = -7$, so the simplified version must also be undefined for $x = -7$.

Warning 12.2.2 Cancel Factors, not Terms. It may be tempting to want to try to simplify $\frac{x+2}{x+3}$ into $\frac{2}{3}$ by canceling each x that appears. But these x's are *terms* (pieces that are added with other pieces), not *factors*. Canceling (an act of division) is only possible with *factors* (an act of multiplication).

The process of canceling factors is key to simplifying rational expressions. If the expression is not given in factored form, then this will be our first step. We'll now look at a few more examples.

Example 12.2.3 Simplify the rational function formula $Q(x) = \frac{3x-12}{x^2+x-20}$ and state the domain of Q.

Explanation.

To start, we'll factor the numerator and denominator. We'll then cancel any factors common to both the numerator and denominator.

$$\begin{aligned} Q(x) &= \frac{3x-12}{x^2+x-20} \\ Q(x) &= \frac{3\cancel{(x-4)}}{(x+5)\cancel{(x-4)}} \\ Q(x) &= \frac{3}{x+5}, \text{ for } x \neq 4 \end{aligned}$$

The domain of this function will incorporate the *explicit* domain restriction $x \neq 4$ that was stated when the factor of $x - 4$ was canceled from both the numerator and denominator. We will also exclude -5 from the domain as this value would make the denominator zero. Thus the domain of Q is $\{x \mid x \neq -5, 4\}$.

Warning 12.2.4 When simplifying the function Q in Example 12.2.3, we cannot simply write $Q(x) = \frac{3}{x+5}$. The reason is that this would result in our simplified version of the function Q having a different domain than the original Q. More specifically, for our original function Q it held that Q(4) was undefined, and this still needs to be true for the simplified form of Q.

Example 12.2.5 Simplify the rational function formula $R(y) = \frac{-y-2y^2}{2y^3-y^2-y}$ and state the domain of R.

Explanation.

$$\begin{aligned}
R(y) &= \frac{-y-2y^2}{2y^3-y^2-y} \\
R(y) &= \frac{-2y^2-y}{y(2y^2-y-1)} \\
R(y) &= \frac{-\cancel{y}\cancel{(2y+1)}}{\cancel{y}\cancel{(2y+1)}(y-1)} \\
R(y) &= -\frac{1}{y-1}, \text{ for } y \neq 0, y \neq -\frac{1}{2}
\end{aligned}$$

The domain of this function will incorporate the explicit restrictions $y \neq 0, y \neq -\frac{1}{2}$ that were stated when the factors of y and $2y+1$ were canceled from both the numerator and denominator. Since the factor $y-1$ is still in the denominator, we also need the restriction that $y \neq 1$. Therefore the domain of R is $\{y \mid y \neq -\frac{1}{2}, 0, 1\}$.

Example 12.2.6 Simplify the expression $\frac{9y+2y^2-5}{y^2-25}$.

Explanation.

To start, we need to recognize that $9y+2y^2-5$ is not written in standard form (where terms are written from highest degree to lowest degree). Before attempting to factor this expression, we'll re-write it as $2y^2+9y-5$.

$$\begin{aligned}
\frac{9y+2y^2-5}{y^2-25} &= \frac{2y^2+9y-5}{y^2-25} \\
&= \frac{(2y-1)\cancel{(y+5)}}{\cancel{(y+5)}(y-5)} \\
&= \frac{2y-1}{y-5}, \text{ for } y \neq -5
\end{aligned}$$

Example 12.2.7 Simplify the expression $\frac{-48z+24z^2-3z^3}{4-z}$.

Explanation. To begin simplifying this expression, we will rewrite each polynomial in descending order. Then we'll factor out the GCF, including the constant -1 from both the numerator and denominator because their leading terms are negative.

$$\begin{aligned}
\frac{-48z+24z^2-3z^3}{4-z} &= \frac{-3z^3+24z^2-48z}{-z+4} \\
&= \frac{-3z(z^2-8z+16)}{-(z-4)} \\
&= \frac{-3z(z-4)^2}{-(z-4)} \\
&= \frac{-3z(z-4)\cancel{(z-4)}}{-\cancel{(z-4)}} \\
&= \frac{3z(z-4)}{1}, \text{ for } z \neq 4 \\
&= 3z(z-4), \text{ for } z \neq 4
\end{aligned}$$

Example 12.2.8 Simplify the rational function formula $f(x) = \frac{8x^3-12x^2+8x-12}{2x^3-3x^2+10x-15}$ and state the domain of f.

Explanation.

To simplify this rational function, we'll first note that both the numerator and denominator have four terms. To factor them we'll need to use factoring by grouping. (Note that if this technique didn't work, very few other approaches would be possible.) Once we've used factoring by grouping, we'll cancel any factors common to both the numerator and denominator and state the associated restrictions.

$$\begin{aligned}
f(x) &= \frac{8x^3 - 12x^2 + 8x - 12}{2x^3 - 3x^2 + 10x - 15} \\
f(x) &= \frac{4(2x^3 - 3x^2 + 2x - 3)}{2x^3 - 3x^2 + 10x - 15} \\
f(x) &= \frac{4(x^2(2x-3) + (2x-3))}{x^2(2x-3) + 5(2x-3)} \\
f(x) &= \frac{4(x^2+1)\cancel{(2x-3)}}{(x^2+5)\cancel{(2x-3)}} \\
f(x) &= \frac{4(x^2+1)}{x^2+5}, \text{ for } x \neq \frac{3}{2}
\end{aligned}$$

In determining the domain of this function, we'll need to account for any implicit and explicit restrictions. When the factor $2x - 3$ was canceled, the explicit statement of $x \neq \frac{3}{2}$ was given. The denominator in the final simplified form of this function has $x^2 + 5$. There is no value of x for which $x^2 + 5 = 0$, so the only restriction is that $x \neq \frac{3}{2}$. Therefore the domain is $\left\{x \mid x \neq \frac{3}{2}\right\}$.

Example 12.2.9 Simplify the expression $\frac{3y-x}{x^2-xy-6y^2}$. In this example, there are two variables. It is still possible that in examples like this, there can be domain restrictions when simplifying rational expressions. However since we are not studying *functions* of more than one variable, this textbook ignores domain restrictions with examples like this one.

Explanation.

$$\begin{aligned}
\frac{3y-x}{x^2-xy-6y^2} &= \frac{-\cancel{(x-3y)}}{\cancel{(x-3y)}(x+2y)} \\
&= \frac{-1}{x+2y}
\end{aligned}$$

12.2.2 Multiplication of Rational Functions and Expressions

Recall the property for multiplying fractions A.2.16, which states that the product of two fractions is equal to the product of their numerators divided by the product of their denominators. We will use this same property for multiplying rational expressions.

When multiplying fractions, one approach is to multiply the numerator and denominator, and then simplify the fraction that results by determining the greatest common factor in both the numerator and denominator, like this:

$$\begin{aligned}
\frac{14}{9} \cdot \frac{3}{10} &= \frac{14 \cdot 3}{9 \cdot 10} \\
&= \frac{42}{90} \\
&= \frac{7 \cdot \cancel{6}}{15 \cdot \cancel{6}} \\
&= \frac{7}{15}
\end{aligned}$$

This approach works great when we can easily identify that 6 is the greatest common factor in both 42 and 90. But in more complicated instances, it isn't always an easy approach. It also won't work particularly well when we have $(x + 2)$ instead of 2 as a factor, as we'll see shortly.

Another approach to multiplying and simplifying fractions involves utilizing the prime factorization of each the numerator and denominator, like this:

$$\begin{aligned}\frac{14}{9}\cdot\frac{3}{10} &= \frac{2\cdot 7}{3^2}\cdot\frac{3}{2\cdot 5}\\ &= \frac{\cancel{2}\cdot 7\cdot \cancel{3}}{\cancel{3}\cdot 3\cdot \cancel{2}\cdot 5}\\ &= \frac{7}{15}\end{aligned}$$

The method for multiplying and simplifying rational expressions is nearly identical, as shown here:

$$\begin{aligned}\frac{x^2+9x+14}{x^2+6x+9}\cdot\frac{x+3}{x^2+7x+10} &= \frac{(x+2)(x+7)}{(x+3)^2}\cdot\frac{x+3}{(x+2)(x+5)}\\ &= \frac{\cancel{(x+2)}(x+7)\cancel{(x+3)}}{\cancel{(x+3)}(x+3)\cancel{(x+2)}(x+5)}\\ &= \frac{(x+7)}{(x+3)(x+5)},\ \text{for } x\neq -2\end{aligned}$$

This process will be used for both multiplying and dividing rational expressions. The main distinctions in various examples will be in the factoring methods required.

Example 12.2.10 Multiply the rational expressions: $\frac{x^2-4x}{x^2-4}\cdot\frac{4-4x+x^2}{20-x-x^2}$.

Explanation. Note that to factor the second rational expression, we'll want to re-write the terms in descending order for both the numerator and denominator. In the denominator, we'll first factor out -1 as the leading term is $-x^2$.

$$\begin{aligned}\frac{x^2-4x}{x^2-4}\cdot\frac{4-4x+x^2}{20-x-x^2} &= \frac{x^2-4x}{x^2-4}\cdot\frac{x^2-4x+4}{-x^2-x+20}\\ &= \frac{x^2-4x}{x^2-4}\cdot\frac{x^2-4x+4}{-(x^2+x-20)}\\ &= \frac{x\cancel{(x-4)}}{(x+2)\cancel{(x-2)}}\cdot\frac{(x-2)\cancel{(x-2)}}{-(x+5)\cancel{(x-4)}}\\ &= -\frac{x(x-2)}{(x+2)(x+5)},\ \text{for } x\neq 2, x\neq 4\end{aligned}$$

Example 12.2.11 Multiply the rational expressions: $\frac{p^2q^4}{3r}\cdot\frac{9r^2}{pq^2}$. Note this book ignores domain restrictions on multivariable expressions.

Explanation. We won't need to factor anything in this example, and can simply multiply across and then simplify.

$$\begin{aligned}\frac{p^2q^4}{3r}\cdot\frac{9r^2}{pq^2} &= \frac{p^2q^2\cdot 9r^2}{3r\cdot pq^2}\\ &= \frac{pq^2\cdot 3r}{1}\\ &= 3pq^2r\end{aligned}$$

12.2.3 Division of Rational Functions and Expressions

We can divide rational expressions using the property for dividing fractions A.2.18, which simply requires that we change dividing by an expression to multiplying by its reciprocal. Let's look at a few examples.

Example 12.2.12 Divide the rational expressions: $\frac{x+2}{x+5} \div \frac{x+2}{x-3}$.

Explanation.

$$\begin{aligned}\frac{x+2}{x+5} \div \frac{x+2}{x-3} &= \frac{\cancel{x+2}}{x+5} \cdot \frac{x-3}{\cancel{x+2}}, \text{ for } x \neq 3\\ &= \frac{x-3}{x+5}, \text{ for } x \neq -2, x \neq 3\end{aligned}$$

Remark 12.2.13 In the first step of 12.2.12, the restriction $x \neq 3$ was used. We hadn't canceled anything yet, so why is there this restriction already? It's because the original expression $\frac{x+2}{x+5} \div \frac{x+2}{x-3}$ had $x-3$ in a denominator, which means that 3 is not a valid input. In the first step of simplifying, the $x-3$ denominator went to the numerator and we lost the information that 3 was not a valid input, so we stated it explicitly. Always be sure to compare the restrictions of the original expression with each step throughout the process.

Example 12.2.14 Simplify the rational expression using division: $\frac{\frac{3x-6}{2x+10}}{\frac{x^2-4}{3x+15}}$.

Explanation. To begin, we'll note that the larger fraction bar is denoting division, so we will use multiplication by the reciprocal. After that, we'll factor each expression and cancel any common factors.

$$\begin{aligned}\frac{\frac{3x-6}{2x+10}}{\frac{x^2-4}{3x+15}} &= \frac{3x-6}{2x+10} \div \frac{x^2-4}{3x+15}\\ &= \frac{3x-6}{2x+10} \cdot \frac{3x+15}{x^2-4}\\ &= \frac{3\cancel{(x-2)}}{2\cancel{(x+5)}} \cdot \frac{3\cancel{(x+5)}}{(x+2)\cancel{(x-2)}}\\ &= \frac{3 \cdot 3}{2(x+2)}, \text{ for } x \neq -5, x \neq 2\\ &= \frac{9}{2x+4}, \text{ for } x \neq -5, x \neq 2\end{aligned}$$

Example 12.2.15 Divide the rational expressions: $\frac{x^2-5x-14}{x^2+7x+10} \div \frac{x-7}{x+4}$.

Explanation.

$$\begin{aligned}\frac{x^2-5x-14}{x^2+7x+10} \div \frac{x-7}{x+4} &= \frac{x^2-5x-14}{x^2+7x+10} \cdot \frac{x+4}{x-7}, \text{ for } x \neq -4\\ &= \frac{\cancel{(x-7)}\cancel{(x+2)}}{(x+5)\cancel{(x+2)}} \cdot \frac{x+4}{\cancel{x-7}}, \text{ for } x \neq -4\\ &= \frac{x+4}{x+5}, \text{ for } x \neq -4, x \neq -2, x \neq 7\end{aligned}$$

Example 12.2.16 Divide the rational expressions: $(p^4 - 16) \div \frac{p^4 - 2p^3}{2p}$.

Explanation.

$$\begin{aligned}(p^4 - 16) \div \frac{p^4 - 2p^3}{2p} &= \frac{p^4 - 16}{1} \cdot \frac{2p}{p^4 - 2p^3} \\ &= \frac{(p^2+4)(p+2)\cancel{(p-2)}}{1} \cdot \frac{2p}{p^3\cancel{(p-2)}} \\ &= \frac{2(p^2+4)(p+2)}{p^2}, \text{ for } p \neq 2\end{aligned}$$

Note here that we *didn't* have to include a restriction in the very first step. That restriction would have been $p \neq 0$, but since 0 *still* cannot be inputted into any of the subsequent expressions, we don't need to explicitly state $p \neq 0$ as a restriction because the expressions tell us that implicitly already.

Example 12.2.17 Divide the rational expressions: $\frac{3x^2}{x^2 - 9y^2} \div \frac{6x^3}{x^2 - 2xy - 15y^2}$. Note this book ignores domain restrictions on multivariable expressions.

Explanation.

$$\begin{aligned}\frac{3x^2}{x^2 - 9y^2} \div \frac{6x^3}{x^2 - 2xy - 15y^2} &= \frac{3x^2}{x^2 - 9y^2} \cdot \frac{x^2 - 2xy - 15y^2}{6x^3} \\ &= \frac{3x^2}{\cancel{(x+3y)}(x-3y)} \cdot \frac{\cancel{(x+3y)}(x-5y)}{6x^3} \\ &= \frac{1}{x-3y} \cdot \frac{x-5y}{2x} \\ &= \frac{x-5y}{2x(x-3y)}\end{aligned}$$

Example 12.2.18 Divide the rational expressions: $\frac{m^2n^2 - 3mn - 4}{2mn} \div (m^2n^2 - 16)$. Note this book ignores domain restrictions on multivariable expressions.

Explanation.

$$\begin{aligned}\frac{m^2n^2 - 3mn - 4}{2mn} \div (m^2n^2 - 16) &= \frac{m^2n^2 - 3mn - 4}{2mn} \cdot \frac{1}{m^2n^2 - 16} \\ &= \frac{\cancel{(mn-4)}(mn+1)}{2mn} \cdot \frac{1}{(mn+4)\cancel{(mn-4)}} \\ &= \frac{mn+1}{2mn} \cdot \frac{1}{mn+4} \\ &= \frac{mn+1}{2mn(mn+4)}\end{aligned}$$

12.2.4 Reading Questions

1. What is the difference between a factor and a term?

2. When canceling pieces of rational function expression to simplify it, what kinds of pieces are the only acceptable pieces to cancel?

3. When you simplify a rational funciton expression, you may need to make note of a $\boxed{}$ $\boxed{}$.

12.2.5 Exercises

Review and Warmup

1. Multiply: $-\frac{3}{13} \cdot \frac{5}{9}$

2. Multiply: $-\frac{9}{11} \cdot \frac{13}{24}$

3. Multiply: $-\frac{8}{9} \cdot \left(-\frac{7}{18}\right)$

4. Multiply: $-\frac{10}{9} \cdot \left(-\frac{19}{4}\right)$

5. Divide: $\frac{3}{5} \div \frac{5}{2}$

6. Divide: $\frac{3}{8} \div \frac{8}{3}$

7. Divide: $\frac{3}{20} \div \left(-\frac{5}{8}\right)$

8. Divide: $\frac{4}{25} \div \left(-\frac{3}{10}\right)$

Factor the given polynomial.

9. $t^2 - 36$

10. $x^2 - 4$

11. $x^2 + 12x + 32$

12. $y^2 + 13y + 40$

13. $y^2 - 3y + 2$

14. $r^2 - 15r + 56$

15. $3r^2 - 15r + 18$

16. $10t^2 - 30t + 20$

17. $2t^{10} + 10t^9 + 12t^8$

18. $6t^5 + 18t^4 + 12t^3$

19. $144x^2 - 24x + 1$

20. $81x^2 - 18x + 1$

Simplifying Rational Expressions with One Variable

21. Simplify the following expressions, and if applicable, write the restricted domain on the simplified expression.

a. $\frac{y+4}{y+4}$

b. $\frac{y+4}{4+y}$

c. $\frac{y-4}{y-4}$

d. $\frac{y-4}{4-y}$

22. Simplify the following expressions, and if applicable, write the restricted domain on the simplified expression.

a. $\frac{y+10}{y+10}$

b. $\frac{y+10}{10+y}$

c. $\frac{y-10}{y-10}$

d. $\frac{y-10}{10-y}$

23. Select all correct simplifications, ignoring possible domain restrictions.

- ☐ $\frac{x+6}{x+7}=\frac{6}{7}$ ☐ $\frac{7x+6}{x+6}=7$
- ☐ $\frac{6}{x+6}=\frac{1}{x}$ ☐ $\frac{x+6}{x}=6$
- ☐ $\frac{7x+6}{7}=x+6$ ☐ $\frac{6}{x+6}=\frac{1}{x+1}$
- ☐ $\frac{x}{7x}=\frac{1}{7}$ ☐ $\frac{6x}{x}=6$ ☐ $\frac{x+6}{x+6}=1$
- ☐ $\frac{x+6}{6}=x$ ☐ $\frac{7(x-6)}{x-6}=7$

24. Select all correct simplifications, ignoring possible domain restrictions.

- ☐ $\frac{x+7}{7}=x$ ☐ $\frac{7x}{x}=7$ ☐ $\frac{x+7}{x+4}=\frac{7}{4}$
- ☐ $\frac{4x+7}{4}=x+7$ ☐ $\frac{x+7}{x+7}=1$
- ☐ $\frac{x+7}{x}=7$ ☐ $\frac{4x+7}{x+7}=4$
- ☐ $\frac{4(x-7)}{x-7}=4$ ☐ $\frac{7}{x+7}=\frac{1}{x}$
- ☐ $\frac{x}{4x}=\frac{1}{4}$ ☐ $\frac{7}{x+7}=\frac{1}{x+1}$

Simplify the following expression, and if applicable, write the restricted domain on the simplified expression.

25. $\frac{t-10}{(t-4)(t-10)}$ **26.** $\frac{t+7}{(t-10)(t+7)}$ **27.** $\frac{3(t-3)}{(t-8)(t-3)}$

28. $\frac{-8(x-9)}{(x-6)(x-9)}$ **29.** $\frac{(x+6)(x-2)}{2-x}$ **30.** $\frac{(y-3)(y-9)}{9-y}$

31. $\frac{9y-63}{y-7}$ **32.** $\frac{-6r+30}{r-5}$ **33.** $\frac{-2r}{r^2+3r}$

34. $\frac{9t}{t^2+8t}$ **35.** $\frac{3t-t^2}{t^2-9t+18}$ **36.** $\frac{t-t^2}{t^2-6t+5}$

37. $\frac{x^2+5x}{25-x^2}$ **38.** $\frac{x^2-3x}{9-x^2}$ **39.** $\frac{-y^2+y}{3-2y-y^2}$

40. $\frac{-y^2+5y}{5+4y-y^2}$ **41.** $\frac{3r^2+5r+2}{-r+4-5r^2}$ **42.** $\frac{5r^2+8r+3}{-r+5-6r^2}$

43. $\frac{r^2+6r+8}{-4r-r^2-4}$ **44.** $\frac{t^2-t-2}{-2t-t^2-1}$ **45.** $\frac{-t^2-11t-30}{t^2-25}$

46. $\frac{-x^2-7x-12}{x^2-9}$ **47.** $\frac{2x^2-x-3}{-11x-5-6x^2}$ **48.** $\frac{5y^2+11y+6}{-11y-5-6y^2}$

49. $\frac{4y^3-y^4}{y^2-2y-8}$ **50.** $\frac{-2r^2-r^3}{r^2-4}$ **51.** $\frac{r^6-3r^5-18r^4}{r^6-11r^5+30r^4}$

52. $\frac{r^5+3r^4-4r^3}{r^5+2r^4-3r^3}$ **53.** $\frac{t^3+8}{t^2-4}$ **54.** $\frac{t^3-125}{t^2-25}$

Simplifying Rational Expressions with More Than One Variable Simplify this expression.

55. $\frac{5xy-x^2y^2}{x^2y^2+xy-30}$ **56.** $\frac{5xr-x^2r^2}{x^2r^2-xr-20}$ **57.** $\frac{4y+16t}{y^2+5yt+4t^2}$

58. $\frac{2y+10t}{y^2+8yt+15t^2}$ **59.** $\frac{-r^2+rx+12x^2}{r^2-16x^2}$ **60.** $\frac{-r^2-rt+12t^2}{r^2-9t^2}$

61. $\frac{2r^2y^2+5ry+3}{-11ry-5-6r^2y^2}$ **62.** $\frac{3t^2x^2+5tx+2}{-7tx-2-5t^2x^2}$

Simplifying Rational Functions Simplify the function formula, and if applicable, write the restricted domain.

63. $G(t) = \dfrac{t+1}{t^2-6t-7}$
Reduced $G(t) =$

64. $h(x) = \dfrac{x-5}{x^2+x-30}$
Reduced $h(x) =$

65. $K(x) = \dfrac{x^3-81x}{x^3+11x^2+18x}$
Reduced $K(x) =$

66. $G(y) = \dfrac{y^3-9y}{y^3+13y^2+30y}$
Reduced $G(y) =$

67. $h(y) = \dfrac{y^4+4y^3+4y^2}{3y^4+5y^3-2y^2}$
Reduced $h(y) =$

68. $K(r) = \dfrac{r^4-8r^3+16r^2}{3r^4-11r^3-4r^2}$
Reduced $K(r) =$

69. $G(r) = \dfrac{3r^3+r^2}{3r^3-11r^2-4r}$
Reduced $G(r) =$

70. $g(r) = \dfrac{5r^3+3r^2}{5r^3-22r^2-15r}$
Reduced $g(r) =$

Multiplying and Dividing Rational Expressions with One Variable

71. Select all correct equations:
☐ $9 \cdot \frac{x}{y} = \frac{9x}{9y}$
☐ $9 \cdot \frac{x}{y} = \frac{x}{9y}$
☐ $9 \cdot \frac{x}{y} = \frac{9x}{y}$
☐ $-\frac{x}{y} = \frac{-x}{-y}$
☐ $-\frac{x}{y} = \frac{-x}{y}$
☐ $-\frac{x}{y} = \frac{x}{-y}$

72. Select all correct equations:
☐ $10 \cdot \frac{x}{y} = \frac{10x}{10y}$
☐ $-\frac{x}{y} = \frac{-x}{y}$
☐ $10 \cdot \frac{x}{y} = \frac{x}{10y}$
☐ $-\frac{x}{y} = \frac{-x}{-y}$
☐ $10 \cdot \frac{x}{y} = \frac{10x}{y}$
☐ $-\frac{x}{y} = \frac{x}{-y}$

73. Simplify the following expressions, and if applicable, write the restricted domain.
$-\dfrac{x^4}{x+4} \cdot x^3$
$-\dfrac{x^4}{x+4} \cdot \dfrac{1}{x^3}$

74. Simplify the following expressions, and if applicable, write the restricted domain.
$-\dfrac{y^4}{y+4} \cdot y^2$
$-\dfrac{y^4}{y+4} \cdot \dfrac{1}{y^2}$

Simplify this expression, and if applicable, write the restricted domain.

75. $\dfrac{y^2-y-2}{y+4} \cdot \dfrac{5y+20}{y+1}$

76. $\dfrac{y^2+7y+12}{y-6} \cdot \dfrac{5y-30}{y+4}$

77. $\dfrac{r^2-9r}{r^2-9} \cdot \dfrac{r^2-3r}{r^2-11r+18}$

78. $\dfrac{r^2-9r}{r^2-9} \cdot \dfrac{r^2-3r}{r^2-7r-18}$

79. $\dfrac{12r-12}{-20-25r-5r^2} \cdot \dfrac{r^2+8r+16}{4r^2-4r}$

80. $\dfrac{6t-24}{28-21t-7t^2} \cdot \dfrac{t^2-2t+1}{2t^2-8t}$

81. $\dfrac{6t^2-11t+5}{20t^3-50t^2} \cdot \dfrac{10t^2-4t^3}{36t^2-25}$

82. $\dfrac{5x^2+(-1)x-4}{126x^2-105x} \cdot \dfrac{15x-18x^2}{25x^2-16}$

83. $\dfrac{x}{x-6} \div 3x^2$

84. $\dfrac{y}{y+10} \div 5y^2$

85. $8y \div \dfrac{2}{y^3}$

86. $12r \div \dfrac{3}{r^2}$

87. $(2r-6) \div (4r-12)$

88. $(4r-12) \div (24r-72)$

89. $\dfrac{25t^2-36}{5t^2+(-9)t+(-18)} \div (6-5t)$

90. $\dfrac{4t^2-49}{2t^2+11t+14} \div (7-2t)$

91. $\dfrac{x^4}{x^2+6x} \div \dfrac{1}{x^2+x-30}$

92. $\dfrac{x^3}{x^2-3x} \div \dfrac{1}{x^2+x-12}$

93. $\dfrac{\frac{5a+1}{a}}{\frac{a+1}{a}}$

94. $\dfrac{\frac{10a+10}{a}}{\frac{a+7}{a}}$

95. $\dfrac{\frac{u}{(u-6)^2}}{\frac{5u}{u^2-36}}$

96. $\dfrac{\frac{r}{(r-3)^2}}{\frac{9r}{r^2-9}}$

97. $\dfrac{x^2+3x}{x^2-16} \div \dfrac{x^2-9}{x^2+2x-8}$

98. $\dfrac{x^2+4x}{x^2-1} \div \dfrac{x^2-16}{x^2-4x-5}$

Multiplying and Dividing Rational Expressions with More Than One Variable Simplify this expression.

99. $\dfrac{8(t+x)}{t-x} \cdot \dfrac{t-x}{2(2t+x)}$

100. $\dfrac{12(x+t)}{x-t} \cdot \dfrac{x-t}{4(2x+t)}$

101. $\dfrac{4x^3y^2}{3x^4} \cdot \dfrac{9x^4y^2}{8y^5}$

102. $\dfrac{5yx}{3y} \cdot \dfrac{3y^2x^3}{25x^5}$

103. $\dfrac{y^2+9yt+20t^2}{y+t} \cdot \dfrac{2y+2t}{y+5t}$

104. $\dfrac{r^2+ry-12y^2}{r+6y} \cdot \dfrac{3r+18y}{r-3y}$

105. $\dfrac{rx^{10}}{4} \div \dfrac{rx^5}{8}$

106. $\dfrac{r^3t^2}{6} \div \dfrac{r^3t}{12}$

107. $(t^4-4t^3y+4t^2y^2) \div (t^5-2t^4y)$

108. $(t^3+8t^2x+16tx^2) \div (t^5+4t^4x)$

109. $\dfrac{1}{x^2-10xr+24r^2} \div \dfrac{x^2}{x^2-4xr}$

110. $\dfrac{1}{x^2+5xy+6y^2} \div \dfrac{x^5}{x^2+2xy}$

111. $\dfrac{y^5}{y^2x-6y} \div \dfrac{1}{y^2x^2-7yx+6}$

112. $\dfrac{y^3}{y^2r-4y} \div \dfrac{1}{y^2r^2+2yr-24}$

113. $\dfrac{36y^4t^2}{y+10t} \div \dfrac{6y^5t}{y^2-100t^2}$

114. $\dfrac{15r^3t^4}{r+9t} \div \dfrac{3r^8t}{r^2-81t^2}$

115. $\dfrac{\frac{p}{q}}{\frac{5p}{4q^2}}$

116. $\dfrac{\frac{m}{n}}{\frac{4m}{3n^2}}$

117. $\dfrac{\frac{mn^2}{10k}}{\frac{m}{6nk}}$

118. $\dfrac{\frac{xy^2}{7z}}{\frac{x}{10yz}}$

Challenge

119. Simplify the following: $\frac{1}{x+1} \div \frac{x+2}{x+1} \div \frac{x+3}{x+2} \div \frac{x+4}{x+3} \div \cdots \div \frac{x+35}{x+34}$. For this exercise, you do not have to write the restricted domain of the simplified expression.

12.3 Addition and Subtraction of Rational Expressions

In the last section, we learned how to multiply and divide rational expressions. In this section, we will learn how to add and subtract rational expressions.

12.3.1 Introduction

Example 12.3.2 Julia is taking her family on a boat trip 12 miles down the river and back. The river flows at a speed of 2 miles per hour and she wants to drive the boat at a constant speed, v miles per hour downstream and back upstream. Due to the current of the river, the actual speed of travel is $v + 2$ miles per hour going downstream, and $v - 2$ miles per hour going upstream. If Julia plans to spend 8 hours for the whole trip, how fast should she drive the boat?

We need to review three forms of the formula for movement at a constant rate:

$$d = vt \qquad v = \frac{d}{t} \qquad t = \frac{d}{v}$$

where d stands for distance, v represents speed, and t stands for time. According to the third form, the time it takes the boat to travel downstream is $\frac{12}{v+2}$, and the time it takes to get back upstream is $\frac{12}{v-2}$.

The function to model the time of the whole trip is

$$t(v) = \frac{12}{v-2} + \frac{12}{v+2}$$

where t stands for time in hours, and v is the boat's speed in miles per hour. Let's look at the graph of this function in Figure 12.3.3. Note that since the speed v and the time $t(v)$ should be positive in context, it's only the first quadrant of Figure 12.3.3 that matters.

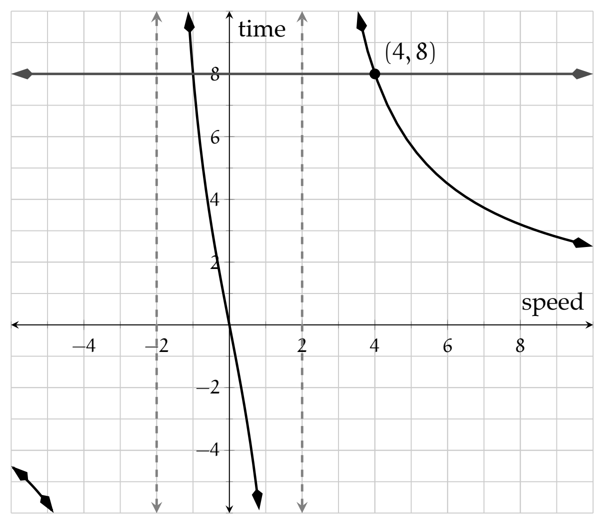


Figure 12.3.3: Graph of $t(v) = \frac{12}{v-2} + \frac{12}{v+2}$ and $t = 8$

To find the speed that Julia should drive the boat to make the round trip last 8 hours we can use graphing technology to solve the equation

$$\frac{12}{v-2} + \frac{12}{v+2} = 8$$

graphically and we see that $v = 4$. This tells us that a speed of 4 miles per hour will give a total time of 8 hours to complete the trip. To go downstream it would take $\frac{12}{v+2} = \frac{12}{4+2} = 2$ hours; and to go upstream it would take $\frac{12}{v-2} = \frac{12}{4-2} = 6$ hours.

The point of this section is to work with expressions like $\frac{12}{v-2} + \frac{12}{v+2}$, where two rational expressions are added (or subtracted). There are times when it is useful to combine them into a single fraction. We will learn that the expression $\frac{12}{v-2} + \frac{12}{v+2}$ is equal to the expression $\frac{24v}{v^2-4}$, and we will learn how to make that simplification.

12.3.2 Addition and Subtraction of Rational Expressions with the Same Denominator

The process of adding and subtracting rational expressions will be very similar to the process of adding and subtracting purely numerical fractions.

If the two expressions have the same denominator, then we can rely on the property of adding and subtracting fractions and simplify that result.

Let's review how to add fractions with the same denominator:

$$\begin{aligned}\frac{1}{10} + \frac{3}{10} &= \frac{1+3}{10}\\ &= \frac{4}{10}\\ &= \frac{2}{5}\end{aligned}$$

We can add and subtract rational expressions in the same way:

$$\begin{aligned}\frac{2}{3x} - \frac{5}{3x} &= \frac{2-5}{3x}\\ &= \frac{-3}{3x}\\ &= -\frac{1}{x}\end{aligned}$$

List 12.3.4: Steps to Adding/Subtracting Rational Expressions

Identify the LCD Determine the least common denominator of all of the denominators.

Build If necessary, build up each expression so that the denominators are the same.

Add/Subtract Combine the numerators using the properties of adding and subtracting fractions.

Simplify Simplify the resulting rational expression as much as possible. This may require factoring the numerator.

Example 12.3.5 Add the rational expressions: $\frac{2x}{x+y} + \frac{2y}{x+y}$.

Explanation. These expressions already have a common denominator:

$$\begin{aligned}\frac{2x}{x+y} + \frac{2y}{x+y} &= \frac{2x+2y}{x+y}\\ &= \frac{2\cancel{(x+y)}}{\cancel{x+y}}\\ &= \frac{2}{1}\\ &= 2\end{aligned}$$

Note that we didn't stop at $\frac{2x+2y}{x+y}$. If possible, we must simplify the numerator and denominator. Since this is a multivariable expression, this textbook ignores domain restrictions while canceling.

12.3.3 Addition and Subtraction of Rational Expressions with Different Denominators

To add rational expressions with different denominators, we'll need to build each fraction to the least common denominator, in the same way we do with numerical fractions. Let's briefly review this process by adding $\frac{3}{5}$ and $\frac{1}{6}$:

$$\begin{aligned}\frac{3}{5}+\frac{1}{6}&=\frac{3}{5}\cdot\frac{6}{6}+\frac{1}{6}\cdot\frac{5}{5}\\&=\frac{18}{30}+\frac{5}{30}\\&=\frac{18+5}{30}\\&=\frac{23}{30}\end{aligned}$$

This exact method can be used when adding rational expressions containing variables. The key is that the expressions *must* have the same denominator before they can be added or subtracted. If they don't have this initially, then we'll identify the least common denominator and build each expression so that it has that denominator.

Let's apply this to adding the two expressions with denominators that are $v-2$ and $v+2$ from Example 12.3.2.

Example 12.3.6 Add the rational expressions and fully simplify the function given by $t(v)=\frac{12}{v-2}+\frac{12}{v+2}$.
Explanation.

$$\begin{aligned}t(v)&=\frac{12}{v-2}+\frac{12}{v+2}\\t(v)&=\frac{12}{v-2}\cdot\frac{v+2}{v+2}+\frac{12}{v+2}\cdot\frac{v-2}{v-2}\\t(v)&=\frac{12v+24}{(v-2)(v+2)}+\frac{12v-24}{(v+2)(v-2)}\\t(v)&=\frac{(12v+24)+(12v-24)}{(v+2)(v-2)}\\t(v)&=\frac{24v}{(v+2)(v-2)}\end{aligned}$$

Example 12.3.7 Add the rational expressions: $\dfrac{2}{5x^2y}+\dfrac{3}{20xy^2}$

Explanation. The least common denominator of $5x^2y$ and $20xy^2$ must include two x's and two y's, as well as 20. Thus it is $20x^2y^2$. We will build both denominators to $20x^2y^2$ before doing addition.

$$\begin{aligned}\frac{2}{5x^2y}+\frac{3}{20xy^2}&=\frac{2}{5x^2y}\cdot\frac{4y}{4y}+\frac{3}{20xy^2}\cdot\frac{x}{x}\\&=\frac{8y}{20x^2y^2}+\frac{3x}{20x^2y^2}\\&=\frac{8y+3x}{20x^2y^2}\end{aligned}$$

Let's look at a few more complicated examples.

Example 12.3.8 Subtract the rational expressions: $\dfrac{y}{y-2}-\dfrac{8y-8}{y^2-4}$

Explanation. To start, we'll make sure each denominator is factored. Then we'll find the least common denominator and build each expression to that denominator. Then we will be able to combine the numerators and simplify the expression.

$$\begin{aligned}
\frac{y}{y-2}-\frac{8y-8}{y^2-4} &= \frac{y}{y-2}-\frac{8y-8}{(y+2)(y-2)}\\
&= \frac{y}{y-2}\cdot\frac{y+2}{y+2}-\frac{8y-8}{(y+2)(y-2)}\\
&= \frac{y^2+2y}{(y+2)(y-2)}-\frac{8y-8}{(y+2)(y-2)}\\
&= \frac{y^2+2y-\overset{\downarrow}{(}8y-8\overset{\downarrow}{)}}{(y+2)(y-2)}\\
&= \frac{y^2+2y-8y+8}{(y+2)(y-2)}\\
&= \frac{y^2-6y+8}{(y+2)(y-2)}\\
&= \frac{\cancel{(y-2)}(y-4)}{(y+2)\cancel{(y-2)}}\\
&= \frac{y-4}{y+2},\ \text{for } y\neq 2
\end{aligned}$$

Note that we must factor the numerator in $\frac{y^2-6y+8}{(y+2)(y-2)}$ and try to reduce the fraction (which we did).

Warning 12.3.9 In Example 12.3.8, be careful to subtract the entire numerator of $8y-8$. When this expression is in the numerator of $\frac{8y-8}{(y+2)(y-2)}$, it's implicitly grouped and doesn't need parentheses. But once $8y-8$ is subtracted from y^2+2y, we need to add parentheses so the entire expression is subtracted.

In the next example, we'll look at adding a rational expression to a polynomial. Much like adding a fraction and an integer, we'll rely on writing that expression as itself over one in order to build its denominator.

Example 12.3.10 Add the expressions: $-\frac{2}{r-1}+r$

Explanation.

$$\begin{aligned}
-\frac{2}{r-1}+r &= -\frac{2}{r-1}+\frac{r}{1}\\
&= -\frac{2}{r-1}+\frac{r}{1}\cdot\frac{r-1}{r-1}\\
&= \frac{-2}{r-1}+\frac{r^2-r}{r-1}\\
&= \frac{-2+r^2-r}{r-1}\\
&= \frac{r^2-r-2}{r-1}\\
&= \frac{(r-2)(r+1)}{r-1}
\end{aligned}$$

Note that we factored the numerator to reduce the fraction if possible. Even though it was not possible in this case, leaving it in factored form makes it easier to see that it is reduced.

Example 12.3.11 Subtract the expressions: $\frac{6}{x^2-2x-8}-\frac{1}{x^2+3x+2}$

Explanation. To start, we'll need to factor each of the denominators. After that, we'll identify the LCD and build each denominator accordingly. Then we can combine the numerators and simplify the resulting expression.

$$\begin{aligned}
\frac{6}{x^2-2x-8}-\frac{1}{x^2+3x+2} &= \frac{6}{(x-4)(x+2)}-\frac{1}{(x+2)(x+1)} \\
&= \frac{6}{(x-4)(x+2)}\cdot\frac{x+1}{x+1}-\frac{1}{(x+2)(x+1)}\cdot\frac{x-4}{x-4} \\
&= \frac{6x+6}{(x-4)(x+2)(x+1)}-\frac{x-4}{(x+2)(x+1)(x-4)} \\
&= \frac{6x+6-(x-4)}{(x-4)(x+2)(x+1)} \\
&= \frac{6x+6-x+4}{(x-4)(x+2)(x+1)} \\
&= \frac{5x+10}{(x-4)(x+2)(x+1)} \\
&= \frac{5\cancel{(x+2)}}{(x-4)\cancel{(x+2)}(x+1)} \\
&= \frac{5}{(x-4)(x+1)},\ \text{for } x \neq -2
\end{aligned}$$

12.3.4 Reading Questions

1. Describe how to add two rational expressions when they have the same denominator.
2. Suppose you are adding two rational expressions where one of them has a quadratic denominator, and the other has a linear denominator. What is the first thing you should try to do with respect to the quadratic denominator?

12.3.5 Exercises

Review and Warmup

1. Add: $\frac{31}{16}+\frac{5}{16}$
2. Add: $\frac{13}{16}+\frac{23}{16}$
3. Add: $\frac{5}{6}+\frac{9}{10}$
4. Add: $\frac{3}{5}+\frac{9}{10}$
5. Subtract: $\frac{25}{27}-\frac{10}{27}$
6. Subtract: $\frac{21}{40}-\frac{17}{40}$
7. Subtract: $\frac{5}{9}-\frac{8}{27}$
8. Subtract: $\frac{4}{9}-\frac{1}{27}$

Factor the given polynomial.

9. x^2-4
10. y^2-81
11. $y^2+14y+40$

12. $y^2 + 5y + 4$

13. $r^2 - 17r + 72$

14. $r^2 - 7r + 12$

15. $3t^2 - 18t + 15$

16. $7t^2 - 28t + 21$

Addition and Subtraction of Rational Expressions with One Variable Add or subtract the rational expressions to a single rational expression and then simplify. If applicable, state the restricted domain.

17. $\dfrac{4x}{x+4} + \dfrac{16}{x+4}$

18. $\dfrac{6x}{x+2} + \dfrac{12}{x+2}$

19. $\dfrac{3y}{y+6} + \dfrac{18}{y+6}$

20. $\dfrac{5y}{y+4} + \dfrac{20}{y+4}$

21. $\dfrac{3}{y^2-4y-5} - \dfrac{y-2}{y^2-4y-5}$

22. $\dfrac{6}{r^2-13r+40} - \dfrac{r-2}{r^2-13r+40}$

23. $\dfrac{6}{r^2-9r-10} - \dfrac{r-4}{r^2-9r-10}$

24. $\dfrac{4}{t^2-3t-40} - \dfrac{t-4}{t^2-3t-40}$

25. $\dfrac{6t}{5} + \dfrac{t}{20}$

26. $\dfrac{3x}{2} + \dfrac{x}{10}$

27. $\dfrac{1}{x+1} + \dfrac{2}{x-1}$

28. $\dfrac{5}{y+5} - \dfrac{1}{y+2}$

29. $\dfrac{5}{y-4} - \dfrac{4}{y-2}$

30. $\dfrac{1}{y+6} + \dfrac{4}{y-3}$

31. $\dfrac{1}{r-2} - \dfrac{4}{r^2-4}$

32. $\dfrac{1}{r+1} + \dfrac{2}{r^2-1}$

33. $\dfrac{1}{t-1} - \dfrac{2}{t^2-1}$

34. $\dfrac{1}{t-2} - \dfrac{4}{t^2-4}$

35. $\dfrac{3}{x-4} - \dfrac{6x}{x^2-16}$

36. $\dfrac{3}{x-3} - \dfrac{6x}{x^2-9}$

37. $\dfrac{3}{y-6} - \dfrac{6y}{y^2-36}$

38. $\dfrac{3}{y+2} - \dfrac{6y}{y^2-4}$

39. $\dfrac{y}{y-6} - \dfrac{10y-24}{y^2-6y}$

40. $\dfrac{r}{r+6} - \dfrac{r+42}{r^2+6r}$

41. $\dfrac{r}{r-8} - \dfrac{4r+32}{r^2-8r}$

42. $\dfrac{t}{t-7} - \dfrac{t+42}{t^2-7t}$

43. $\dfrac{2}{t^2-1} + \dfrac{1}{t+1} + \dfrac{3}{t-1}$

44. $-\dfrac{4}{x^2-4} - \dfrac{4}{x+2} + \dfrac{1}{x-2}$

45. $-\dfrac{9x}{x^2-7x+10} - \dfrac{3x}{x-2}$

46. $-\dfrac{12y}{y^2-y-2} + \dfrac{4y}{y-2}$

47. $\dfrac{12y}{y^2+8y+12} - \dfrac{3y}{y+2}$

48. $-\dfrac{18y}{y^2+4y-5} - \dfrac{3y}{y+5}$

49. $\dfrac{r^2+8}{r^2+4r} - \dfrac{r+2}{r}$

50. $\dfrac{r^2+8}{r^2-4r} - \dfrac{r-2}{r}$

51. $\dfrac{2}{t-5} - 3$

52. $\dfrac{4}{t+1} + 5$

53. $\dfrac{4x}{x+4} + \dfrac{x}{x-4} - 5$

54. $\dfrac{6x}{x+2} + \dfrac{x}{x-2} - 7$

Addition and Subtraction of Rational Expressions with More Than Variable Add or subtract the rational expressions to a single rational expression and then simplify.

55. $\dfrac{16y^2}{4y-3x} - \dfrac{9x^2}{4y-3x}$

56. $\dfrac{64y^2}{8y+3t} - \dfrac{9t^2}{8y+3t}$

57. $\dfrac{y}{6x} - \dfrac{5y}{3x}$

58. $\dfrac{r}{20x} - \dfrac{4r}{5x}$

59. $\dfrac{6r}{5t^4} + \dfrac{4}{3rt}$

60. $-\dfrac{5t}{4y^2} + \dfrac{5}{3ty}$

61. $\dfrac{2}{tx-5} - \dfrac{4tx}{t^2x^2-25}$

62. $\dfrac{2}{xr-2} - \dfrac{4xr}{x^2r^2-4}$

63. $-\dfrac{24xy}{x^2+8xy+12y^2} - \dfrac{6x}{x+6y}$

64. $\dfrac{2xt}{x^2+9xt+20t^2} - \dfrac{2x}{x+4t}$

12.4 Complex Fractions

In this section, we will learn how to simplify complex fractions, which have fractions in the numerator and/or denominator of another fraction.

12.4.1 Simplifying Complex Fractions

Consider the rational expression

$$\frac{\frac{6}{x-4}}{\frac{6}{x-4}+3}.$$

It's difficult to quickly evaluate this expression, or determine the important information such as its domain. This type of rational expression, which contains a "fraction within a fraction," is referred to as a **complex fraction**. Our goal is to simplify such a fraction so that it has a *single* numerator and a *single* denominator, neither of which contain any fractions themselves.

A complex fraction may have fractions in its numerator and/or denominator. Here is an example to show how we use division to simplify a complex fraction.

$$\begin{aligned}\frac{\frac{1}{2}}{3} &= \frac{1}{2} \div 3\\ &= \frac{1}{2} \div \frac{3}{1}\\ &= \frac{1}{2} \cdot \frac{1}{3}\\ &= \frac{1}{6}\end{aligned}$$

What if the expression had something more complicated in the denominator, like $\frac{\frac{1}{2}}{\frac{1}{3}+\frac{1}{4}}$? We would no longer be able to simply multiply by the reciprocal of the denominator, since we don't immediately know the reciprocal of that denominator. Instead, we could multiply the "main" numerator and denominator by something that eliminates all of the "internal" denominators. (We'll use the LCD to determine this). For example, with $\frac{\frac{1}{2}}{3}$, we can multiply by $\frac{2}{2}$:

$$\begin{aligned}\frac{\frac{1}{2}}{3} &= \frac{\frac{1}{\cancel{2}}}{3} \cdot \frac{\cancel{2}}{2}\\ &= \frac{1}{6}\end{aligned}$$

Remark 12.4.2 In the last example, it's important to identify which fraction bar is the "main" fraction bar, and which fractions are "internal." Comparing the two expressions below, both of which are "one over two over three", we see that they are not equivalent.

$$\begin{aligned}\frac{\frac{1}{2}}{3} &= \frac{\frac{1}{\cancel{2}}}{3} \cdot \frac{\cancel{2}}{2}\\ &= \frac{1}{6}\end{aligned}$$

versus

$$\begin{aligned}\frac{1}{\frac{2}{3}} &= \frac{1}{\frac{2}{\cancel{3}}} \cdot \frac{3}{\cancel{3}}\\ &= \frac{3}{2}\end{aligned}$$

For the first of these, the "main" fraction bar is above the 3, but for the second of these, the "main" fraction bar is above the $\frac{2}{3}$.

To attack multiple fractions in a complex fraction, we need to multiply the numerator and denominator by the LCD of all the internal fractions, as we will show in the next example.

Example 12.4.3 Simplify the complex fraction $\frac{\frac{1}{2}}{\frac{1}{3}+\frac{1}{4}}$.

Explanation.

The internal denominators are 2, 3, and 4, so the LCD is 12. We will thus multiply the main numerator and denominator by 12 and simplify the result:

$$\begin{aligned}\frac{\frac{1}{2}}{\frac{1}{3}+\frac{1}{4}} &= \frac{\frac{1}{2}}{\frac{1}{3}+\frac{1}{4}}\cdot\frac{12}{12}\\ &= \frac{\frac{1}{2}\cdot 12}{\left(\frac{1}{3}+\frac{1}{4}\right)\cdot 12}\\ &= \frac{\frac{1}{2}\cdot 12}{\frac{1}{3}\cdot 12+\frac{1}{4}\cdot 12}\\ &= \frac{6}{4+3}\\ &= \frac{6}{7}\end{aligned}$$

Next we will evaluate a function whose formula is a complex fraction and then simplify the result.

Example 12.4.4 Find each function value for $f(x)=\frac{\frac{x+2}{x+3}}{\frac{2}{x+3}-\frac{3}{x-1}}$.

a. $f(4)$ b. $f(0)$ c. $f(-3)$ d. $f(-11)$

Explanation. We will determine each function value by replacing x with the specified number and then simplify the complex fraction:

a.
$$\begin{aligned}f(4) &= \frac{\frac{4+2}{4+3}}{\frac{2}{4+3}-\frac{3}{4-1}}\\ &= \frac{\frac{6}{7}}{\frac{2}{7}-\frac{3}{3}}\\ &= \frac{\frac{6}{7}}{\frac{2}{7}-1}\cdot\frac{7}{7}\\ &= \frac{6}{2-7}\\ &= -\frac{6}{5}\end{aligned}$$

b.
$$\begin{aligned}f(0) &= \frac{\frac{0+2}{0+3}}{\frac{2}{0+3}-\frac{3}{0-1}}\\ &= \frac{\frac{2}{3}}{\frac{2}{3}-\frac{3}{-1}}\\ &= \frac{\frac{2}{3}}{\frac{2}{3}+3}\cdot\frac{3}{3}\\ &= \frac{2}{2+9}\\ &= \frac{2}{11}\end{aligned}$$

c. When evaluating f at -3, we can quickly see that this results in division by zero:

$$\begin{aligned}f(-3) &= \frac{\frac{-3+2}{-3+3}}{\frac{2}{-3+3}-\frac{3}{-3-1}}\\ &= \frac{\frac{2}{0}}{\frac{2}{0}-\frac{3}{-4}}\end{aligned}$$

Thus $f(-3)$ is undefined.

d.
$$\begin{aligned}f(-11) &= \frac{\frac{-11+2}{-11+3}}{\frac{2}{-11+3}-\frac{3}{-11-1}}\\ &= \frac{\frac{-8}{-9}}{\frac{2}{-8}-\frac{3}{-12}}\\ &= \frac{\frac{8}{9}}{-\frac{1}{4}+\frac{1}{4}}\\ &= \frac{\frac{8}{9}}{0}\end{aligned}$$

Therefore $f(-11)$ is undefined.

We have simplified complex fractions involving numbers and now we will apply the same concept to complex fractions with variables.

Example 12.4.5 Simplify the complex fraction $\frac{3}{\frac{1}{y}+\frac{5}{y^2}}$.

Explanation.

To start, we look at the internal denominators and identify the LCD as y^2. We'll multiply the main numerator and denominator by the LCD, and then simplify. Since we are multiplying by $\frac{y^2}{y^2}$, it is important to note that y cannot be 0, since $\frac{0}{0}$ is undefined.

$$\begin{aligned}\frac{3}{\frac{1}{y}+\frac{5}{y^2}} &= \frac{3}{\frac{1}{y}+\frac{5}{y^2}}\cdot\frac{y^2}{y^2}\\ &= \frac{3\cdot y^2}{\frac{1}{y}\cdot y^2+\frac{5}{y^2}\cdot y^2}\\ &= \frac{3y^2}{y+5},\ \text{for } y\neq 0\end{aligned}$$

Example 12.4.6 Simplify the complex fraction $\frac{\frac{5x-6}{2x+1}}{\frac{3x+2}{2x+1}}$.

Explanation.

The internal denominators are both $2x+1$, so this is the LCD and we will multiply the main numerator and denominator by this expression. Since we are multiplying by $\frac{2x+1}{2x+1}$, what x-value would cause $2x+1$ to equal 0? Solving $2x+1=0$ leads to $x=-\frac{1}{2}$. So x cannot be $-\frac{1}{2}$, since $\frac{0}{0}$ is undefined.

$$\begin{aligned}\frac{\frac{5x-6}{2x+1}}{\frac{3x+2}{2x+1}} &= \frac{\frac{5x-6}{\cancel{2x+1}}}{\frac{3x+2}{\cancel{2x+1}}}\cdot\frac{\cancel{2x+1}}{\cancel{2x+1}}\\ &= \frac{5x-6}{3x+2},\ \text{for } x\neq-\frac{1}{2}\end{aligned}$$

Example 12.4.7 Completely simplify the function defined by $f(x)=\frac{\frac{x+2}{x+3}}{\frac{2}{x+3}-\frac{3}{x-1}}$. Then determine the domain of this function.

Explanation. The LCD of the internal denominators is $(x+3)(x-1)$. We will thus multiply the main numerator and denominator by the expression $(x+3)(x-1)$ and then simplify the resulting expression.

$$\begin{aligned}f(x) &= \frac{\frac{x+2}{x+3}}{\frac{2}{x+3}-\frac{3}{x-1}}\\ &= \frac{\frac{x+2}{x+3}}{\frac{2}{x+3}-\frac{3}{x-1}}\cdot\frac{(x+3)(x-1)}{(x+3)(x-1)}\\ &= \frac{\frac{x+2}{x+3}\cdot(x+3)(x-1)}{\left(\frac{2}{x+3}-\frac{3}{x-1}\right)\cdot(x+3)(x-1)}\\ &= \frac{\frac{x+2}{\cancel{x+3}}\cancel{(x+3)}(x-1)}{\frac{2}{\cancel{x+3}}\cancel{(x+3)}(x-1)-\frac{3}{\cancel{x-1}}(x+3)\cancel{(x-1)}}\\ &= \frac{(x+2)(x-1)}{2(x-1)-3(x+3)},\ \text{for } x\neq-3, x\neq1\\ &= \frac{(x+2)(x-1)}{2x-2-3x-9},\ \text{for } x\neq-3, x\neq1\\ &= \frac{(x+2)(x-1)}{-x-11},\ \text{for } x\neq-3, x\neq1\\ &= \frac{(x+2)(x-1)}{-(x+11)},\ \text{for } x\neq-3, x\neq1\end{aligned}$$

In the original (unsimplified) function, we could see that $x\neq-3$ and $x\neq1$. In the simplified function, we need $x+11\neq0$, so we can also see that $x\neq-11$. Therefore the domain of the function f is $\{x \mid x\neq-11,-3,1\}$.

Example 12.4.8 Simplify the complex fraction $\dfrac{2\left(\frac{-4x+3}{x-2}\right)+3}{\frac{-4x+3}{x-2}+4}$.

Explanation. The only internal denominator is $x-2$, so we will begin by multiplying the main numerator and denominator by this. Then we'll simplify the resulting expression.

$$\begin{aligned}
\frac{2\left(\frac{-4x+3}{x-2}\right)+3}{\frac{-4x+3}{x-2}+4} &= \frac{2\left(\frac{-4x+3}{x-2}\right)+3}{\frac{-4x+3}{x-2}+4}\cdot\frac{x-2}{x-2}\\
&= \frac{2\left(\frac{-4x+3}{\cancel{x-2}}\right)\cancel{(x-2)}+3(x-2)}{\left(\frac{-4x+3}{\cancel{x-2}}\right)\cancel{(x-2)}+4(x-2)}\\
&= \frac{2(-4x+3)+3(x-2)}{(-4x+3)+4(x-2)},\ \text{for } x\neq 2\\
&= \frac{-8x+6+3x-6}{-4x+3+4x-8},\ \text{for } x\neq 2\\
&= \frac{-5x}{-5},\ \text{for } x\neq 2\\
&= x,\ \text{for } x\neq 2
\end{aligned}$$

Example 12.4.9 Simplify the complex fraction $\dfrac{\frac{5}{x}+\frac{4}{y}}{\frac{3}{x}-\frac{2}{y}}$. Recall that with a multivariable expression, this textbook ignores domain restrictions.

Explanation.

We multiply the numerator and denominator by the common denominator of x and y, which is xy:

$$\begin{aligned}
\frac{\frac{5}{x}+\frac{4}{y}}{\frac{3}{x}-\frac{2}{y}} &= \frac{\frac{5}{x}+\frac{4}{y}}{\frac{3}{x}-\frac{2}{y}}\cdot\frac{xy}{xy}\\
&= \frac{\left(\frac{5}{x}+\frac{4}{y}\right)xy}{\left(\frac{3}{x}-\frac{2}{y}\right)xy}\\
&= \frac{\frac{5}{\cancel{x}}\cancel{x}y+\frac{4}{\cancel{y}}x\cancel{y}}{\frac{3}{\cancel{x}}\cancel{x}y-\frac{2}{\cancel{y}}x\cancel{y}}\\
&= \frac{5y+4x}{3y-2x}
\end{aligned}$$

Example 12.4.10 Simplify the complex fraction $\dfrac{\frac{t}{t+3}+\frac{2}{t-3}}{1-\frac{t}{t^2-9}}$.

Explanation. First, we check all quadratic polynomials to see if they can be factored and factor them:

$$\frac{\frac{t}{t+3}+\frac{2}{t-3}}{1-\frac{t}{t^2-9}} = \frac{\frac{t}{t+3}+\frac{2}{t-3}}{1-\frac{t}{(t-3)(t+3)}}$$

Next, we identify the common denominator of the three fractions, which is $(t+3)(t-3)$. We then multiply the main numerator and denominator by that expression:

$$\frac{\frac{t}{t+3}+\frac{2}{t-3}}{1-\frac{t}{t^2-9}} = \frac{\frac{t}{t+3}+\frac{2}{t-3}}{1-\frac{t}{(t-3)(t+3)}}\cdot\frac{(t+3)(t-3)}{(t+3)(t-3)}$$

$$
\begin{aligned}
&= \frac{\frac{t}{\cancel{t+3}}\cancel{(t+3)}(t-3)+\frac{2}{\cancel{t-3}}(t+3)\cancel{(t-3)}}{1(t+3)(t-3)-\frac{t}{\cancel{(t-3)(t+3)}}\cancel{(t+3)(t-3)}}\\
&= \frac{t(t-3)+2(t+3)}{(t+3)(t-3)-t} \text{ for } t \neq -3, t \neq 3\\
&= \frac{t^2-3t+2t+6}{t^2-9-t} \text{ for } t \neq -3, t \neq 3\\
&= \frac{t^2-t+6}{t^2-t-9} \text{ for } t \neq -3, t \neq 3
\end{aligned}
$$

Note that since both the numerator and denominator are prime trinomials, this expression can neither factor nor simplify any further.

12.4.2 Reading Questions

1. What does it mean for a fraction to be a "complex" fraction?

2. When simplifying a complex fraction, why is it necessary to keep track of domain restrictions?

12.4.3 Exercises

Review and Warmup Calculate the following. Use an improper fraction in your answer.

1. a. $\frac{\frac{5}{2}}{\frac{5}{7}}$ b. $\frac{\frac{y}{r}}{\frac{x}{t}}$

2. a. $\frac{\frac{25}{3}}{\frac{5}{4}}$ b. $\frac{\frac{y}{t}}{\frac{x}{r}}$

3. a. $\frac{4}{\frac{9}{4}}$ b. $\frac{\frac{4}{9}}{4}$

4. a. $\frac{5}{\frac{6}{7}}$ b. $\frac{\frac{5}{6}}{7}$

5. $\frac{\frac{1}{6}-\frac{2}{3}}{\frac{3}{2}}$

6. $\frac{\frac{1}{6}-\frac{3}{4}}{\frac{3}{5}}$

7. $\frac{1}{\frac{4}{3}-\frac{3}{4}}$

8. $\frac{1}{\frac{2}{3}-\frac{3}{5}}$

Simplifying Complex Fractions with One Variable Simplify this expression, and if applicable, write the restricted domain.

9. $\frac{\frac{9a+7}{a}}{\frac{a+10}{a}}$

10. $\frac{\frac{6a-6}{a}}{\frac{a-5}{a}}$

11. $\frac{\frac{u}{(u-2)^2}}{\frac{7u}{u^2-4}}$

12. $\frac{\frac{u}{(u-9)^2}}{\frac{2u}{u^2-81}}$

13. $\frac{6+\frac{1}{p}}{p+6}$

14. $\frac{2+\frac{1}{p}}{p+10}$

15. $\frac{3}{\frac{2}{t}-\frac{4}{t+5}}$

16. $\frac{4}{\frac{6}{x}+\frac{3}{x-6}}$

17. $\frac{2+\frac{1}{y-2}}{\frac{1}{y-2}-\frac{1}{8}}$

18. $\frac{8+\frac{1}{b-6}}{\frac{1}{b-6}-\frac{1}{6}}$

19. $\frac{\frac{1}{c+4}+\frac{10}{c-4}}{3-\frac{1}{c-4}}$

20. $\frac{\frac{1}{u+1}+\frac{5}{u-1}}{10-\frac{1}{u-1}}$

21. $\frac{\frac{1}{s-7}+\frac{9}{s-7}}{8-\frac{1}{s+7}}$

22. $\frac{\frac{1}{p-4}+\frac{4}{p-4}}{5-\frac{1}{p+4}}$

23. $\frac{\frac{10}{q-1}-7}{\frac{1}{q-1}+\frac{1}{q-3}}$

24. $\dfrac{\frac{7}{n-1}-2}{\frac{1}{n-1}+\frac{1}{n-9}}$

25. $\dfrac{\frac{4x}{x^2-25}+1}{\frac{2}{x+5}-\frac{5}{x-5}}$

26. $\dfrac{\frac{6x}{x^2-25}-5}{\frac{2}{x+5}+\frac{1}{x-5}}$

27. $\dfrac{\frac{c}{c^2-36}-\frac{1}{c^2-36}}{\frac{1}{c+36}}$

28. $\dfrac{\frac{c}{c^2-9}-\frac{1}{c^2-9}}{\frac{1}{c+9}}$

Simplifying Complex Fractions with More Than One Variable Simplify this expression.

29. $\dfrac{\frac{s}{t}}{\frac{6s}{5t^2}}$

30. $\dfrac{\frac{s}{t}}{\frac{5s}{4t^2}}$

31. $\dfrac{\frac{pq^2}{4r}}{\frac{p}{10qr}}$

32. $\dfrac{\frac{pq^2}{9r}}{\frac{p}{5qr}}$

33. a. $\dfrac{\frac{t}{r}}{y}$

b. $\dfrac{t}{\frac{r}{y}}$

34. a. $\dfrac{\frac{x}{r}}{t}$

b. $\dfrac{x}{\frac{r}{t}}$

35. $\dfrac{\frac{4}{x}}{20+4t}$

36. $\dfrac{\frac{2}{x}}{8+2y}$

37. $\dfrac{\frac{2}{y}+\frac{2}{x}}{\frac{2}{y}-\frac{12}{x}}$

38. $\dfrac{\frac{2}{y}+\frac{10}{t}}{\frac{12}{y}+\frac{2}{t}}$

12.5 Solving Rational Equations

12.5.1 Solving Rational Equations

We open this section looking back on Example 12.3.2. Julia is taking her family on a boat trip 12 miles down the river and back. The river flows at a speed of 2 miles per hour and she wants to drive the boat at a constant speed, v miles per hour downstream and back upstream. Due to the current of the river, the actual speed of travel is $v+2$ miles per hour going downstream, and $v-2$ miles per hour going upstream. If Julia plans to spend 8 hours for the whole trip, how fast should she drive the boat?

The time it takes Julia to drive the boat downstream is $\frac{12}{v+2}$ hours, and upstream is $\frac{12}{v-2}$ hours. The function to model the whole trip's time is

$$t(v) = \frac{12}{v-2} + \frac{12}{v+2}$$

where t stands for time in hours. The trip will take 8 hours, so we want $t(v)$ to equal 8, and we have:

$$\frac{12}{v-2} + \frac{12}{v+2} = 8.$$

Instead of using the function's graph, we will solve this equation algebraically. You may wish to review the technique of eliminating denominators discussed in Subsection 2.3.2. We can use the same technique with variable expressions in the denominators. To remove the fractions in this equation, we will multiply both sides of the equation by the least common denominator $(v-2)(v+2)$, and we have:

$$\begin{aligned}
\frac{12}{v-2} + \frac{12}{v+2} &= 8 \\
(v+2)(v-2)\cdot\left(\frac{12}{v-2} + \frac{12}{v+2}\right) &= (v+2)(v-2)\cdot 8 \\
(v+2)\cancel{(v-2)}\cdot\frac{12}{\cancel{v-2}} + \cancel{(v+2)}(v-2)\cdot\frac{12}{\cancel{v+2}} &= (v+2)(v-2)\cdot 8 \\
12(v+2) + 12(v-2) &= 8(v^2-4) \\
12v + 24 + 12v - 24 &= 8v^2 - 32 \\
24v &= 8v^2 - 32 \\
0 &= 8v^2 - 24v - 32 \\
0 &= 8(v^2 - 3v - 4) \\
0 &= 8(v-4)(v+1)
\end{aligned}$$

$$\begin{aligned}
v - 4 &= 0 \quad &&\text{or} \quad & v + 1 &= 0 \\
v &= 4 \quad &&\text{or} \quad & v &= -1
\end{aligned}$$

Remark 12.5.2 At this point, logically all that we know is that the only *possible* solutions are -1 and 4. Because of the step where factors were canceled, it's possible that these might not actually be solutions to the original equation. They each might be what is called an **extraneous solution**. An extraneous solution is a number that would appear to be a solution based on the solving process, but actually does not make the original equation true. Because of this, it is important that these proposed solutions be checked. Note that we're not checking to see if we made a calculation error, but are instead checking to see if the proposed solutions actually solve the original equation.

We check these values.

$$\begin{aligned} \frac{12}{-1-2}+\frac{12}{-1+2} &\stackrel{?}{=} 8 \\ \frac{12}{-3}+\frac{12}{1} &\stackrel{?}{=} 8 \\ -4+12 &\stackrel{\checkmark}{=} 8 \end{aligned} \qquad \begin{aligned} \frac{12}{4-2}+\frac{12}{4+2} &\stackrel{?}{=} 8 \\ \frac{12}{2}+\frac{12}{6} &\stackrel{?}{=} 8 \\ 6+2 &\stackrel{\checkmark}{=} 8 \end{aligned}$$

Algebraically, both values do check out to be solutions. In the context of this scenario, the boat's speed can't be negative, so we only take the solution 4. If Julia drives at 4 miles per hour, the whole trip would take 8 hours. This result matches the solution in Example 12.3.2.

Definition 12.5.3 Rational Equation. A rational equation is an equation involving one or more rational expressions. Usually, we consider these to be equations that have the variable in the denominator of at least one term. ◊

Let's look at another application problem.

Example 12.5.4 It takes Ku 3 hours to paint a room and it takes Jacob 6 hours to paint the same room. If they work together, how long would it take them to paint the room?

Explanation. Since it takes Ku 3 hours to paint the room, he paints $\frac{1}{3}$ of the room each hour. Similarly, Jacob paints $\frac{1}{6}$ of the room each hour. If they work together, they paint $\frac{1}{3}+\frac{1}{6}$ of the room each hour.

Assume it takes x hours to paint the room if Ku and Jacob work together. This implies they paint $\frac{1}{x}$ of the room together each hour. Now we can write this equation:

$$\frac{1}{3}+\frac{1}{6}=\frac{1}{x}.$$

To clear away denominators, we multiply both sides of the equation by the common denominator of 3, 6 and x, which is $6x$:

$$\begin{aligned} \frac{1}{3}+\frac{1}{6} &= \frac{1}{x} \\ 6x\cdot\left(\frac{1}{3}+\frac{1}{6}\right) &= 6\cancel{x}\cdot\frac{1}{\cancel{x}} \\ 6x\cdot\frac{1}{3}+6x\cdot\frac{1}{6} &= 6 \\ 2x+x &= 6 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

Does the possible solution $x=2$ check as an actual solution?

$$\begin{aligned} \frac{1}{3}+\frac{1}{6} &\stackrel{?}{=} \frac{1}{2} \\ \frac{2}{6}+\frac{1}{6} &\stackrel{?}{=} \frac{1}{2} \\ \frac{3}{6} &\stackrel{\checkmark}{=} \frac{1}{2} \end{aligned}$$

It does, so it is a solution. If Ku and Jacob work together, it would take them 2 hours to paint the room.

We are ready to outline a general process for solving a rational equation.

Process 12.5.5 Solving Rational Equations. *To solve a rational equation,*

1. *Find the least common denominator for all terms in the equation.*
2. *Multiply* every term *in the equation by the least common denominator*
3. *Every denominator should cancel leaving a simpler kind of equation to solve. Use previous method to solve that equation.*

Let's look at a few more examples of solving rational equations.

Example 12.5.6 Solve for y in $\frac{2}{y+1} = \frac{3}{y}$.

Explanation. The common denominator is $y(y+1)$. We will multiply both sides of the equation by $y(y+1)$:

$$\begin{aligned} \frac{2}{y+1} &= \frac{3}{y} \\ y\cancel{(y+1)} \cdot \frac{2}{\cancel{y+1}} &= \cancel{y}(y+1) \cdot \frac{3}{\cancel{y}} \\ 2y &= 3(y+1) \\ 2y &= 3y+3 \\ -y &= 3 \\ y &= -3 \end{aligned}$$

Does the possible solution $y = -3$ check as an actual solution?

$$\begin{aligned} \frac{2}{-3+1} &\stackrel{?}{=} \frac{3}{-3} \\ \frac{2}{-2} &\stackrel{\checkmark}{=} -1 \end{aligned}$$

It checks, so -3 is a solution. We write the solution set as $\{-3\}$.

Example 12.5.7 Solve for z in $z + \frac{1}{z-4} = \frac{z-3}{z-4}$.

Explanation. The common denominator is $z-4$. We will multiply both sides of the equation by $z-4$:

$$\begin{aligned} z + \frac{1}{z-4} &= \frac{z-3}{z-4} \\ (z-4) \cdot \left(z + \frac{1}{z-4}\right) &= \cancel{(z-4)} \cdot \frac{z-3}{\cancel{z-4}} \\ (z-4) \cdot z + \cancel{(z-4)} \cdot \frac{1}{\cancel{z-4}} &= z-3 \\ (z-4) \cdot z + 1 &= z-3 \\ z^2 - 4z + 1 &= z-3 \\ z^2 - 5z + 4 &= 0 \\ (z-1)(z-4) &= 0 \end{aligned}$$

$z - 1 = 0$ or $z - 4 = 0$

$$z = 1 \quad \text{or} \quad z = 4$$

Do the possible solutions $z = 1$ and $z = 4$ check as actual solutions?

$$\begin{aligned} 1 + \frac{1}{1-4} &\stackrel{?}{=} \frac{1-3}{1-4} \\ 1 - \frac{1}{3} &\stackrel{\checkmark}{=} \frac{-2}{-3} \end{aligned} \qquad \begin{aligned} 4 + \frac{1}{4-4} &\stackrel{?}{=} \frac{4-3}{4-4} \\ 4 + \frac{1}{0} &\stackrel{\text{no}}{=} \frac{1}{0} \end{aligned}$$

The possible solution $z = 4$ does not actually work, since it leads to division by 0 in the equation. It is an extraneous solution. However, $z = 1$ is a valid solution. The only solution to the equation is 1, and thus we can write the solution set as $\{1\}$.

Example 12.5.8 Solve for p in $\frac{3}{p-2} + \frac{5}{p+2} = \frac{12}{p^2-4}$.

Explanation. To find the common denominator, we need to factor all denominators if possible:

$$\frac{3}{p-2} + \frac{5}{p+2} = \frac{12}{(p+2)(p-2)}$$

Now we can see the common denominator is $(p+2)(p-2)$. We will multiply both sides of the equation by $(p+2)(p-2)$:

$$\begin{aligned} \frac{3}{p-2} + \frac{5}{p+2} &= \frac{12}{p^2-4} \\ \frac{3}{p-2} + \frac{5}{p+2} &= \frac{12}{(p+2)(p-2)} \\ (p+2)(p-2) \cdot \left(\frac{3}{p-2} + \frac{5}{p+2}\right) &= (p+2)(p-2) \cdot \frac{12}{(p+2)(p-2)} \\ (p+2)\cancel{(p-2)} \cdot \frac{3}{\cancel{p-2}} + \cancel{(p+2)}(p-2) \cdot \frac{5}{\cancel{p+2}} &= \cancel{(p+2)}\cancel{(p-2)} \cdot \frac{12}{\cancel{(p+2)}\cancel{(p-2)}} \\ 3(p+2) + 5(p-2) &= 12 \\ 3p + 6 + 5p - 10 &= 12 \\ 8p - 4 &= 12 \\ 8p &= 16 \\ p &= 2 \end{aligned}$$

Does the possible solution $p = 2$ check as an actual solution?

$$\begin{aligned} \frac{3}{2-2} + \frac{5}{2+2} &\stackrel{?}{=} \frac{12}{2^2-4} \\ \frac{3}{0} + \frac{5}{4} &\stackrel{\text{no}}{=} \frac{12}{0} \end{aligned}$$

The possible solution $p = 2$ does not actually work, since it leads to division by 0 in the equation. So this is an extraneous solution, and the equation actually has no solution. We could say that its solution set is the empty set, $\emptyset$.

Example 12.5.9 Solve $C(t) = 0.35$, where $C(t) = \frac{3t}{t^2+8}$ gives a drug's concentration in milligrams per liter t hours since an injection. (This function was explored in the introduction of Section 12.1.)

Explanation. To solve $C(t) = 0.35$, we'll begin by setting up $\frac{3t}{t^2+8} = 0.35$. We'll begin by identifying that the LCD is $t^2 + 8$, and multiply each side of the equation by this:

$$\begin{aligned}
\frac{3t}{t^2+8} &= 0.35 \\
\frac{3t}{\cancel{t^2+8}} \cdot \cancel{\left(t^2+8\right)} &= 0.35 \cdot \left(t^2+8\right) \\
3t &= 0.35\left(t^2+8\right) \\
3t &= 0.35t^2 + 2.8
\end{aligned}$$

This results in a quadratic equation so we will put it in standard form and use the quadratic formula:

$$\begin{aligned}
&0 = 0.35t^2 - 3t + 2.8 \\
&t = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(0.35)(2.8)}}{2(0.35)} \\
&t = \frac{3 \pm \sqrt{5.08}}{0.7} \\
&t \approx 1.066 \text{ or } t \approx 7.506
\end{aligned}$$

Each of these answers should be checked in the original equation; they both work. In context, this means that the drug concentration will reach 0.35 milligrams per liter about 1.066 hours after the injection was given, and again 7.506 hours after the injection was given.

12.5.2 Solving Rational Equations for a Specific Variable

Rational equations can contain many variables and constants and we can solve for any one of them. The process for solving still involves multiplying each side of the equation by the LCD. Instead of having a numerical answer though, our final result will contain other variables and constants.

Example 12.5.10 In physics, when two resistances, R_1 and R_2, are connected in a parallel circuit, the combined resistance, R, can be calculated by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Solve for R in this formula.

Explanation. The common denominator is RR_1R_2. We will multiply both sides of the equation by RR_1R_2:

$$\begin{aligned}
\frac{1}{R} &= \frac{1}{R_1} + \frac{1}{R_2} \\
\cancel{R}R_1R_2 \cdot \frac{1}{\cancel{R}} &= RR_1R_2 \cdot \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \\
R_1R_2 &= R\cancel{R_1}R_2 \cdot \frac{1}{\cancel{R_1}} + RR_1\cancel{R_2} \cdot \frac{1}{\cancel{R_2}} \\
R_1R_2 &= RR_2 + RR_1
\end{aligned}$$

$$\begin{aligned}
R_1R_2 &= R\left(R_2+R_1\right)\\
\frac{R_1R_2}{R_2+R_1} &= R\\
R &= \frac{R_1R_2}{R_1+R_2}
\end{aligned}$$

Example 12.5.11 Here is the slope formula

$$m=\frac{y_2-y_1}{x_2-x_1}.$$

Solve for x_1 in this formula.

Explanation. The common denominator is x_2-x_1. We will multiply both sides of the equation by x_2-x_1:

$$\begin{aligned}
m &= \frac{y_2-y_1}{x_2-x_1}\\
\left(x_2-x_1\right)\cdot m &= \cancel{\left(x_2-x_1\right)}\cdot\frac{y_2-y_1}{\cancel{x_2-x_1}}\\
mx_2-mx_1 &= y_2-y_1\\
-mx_1 &= y_2-y_1-mx_2\\
\frac{-mx_1}{-m} &= \frac{y_2-y_1-mx_2}{-m}\\
x_1 &= -\frac{y_2-y_1-mx_2}{m}
\end{aligned}$$

Example 12.5.12 Solve the rational equation $x=\frac{4y-1}{2y-3}$ for y.

Explanation. Our first step will be to multiply each side by the LCD, which is simply $2y-3$. After that, we'll isolate all terms containing y, factor out y, and then finish solving for that variable.

$$\begin{aligned}
x &= \frac{4y-1}{2y-3}\\
x\cdot(2y-3) &= \frac{4y-1}{\cancel{2y-3}}\cdot\cancel{(2y-3)}\\
2xy-3x &= 4y-1\\
2xy &= 4y-1+3x\\
2xy-4y &= -1+3x\\
y(2x-4) &= 3x-1\\
\frac{y(2x-4)}{2x-4} &= \frac{3x-1}{2x-4}\\
y &= \frac{3x-1}{2x-4}
\end{aligned}$$

12.5.3 Solving Rational Equations Using Technology

In some instances, it may be difficult to solve rational equations algebraically. We can instead use graphing technology to obtain approximate solutions. Let's look at one such example.

Example 12.5.13 Solve the equation $\frac{2}{x-3} = \frac{x^3}{8}$ using graphing technology.

Explanation.

We will define $f(x) = \frac{2}{x-3}$ and $g(x) = \frac{x^3}{8}$, and then look for the points of intersection.

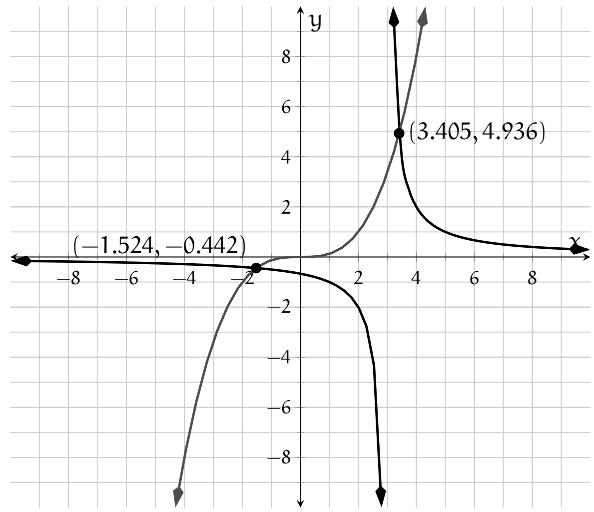


Figure 12.5.14: Graph of $f(x) = \frac{2}{x-3}$ and $g(x) = \frac{x^3}{8}$

Since the two functions intersect at approximately $(-1.524, -0.442)$ and $(3.405, 4.936)$, the solutions to $\frac{2}{x-3} = \frac{x^3}{8}$ are approximately -1.524 and 3.405. We can write the solution set as $\{-1.524\ldots, 3.405\ldots\}$ or in several other forms. It may be important to do *something* to communicate that these solutions are approximations. Here we used "...", but you could also just say in words that the solutions are approximate.

12.5.4 Reading Questions

1. Describe what an "extraneous solution" to a rational equation is.

2. In general, when solving a rational equation, multiplying through by the $\boxed{}$ will leave you with a simpler equation to solve.

3. When you believe you have the solutions to a rational equation, what is more important than usual (compared to other kinds of equations) for you to do?

12.5.5 Exercises

Review and Warmup Solve the equation.

1. $8A + 4 = A + 60$

2. $6C + 8 = C + 33$

3. $50 = 8 - 3(n - 6)$

4. $-3 = 4 - 7(p - 6)$

5. $2(x + 10) - 8(x - 2) = 36$

6. $4(y + 5) - 7(y - 7) = 72$

7. $(x - 6)^2 = 25$

8. $(x - 3)^2 = 144$

9. $x^2 + 2x - 3 = 0$

10. $x^2 - 3x - 10 = 0$

11. $x^2 + 15x + 61 = 17$

12. $x^2 + 9x + 31 = 17$

13. Recall the time that Filip traveled with his kids to kick a soccer ball on Mars? We should examine one more angle to our soccer kick question. The formula $H(t) = -6.07t^2 + 27.1t$ finds the height of the soccer ball in feet above the ground at a time t seconds after being kicked.

(a) Using technology, find out what the maximum height of the ball was and when it reached that height.

(b) Using technology, solve for when $H(t) = 20$ and interpret the meaning of this in a complete sentence.

(c) Using technology, solve for when $H(t) = 0$ and interpret the meaning of this in a complete sentence.

Solving Rational Equations Solve the equation.

14. $\dfrac{20}{t} = -5$

15. $\dfrac{40}{x} = 8$

16. $\dfrac{x}{x+3} = 4$

17. $\dfrac{x}{x+4} = -3$

18. $\dfrac{y+10}{3y-8} = \dfrac{9}{8}$

19. $\dfrac{y-4}{3y-2} = \dfrac{3}{4}$

20. $\dfrac{-7r+4}{r-4} = -\dfrac{7r}{r-8}$

21. $\dfrac{2r+9}{r+8} = \dfrac{2r}{r+7}$

22. $\dfrac{3}{t} = 4 + \dfrac{23}{t}$

23. $\dfrac{3}{t} = -2 + \dfrac{13}{t}$

24. $\dfrac{5}{4x} + \dfrac{1}{3x} = -5$

25. $\dfrac{3}{5y} + \dfrac{4}{3y} = -5$

26. $\dfrac{x}{5x-30} - \dfrac{2}{x-6} = 1$

27. $\dfrac{y}{2y+12} + \dfrac{6}{y+6} = 2$

28. $\dfrac{y-2}{y^2+2} = 0$

29. $\dfrac{r-9}{r^2+6} = 0$

30. $\dfrac{3}{r} = 0$

31. $-\dfrac{3}{t} = 0$

32. $\dfrac{t+9}{t^2+15t+54} = 0$

33. $\dfrac{t+3}{t^2-2t-15} = 0$

34. $-\dfrac{2}{x} + \dfrac{8}{x+7} = -1$

35. $-\dfrac{8}{x} - \dfrac{5}{x+9} = 1$

36. $\dfrac{1}{y+2} + \dfrac{2}{y^2+2y} = \dfrac{1}{5}$

37. $\dfrac{1}{y-5} - \dfrac{5}{y^2-5y} = -\dfrac{1}{9}$

38. $\dfrac{1}{r+9} - \dfrac{5}{r^2+9r} = \dfrac{1}{2}$

39. $\dfrac{1}{r-7} - \dfrac{4}{r^2-7r} = \dfrac{1}{2}$

40. $\dfrac{t-4}{t+8} + \dfrac{2}{t+3} = -4$

41. $\dfrac{t+9}{t-3} + \dfrac{4}{t-8} = 3$

Solve the equation.

42. $\dfrac{2}{t+1} = \dfrac{3}{t-1} - \dfrac{2}{t^2-1}$

43. $-\dfrac{4}{x+3} = -\left(\dfrac{2}{x-3} + \dfrac{6}{x^2-9}\right)$

44. $\dfrac{8}{x+6} - \dfrac{9}{x+9} = -\dfrac{9}{x^2+15x+54}$

45. $\dfrac{2}{y+5} - \dfrac{5}{y+1} = -\dfrac{2}{y^2+6y+5}$

46. $-\dfrac{8}{y-6} + \dfrac{4y}{y-5} = -\dfrac{8}{y^2-11y+30}$

47. $\dfrac{4}{r-7} + \dfrac{2r}{r-5} = \dfrac{8}{r^2-12r+35}$

48. $-\dfrac{6}{r-3} + \dfrac{8r}{r+9} = -\dfrac{4}{r^2+6r-27}$

49. $\dfrac{6}{t-7} + \dfrac{2t}{t-3} = -\dfrac{8}{t^2-10t+21}$

Solving Rational Equations for a Specific Variable

50. Solve this equation for a:
$p = \frac{m}{a}$

51. Solve this equation for m:
$q = \frac{b}{m}$

52. Solve this equation for x:
$y = \frac{x}{r}$

53. Solve this equation for C:
$r = \frac{C}{B}$

54. Solve this equation for a:
$\frac{1}{2a} = \frac{1}{x}$

55. Solve this equation for c:
$\frac{1}{8c} = \frac{1}{q}$

56. Solve this equation for B:
$\frac{1}{A} = \frac{8}{B+2}$

57. Solve this equation for a:
$\frac{1}{C} = \frac{8}{a+6}$

Solving Rational Equations Using Technology Use technology to solve the equation.

58.
$$\frac{10}{x^2+3} = \frac{x+1}{x+5}.$$

59.
$$\frac{x-9}{x^5+1} = -3x-7.$$

60.
$$\frac{1}{x} + \frac{1}{x^2} = \frac{1}{x^3}.$$

61.
$$\frac{12x}{x-5} + \frac{3}{x+1} = \frac{x-5}{x^2}.$$

62.
$$2x - \frac{1}{x+4} = \frac{3}{x+6}.$$

63.
$$\frac{1}{x^2-1} - \frac{2}{x-4} = \frac{3}{x-2}.$$

Application Problems

64. Scot and Jay are working together to paint a room. If Scot paints the room alone, it would take him 18 hours to complete the job. If Jay paints the room alone, it would take him 12 hours to complete the job. Answer the following question:

If they work together, it would take them ______ hours to complete the job. Use a decimal in your answer if needed.

65. There are three pipes at a tank. To fill the tank, it would take Pipe A 3 hours, Pipe B 12 hours, and Pipe C 4 hours. Answer the following question:

If all three pipes are turned on, it would take ______ hours to fill the tank.

66. Casandra and Tien are working together to paint a room. Casandra works 1.5 times as fast as Tien does. If they work together, it took them 9 hours to complete the job. Answer the following questions:

If Casandra paints the room alone, it would take her ______ hours to complete the job.

If Tien paints the room alone, it would take him ______ hours to complete the job.

67. Two pipes are being used to fill a tank. Pipe A can fill the tank 4.5 times as fast as Pipe B does. When both pipes are turned on, it takes 18 hours to fill the tank. Answer the following questions:

If only Pipe A is turned on, it would take ______ hours to fill the tank.

If only Pipe B is turned on, it would take ______ hours to fill the tank.

68. Kandace and Jenny worked together to paint a room, and it took them 2 hours to complete the job. If they work alone, it would take Jenny 3 more hours than Kandace to complete the job. Answer the following questions:

If Kandace paints the room alone, it would take her ______ hours to complete the job.

If Jenny paints the room alone, it would take her $\boxed{}$ hours to complete the job.

69. If both Pipe A and Pipe B are turned on, it would take 2 hours to fill a tank. If each pipe is turned on alone, it takes Pipe B 3 fewer hours than Pipe A to fill the tank. Answer the following questions:

If only Pipe A is turned on, it would take $\boxed{}$ hours to fill the tank.

If only Pipe B is turned on, it would take $\boxed{}$ hours to fill the tank.

70. Town A and Town B are 570 miles apart. A boat traveled from Town A to Town B, and then back to Town A. Since the river flows from Town B to Town A, the boat's speed was 25 miles per hour faster when it traveled from Town B to Town A. The whole trip took 19 hours. Answer the following questions:

The boat traveled from Town A to Town B at the speed of $\boxed{}$ miles per hour.

The boat traveled from Town B back to Town A at the speed of $\boxed{}$ miles per hour.

71. A river flows at 7 miles per hour. A boat traveled with the current from Town A to Town B, which are 260 miles apart. Then, the boat turned around, and traveled against the current to reach Town C, which is 160 miles away from Town B. The second leg of the trip (Town B to Town C) took the same time as the first leg (Town A to Town B). During this whole trip, the boat was driving at a constant still-water speed. Answer the following question:

During this trip, the boat's speed on still water was $\boxed{}$ miles.

72. A river flows at 5 miles per hour. A boat traveled with the current from Town A to Town B, which are 100 miles apart. The boat stayed overnight at Town B. The next day, the water's current stopped, and boat traveled on still water to reach Town C, which is 190 miles away from Town B. The second leg of the trip (Town B to Town C) took 9 hours longer than the first leg (Town A to Town B). During this whole trip, the boat was driving at a constant still-water speed. Find this speed.

Note that you should not consider the unreasonable answer.

During this trip, the boat's speed on still water was $\boxed{}$ miles per hour.

73. Town A and Town B are 600 miles apart. With a constant still-water speed, a boat traveled from Town A to Town B, and then back to Town A. During this whole trip, the river flew from Town A to Town B at 20 miles per hour. The whole trip took 16 hours. Answer the following question:

During this trip, the boat's speed on still water was $\boxed{}$ miles per hour.

74. Town A and Town B are 350 miles apart. With a constant still-water speed of 24 miles per hour, a boat traveled from Town A to Town B, and then back to Town A. During this whole trip, the river flew from Town B to Town A at a constant speed. The whole trip took 30 hours. Answer the following question:

During this trip, the river's speed was $\boxed{}$ miles per hour.

75. Suppose that a large pump can empty a swimming pool in 43 hr and that a small pump can empty the same pool in 53 hr. If both pumps are used at the same time, how long will it take to empty the pool?

If both pumps are used at the same time, it will take $\boxed{}$ to empty the pool.

76. The winner of a 9 mi race finishes 14.73 min ahead of the second-place runner. On average, the winner ran 0.6 $\frac{\text{mi}}{\text{hr}}$ faster than the second place runner. Find the average running speed for each runner.

The winner's average speed was [ ] and the second-place runner's average speed was [ ].

77. In still water a tugboat can travel 15 $\frac{\text{mi}}{\text{hr}}$. It travels 42 mi upstream and then 42 mi downstream in a total time of 5.96 hr. Find the speed of the current.

The current's speed is [ ].

78. Without any wind an airplane flies at 300 $\frac{\text{mi}}{\text{hr}}$. The plane travels 600 mi into the wind and then returns with the wind in a total time of 4.04 hr. Find the average speed of the wind.

The wind's speed is [ ].

79. When there is a 11.8 $\frac{\text{mi}}{\text{hr}}$ wind, an airplane can fly 770 mi with the wind in the same time that it can fly 702 mi against the wind. Find the speed of the plane when there is no wind.

The plane's airspeed is [ ].

80. It takes one employee 2.5 hr longer to mow a football field than it does a more experienced employee. Together they can mow the grass in 1.9 hr. How long does it take each person to mow the football field working alone?

The more experienced worker takes [ ] to mow the field alone, and the less experienced worker takes [ ].

81. It takes one painter 13 hr longer to paint a house than it does a more experienced painter. Together they can paint the house in 30 hr. How long does it take for each painter to paint the house working alone?

The more experienced painter takes [ ] to paint the house alone, and the less experienced painter takes [ ].

12.6 Rational Functions and Equations Chapter Review

12.6.1 Introduction to Rational Functions

In Section 12.1 we learned about rational functions and explored them with tables and graphs.

Example 12.6.1 Graphs of Rational Functions. In an apocalypse, a zombie infestation begins with 1 zombie and spreads rapidly. The population of zombies can be modeled by $Z(x) = \frac{200000x+100}{5x+100}$, where x is the number of days after the apocalypse began. Use technology to graph the function and answer these questions:

a. How many zombies are there 2 days after the apocalypse began?

b. After how many days will the zombie population be 20,000?

c. As time goes on, the population will level off at about how many zombies?

Explanation. We will graph the function with technology. After adjusting window settings, we have:

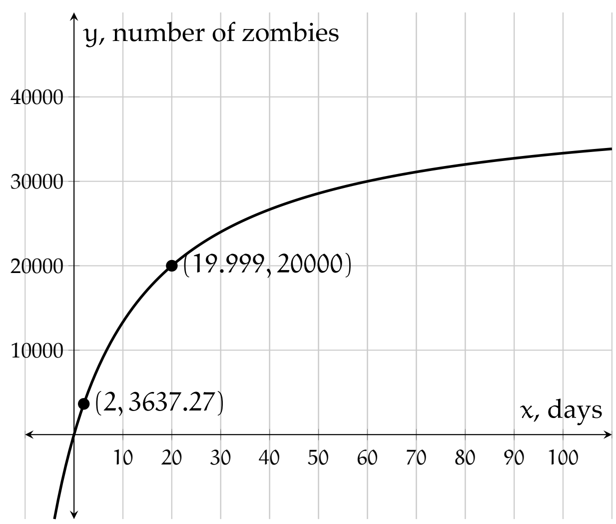


Figure 12.6.2: Graph of $y = Z(x) = \frac{200000x+100}{5x+100}$

a. To find the number of zombies after 2 days, we locate the point $(2, 3637.27)$. Since we can only have a whole number of zombies, we round to 3,637 zombies.

b. To find the number of days it will take for the zombie population reach 20,000, we locate the point $(19.999, 20000)$ so it will take about 20 days.

c. When we look far to the right on the graph using technology we can see that the population will level off at about 40,000 zombies.

12.6.2 Multiplication and Division of Rational Expressions

In Section 12.2 we covered how to simplify rational expressions. It is very important to list any domain restrictions from factors that are canceled. We also multiplied and divided rational expressions.

Example 12.6.3 Simplifying Rational Expressions. Simplify the expression $\frac{8t+4t^2-12t^3}{1-t}$.

Explanation. To begin simplifying this expression, we will rewrite each polynomial in descending order. Then we'll factor out the GCF, including the constant -1 from both the numerator and denominator because their leading terms are negative.

$$\begin{aligned}\frac{8t+4t^2-12t^3}{1-t} &= \frac{-12t^3+4t^2+8t}{-t+1}\\ &= \frac{-4t(3t^2-t-2)}{-(t-1)}\\ &= \frac{-4t(3t+2)(t-1)}{-(t-1)}\\ &= \frac{-4t(3t+2)\cancel{(t-1)}}{-\cancel{(t-1)}}\\ &= \frac{-4t(3t+2)}{-1},\ \text{for } t\neq 1\\ &= 4t(3t+2),\ \text{for } t\neq 1\end{aligned}$$

Example 12.6.4 Multiplication of Rational Functions and Expressions. Multiply the rational expressions: $\frac{r^3s}{4t}\cdot\frac{2t^2}{r^2s^3}$.

Explanation. Note that we won't need to factor anything in this problem, and can simply multiply across and then simplify. With multivariable expressions, this textbook ignores domain restrictions.

$$\begin{aligned}\frac{r^3s}{4t}\cdot\frac{2t^2}{r^2s^3} &= \frac{r^3s\cdot 2t^2}{4t\cdot r^2s^3}\\ &= \frac{2r^3st^2}{4r^2s^3t}\\ &= \frac{rt}{2s^2}\end{aligned}$$

Example 12.6.5 Division of Rational Functions and Expressions. Divide the rational expressions: $\frac{2x^2+8xy}{x^2-4x+3}\div\frac{x^3+4x^2y}{x^2+4x-5}$.

Explanation. To divide rational expressions, we multiply by the reciprocal of the second fraction. Then we will factor and cancel any common factors. With multivariable expressions, this textbook ignores domain restrictions.

$$\begin{aligned}\frac{2x^2+8xy}{x^2-4x+3}\div\frac{x^3+4x^2y}{x^2+4x-5} &= \frac{2x^2+8xy}{x^2-4x+3}\cdot\frac{x^2+4x-5}{x^3+4x^2y}\\ &= \frac{2x\cancel{(x+4y)}}{\cancel{(x-1)}(x-3)}\cdot\frac{\cancel{(x-1)}(x+5)}{x^2\cancel{(x+4y)}}\\ &= \frac{2x}{x-3}\cdot\frac{x+5}{x^2}\end{aligned}$$

$$= \frac{2(x+5)}{x(x-3)}$$

12.6.3 Addition and Subtraction of Rational Expressions

In Section 12.3 we covered how to add and subtract rational expressions.

Example 12.6.6 Addition and Subtraction of Rational Expressions with the Same Denominator. Add the rational expressions: $\frac{5x}{x+5} + \frac{25}{x+5}$.

Explanation. These expressions already have a common denominator:

$$\begin{aligned}\frac{5x}{x+5} + \frac{25}{x+5} &= \frac{5x+25}{x+5}\\ &= \frac{5\cancel{(x+5)}}{\cancel{x+5}}\\ &= \frac{5}{1}, \text{ for } x \neq -5\\ &= 5, \text{ for } x \neq -5\end{aligned}$$

Note that we didn't stop at $\frac{5x+25}{x+5}$. If possible, we must simplify the numerator and denominator.

Example 12.6.7 Addition and Subtraction of Rational Expressions with Different Denominators. Add and subtract the rational expressions: $\frac{6y}{y+2} + \frac{y}{y-2} - 7$

Explanation. The denominators can't be factored, so we'll find the least common denominator and build each expression to that denominator. Then we will be able to combine the numerators and simplify the expression.

$$\begin{aligned}\frac{6y}{y+2} + \frac{y}{y-2} - 7 &= \frac{6y}{y+2} \cdot \frac{y-2}{y-2} + \frac{y}{y-2} \cdot \frac{y+2}{y+2} - 7 \cdot \frac{(y-2)(y+2)}{(y-2)(y+2)}\\ &= \frac{6y(y-2)}{(y-2)(y+2)} + \frac{y(y+2)}{(y-2)(y+2)} - \frac{7(y-2)(y+2)}{(y-2)(y+2)}\\ &= \frac{6y^2 - 12y + y^2 + 2y - (7(y^2-4))}{(y-2)(y+2)}\\ &= \frac{6y^2 - 12y + y^2 + 2y - 7y^2 + 28}{(y-2)(y+2)}\\ &= \frac{-10y+28}{(y-2)(y+2)}\\ &= \frac{-2(5y-14)}{(y-2)(y+2)}\end{aligned}$$

12.6.4 Complex Fractions

In Section 12.4 we covered how to simplify a rational expression that has fractions in the numerator and/or denominator.

Example 12.6.8 Simplifying Complex Fractions. Simplify the complex fraction $\frac{\frac{2t}{t^2-9}+3}{\frac{6}{t+3}+\frac{1}{t-3}}$.

Explanation. First, we check all quadratic polynomials to see if they can be factored and factor them:

$$\frac{\frac{2t}{t^2-9}+3}{\frac{6}{t+3}+\frac{1}{t-3}} = \frac{\frac{2t}{(t-3)(t+3)}+3}{\frac{6}{t+3}+\frac{1}{t-3}}$$

Next, we identify the common denominator of the three fractions, which is $(t+3)(t-3)$. We then multiply the main numerator and denominator by that expression:

$$\begin{aligned}
\frac{\frac{2t}{(t-3)(t+3)}+3}{\frac{6}{t+3}+\frac{1}{t-3}} &= \frac{\frac{2t}{(t-3)(t+3)}+3}{\frac{6}{t+3}+\frac{1}{t-3}} \cdot \frac{(t-3)(t+3)}{(t-3)(t+3)}\\
&= \frac{\frac{2t}{\cancel{(t-3)(t+3)}}\cancel{(t-3)(t+3)}+3(t-3)(t+3)}{\frac{6}{\cancel{t+3}}(t-3)\cancel{(t+3)}+\frac{1}{\cancel{t-3}}\cancel{(t-3)}(t+3)}\\
&= \frac{2t+3(t-3)(t+3)}{6(t-3)+1(t+3)} \text{ for } t \neq -3, t \neq 3\\
&= \frac{2t+3(t^2-9)}{6t-18+t+3} \text{ for } t \neq -3, t \neq 3\\
&= \frac{2t+3t^2-27}{7t-15} \text{ for } t \neq -3, t \neq 3\\
&= \frac{3t^2+2t-27}{7t-15} \text{ for } t \neq -3, t \neq 3
\end{aligned}$$

Both the numerator and denominator are prime polynomials so this expression can neither factor nor simplify any further.

12.6.5 Solving Rational Equations

In Section 12.5 we covered how to solve rational equations. We looked at rate problems, solved for a specified variable and used technology to solve rational equations.

Example 12.6.9 Solving Rational Equations. Two pipes are being used to fill a large tank. Pipe B can fill the tank twice as fast as Pipe A can. When both pipes are turned on, it takes 12 hours to fill the tank. Write and solve a rational equation to answer the following questions:

a. If only Pipe A is turned on, how many hours would it take to fill the tank?

b. If only Pipe B is turned on, how many hours would it take to fill the tank?

Explanation. Since both pipes can fill the tank in 12 hours, they fill $\frac{1}{12}$ of the tank together each hour. We will let a represent the number of hours it takes pipe A to fill the tank alone, so pipe A will fill $\frac{1}{a}$ of the tank each hour. Pipe B can fill the tank twice as fast so it fills $2 \cdot \frac{1}{a}$ of the tank each hour or $\frac{2}{a}$. When they are both turned on, they fill $\frac{1}{a} + \frac{2}{a}$ of the tank each hour.

Now we can write this equation:

$$\frac{1}{a} + \frac{2}{a} = \frac{1}{12}$$

To clear away denominators, we multiply both sides of the equation by the common denominator of 12 and a, which is $12a$:

$$\begin{aligned} \frac{1}{a} + \frac{2}{a} &= \frac{1}{12} \\ 12a \cdot \left(\frac{1}{a} + \frac{2}{a}\right) &= 12a \cdot \frac{1}{12} \\ 12a \cdot \frac{1}{a} + 12a \cdot \frac{2}{a} &= 12a \cdot \frac{1}{12} \\ 12 + 24 &= a \\ 36 &= a \\ a &= 36 \end{aligned}$$

The possible solution $a = 36$ should be checked

$$\begin{aligned} \frac{1}{36} + \frac{2}{36} &\stackrel{?}{=} \frac{1}{12} \\ \frac{3}{36} &\stackrel{\checkmark}{=} \frac{1}{12} \end{aligned}$$

So it is a solution.

a. If only Pipe A is turned on, it would take 36 hours to fill the tank.

b. Since Pipe B can fill the tank twice as fast, it would take half the time, or 18 hours to fill the tank.

Example 12.6.10 Solving Rational Equations for a Specific Variable. Solve the rational equation $y = \frac{2x+5}{3x-1}$ for x.

Explanation. To get the x out of the denominator, our first step will be to multiply each side by the LCD, which is $3x-1$. Then we'll isolate all terms containing x, factor out x, and then finish solving for that variable.

$$\begin{aligned} y &= \frac{2x+5}{3x-1} \\ y \cdot (3x-1) &= \frac{2x+5}{\cancel{3x-1}} \cdot \cancel{(3x-1)} \\ 3xy - y &= 2x + 5 \\ 3xy &= 2x + 5 + y \\ 3xy - 2x &= y + 5 \\ x(3y-2) &= y + 5 \\ \frac{x(3y-2)}{3y-2} &= \frac{y+5}{3y-2} \\ x &= \frac{y+5}{3y-2} \end{aligned}$$

Example 12.6.11 Solving Rational Equations Using Technology. Solve the equation $\frac{1}{x+2}+1=\frac{10x}{x^2+5}$ using graphing technology.

Explanation.

We will define $f(x)=\frac{1}{x+2}+1$ and $g(x)=\frac{10x}{x^2+5}$, and then find a window where we can see all of the points of intersection.

Since the two functions intersect at approximately $(-2.309,-2.235)$, $(0.76,1.362)$ and $(8.549,1.095)$, the solutions to $\frac{1}{x+2}+1=\frac{10x}{x^2+5}$ are approximately -2.309, 0.76 and 8.549. The solution set is approximately $\{-2.309\ldots, 0.76\ldots, 8.549\ldots\}$.

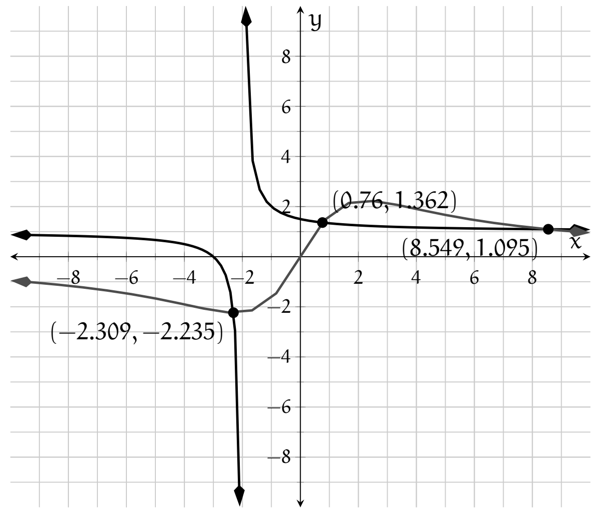
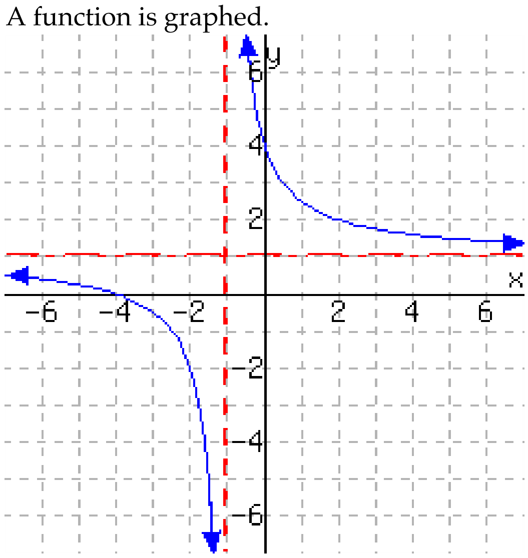


Figure 12.6.12: Graph of $f(x)=\frac{1}{x+2}+1$ and $g(x)=\frac{10x}{x^2+5}$

12.6.6 Exercises

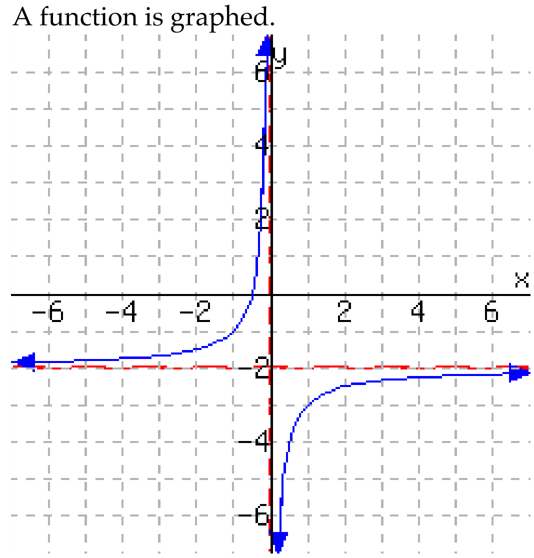
Introduction to Rational Functions

1. A function is graphed.



This function has domain [ ] and range [ ].

2. A function is graphed.



This function has domain [ ] and range [ ].

3. The population of deer in a forest can be modeled by

$$P(x) = \frac{3220x + 2940}{7x + 6}$$

where x is the number of years in the future. Answer the following questions.

a. How many deer live in this forest this year?

b. How many deer will live in this forest 17 years later? Round your answer to an integer.

c. After how many years, the deer population will be 461? Round your answer to an integer.

d. Use a calculator to answer this question: As time goes on, the population levels off at about how many deer?

4. The population of deer in a forest can be modeled by

$$P(x) = \frac{2400x + 1850}{8x + 5}$$

where x is the number of years in the future. Answer the following questions.

a. How many deer live in this forest this year?

b. How many deer will live in this forest 5 years later? Round your answer to an integer.

c. After how many years, the deer population will be 302? Round your answer to an integer.

d. Use a calculator to answer this question: As time goes on, the population levels off at about how many deer?

5. In a certain store, cashiers can serve 55 customers per hour on average. If x customers arrive at the store in a given hour, then the average number of customers C waiting in line can be modeled by the function

$$C(x) = \frac{x^2}{3025 - 55x}$$

where $x < 55$.
Answer the following questions with a graphing calculator. Round your answers to integers.

a. If 41 customers arrived in the store in the past hour, there are approximately ________ customers waiting in line.

b. If there are 7 customers waiting in line, approximately ________ customers arrived in the past hour.

6. In a certain store, cashiers can serve 60 customers per hour on average. If x customers arrive at the store in a given hour, then the average number of customers C waiting in line can be modeled by the function

$$C(x) = \frac{x^2}{3600 - 60x}$$

where $x < 60$.
Answer the following questions with a graphing calculator. Round your answers to integers.

a. If 48 customers arrived in the store in the past hour, there are approximately ________ customers waiting in line.

b. If there are 8 customers waiting in line, approximately ________ customers arrived in the past hour.

7. The concentration of a drug in a patient's blood stream, in milligrams per liter, can be modeled by the function $C(t) = \frac{2t}{t^2+7}$, where t is the number of hours since the drug is injected. Answer the following question with technology. Round your answer to two decimal places if needed.

[ ] hours since injection, the drug's concentration is at the maximum value of [ ] milligrams per liter.

8. The concentration of a drug in a patient's blood stream, in milligrams per liter, can be modeled by the function $C(t) = \frac{3t}{t^2+5}$, where t is the number of hours since the drug is injected. Answer the following question with technology. Round your answer to two decimal places if needed.

[ ] hours since injection, the drug's concentration is at the maximum value of [ ] milligrams per liter.

Multiplication and Division of Rational Expressions

9. Simplify this expression.
$$\frac{-y^2 - 2yt + 3t^2}{y^2 - 9t^2}$$

10. Simplify this expression.
$$\frac{-y^2 + 5yx - 6x^2}{y^2 - 4x^2}$$

11. Simplify the function formula, and if applicable, write the restricted domain.
$$f(r) = \frac{r^4 + 8r^3 + 16r^2}{5r^4 + 19r^3 - 4r^2}$$
Reduced $f(r) =$ [ ]

12. Simplify the function formula, and if applicable, write the restricted domain.
$$H(r) = \frac{r^4 - 6r^3 + 9r^2}{3r^4 - 10r^3 + 3r^2}$$
Reduced $H(r) =$
[ ]

13. Simplify this expression, and if applicable, write the restricted domain.
$$\frac{t^2 - 16t}{t^2 - 16} \cdot \frac{t^2 - 4t}{t^2 - 13t - 48}$$

14. Simplify this expression, and if applicable, write the restricted domain.
$$\frac{t^2 - 4t}{t^2 - 4} \cdot \frac{t^2 - 2t}{t^2 + 5t - 36}$$

15. Simplify this expression, and if applicable, write the restricted domain.
$$\frac{9t^2 - 49}{3t^2 + t + (-14)} \div (7 - 3t)$$

16. Simplify this expression, and if applicable, write the restricted domain.
$$\frac{25x^2 - 16}{5x^2 + (-1)x + (-4)} \div (4 - 5x)$$

17. Simplify this expression.
$$\frac{x^5}{x^2y - 5x} \div \frac{1}{x^2y^2 - 8xy + 15}$$

18. Simplify this expression.
$$\frac{y^3}{y^2r + 3y} \div \frac{1}{y^2r^2 + 7yr + 12}$$

Addition and Subtraction of Rational Expressions Add or subtract the rational expressions to a single rational expression and then simplify. If applicable, state the restricted domain.

19. $\dfrac{1}{y+1} + \dfrac{2}{y^2 - 1}$

20. $\dfrac{1}{r+2} + \dfrac{4}{r^2 - 4}$

21. $-\dfrac{15r}{r^2 + 9r + 18} + \dfrac{5r}{r+3}$

22. $-\dfrac{12t}{t^2 + t - 2} - \dfrac{4t}{t+2}$

23. $\dfrac{t^2 - 20}{t^2 - 4t} - \dfrac{t+5}{t}$

24. $\dfrac{t^2 + 15}{t^2 - 3t} - \dfrac{t-5}{t}$

Add or subtract the rational expressions to a single rational expression and then simplify.

25. $-\frac{4x}{3y^4} + \frac{2}{5xy}$

26. $-\frac{2x}{3t^5} - \frac{6}{5xt}$

27. $-\frac{20yr}{y^2 - 8yr + 12r^2} + \frac{5y}{y - 6r}$

28. $\frac{10yx}{y^2 + 7yx + 6x^2} - \frac{2y}{y + x}$

Complex Fractions

29. Calculate the following. Use an improper fraction in your answer.

a. $\frac{\frac{10}{9}}{\frac{5}{4}}$

b. $\frac{\frac{r}{t}}{\frac{y}{x}}$

30. Calculate the following. Use an improper fraction in your answer.

a. $\frac{\frac{25}{7}}{\frac{5}{4}}$

b. $\frac{\frac{r}{t}}{\frac{y}{x}}$

31. Simplify this expression, and if applicable, write the restricted domain.
$\frac{\frac{2}{q-1} - 3}{\frac{1}{q-1} + \frac{1}{q-3}}$

32. Simplify this expression, and if applicable, write the restricted domain.
$\frac{\frac{9}{n-1} - 7}{\frac{1}{n-1} + \frac{1}{n-10}}$

33. Simplify this expression, and if applicable, write the restricted domain.
$\frac{\frac{2t}{t^2-36} - 5}{\frac{3}{t+6} + \frac{4}{t-6}}$

34. Simplify this expression, and if applicable, write the restricted domain.
$\frac{\frac{3x}{x^2-9} - 2}{\frac{2}{x+3} - \frac{3}{x-3}}$

35. Simplify this expression.
$\frac{\frac{x}{y}}{\frac{6x}{5y^2}}$

36. Simplify this expression.
$\frac{\frac{a}{b}}{\frac{4a}{3b^2}}$

37. Simplify this expression.
$\frac{\frac{5}{y}}{20 - 5x}$

38. Simplify this expression.
$\frac{\frac{3}{r}}{3 - \frac{3x}{2}}$

Solving Rational Equations Solve the equation.

39. $\frac{3}{r+3} - \frac{5}{r+9} = \frac{4}{r^2 + 12r + 27}$

40. $\frac{5}{r+2} - \frac{7}{r+9} = -\frac{1}{r^2 + 11r + 18}$

41. $\frac{1}{t+4} + \frac{4}{t^2 + 4t} = \frac{1}{4}$

42. $\frac{1}{t-3} - \frac{3}{t^2 - 3t} = \frac{1}{8}$

43. $-\frac{2}{x-2} + \frac{2x}{x+7} = \frac{6}{x^2 + 5x - 14}$

44. $\frac{6}{x-8} + \frac{9x}{x+4} = \frac{3}{x^2 - 4x - 32}$

45. $\frac{y-6}{y-9} - \frac{7}{y+3} = 2$

46. $\frac{y-3}{y+9} + \frac{5}{y+7} = 2$

47. Solve this equation for B:
$\frac{1}{A} = \frac{9}{B+8}$

48. Solve this equation for t:
$\frac{1}{B} = \frac{4}{t+7}$

49. Use technology to solve the equation

$$2x - \frac{1}{x+4} = \frac{3}{x+6}.$$

50. Use technology to solve the equation

$$\frac{1}{x^2-1} - \frac{2}{x-4} = \frac{3}{x-2}.$$

51. Two pipes are being used to fill a tank. Pipe A can fill the tank 4.5 times as fast as Pipe B does. When both pipes are turned on, it takes 18 hours to fill the tank. Answer the following questions:

If only Pipe A is turned on, it would take ______ hours to fill the tank.

If only Pipe B is turned on, it would take ______ hours to fill the tank.

52. Two pipes are being used to fill a tank. Pipe A can fill the tank 5.5 times as fast as Pipe B does. When both pipes are turned on, it takes 11 hours to fill the tank. Answer the following questions:

If only Pipe A is turned on, it would take ______ hours to fill the tank.

If only Pipe B is turned on, it would take ______ hours to fill the tank.

53. Town A and Town B are 580 miles apart. A boat traveled from Town A to Town B, and then back to Town A. Since the river flows from Town B to Town A, the boat's speed was 30 miles per hour faster when it traveled from Town B to Town A. The whole trip took 29 hours. Answer the following questions:

The boat traveled from Town A to Town B at the speed of ______ miles per hour.

The boat traveled from Town B back to Town A at the speed of ______ miles per hour.

54. Town A and Town B are 390 miles apart. A boat traveled from Town A to Town B, and then back to Town A. Since the river flows from Town B to Town A, the boat's speed was 25 miles per hour faster when it traveled from Town B to Town A. The whole trip took 13 hours. Answer the following questions:

The boat traveled from Town A to Town B at the speed of ______ miles per hour.

The boat traveled from Town B back to Town A at the speed of ______ miles per hour.

Chapter 13

Graphs and Equations

13.1 Overview of Graphing

In this section, we will review how to graph lines and general functions which will be useful when we graph parabolas in the next section.

13.1.1 Graphing Lines by Plotting Points

Sometimes, the easiest way to make a graph of an equation is by making a table and plotting points. (This was the approach in Section 3.2.) Let's refresh ourselves on how this works.

Example 13.1.2 A bathtub is holding 12 gallons of water. The drain starts to leak water at a constant rate of 0.6 gallons per second. A linear function with formula $W(x) = -0.6x + 12$ can be used to model the amount of water, in gallons, in the tub x seconds after it started draining. Let's make a graph of this function. The most straightforward method to graph any function is to build a table of x- and y-values, and then plot the points.

x	$W(x) = -0.6x + 12$	Point	Interpretation
0	$-0.6(0) + 12$ $= 12$	$(0, 12)$	There were 12 gallons of water in the tub when the tub started to drain.
1	$-0.6(1) + 12$ $= 11.4$	$(1, 11.4)$	There were 11.4 gallons in the tub 1 second after the tub started to drain.
2	$-0.6(2) + 12$ $= 10.8$	$(2, 10.8)$	There were 10.8 gallons in the tub 2 seconds after the tub started to drain.
3	$-0.6(3) + 12$ $= 10.2$	$(3, 10.2)$	There were 10.2 gallons in the tub 3 seconds after the tub started to drain.
4	$-0.6(4) + 12$ $= 9.6$	$(4, 9.6)$	There were 9.6 gallons in the tub 4 seconds after the tub started to drain.

Figure 13.1.3: A table of values for $W(x) = -0.6x + 12$

Could we have made a more helpful table? Maybe. The y-values are close together and for the most part they are decimals which can be difficult to plot accurately. No matter, for now we use these points and make a plot.

The advantage of plotting points is that it is a universal method to graph any function. It is easy to forget about this method after learning faster ways to graph functions, so to keep this method in your mathematical tool box in case you come across something that you don't know or remember how to graph.

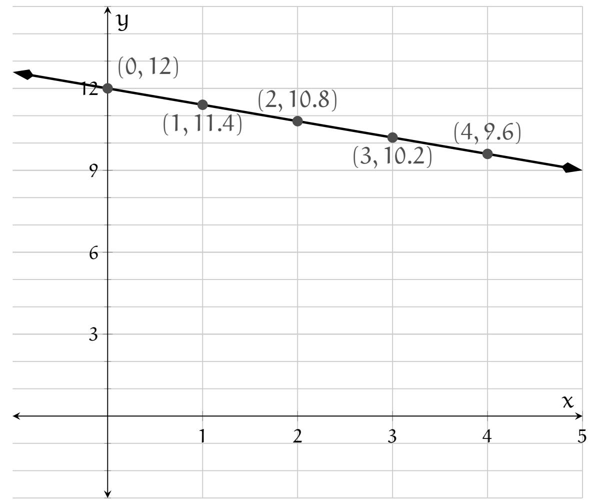


Figure 13.1.4: A graph of $W(x) = -0.6x + 12$

Checkpoint 13.1.5 Make a table for the equation.

x	$y = \frac{11}{5}x - 8$

Explanation. Since this equation has a fractional coefficient for x with denominator 5, it would be wise to choose our own x-values that are multiples of 5. Then when we use them to solve for y, the denominator will be cleared, and we will not need to continue with fraction arithmetic.

This solution will use the x-values -5, 0, 5, 10 and 15. The choice to use these x-values is arbitrary, but they are small multiples of 5, which will make computation easier.

One at a time, we substitute these x-values into the equation $y = \frac{11}{5}x - 8$, and solve for y:

$$\begin{aligned}
y &= \frac{11}{5}(-5) - 8 \implies y = -19\\
y &= \frac{11}{5}(0) - 8 \implies y = -8\\
y &= \frac{11}{5}(5) - 8 \implies y = 3\\
y &= \frac{11}{5}(10) - 8 \implies y = 14\\
y &= \frac{11}{5}(15) - 8 \implies y = 25
\end{aligned}$$

So the table may be completed as:

x	y
-5	-19
0	-8
5	3
10	14
15	25

13.1.2 Graphing Lines in Slope-Intercept Form

Recall that the slope-intercept form (3.5.1) of a line equation is $y = mx + b$ where m is the slope and $(0, b)$ is the vertical intercept.

Example 13.1.6 An efficient method to graph $y = -0.6x + 12$ is to use the fact that it is in slope-intercept form. To quickly make a graph, examine the equation and pick out the slope (in this case -0.6) and vertical intercept (in this case $(0, 12)$), and then plot slope-triangles from the intercept to locate more points on the line. One key point here is that it helps to have the slope written as a fraction. In this case,

$$-0.6 = -\frac{6}{10} = -\frac{3}{5}.$$

So start our graph at $(0, 12)$ and go forward 5 units and then down 3 units to reach more points.

Since we know that we will go forward 5 units and then down 3 units, and that we will start our graph at $(0, 12)$, we can choose to orient and scale our axes to see a more complete picture of W than we achieved by plotting convenient points in Example 13.1.2.

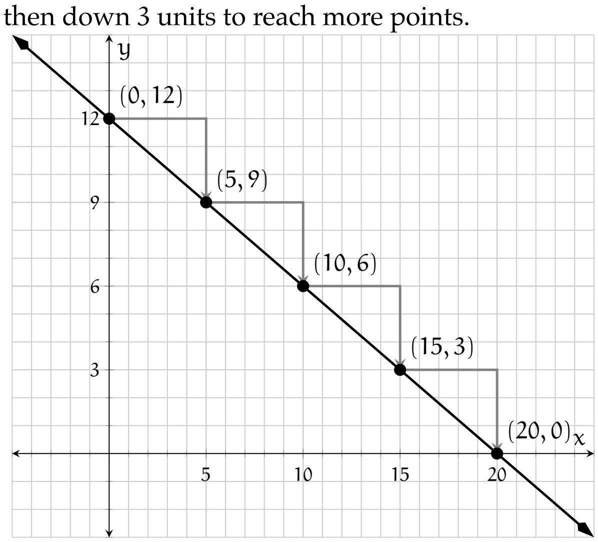


Figure 13.1.7: A graph of $W(x) = -0.6x + 12$

Example 13.1.8 Find the slope and vertical intercept of $y = h(x)$, where $h(x) = \frac{5}{3}x - 4$. Then use slope triangles to find two more points on the line and sketch it.

Explanation.

The slope is $\frac{5}{3}$ and the vertical intercept is $(0, -4)$. Starting at $(0, -4)$, we go forward 3 units and up 5 units to reach more points: $(3, 1)$ and $(6, 6)$.

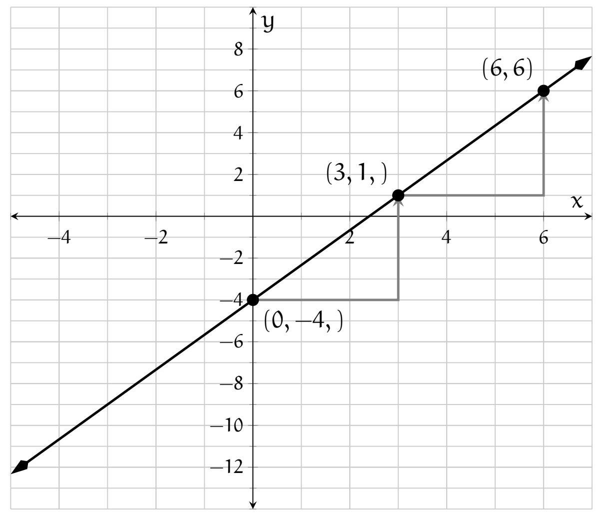


Figure 13.1.9: A graph of $h(x) = \frac{5}{3}x - 4$

13.1.3 Graphing Lines in Point-Slope Form

Recall that the point-slope form (3.6.1) of a line equation is $y = m(x - x_0) + y_0$ where m is the slope and (x_0, y_0) is a point on the line. The reason that (x_0, y_0) is a point on the line is because you can substitute in x_0 for x and then y_0 is the result for y.

$$\underset{\substack{\downarrow \\ y_0}}{y} = m(\overset{\substack{x_0 \\ \downarrow}}{x} - x_0) + y_0$$

Example 13.1.10 The population of Monarch butterflies has been on an overall downward trajectory since the 1980s, as have populations of many migratory animals. Efforts to restore the population haven't had great success yet. There are several distinct populations of Monarchs that probably never meet each other: the Hawaii population, the Florida Keys population, the Western population, and the Eastern population. Of these, the Eastern population is by far the largest and we can model this population of Monarch butterflies with a simple linear function.

$$M(x) = -(x - 2006) + 15$$

approximates the total number of acres of Mexican forest that the Eastern population of Monarchs hibernates in during winter in year x. This formula is only valid from 1995 to 2018, the years that the population has been well studied.

Let's make graph of this equation given the information provided, but only between 1995 and 2018.

Since this formula is linear and given in point-slope form, we can easily read that the slope of the line is -1, and the point given by the equation is $(2006, 15)$. This means that we should scale our graph appropriately to be able to see these details. We can interpret the point $(2006, 15)$ to mean that in the year 2006, the Monarchs overwintered in 15 acres of Mexican forest. The slope means that for every one year that goes by, the overwintering population takes up about one less acre of forest.

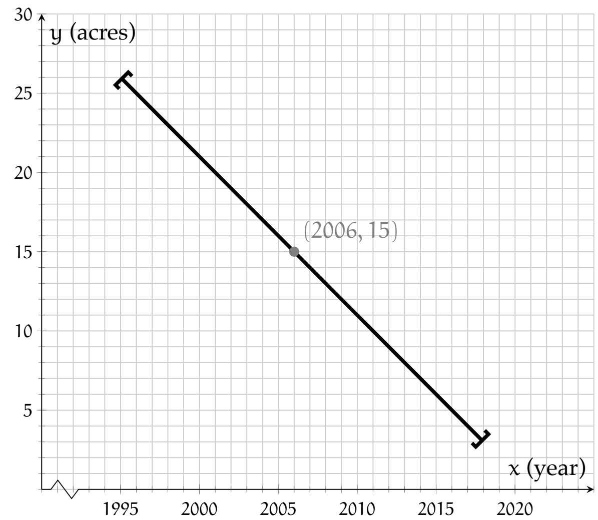


Figure 13.1.11: A graph of $M(x) = -(x - 2006) + 15$

Example 13.1.12 Find the slope and a point on the graph of $y = m(x)$, where $m(x) = -\frac{9}{5}(x+1) - 3$. Then use slope triangles to find two more points on the line and sketch it.

Explanation.

The slope of the line is $-\frac{9}{5}$, and the point given by the equation is $(-1, -3)$. So to graph h, start at $(-1, -3)$, and the go forward 5 units and down 9 units to reach more points: $(4, -12)$ and $(9, -21)$.

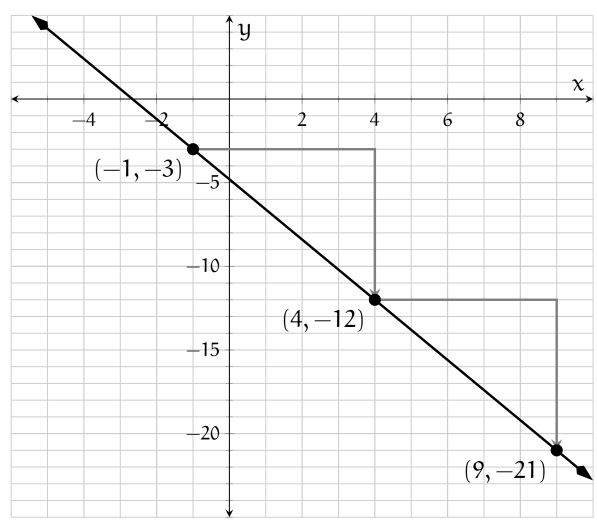


Figure 13.1.13: A graph of $m(x) = -\frac{9}{5}(x+1) - 3$

13.1.4 Graphing Lines Using Intercepts

Recall that the standard form (3.7.1) of a line equation is $Ax + By = C$ where where A, B, and C are three numbers (each of which might be 0, although at least one of A and B must be nonzero).

Example 13.1.14 Recall our bathtub draining problem from Example 13.1.2, where $W(x) = -0.6x + 12$ modeled the amount of water, in gallons, in the tub x seconds after it started draining. Let's write the line equation $y = -0.6x + 12$ in standard form.

To find the standard form of the equation, we do as in Subsection 3.7.3. First, we will replace the variable $W(x)$ with y because standard form relates x and y and does not use function notation. So $W(x) = -0.6x+12$ becomes $y = -0.6x + 12$. Now to convert to standard form, move both x and y to the left-hand side.

$$\begin{aligned} y &= -0.6x + 12 \\ 0.6x + y &= 12 \end{aligned}$$

The equation is in standard form written as $0.6x + y = 12$.

If a linear function is given in standard form, we can relative easily find the equation's x- and y-intercepts by substituting in $y = 0$ and $x = 0$, respectively.

Example 13.1.15 Let's find the intercepts of $0.6x+y = 12$, still relating back to Example 13.1.2. Then we may graph the equation using those intercepts.

To find the x-intercept, set $y = 0$ and solve for x.

$$\begin{aligned} 0.6x + (0) &= 12 \\ 0.6x &= 12 \\ x &= 20 \end{aligned}$$

So the x-intercept is the point $(20, 0)$. In context, this means that 20 minutes after the tub started to drain, 0 gallons of water remained. This is telling us that the tub is empty!

Now with the x- and y-intercepts known, we may plot these points and draw the line that runs through them.

To find the y-intercept, set $x = 0$ and solve for y.

$$\begin{aligned} 0.6(0) + y &= 12 \\ y &= 12 \end{aligned}$$

So, the y-intercept is the point $(0, 12)$. In context, this means that 0 minutes after the tub started to drain, 12 gallons of water remained. This is telling us how much water was initially in the tub.

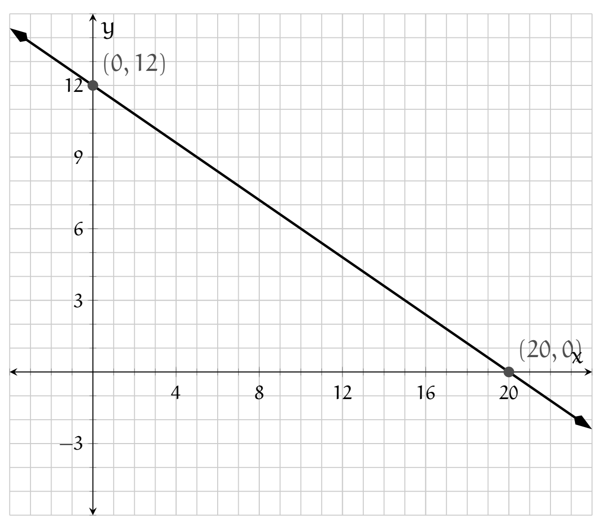


Figure 13.1.16: A graph of $3x + 5y = 60$

Checkpoint 13.1.17 Find the y-intercept and x-intercept of the line given by the equation. If a particular intercept does not exist, enter none into all the answer blanks for that row.

$$2x + 5y = -20$$

	x-value	y-value	Location (as an ordered pair)
y-intercept	______	______	____________
x-intercept	______	______	____________

Explanation. A line's y-intercept is on the y-axis, implying that its x-value must be 0. To find a line's y-intercept, we substitute in $x = 0$. In this problem we have:

$$\begin{aligned} 2x + 5y &= -20 \\ 2(0) + 5y &= -20 \\ 5y &= -20 \\ \frac{5y}{5} &= \frac{-20}{5} \\ y &= -4 \end{aligned}$$

This line's y-intercept is $(0, -4)$.

Next, a line's x-intercept is on the x-axis, implying that its y-value must be 0. To find a line's x-intercept, we substitute in $y = 0$. In this problem we have:

$$\begin{aligned} 2x + 5y &= -20 \\ 2x + 5(0) &= -20 \\ 2x &= -20 \\ \frac{2x}{2} &= \frac{-20}{2} \\ x &= -10 \end{aligned}$$

The line's x-intercept is $(-10, 0)$.

The entries for the table are:

	x-value	y-value	Location
y-intercept	0	-4	$(0, -4)$
x-intercept	-10	0	$(-10, 0)$

13.1.5 Graphing Functions by Plotting Points

Any function, linear or not, can be graphed by building a table of x- and y-values and plotting points. Let's look at a few more examples.

Example 13.1.18 Imagine a company called Corduroy's-Я-Us that makes pants. Their profit from their Royal Blue Corduroys, in thousands of dollars, can be modeled by the function $P(x) = -0.5x^2 + 33x - 200$ where x is the price of each pair of Royal Blue pants that they sell. Let's build a table of values and plot the function's graph.

In this context, the value of x must be positive. Furthermore, we shouldn't really consider x-values like 1, 2, etc., because it is not realistic that the price of a pair of new pants would be so low. Instead we try multiples of 10: 10, 20, etc.

x	$P(x) = -0.5x^2 + 33x - 200$	Point	Interpretation
0	$-0.5(0)^2 + 33(0) - 200$ $= -200$	$(0, -200)$	If each pair costs \$0, there is a loss of \$200,000.
10	$-0.5(10)^2 + 33(10) - 200$ $= 80$	$(10, 80)$	If each pair costs \$10, the profit is \$80,000.
20	$-0.5(20)^2 + 33(20) - 200$ $= 260$	$(20, 260)$	If each pair costs \$20, the profit is \$260,000.
30	$-0.5(30)^2 + 33(30) - 200$ $= 340$	$(30, 340)$	If each pair costs \$30, the profit is \$340,000.
40	$-0.5(40)^2 + 33(40) - 200$ $= 320$	$(40, 320)$	If each pair costs \$40, the profit is \$320,000.
50	$-0.5(50)^2 + 33(50) - 200$ $= 200$	$(50, 200)$	If each pair costs \$50, the profit is \$200,000.
60	$-0.5(60)^2 + 33(60) - 200$ $= -20$	$(60, -20)$	If each pair costs \$60, there is a loss of \$20,000.

Figure 13.1.19: A table of values for $P(x) = -0.5x^2 + 33x - 200$

With the values in Table 13.1.19, we can sketch the graph. Note that we have to estimate the how the graph curves which is a limitation of graphing a function by plotting points compared with using algebraic techniques.

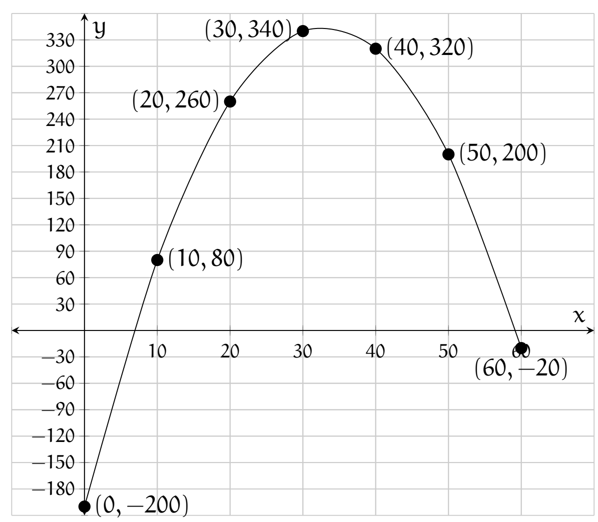


Figure 13.1.20: $P(x) = -0.5x^2 + 33x - 200$

Checkpoint 13.1.21 Make a table of solutions for the equation $y = -0.6x^2$. Then graph the equation.

x	y

Explanation. This solution will use the x values -2, -1, 0, 1 and 2. The choice to use these x-values is arbitrary. Since they are small numbers, they might make calculations easier. It's important to include negative numbers.

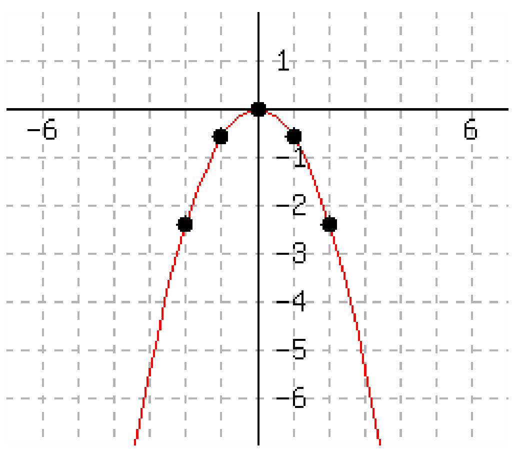
One at a time, we substitute these x-values into the equation $y = -0.6x^2$, and solve for y

$$\begin{aligned} y = -0.6(-2)^2 &\implies y = -2.4 \\ y = -0.6(-1)^2 &\implies y = -0.6 \\ y = -0.6 \cdot 0^2 &\implies y = 0 \\ y = -0.6 \cdot 1^2 &\implies y = -0.6 \\ y = -0.6 \cdot 2^2 &\implies y = -2.4 \end{aligned}$$

So the table may be completed as:

x	y
-2	-2.4
-1	-0.6
0	0
1	-0.6
2	-2.4

Using the values in the table, we can plot the following graph.



Example 13.1.22 Human-initiated global warming has been the subject of some debate. However, one aspect of the debate is undeniable fact: the amount of atmospheric carbon dioxide (CO_2: a greenhouse gas [1]) is being regularly and carefully measured [2] and is increasing faster and faster. The measured yearly average atmospheric carbon dioxide levels in parts per million (ppm) since 1958 can be very closely approximated by the function $C(x) = 244 + 29 \cdot 1.0148^x$ where x represents the number of years since the year 1900. Before 1958, the greenhouse gases weren't regularly measured. Create a table of values rounded to the nearest whole number for the carbon dioxide levels since 1958.

Explanation. Since 1958 is 58 years since 1900, we will start our table at $x = 58$ and go by 10s up through $x = 118$, which would stand for the year 2018.

x	$C(x)$	Point	Interpretation
58	$w(58) \approx 312$	$(5, 312)$	In 1958, the atmosphere was about 312 ppm CO_2.
68	$w(68) \approx 323$	$(68, 323)$	In 1968, the atmosphere was about 323 ppm CO_2.
78	$w(78) \approx 335$	$(78, 335)$	In 1978, the atmosphere was about 335 ppm CO_2.
88	$w(88) \approx 350$	$(88, 350)$	In 1988, the atmosphere was about 350 ppm CO_2.
98	$w(98) \approx 366$	$(98, 366)$	In 1998, the atmosphere was about 366 ppm CO_2.
108	$w(108) \approx 386$	$(108, 386)$	In 2008, the atmosphere was about 386 ppm CO_2.
118	$w(118) \approx 408$	$(118, 408)$	In 2018, the atmosphere was about 408 ppm CO_2.

Figure 13.1.23: A table of values for $C(x) = 244 + 29 \cdot 1.0148^x$

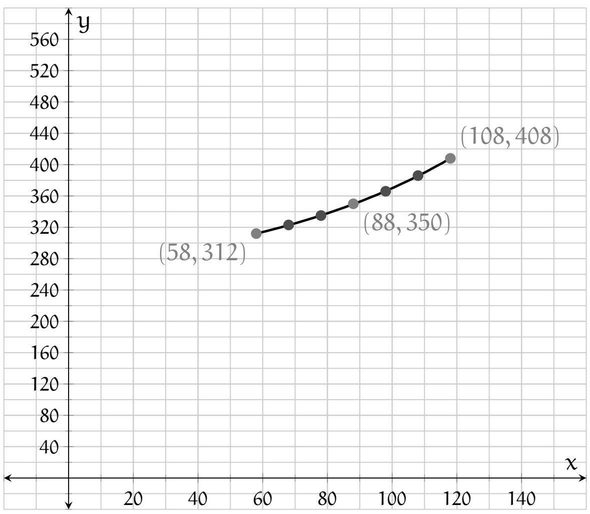


Figure 13.1.24: A graph of $C(x) = 244 + 29 \cdot 1.0148^x$

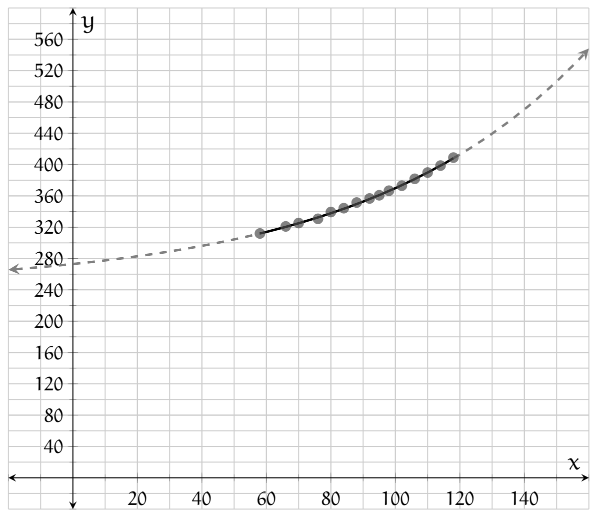


Figure 13.1.25: A graph of C with the ESRL overlaid and the function extrapolated beyond known dates.

13.1.6 Reading Questions

1. What are the four methods we recalled to graph lines in this section?

2. Why might it be better to represent a line in point-slope form than slope intercept form?

3. Explain how an equation for a line given in slope-intercept or point-slope form can be graphed without creating a table of values.

4. Describe one or more possible issues you might encounter after creating a table of points for a function and trying to use those points to make a graph.

[1] epa.gov/ghgemissions/overview-greenhouse-gases
[2] esrl.noaa.gov/gmd/ccgg/trends/graph.html

13.1.7 Exercises

Graphing Lines by Plotting Points Create a table of ordered pairs and then make a plot of the equation.

1. $y = 2x + 3$

2. $y = 3x + 5$

3. $y = -\frac{2}{5}x - 3$

4. $y = -\frac{3}{4}x + 2$

Graphing Lines in Slope-Intercept Form

5. Graph the equation $y = \frac{2}{3}x + 4$.

6. Graph the equation $y = \frac{3}{2}x - 5$.

7. Graph the equation $y = -\frac{3}{5}x - 1$.

8. Graph the equation $y = -\frac{1}{5}x + 1$.

Graphing Lines in Point-Slope Form

9. Graph the linear equation $y = -\frac{8}{3}(x - 4) - 5$ by identifying the slope and one point on this line.

10. Graph the linear equation $y = \frac{5}{7}(x + 3) + 2$ by identifying the slope and one point on this line.

11. Graph the linear equation $y = \frac{3}{4}(x + 2) + 1$ by identifying the slope and one point on this line.

12. Graph the linear equation $y = -\frac{5}{2}(x - 1) - 5$ by identifying the slope and one point on this line.

13. Graph the linear equation $y = -3(x - 9) + 4$ by identifying the slope and one point on this line.

14. Graph the linear equation $y = 7(x + 3) - 10$ by identifying the slope and one point on this line.

Graphing Lines Using Intercepts

15. Find the x- and y-intercepts of the line with equation $5x - 2y = 10$. Then find one other point on the line. Use your results to graph the line.

16. Find the x- and y-intercepts of the line with equation $5x - 6y = -90$. Then find one other point on the line. Use your results to graph the line.

17. Find the x- and y-intercepts of the line with equation $x + 5y = -15$. Then find one other point on the line. Use your results to graph the line.

18. Find the x- and y-intercepts of the line with equation $6x + y = -18$. Then find one other point on the line. Use your results to graph the line.

19. Make a graph of the line $-5x - y = -3$.

20. Make a graph of the line $x + 5y = 5$.

21. Make a graph of the line $20x - 4y = 8$.

22. Make a graph of the line $3x + 5y = 10$.

Graphing Functions by Plotting Points Create a table of ordered pairs and then make a plot of the equation.

23. $y = -3x^2$

24. $y = -x^2 - 2x - 3$

25. $y = \frac{1}{2}x^3 - x$

26. $y = \frac{1}{4}x^3 + x + 2$

27. $y = \sqrt{x + 5}$

28. $y = 3 - \sqrt{x + 2}$

13.2 Quadratic Graphs and Vertex Form

In this section, we will explore quadratic functions using graphing technology and learn the vertex and factored forms of a quadratic function's formula. We will also see how parabola graphs can be shifted.

13.2.1 Exploring Quadratic Functions with Graphing Technology

Graphing technology is very important and useful for applications and for finding points quickly. Let's explore some quadratic functions with graphing technology.

Example 13.2.2 Use technology to graph and make a table of the quadratic function f defined by $f(x) = 2x^2 + 4x - 3$ and find each of the key points or features.

a. Find the vertex.

b. Find the vertical intercept (i.e. y-intercept).

c. Find the horizontal or (i.e. x-intercept(s)).

d. Find $f(-2)$.

e. Solve $f(x) = 3$ using the graph.

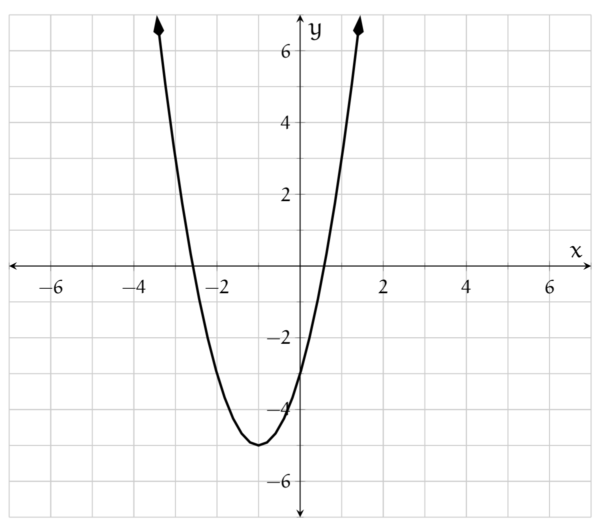
f. Solve $f(x) \leq 3$ using the graph.

g. State the domain and range of the function.

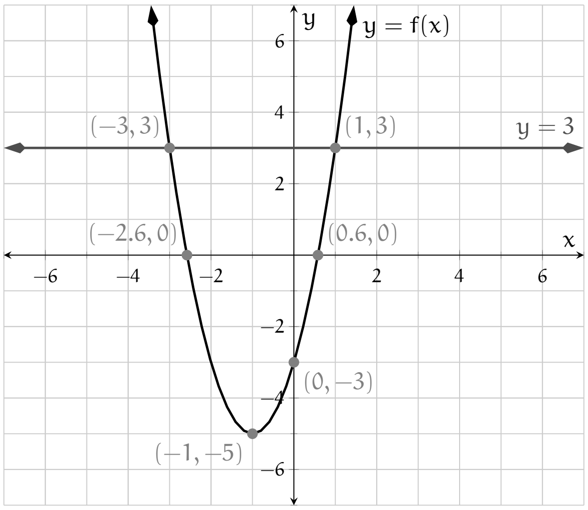
Explanation.

The specifics of how to use any one particular technology tool vary. Whether you use an app, a physical calculator, or something else, a table and graph should look like:

x	$f(x)$
-2	-3
-1	-5
0	-3
1	3
2	13



Additional features of your technology tool can enhance the graph to help answer these questions. You may be able to make the graph appear like:



a. The vertex is $(-1, -5)$.

b. The vertical intercept is $(0, -3)$.

c. The horizontal intercepts are approximately $(-2.6, 0)$ and $(0.6, 0)$.

d. When $x = -2$, $y = -3$, so $f(-2) = -3$.

e. The solutions to $f(x) = 3$ are the x-values where $y = 3$. We graph the horizontal line $y = 3$ and find the x-values where the graphs intersect. The solution set is $\{-3, 1\}$.

f. The solutions are all of the x-values where the function's graph is below (or touching) the line $y = 3$. The interval is $[-3, 1]$.

g. The domain is $(-\infty, \infty)$ and the range is $[-5, \infty)$.

Now we will look at an application with graphing technology and put the points of interest in context.

Example 13.2.3 A reduced-gravity aircraft[1] is a fixed-wing airplane that astronauts use for training. The airplane flies up and then down in a parabolic path to simulate the feeling of weightlessness. In one training flight, the pilot will fly 40 to 60 parabolic maneuvers.

For the first parabolic maneuver, the altitude of the plane, in feet, at time t, in seconds since the maneuver began, is given by $H(t) = -16t^2 + 400t + 30500$.

a. Determine the starting altitude of the plane for the first maneuver.

b. What is the altitude of the plane 10 seconds into the maneuver?

c. Determine the maximum altitude of the plane and how long it takes to reach that altitude.

d. The zero-gravity effect is experienced when the plane begins the parabolic path until it gets back down to 30,500 feet. Write an inequality to express this and solve it using the graph. Write the times of the zero-gravity effect as an interval and determine how long the astronauts experience weightlessness during each cycle.

e. Use technology to make a table for H with t-values from 0 to 25 seconds. Use an increment of 5 seconds and then use the table to solve $H(t) = 32100$.

f. State the domain and range for this context.

Explanation. We can answer the questions based on the information in the graph.

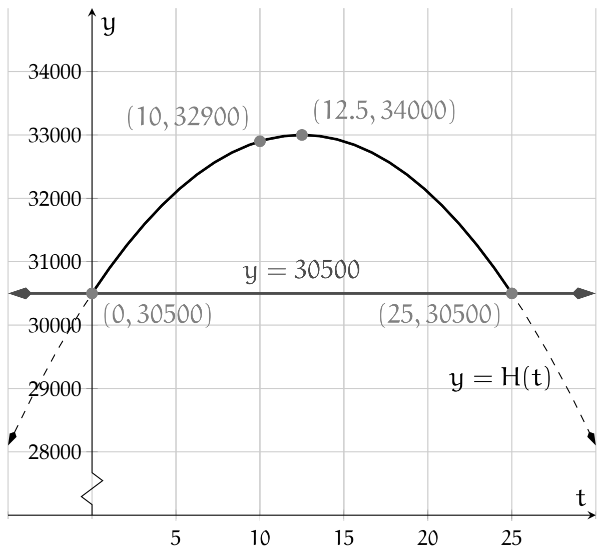


Figure 13.2.4: Graph of $H(t) = -16t^2 + 400t + 30500$ with $y = 30500$

a. The starting altitude can be read from the vertical intercept, which is $(0, 30500)$. The feeling of weightlessness begins at 30,500 feet.

b. After 10 seconds, the altitude of the plane is 32,900 feet.

c. For the maximum altitude of the plane we look at the vertex, which is approximately $(12.5, 33000)$. This tells us that after 12.5 seconds the plane will be at its maximum altitude of 33,000 feet.

d. We can write an inequality to describe when the plane is at or above 30,500 feet and solve it graphically.

$$\begin{aligned} H(t) &\geq 30500 \\ -16t^2 + 400t + 30500 &\geq 30500. \end{aligned}$$

We graph the line $y = 30500$ and find the points of intersection with the parabola. The astronauts experience weightlessness from 0 seconds to 25 seconds into the maneuver, or $[0, 25]$ seconds. They experience weightlessness for 25 seconds in each cycle.

e. To solve $H(t) = 32100$ using the table, we look for where the H-values are equal to 32100.

t	0	5	10	15	20	25
$H(t)$	30500	32100	32900	32900	32100	30500

There are two solutions, 5 seconds and 20 seconds. The solution set is $\{5, 20\}$.

f. When we use technology we see the entire function but in this context the plane is only on a parabolic path from $t = 0$ to $t = 25$ seconds. So the domain is $[0, 25]$, and the range is the set of corresponding y-values which is $[30500, 33000]$ feet.

Let's look at the remote-controlled airplane dive from Example 9.3.18. This time we will use technology to answer the questions.

Example 13.2.5 Maia has a remote-controlled airplane and she is going to do a stunt dive where the plane dives toward the ground and back up along a parabolic path. The altitude or height of the plane is given by the function H where $H(t) = 0.7t^2 - 23t + 200$, for $0 \leq t \leq 30$. The height is measured in feet and the time, t, is measured in seconds since the stunt began.

a. Determine the starting height of the plane as the dive begins.

b. Determine the height of the plane after 5 seconds.

c. Will the plane hit the ground, and if so, at what time?

d. If the plane does not hit the ground, what is the closest it gets to the ground, and at what time?

e. At what time(s) will the plane have an altitude of 50 feet?

f. State the domain and the range of the function (in context).

Explanation. We have graphed the function and we will find the key information and put it in context.

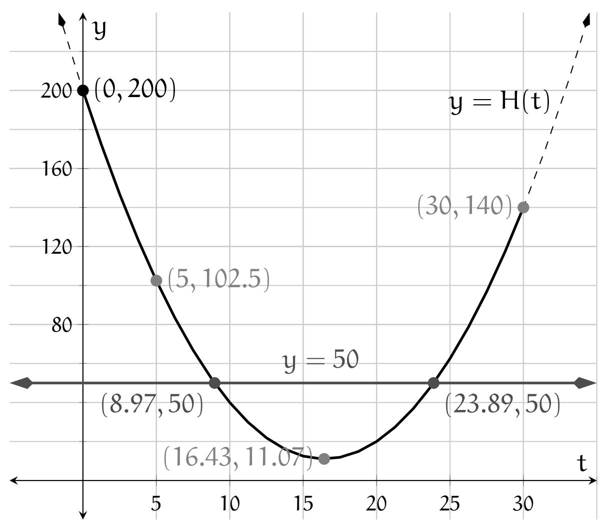


Figure 13.2.6: Graph of $H(t) = 0.7t^2 - 23t + 200$

a. The starting altitude can be read from the vertical intercept, which is $(0, 200)$. When the stunt begins, the plane has a altitude of 200 feet.

b. When $x = 5$, the y-value is 102.5. So $H(5) = 102.5$. This means that after 5 seconds, the plane is 102.5 feet above the ground.

c. From the graph we can see that the parabola does not touch or cross the x-axis, which represents the ground. This means the plane does not hit the ground and there are no real solutions to the equation $H(t) = 0$.

[1] en.wikipedia.org/wiki/Reduced-gravity_aircraft

d. The lowest point is the vertex, which is approximately $(16.43, 11.07)$. The minimum altitude of the plane is about 11 feet, which occurs after about 16.4 seconds.

e. We graph the horizontal line $y = 50$ and look for the points of intersection. The plane will be 50 feet above the ground about 9 seconds after the plane begins the stunt, and again at about 24 seconds.

f. The domain for this function is given in the problem statement because only part of the parabola represents the path of the plane. The domain is $[0, 30]$. For the range we look at the possible altitudes of the plane and see that it is $[11.07\ldots, 200]$. The plane is doing this stunt from 0 to 30 seconds and its height ranges from about 11 to 200 feet above the ground.

13.2.2 The Vertex Form of a Parabola

We have learned the standard form of a quadratic function's formula, which is $f(x) = ax^2 + bx + c$. In this subsection, we will learn another form called the "vertex form".

Using graphing technology, consider the graphs of $f(x) = x^2 - 6x + 7$ and $g(x) = (x - 3)^2 - 2$ on the same axes.

We see only one parabola because these are two different forms of the same function. Indeed, if we convert $g(x)$ into standard form:

$$g(x) = (x - 3)^2 - 2$$
$$g(x) = x^2 - 6x + 9 - 2$$
$$g(x) = x^2 - 6x + 7$$

it is clear that f and g are the same function.

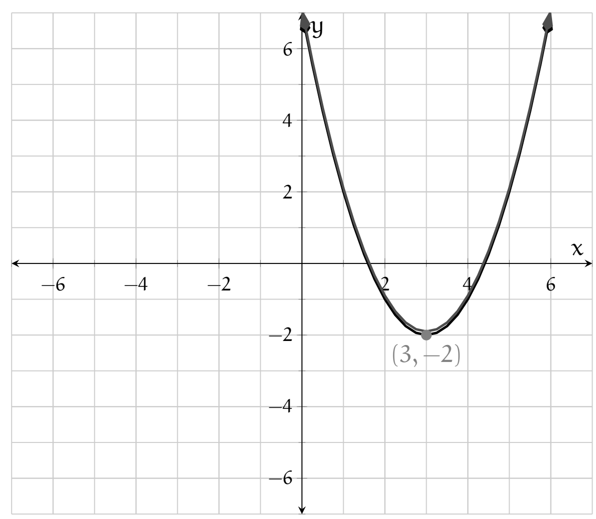


Figure 13.2.7: Graph of $f(x) = x^2 - 6x + 7$ and $g(x) = (x - 3)^2 - 2$

The formula given for g is said to be in "vertex form" because it allows us to read the vertex without doing any calculations. The vertex of the parabola is $(3, -2)$. We can see those numbers in $g(x) = (x - 3)^2 - 2$. The x-value is the solution to $(x - 3) = 0$, and the y-value is the constant *added* at the end.

Here are the graphs of three more functions with formulas in vertex form. Compare each function with the vertex of its graph.

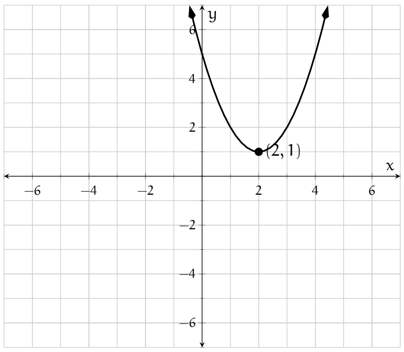


Figure 13.2.8: $r(x) = (x-2)^2 + 1$

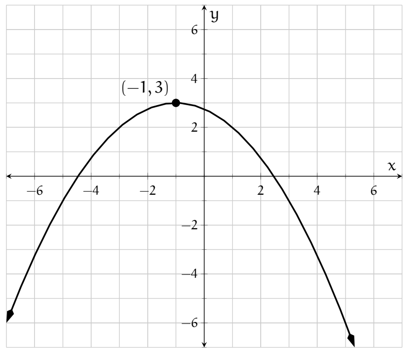


Figure 13.2.9: $s(x) = -\frac{1}{4}(x+1)^2 + 3$

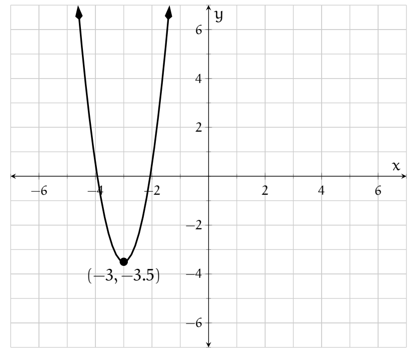


Figure 13.2.10: $t(x) = 4(x+3)^2 - 3.5$

Notice that the x-coordinate of the vertex has the opposite sign as the value in the function formula. On the other hand, the y-coordinate of the vertex has the same sign as the value in the function formula. Let's look at an example to understand why. We will evaluate $r(2)$.

$$\begin{aligned} r(2) &= (2-2)^2 + 1 \\ &= 1 \end{aligned}$$

The x-value is the solution to $(x-2) = 0$, which is positive 2. When we substitute 2 for x we get the value $y = 1$. Note that these coordinates create the vertex at $(2, 1)$. Now we can define the vertex form of a quadratic function.

Fact 13.2.11 Vertex Form of a Quadratic Function. *A quadratic function with the vertex at the point* (h, k) *is given by* $f(x) = a(x-h)^2 + k$.

Checkpoint 13.2.12 Find the vertex of each quadratic function.

a. $r(x) = -2(x+4)^2 + 10$

b. $s(x) = 5(x-1)^2 + 2$

c. $t(x) = (x-10)^2 - 5$

d. $u(x) = 3(x+7)^2 - 13$

Explanation.

a. The vertex of $r(x) = -2(x+4)^2 + 10$ is $(-4, 10)$.

b. The vertex of $s(x) = 5(x-1)^2 + 2$ is $(1, 2)$.

c. The vertex of $t(x) = (x-10)^2 - 5$ is $(10, -5)$.

d. The vertex of $u(x) = 3(x+7)^2 - 13$ is $(-7, -13)$.

Now let's do the reverse. When given the vertex and the value of a, we can write the function in vertex form.

Example 13.2.13 Write a formula for the quadratic function f with the given vertex and value of a.

a. Vertex $(-2, 8)$, $a = 1$

b. Vertex $(4, -9)$, $a = -4$

c. Vertex $(-3, -1)$, $a = 2$

d. Vertex $(5, 12)$, $a = -3$

Explanation.

a. The vertex form is $f(x) = (x + 2)^2 + 8$.

b. The vertex form is $f(x) = -4(x - 4)^2 - 9$.

c. The vertex form is $f(x) = 2(x + 3)^2 - 1$.

d. The vertex form is $f(x) = -3(x - 5)^2 + 12$.

Once we read the vertex we can also state the domain and range. All quadratic functions have a domain of $(-\infty, \infty)$ because we can put in any value to a quadratic function. The range, however, depends on the y-value of the vertex and whether the parabola opens upward or downward. When we have a quadratic function in vertex form we can read the range from the formula. Let's look at the graph of f, where $f(x) = 2(x-3)^2 - 5$, as an example.

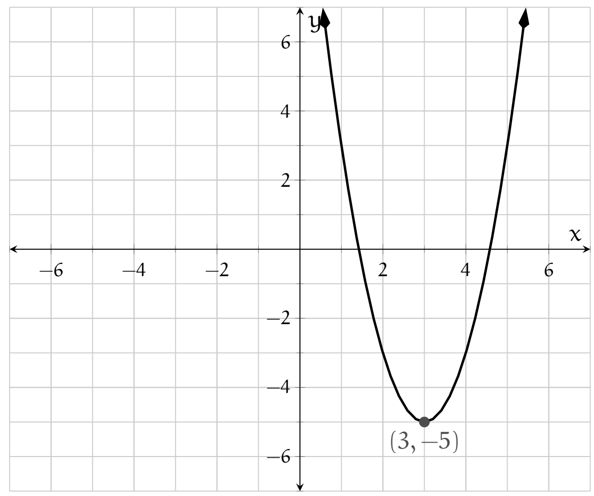


Figure 13.2.14: The graph of $f(x) = 2(x - 3)^2 - 5$

The domain is $(-\infty, \infty)$. The graph of f opens upward (which we know because $a = 2$ is positive) so the vertex is the minimum point. The y-value of -5 is the minimum. The range is $[-5, \infty)$.

Example 13.2.15 Identify the domain and range of g, where $g(x) = -3(x + 1)^2 + 6$.

Explanation.

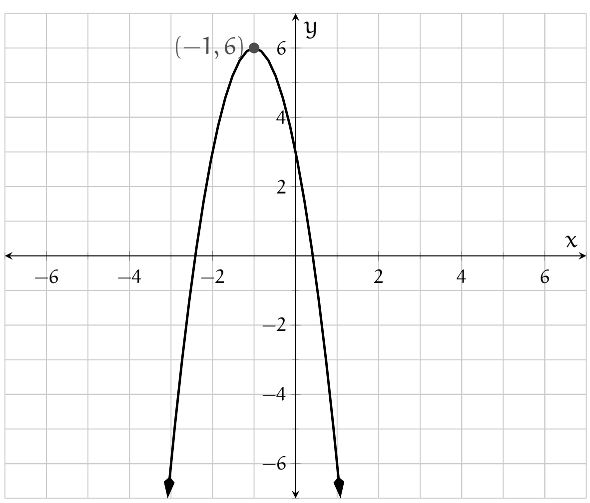


Figure 13.2.16: $g(x) = -3(x + 1)^2 + 6$

The domain is $(-\infty, \infty)$. The graph of g opens downward (which we know because $a = -3$ is negative) so the vertex is the maximum point. The y-value of 6 is the maximum. The range is $(-\infty, 6]$.

Checkpoint 13.2.17 Identify the domain and range of each quadratic function.

a. $w(x) = -3(x+10)^2 - 11$

The domain is ☐ and the range is ☐.

b. $u(x) = 4(x-7)^2 + 20$

The domain is ☐ and the range is ☐.

c. $y(x) = -(x-1)^2$

The domain is ☐ and the range is ☐.

d. $z(x) = 3(x+9)^2 - 4$

The domain is ☐ and the range is ☐.

Explanation.

a. The domain of w is $(-\infty, \infty)$. The parabola opens downward so the range is $(-\infty, -11]$.

b. The domain of u is $(-\infty, \infty)$. The parabola opens upward so the range is $[20, \infty)$.

c. The domain of y is $(-\infty, \infty)$. The parabola opens downward so the range is $(-\infty, 0]$.

d. The domain of z is $(-\infty, \infty)$. The parabola opens upward so the range is $[-4, \infty)$.

13.2.3 Horizontal and Vertical Shifts

Let $f(x) = x^2$ and $g(x) = (x-4)^2 + 1$. The graph of $y = f(x)$ has its vertex at the point $(0, 0)$. Now we will compare this with the graph of $y = g(x)$ on the same axes.

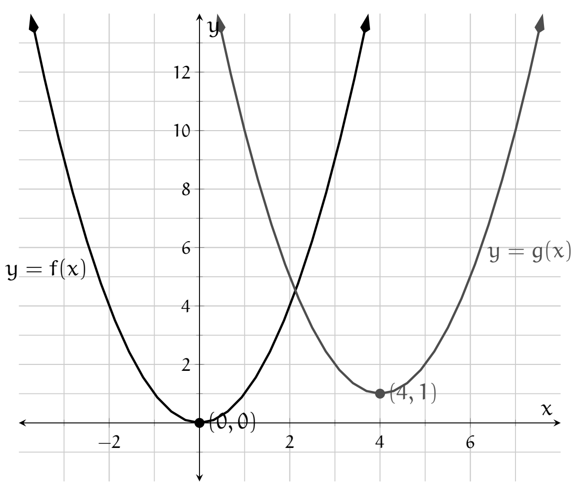


Figure 13.2.18: The graph of f and g

Both graphs open upward and have the same shape. Notice that the graph of g is the same as the graph of f but is shifted to the right by 4 units and up by 1 units because its vertex is $(4, 1)$.

Let's look at another graph. Let $h(x) = -x^2$ and let $j(x) = -(x+3)^2 + 4$.

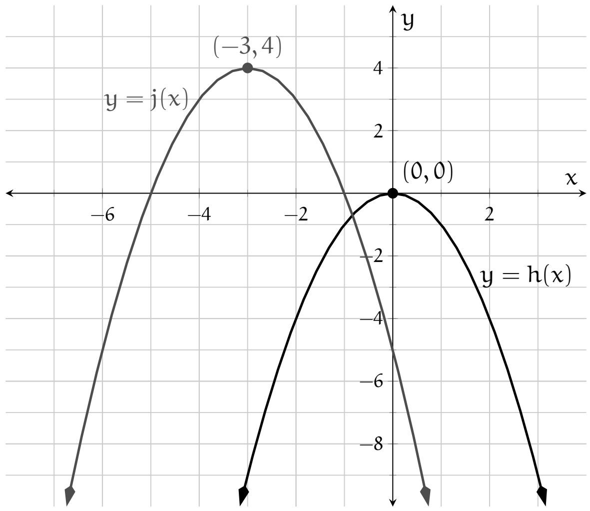


Figure 13.2.19: The graph of h and j

Both parabolas open downward and have the same shape. The graph of j is the same as the graph of h but it has been shifted to the left by 3 units and up by 4 units making its vertex $(-3, 4)$.

To summarize, when a quadratic function is written in vertex form, the h-value is the horizontal shift of its graph from the graph of $y = ax^2$ and the k-value is the vertical shift of its graph from the graph of $y = ax^2$.

Example 13.2.20 Identify the horizontal and vertical shifts compared with $y = x^2$.

a. $m(x) = (x + 7)^2 + 3$

b. $n(x) = (x - 1)^2 + 6$

c. $o(x) = (x - 5)^2 - 1$

d. $p(x) = (x + 3)^2 - 11$

Explanation.

a. The graph of $y = m(x)$ has vertex at $(-7, 3)$. Therefore the graph is the same as $y = x^2$ shifted to the left 7 units and up 3 units.

b. The graph of $y = n(x)$ has vertex at $(1, 6)$. Therefore the graph is the same as $y = x^2$ shifted to the right 1 unit and up 6 units.

c. The graph of $y = o(x)$ has vertex at $(5, -1)$. Therefore the graph is the same as $y = x^2$ shifted to the right 5 units and down 1 unit.

d. The graph of $y = p(x)$ has vertex at $(-3, -11)$. Therefore the graph is the same as $y = x^2$ shifted to the left 3 units and down 11 units.

13.2.4 The Factored Form of a Parabola

There is another form of a quadratic function's formula, called "factored form", which we will explore next. Let's consider the two functions $q(x) = -x^2 + 3x + 4$ and $s(x) = -(x-4)(x+1)$. Using graphing technology, we will graph $y = q(x)$ and $y = s(x)$ on the same axes.

These graphs coincide because the functions are actually the same. We can tell by multiplying out the formula for g to get back to the formula for f.

$$\begin{aligned} g(x) &= -(x-4)(x+1) \\ g(x) &= -(x^2-3x-4) \\ g(x) &= -x^2+3x+4 \end{aligned}$$

Now we can see that f and g are really the same function.

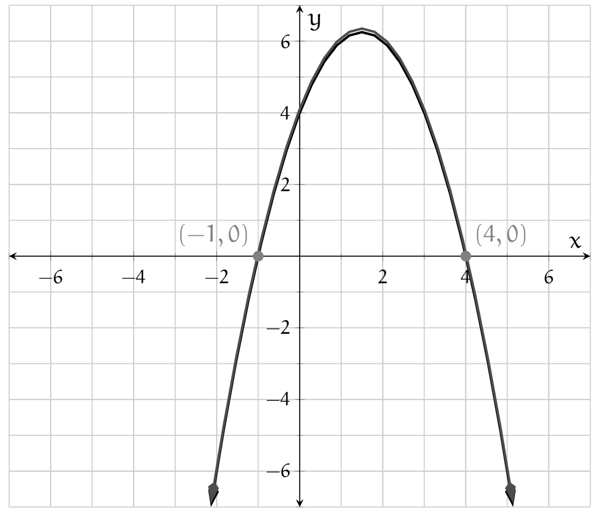


Figure 13.2.21: Graph of q and s

Factored form is very useful because we can read the x-intercepts directly from the function, which in this case are $(4,0)$ and $(-1,0)$. We find these by looking for the values that make the factors equal to 0, so the x-values have the opposite signs as are shown in the formula. To demonstrate this, we will find the roots by solving $g(x)=0$.

$$\begin{aligned} g(x) &= -(x-4)(x+1) \\ 0 &= -(x-4)(x+1) \end{aligned}$$

$$\begin{aligned} x-4 &= 0 \qquad \text{or} \qquad & x+1 &= 0 \\ x &= 4 \qquad \text{or} \qquad & x &= -1 \end{aligned}$$

This shows us that the x-intercepts are $(4,0)$ and $(-1,0)$.

The x-values of the x-intercepts are also called **zeros** or **roots**. The zeros or roots of the function g are -1 and 4.

Fact 13.2.22 Factored Form of a Quadratic Function. *A quadratic function with horizontal intercepts at $(r,0)$ and $(s,0)$ has the formula $f(x)=a(x-r)(x-s)$.*

Checkpoint 13.2.23 Write the horizontal intercepts of each function.

a. $t(x) = -(x+2)(x-4)$

b. $u(x) = 6(x-7)(x-5)$

c. $v(x) = -2(x+1)(x+4)$

d. $w(x) = 10(x-8)(x+3)$

Explanation.

a. The horizontal intercepts of t are $(-2,0)$ and $(4,0)$.

b. The horizontal intercepts of u are $(7,0)$ and $(5,0)$.

c. The horizontal intercepts of v are $(-1,0)$ and $(-4,0)$.

d. The horizontal intercepts of w are $(8,0)$ and $(-3,0)$.

Let's summarize the three forms of a quadratic function's formula:

Standard Form $f(x) = ax^2 + bx + c$, with y-intercept $(0, c)$.

Vertex Form $f(x) = a(x - h)^2 + k$, with vertex (h, k).

Factored Form $f(x) = a(x - r)(x - s)$, with x-intercepts $(r, 0)$ and $(s, 0)$.

13.2.5 Reading Questions

1. With the vertex form of a quadratic function, the formula shows you a point on the graph (without having to do any calculation). What is the name of that point?

2. With the standard form of a quadratic function, the formula shows you a point on the graph (without having to do any calculation). What is the name of that point?

3. What makes the vertex form of a quadratic function nicer for graphing compared to standard form?

4. If a fellow student attempted to graph the equation $y = (x - 4)^2 + 6$ and put the vertex at (-4,6), how would you explain to them that they have made an error?

13.2.6 Exercises

Review and Warmup

1. Multiply the polynomials.
$(t + 7)(t - 4)$

2. Multiply the polynomials.
$(t + 4)(t - 10)$

3. Multiply the polynomials.
$(10t - 5)(t + 7)$

4. Multiply the polynomials.
$(7x - 1)(x + 5)$

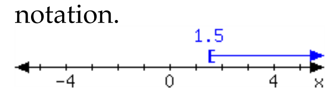
5. Factor the given polynomial.
$x^2 + 7x + 12$

6. Factor the given polynomial.
$y^2 + 18y + 80$

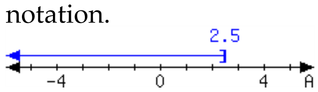
7. Factor the given polynomial.
$8y^2 + 24y + 16$

8. Factor the given polynomial.
$2r^2 + 12r + 10$

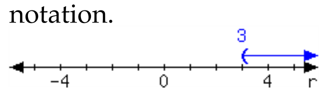
9. For the interval expressed in the number line, write it using set-builder notation and interval notation.



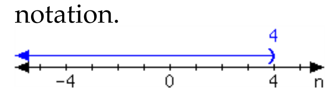
10. For the interval expressed in the number line, write it using set-builder notation and interval notation.



11. For the interval expressed in the number line, write it using set-builder notation and interval notation.



12. For the interval expressed in the number line, write it using set-builder notation and interval notation.



Technology and Tables

13. Let $f(x) = x^2 + x - 1$. Use technology to make a table of values f.

x	$f(x)$

14. Let $g(x) = x^2 - 3x - 3$. Use technology to make a table of values g.

x	$g(x)$

15. Let $h(x) = -x^2 + 3x - 1$. Use technology to make a table of values h.

x	$h(x)$

16. Let $F(x) = -x^2 + 4x - 3$. Use technology to make a table of values F.

x	$F(x)$

17. Let $F(x) = -2x^2 + 5x + 1$. Use technology to make a table of values F.

x	$F(x)$

18. Let $G(x) = 3x^2 - 5x - 4$. Use technology to make a table of values G.

x	$G(x)$

19. Let $H(x) = -2x^2 - 4x + 30$. Use technology to make a table of values H.

x	$H(x)$

20. Let $K(x) = 3x^2 - 8x + 53$. Use technology to make a table of values K.

x	$K(x)$

Technology and Graphs

21. Use technology to make a graph of f where $f(x) = x^2 + 3x - 2$.

22. Use technology to make a graph of f where $f(x) = x^2 - 2x - 1$.

23. Use technology to make a graph of f where $f(x) = -x^2 + 3x + 2$.

24. Use technology to make a graph of f where $f(x) = -x^2 + x + 2$.

25. Use technology to make a graph of f where $f(x) = 3x^2 - 6x - 5$.

26. Use technology to make a graph of f where $f(x) = -3x^2 - 8x + 3$.

27. Use technology to make a graph of f where $f(x) = -3x^2 + 4x + 49$.

28. Use technology to make a graph of f where $f(x) = 2x^2 - 2x + 41$.

Technology and Features of Quadratic Function Graphs Use technology to find features of a quadratic function and its graph.

29. Let $K(x) = x^2 - 4x + 2$. Use technology to find the following.

a. The vertex is ☐.

b. The y-intercept is ☐.

c. The x-intercept(s) is/are ☐.

d. The domain of K is ☐.

e. The range of K is ☐.

f. Calculate $K(1)$. ☐.

g. Solve $K(x) = 2$. ☐

h. Solve $K(x) \geq 2$. ☐

30. Let $f(x) = -x^2 - x - 1$. Use technology to find the following.

a. The vertex is ☐.

b. The y-intercept is ☐.

c. The x-intercept(s) is/are ☐.

d. The domain of f is ☐.

e. The range of f is ☐.

f. Calculate $f(1)$. ☐.

g. Solve $f(x) = -3$. ☐

h. Solve $f(x) \geq -3$. ☐

31. Let $g(x) = 1.1x^2 - 2.1x + 4.2$. Use technology to find the following.

a. The vertex is ☐.

b. The y-intercept is ☐.

c. The x-intercept(s) is/are ☐.

d. The domain of g is ☐.

e. The range of g is ☐.

f. Calculate $g(2)$. ☐.

g. Solve $g(x) = 5$. ☐

h. Solve $g(x) > 5$. ☐

32. Let $h(x) = 1.1x^2 - 2.8x + 3.7$. Use technology to find the following.

a. The vertex is ☐.

b. The y-intercept is ☐.

c. The x-intercept(s) is/are ☐.

d. The domain of h is ☐.

e. The range of h is ☐.

f. Calculate $h(5)$. ☐.

g. Solve $h(x) = 3$. ☐

h. Solve $h(x) \leq 3$. ☐

33. Let $F(x) = \frac{x^2}{3} + 2.3x + 0.9$. Use technology to find the following.

a. The vertex is ______.

b. The y-intercept is ______.

c. The x-intercept(s) is/are ______.

d. The domain of F is ______.

e. The range of F is ______.

f. Calculate $F(-3)$. ______.

g. Solve $F(x) = -1$. ______

h. Solve $F(x) < -1$. ______

34. Let $F(x) = \frac{x^2}{4} + 2.2x + 3.2$. Use technology to find the following.

a. The vertex is ______.

b. The y-intercept is ______.

c. The x-intercept(s) is/are ______.

d. The domain of F is ______.

e. The range of F is ______.

f. Calculate $F(1)$. ______.

g. Solve $F(x) = 7$. ______

h. Solve $F(x) \leq 7$. ______

Applications

35. An object was launched from the top of a hill with an upward vertical velocity of 150 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 150t + 250$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Find the answer using graphing technology.

The object's height was ______ when it was launched.

36. An object was launched from the top of a hill with an upward vertical velocity of 170 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 170t + 150$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Find the answer using graphing technology.

Use a table to list the object's height within the first second after it was launched, at an increment of 0.1 second. Fill in the blanks. Round your answers to two decimal places when needed.

Time in Seconds	Height in Feet
0.1	______
0.2	______
0.3	______

37. An object was launched from the top of a hill with an upward vertical velocity of 190 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 190t + 300$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Use technology to find the answer.

The object was ______ feet in the air 5 seconds after it was launched.

38. An object was launched from the top of a hill with an upward vertical velocity of 200 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 200t + 200$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Find the answer using technology.

[ ] seconds after its launch, the object reached its maximum height of [ ] feet.

39. An object was launched from the top of a hill with an upward vertical velocity of 60 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 60t + 150$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Find the answer using technology.

[ ] seconds after its launch, the object fell to the ground at sea level.

40. An object was launched from the top of a hill with an upward vertical velocity of 80 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 80t + 300$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Find the answer using technology. Round your answers to two decimal places. If there is more than one answer, use a comma to separate them.

The object was 359 feet high at the following number of seconds after it was launched: [ ].

41. In a race, a car drove through the starting line at the speed of 7 meters per second. It was accelerating at 2.3 meters per second squared. Its distance from the starting position can be modeled by the function $d(t) = 1.15t^2 + 7t$. Find the answer using technology.

After [ ] seconds, the car was 63.75 meters away from the starting position.

42. In a race, a car drove through the starting line at the speed of 4 meters per second. It was accelerating at 2.7 meters per second squared. Its distance from the starting position can be modeled by the function $d(t) = 1.35t^2 + 4t$. Find the answer using technology.

After [ ] seconds, the car was 242.4 meters away from the starting position.

43. A farmer purchased 520 meters of fencing, and will build a rectangular pen with it. To enclose the largest possible area, what should the pen's length and width be? Model the pen's area with a function, and then find its maximum value.
Use a comma to separate your answers.

To enclose the largest possible area, the pen's length and width should be [ ] meters.

44. A farmer purchased 310 meters of fencing, and will build a rectangular pen along a river. This implies the pen has only 3 fenced sides. To enclose the largest possible area, what should the pen's length and width be? Model the pen's area with a function, and then find its maximum value.

To enclose the largest possible area, the pen's length and width should be [ ] meters.

Quadratic Functions in Vertex Form

45. Find the vertex of the graph of $y = 5(x+7)^2 + 4$.

46. Find the vertex of the graph of $y = 8(x-7)^2 - 8$.

47. Find the vertex of the graph of $y = 10(x+1)^2 + 4$.

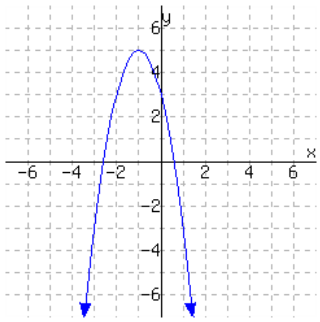
48. Find the vertex of the graph of $y = -9(x+8)^2 + 9$.

49. Find the vertex of the graph of $y = -5.9(x-5.4)^2 - 2.9$.

50. Find the vertex of the graph of $y = -3.7(x+1.5)^2 + 5.5$.

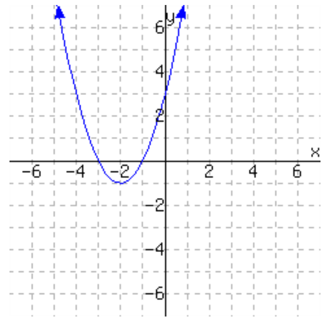
A graph of a function f is given. Use the graph to write a formula for f in vertex form. You will need to identify the vertex and also one more point on the graph to find the leading coefficient a.

51.



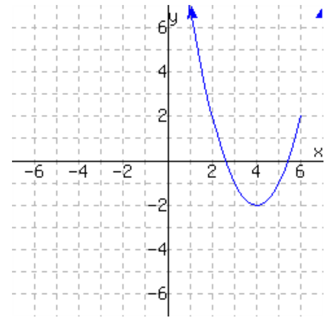
f(x)

52.



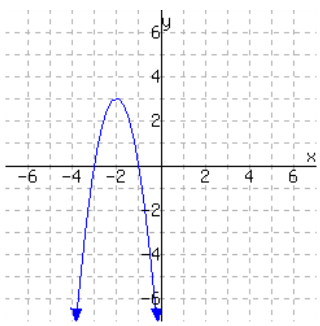
f(x)

53.



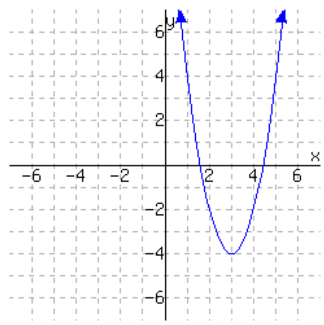
f(x)

54.



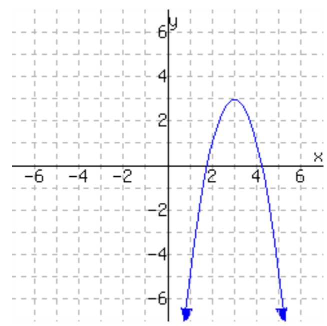
f(x)

55.



f(x)

56.



f(x)

57. Write the vertex form for the quadratic function f, whose vertex is $(9, 4)$ and has leading coefficient $a = -8$.

58. Write the vertex form for the quadratic function f, whose vertex is $(3, -7)$ and has leading coefficient $a = -6$.

59. Write the vertex form for the quadratic function f, whose vertex is $(-4, -2)$ and has leading coefficient $a = -4$.

60. Write the vertex form for the quadratic function f, whose vertex is $(8, -7)$ and has leading coefficient $a = -1$.

61. Let F be defined by $F(x) = (x - 2)^2 - 3$.

a. What is the domain of F?

b. What is the range of F?

62. Let G be defined by $G(x) = (x + 5)^2 + 5$.

a. What is the domain of G?

b. What is the range of G?

63. Let H be defined by $H(x) = 9.1(x + 6)^2 - 4$.

a. What is the domain of H?

b. What is the range of H?

64. Let K be defined by $K(x) = 5.5(x - 2)^2 + 9$.

a. What is the domain of K?

b. What is the range of K?

65. Let K be defined by $K(x) = -6(x+9)^2 + 4$.

a. What is the domain of K?

b. What is the range of K?

66. Let f be defined by $f(x) = 7(x+1)^2 - 1$.

a. What is the domain of f?

b. What is the range of f?

67. Let g be defined by $g(x) = 6\left(x - \frac{3}{4}\right)^2 + 1$.

a. What is the domain of g?

b. What is the range of g?

68. Let h be defined by $h(x) = -3\left(x + \frac{7}{9}\right)^2 - \frac{4}{5}$.

a. What is the domain of h?

b. What is the range of h?

69. Let h be defined by $h(x) = 8\left(x + \frac{2}{9}\right)^2 + 2$.

a. What is the domain of h?

b. What is the range of h?

70. Let F be defined by $F(x) = -5\left(x - \frac{1}{3}\right)^2 + (-4)$.

a. What is the domain of F?

b. What is the range of F?

71. Consider the graph of the equation $y = (x-3)^2 - 7$. Compared to the graph of $y = x^2$, the vertex has been shifted ☐ units (☐ left ☐ right) and ☐ units (☐ down ☐ up) .

72. Consider the graph of the equation $y = (x-8)^2 + 5$. Compared to the graph of $y = x^2$, the vertex has been shifted ☐ units (☐ left ☐ right) and ☐ units (☐ down ☐ up) .

73. Consider the graph of the equation $y = (x-71.6)^2 - 14.5$. Compared to the graph of $y = x^2$, the vertex has been shifted ☐ units (☐ left ☐ right) and ☐ units (☐ down ☐ up) .

74. Consider the graph of the equation $y = (x-93.5)^2 - 83.4$. Compared to the graph of $y = x^2$, the vertex has been shifted ☐ units (☐ left ☐ right) and ☐ units (☐ down ☐ up) .

75. Consider the graph of the equation $y = \left(x - \frac{9}{5}\right)^2 + \frac{3}{2}$. Compared to the graph of $y = x^2$, the vertex has been shifted ☐ units (☐ left ☐ right) and ☐ units (☐ down ☐ up) .

76. Consider the graph of the equation $y = \left(x - \frac{2}{3}\right)^2 + \frac{9}{8}$. Compared to the graph of $y = x^2$, the vertex has been shifted ☐ units (☐ left ☐ right) and ☐ units (☐ down ☐ up) .

Three Forms of Quadratic Functions

77. The quadratic expression $(x-2)^2 - 1$ is written in vertex form.

a. Write the expression in standard form.

b. Write the expression in factored form.

78. The quadratic expression $(x-3)^2 - 25$ is written in vertex form.

a. Write the expression in standard form.

b. Write the expression in factored form.

79. The quadratic expression $(x-4)^2 - 81$ is written in vertex form.

a. Write the expression in standard form.

b. Write the expression in factored form.

80. The quadratic expression $(x-1)^2 - 36$ is written in vertex form.

a. Write the expression in standard form.

b. Write the expression in factored form.

Factored Form and Intercepts

81. The formula for a quadratic function G is $G(x) = (x-5)(x-3)$.

a. The y-intercept is ______.

b. The x-intercept(s) is/are ______.

82. The formula for a quadratic function f is $f(x) = (x-7)(x+4)$.

a. The y-intercept is ______.

b. The x-intercept(s) is/are ______.

83. The formula for a quadratic function H is $H(x) = 9(x-9)(x-1)$.

a. The y-intercept is ______.

b. The x-intercept(s) is/are ______.

84. The formula for a quadratic function F is $F(x) = -8(x-2)(x-9)$.

a. The y-intercept is ______.

b. The x-intercept(s) is/are ______.

85. The formula for a quadratic function h is $h(x) = -6x(x+5)$.

a. The y-intercept is ______.

b. The x-intercept(s) is/are ______.

86. The formula for a quadratic function H is $H(x) = -4(x-8)\,x$.

a. The y-intercept is ______.

b. The x-intercept(s) is/are ______.

87. The formula for a quadratic function g is $g(x) = -2(x-1)(x-1)$.

a. The y-intercept is ______.

b. The x-intercept(s) is/are .

88. The formula for a quadratic function h is $h(x) = -5(x-2)(x-2)$.

a. The y-intercept is ______.

b. The x-intercept(s) is/are ______.

89. The formula for a quadratic function g is $g(x) = 3(8x+9)(2x-5)$.

a. The y-intercept is ______.

b. The x-intercept(s) is/are ______.

90. The formula for a quadratic function G is $G(x) = 5(6x+1)(8x-5)$.

a. The y-intercept is ______.

b. The x-intercept(s) is/are ______.

13.3 Completing the Square

In this section, we will learn how to "complete the square" with a quadratic expression. This topic is useful for solving quadratic equations and putting quadratic functions in vertex form.

13.3.1 Solving Quadratic Equations by Completing the Square

When we have an equation like $(x+5)^2 = 4$, we can solve it quickly using the square root property:

$$(x+5)^2 = 4$$

$$\begin{aligned} x+5 &= -2 && \text{or} && x+5 = 2 \\ x &= -7 && \text{or} && x = -3 \end{aligned}$$

The method of **completing the square** allows us to solve *any* quadratic equation using the square root property.

Suppose you have a small quadratic expression in the form $x^2 + bx$. It can be visualized as an "L"-shape as in Figure 13.3.2.

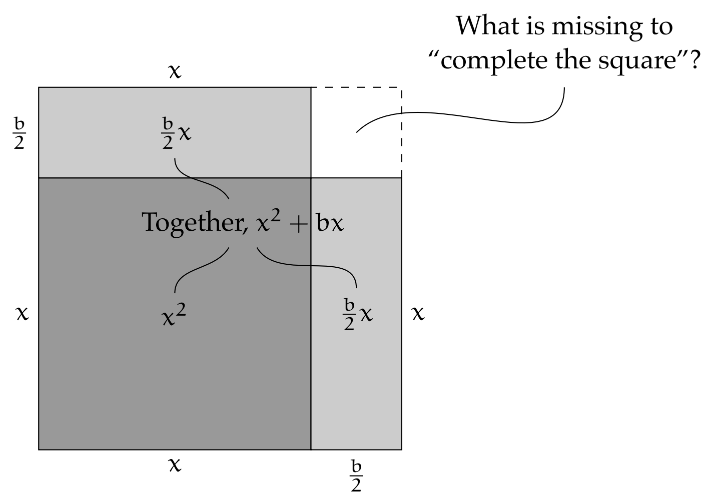


Figure 13.3.2

The "missing" square in the upper right corner of Figure 13.3.2 is $\frac{b}{2}$ on each side, so its area is $\left(\frac{b}{2}\right)^2$. This means that if we have $x^2 + bx$ and add $\left(\frac{b}{2}\right)^2$, we are "completing" the larger square.

Fact 13.3.3 The Term that Completes the Square. *For a polynomial $x^2 + bx$, the constant term needed to make a perfect square trinomial is $\left(\frac{b}{2}\right)^2$.*

Process 13.3.4 Completing the Square. *For a quadratic equation simplified to the form $x^2 + bx = c$, to solve for x by completing the square,*

1. *Use Fact 13.3.3 to find the number to add to both sides of the equation to make the left hand side a perfect square.*

This number is always $\left(\frac{b}{2}\right)^2$.

2. *Add that number to both sides of* $x^2 + bx = c$ *to get*

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

3. *The left hand side is now a perfect square that factors as* $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$*, so the equation becomes*

$$\left(x + \frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

4. *Solve remaining equation using the Square Root Property.*

Example 13.3.5 Solve the quadratic equation $x^2 + 6x = 16$ by completing the square.

Explanation. To solve the quadratic equation $x^2 + 6x = 16$, on the left side we can complete the square by adding $\left(\frac{b}{2}\right)^2$; note that $b = 6$ in this case, which makes $\left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 3^2 = 9$. We add it to both sides to maintain equality.

$$\begin{aligned} x^2 + 6x + 9 &= 16 + 9 \\ x^2 + 6x + 9 &= 25 \\ (x+3)^2 &= 25 \end{aligned}$$

Now that we have completed the square, we can solve the equation using the square root property.

$$\begin{aligned} x + 3 &= -5 \qquad \text{or} \qquad & x + 3 &= 5 \\ x &= -8 \qquad \text{or} \qquad & x &= 2 \end{aligned}$$

The solution set is $\{-8, 2\}$.

Now let's see the process for completing the square when the quadratic equation is given in standard form.

Example 13.3.6 Solve $x^2 - 14x + 11 = 0$ by completing the square.

Explanation. We see that the polynomial on the left side is not a perfect square trinomial, so we need to complete the square. We subtract 11 from both sides so we can add the missing term on the left.

$$\begin{aligned} x^2 - 14x + 11 &= 0 \\ x^2 - 14x &= -11 \end{aligned}$$

Next comes the completing-the-square step. We need to add the correct number to both sides of the equation to make the left side a perfect square. Remember that Fact 13.3.3 states that we need to use $\left(\frac{b}{2}\right)^2$ for this. In our case, $b = -14$, so $\left(\frac{b}{2}\right)^2 = \left(\frac{-14}{2}\right)^2 = 49$

$$\begin{aligned} x^2 - 14x + 49 &= -11 + 49 \\ (x-7)^2 &= 38 \end{aligned}$$

$$x - 7 = -\sqrt{38} \qquad \text{or} \qquad x - 7 = \sqrt{38}$$

$$x = 7 - \sqrt{38} \quad \text{or} \quad x = 7 + \sqrt{38}$$

The solution set is $\{7 - \sqrt{38}, 7 + \sqrt{38}\}$.

Checkpoint 13.3.7 Complete the square to solve for y in $y^2 - 20y - 21 = 0$.

Explanation. To complete the square, first move the constant term to the right side of the equation. Then use Fact 13.3.3 to find $\left(\frac{b}{2}\right)^2$ to add to both sides.

$$\begin{aligned} y^2 - 20y - 21 &= 0 \\ y^2 - 20y &= 21 \end{aligned}$$

In our case, $b = -20$, so $\left(\frac{b}{2}\right)^2 = \left(\frac{-20}{2}\right)^2 = 100$

$$\begin{aligned} y^2 - 20y + 100 &= 21 + 100 \\ (y - 10)^2 &= 121 \end{aligned}$$

$$\begin{aligned} y - 10 = -11 \quad &\text{or} \quad y - 10 = 11 \\ y = -1 \quad &\text{or} \quad y = 21 \end{aligned}$$

The solution set is $\{-1, 21\}$.

So far, the value of b has been even each time, which makes $\frac{b}{2}$ a whole number. When b is odd, we end up adding a fraction to both sides. Here is an example.

Example 13.3.8 Complete the square to solve for z in $z^2 - 3z - 10 = 0$.

Explanation. First move the constant term to the right side of the equation:

$$\begin{aligned} z^2 - 3z - 10 &= 0 \\ z^2 - 3z &= 10 \end{aligned}$$

Next, to complete the square, we need to find the right number to add to both sides. According to Fact 13.3.3, we need to divide the value of b by 2 and then square the result to find the right number. First, divide by 2:

$$\frac{b}{2} = \frac{-3}{2} = -\frac{3}{2} \tag{13.3.1}$$

and then we square that result:

$$\left(-\frac{3}{2}\right)^2 = \frac{9}{4} \tag{13.3.2}$$

Now we can add the $\frac{9}{4}$ from Equation (13.3.2) to both sides of the equation to complete the square.

$$z^2 - 3z + \frac{9}{4} = 10 + \frac{9}{4}$$

Now, to factor the seemingly complicated expression on the left, just know that it should always factor using the number from the first step in the completing the square process, Equation (13.3.1).

$$\left(z - \frac{3}{2}\right)^2 = \frac{49}{4}$$

$$
\begin{aligned}
z - \frac{3}{2} &= -\frac{7}{2} & \text{or} && z - \frac{3}{2} &= \frac{7}{2} \\
z &= \frac{3}{2} - \frac{7}{2} & \text{or} && z &= \frac{3}{2} + \frac{7}{2} \\
z &= -\frac{4}{2} & \text{or} && z &= \frac{10}{2} \\
z &= -2 & \text{or} && z &= 5
\end{aligned}
$$

The solution set is $\{-2, 5\}$.

In each of the previous examples, the value of a was equal to 1. This is necessary for our missing term formula to work. When a is not equal to 1 we will divide both sides by a. Let's look at an example of that.

Example 13.3.9 Solve for r in $2r^2 + 2r = 3$ by completing the square.

Explanation. Because there is a leading coefficient of 2, we divide both sides by 2.

$$
\begin{aligned}
2r^2 + 2r &= 3 \\
\frac{2r^2}{2} + \frac{2r}{2} &= \frac{3}{2} \\
r^2 + r &= \frac{3}{2}
\end{aligned}
$$

Next, we complete the square. Since $b = 1$, first,

$$\frac{b}{2} = \frac{1}{2} \tag{13.3.3}$$

and next, squaring that, we have

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}. \tag{13.3.4}$$

So we add $\frac{1}{4}$ from Equation (13.3.4) to both sides of the equation:

$$
\begin{aligned}
r^2 + r + \frac{1}{4} &= \frac{3}{2} + \frac{1}{4} \\
r^2 + r + \frac{1}{4} &= \frac{6}{4} + \frac{1}{4}
\end{aligned}
$$

Here, remember that we always factor with the number found in the first step of completing the square, Equation (13.3.3).

$$\left(r + \frac{1}{2}\right)^2 = \frac{7}{4}$$

$$
\begin{aligned}
r + \frac{1}{2} &= -\frac{\sqrt{7}}{2} & \text{or} && r + \frac{1}{2} &= \frac{\sqrt{7}}{2} \\
r &= -\frac{1}{2} - \frac{\sqrt{7}}{2} & \text{or} && r &= -\frac{1}{2} + \frac{\sqrt{7}}{2} \\
r &= \frac{-1 - \sqrt{7}}{2} & \text{or} && r &= \frac{-1 + \sqrt{7}}{2}
\end{aligned}
$$

The solution set is $\left\{\frac{-1-\sqrt{7}}{2}, \frac{-1+\sqrt{7}}{2}\right\}$.

13.3.2 Deriving the Quadratic Formula by Completing the Square

In Section 7.2, we learned the Quadratic Formula. You may have wondered where the formula comes from, and now that we know how to complete the square, we can derive it. We will solve the standard form equation $ax^2 + bx + c = 0$ for x.

First, we subtract c from both sides and divide both sides by a.

$$\begin{aligned} ax^2 + bx + c &= 0 \\ ax^2 + bx &= -c \\ \frac{ax^2}{a} + \frac{bx}{a} &= -\frac{c}{a} \\ x^2 + \frac{b}{a}x &= -\frac{c}{a} \end{aligned}$$

Next, we complete the square by taking half of the middle coefficient and squaring it. First,

$$\frac{\frac{b}{a}}{2} = \frac{b}{2a} \tag{13.3.5}$$

and then squaring that we have

$$\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} \tag{13.3.6}$$

We add the $\frac{b^2}{4a^2}$ from Equation (13.3.6) to both sides of the equation:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = +\frac{b^2}{4a^2} - \frac{c}{a}$$

Remember that the left side always factors with the value we found in Equation (13.3.5). So we have:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

To find a common denominator on the right, we multiply by $4a$ in the numerator and denominator on the second term.

$$\begin{aligned} \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} \cdot \frac{4a}{4a} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \end{aligned}$$

Now that we have completed the square, we can see that the x-value of the vertex is $-\frac{b}{2a}$. That is the vertex formula. Next, we solve the equation using the square root property to find the Quadratic Formula.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\begin{aligned} x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2-4ac}}{2a} \\ x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} \\ x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \end{aligned}$$

This shows us that the solutions to the equation $ax^2 + bx + c = 0$ are $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

13.3.3 Putting Quadratic Functions in Vertex Form

In Section 13.2, we learned about the vertex form of a parabola, which allows us to quickly read the coordinates of the vertex. We can now use the method of completing the square to put a quadratic function in vertex form. Completing the square with a function is a little different than with an equation so we will start with an example.

Example 13.3.10 Write a formula in vertex form for the function q defined by $q(x) = x^2 + 8x$

Explanation. The formula is in the form $x^2 + bx$, so we need to add $\left(\frac{b}{2}\right)^2$ to complete the square by Fact 13.3.3. When we had an equation, we could add the same quantity to both sides. But now we do not wish to change the left side, since we are trying to end up with a formula that still says $q(x) = \ldots$. Instead, we add *and subtract* the term from the right side in order to maintain equality. In this case,

$$\begin{aligned} \left(\frac{b}{2}\right)^2 &= \left(\frac{8}{2}\right)^2 \\ &= 4^2 \\ &= 16 \end{aligned}$$

To maintain equality, we both add *and* subtract 16 on the same side of the equation. It is functionally the same as adding 0 on the right, but the 16 makes it possible to factor the expression in a particular way:

$$\begin{aligned} q(x) &= x^2 + 8x + 16 - 16 \\ &= \left(x^2 + 8x + 16\right) - 16 \\ &= (x+4)^2 - 16 \end{aligned}$$

Now that we have completed the square, our function is in vertex form. The vertex is $(-4, -16)$. One way to verify that our work is correct is to graph the original version of the function and check that the vertex is where it should be.

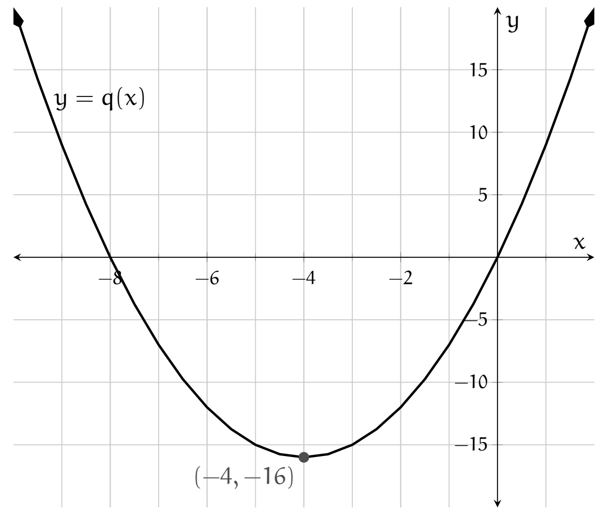


Figure 13.3.11: Graph of $y = x^2 + 8x$

Let's look at a function that has a constant term and see how to complete the square.

Example 13.3.12 Write a formula in vertex form for the function f defined by $f(x) = x^2 - 12x + 3$

Explanation. To complete the square, we need to add and subtract $\left(-\frac{12}{2}\right)^2 = (-6)^2 = 36$ on the right side.

$$\begin{aligned} f(x) &= x^2 - 12x + 36 - 36 + 3 \\ &= \left(x^2 - 12x + 36\right) - 36 + 3 \\ &= (x - 6)^2 - 33 \end{aligned}$$

The vertex is $(6, -33)$.

In the first two examples, a was equal to 1. When a is not equal to one, we have an additional step. Since we are working with an expression where we intend to preserve the left side as $f(x) = \ldots$, we cannot divide both sides by a. Instead we factor a out of the first two terms. Let's look at an example of that.

Example 13.3.13 Write a formula in vertex form for the function g defined by $g(x) = 5x^2 + 20x + 25$

Explanation. Before we can complete the square, we factor the 5 out of the first two terms.

$$g(x) = 5\left(x^2 + 4x\right) + 25$$

Now we complete the square inside the parentheses by adding and subtracting $\left(\frac{4}{2}\right)^2 = 2^2 = 4$.

$$g(x) = 5\left(x^2 + 4x + 4 - 4\right) + 25$$

Notice that the constant that we subtracted is inside the parentheses, but it will not be part of our perfect square trinomial. In order to bring it outside, we need to multiply it by 5. We are distributing the 5 to that term so we can combine it with the outside term.

$$\begin{aligned} g(x) &= 5\left(\left(x^2 + 4x + 4\right) - 4\right) + 25 \\ &= 5\left(x^2 + 4x + 4\right) - 5 \cdot 4 + 25 \end{aligned}$$

$$= 5\left(x+2\right)^2 - 20 + 25$$
$$= 5\left(x+2\right)^2 + 5$$

The vertex is $(-2, 5)$.

Here is an example that includes fractions.

Example 13.3.14 Write a formula in vertex form for the function h defined by $h(x) = -3x^2 - 4x - \frac{7}{4}$

Explanation. First, we factor the leading coefficient out of the first two terms.

$$\begin{aligned} h(x) &= -3x^2 - 4x - \frac{7}{4} \\ &= -3\left(x^2 + \frac{4}{3}x\right) - \frac{7}{4} \end{aligned}$$

Next, we complete the square for $x^2 + \frac{4}{3}x$ inside the grouping symbols by adding and subtracting the right number. To find that number, we divide the value of b by two and square the result. That looks like:

$$\frac{b}{2} = \frac{\frac{4}{3}}{2} = \frac{4}{3} \cdot \frac{1}{2} = \frac{2}{3} \tag{13.3.7}$$

and then,

$$\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9} \tag{13.3.8}$$

Adding and subtracting the value from Equation (13.3.8), we have:

$$\begin{aligned} h(x) &= -3\left(x^2 + \frac{4}{3}x + \frac{4}{9} - \frac{4}{9}\right) - \frac{7}{4} \\ &= -3\left(\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) - \frac{4}{9}\right) - \frac{7}{4} \\ &= -3\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) - \left(3 \cdot -\frac{4}{9}\right) - \frac{7}{4} \end{aligned}$$

Remember that when completing the square, the expression should always factor with the number found in the first step of the completing-the-square process, Equation (13.3.7).

$$\begin{aligned} &= -3\left(x + \frac{2}{3}\right)^2 + \frac{4}{3} - \frac{7}{4} \\ &= -3\left(x + \frac{2}{3}\right)^2 + \frac{16}{12} - \frac{21}{12} \\ &= -3\left(x + \frac{2}{3}\right)^2 - \frac{5}{12} \end{aligned}$$

The vertex is $\left(-\frac{2}{3}, -\frac{5}{12}\right)$.

Completing the square can also be used to find a minimum or maximum in an application.

Example 13.3.15 In Example 5.4.16, we learned that artist Tyrone's annual income from paintings can be modeled by $I(x) = -100x^2+1000x+20000$, where x is the number of times he will raise the price per painting by \$20.00. To maximize his income, how should Tyrone set his price per painting? Find the maximum by completing the square.

Explanation. To find the maximum is essentially the same as finding the vertex, which we can find by completing the square. To complete the square for $I(x) = -100x^2 + 1000x + 20000$, we start by factoring out the -100 from the first two terms:

$$\begin{aligned} I(x) &= -100x^2 + 1000x + 20000 \\ &= -100\left(x^2 - 10x\right) + 20000 \end{aligned}$$

Next, we complete the square for $x^2 - 10x$ by adding and subtracting $\left(-\frac{10}{2}\right)^2 = (-5)^2 = 25$.

$$\begin{aligned} I(x) &= -100\left(x^2 - 10x + 25 - 25\right) + 20000 \\ &= -100\left(\left(x^2 - 10x + 25\right) - 25\right) + 20000 \\ &= -100\left(x^2 - 10x + 25\right) - (100 \cdot -25) + 20000 \\ &= -100(x-5)^2 + 2500 + 20000 \\ &= -100(x-5)^2 + 22500 \end{aligned}$$

The vertex is the point $(5, 22500)$. This implies Tyrone should raise the price per painting 5 times, which is $5 \cdot 20 = 100$ dollars. He would sell $100 - 5(5) = 75$ paintings. This would make the price per painting $200 + 100 = 300$ dollars, and his annual income from paintings would become \$22,500 by this model.

13.3.4 Graphing Quadratic Functions by Hand

Now that we know how to put a quadratic function in vertex form, let's review how to graph a parabola by hand.

Example 13.3.16 Graph the function h defined by $h(x) = 2x^2 + 4x - 6$ by determining its key features algebraically.

Explanation. To start, we'll note that this function opens upward because the leading coefficient, 2, is positive.

Now we may complete the square to find the vertex. We factor the 2 out of the first two terms, and then add and subtract $\left(\frac{2}{2}\right)^2 = 1^2 = 1$ on the right side.

$$\begin{aligned} h(x) &= 2\left(x^2 + 2x\right) - 6 \\ &= 2\left[x^2 + 2x + 1 - 1\right] - 6 \\ &= 2\left[\left(x^2 + 2x + 1\right) - 1\right] - 6 \\ &= 2\left(x^2 + 2x + 1\right) - (2 \cdot 1) - 6 \\ &= 2\left(x + 1\right)^2 - 2 - 6 \\ &= 2\left(x + 1\right)^2 - 8 \end{aligned}$$

The vertex is $(-1, -8)$ so the axis of symmetry is the line $x = -1$.

To find the y-intercept, we'll replace x with 0 or read the value of c from the function in standard form:

$$\begin{aligned} h(0) &= 2(0)^2 + 2(0) - 6 \\ &= -6 \end{aligned}$$

The y-intercept is $(0, -6)$ and we can find its symmetric point on the graph, which is $(-2, -6)$.

Next, we'll find the horizontal intercepts. We see this function factors so we write the factored form to get the horizontal intercepts.

$$\begin{aligned} h(x) &= 2x^2 + 4x - 6 \\ &= 2\left(x^2 + 2x - 3\right) \\ &= 2(x-1)(x+3) \end{aligned}$$

The x-intercepts are $(1, 0)$ and $(-3, 0)$.

Now we plot all of the key points and draw the parabola.

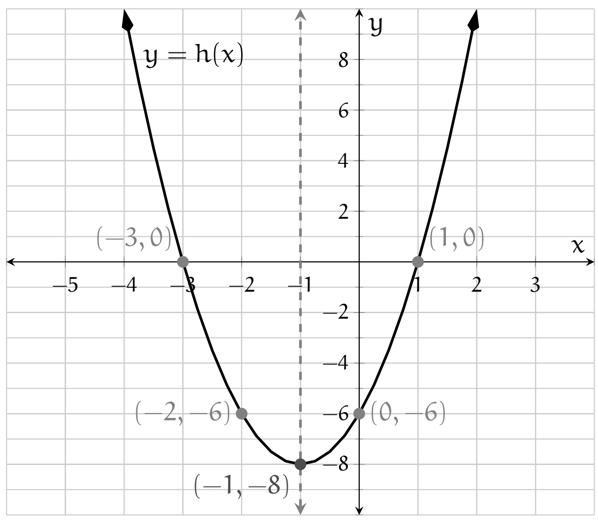


Figure 13.3.17: The graph of $y = 2x^2 + 4x - 6$.

Example 13.3.18 Write a formula in vertex form for the function p defined by $p(x) = -x^2 - 4x - 1$, and find the graph's key features algebraically. Then sketch the graph.

Explanation. In this function, the leading coefficient is negative so it will open downward. To complete the square we first factor -1 out of the first two terms.

$$\begin{aligned} p(x) &= -x^2 - 4x - 1 \\ &= -\left(x^2 + 4x\right) - 1 \end{aligned}$$

Now, we add and subtract the correct number on the right side of the function: $\left(\frac{b}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 2^2 = 4$.

$$\begin{aligned} p(x) &= -\left(x^2 + 4x + 4 - 4\right) - 1 \\ &= -\left(\left(x^2 + 4x + 4\right) - 4\right) - 1 \end{aligned}$$

$$
\begin{aligned}
&= -\left(x^2 + 4x + 4\right) - (-4) - 1 \\
&= -(x+2)^2 + 4 - 1 \\
&= -(x+2)^2 + 3
\end{aligned}
$$

The vertex is $(-2, 3)$ so the axis of symmetry is the line $x = -2$.

We find the y-intercept by looking at the value of c, which is -1. So, the y-intercept is $(0, -1)$ and we can find its symmetric point on the graph, $(-4, -1)$.

The original expression, $-x^2 - 4x - 1$, does not factor so to find the x-intercepts we need to set $p(x) = 0$ and complete the square or use the quadratic formula. Since we just went through the process of completing the square above, we can use that result to save several repetitive steps.

$$
\begin{aligned}
p(x) &= 0 \\
-(x+2)^2 + 3 &= 0 \\
-(x+2)^2 &= -3 \\
(x+2)^2 &= 3
\end{aligned}
$$

$$
\begin{aligned}
x + 2 &= -\sqrt{3} &\quad \text{or} \quad && x + 2 &= \sqrt{3} \\
x &= -2 - \sqrt{3} &\quad \text{or} \quad && x &= -2 + \sqrt{3} \\
x &\approx -3.73 &\quad \text{or} \quad && x &\approx -0.268
\end{aligned}
$$

The x-intercepts are approximately $(-3.7, 0)$ and $(-0.3, 0)$. Now we can plot all of the points and draw the parabola.

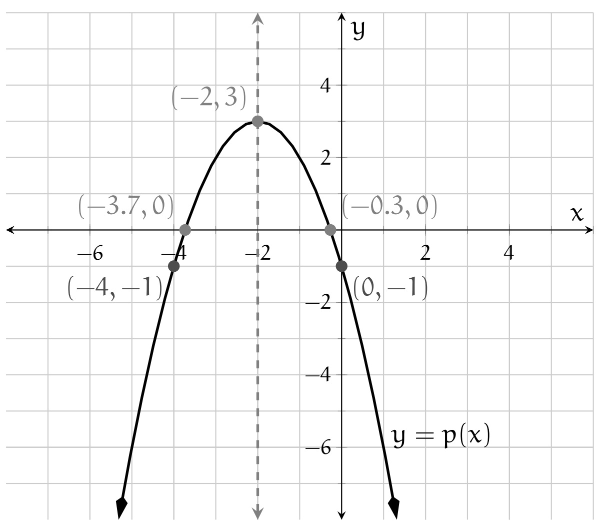


Figure 13.3.19: The graph of $y = -x^2 - 4x - 1$.

13.3.5 Reading Questions

1. For the expression $y = x^2 + 10x - 9$, explain in words what is the next step to complete the square.

2. Why is completing the square called completing the square?

3. How can you check that they completed the square correctly?

13.3.6 Exercises

Review and Warmup

1. Use a square root to solve $(t+6)^2 = 16$.

2. Use a square root to solve $(x-1)^2 = 49$.

3. Use a square root to solve $(3x-7)^2 = 49$.

4. Use a square root to solve $(8y+1)^2 = 25$.

5. Use a square root to solve $(y-1)^2 = 19$.

6. Use a square root to solve $(r-8)^2 = 7$.

7. Use a square root to solve $r^2 + 8r + 16 = 64$.

8. Use a square root to solve $r^2 - 4r + 4 = 9$.

9. Use a square root to solve $4t^2 + 8t + 4 = 4$.

10. Use a square root to solve $49t^2 - 126t + 81 = 49$.

11. Use a square root to solve $16x^2 - 8x + 1 = 10$.

12. Use a square root to solve $81x^2 + 126x + 49 = 5$.

Completing the Square to Solve Equations Solve the equation by completing the square.

13. $y^2 - 6y = 16$

14. $y^2 - 4y = -3$

15. $r^2 - 5r = -6$

16. $r^2 - 3r = -2$

17. $r^2 + 6r = -2$

18. $t^2 + 10t = 3$

19. $t^2 - 6t + 5 = 0$

20. $x^2 - 10x + 9 = 0$

21. $x^2 - 9x + 14 = 0$

22. $y^2 + 9y + 8 = 0$

23. $y^2 - 8y - 1 = 0$

24. $r^2 + 8r + 8 = 0$

25. $12r^2 - 44r + 35 = 0$

26. $3r^2 - 2r - 1 = 0$

27. $2t^2 + 5t - 4 = 0$

28. $2t^2 - 2t - 5 = 0$

Converting to Vertex Form

29. Consider $f(x) = x^2 + 8x - 1$.

a. Give the formula for f in vertex form.

b. What is the vertex of the parabola graph of f?

30. Consider $g(r) = r^2 - 2r - 3$.

a. Give the formula for g in vertex form.

b. What is the vertex of the parabola graph of g?

31. Consider $h(y) = y^2 + 5y + 5$.

a. Give the formula for h in vertex form.

b. What is the vertex of the parabola graph of h?

32. Consider $h(x) = x^2 - 5x + 2$.

a. Give the formula for h in vertex form.

b. What is the vertex of the parabola graph of h?

33. Consider $F(r) = 6r^2 - 12r - 2$.

a. Give the formula for F in vertex form.

b. What is the vertex of the parabola graph of F?

34. Consider $G(y) = 2y^2 - 16y - 2$.

a. Give the formula for G in vertex form.

b. What is the vertex of the parabola graph of G?

Domain and Range Complete the square to convert the quadratic function from standard form to vertex form, and use the result to find the function's domain and range.

35. $f(x) = x^2 - 10x + 15$

36. $f(x) = x^2 - 14x + 52$

37. $f(x) = -x^2 + 18x - 85$

38. $f(x) = -x^2 - 18x - 71$

39. $f(x) = 2x^2 + 28x + 100$

40. $f(x) = 4x^2 + 40x + 95$

41. $f(x) = -5x^2 - 20x - 11$

42. $f(x) = -5x^2 + 10x - 8$

Sketching Graphs of Quadratic Functions Graph each function by algebraically determining its key features. Then state the domain and range of the function.

43. $f(x) = x^2 - 7x + 12$

44. $f(x) = x^2 + 5x - 14$

45. $f(x) = -x^2 - x + 20$

46. $f(x) = -x^2 + 4x + 21$

47. $f(x) = x^2 - 8x + 16$

48. $f(x) = x^2 + 6x + 9$

49. $f(x) = x^2 - 4$

50. $f(x) = x^2 - 9$

51. $f(x) = x^2 + 6x$

52. $f(x) = x^2 - 8x$

53. $f(x) = -x^2 + 5x$

54. $f(x) = -x^2 + 16$

55. $f(x) = x^2 + 4x + 7$

56. $f(x) = x^2 - 2x + 6$

57. $f(x) = x^2 + 2x - 5$

58. $f(x) = x^2 - 6x + 2$

59. $f(x) = -x^2 + 4x - 1$

60. $f(x) = -x^2 - x + 3$

61. $f(x) = 2x^2 - 4x - 30$

62. $f(x) = 3x^2 + 21x + 36$

Information from Vertex Form

63. Find the minimum value of the function

$$f(x) = 7x^2 - 6x + 6$$

64. Find the minimum value of the function

$$f(x) = 8x^2 + 8x - 6$$

65. Find the maximum value of the function

$$f(x) = x - 9x^2 + 3$$

66. Find the maximum value of the function

$$f(x) = -\left(10x^2 + 7x + 10\right)$$

67. Find the range of the function

$$f(x) = 7x - x^2 - 1$$

68. Find the range of the function

$$f(x) = 8 - 2x^2$$

69. Find the range of the function

$$f(x) = 3x^2 - 8x - 4$$

70. Find the range of the function

$$f(x) = 4x^2 + 6x + 4$$

71. If a ball is throw straight up with a speed of 60 $\frac{\text{ft}}{\text{s}}$, its height at time t (in seconds) is given by

$$h(t) = -8t^2 + 60t + 2$$

Find the maximum height the ball reaches.

72. If a ball is throw straight up with a speed of 62 $\frac{\text{ft}}{\text{s}}$, its height at time t (in seconds) is given by

$$h(t) = -8t^2 + 62t + 2$$

Find the maximum height the ball reaches.

Challenge

73. Let $f(x) = x^2 + bx + c$. Let b and c be real numbers. Complete the square to find the vertex of $f(x) = x^2 + bx + c$. Write $f(x)$ in vertex form and then state the vertex.

13.4 Absolute Value Equations

Whether it's a washer, nut, bolt, or gear, when a machine part is made, it must be made to fit with all of the other parts of the system. Since no manufacturing process is perfect, there are small deviations from the norm when each piece is made. In fact, manufacturers have a *range* of acceptable values for each measurement of every screw, bolt, etc.

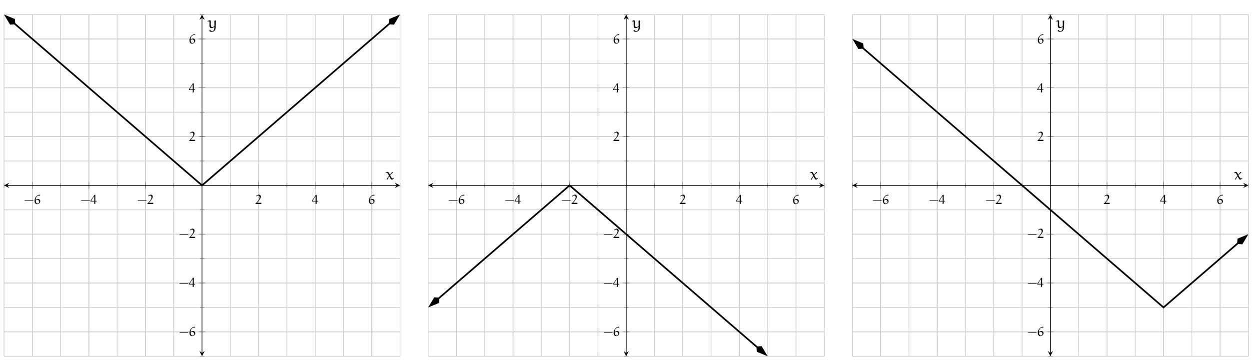
Let's say we were examining some new bolts just out of the factory. The manufacturer specifies that each bolt should be within a *tolerance* of 0.04 mm to 10 mm in diameter. So the lowest diameter that the bolt could be to make it through quality assurance is 0.04 mm smaller than 10 mm, which is 9.96 mm. Similarly, the largest diameter that the bolt could be is 0.04 mm larger than 10 mm, which is 10.04 mm.

To write an equation that describes the minimum and maximum deviation from average, we want the difference between the actual diameter and the specification to be equal to 0.04 mm. Since absolute values are used to describe distances, we can summarize our thoughts mathematically as $|x - 10| = 0.04$, where x represents the diameter of an acceptably sized bolt, in millimeters. This equation says the same thing as the lowest diameter that the bolt could be to make it through quality assurance is 9.96 mm and the largest diameter that the bolt could be is 10.04 mm.

In this section we will examine a variety of problems that relate to this sort of math with absolute values.

13.4.1 Graphs of Absolute Value Functions

Absolute value functions have generally the same shape. They are usually described as "V"-shaped graphs and the tip of the "V" is called the **vertex**. A few graphs of various absolute value functions are shown in Figure 13.4.2. In general, the domain of an absolute value function (where there is a polynomial inside the absolute value) is $(-\infty, \infty)$.



(a) $y = |x|$ **(b)** $y = -|x + 2|$ **(c)** $y = |x - 4| - 5$

Figure 13.4.2

Example 13.4.3 Let $h(x) = -2|x - 3| + 5$. Using technology, create table of values with x-values from -3 to 3, using an increment of 1. Then sketch a graph of $y = h(x)$. State the domain and range of h.

Explanation.

x	y
-3	-7
-2	-5
-1	-3
0	-1
1	1
2	3
3	5

Figure 13.4.4: Table for $y = h(x)$.

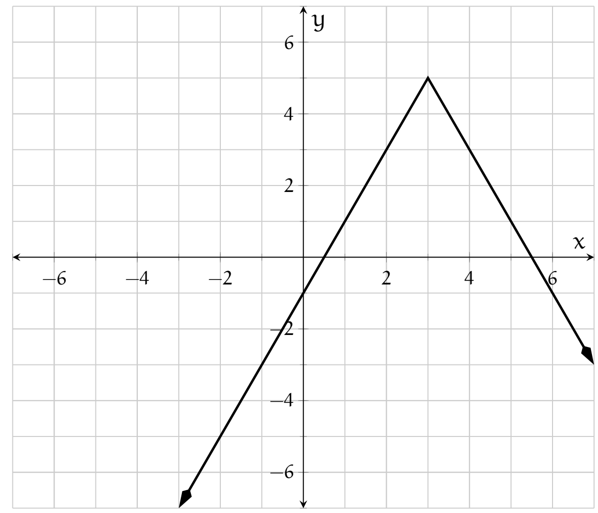


Figure 13.4.5: Graph of $y = h(x)$

The graph indicates that the domain is (∞, ∞) as it goes to the right and left indefinitely. The range is $(-\infty, 5]$.

Example 13.4.6 Let $j(x) = \big||x+1| - 2\big| - 1$. Using technology, create table of values with x-values from -5 to 5, using an increment of 1 and sketch a graph of $y = j(x)$. State the domain and range of j.

Explanation. This is a strange one because it has an absolute value within an absolute value.

x	y
-5	1
-4	0
-3	-1
-2	0
-1	1
0	0
1	-1
2	0
3	1
4	2
5	3

Figure 13.4.7: A table of values for $y = j(x)$.

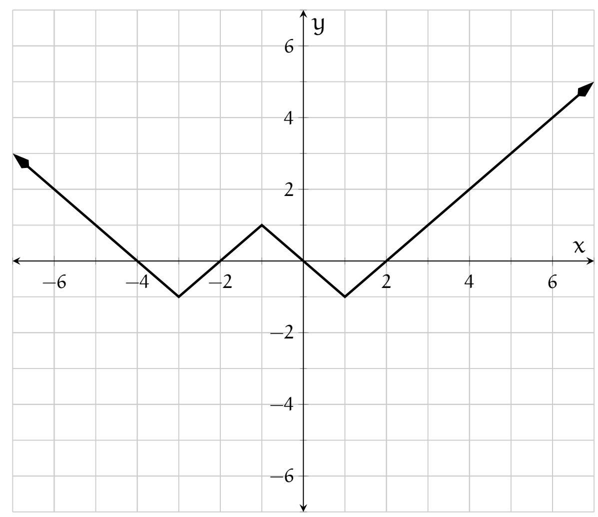


Figure 13.4.8: $y = \big||x+1| - 2\big| - 1$

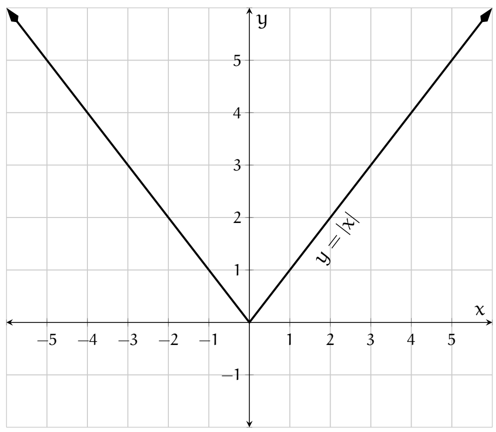
The graph indicates that the domain is (∞, ∞) as it goes to the right and left indefinitely. The range is $[-1, \infty)$.

13.4.2 Solving Absolute Value Equations with One Absolute Value

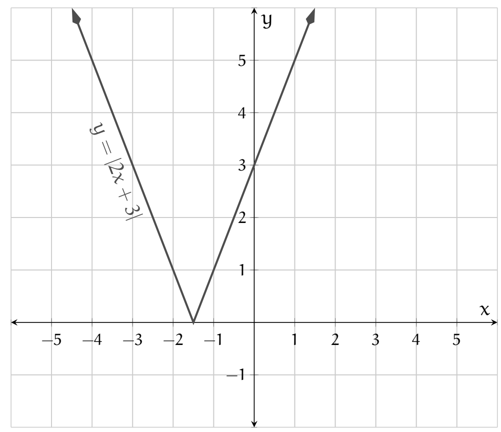
We can solve absolute value equations graphically.

Example 13.4.9 Solve the equations graphically using the graphs provided.

a. $|x| = 3$

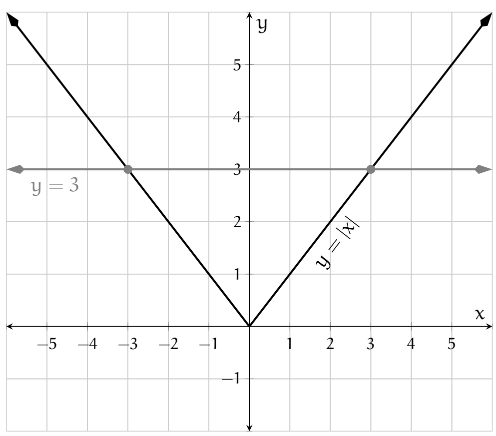


b. $|2x + 3| = 5$



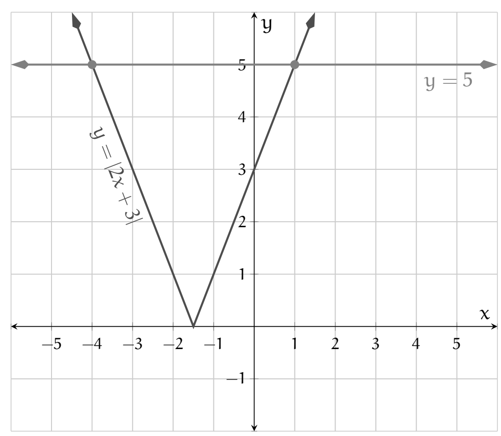
Explanation. To solve the equations graphically, first we need to graph the right sides of the equations also.

a. $|x| = 3$



Since the graph of $y = |x|$ crosses $y = 3$ at the x-values -3 and 3, the solution set to the equation $|x| = 3$ must be $\{-3, 3\}$.

b. $|2x + 3| = 5$



Since the graph of $y = |2x + 3|$ crosses $y = 5$ at the x-values -4 and 1, the solution set to the equation $|2x + 3| = 5$ must be $\{-4, 1\}$.

Remark 13.4.10 Please note that there is a big difference between the expression $|3|$ and the equation $|x| = 3$.

1. The expression $|3|$ is describing the distance from 0 to the number 3. The distance is just 3. So $|3| = 3$.
2. The equation $|x| = 3$ is asking you to find the numbers that are a distance of 3 from 0. These two numbers are 3 and -3.

Let's solve some absolute value equations algebraically. To motivate this, we will think about what an absolute value equation means in terms of the "distance from zero" definition of absolute value. If

$$|X| = n,$$

where $n \geq 0$, then this means that we want all of the numbers, X, that are a distance n from 0. Since we can only go left or right along the number line, this is describing both $X = n$ as well as $X = -n$.

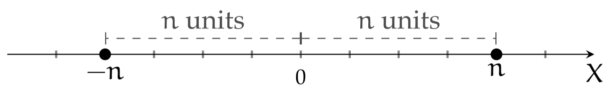


Figure 13.4.11: A Numberline with Points a Distance n from 0

Let's summarize this.

Fact 13.4.12 Equations with an Absolute Value Expression. *Let n be a non-negative number and X be an algebraic expression. Then the equation*

$$|X| = n$$

has the same solutions as

$$X = n \text{ or } X = -n.$$

Example 13.4.13 Solve the absolute value equations using Fact 13.4.12. Write solutions in a solution set.

a. $|x| = 6$

b. $|x| = -4$

c. $|5x - 7| = 23$

d. $|14 - 3x| = 8$

e. $|3 - 4x| = 0$

Explanation.

a. Fact 13.4.12 says that the equation $|x| = 6$ is the same as

$$x = 6 \text{ or } x = -6.$$

Thus the solution set is $\{6, -6\}$.

b. Fact 13.4.12 doesn't actually apply to the equation $|x| = -4$ because the value on the right side is *negative*. How often is an absolute value of a number negative? Never! Thus, there are no solutions and the solution set is the empty set, denoted $\emptyset$.

c. The equation $|5x - 7| = 23$ breaks into two pieces, each of which needs to be solved independently.

$$\begin{aligned} 5x - 7 &= 23 & &\text{or} & 5x - 7 &= -23 \\ 5x &= 30 & &\text{or} & 5x &= -16 \\ x &= 6 & &\text{or} & x &= -\frac{16}{5} \end{aligned}$$

Thus the solution set is $\left\{6, -\frac{16}{5}\right\}$.

d. The equation $|14 - 3x| = 8$ breaks into two pieces, each of which needs to be solved independently.

$$\begin{aligned} 14 - 3x &= 8 & &\text{or} & 14 - 3x &= -8 \\ -3x &= -6 & &\text{or} & -3x &= -22 \end{aligned}$$

$$x = 2 \qquad \text{or} \qquad x = \frac{22}{3}$$

Thus the solution set is $\left\{2, \frac{22}{3}\right\}$.

e. The equation $|3 - 4x| = 0$ breaks into two pieces, each of which needs to be solved independently.

$$3 - 4x = 0 \qquad \text{or} \qquad 3 - 4x = -0$$

Since these are identical equations, all we have to do is solve one equation.

$$\begin{aligned} 3 - 4x &= 0 \\ -4x &= -3 \\ x &= \frac{3}{4} \end{aligned}$$

Thus, the equation $|3 - 4x| = 0$ only has one solution, and the solution set is $\left\{\frac{3}{4}\right\}$.

13.4.3 Solving Absolute Value Equations with Two Absolute Values

Example 13.4.14 Let's graphically solve an equation with an absolute value expression on each side: $|x| = |2x + 6|$. Since $|x| = 3$ had two solutions as we saw in Example 13.4.9, you might be wondering how many solutions $|x| = |2x + 6|$ will have. Make a graph to find out what the solutions of the equation are.

Explanation.

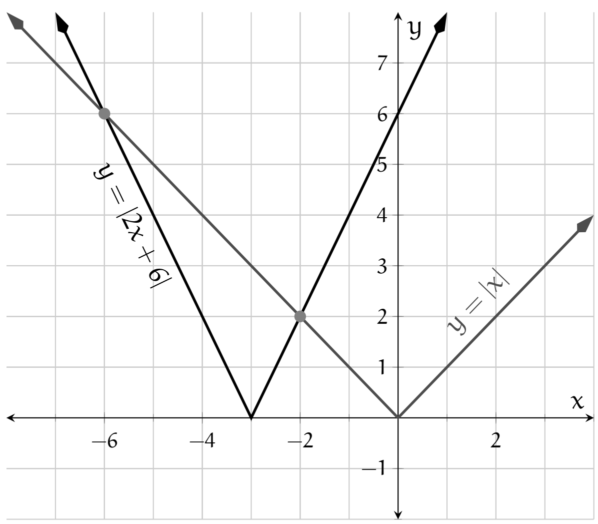


Figure 13.4.15: $y = |x|$ and $y = |2x + 6|$

Figure 13.4.15 shows that there are also two points of intersection between the graphs of $y = |x|$ and $y = |2x + 6|$. The solutions to the equation $|x| = |2x + 6|$ are the x-values where the graphs cross. So, the solution set is $\{-6, -2\}$.

Example 13.4.16 Solve the equation $|x+1| = |2x-4|$ graphically.

Explanation.

First break up the equation into the left side and the right side and graph each separately, as in $y = |x+1|$ and $y = |2x-4|$. We can see in the graph that the graphs intersect twice. The x-values of those intersections are 1 and 5 so the solution set to the equation $|x+1| = |2x-4|$ is $\{1, 5\}$.

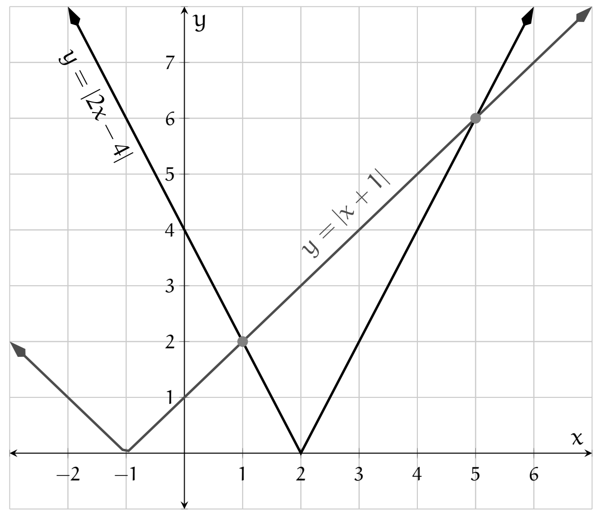


Figure 13.4.17: $y = |x+1|$ and $y = |2x-4|$

Fortunately, this kind of equation also has a rule to solve these types of equations algebraically that is similar to the rule for equations with one absolute value.

Fact 13.4.18 Equations with Two Absolute Value Expressions. *Let X and Y be linear algebraic expressions. Then, the equation*

$$|X| = |Y|$$

has the same solutions as

$$X = Y \text{ or } X = -Y.$$

Remark 13.4.19 You might wonder why the negative sign "has" to go on the right side of the equation in $X = -Y$. It doesn't; it can go on either side of the equation. The equations $X = -Y$ and $-X = Y$ are equivalent. Similarly, $-X = -Y$ is equivalent to $X = Y$. That's why we only need to solve two of the four possible equations.

Example 13.4.20 Solve the equations using Fact 13.4.18.

a. $|x-4| = |3x-2|$

b. $\left|\frac{1}{2}x+1\right| = \left|\frac{1}{3}x+2\right|$

c. $|x-2| = |x+1|$

d. $|x-1| = |1-x|$

Explanation.

a. The equation $|x-4| = |3x-2|$ breaks down into two pieces:

$$\begin{aligned} x-4 &= 3x-2 && \text{or} && x-4 = -(3x-2) \\ x-4 &= 3x-2 && \text{or} && x-4 = -3x+2 \\ -2 &= 2x && \text{or} && 4x = 6 \\ \frac{-2}{2} &= \frac{2x}{2} && \text{or} && \frac{4x}{4} = \frac{6}{4} \end{aligned}$$

$$
\begin{aligned}
-1 &= x \qquad &\text{or} \qquad && x &= \frac{3}{2}
\end{aligned}
$$

So, the solution set is $\left\{-1, \frac{3}{2}\right\}$.

b. The equation $\left|\frac{1}{2}x + 1\right| = \left|\frac{1}{3}x + 2\right|$ breaks down into two pieces:

$$
\begin{aligned}
\frac{1}{2}x + 1 &= \frac{1}{3}x + 2 &\quad \text{or} \quad && \frac{1}{2}x + 1 &= -\left(\frac{1}{3}x + 2\right) \\
\frac{1}{2}x + 1 &= \frac{1}{3}x + 2 &\quad \text{or} \quad && \frac{1}{2}x + 1 &= -\frac{1}{3}x - 2 \\
6 \cdot \left(\frac{1}{2}x + 1\right) &= 6 \cdot \left(\frac{1}{3}x + 2\right) &\quad \text{or} \quad && 6 \cdot \left(\frac{1}{2}x + 1\right) &= 6 \cdot \left(-\frac{1}{3}x - 2\right) \\
3x + 6 &= 2x + 12 &\quad \text{or} \quad && 3x + 6 &= -2x - 12 \\
x &= 6 &\quad \text{or} \quad && 5x &= -18 \\
x &= 6 &\quad \text{or} \quad && x &= -\frac{18}{5}
\end{aligned}
$$

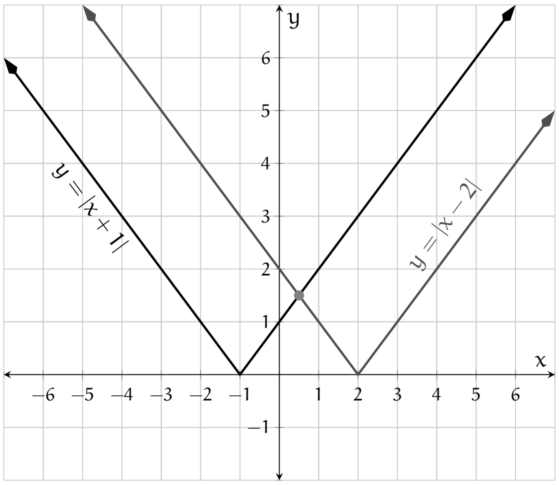
So, the solution set is $\left\{6, -\frac{18}{5}\right\}$.

c. The equation $|x - 2| = |x + 1|$ breaks down into two pieces:

$$
\begin{aligned}
x - 2 &= x + 1 &\quad \text{or} \quad && x - 2 &= -(x + 1) \\
x - 2 &= x + 1 &\quad \text{or} \quad && x - 2 &= -x - 1 \\
x &= x + 3 &\quad \text{or} \quad && 2x &= 1 \\
0 &= 3 &\quad \text{or} \quad && x &= \frac{1}{2}
\end{aligned}
$$

Note that one of the two pieces gives us an equation with no solutions. Since $0 \neq 3$, we can safely ignore this piece. Thus the only solution is $\frac{1}{2}$.

We should visualize this equation graphically because our previous assumption was that two absolute value graphs would cross twice. The graph shows why there is only one crossing: the left and right sides of each "V" are parallel.



d. The equation $|x-1| = |1-x|$ breaks down into two pieces:

$$\begin{aligned} x-1 &= 1-x & \quad\text{or}\quad && x-1 &= -(1-x) \\ x-1 &= 1-x & \quad\text{or}\quad && x-1 &= -1+x \\ 2x &= 2 & \quad\text{or}\quad && x &= 0+x \\ x &= 1 & \quad\text{or}\quad && 0 &= 0 \end{aligned}$$

Note that our second equation is an identity so recall from Section 2.4 that the solution set is "all real numbers."

So, our two pieces have solutions 1 and "all real numbers." Since 1 *is* a real number and we have an *or* statement, our overall solution set is $(-\infty, \infty)$. The graph confirms our answer since the two "V" graphs are coinciding.

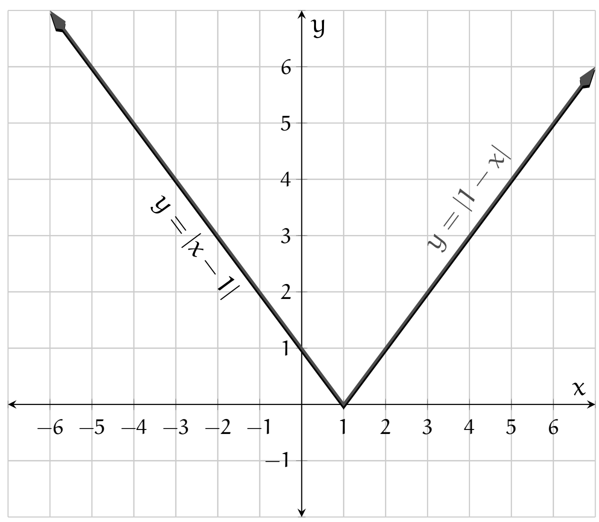


Figure 13.4.21: $y = |x-1|$ and $y = |1-x|$

13.4.4 Reading Questions

1. How many solutions does an absolute value equation typically have?

2. The graph of an absolute value function is typically shaped like which letter?

3. Solving an absolute value equation like $|2x+1| = 3$ is "easy" because we can turn it into two equations of what simpler type?

13.4.5 Exercises

Review and Warmup Solve the equation.

1. $\dfrac{n}{5} - 6 = \dfrac{n}{8}$

2. $\dfrac{q}{3} - 2 = \dfrac{q}{9}$

3. $56 = -8(x+3)$

4. $-60 = -5(r+7)$

5. $2t + 7 = 9t + 6$

6. $2b + 3 = 8b + 2$

Solving Absolute Value Equations Algebraically

7. a. Write the equation $5 = |7x| - 4$ as two separate equations. Neither of your equations should use absolute value.

b. Solve both equations above.

8. a. Write the equation $6 = |4x| - 7$ as two separate equations. Neither of your equations should use absolute value.

b. Solve both equations above.

9. a. Write the equation $\left|6 - \frac{r}{5}\right| = 7$ as two separate equations. Neither of your equations should use absolute value.

b. Solve both equations above.

10. a. Write the equation $\left|8 - \frac{r}{3}\right| = 7$ as two separate equations. Neither of your equations should use absolute value.

b. Solve both equations above.

11.

(a) Verify that the value -1 is a solution to the absolute value equation $\left|\frac{x-3}{2}\right| = 2$.

(b) Verify that the value $\frac{2}{3}$ is a solution to the absolute value inequality $|6x - 5| < 4$.

12.

(a) Verify that the value 8 is a solution to the absolute value equation $\left|\frac{1}{2}x - 2\right| = 2$.

(b)

13. Solve the following equation. $|10x + 9| = 6$

14. Solve the following equation. $|x + 1| = 10$

15. Solve the equation $|2x - 2| = 14$.

16. Solve the equation $|3x + 3| = 18$.

17. Solve: $|b| = 7$

18. Solve: $|t| = 3$

19. Solve: $|x - 7| = 9$

20. Solve: $|x - 3| = 13$

21. Solve: $|2y + 1| = 17$

22. Solve: $|2y + 7| = 11$

23. Solve: $\left|\dfrac{2a - 3}{7}\right| = 1$

24. Solve: $\left|\dfrac{2a - 1}{3}\right| = 3$

25. Solve: $|b| = -6$

26. Solve: $|b| = -8$

27. Solve: $|t + 2| = 0$

28. Solve: $|x + 4| = 0$

29. Solve: $|2 - 3x| = 7$

30. Solve: $|2 - 3y| = 11$

31. Solve: $\left|\frac{1}{4}y + 1\right| = 5$

32. Solve: $\left|\frac{1}{2}a + 5\right| = 3$

33. Solve: $|0.9 - 0.8a| = 2$

34. Solve: $|0.6 - 0.2b| = 5$

35. Solve: $|b + 3| - 6 = 6$

36. Solve: $|t + 9| - 2 = 4$

37. Solve: $|4t - 12| + 2 = 2$

38. Solve: $|2x - 10| + 7 = 7$

39. Solve: $|y + 9| + 7 = 4$

40. Solve: $|y + 5| + 8 = 6$

41. Solve: $|6a + 1| + 3 = 2$

42. Solve: $|6a + 9| + 8 = 4$

43. Solve the equation *by inspection* (meaning in your head). $|5x + 15| = 0$

44. Solve the equation *by inspection* (meaning in your head). $|5x + 10| = 0$

45. The equation $|x| = |y|$ is satisfied if $x = y$ or $x = -y$. Use this fact to solve the following equation.
$|3x + 4| = |4x + 3|$

46. The equation $|x| = |y|$ is satisfied if $x = y$ or $x = -y$. Use this fact to solve the following equation.
$|4x - 4| = |-x + 4|$

47. The equation $|x| = |y|$ is satisfied if $x = y$ or $x = -y$. Use this fact to solve the following equation.
$|x + 6| = |x - 5|$

48. The equation $|x| = |y|$ is satisfied if $x = y$ or $x = -y$. Use this fact to solve the following equation.
$|x + 6| = |x - 1|$

49. Solve the equation: $|8x - 5| = |7x + 8|$

50. Solve the equation: $|2x - 2| = |9x + 6|$

51. Solve the following equation.
$|x + 4| = |7x - 5|$

52. Solve the following equation.
$|2x - 3| = |10x + 10|$

Challenge

53. Algebraically, solve for x in the equation:

$$5 = |x - 5| + |x - 10|$$

13.5 Solving Mixed Equations

In this section, we will learn to differentiate between different types of equations and recall the various methods used to solve them. Real life doesn't come with instructions, so it is important to develop the skills in this section. One day, you might be faced with a geometry problem in your home or a rational equation in a lab, and it will be your challenge to solve the equation with some strategy.

We have solved a variety of equations throughout this book, and we covered some general equation-solving strategies in Solving Equations in General. Since then, we have covered even more topics, and it seems time to refresh ourselves on everything that we have done so far. Here is a reference guide to all of the sections that cover solving equations. We hope that this section will help pinpoint those that you need help with.

Section 2.1 Solving linear equations.

Section 2.5 Solving linear equations with more than one variable.

Sections 4.2, 4.3 Algebraically solving systems of linear equations.

Section 6.4 Solving equations with roots in them.

Sections 7.1, 7.2, 10.7, 13.3 Solving quadratic equations.

Section 12.5 Solving rational equations.

13.5.1 Types of Equations

The point of this section isn't to compartmentalize your knowledge to help learn small pieces, it's to put all the pieces that we've learned previously together and how to differentiate those pieces from one another. To do that, we need to recall the different types of equations that we have had before.

Linear Equation This is a type of equation where the variable that we are solving for only appears with addition, subtraction, multiplication and division by constant numbers. Examples are $7(x-2) = \frac{3}{5}x+1$ and $rt = 4t+3r-1$ where r is the variable and t is considered a constant. Use the Steps to Solve Linear Equations.

System of Linear Equations This is a grouping of two linear equations. An example is

$$\begin{cases} y = 3x + 1 \\ y = 2x - 5 \end{cases}$$

One can use either substitution or elimination to solve these systems.

Quadratic Equation This is a type of equation where at least one side of the equation is a quadratic function, and the other side is either constant, linear, or quadratic with a different leading coefficient. Examples are $3x^2 + 2x - 4 = 0$ and $6(y - 2)^2 - 1 = 7$. There are several methods to solve quadratic equations including using the square root method, the quadratic formula, factoring, and completing the square.

Radical Equation This is a type of equation where the variable is inside a root of some kind. We usually solve radical equations by isolating the radical and raising both sides to a power to cancel the radical.

Rational Equation This is a type of equation where both sides of the equation are rational functions, although it's possible one side is a very simple rational function like a constant function. Solving these equations involves clearing the denominators and solving the equation that remains.

Absolute Value Equation This is a type of equation where the variable is inside absolute value bars. Solving these equations involves using a rule to convert an equation from absolute value into two separate equations without absolute values.

For all of these equation types, in this section we only concern ourselves with equations in one variable, i.e. the solution will be a number or expression rather than points. For example, the equation $3 = 2x + 5$ has a single solution, -1, whereas the equation $y = 2x + 5$ has infinitely many solutions, all points, that make up the line with slope 2 and vertical intercept $(0, 5)$. The only exceptions that we will be covering are systems of linear equations, which have a point or points as solutions.

Example 13.5.2 Identify the type of equation as linear, a system of linear equations, quadratic, radical, rational, absolute value, or something else.

a. $3 - \sqrt{2x - 3} = x$

b. $2x^2 + 3x = 7$

c. $7 - 2(3x - 5) = x + \sqrt{2}$

d. $\frac{1}{x-2} + \frac{x}{x^2-4} = \frac{3}{x+2}$

e. $|5x - 9| + 2 = 7$

f. $(4x - 1)^2 + 9 = 16$

g. $\sqrt[3]{6x - 5} = 2$

h. $6x^2 - 7x = 20$

i. $\begin{cases} 4x + 2y = 8 \\ 3x - y = 11 \end{cases}$

j. $3^x + 2^x = 1$

Explanation.

a. The equation $3 - \sqrt{2x - 3} = x$ is a radical equation since the variable appears inside the radical.

b. The equation $2x^2 + 3x = 7$ is a quadratic equation since the variable is being squared (but doesn't have any higher power).

c. The equation $7 - 2(3x - 5) = x + \sqrt{2}$ is a linear equation since the variable is only to the first power. The square root in the equation is only on the number 2 and not x, so it doesn't make it a radical equation.

d. The equation $\frac{1}{x-2} + \frac{x}{x^2-4} = \frac{3}{x+2}$ is a rational equation since the variable is present in a denominator.

e. The equation $|5x - 9| + 2 = 7$ is an absolute value equation since the variable is inside an absolute value.

f. The equation $(4x - 1)^2 + 9 = 16$ is a quadratic equation since if we were to distribute everything out, we would have a term with x^2.

g. The equation $\sqrt[3]{6x - 5} = 2$ is a radical equation since the variable is inside the radical.

h. The equation $6x^2 - 7x = 20$ is a quadratic equation since there is a degree-two term.

i. This is a system of linear equations.

j. The equation $3^x + 2^x = 1$ is an equation type that we have not covered and is not listed above.

13.5.2 Solving Mixed Equations

After you have identified which type of equation confronts you, the next step is to consider the methods for solving that type of equation.

Example 13.5.3 Solve the equations using appropriate techniques.

a. $3-\sqrt{2x-3}=x$

b. $2x^2+3x=7$

c. $7-2(3x-5)=x+\sqrt{2}$

d. $\frac{1}{x-2}+\frac{x}{x^2-4}=\frac{3}{x+2}$

e. $|5x-9|+2=7$

f. $(4x-1)^2+9=16$

g. $\sqrt[3]{6x-5}=2$

h. $6x^2-7x=20$

i. $\begin{cases} 4x+2y = 8 \\ 3x-\ \ y = 11 \end{cases}$

j. $2x^2-12x=7$ (using completing the square)

Explanation.

a. Since the equation $3-\sqrt{2x-3}=x$ is a radical equation, we can isolate the radical and then square both sides to cancel the square root. After that, we will solve whatever remains.

$$\begin{aligned} 3-\sqrt{2x-3} &= x \\ -\sqrt{2x-3} &= x-3 \\ \sqrt{2x-3} &= -x+3 \\ \left(\sqrt{2x-3}\right)^2 &= (-x+3)^2 \\ 2x-3 &= x^2-6x+9 \\ 0 &= x^2-8x+12 \end{aligned}$$

We now have a quadratic equation. We will solve by factoring.

$$0=(x-2)(x-6)$$

$$\begin{aligned} x-2&=0 &\quad\text{or}\quad&& x-6&=0 \\ x&=2 &\quad\text{or}\quad&& x&=6 \end{aligned}$$

Every potential solution to a radical equation should be verified to check for any "extraneous solutions".

$$\begin{aligned} 3-\sqrt{2(2)-3} &\stackrel{?}{=} 2 &\quad\text{or}\quad&& 3-\sqrt{2(6)-3} &\stackrel{?}{=} 6 \\ 3-\sqrt{1} &\stackrel{?}{=} 2 &\quad\text{or}\quad&& 3-\sqrt{9} &\stackrel{?}{=} 6 \\ 3-1 &\stackrel{\checkmark}{=} 2 &\quad\text{or}\quad&& 3-3 &\stackrel{\text{no}}{=} 6 \end{aligned}$$

So the solution set is $\{2\}$.

b. Since the equation $2x^2+3x=7$ is quadratic we should consider the square root method, the quadratic formula, factoring, and completing the square. In this case, we will start with the quadratic formula. First, note that we should rearrange the terms in equation into standard form.

$$\begin{aligned} 2x^2+3x&=7\\ 2x^2+3x-7&=0 \end{aligned}$$

Note that $a=2$, $b=3$, and $c=-7$.

$$\begin{aligned} x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\ x&=\frac{-(3)\pm\sqrt{(3)^2-4(2)(-7)}}{2(2)}\\ x&=\frac{-3\pm\sqrt{9+56}}{4}\\ x&=\frac{-3\pm\sqrt{65}}{4} \end{aligned}$$

The solution set is $\left\{\frac{-3+\sqrt{65}}{4}, \frac{-3-\sqrt{65}}{4}\right\}$.

c. Since the equation $7-2(3x-5)=x+\sqrt{2}$ is a linear equation, we isolate the variable step-by-step.

$$\begin{aligned} 7-2(3x-5)&=x+\sqrt{2}\\ 7-6x+10&=x+\sqrt{2}\\ 17-6x&=x+\sqrt{2}\\ 17&=7x+\sqrt{2}\\ 17-\sqrt{2}&=7x\\ \frac{17-\sqrt{2}}{7}&=x \end{aligned}$$

The solution set is $\left\{\frac{17-\sqrt{2}}{7}\right\}$.

d. Since the equation $\frac{1}{x-2}+\frac{x}{x^2-4}=\frac{3}{x+2}$ is a rational equation, we first need to cancel the denominators after factoring and finding the least common denominator.

$$\begin{aligned} \frac{1}{x-2}+\frac{x}{x^2-4}&=\frac{3}{x+2}\\ \frac{1}{x-2}+\frac{x}{(x-2)(x+2)}&=\frac{3}{x+2} \end{aligned}$$

At this point, we note that the least common denominator is $(x-2)(x+2)$. We need to multiply every term by this least common denominator.

$$\frac{1}{x-2}\cdot(x-2)(x+2)+\frac{x}{(x-2)(x+2)}\cdot(x-2)(x+2)=\frac{3}{x+2}\cdot(x-2)(x+2)$$

$$\begin{aligned}\frac{1}{\cancel{x-2}}\cdot(\cancel{x-2})(x+2)+\frac{x}{(\cancel{x-2})(\cancel{x+2})}\cdot(\cancel{x-2})(\cancel{x+2}) &= \frac{3}{\cancel{x+2}}\cdot(x-2)(\cancel{x+2})\\ 1(x+2)+x &= 3(x-2)\\ x+2+x &= 3x-6\\ 2x+2 &= 3x-6\\ 8 &= x\end{aligned}$$

We always check solutions to rational equations to ensure we don't have any "extraneous solutions".

$$\begin{aligned}\frac{1}{8-2}+\frac{8}{8^2-4} &\stackrel{?}{=} \frac{3}{8+2}\\ \frac{1}{6}+\frac{8}{60} &\stackrel{?}{=} \frac{3}{10}\\ \frac{10}{60}+\frac{8}{60} &\stackrel{?}{=} \frac{3}{10}\\ \frac{18}{60} &\stackrel{\checkmark}{=} \frac{3}{10}\end{aligned}$$

So, the solution set is $\{8\}$.

e. Since the equation $|5x-9|+2=7$ is an absolute value equation, we will first isolate the absolute value and then use Equations with an Absolute Value Expression to solve the remaining equation.

$$\begin{aligned}|5x-9|+2 &= 7\\ |5x-9| &= 5\end{aligned}$$

$$\begin{array}{rcr} 5x-9=5 & \text{or} & 5x-9=-5\\ 5x=14 & \text{or} & 5x=4\\ x=\dfrac{14}{5} & \text{or} & x=\dfrac{4}{5}\end{array}$$

The solution set is $\left\{\frac{14}{5},\frac{4}{5}\right\}$.

f. Since the equation $(4x-1)^2+9=16$ is a quadratic equation, we again have several options. Since the variable only appears once in this equation we will use the the square root method to solve.

$$\begin{aligned}(4x-1)^2+9 &= 16\\ (4x-1)^2 &= 7\end{aligned}$$

$$\begin{array}{rcr} 4x-1=\sqrt{7} & \text{or} & 4x-1=-\sqrt{7}\\ 4x=1+\sqrt{7} & \text{or} & 4x=1-\sqrt{7}\\ x=\dfrac{1+\sqrt{7}}{4} & \text{or} & x=\dfrac{1-\sqrt{7}}{4}\end{array}$$

The solution set is $\left\{\frac{1+\sqrt{7}}{4},\frac{1-\sqrt{7}}{4}\right\}$.

g. Since the equation $\sqrt[3]{6x-5}=2$ is a radical equation, we will isolate the radical (which is already done) and then raise both sides to the third power to cancel the cube root.

$$\begin{aligned}\sqrt[3]{6x-5} &= 2\\ \left(\sqrt[3]{6x-5}\right)^3 &= 2^3\\ 6x-5 &= 8\\ 6x &= 13\\ x &= \frac{13}{6}\end{aligned}$$

The solution set is $\left\{\frac{13}{6}\right\}$.

h. Since the equation $6x^2-7x=20$ is a quadratic equation, we again have several options to consider. We will try factoring on this one after first converting it to standard form.

$$\begin{aligned}6x^2-7x &= 20\\ 6x^2-7x-20 &= 0\end{aligned}$$

Here, $ac=-120$ and two numbers that multiply to be -120 but add to be -7 are 8 and -15.

$$\begin{aligned}6x^2+8x-15x-20 &= 0\\ \left(6x^2+8x\right)+(-15x-20) &= 0\\ 2x(3x+4)-5(3x+4) &= 0\\ (2x-5)(3x+4) &= 0\end{aligned}$$

$$\begin{array}{ccc} 2x-5=0 & \text{or} & 3x+4=0\\ x=\dfrac{5}{2} & \text{or} & x=-\dfrac{4}{3}\end{array}$$

The solution set is $\left\{\frac{5}{2},-\frac{4}{3}\right\}$.

i. Since

$$\begin{cases}4x+2y=\ \ 8\\ 3x-\ \ y=11\end{cases}$$

is a system of linear equations, we can either use substitution or elimination to solve. Here we will use elimination. To use elimination, we need to make one variable have equal but opposite sign in the two equations. We will accomplish this by multiplying the second equation by 2.

$$\begin{aligned}3x-y &= 11\\ 2\cdot(3x-y) &= 2\cdot 11\\ 6x-2y &= 22\end{aligned}$$

So our original system becomes:

$$\begin{cases}4x+2y=\ \ 8\\ 6x-2y=22\end{cases}$$

Adding the sides of the equations, we get:

$$\begin{aligned} 10x &= 30 \\ x &= 3 \end{aligned}$$

Now that we have found x, we can substitute that back into one of the equations to find y. We will substitute into the first equation.

$$\begin{aligned} 4(3) + 2y &= 8 \\ 12 + 2y &= 8 \\ 2y &= -4 \\ y &= -2 \end{aligned}$$

So, the solution must be the point $(3, -2)$.

j. Since the equation $2x^2 - 10x = 7$ is quadratic and we are instructed to solve by using completing the square, we should recall how to complete the square, after we have sufficiently simplified. Let's start by dividing all of the terms by 2.

$$\begin{aligned} 2x^2 - 12x &= 7 \\ x^2 - 6x &= \frac{7}{2} \end{aligned}$$

Next, we need to add $\left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 9$ to both sides of the equation.

$$\begin{aligned} x^2 - 6x + 9 &= \frac{7}{2} + 9 \\ (x-3)^2 &= \frac{7}{2} + \frac{18}{2} \\ (x-3)^2 &= \frac{25}{2} \\ x - 3 &= \pm\sqrt{\frac{25}{2}} \\ x - 3 &= \pm\frac{\sqrt{25}}{\sqrt{2}} \\ x - 3 &= \pm\frac{5}{\sqrt{2}} \\ x - 3 &= \pm\frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ x - 3 &= \pm\frac{5\sqrt{2}}{2} \\ x &= 3 \pm \frac{5\sqrt{2}}{2} \end{aligned}$$

So, our solution set is $\left\{3 + \frac{5\sqrt{2}}{2}, 3 - \frac{5\sqrt{2}}{2}\right\}$

13.5.3 Reading Questions

1. What are all the types of equation that *you* know how to solve? (Don't worry about which types you think you *should* know how to solve. Just try to list all the kinds of equations that you know you know.)

2. There are two types of equation that have been covered in this book where it is especially important to verify solutions. What are those two types of equation?

3. What are all the ways that you know of for solving a quadratic equation? (This book has covered five general methods, but answer with as many mehtods as you know you know).

13.5.4 Exercises

Solving Mixed Equations Solve the equation.

1. $\sqrt{y}+72=y$

2. $\sqrt{r}+30=r$

3. $5+8(C-9)=-72-(4-5C)$

4. $4+10(n-7)=-72-(9-5n)$

5. $x^2+6x=27$

6. $x^2-2x=80$

7. $-8-5r+7=-r+6-4r$

8. $-6-8t+3=-t+4-7t$

Solve the equation by completing the square.

9. $2y^2-6y-3=0$

10. $2y^2+8y-3=0$

Solve the equation.

11. $x^2+2x-7=0$

12. $x^2+7x+1=0$

13. Solve: $\left|\dfrac{2y-3}{7}\right|=1$

14. Solve: $\left|\dfrac{2y-1}{3}\right|=3$

Solve the equation.

15. $\dfrac{x+6}{x-4}+\dfrac{9}{x-6}=2$

16. $\dfrac{x+8}{x+2}+\dfrac{9}{x+8}=2$

17. $12-2(y-7)^2=10$

18. $49-5(y-7)^2=4$

19. $14=\dfrac{c}{5}+\dfrac{c}{2}$

20. $3=\dfrac{B}{3}+\dfrac{B}{6}$

21. $r=\sqrt{r+4}+86$

22. $t=\sqrt{t+2}+40$

23. $x^2+8x-9=0$

24. $x^2-3x-70=0$

25. Solve the equation: $|2x-4|=|7x+8|$

26. Solve the equation: $|2x-9|=|3x+5|$

Solve the equation.

27. $x^2+11x=-28$

28. $x^2+8x=-15$

29. $\dfrac{1}{r+8} + \dfrac{8}{r^2+8r} = -\dfrac{1}{4}$

30. $\dfrac{1}{r+2} + \dfrac{2}{r^2+2r} = \dfrac{1}{7}$

31. $x^2 = -6x$

32. $x^2 = -9x$

33. Solve: $|8a+5| + 9 = 6$

34. Solve: $|8a+1| + 6 = 4$

Solve the equation.

35. $59x^2 + 11 = 0$

36. $29x^2 + 17 = 0$

37. Solve: $|t-1| = 15$

38. Solve: $|t-7| = 9$

Solve the equation.

39. $5x^2 = -31x - 44$

40. $5x^2 = -52x - 20$

41. $t = \sqrt{t+7} + 5$

42. $x = \sqrt{x+5} + 7$

43. $\dfrac{1}{x-7} + \dfrac{5}{x+6} = -\dfrac{5}{x^2-x-42}$

44. $\dfrac{1}{x-5} + \dfrac{4}{x+3} = -\dfrac{7}{x^2-2x-15}$

Solve the equation by completing the square.

45. $y^2 - 6y = 27$

46. $y^2 - 14y = -45$

13.6 Compound Inequalities

On the newest version of the SAT (an exam that often qualifies students for colleges) the minimum score that you can earn is 400 and the maximum score that you can earn is 1600. This means that only numbers between 400 and 1600, including these endpoints, are possible scores. To plot all of these values on a number line would look something like:

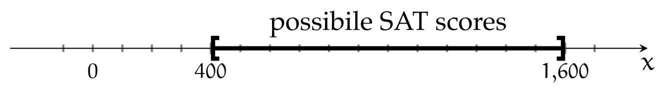


Figure 13.6.1: Possible SAT Scores

Going back to the original statement, "the minimum score that you can earn is 400 and the maximum score that you can earn is 1600," this really says two things. First, it says that (a SAT score) ≥ 400, and second, that (a SAT score) ≤ 1600. When we combine two inequalities like this into a single problem, it becomes a **compound inequality**.

Our lives are often constrained by the compound inequalities of reality: you need to buy enough materials to complete your project, but you can only fit so much into your vehicle; you would like to finish your degree early, but only have so much money and time to put toward your courses; you would like a vegetable garden big enough to supply you with veggies all summer long, but your yard or balcony only gets so much sun. In the rest of the section we hope to illuminate how to think mathematically about problems like these.

Before continuing, a review on how notation for intervals works may be useful, and you may benefit from revisiting Section 1.3. Then a refresher on solving linear inequalities may also benefit you, which you can revisit in Section 2.2 and Section 2.3.

13.6.1 Unions of Intervals

Definition 13.6.3 The **union** of two sets, A and B, is the set of all elements contained in either A or B (or both). We write $A \cup B$ to indicate the union of the two sets.

In other words, the union of two sets is what you get if you toss every number in both sets into a bigger set. ◊

Example 13.6.4 The union of sets $\{1, 2, 3, 4\}$ and $\{3, 4, 5, 6\}$ is the set of all elements from either set. So $\{1, 2, 3, 4\} \cup \{3, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$. Note that we don't write duplicates.

Example 13.6.5 Visualize the union of the sets $(-\infty, 4)$ and $[7, \infty)$.

Explanation. First we make a number line with both intervals drawn to understand what both sets mean.

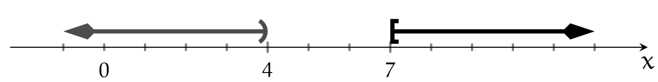


Figure 13.6.6: A number line sketch of $(-\infty, 4)$ as well as $[7, \infty)$

The two intervals should be viewed as a single object when stating the union, so here is the picture of the union. It looks the same, but now it is a graph of a single set.

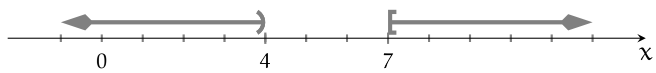


Figure 13.6.7: A number line sketch of $(-\infty, 4) \cup [7, \infty)$

Definition 13.6.8 The **intersection** of two sets, A and B, is the set of all elements that are in A **and** B. We write $A \cap B$ to indicate the intersection of the two sets.

In other words, the intersection of two sets is where the two sets overlap. $\Diamond$

Example 13.6.9 The intersection of sets $\{1, 2, 3, 4\}$ and $\{3, 4, 5, 6\}$ is the set of all elements that are in common to both sets. So $\{1, 2, 3, 4\} \cap \{3, 4, 5, 6\} = \{3, 4\}$.

Example 13.6.10 Find the intersection of the sets $(-\infty, 5)$ and $[3, \infty)$.

Explanation. To find the intersection of the sets $(-\infty, 5)$ and $[3, \infty)$, first we draw a number line with both intervals drawn to visualize where the sets overlap.

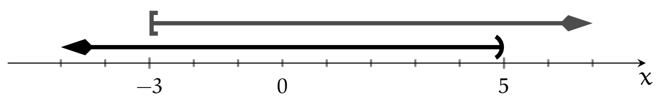


Figure 13.6.11: A number line sketch of $(-\infty, 5)$ and $[-3, \infty)$

Recall that the intersection of two sets is the set of the numbers in common to both sets. In English, we might say that the lines overlap at every number between -3 and 5. This description is the same as the interval $[-3, 5)$.

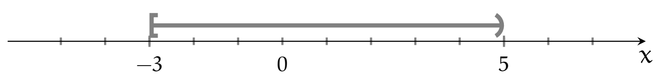


Figure 13.6.12: A number line sketch of $[-3, 5)$

In conclusion,

$$[-3, \infty) \cap (-\infty, 5) = [-3, 5).$$

Remark 13.6.13 Note that every intersection of two intervals can and should be simplified in some way. On the other hand, there *are* unions which cannot be algebraically simplified. For example, if the two sets have nothing in common, as in $(-\infty, 4)$ and $[7, \infty)$ again, then the union is simply $(-\infty, 4) \cup [7, \infty)$ which is our final simplification.

Example 13.6.14 Simplify the intersections and unions.

a. $(-\infty, 12) \cup [-3, \infty)$

b. $(-\infty, 12) \cap [-3, \infty)$

c. $(-\infty, -2] \cup [4, \infty)$

d. $(-\infty, -2] \cap [4, \infty)$

Explanation.

a. $(-\infty, 12) \cup [-3, \infty) = \mathbb{R}$

b. $(-\infty, 12) \cap [-3, \infty) = [-3, 12)$

c. $(-\infty, -2] \cup [4, \infty) = (-\infty, -2] \cup [4, \infty)$

This union cannot be simplified because the two sets have nothing in common.

d. $(-\infty, -2] \cap [4, \infty) = \emptyset$

Since the two sets have nothing in common, their intersection is empty.

Remark 13.6.15 In this section, we mostly use interval notation to answer questions. Recall that we can also use set builder notation. For example, the set $[3, \infty)$ can also be written as $\{x \mid x \geq 3\}$.

13.6.2 "Or" Compound Inequalities

Definition 13.6.16 A **compound inequality** is a grouping of two or more inequalities into a larger inequality statement. These usually come in two flavors: "or" and "and" inequalities. For an example of an "or" compound inequality, you might get a discount at the movie theater if your age is less than 13 *or* greater than 64. For an example of an "and" compound inequality, to purchase a drink at a bar in Oregon, you need to be over 21 years old *and* be have money for your drink. You need to fulfill *both* requirements. ◊

In math, the technical term **or** means "either or both." So, mathematically, if we asked if you would like "chocolate cake *or* apple pie" for dessert, your choices are either "chocolate cake," "apple pie," or "both chocolate cake and apple pie." This is slightly different than the English "or" which usually means "one or the other but not both."

"Or" shows up in math between equations (as in when solving a quadratic equation, you might end up with "$x = 2$ or $x = -3$") or between inequalities (which is what we're about to discuss).

Remark 13.6.17 The definition of "or" is very close to the definition of a union where you combine elements from either or both sets together. In fact, when you have an "or" between inequalities in a compound inequality, to find the solution set of the compound inequality, you find the union of the the solutions sets of each of the pieces.

Example 13.6.18 Solve the compound inequality.

$$x \leq 1 \quad \text{or} \quad x > 4$$

Explanation.

Writing the solution set to this compound inequality doesn't require any algebra beforehand because each of the inequalities is already solved for x. The first thing we should do is understand what each inequality is saying using a graph.

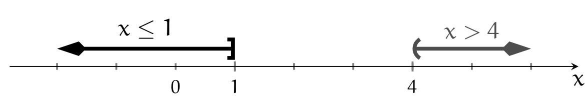


Figure 13.6.19: A number line sketch of solutions to $x \leq 1$ as well as to $x > 4$

An "or" statement becomes a union of solution sets, so the solution set to the compound inequality must be:

$$(-\infty, 1] \cup (4, \infty).$$

Example 13.6.20 Solve the compound inequality.

$$3 - 5x > -7 \quad \text{or} \quad 2 - x \leq -3$$

Explanation. First we need to do some algebra to isolate x in each piece. Note that we are going to do algebra on both pieces simultaneously. Also note that the mathematical symbol "or" should be written on each line.

$$\begin{array}{rcr}
3-5x > -7 & \text{or} & 2-x \leq -3 \\
-5x > -10 & \text{or} & -x \leq -5 \\
\dfrac{-5x}{-5} < \dfrac{-10}{-5} & \text{or} & \dfrac{-x}{-1} \geq \dfrac{-5}{-1} \\
x < 2 & \text{or} & x \geq 5
\end{array}$$

The solution set for the compound inequality $x < 2$ is $(-\infty, 2)$ and the solution set to $x \geq 5$ is $[5, \infty)$. To do the "or" portion of the problem, we need to take the union of these two sets. Let's first make a graph of the solution sets to visualize the problem.

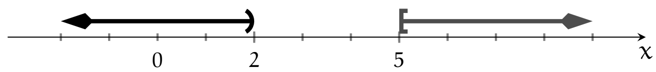


Figure 13.6.21: A number line sketch of $(-\infty, 2)$ as well as $[5, \infty)$

The union combines both solution sets into one, and so

$$(-\infty, 2) \cup [5, \infty)$$

We have finished the problem, but for the sake of completeness, let's try to verify that our answer is reasonable.

- First, let's choose a number that is *not* in our proposed solution set. We will arbitrarily choose 3.

$$\begin{array}{rcr}
3-5x > -7 & \text{or} & 2-x \leq -3 \\
3-5(3) \stackrel{?}{>} -7 & \text{or} & 2-(3) \stackrel{?}{\leq} -3 \\
-9 \stackrel{\text{no}}{>} -7 & \text{or} & -1 \stackrel{\text{no}}{\leq} -3
\end{array}$$

  This value made *both* inequalities false which is why 3 isn't in our solution set.

- Next, let's choose a number that *is* in our solution region. We will arbitrarily choose 1.

$$\begin{array}{rcr}
3-5x > -7 & \text{or} & 2-x \leq -3 \\
3-5(1) \stackrel{?}{>} -7 & \text{or} & 2-(1) \stackrel{?}{\leq} -3 \\
-12 \stackrel{\checkmark}{<} -7 & \text{or} & -1 \stackrel{\text{no}}{\leq} -3
\end{array}$$

  This value made *one* of the inequalities true. Since this is an "or" statement, only one *or* the other piece has to be true to make the compound inequality true.

- Last, what will happen if we choose a value that was in the other solution region in Figure 13.6.21, like the number 6?

$$\begin{array}{rcr}
3-5x > -7 & \text{or} & 2-x \leq -3
\end{array}$$

$$3-5(6) \stackrel{?}{>} -7 \qquad \text{or} \qquad 2-(6) \stackrel{?}{\leq} -3$$
$$-27 \stackrel{\text{no}}{>} -7 \qquad \text{or} \qquad -4 \stackrel{\checkmark}{\leq} -3$$

This solution made the *other* inequality piece true.

This completes the check. Numbers from within the solution region make the compound inequality true and numbers outside the solution region make the compound inequality false.

Example 13.6.22 Solve the compound inequality.

$$\frac{3}{4}t+2 \leq \frac{5}{2} \quad \text{or} \quad -\frac{1}{2}(t-3) < -2$$

Explanation. First we will solve each inequality for t. Recall that we usually try to clear denominators by multiplying both sides by the least common denominator.

$$\begin{aligned}
\frac{3}{4}t+2 &\leq \frac{5}{2} & &\text{or} & -\frac{1}{2}(t-3) &< -2\\
4\cdot\left(\frac{3}{4}t+2\right) &\leq 4\cdot\frac{5}{2} & &\text{or} & 2\cdot\left(-\frac{1}{2}(t-3)\right) &< 2\cdot(-2)\\
3t+8 &\leq 10 & &\text{or} & -t+3 &< -4\\
3t &\leq 2 & &\text{or} & -t &< -7\\
\frac{3t}{3} &\leq \frac{2}{3} & &\text{or} & \frac{-t}{-1} &> \frac{-7}{-1}\\
t &\leq \frac{2}{3} & &\text{or} & t &> 7
\end{aligned}$$

The solution set to $t \leq \frac{2}{3}$ is $\left(-\infty, \frac{2}{3}\right]$ and the solution set to $t > 7$ is $(7,\infty)$. Figure 13.6.23 shows these two sets.

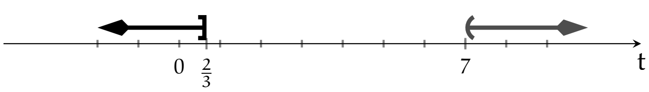


Figure 13.6.23: A number line sketch of $\left(-\infty, \frac{2}{3}\right]$ and also $(7,\infty)$

Note that the two sets do not overlap so there will be no way to simplify the union. Thus the solution set to the compound inequality is:

$$\left(-\infty, \frac{2}{3}\right] \cup (7,\infty)$$

Example 13.6.24 Solve the compound inequality.

$$3y-15 > 6 \quad \text{or} \quad 7-4y \geq y-3$$

Explanation. First we solve each inequality for y.

$$3y-15 > 6 \qquad \text{or} \qquad 7-4y \geq y-3$$

$$\begin{array}{ccc} 3y > 21 & \text{or} & -5y \geq -10 \\ \dfrac{3y}{3} > \dfrac{21}{3} & \text{or} & \dfrac{-5y}{-5} \leq \dfrac{-10}{-5} \\ y > 7 & \text{or} & y \leq 2 \end{array}$$

The solution set to $y > 7$ is $(7, \infty)$ and the solution set to $y \leq 2$ is $(-\infty, 2]$. Figure 13.6.25 shows these two sets.

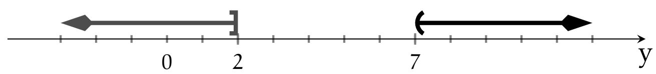


Figure 13.6.25: A number line sketch of $(7, \infty)$ as well as $(-\infty, 2]$

So the solution set to the compound inequality is:

$$(-\infty, 2] \cup (7, \infty)$$

13.6.3 Three-Part Inequalities

There are two different kinds of "and" compound inequalities. One type has an expression that is "between" two values, like $A < B \leq C$, that we will call "three-part inequalities". The other type has two inequalities joined by the word "and," as in $A < B$ and $C \geq D$. We will start with the three-part inequalities.

The inequality $1 \leq 2 < 3$ says a lot more than you might think. It actually says four different single inequalities which are highlighted for you to see.

$$1 \leq 2 < 3 \quad 1 \leq 2 < 3 \quad 1 \leq 2 < 3 \quad 1 \leq 2 < 3$$

This might seem trivial at first, but if you are presented with an inequality like $-1 < 3 \geq 2$, at first it might look sensible; however, in reality, you need to check that *all four* linear inequalities make sense. Those are highlighted here.

$$-1 < 3 \geq 2 \quad -1 < 3 \geq 2 \quad -1 < 3 \geq 2 \quad -1 < 3 \geq 2$$

One of these inequalities is false: $-1 \not\geq 2$. This implies that the entire original inequality, $-1 < 3 \geq 2$, is nonsense.

Example 13.6.26 Decide whether or not the following inequalities are true or false.

a. True or False: $-5 < 7 \leq 12$?

b. True or False: $-7 \leq -10 < 4$?

c. True or False: $-2 \leq 0 \geq 1$?

d. True or False: $5 > -3 \geq -9$?

e. True or False: $3 < 3 \leq 5$?

f. True or False: $9 > 1 < 5$?

g. True or False: $3 < 8 \leq -2$?

h. True or False: $-9 < -4 \leq -2$?

Explanation. We need to go through all four single inequalities for each. If the inequality is false, for simplicity's sake, we will only highlight the one single inequality that makes the inequality false.

a. True: $-5 < 7 \leq 12$.

b. False: $-7 \overset{\text{no}}{\leq} -10 < 4$.

c. False: $-2 \leq 0 \overset{\text{no}}{\geq} 1$.

d. True: $5 > -3 \geq -9$.

e. False: $3 \overset{\text{no}}{<} 3 \leq 5$.

f. False: $9 > 1 \overset{\text{no}}{<} 5$.

g. False: $3 < 8 \overset{\text{no}}{\leq} -2$.

h. True: $-9 < -4 \leq -2$.

As a general hint, no (nontrivial) three-part inequality can ever be true if the inequality signs are not pointing in the same direction. So no matter what numbers a, b, and c are, both $a < b \geq c$ and $a \geq b < c$ cannot be true! Soon you will be writing inequalities like $2 < x \leq 4$ and you need to be sure to check that your answer is feasible. You will know that if you get $2 > x \leq 4$ or $2 < x \geq 4$ that something went wrong in the solving process. The only exception is that something like $1 \leq 1 \geq 1$ is true because $1 = 1 = 1$, although this shouldn't come up very often!

Example 13.6.27 Write the solution set to the compound inequality.

$$-7 < x \leq 5$$

Explanation. The solutions to the three-part inequality $-7 < x \leq 5$ are those numbers that are trapped between -7 and 5, including 5 but not -7. Keep in mind that there are infinitely many decimal numbers and irrational numbers that satisfy this inequality like -2.781828 and π. We will write these numbers in interval notation as $(-7, 5]$ or in set builder notation as $\{x \mid -7 < x \leq 5\}$.

Example 13.6.28 Solve the compound inequality.

$$4 \leq 9x + 13 < 20$$

Explanation.

This is a three-part inequality which we can treat just as a regular inequality with three "sides." The goal is to isolate x in the middle and whatever you do to one "side," you have to do to the other two "sides."

$$\begin{aligned} 4 &\leq 9x + 13 < 20 \\ 4 - 13 &\leq 9x + 13 - 13 < 20 - 13 \\ -9 &\leq 9x < 7 \\ \frac{-9}{9} &\leq \frac{9x}{9} < \frac{7}{9} \\ -1 &\leq x < \frac{7}{9} \end{aligned}$$

The solutions to the three-part inequality $-1 \leq x < \frac{7}{9}$ are those numbers that are trapped between -1 and $\frac{7}{9}$, including -1 but not $\frac{7}{9}$. The solution set in interval notation is $\left[-1, \frac{7}{9}\right)$.

Example 13.6.29 Solve the compound inequality.

$$-13 < 7 - \frac{4}{3}x \leq 15$$

Explanation.

This is a three-part inequality which we can treat just as a regular inequality with three "sides." The goal is to isolate x in the middle and whatever you do to one "side," you have to do to the other two "sides." We will begin by canceling the fraction by multiplying each part by the least common denominator.

At the end we reverse the entire statement to go from smallest to largest. The solution set is $[-6, 15)$.

$$\begin{aligned}
-13 &< 7 - \frac{4}{3}x \leq 15 \\
-13 \cdot 3 &< \left(7 - \frac{4}{3}x\right) \cdot 3 \leq 15 \cdot 3 \\
-39 &< 21 - 4x \leq 45 \\
-60 &< -4x \leq 24 \\
\frac{-60}{-4} &> \frac{-4x}{-4} \geq \frac{24}{-4} \\
15 &> x \geq -6 \\
-6 &\leq x < 15
\end{aligned}$$

13.6.4 Solving "And" Inequalities

Here we will deal with the other kind of compound inequality: the "and" variety.

Remark 13.6.30 An "and" statement means that you need both inequalities to be true simultaneously. In English, if you say, "I need Khaleem *and* Freja to paint the fence," then the only way you will be happy is if *both* people are working simultaneously on the fence. This statement that both things happen at the same time should be very reminiscent of our discussion of intersections earlier in this section. In fact, every "and" statement will result in the intersection of the solution sets of the pieces.

Example 13.6.31 Solve the compound inequality.

$$4 - 2t > -2 \quad \text{and} \quad 3t + 1 \geq -2$$

Explanation.

$$\begin{array}{ccc}
4 - 2t > -2 & \text{and} & 3t + 1 \geq -2 \\
4 - 2t - 4 > -2 - 4 & \text{and} & 3t + 1 - 1 \geq -2 - 1 \\
-2t > -6 & \text{and} & 3t \geq -3 \\
\dfrac{-2t}{-2} < \dfrac{-6}{-2} & \text{and} & \dfrac{3t}{3} \geq \dfrac{-3}{3} \\
t < 3 & \text{and} & t \geq -1
\end{array}$$

The solution set to $t < 3$ is $(-\infty, 3)$ and the solution set to $t \geq -1$ is $[-1, \infty)$. Shown is a graph of these solution sets.

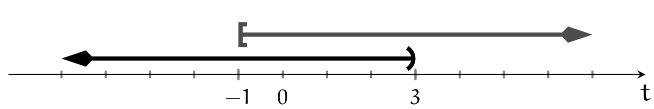


Figure 13.6.32: A number line sketch of $(-\infty, 3)$ and also $[-1, \infty)$

Recall that an "and" problem finds the intersection of the solution sets. Intersection finds the t-values where the two lines overlap, so the solution to the compound inequality must be

$$(-\infty, 3) \cap [-1, \infty) = [-1, 3).$$

We have finished the problem, but for the sake of completeness, let's try to "verify" that our answer is reasonable.

- First, choose a number within our solution region and test that it makes both original inequalities true. We will arbitrarily choose 1.

$$\begin{aligned} 4-2t &> -2 & \text{and} && 3t+1 &\geq -2 \\ 4-2(1) &\stackrel{?}{>} -2 & \text{and} && 3(1)+1 &\stackrel{?}{\geq} -2 \\ 2 &\stackrel{\checkmark}{>} -2 & \text{and} && 4 &\stackrel{\checkmark}{\geq} -2 \end{aligned}$$

- Next, choose a value outside the solution set and test that it makes *at least* one of the inequalities false. We will arbitrarily choose 4.

$$\begin{aligned} 4-2t &> -2 & \text{and} && 3t+1 &\geq -2 \\ 4-2(4) &\stackrel{?}{>} -2 & \text{and} && 3(4)+1 &\stackrel{?}{\geq} -2 \\ -4 &\stackrel{\text{no}}{>} -2 & \text{and} && 13 &\stackrel{\checkmark}{\geq} -2 \end{aligned}$$

  Since one of the inequalities is false and this is an "and" question, the compound inequality is false for this value which is what expected by picking a number outside the solution set.

- Last, we should choose a number that is not a solution that is on the "other side" of the solution set. We will arbitrarily choose -2.

$$\begin{aligned} 4-2t &> -2 & \text{and} && 3t+1 &\geq -2 \\ 4-2(-2) &\stackrel{?}{>} -2 & \text{and} && 3(-2)+1 &\stackrel{?}{\geq} -2 \\ 8 &\stackrel{\checkmark}{>} -2 & \text{and} && -5 &\stackrel{\text{no}}{\geq} -2 \end{aligned}$$

  Again, since one of the inequalities is false and this is an "and" question, the compound inequality is false for -2.

So, numbers outside the proposed solution region make the compound inequality false, and numbers inside the region make the compound inequality true. We have verified our solution set.

Checkpoint 13.6.33 Solve the compound inequality.

$$-6 \geq 3x+3 \quad \text{and} \quad 3x+9 > -6$$

Explanation.

$$\begin{aligned} -6 &\geq 3x+3 & \text{and} && 3x+9 &> -6 \\ -9 &\geq 3x & \text{and} && 3x &> -15 \\ -3 &\geq x & \text{and} && x &> -5 \\ x &\leq -3 & \text{and} && x &> -5 \end{aligned}$$

The solution set to $x \le -3$ is $(-\infty, -3]$ and the solution set to $x > -5$ is $(-5, \infty)$. Shown is a graph of these solution sets on a number line.

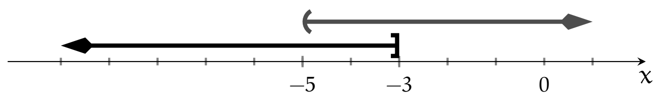


Figure 13.6.34: A number line sketch of $(-\infty, -3]$ and $(-5, \infty)$

Recall that "and" statements of inequalities become intersections of the solution sets. Since intersections refer to where the sets overlap, and these sets overlap between -5 (exclusive) and -3 (inclusive), we would say

$$(-\infty, -3] \cap (-5, \infty) = (-5, -3].$$

13.6.5 Applications of Compound inequalities

Example 13.6.35 Raphael's friend is getting married and he's decided to give them some dishes from their registry. Raphael doesn't want to seem cheap but isn't a wealthy man either, so he wants to buy "enough" but not "too many." He's decided that he definitely wants to spend at least \$150 on his friend, but less than \$250. Each dish is \$21.70 and shipping on an order of any size is going to be \$19.99. Given his budget, set up and algebraically solve a compound inequality to find out what his different options are for the number of dishes that he can buy.

Explanation. First, we should define our variable. Let x represent the number of dishes that Raphael can afford. Next we should write a compound inequality that describes this situation. In this case, Raphael wants to spend between \$150 and \$250 and, since he's buying x dishes, the price that he will pay is $21.70x + 19.99$. All of this translates to a triple inequality

$$150 < 21.70x + 19.99 < 250$$

Now we have to solve this inequality in the usual way.

$$\begin{aligned} 150 &< 21.70x + 19.99 < 250 \\ 150 - 19.99 &< 21.70x + 19.99 - 19.99 < 250 - 19.99 \\ 130.01 &< 21.70x < 230.01 \\ \frac{130.01}{21.70} &< \frac{21.70x}{21.70} < \frac{230.01}{21.70} \\ 5.991 &< x < 10.6 && \text{(note: these values are approximate)} \end{aligned}$$

The interpretation of this inequality is a little tricky. Remember that x represents the number of dishes Raphael can afford. Since you cannot buy 5.991 dishes (manufacturers will typically only ship whole number amounts of tableware) his minimum purchase must be 6 dishes. We have a similar problem with his maximum purchase: clearly he cannot buy 10.6 dishes. So, should we round up or down? If we rounded up, that would be 11 dishes and that would cost $\$21.70 \cdot 11 + \$19.99 = \$258.69$, which is outside his price range. Therefore, we should actually round *down* in this case.

In conclusion, Raphael should buy somewhere between 6 and 10 dishes for his friend to stay within his budget.

Example 13.6.36 Oak Ridge National Laboratory, a renowned scientific research facility, compiled some data[1] on fuel efficiency of a mid-size hybrid car versus the speed that the car was driven. A model for the fuel efficiency $e(x)$ (in miles per gallon, mpg) at a speed x (in miles per hour, mph) is $e(x) = 88 - 0.7x$.

a. Evaluate and interpret $e(60)$ in the context of the problem.

b. Note that this model only applies between certain speeds. The maximum fuel efficiency for which this formula applies is 55 mpg and the minimum fuel efficiency for which it applies is 33 mpg. Set up and algebraically solve a compound inequality to find the range of speeds for which this model applies.

Explanation.

a. Let's evaluate $e(60)$ first.

$$\begin{aligned} e(x) &= 88 - 0.7x \\ e(60) &= 88 - 0.7(60) \\ &= 46 \end{aligned}$$

So, when the hybrid car travels at a speed of 60 mph, it has a fuel efficiency of 46 mpg.

b. In this case, the minimum efficiency is 33 mpg and the maximum efficiency is 55 mpg. We need to trap our formula between these two values to solve for the respective speeds.

$$\begin{aligned} &33 < 88 - 0.7x < 55 \\ &33 - 88 < 88 - 0.7x - 88 < 55 - 88 \\ &-55 < -0.7x < -33 \\ &\frac{-55}{-0.7} > \frac{-0.7x}{-0.7} > \frac{-33}{-0.7} \\ &78.57 > x > 47.14 \qquad \text{(note: these values are approximate)} \end{aligned}$$

This inequality says that our model is applicable when the car's speed is between about 47 mph and about 79 mph.

13.6.6 Reading Questions

1. What is the difference between an "inequality" and a "compound inequality"?

2. What is the difference between a union and an intersection?

3. Explain why $-3 < 5 \geq 2$ doesn't make mathematical sense.

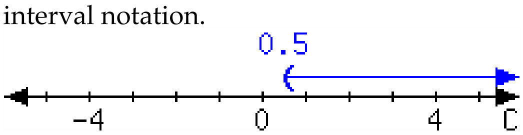
4. If you solve a compound inequality and your final simplification is "$x > 7$ and $x < 12$", how many solutions are in your solution set? How would you write those solutions?

[1] tedb.ornl.gov/data/

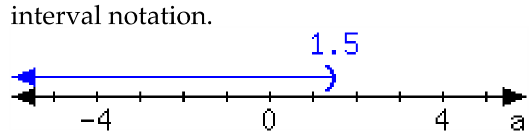
13.6.7 Exercises

Review and Warmup

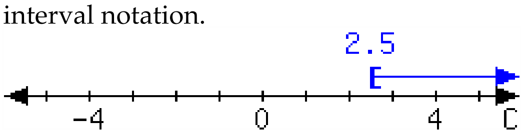
1. For the interval expressed in the number line, write it using set-builder notation and interval notation.



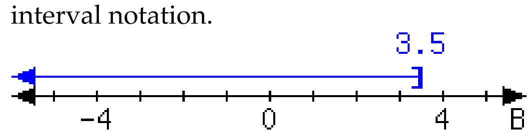
2. For the interval expressed in the number line, write it using set-builder notation and interval notation.



3. For the interval expressed in the number line, write it using set-builder notation and interval notation.



4. For the interval expressed in the number line, write it using set-builder notation and interval notation.



5. Solve this inequality.
$1 > x + 7$

6. Solve this inequality.
$1 > x + 10$

7. Solve this inequality.
$-2x \geq 6$

8. Solve this inequality.
$-3x \geq 6$

9. Solve this inequality.
$5 \geq -6x + 5$

10. Solve this inequality.
$4 \geq -7x + 4$

11. Solve this inequality.
$9t + 3 < 5t + 43$

12. Solve this inequality.
$9t + 9 < 3t + 51$

Check Solutions Decide whether the given value for the variable is a solution.

13. a. $x > 9$ and $x \leq 5$ $x = 8$The given value (☐ is ☐ is not) a solution.

b. $x < 4$ or $x \geq 6$ $x = 1$The given value (☐ is ☐ is not) a solution.

c. $x \geq -2$ and $x \leq 8$ $x = -2$The given value (☐ is ☐ is not) a solution.

d. $-3 \leq x \leq 3$ $x = 2$The given value (☐ is ☐ is not) a solution.

14. a. $x > 1$ and $x \leq 2$ $x = 2$The given value (☐ is ☐ is not) a solution.

b. $x < 1$ or $x \geq 6$ $x = 6$The given value (☐ is ☐ is not) a solution.

c. $x \geq -2$ and $x \leq 3$ $x = 4$The given value (☐ is ☐ is not) a solution.

d. $-1 \leq x \leq 1$ $x = 1$The given value (☐ is ☐ is not) a solution.

Compound Inequalities and Interval Notation

15. Solve the compound inequality. Write the solution set in interval notation.
$-9 < x \leq 9$

16. Solve the compound inequality. Write the solution set in interval notation.
$-8 < x \leq 5$

17. Solve the compound inequality. Write the solution set in interval notation.
$-7 > x$ or $x \geq 2$

18. Solve the compound inequality. Write the solution set in interval notation.
$-6 > x$ or $x \geq 8$

19. Express the following inequality using interval notation.
$x < -5$ or $x \leq 5$

20. Express the following inequality using interval notation.
$x < -4$ or $x \leq 1$

21. Express the following inequality using interval notation.
$-3 < x$ and $x \geq 8$

22. Express the following inequality using interval notation.
$-2 < x$ and $x \geq 4$

23. Express the following inequality using interval notation.
$-10 \leq x$ and $x < 1$

24. Express the following inequality using interval notation.
$-9 \leq x$ and $x < 7$

Solving a Compound Inequality Algebraically Solve the compound inequality algebraically.

25. $-5 < 7 - x \leq 0$

26. $-7 < 20 - x \leq -2$

27. $10 \leq x + 13 < 15$

28. $12 \leq x + 6 < 17$

29. $21 \leq \frac{5}{9}(F - 32) \leq 49$

F is in $\boxed{}$

30. $24 \leq \frac{5}{9}(F - 32) \leq 42$

F is in $\boxed{}$

31. $-16x + 11 \leq 1 \quad \text{or} \quad -14x - 13 \geq -13$

32. $16x + 10 \leq -17 \quad \text{and} \quad 5x - 1 < 3$

33. $2x - 14 \leq 13 \quad \text{and} \quad -16x + 3 \leq 20$

34. $-12x + 3 \leq 1 \quad \text{and} \quad 5x + 7 \geq -5$

35. $-8x - 5 \geq -17 \quad \text{and} \quad 19x + 20 \geq -14$

36. $-4x + 20 > -7 \quad \text{and} \quad 14x - 13 \geq -5$

37. $10x + 11 \leq -18 \quad \text{or} \quad -11x + 10 \leq -10$

38. $5x - 20 \geq -4 \quad \text{or} \quad -4x - 15 \geq 15$

39. $8 < \frac{4}{3}x < 36$

40. $5 < \frac{5}{2}x < 50$

41. $11 > 1 - \frac{2}{7}x \geq -3$

42. $12 > -3 - \frac{5}{4}x \geq -23$

Applications

43. As dry air moves upward, it expands. In so doing, it cools at a rate of about 1°C for every 100 m rise, up to about 12 km.

a. If the ground temperature is 17°C, write a formula for the temperature at height x km. $T(x) =$ $\boxed{}$

b. What range of temperature will a plane be exposed to if it takes off and reaches a maximum height of 5 km? Write answer in interval notation.

The range is $\boxed{}$.

Challenge

44. Algebraically, solve for x in the equation:

$$5 = |x - 5| + |x - 10|$$

13.7 Solving Inequalities Graphically

In this text, we have mostly focused on solving inequalities algebraically. While we have had some practice solving inequalities graphically 11.3.3 with technology, we want to solidify those skills. Solving using graphing is special because the graphing utility we use can do much of the heavy lifting and all that is left is to analyze the graph that is shown to us. So let's let our favorite graphing program make some graphs for us and then we can interpret the results.

13.7.1 Solving Inequalities Graphically

Example 13.7.1
Business leaders and professionals around the world concern themselves with money and how to grow their wealth. While a vast majority of people who live in the United States own few or no stocks, the stock market is important to learn about for anyone interested in earning a retirement. Stock owners need to know when to buy or sell their stocks to make a profit and the most essential tool to do so is the ability to read a graph. Let's examine a graph of the actual closing value of Apple (AAPL) stock from June 3, 2019 to August 6, 2019.

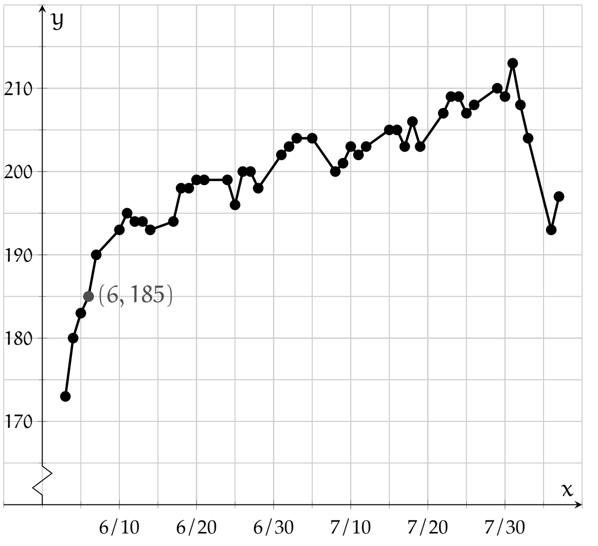


Figure 13.7.2: A Graph of AAPL Stock Data

If a person bought the stock on June 6, when the stock was valued at $185 per share, and they wanted at least a $20 per share profit, during what days could they have sold that stock?

If we want a \$20 per share profit, then we should be interested in stock prices of \$205 or more. Let's draw a line at $y = 205$, representing the price of \$205, and find any days when the stock was on or above that line. According to the graph, there are several dates in question in starting in late July. Let's zoom in on those dates to read our solutions better.

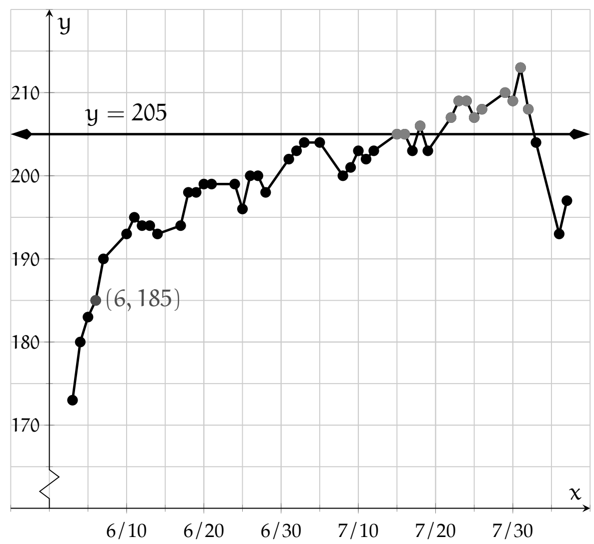


Figure 13.7.3: A Graph of AAPL Stock Data with $y = 205$

With a zoomed-in and rescaled graph, we can clearly see the dates that would have resulted in a \$20 per share profit. Those dates were July 15, 16, 18, 22, 23, 24, 25, 26, 29, 30, 31, and August 1. Keep in mind that the stock market is closed on weekends and holidays, so we are only counting the solid dots as our solutions.

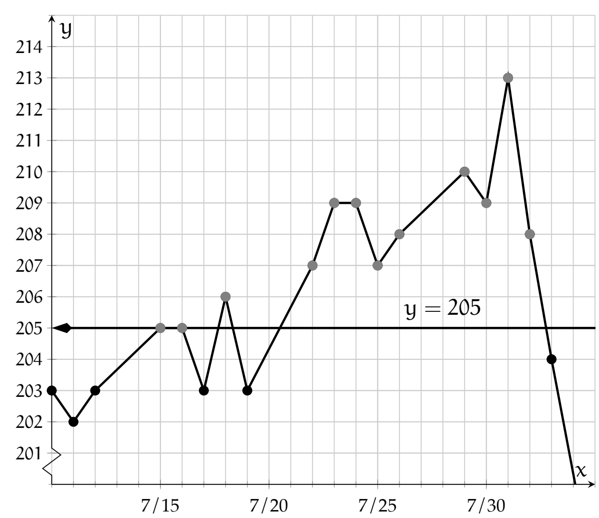


Figure 13.7.4: A Zoomed-In Graph of AAPL Stock Data with $y = 205$

Let's turn to an example involving a linear equation.

Example 13.7.5 Solve the inequality $3x - 2 < 7$ graphically.

Explanation.

To solve any inequality (or equation) graphically, we first take each side of the equation and graph $y =$ "left hand side" and $y =$ "right hand side". In this case, that would be $y = 3x - 2$ and $y = 7$. Now we can see that the graphs of $y = 3x - 2$ and $y = 7$ intersect at the point $(3, 7)$.

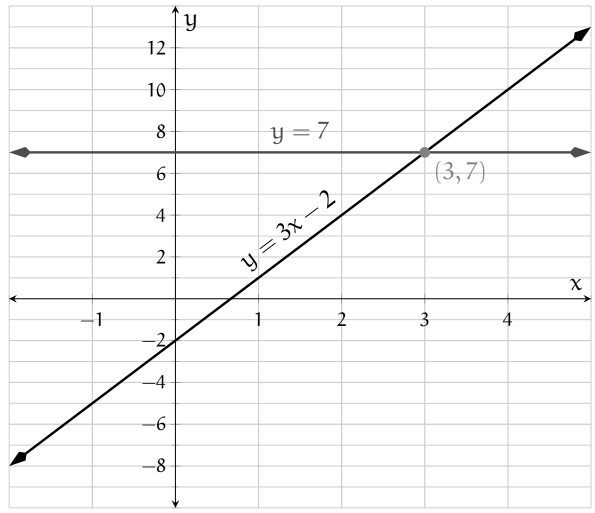


Figure 13.7.6: A Graph of Both $y = 3x - 2$ and $y = 7$

Since we are trying to solve the inequality $3x - 2 < 7$, we need to examine the graph for where (what x-values) the graph of $y = 3x - 2$ is below the graph of $y = 7$. This happens for x-values less than 3. So we would say that the solution set to $3x - 2 < 7$ is $(-\infty, 3)$. It is review to solve the inequality algebraically to verify our result.

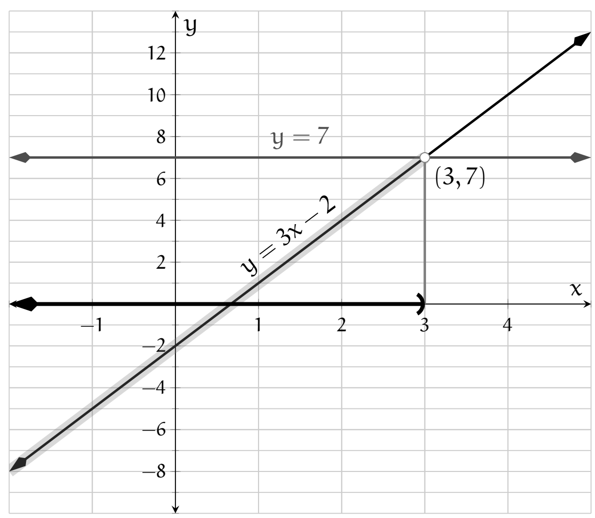


Figure 13.7.7: A Graph of Both $y = 3x - 2$ and $y = 7$

13.7.2 Solving Absolute Value and Quadratic Inequalities Graphically

Recall in Section 13.4 that we learned that graphs of absolute value function are in general shaped like "V"s. We can now solve some absolute value inequalities graphically.

Example 13.7.8
Graphically solving the inequality $|2x-1| \leq 5$ means looking for the x-values where the graph of $y=|2x-1|$ is below (or touching) the line $y=5$. On the graph the highlighted region of $y=|2x-1|$ is the portion that is below the line $y=5$, and the x-values in that region are $[-2,5]$.

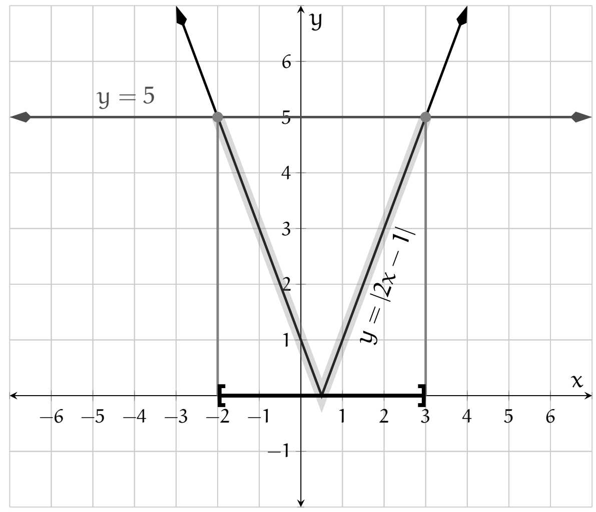


Figure 13.7.9: $y=|2x-1|$ and $y=5$

Example 13.7.10 Solve the inequality $\left|\frac{2}{3}x+1\right| < 3$ graphically.

Explanation. To solve the inequality $\left|\frac{2}{3}x+1\right| < 3$, we will start by making a graph with both $y=\left|\frac{2}{3}x+1\right|$ and $y=3$.

The portion of the graph of $y=\left|\frac{2}{3}x+1\right|$ that is below $y=3$ is highlighted and the x-values of that highlighted region are trapped between -6 and 3: $-6<x<3$. That means that the solution set is $(-6,3)$. Note that we shouldn't include the endpoints of the interval because at those values, the two graphs are *equal* whereas the original inequality was only *less than* and not equal.

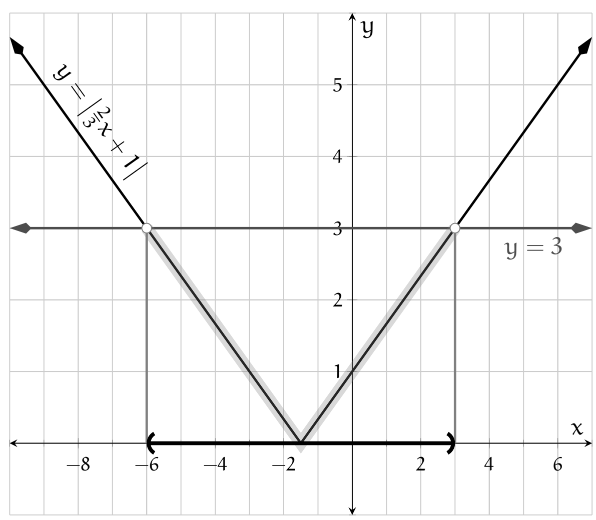


Figure 13.7.11: $y=\left|\frac{2}{3}x+1\right|$ and $y=3$

The last examples had absolute value expressions being *less than* some value. We now need to investigate what happens when we have an absolute value expression that is *greater than* a value.

Example 13.7.12 To graphically solve the inequality $|x-1|>3$ would mean looking for the x-values where the graph of $y=|x-1|$ is *above* the line $y=3$.

On the graph the highlighted region of $y=|x-1|$ is the portion that is above the line $y=3$ and the x-values in that region can be represented by $(-\infty,-2)\cup(4,\infty)$.

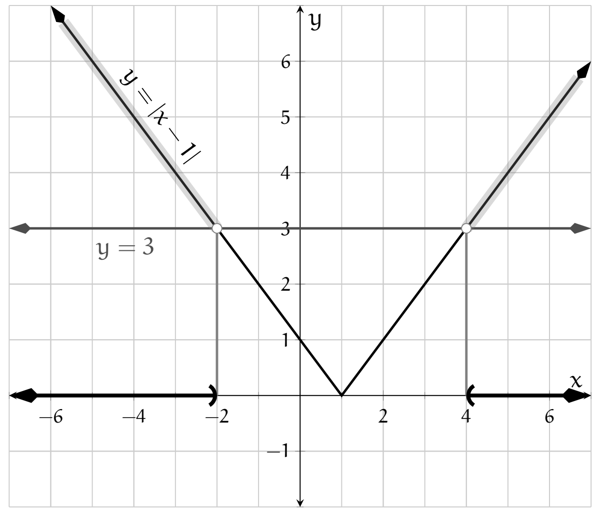


Figure 13.7.13: $y=|x-1|$ and $y=3$

Example 13.7.14 Solve the inequality $\left|\frac{1}{3}x+2\right|\geq 6$ graphically.

Explanation. To solve the inequality $\left|\frac{1}{3}x+2\right|\geq 6$, we will start by making a graph with both $y=\left|\frac{1}{3}x+2\right|$ and $y=6$.

The portion of the graph of $y=\left|\frac{1}{3}x+2\right|$ that is above $y=6$ is highlighted and the x-values of that highlighted region are those below (or equal to) -24 and those above (or equal to) 12: $x\leq -24$ or $x\geq 12$. That means that the solution set is $(-\infty,-24)\cup(12,\infty)$.

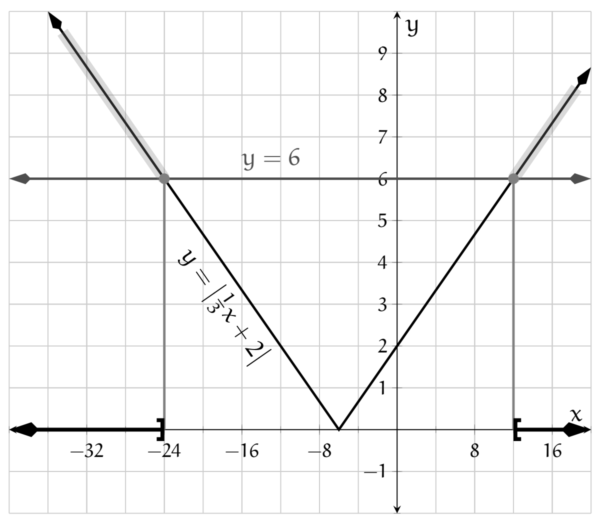


Figure 13.7.15: $y=\left|\frac{1}{3}x+2\right|$ and $y=3$

Solving inequalities with quadratic expressions graphically is very similar to solving absolute value inequalities graphically.

Example 13.7.16 Graphically solve the following quadratic inequalities.

a. $42(x-2)^2-60 \geq 21x-39$

b. $42(x-2)^2-60 < 21x-39$

Explanation.

For both parts of this example, we start by graphing the equations $y = 42(x-2)^2-60$ and $y = 21x-39$ using graphing technology, and determining the points of intersection.

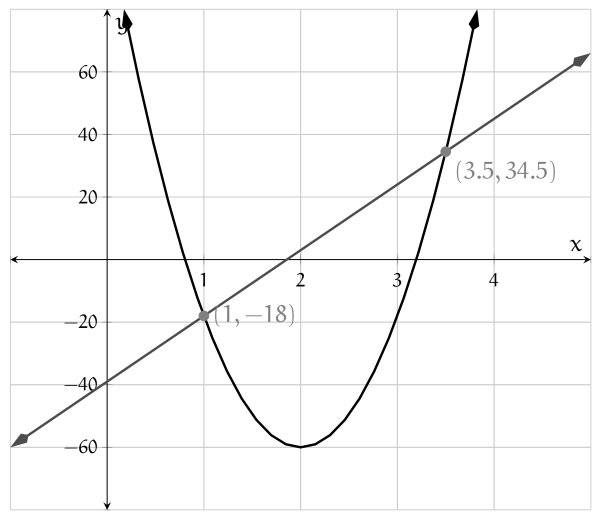


Figure 13.7.17: Points of intersection for $y = 42(x-2)^2-60$ and $y = 21x-39$

a. To solve $42(x-2)^2-60 \geq 21x-39$, we need to determine where the y-values of the parabola are higher than (or equal to) those of the line. This region is highlighted in Figure 13.7.18.

   We can see that $42(x-2)^2-60 \geq 21x-39$ for all values of x where $x \leq 1$ or $x \geq 3.5$. We can write this solution set in interval notation as $(-\infty, 1] \cup [3.5, \infty)$ or in set-builder notation as $\{x \mid x \leq 1 \text{ or } x \geq 3.5\}$.

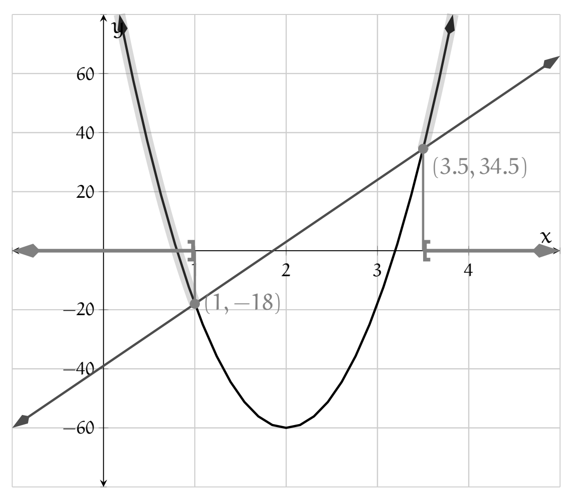


Figure 13.7.18: Where $42(x-2)^2-60 \geq 21x-39$

b. To now solve $42(x-2)^2-60 < 21x-39$, we will need to determine where the y-values of the parabola are *less* than those of the line. This region is highlighted in Figure 13.7.19.

So the solutions to this inequality include all values of x for which $1 < x < 3.5$. We can write this solution set in interval notation as $(1, 3.5)$ or in set-builder notation as $\{x \mid 1 < x < 3.5\}$.

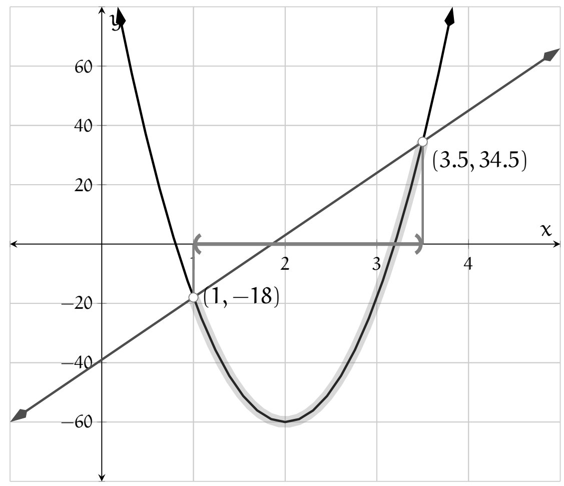


Figure 13.7.19: Where $42(x-2)^2-60 < 21x-39$

13.7.3 Solving Compound Inequalities Graphically

Example 13.7.20
Figure 13.7.21 shows a graph of $y = f(x)$. Use the graph to solve the inequality $2 \leq f(x) < 6$.

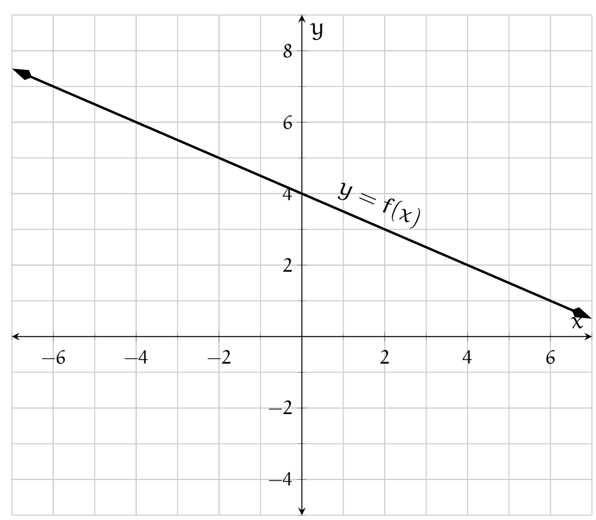


Figure 13.7.21: Graph of $y = f(x)$

Explanation.

To solve the inequality $2 \leq f(x) < 6$ means to find the x-values that give function values between 2 and 6, not including 6. We draw the horizontal lines $y = 2$ and $y = 6$. Then we look for the points of intersection and find their x-values. We see that when x is between -4 and 4, not including -4, the inequality will be true. We have drawn the interval $(-4, 4]$ along the x-axis, which is the solution set.

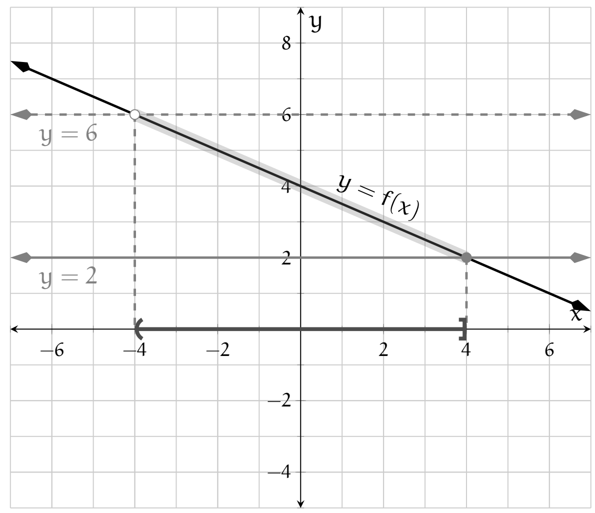


Figure 13.7.22: Graph of $y = f(x)$ and the solution set to $2 \leq f(x) < 6$

Example 13.7.23 Figure 13.7.24 shows a graph of $y = g(x)$. Use the graph to solve the inequality $-4 < g(x) \leq 3$.

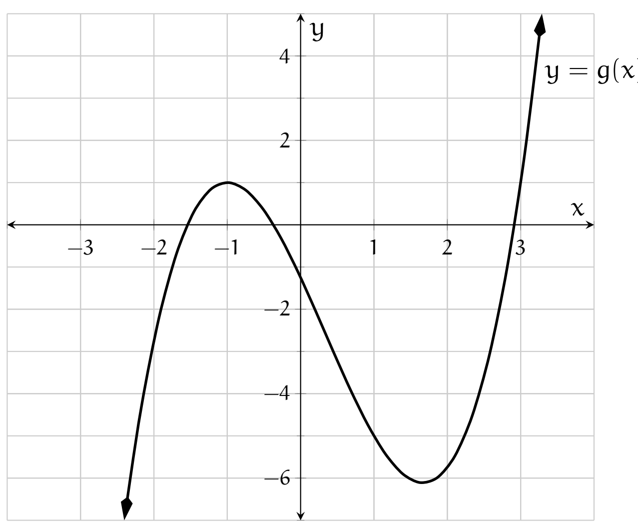


Figure 13.7.24: Graph of $y = g(x)$

Explanation. To solve $-4 < g(x) \leq 3$, we first draw the horizontal lines $y = -4$ and $y = 3$. To solve this inequality we notice that there are two pieces of the function g that are trapped between the y-values -4 and 3.

The solution set is the compound inequality $(-2.1, 0.7) \cup (2.4, 3.2]$.

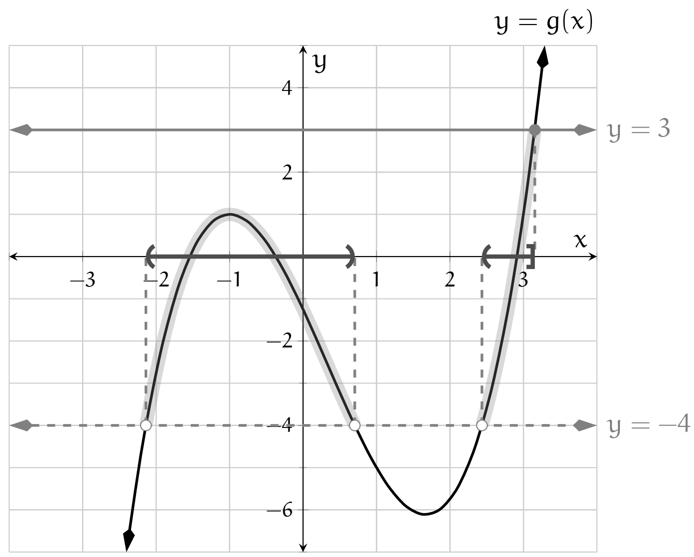


Figure 13.7.25: Graph of $y = g(x)$ and solution set to $-4 < g(x) \leq 3$

Example 13.7.26 Phuong is taking the standard climbing route on Mount Hood from Timberline Lodge up the Southside Hogsback to the summit and back down the same way. Her altitude can be very closely modeled by an absolute value function since the angle of ascent is nearly constant. Let x represent the number of miles walked from Timberline Lodge, and let $f(x)$ represent the altitude, in miles, after walking for a distance x. The altitude can be modeled by $f(x) = 2.1 - 0.3077 \cdot |x - 3.25|$. Note that below Timberline Lodge this model fails to be accurate.

a. Solve the equation $f(x) = 1.1$ graphically and interpret the results in the context of the problem.

b. Altitude sickness can occur at or above altitudes 1.5 miles. Set up and solve an inequality graphically to find out how far Phuong can walk the trail and still be under 1.5 miles of elevation.

Explanation.

a. First, we substitute the formula for $f(x)$ and simplify the equation.

$$f(x) = 1.1$$
$$2.1 - 0.3077 \cdot |x - 3.25| = 1.1$$

At this point, we should make a graph of both $y = 2.1 - 0.3077 \cdot |x - 3.25|$ and $y = 1.1$ and find their intersections.

Next, we should note that we are looking for the x-values of the intersections. These solutions are 0 and 6.5. According to the model, Phuong will be at 1.1 miles of elevation after walking about 0 miles as well as about 6.5 miles along the trail. This implies that Timberline Lodge is at 1.1 miles of elevation. In addition, it implies that the entire hike is 6.5 miles round trip, ending at Timberline Lodge again.

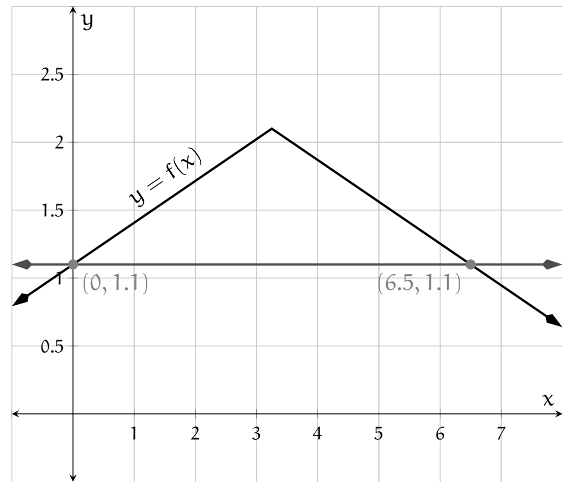


Figure 13.7.27: $y = f(x)$ and $y = 1.1$

b. The inequality we are looking for will describe when the altitude is below 1.5 miles, but also above 1.1 miles based on the reality of the situation (since the model only works above Timberline lodge at 1.1 miles of altitude). Since $f(x)$ is the altitude, the inequality we need is $1.1 \leq f(x) < 1.5$, which becomes $1.1 \leq 2.1 - 0.3077 \cdot |x - 3.25| < 1.5$.

Let's examine the graph again to solve this inequality: We are looking for places on the graph where the y-value is above 1.1, but also where the graph is below 1.5. To find this, we will draw in lines at both of those y-values and find intersections with f.

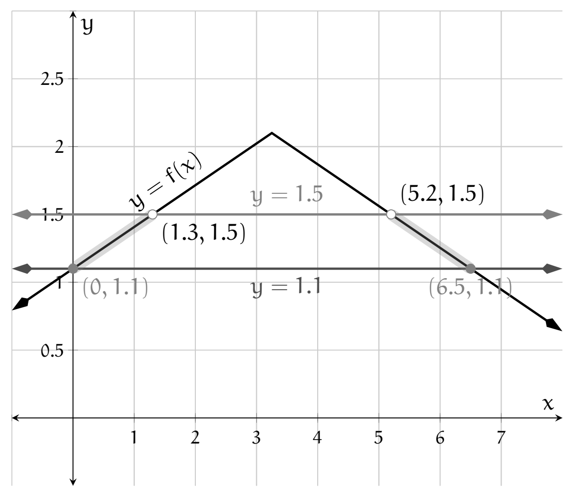


Figure 13.7.28: $y = f(x)$, the Graph of the Mt Hood Ascent and Descent

The highlighted portions of the graph have x-values that satisfy the inequalities $0 \leq x < 1.3$ or $5.2 < x \leq 6.5$.

In conclusion, based both on our math and the reality of the situation, regions of the trail that are below 1.5 miles are those that are from Timberline Lodge (at 0 miles on the trail), to 1.3 miles along the trail and then also from 5.2 miles along the trail (and by now we are on our way back down) to 6.5 miles along the trail (back at Timberline Lodge). If we wanted to write this in interval notation, we might write $[0, 1.3) \cup (5.2, 6.5]$. There is a big portion along the trail (from 1.3 miles to 5.2 miles) that Phuong will be above the 1.5 mile altitude and should watch for signs of altitude sickness.

13.7.4 Reading Questions

1. The graph of the function f is above the graph of the function g between $x = 6$ and $x = 9$. How many solutions does the inequality $f(x) > g(x)$ have?

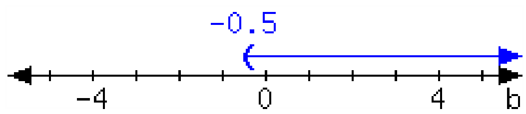
2. Can the solution set to the inequality $h(x) > k(x)$ be the set of all real numbers? Why or why not?

3. Can the solution set to the inequality $h(x) > k(x)$ be the empty set (i.e., the inequality has no solutions)? Why or why not?

13.7.5 Exercises

Review and Warmup For the interval expressed in the number line, write it using set-builder notation and interval notation.

1.



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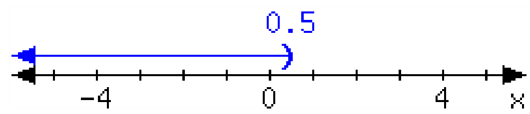
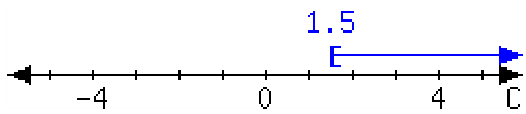


3.



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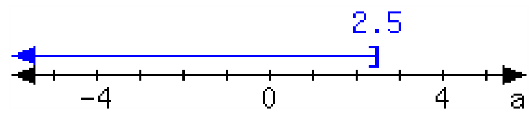


5. For the function L defined by

$$L(x) = 3000x^2 + 10x + 4,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.

b. The vertex.

c. The domain.

d. The range.

6. For the function M defined by

$$M(x) = -(300x - 2950)^2,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.

b. The vertex.

c. The domain.

d. The range.

7. For the function N defined by

$$N(x) = (300x - 1.05)^2,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.

b. The vertex.

c. The domain.

d. The range.

8. For the function B defined by

$$B(x) = x^2 - 0.05x + 0.0006,$$

use technology to determine the following. Round answers as necessary.

a. Any intercepts.

b. The vertex.

c. The domain.

d. The range.

Solving Inequalities Graphically

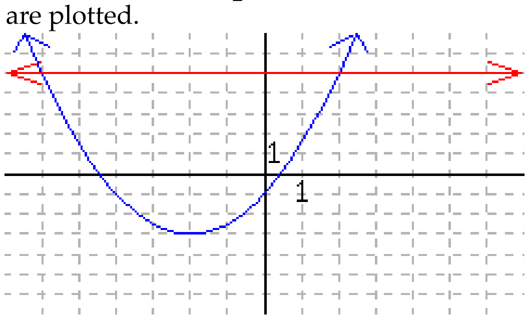
9. Solve the equations and inequalities graphically. Use interval notation when applicable.
a. $\left|\frac{2}{3}x+2\right|=4$ b. $\left|\frac{2}{3}x+2\right|>4$ c. $\left|\frac{2}{3}x+2\right|\le 4$

10. Solve the equations and inequalities graphically. Use interval notation when applicable.
a. $\left|\frac{11-2x}{5}\right|=4$ b. $\left|\frac{11-2x}{5}\right|>4$ c. $\left|\frac{11-2x}{5}\right|\le 4$

11. Solve the equations and inequalities graphically. Use interval notation when applicable.
a. $x^2-3=1$ b. $x^2-3>1$ c. $x^2-3\le 1$

12. Solve the equations and inequalities graphically. Use interval notation when applicable.
a. $x^2-x-3=x$ b. $x^2-x-3>x$ c. $x^2-x-3\le x$

13. The equations $y=\frac{1}{2}x^2+2x-1$ and $y=5$ are plotted.

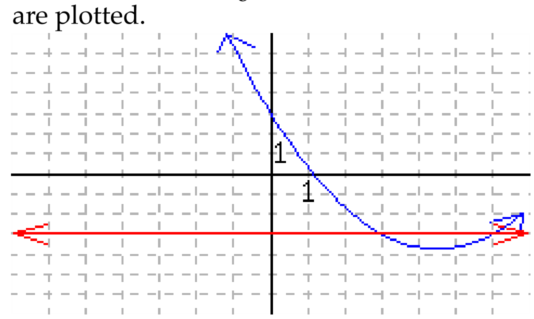


a. What are the points of intersection?

b. Solve $\frac{1}{2}x^2+2x-1=5$.

c. Solve $\frac{1}{2}x^2+2x-1>5$.

14. The equations $y=\frac{1}{3}x^2-3x+3$ and $y=-3$ are plotted.

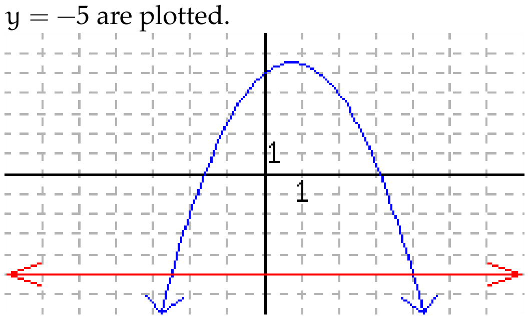


a. What are the points of intersection?

b. Solve $\frac{1}{3}x^2-3x+3=-3$.

c. Solve $\frac{1}{3}x^2-3x+3>-3$.

15. The equations $y=-x^2+1.5x+5$ and $y=-5$ are plotted.

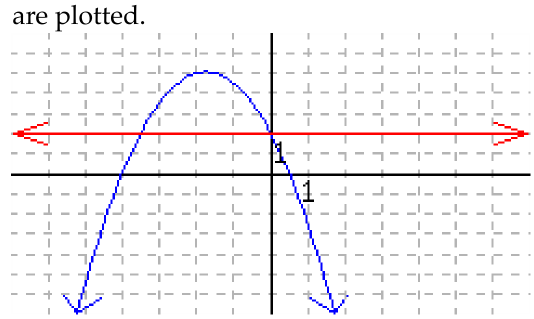


a. What are the points of intersection?

b. Solve $-x^2+1.5x+5=-5$.

c. Solve $-x^2+1.5x+5>-5$.

16. The equations $y=-x^2-3.5x+2$ and $y=2$ are plotted.

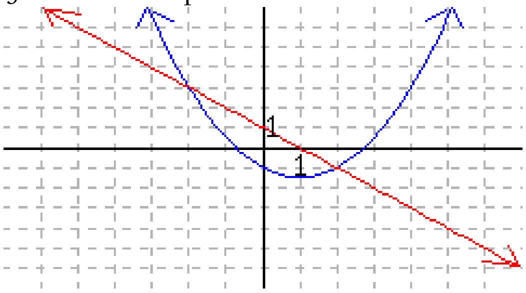


a. What are the points of intersection?

b. Solve $-x^2-3.5x+2=2$.

c. Solve $-x^2-3.5x+2>2$.

17. The equations $y = \frac{1}{2}x^2 - x - 1$ and $y = -x + 1$ are plotted.

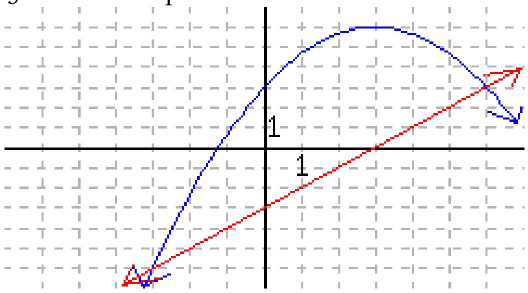


a. What are the points of intersection?

b. Solve $\frac{1}{2}x^2 - x - 1 = -x + 1$.

c. Solve $\frac{1}{2}x^2 - x - 1 > -x + 1$.

18. The equations $y = \frac{-1}{3}x^2 + 2x + 3$ and $y = x - 3$ are plotted.

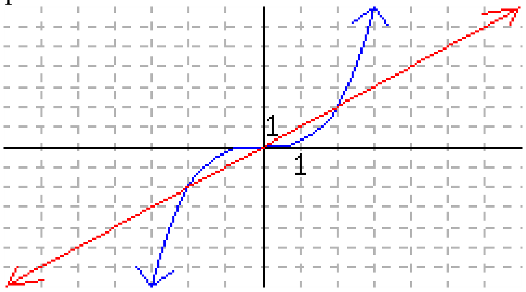


a. What are the points of intersection?

b. Solve $\frac{-1}{3}x^2 + 2x + 3 = x - 3$.

c. Solve $\frac{-1}{3}x^2 + 2x + 3 > x - 3$.

19. The equations $y = \frac{1}{4}x^3$ and $y = x$ are plotted.

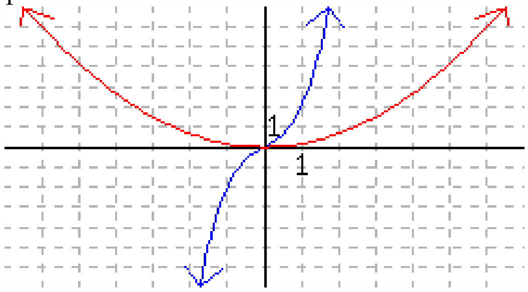


a. What are the points of intersection?

b. Solve $\frac{1}{4}x^3 = x$.

c. Solve $\frac{1}{4}x^3 > x$.

20. The equations $y = x^3 + x$ and $y = \frac{1}{6}x^2$ are plotted.

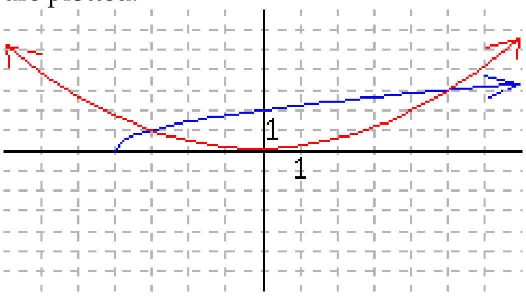


a. What are the points of intersection?

b. Solve $x^3 + x = \frac{1}{6}x^2$.

c. Solve $x^3 + x > \frac{1}{6}x^2$.

21. The equations $y = \sqrt{x+4}$ and $y = \frac{4x^2+x+3}{36}$ are plotted.

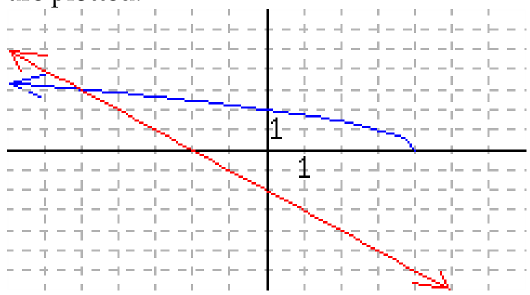


a. What are the points of intersection?

b. Solve $\sqrt{x+4} = \frac{4x^2+x+3}{36}$.

c. Solve $\sqrt{x+4} > \frac{4x^2+x+3}{36}$.

22. The equations $y = \sqrt{4-x}$ and $y = -2-x$ are plotted.

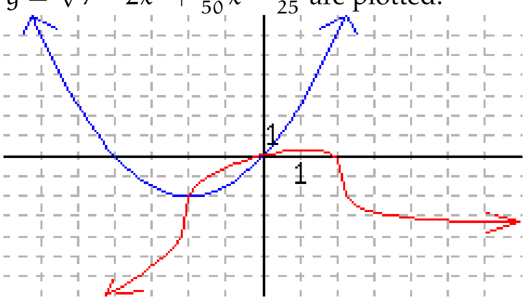


a. What are the points of intersection?

b. Solve $\sqrt{4-x} = -2-x$.

c. Solve $\sqrt{4-x} > -2-x$.

23. The equations $y = \frac{1}{2}x^2 + 2x$ and $y = \sqrt[3]{9-2x^2} + \frac{23}{50}x - \frac{52}{25}$ are plotted.

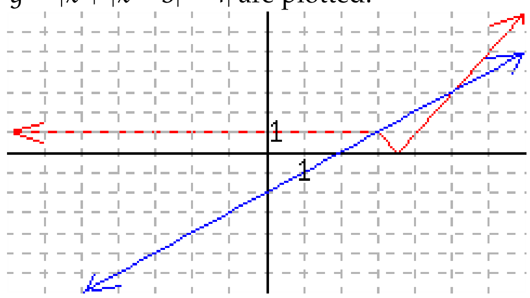


a. What are the points of intersection?

b. Solve $\frac{1}{2}x^2 + 2x = \sqrt[3]{9-2x^2} + \frac{23}{50}x - \frac{52}{25}$.

c. Solve $\frac{1}{2}x^2 + 2x > \sqrt[3]{9-2x^2} + \frac{23}{50}x - \frac{52}{25}$.

24. The equations $y = x - 2$ and $y = |x + |x-3| - 4|$ are plotted.



a. What are the points of intersection?

b. Solve $x - 2 = |x + |x-3| - 4|$.

c. Solve $x - 2 > |x + |x-3| - 4|$.

Solving Equations and Inequalities Graphically Using Technology

25. Let $s(x) = \frac{1}{5}x^2 - 2x + 10$ and $t(x) = -x + 40$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $s(x) = t(x)$.

c. Solve $s(x) > t(x)$.

d. Solve $s(x) \leq t(x)$.

26. Let $w(x) = \frac{1}{4}x^2 - 3x - 8$ and $m(x) = x + 12$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $w(x) = m(x)$.

c. Solve $w(x) > m(x)$.

d. Solve $w(x) \leq m(x)$.

27. Let $f(x) = 4x^2 + 5x - 1$ and $g(x) = 5$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $f(x) = g(x)$.

c. Solve $f(x) < g(x)$.

d. Solve $f(x) \geq g(x)$.

28. Let $p(x) = 6x^2 - 3x + 4$ and $k(x) = 7$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $p(x) = k(x)$.

c. Solve $p(x) < k(x)$.

d. Solve $p(x) \geq k(x)$.

29. Let $q(x) = -4x^2 - 24x + 10$ and $r(x) = 2x + 22$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $q(x) = r(x)$.

c. Solve $q(x) > r(x)$.

d. Solve $q(x) \leq r(x)$.

30. Let $h(x) = -10x^2 - 5x + 3$ and $j(x) = -3x - 9$. Use graphing technology to determine the following.

a. What are the points of intersection for these two functions?

b. Solve $h(x) = j(x)$.

c. Solve $h(x) > j(x)$.

d. Solve $h(x) \leq j(x)$.

31. Use graphing technology to solve the equation $(200 + 5x)(100 - 2x) = 15000$. Approximate the solution(s) if necessary.

32. Use graphing technology to solve the inequality $2x^2 + 5x - 3 > -5$. State the solution set using interval notation, and approximate if necessary.

33. Use graphing technology to solve the inequality $-x^2 + 4x - 7 > -12$. State the solution set using interval notation, and approximate if necessary.

34. Use graphing technology to solve the inequality $10x^2 - 11x + 7 \leq 7$. State the solution set using interval notation, and approximate if necessary.

35. Use graphing technology to solve the inequality $-10x^2 - 15x + 4 \leq 9$. State the solution set using interval notation, and approximate if necessary.

36. Use graphing technology to solve the inequality $-x^2 - 6x + 1 > x + 5$. State the solution set using interval notation, and approximate if necessary.

37. Use graphing technology to solve the inequality $3x^2 + 5x - 4 > -2x + 1$. State the solution set using interval notation, and approximate if necessary.

38. Use graphing technology to solve the inequality $-10x + 4 \leq 20x^2 - 34x + 6$. State the solution set using interval notation, and approximate if necessary.

39. Use graphing technology to solve the inequality $-15x^2 - 6 \leq 10x - 4$. State the solution set using interval notation, and approximate if necessary.

40. Use graphing technology to solve the inequality $\frac{1}{2}x^2 + \frac{3}{2}x \geq \frac{1}{2}x - \frac{3}{2}$. State the solution set using interval notation, and approximate if necessary.

41. Use graphing technology to solve the inequality $\frac{3}{4}x \geq \frac{1}{4}x^2 - 3x$. State the solution set using interval notation, and approximate if necessary.

13.8 Graphs and Equations Chapter Review

13.8.1 Overview of Graphing

In Section 13.1 we reviewed several ways of making graphs of both lines (by hand) and general functions (using technology).

Example 13.8.1 Graphing Lines by Plotting Points. Graph the equation $y = \frac{5}{3}x - 3$ by creating a table of values and plotting those points.

Explanation. To make a good table for this line, we should have x-values that are multiples of 3 to make sure that the fraction cancels nicely for the outputs.

x	$y = \frac{5}{3}x - 3$	Point
-3	$\frac{5}{3}(-3) - 3 = -8$	$(1, -8)$
0	$\frac{5}{3}(0) - 3 = -3$	$(2, -3)$
3	$\frac{5}{3}(3) - 3 = 2$	$(3, 2)$
6	$\frac{5}{3}(6) - 3 = 7$	$(4, 7)$

Figure 13.8.2: A table of values for $y = \frac{5}{3}x - 3$

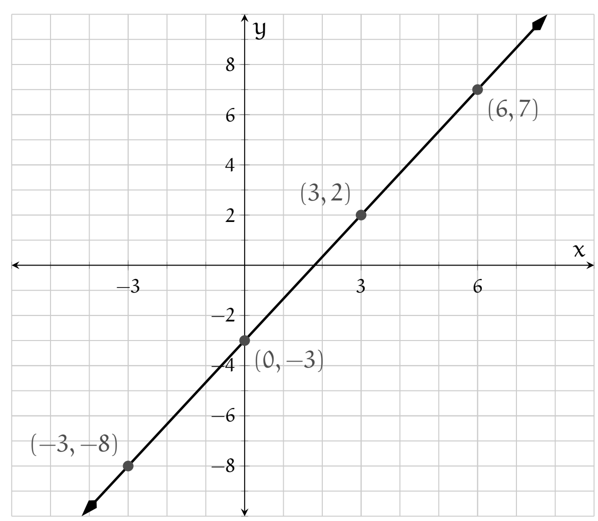


Figure 13.8.3: A graph of $y = \frac{5}{3}x - 3$

Example 13.8.4 Graphing Lines in Slope-Intercept Form. Find the slope and vertical intercept of $j(x) = -\frac{7}{2}x + 5$, and then use slope triangles to find the next two points on the line. Draw the line.

Explanation. The slope of $j(x) = -\frac{7}{2}x + 5$ is $-\frac{7}{2}$, and the vertical intercept is $(0, 5)$. Starting at $(0, 5)$, we go down 7 units and right 2 units to reach more points.

From the graph, we can read that two more points that $j(x) = -\frac{7}{2}x + 5$ passes through are $(2, -2)$ and $(4, 9)$.

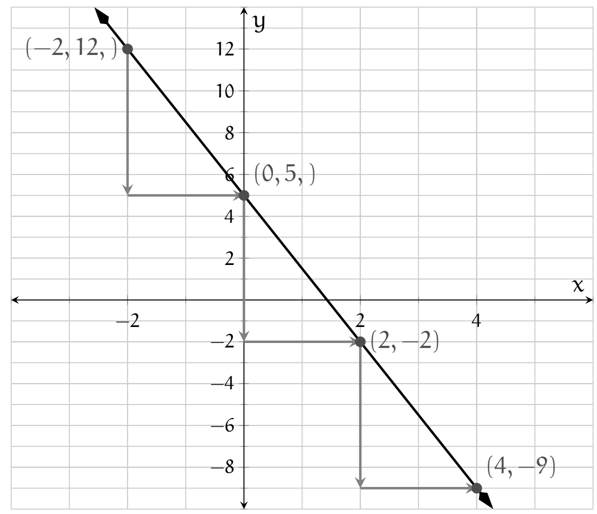


Figure 13.8.5: A graph of $j(x) = -\frac{7}{2}x + 5$

Example 13.8.6 Graphing Lines in Point-Slope Form. From the equation, find the slope and a point on the graph of $k(x) = \frac{3}{4}(x - 2) - 5$, and then use slope triangles to find the next two points on the line. Draw the line.

Explanation. The slope of $k(x) = \frac{3}{4}(x - 2) - 5$ is $\frac{3}{4}$ and the point on the graph given in the equation is $(2, -5)$. So to graph k, start at $(2, -5)$, and the go up 3 units and right 4 units (or down 3 left 4) to reach more points.

From the graph, we can read that two more points that $k(x) = \frac{3}{4}(x - 2) - 5$ passes through are $(6, -2)$ and $(10, 1)$.

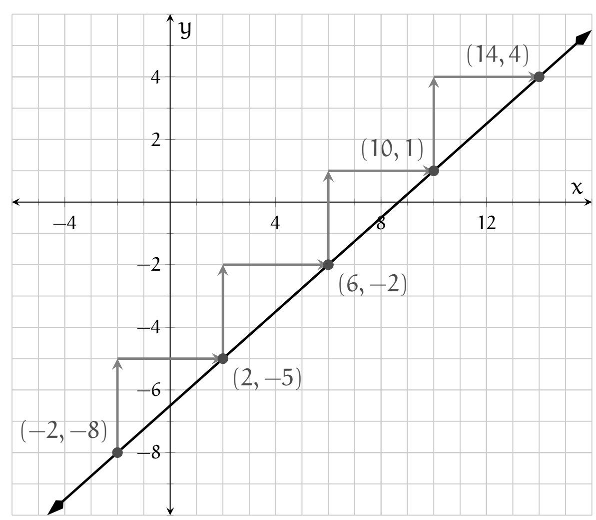


Figure 13.8.7: A graph of $k(x) = \frac{3}{4}(x - 2) - 5$

Example 13.8.8 Graphing Lines Using Intercepts. Use the intercepts of $4x-2y=16$ to graph the equation.

Explanation.

To find the x-intercept, set $y=0$ and solve for x.

$$\begin{aligned} 4x-2(0)&=16\\ 4x&=16\\ x&=4 \end{aligned}$$

The x-intercept is the point $(4,0)$.

To find the y-intercept, set $x=0$ and solve for y.

$$\begin{aligned} 4(0)-2y&=16\\ -2y&=16\\ y&=-8 \end{aligned}$$

The y-intercept is the point $(0,-8)$.

Next, we just plot these points and draw the line that runs through them.

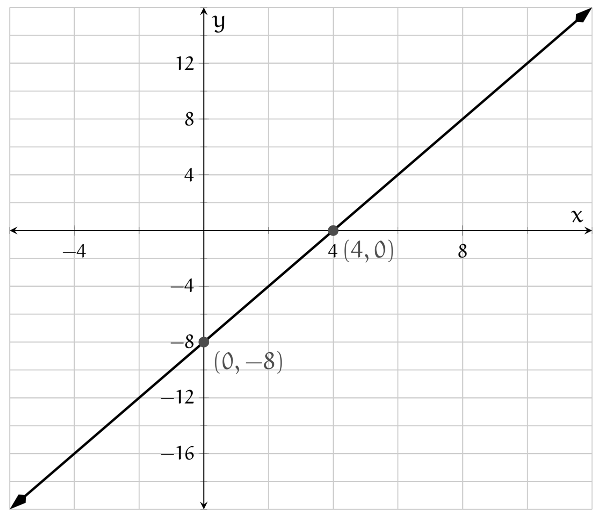


Figure 13.8.9: A graph of $4x-2y=16$

Example 13.8.10 Graphing Functions by Plotting Points. The wind chill[1] is how cold it *feels* outside due to the wind. Imagine a chilly 32 °F day with a breeze blowing over the snowy ground. The wind chill, $w(v)$, at this temperature can be approximated by the formula $w(v)=54.5-21\sqrt[6]{v}$ where v is the speed of the wind in miles per hour. This formula only approximates the wind chill for reasonable wind values of about 5 mph to 60 mph. Create a table of values rounded to the nearest tenth for the wind chill at realistic wind speeds and make a graph of w.

Explanation. Typical wind speeds vary between 5 and 20 mph, with gusty conditions up to 60 mph, depending on location. A good way to enter the sixth root into a calculator is to recall that $\sqrt[6]{x}=x^{\frac{1}{6}}$.

v	$w(v) = 54.5 - 21\sqrt[6]{v}$	Point	Interpretation
5	$w(5) \approx 27.0$	$(5, 27.0)$	A 5 mph wind causes a wind chill of 27.0 °F.
10	$w(10) \approx 23.7$	$(10, 23.7)$	A 10 mph wind causes a wind chill of 23.7 °F.
20	$w(20) \approx 19.9$	$(20, 19.9)$	A 20 mph wind causes a wind chill of 19.9 °F.
30	$w(30) \approx 17.5$	$(30, 17.5)$	A 30 mph wind causes a wind chill of 17.5 °F.
40	$w(40) \approx 15.7$	$(40, 15.7)$	A 40 mph wind causes a wind chill of 15.7 °F.
50	$w(50) \approx 14.2$	$(50, 14.2)$	A 50 mph wind causes a wind chill of 14.2 °F.
60	$w(60) \approx 12.9$	$(60, 12.9)$	A 60 mph wind causes a wind chill of 12.9 °F.

Figure 13.8.11: A table of values for $w(v) = 54.5 - 21\sqrt[6]{v}$

With the values in Table 13.8.11, we can sketch the graph.

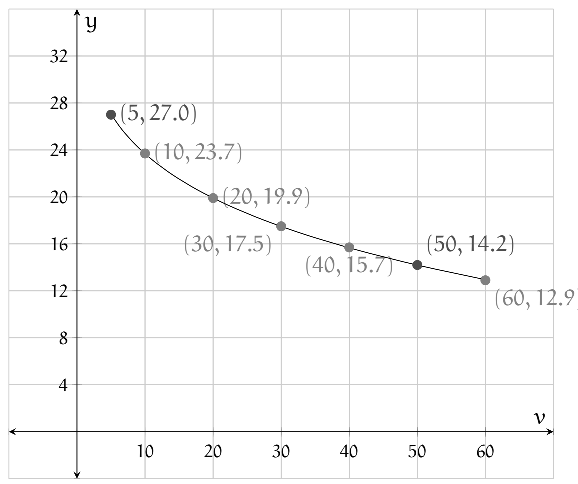


Figure 13.8.12: A graph of $w(v) = 54.5 - 21\sqrt[6]{v}$

13.8.2 Quadratic Graphs and Vertex Form

In Section 13.2 we covered the use of technology in analyzing quadratic functions, the vertex form of a quadratic function and how it affects horizontal and vertical shifts of the graph of a parabola, and the factored form of a quadratic function.

Example 13.8.13 Exploring Quadratic Functions with Graphing Technology. Use technology to graph and make a table of the quadratic function g defined by $g(x) = -x^2 + 5x - 6$ and find each of the key points or features.

a. Find the vertex.

b. Find the vertical intercept.

c. Find the horizontal intercept(s).

d. Find $g(-1)$.

e. Solve $g(x) = -6$ using the graph.

f. Solve $g(x) \leq -6$ using the graph.

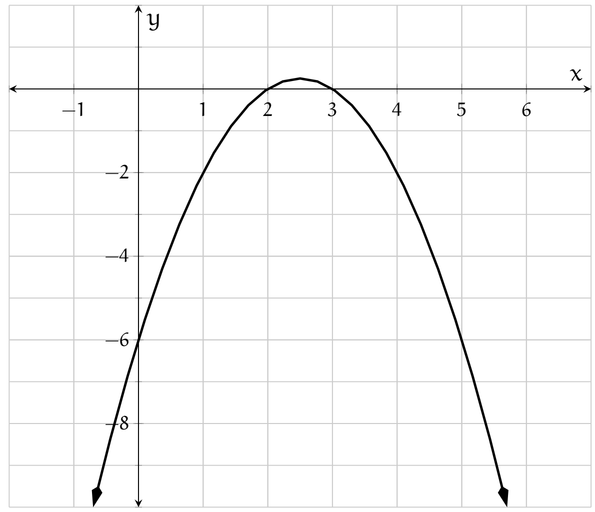
g. State the domain and range of the function.

[1] en.wikipedia.org/wiki/Wind_chill

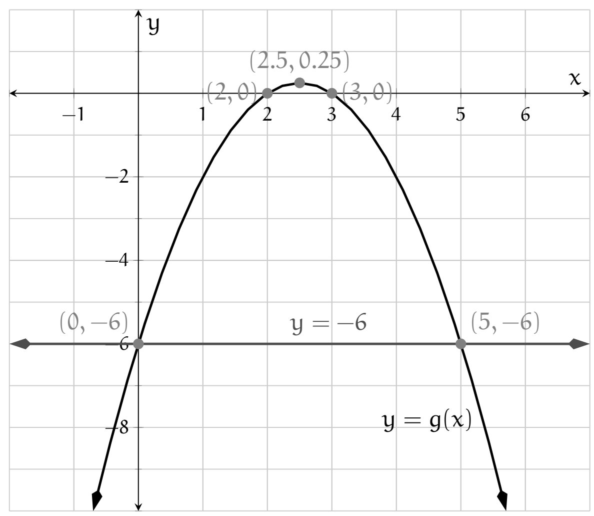
Explanation.

The specifics of how to use any one particular technology tool vary. Whether you use an app, a physical calculator, or something else, a table and graph should look like:

x	$g(x)$
-1	-12
0	-6
1	-2
2	0
2	0
3	0
4	-2



Additional features of your technology tool can enhance the graph to help answer these questions. You may be able to make the graph appear like:



a. The vertex is $(2.5, 0.25)$.

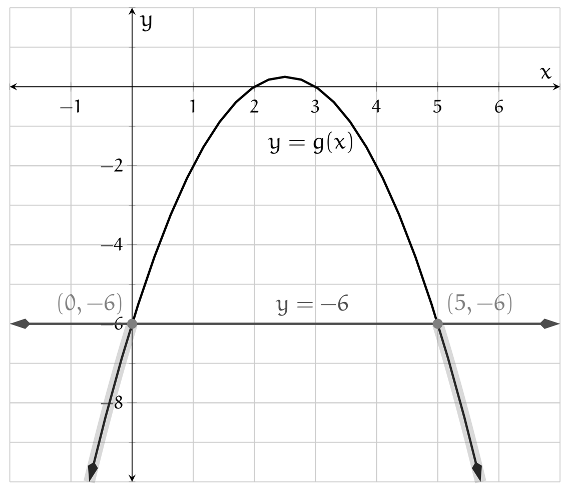
b. The vertical intercept is $(0, -6)$.

c. The horizontal intercepts are $(2, 0)$ and $(3, 0)$.

d. $g(-1) = -2$.

e. The solutions to $g(x) = -6$ are the x-values where $y = 6$. We graph the horizontal line $y = -6$ and find the x-values where the graphs intersect. The solution set is $\{0, 5\}$.

f. The solutions are all x-values where the function below (or touching) the line $y = -6$. The solution set is $(-\infty, 0] \cup [5, \infty)$.



g. The domain is $(-\infty, \infty)$ and the range is $(-\infty, 0.25]$.

Example 13.8.14 The Vertex Form of a Parabola. Recall that the vertex form of a quadratic function tells us the location of the vertex of a parabola.

a. State the vertex of the quadratic function $r(x) = -8(x + 1)^2 + 7$.

b. State the vertex of the quadratic function $u(x) = 5(x - 7)^2 - 3$.

c. Write the formula for a parabola with vertex $(-5, 3)$ and $a = 2$.

d. Write the formula for a parabola with vertex $(1, -17)$ and $a = -4$.

Explanation.

a. The vertex of the quadratic function $r(x) = -8(x + 1)^2 + 7$ is $(-1, 7)$.

b. The vertex of the quadratic function $u(x) = 5(x - 7)^2 - 3$ is $(7, -3)$.

c. The formula for a parabola with vertex $(-5, 3)$ and $a = 2$ is $y = 2(x + 5)^2 + 3$.

d. The formula for a parabola with vertex $(1, -17)$ and $a = -4$ is $y = 4(x - 1)^2 - 17$.

Example 13.8.15 Horizontal and Vertical Shifts. Identify the horizontal and vertical shifts compared with $f(x) = x^2$.

a. $s(x) = (x + 1)^2 + 7$.

b. $v(x) = (x - 7)^2 - 3$.

Explanation.

a. The graph of the quadratic function $s(x) = -8(x + 1)^2 + 7$ is the same as the graph of $f(x) = x^2$ shifted to the left 1 unit and up 7 units.

b. The graph of the quadratic function $v(x) = 5(x - 7)^2 - 3$ is the same as the graph of $f(x) = x^2$ shifted to the right 7 units and down 3 units.

Example 13.8.16 The Factored Form of a Parabola. Recall that the factored form of a quadratic function tells us the horizontal intercepts very quickly.

a. $n(x) = 13(x - 1)(x + 6)$.

b. $p(x) = -6(x - \frac{2}{3})(x + \frac{1}{2})$.

Explanation.

a. The horizontal intercepts of n are $(1, 0)$ and $(-6, 0)$.

b. The horizontal intercepts of p are $(\frac{2}{3}, 0)$ and $(-\frac{1}{2}, 0)$.

13.8.3 Completing the Square

In Section 13.3 we covered how to complete the square to both solve quadratic equations in one variable and to put quadratic functions into vertex form.

Example 13.8.17 Solving Quadratic Equations by Completing the Square. Solve the equations by completing the square.

a. $k^2 - 18k + 1 = 0$

b. $4p^2 - 3p = 2$

Explanation.

a. To complete the square in the equation $k^2 - 18k + 1 = 0$, we first we will first move the constant term to the right side of the equation. Then we will use Fact 13.3.3 to find $\left(\frac{b}{2}\right)^2$ to add to both sides.

$$\begin{aligned} k^2 - 18k + 1 &= 0 \\ k^2 - 18k &= -1 \end{aligned}$$

In our case, $b = -18$, so $\left(\frac{b}{2}\right)^2 = \left(\frac{-18}{2}\right)^2 = 81$

$$\begin{aligned} k^2 - 18k + 81 &= -1 + 81 \\ (k-9)^2 &= 80 \end{aligned}$$

$$\begin{aligned} k - 9 &= -\sqrt{80} &\quad \text{or} \quad && k - 9 &= \sqrt{80} \\ k - 9 &= -4\sqrt{5} &\quad \text{or} \quad && k - 9 &= 4\sqrt{5} \\ k &= 9 - 4\sqrt{5} &\quad \text{or} \quad && k &= 9 + 4\sqrt{5} \end{aligned}$$

The solution set is $\{9 + 4\sqrt{5}, 9 - 4\sqrt{5}\}$.

b. To complete the square in the equation $4p^2 - 3p = 2$, we first divide both sides by 4 since the leading coefficient is 4.

$$\begin{aligned} \frac{4p^2}{4} - \frac{3p}{4} &= \frac{2}{4} \\ p^2 - \frac{3}{4}p &= \frac{1}{2} \\ p^2 - \frac{3}{4}p &= \frac{1}{2} \end{aligned}$$

Next, we will complete the square. Since $b = -\frac{3}{4}$, first,

$$\frac{b}{2} = \frac{-\frac{3}{4}}{2} = -\frac{3}{8} \qquad (13.8.1)$$

and next, squaring that, we have

$$\left(-\frac{3}{8}\right)^2 = \frac{9}{64}. \tag{13.8.2}$$

So we will add $\frac{9}{64}$ from Equation (13.8.2) to both sides of the equation:

$$\begin{aligned} p^2 - \frac{3}{4}p + \frac{9}{64} &= \frac{1}{2} + \frac{9}{64} \\ p^2 - \frac{3}{4}p + \frac{9}{64} &= \frac{32}{64} + \frac{9}{64} \\ p^2 - \frac{3}{4}p + \frac{9}{64} &= \frac{41}{64} \end{aligned}$$

Here, remember that we always factor with the number found in the first step of completing the square, Equation (13.8.1).

$$\left(p - \frac{3}{8}\right)^2 = \frac{41}{64}$$

$$\begin{aligned} p - \frac{3}{8} &= -\frac{\sqrt{41}}{8} &\quad\text{or}\quad&& p - \frac{3}{8} &= \frac{\sqrt{41}}{8} \\ p &= \frac{3}{8} - \frac{\sqrt{41}}{8} &\quad\text{or}\quad&& p &= \frac{3}{8} + \frac{\sqrt{41}}{8} \\ p &= \frac{3-\sqrt{41}}{8} &\quad\text{or}\quad&& p &= \frac{3+\sqrt{41}}{8} \end{aligned}$$

The solution set is $\left\{\frac{3-\sqrt{41}}{8}, \frac{3+\sqrt{41}}{8}\right\}$.

Example 13.8.18 Putting Quadratic Functions in Vertex Form. Write a formula in vertex form for the function T defined by $T(x) = 4x^2 + 20x + 24$.

Explanation. Before we can complete the square, we will factor the 4 out of the first two terms. Don't be tempted to factor the 4 out of the constant term.

$$T(x) = 4\left(x^2 + 5x\right) + 24$$

Now we will complete the square inside the parentheses by adding and subtracting $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$.

$$T(x) = 4\left(x^2 + 5x + \frac{25}{4} - \frac{25}{4}\right) + 24$$

Notice that the constant that we subtracted is inside the parentheses, but it will not be part of our perfect square trinomial. In order to bring it outside, we need to multiply it by 4. We are distributing the 4 to that term so we can combine it with the outside term.

$$\begin{aligned} T(x) &= 4\left(\left(x^2 + 5x + \frac{25}{4}\right) - \frac{25}{4}\right) + 24 \\ &= 4\left(x^2 + 5x + \frac{25}{4}\right) - 4 \cdot \frac{25}{4} + 24 \end{aligned}$$

$$
\begin{aligned}
&= 4\left(x+\frac{5}{2}\right)^2 - 25 + 24 \\
&= 4\left(x+\frac{5}{2}\right)^2 - 1
\end{aligned}
$$

Note that The vertex is $\left(-\frac{5}{2}, -1\right)$.

Example 13.8.19 Graphing Quadratic Functions by Hand. Graph the function H defined by $H(x) = -x^2 - 8x - 15$ by determining its key features algebraically.

Explanation. To start, we'll note that this function opens downward because the leading coefficient, -1, is negative.

Now we will complete the square to find the vertex. We will factor the -1 out of the first two terms, and then add and subtract $\left(\frac{8}{2}\right)^2 = 4^2 = 16$ on the right side.

$$
\begin{aligned}
H(x) &= -\left[x^2 + 8x\right] - 15 \\
&= -\left[x^2 + 8x + 16 - 16\right] - 15 \\
&= -\left[\left(x^2 + 8x + 16\right) - 16\right] - 15 \\
&= -\left(x^2 + 8x + 16\right) - (-1 \cdot 16) - 15 \\
&= -\left(x+4\right)^2 + 16 - 15 \\
&= -\left(x+4\right)^2 + 1
\end{aligned}
$$

The vertex is $(-4, 1)$ so the axis of symmetry is the line $x = -4$.

To find the y-intercept, we'll replace x with 0 or read the value of c from the function in standard form:

$$
\begin{aligned}
H(0) &= -(0)^2 - 8(0) - 15 \\
&= -15
\end{aligned}
$$

The y-intercept is $(0, -15)$ and we will find its symmetric point on the graph, which is $(-8, -15)$.

Next, we'll find the horizontal intercepts. We see this function factors so we will write the factored form to get the horizontal intercepts.

$$
\begin{aligned}
H(x) &= -x^2 - 8x - 15 \\
&= -\left(x^2 + 8x + 15\right) \\
&= -(x+3)(x+5)
\end{aligned}
$$

The x-intercepts are $(-3, 0)$ and $(-5, 0)$.

Now we will plot all of the key points and draw the parabola.

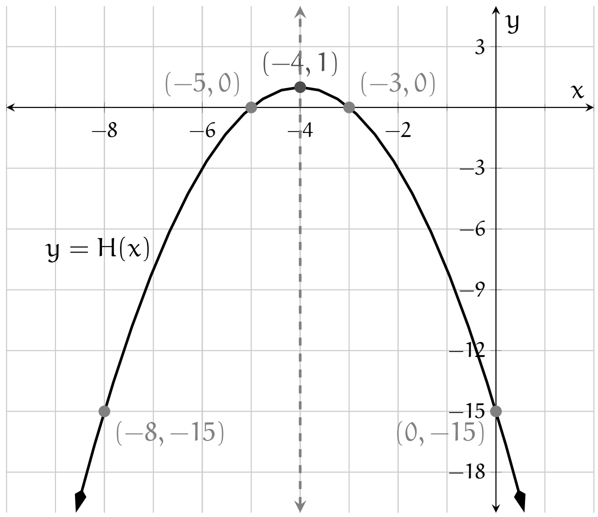


Figure 13.8.20: The graph of $y = -x^2 - 8x - 15$.

13.8.4 Absolute Value Equations

In Section 13.4 we covered how to solve equations when an absolute value is equal to a number and when an absolute value is equal to an absolute value.

Example 13.8.21 Solving an Equation with an Absolute Value. Solve the absolute value equation $|9 - 4x| = 17$ using Fact 13.4.12.

Explanation. The equation $|9 - 4x| = 17$ breaks into two pieces, each of which needs to be solved independently.

$$\begin{aligned} 9 - 4x &= 17 & &\text{or} & 9 - 4x &= -17 \\ -4x &= 8 & &\text{or} & -4x &= -26 \\ \frac{-4x}{-4} &= \frac{8}{-4} & &\text{or} & \frac{-4x}{-4} &= \frac{-26}{-4} \\ x &= -2 & &\text{or} & x &= \frac{13}{2} \end{aligned}$$

The solution set is $\left\{-2, \frac{13}{2}\right\}$.

Example 13.8.22 Solving an Equation with Two Absolute Values. Solve the absolute value equation $|7 - 3x| = |6x - 5|$ using Fact 13.4.18.

Explanation. The equation $|7 - 3x| = |6x - 5|$ breaks into two pieces, each of which needs to be solved independently.

$$\begin{aligned} 7 - 3x &= 6x - 5 & &\text{or} & 7 - 3x &= -(6x - 5) \\ 7 - 3x &= 6x - 5 & &\text{or} & 7 - 3x &= -6x + 5 \\ 12 - 3x &= 6x & &\text{or} & 2 - 3x &= -6x \\ 12 &= 9x & &\text{or} & 2 &= -3x \end{aligned}$$

$$\frac{12}{9} = \frac{9x}{9} \quad \text{or} \quad \frac{2}{-3} = \frac{-3x}{-3}$$
$$\frac{4}{3} = x \quad \text{or} \quad -\frac{2}{3} = x$$

The solution set is $\left\{\frac{4}{3}, -\frac{2}{3}\right\}$.

13.8.5 Solving Mixed Equations

In Section 13.5 we reviewed all of the various solving methods covered so far including solving linear equations with one variable and for a specified variable; solving systems of linear equations using substitution and elimination; solving equations with radicals; solving quadratic equations using the square root method, the quadratic formula, factoring, completing the square; and solving rational equations.

Example 13.8.23 Types of equations. Identify the type of equation as linear, a system of linear equations, quadratic, radical, rational, absolute value, or something else.

a. $5x^2 - 2x = 9$

b. $\pi x - 3 = 4(x + 1)$

c. $8x^2 = x + 9$

d. $\frac{2}{x+2} + \frac{3}{2x+4} = \frac{7}{x+8}$

e. $|2x - 7| + 2 = 3$

f. $\sqrt{x+2} = x - 4$

g. $(3 - 2x)^2 - 9 = 9$

h. $3 = \sqrt[3]{5 - 2x}$

i.
$$\begin{cases} 5x - y = -6 \\ 2x - 3y = 8 \end{cases}$$

j. $x^x = x - \left|x - x^2 - 3\right|$

Explanation.

a. The equation $5x^2 - 2x = 9$ is a quadratic equation since the variable is being squared (but doesn't have any higher power).

b. The equation $\pi x - 3 = 4(x + 1)$ is a linear equation since the variable is only to the first power.

c. The equation $8x^2 = x + 9$ is a quadratic equation since there is a degree-two term.

d. The equation $\frac{2}{x+2} + \frac{3}{2x+4} = \frac{7}{x+8}$ is a rational equation since the variable exists in the denominator.

e. The equation $|2x - 7| + 2 = 3$ is an absolute value equation since the variable is inside an absolute value.

f. The equation $\sqrt{x+2} = x - 4$ is a radical equation since the variable appears inside the radical.

g. The equation $(3 - 2x)^2 - 9 = 9$ is a quadratic equation since if we were to distribute everything out, we would have a term with x^2.

h. The equation $3 = \sqrt[3]{5 - 2x}$ is a radical equation since the variable is inside the radical.

i. The system

$$\begin{cases} 5x - y = -6 \\ 2x - 3y = 8 \end{cases}$$

is a system of linear equations.

j. The equation $x^x = x - |x - x^2 - 3|$ is an equation type that we have not covered and is not listed above.

Example 13.8.24 Solving Mixed Equations. Solve the equations using appropriate techniques.

a. $5x^2 - 2x = 9$

b. $\pi x - 3 = 4(x+1)$

c. $8x^2 = x + 9$

d. $\frac{2}{x+2} + \frac{3}{2x+4} = \frac{7}{x+8}$

e. $|2x - 7| + 2 = 3$

f. $\sqrt{x+2} = x - 4$

g. $(3 - 2x)^2 - 9 = 9$

h. $3 = \sqrt[3]{5 - 2x}$

i.
$$\begin{cases} 5x - y = -6 \\ 2x - 3y = 8 \end{cases}$$

j. $x^2 + 10x = 12$ (using completing the square)

Explanation.

a. Since the equation $5x^2 - 2x = 9$ is a quadratic we should consider the square root method, the quadratic formula, factoring, and completing the square. In this case, we will start with the quadratic formula. First, note that we should rearrange the terms in equation into standard form.

$$\begin{aligned} 5x^2 - 2x &= 9 \\ 5x^2 - 2x - 9 &= 0 \end{aligned}$$

Note that $a = 5$, $b = -2$, and $c = -9$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(5)(-9)}}{2(5)} \\ x &= \frac{2 \pm \sqrt{4 + 180}}{10} \\ x &= \frac{2 \pm \sqrt{184}}{10} \\ x &= \frac{2 \pm \sqrt{4 \cdot 46}}{10} \\ x &= \frac{2 \pm 2\sqrt{46}}{10} \\ x &= \frac{1 \pm \sqrt{46}}{5} \end{aligned}$$

The solution set is $\left\{\frac{1+\sqrt{46}}{5}, \frac{1-\sqrt{46}}{5}\right\}$.

b. Since the equation $\pi x - 3 = 4(x+1)$ is a linear equation, we isolate the variable step-by-step.

$$\begin{aligned} \pi x - 3 &= 4(x+1) \\ \pi x - 3 &= 4x + 4 \end{aligned}$$

$$\begin{aligned}\pi x - 4x &= 7\\ x(\pi-4) &= 7\\ x &= \frac{7}{\pi-4}\end{aligned}$$

The solution set is $\left\{\frac{7}{\pi-4}\right\}$.

c. Since the equation $8x^2 = x+9$ is a quadratic equation, we again have several options to consider. We will try factoring on this one first after converting it to standard form.

$$\begin{aligned}8x^2 &= x+9\\ 8x^2 - x - 9 &= 0\end{aligned}$$

Here, $ac = -72$ and two numbers that multiply to be -72 but add to be -1 are 8 and -9.

$$\begin{aligned}8x^2 + 8x - 9x - 9 &= 0\\ \left(8x^2+8x\right) + (-9x-9) &= 0\\ 8x(x+1) - 9(x+1) &= 0\\ (8x-9)(x+1) &= 0\end{aligned}$$

$$\begin{aligned}8x-9 &= 0 &\quad\text{or}\quad && x+1 &= 0\\ x &= \frac{9}{8} &\quad\text{or}\quad && x &= -1\end{aligned}$$

The solution set is $\left\{\frac{9}{8}, -1\right\}$

d. Since the equation $\frac{2}{x+2} + \frac{3}{2x+4} = \frac{7}{x+8}$ is a rational we first need to cancel the denominators after factoring and finding the least common denominator.

$$\begin{aligned}\frac{2}{x+2} + \frac{3}{2x+4} &= \frac{7}{x+8}\\ \frac{2}{x+2} + \frac{3}{2(x+2)} &= \frac{7}{x+8}\end{aligned}$$

At this point, we note that the least common denominator is $2(x+2)(x+8)$. We need to multiply every term by this least common denominator.

$$\begin{aligned}\frac{2}{x+2}\cdot 2(x+2)(x+8) + \frac{3}{2(x+2)}\cdot 2(x+2)(x+8) &= \frac{7}{x+8}\cdot 2(x+2)(x+8)\\ \frac{2}{\cancel{x+2}}\cdot 2(\cancel{x+2})(x+8) + \frac{3}{\cancel{2}(\cancel{x+2})}\cdot \cancel{2}(\cancel{x+2})(x+8) &= \frac{7}{\cancel{x+8}}\cdot 2(x+2)(\cancel{x+8})\\ 2\cdot 2(x+8) + 3(x+8) &= 7\cdot 2(x+2)\\ 4(x+8) + 3(x+8) &= 14(x+2)\\ 4x + 32 + 3x + 24 &= 14x + 28\\ 7x + 56 &= 14x + 28\end{aligned}$$

$$\begin{aligned} 28 &= 7x \\ 4 &= x \end{aligned}$$

We always check solutions to rational equations to ensure we don't have any "extraneous solutions".

$$\begin{aligned} \frac{2}{(4)+2} + \frac{3}{2(4)+4} &\stackrel{?}{=} \frac{7}{(4)+8} \\ \frac{2}{6} + \frac{3}{12} &\stackrel{?}{=} \frac{7}{12} \\ \frac{4}{12} + \frac{3}{12} &\stackrel{?}{=} \frac{7}{12} \\ \frac{7}{12} &\stackrel{\checkmark}{=} \frac{7}{12} \end{aligned}$$

So, the solution set is $\{4\}$.

e. Since the equation $|2x-7|+2=3$ is an absolute value equation, we will first isolate the absolute value and then use Equations with an Absolute Value Expression to solve the remaining equation.

$$\begin{aligned} |2x-7|+2 &= 3 \\ |2x-7| &= 1 \end{aligned}$$

$$\begin{aligned} 2x-7 &= 5 &\quad \text{or} \quad&& 2x-7 &= -5 \\ 2x &= 12 &\quad \text{or} \quad&& 2x &= 2 \\ x &= 6 &\quad \text{or} \quad&& x &= 1 \end{aligned}$$

The solution set is $\{6, 1\}$.

f. Since the equation $\sqrt{x+2} = x-4$ is a radical equation, we will have to isolate the radical (which is already done), then square both sides to cancel the square root. After that, we will solve whatever remains.

$$\begin{aligned} \sqrt{x+2} &= x-4 \\ \sqrt{x+2} &= x-4 \\ (\sqrt{x+2})^2 &= (x-4)^2 \\ x+2 &= (x-4)(x-4) \\ x+2 &= x^2-8x+16 \\ 0 &= x^2-9x+14 \end{aligned}$$

We now have a quadratic equation. We will solve by factoring.

$$0 = (x-2)(x-7)$$

$$\begin{aligned} x-2 &= 0 &\quad \text{or} \quad&& x-7 &= 0 \\ x &= 2 &\quad \text{or} \quad&& x &= 7 \end{aligned}$$

Every potential solution to a radical equation should be verified to check for any "extraneous solutions".

$$\begin{aligned} \sqrt{2+2} &\stackrel{?}{=} 2-4 &\quad\text{or}\quad&& \sqrt{7+2} &\stackrel{?}{=} 7-4 \\ \sqrt{4} &\stackrel{?}{=} -2 &\quad\text{or}\quad&& \sqrt{9} &\stackrel{?}{=} 3 \\ 2 &\stackrel{\text{no}}{=} -2 &\quad\text{or}\quad&& 3 &\stackrel{\checkmark}{=} 3 \end{aligned}$$

So the solution set is $\{7\}$

g. Since the equation $(3-2x)^2 - 9 = 9$ is a quadratic equation, we again have several options. Since the variable only appears once in this equation we will use the the square root method to solve.

$$\begin{aligned} (3-2x)^2 - 9 &= 9 \\ (3-2x)^2 &= 18 \end{aligned}$$

$$\begin{aligned} 3-2x &= \sqrt{18} &\quad\text{or}\quad&& 3-2x &= -\sqrt{18} \\ 3-2x &= 3\sqrt{2} &\quad\text{or}\quad&& 3-2x &= -3\sqrt{2} \\ -2x &= -3+3\sqrt{2} &\quad\text{or}\quad&& -2x &= -3-3\sqrt{2} \\ x &= \frac{-3+3\sqrt{2}}{-2} &\quad\text{or}\quad&& x &= \frac{-3-3\sqrt{2}}{-2} \\ x &= \frac{3-3\sqrt{2}}{2} &\quad\text{or}\quad&& x &= \frac{3+3\sqrt{2}}{2} \end{aligned}$$

The solution set is $\left\{\frac{3-3\sqrt{2}}{2}, \frac{3+3\sqrt{2}}{2}\right\}$.

h. Since the equation $3 = \sqrt[3]{5-2x}$ is a radical equation, we will isolate the radical (which is already done) and then raise both sides to the third power to cancel the cube root.

$$\begin{aligned} 3 &= \sqrt[3]{5-2x} \\ (3)^3 &= \left(\sqrt[3]{6x-5}\right)^3 \\ 27 &= 6x-5 \\ 32 &= 6x \\ x &= \frac{32}{6} \\ x &= \frac{16}{3} \end{aligned}$$

The solution set is $\left\{\frac{16}{3}\right\}$.

i. Since

$$\begin{cases} 5x - y = -6 \\ 2x - 3y = 8 \end{cases}$$

is a system of linear equations, we can either use substitution or elimination to solve. Here we will use substitution. To use substitution, we need to solve one of the equations for one of the variables. We will solve the top equation for y.

$$\begin{aligned} 5x - y &= -6 \\ -y &= -5x - 6 \\ y &= 5x + 6 \end{aligned}$$

Now, we substitute $5x + 6$ where ever we see y in the other equation.

$$\begin{aligned} 2x - 3y &= 8 \\ 2x - 3(5x + 6) &= 8 \\ 2x - 15x - 18 &= 8 \\ -13x - 18 &= 8 \\ -13x &= 26 \\ x &= -2 \end{aligned}$$

Now that we have found x, we can substitute that back into one of the equations to find y. We will substitute into the first equation.

$$\begin{aligned} 5(-2) - y &= -6 \\ -10 - y &= -6 \\ -y &= 4 \\ y &= -4 \end{aligned}$$

So, the solution must be the point $(-2, -4)$.

j. Since the equation $x^2 + 10x = 12$ is quadratic and we are instructed to solve by using completing the square, we should recall that Fact 13.3.3 tells us how to complete the square, after we have sufficiently simplified. Since our equation is already in a simplified state, we need to add $\left(\frac{b}{2}\right)^2 = \left(\frac{10}{2}\right)^2 = 25$ to both sides of the equation.

$$\begin{aligned} x^2 + 10x &= 12 \\ x^2 + 10x + 25 &= 12 + 25 \\ (x + 5)^2 &= 37 \\ x + 5 &= \pm\sqrt{37} \\ x &= -5 \pm \sqrt{37} \end{aligned}$$

So, our solution set is $\left\{-5 + \sqrt{37}, -5 - \sqrt{37}\right\}$

13.8.6 Compound Inequalities

In Section 13.6 we defined the union of intervals, what compound inequalities are, and how to solve both "or" inequalities and triple inequalities.

Example 13.8.25 Unions of Intervals. Draw a representation of the union of the sets $(-\infty, -1]$ and $(2, \infty)$.

Explanation. First we make a number line with both intervals drawn to understand what both sets mean.

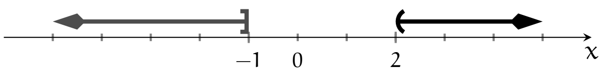


Figure 13.8.26: A number line sketch of $(-\infty, -1]$ as well as $(2, \infty)$

The two intervals should be viewed as a single object when stating the union, so here is the picture of the union. It looks the same, but now it is a graph of a single set.

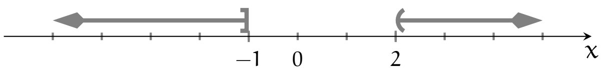


Figure 13.8.27: A number line sketch of $(-\infty, -1] \cup (2, \infty)$

Example 13.8.28 "Or" Compound Inequalities. Solve the compound inequality.

$$5z + 12 \leq 7 \text{ or } 3 - 9z < -2$$

Explanation. First we will solve each inequality for z.

$$\begin{aligned} 5z + 12 &\leq 7 & &\text{or} & 3 - 9z &< -2 \\ 5z &\leq -5 & &\text{or} & -9z &< -5 \\ z &\leq -1 & &\text{or} & z &> \frac{5}{9} \end{aligned}$$

The solution set to the compound inequality is:

$$(-\infty, -1] \cup \left(\frac{5}{9}, \infty\right)$$

Example 13.8.29 Three-Part Inequalities. Solve the three-part inequality $-4 \leq 20 - 6x < 32$.

Explanation. This is a three-part inequality. The goal is to isolate x in the middle and whatever you do to one "side," you have to do to the other two "sides."

$$\begin{gathered} -4 \leq 20 - 6x < 32 \\ -4 - 20 \leq 20 - 6x - 20 < 32 - 20 \\ -24 \leq -6x < 12 \\ \frac{-24}{-6} \geq \frac{-6x}{-6} > \frac{12}{-6} \\ 4 \geq x > -2 \end{gathered}$$

The solutions to the three-part inequality $4 \geq x > -2$ are those numbers that are trapped between -2 and 4, including 4 but not -2. The solution set in interval notation is $(-2, 4]$.

Example 13.8.30 Solving "And" Inequalities. Solve the compound inequality.

$$5 - 3t < 3 \text{ and } 4t + 1 \leq 6$$

Explanation. This is an "and" inequality. We will solve each part of the inequality and then combine the two solutions sets with an intersection.

$$\begin{aligned} 5 - 3t &< 3 & &\text{and} & 4t + 1 &\leq 6 \\ -3t &< -2 & &\text{and} & 4t &\leq 5 \\ \frac{-3t}{-3} &> \frac{-2}{-3} & &\text{and} & \frac{4t}{4} &\leq \frac{5}{4} \\ t &> \frac{2}{3} & &\text{and} & t &\leq \frac{5}{4} \end{aligned}$$

The solution set to $t > \frac{2}{3}$ is $\left(\frac{2}{3}, \infty\right)$ and the solution set to $t \leq \frac{5}{4}$ is $\left(-\infty, \frac{5}{4}\right]$. Shown is a graph of these solution sets.

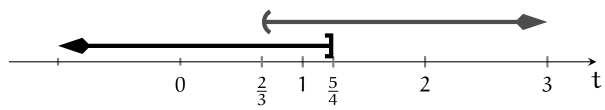


Figure 13.8.31: A number line sketch of $\left(\frac{2}{3}, \infty\right)$ and also $\left(-\infty, \frac{5}{4}\right]$

Recall that an "and" problem finds the intersection of the solution sets. Intersection finds the t-values where the two lines overlap, so the solution to the compound inequality must be

$$\left(\frac{2}{3}, \infty\right) \cap \left(-\infty, \frac{5}{4}\right] = \left(\frac{2}{3}, \frac{5}{4}\right].$$

Example 13.8.32 Application of Compound Inequalities. Mishel wanted to buy some mulch for their spring garden. Each cubic yard of mulch cost \$27 and delivery for any size load was \$40. If they wanted to spend between \$200 and \$300, set up and solve a compound inequality to solve for the number of cubic yards, x, that they could buy.

Explanation. Since the mulch costs \$27 per cubic yard and delivery is \$40, the formula for the cost of x yards of mulch is $27x + 40$. Since Mishel wants to spend between \$200 and \$300, we just trap their cost between these two values.

$$\begin{aligned} 200 &< 27x + 40 < 300 \\ 200 - 40 &< 27x + 40 - 40 < 300 - 40 \\ 160 &< 27x < 260 \\ \frac{160}{27} &< \frac{27x}{27} < \frac{260}{27} \\ 5.93 &< x < 9.63 \end{aligned}$$

Note: these values are approximate

Most companies will only sell whole number cubic yards of mulch, so we have to round appropriately. Since Mishel wants to spend more than \$200, we have to round our lower value from 5.93 up to 6 cubic yards.

If we round the 9.63 up to 10, then the total cost will be $27 \cdot 10 + 40 = 310$ (which represents $310), which is more than Mishel wanted to spend. So we actually have to round down to 9cubic yards to stay below the $300 maximum.

In conclusion, Mishel could buy 6, 7, 8, or 9 cubic yards of mulch to stay between $200 and $300.

13.8.7 Solving Inequalities Graphically

Example 13.8.33 Solving Absolute Value Inequalities Graphically. Solve the inequality $|4 - 2x| < 10$ graphically.

Explanation. To solve the inequality $|4 - 2x| < 10$, we will start by making a graph with both $y = |4 - 2x|$ and $y = 10$.

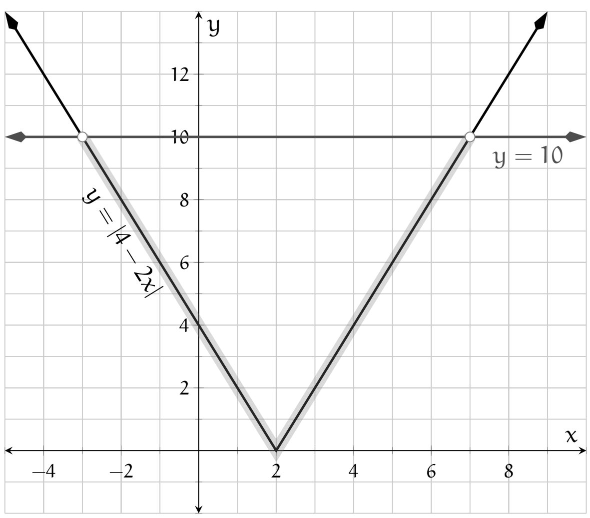


Figure 13.8.34: $y = |4 - 2x|$ and $y = 3$

The portion of the graph of $y = |4 - 2x|$ that is below $y = 10$ is highlighted and the x-values of that highlighted region are trapped between -3 and 7: $-3 < x < 7$. That means that the solution set is $(-3, 7)$. Note that we shouldn't include the endpoints of the interval because at those values, the two graphs are *equal* whereas the original inequality was only *less than* and not equal.

Example 13.8.35 Solving Compound Inequalities Graphically. Figure 13.8.36 shows a graph of $y = G(x)$. Use the graph do the following.

a. Solve $G(x) < -2$.

b. Solve $G(x) \geq 1$.

c. Solve $-1 \leq G(x) < 1$.

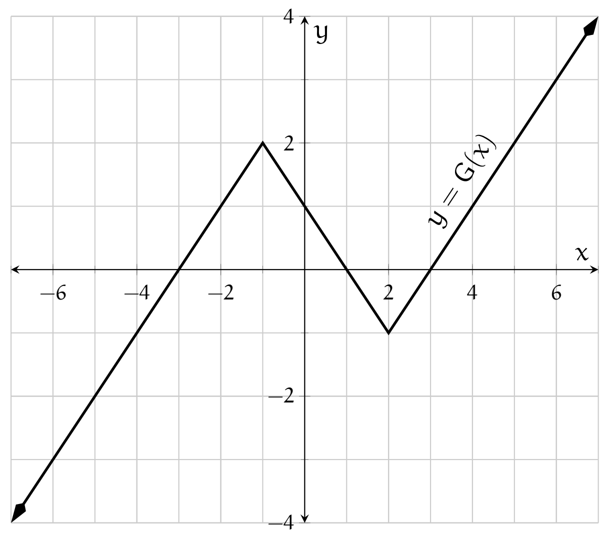


Figure 13.8.36: Graph of $y = G(x)$

Explanation.

a. To solve $G(x) < -2$, we first draw a dotted line (since it's a less-than, not a less-than-or-equal) at $y = -2$. Then we examine the graph to find out where the graph of $y = G(x)$ is underneath the line $y = -2$. Our graph is below the line $y = -2$ for x-values less than -5. So the solution set is $(-\infty, -5)$.

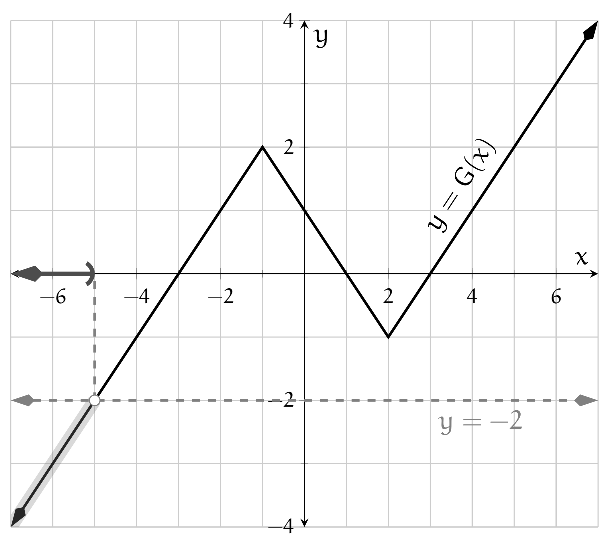


Figure 13.8.37: Graph of $y = G(x)$ and solution set to $G(x) < -2$

b. To solve $G(x) \geq 1$, we first draw a solid line (since it's a greater-than-or-equal) at $y = 1$. Then we examine the graph to find out what parts of the graph of $y = G(x)$ are above the line $y = 1$. Our graph is above (or on) the line $y = 1$ for x-values between -2 and 0 as well as x-values bigger than 4. So the solution set is $[-2, 0] \cup [4, \infty)$.

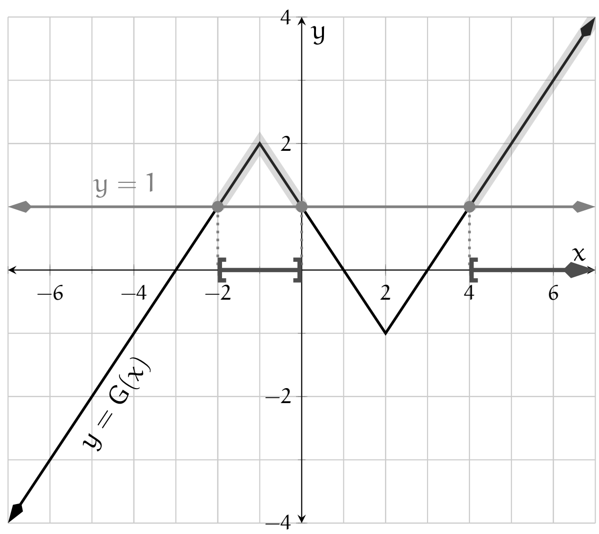


Figure 13.8.38: Graph of $y = G(x)$ and solution set to $G(x) \geq 1$

c. To solve $-1 < G(x) \leq 1$, we first draw a solid line at $y = 1$ and dotted line at $y = -1$. Then we examine the graph to find out what parts of the graph of $y = G(x)$ are trapped between the two lines we just drew. Our graph is between those values for x-values between -4 and -2 as well as x-values between 0 and 2 as well as as well as x-values between 2 and 4. We use the solid and hollow dots on the graph to decide whether or not to include those values. So the solution set is $(-4, -2] \cup [0, 2) \cup (2, 4]$.

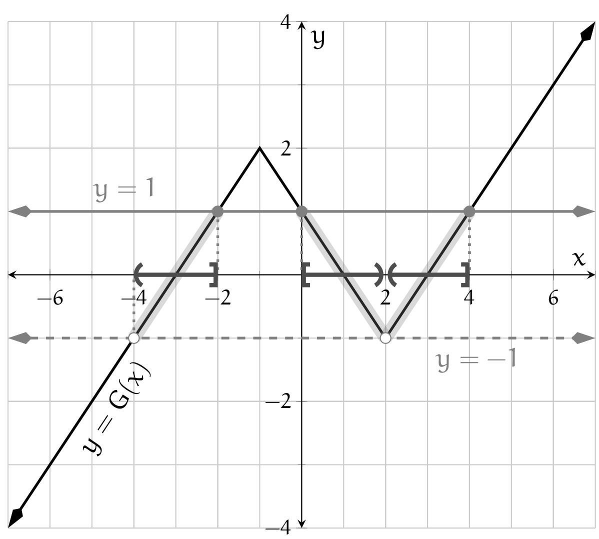


Figure 13.8.39: Graph of $y = G(x)$ and solution set to $-1 < G(x) \leq 1$

13.8.8 Exercises

Overview of Graphing

1. Create a table of ordered pairs and then make a plot of the equation $y = -\frac{2}{5}x - 3$.

2. Create a table of ordered pairs and then make a plot of the equation $y = -\frac{3}{4}x + 2$.

3. Graph the equation $y = \frac{2}{3}x + 4$.

4. Graph the equation $y = \frac{3}{2}x - 5$.

5. Graph the linear equation $y = -\frac{8}{3}(x - 4) - 5$ by identifying the slope and one point on this line.

6. Graph the linear equation $y = \frac{5}{7}(x + 3) + 2$ by identifying the slope and one point on this line.

7. Make a graph of the line $20x - 4y = 8$.

8. Make a graph of the line $3x + 5y = 10$.

9. Create a table of ordered pairs and then make a plot of the equation $y = -3x^2$.

10. Create a table of ordered pairs and then make a plot of the equation $y = -x^2 - 2x - 3$.

Quadratic Graphs and Vertex Form

11. Let $F(x) = 2x^2 - 2x + 3$. Use technology to find the following.

a. The vertex is ______.

b. The y-intercept is ______.

c. The x-intercept(s) is/are ______.

d. The domain of F is ______.

e. The range of F is ______.

f. Calculate $F(1)$. ______.

g. Solve $F(x) = 6$. ______

h. Solve $F(x) \geq 6$. ______

12. Let $G(x) = -2x^2 + 4x - 1$. Use technology to find the following.

a. The vertex is ______.

b. The y-intercept is ______.

c. The x-intercept(s) is/are ______.

d. The domain of G is ______.

e. The range of G is ______.

f. Calculate $G(1)$. ______.

g. Solve $G(x) = -6$. ______

h. Solve $G(x) < -6$. ______

13. An object was launched from the top of a hill with an upward vertical velocity of 170 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 170t + 300$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Find the answer using technology.

[ ] seconds after its launch, the object reached its maximum height of [ ] feet.

14. An object was launched from the top of a hill with an upward vertical velocity of 180 feet per second. The height of the object can be modeled by the function $h(t) = -16t^2 + 180t + 200$, where t represents the number of seconds after the launch. Assume the object landed on the ground at sea level. Find the answer using technology.

[ ] seconds after its launch, the object fell to the ground at sea level.

15. Consider the graph of the equation $y = (x-9)^2 - 7$.
Compared to the graph of $y = x^2$, the vertex has been shifted [ ] units (☐ left ☐ right) and [ ] units (☐ down ☐ up) .

16. Consider the graph of the equation $y = (x+8)^2 + 5$.
Compared to the graph of $y = x^2$, the vertex has been shifted [ ] units (☐ left ☐ right) and [ ] units (☐ down ☐ up) .

17. The quadratic expression $(x-1)^2$ is written in vertex form.

a. Write the expression in standard form.

b. Write the expression in factored form.

18. The quadratic expression $(x-2)^2 - 9$ is written in vertex form.

a. Write the expression in standard form.

b. Write the expression in factored form.

19. The formula for a quadratic function K is $K(x) = (x+1)(x-4)$.

a. The y-intercept is [ ].

b. The x-intercept(s) is/are [ ].

20. The formula for a quadratic function h is $h(x) = (x-1)(x+2)$.

a. The y-intercept is [ ].

b. The x-intercept(s) is/are [ ].

Completing the Square

21. Solve $r^2 - r - 2 = 0$ by completing the square.

22. Solve $t^2 + 5t - 6 = 0$ by completing the square.

23. Solve $3t^2 - 14t + 15 = 0$ by completing the square.

24. Solve $12t^2 + 20t + 7 = 0$ by completing the square.

25. Complete the square to convert the quadratic function from standard form to vertex form, and use the result to find the function's domain and range.
$f(x) = -3x^2 - 54x - 240$

26. Complete the square to convert the quadratic function from standard form to vertex form, and use the result to find the function's domain and range.
$f(x) = -5x^2 - 60x - 184$

27. Graph $f(x) = x^2 - 7x + 12$ by algebraically determining its key features. Then state the domain and range of the function.

28. Graph $f(x) = -x^2 + 4x + 21$ by algebraically determining its key features. Then state the domain and range of the function.

29. Graph $f(x) = x^2 - 8x + 16$ by algebraically determining its key features. Then state the domain and range of the function.

30. Graph $f(x) = x^2 + 6x + 9$ by algebraically determining its key features. Then state the domain and range of the function.

Absolute Value Equations

31. Solve the following equation.
$|4x + 9| = 3$

32. Solve the following equation.
$|5x + 2| = 7$

33. Solve: $\left|\dfrac{2t-3}{3}\right| = 1$

34. Solve: $\left|\dfrac{2x-7}{7}\right| = 3$

35. Solve: $\left|\frac{1}{4}x + 7\right| = 5$

36. Solve: $\left|\frac{1}{2}y + 5\right| = 3$

37. Solve: $|5y - 20| + 2 = 2$

38. Solve: $|4a - 4| + 7 = 7$

39. Solve the equation: $|2x - 3| = |7x + 6|$

40. Solve the equation: $|4x - 8| = |3x + 4|$

41. Solve the following equation.
$|3x - 1| = |8x - 10|$

42. Solve the following equation.
$|3x - 8| = |10x + 5|$

Solving Mixed Equations

43. Solve the equation.
$47x^2 + 41 = 0$

44. Solve the equation.
$23x^2 + 47 = 0$

45. Solve: $|y - 1| = 9$

46. Solve: $|y - 5| = 13$

47. Solve the equation.
$2x^2 = -25x - 50$

48. Solve the equation.
$2x^2 = -11x - 5$

49. Solve the equation.
$y = \sqrt{y + 10} + 2$

50. Solve the equation.
$y = \sqrt{y + 8} - 2$

51. Solve the equation.
$\dfrac{8}{r+1} - \dfrac{7}{r-6} = \dfrac{5}{r^2 - 5r - 6}$

52. Solve the equation.
$\dfrac{5}{r-2} - \dfrac{7}{r+9} = \dfrac{9}{r^2 + 7r - 18}$

53. Solve the equation by completing the square.
$t^2 + 6t = -8$

54. Solve the equation by completing the square.
$t^2 - 16t = -63$

Compound Inequalities Solve the compound inequality algebraically.

55. $-12x - 15 < -6 \quad \text{or} \quad -8x - 10 > 18$

56. $-10x + 7 < 15 \quad \text{and} \quad 15x + 3 \geq 2$

57. $17x - 17 < 4 \quad \text{and} \quad -6x + 7 < 18$

58. $3x + 1 < -8 \quad \text{and} \quad 15x + 11 \leq -6$

59. $2x + 10 \geq -19 \quad \text{and} \quad -11x + 9 < 20$

60. $20x - 5 \geq -20 \quad \text{or} \quad -9x - 7 \geq 17$

61. $x + 11 < -1 \quad \text{or} \quad -6x - 19 \leq 3$

62. $16x + 5 > 20 \quad \text{or} \quad -5x + 1 \geq 11$

63. $18 \leq x + 17 < 23$

64. $20 \leq x + 10 < 25$

65. $3 \leq \dfrac{5}{9}(F - 32) \leq 33$

F is in $\boxed{}$

66. $6 \leq \dfrac{5}{9}(F - 32) \leq 46$

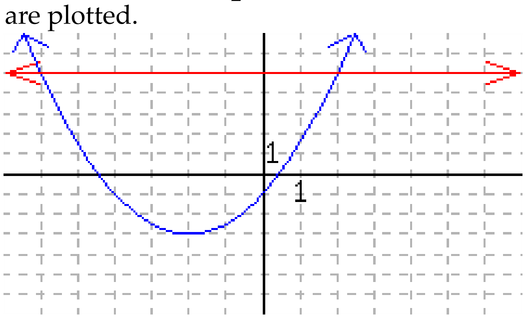
F is in $\boxed{}$

Solving Inequalities Graphically

67. Solve the equations and inequalities graphically. Use interval notation when applicable.
a. $\left|\frac{2}{3}x+2\right|=4$ b. $\left|\frac{2}{3}x+2\right|>4$ c. $\left|\frac{2}{3}x+2\right|\leq 4$

68. Solve the equations and inequalities graphically. Use interval notation when applicable.
a. $\left|\frac{11-2x}{5}\right|=4$ b. $\left|\frac{11-2x}{5}\right|>4$ c. $\left|\frac{11-2x}{5}\right|\leq 4$

69. The equations $y=\frac{1}{2}x^2+2x-1$ and $y=5$ are plotted.

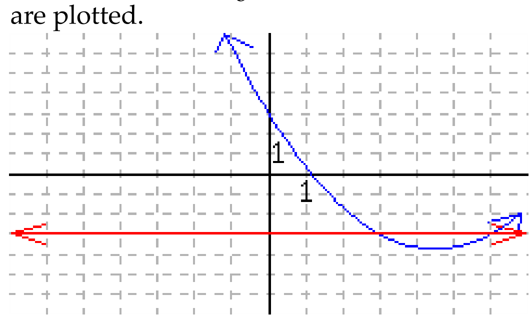


a. What are the points of intersection?

b. Solve $\frac{1}{2}x^2+2x-1=5$.

c. Solve $\frac{1}{2}x^2+2x-1>5$.

70. The equations $y=\frac{1}{3}x^2-3x+3$ and $y=-3$ are plotted.

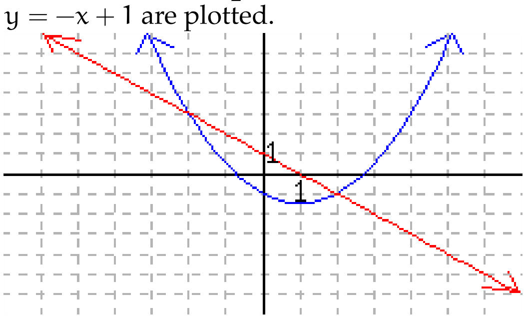


a. What are the points of intersection?

b. Solve $\frac{1}{3}x^2-3x+3=-3$.

c. Solve $\frac{1}{3}x^2-3x+3>-3$.

71. The equations $y=\frac{1}{2}x^2-x-1$ and $y=-x+1$ are plotted.

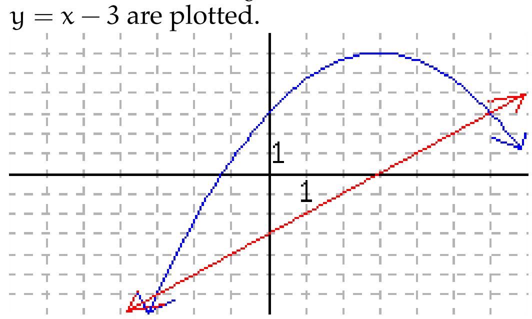


a. What are the points of intersection?

b. Solve $\frac{1}{2}x^2-x-1=-x+1$.

c. Solve $\frac{1}{2}x^2-x-1>-x+1$.

72. The equations $y=\frac{-1}{3}x^2+2x+3$ and $y=x-3$ are plotted.

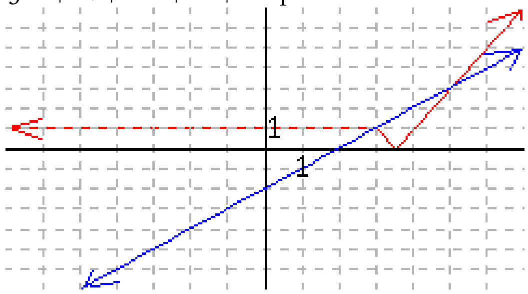


a. What are the points of intersection?

b. Solve $\frac{-1}{3}x^2+2x+3=x-3$.

c. Solve $\frac{-1}{3}x^2+2x+3>x-3$.

73. The equations $y = x - 2$ and $y = |x + |x - 3| - 4|$ are plotted.



a. What are the points of intersection?

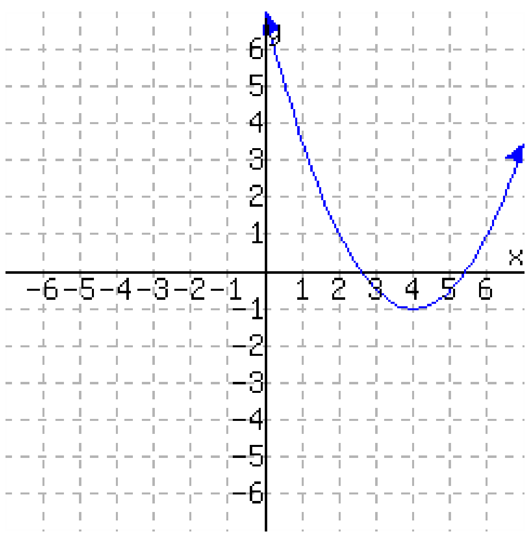
b. Solve $x - 2 = |x + |x - 3| - 4|$.

c. Solve $x - 2 > |x + |x - 3| - 4|$.

74. Use graphing technology to solve the inequality $\frac{3}{4}x \geq \frac{1}{4}x^2 - 3x$. State the solution set using interval notation, and approximate if necessary.

A graph of f is given. Use the graph alone to solve the compound inequalities.

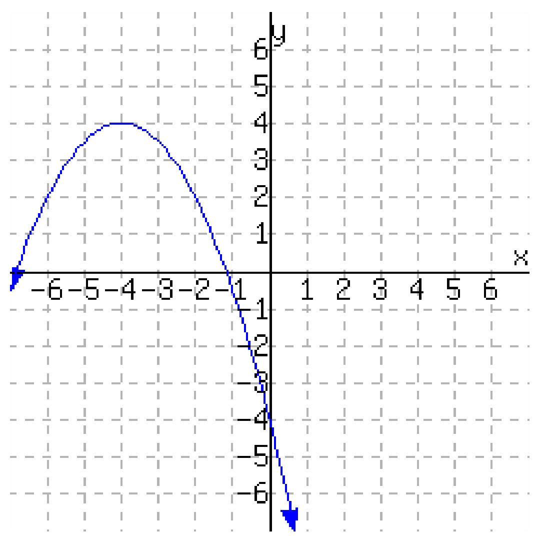
75.



a. $f(x) > 1$

b. $f(x) \leq 1$

76.



a. $f(x) > 2$

b. $f(x) \leq 2$

Appendix A

Basic Math Review

This appendix is *mostly* intended to *review* topics from a basic math course, especially Sections A.1–A.5. These topics are covered differently than they would be covered for a student seeing them for the first time.

A.1 Arithmetic with Negative Numbers

Adding, subtracting, multiplying, dividing, and raising to powers each have peculiarities when using negative numbers. This section reviews arithmetic with signed (both positive and negative) numbers.

A.1.1 Signed Numbers

Is it valid to subtract a large number from a smaller one? It may be hard to imagine what it would mean physically to subtract 8 cars from your garage if you only have 1 car in there in the first place. Nevertheless, mathematics has found a way to give meaning to expressions like $1 - 8$ using **signed numbers**.

In daily life, the signed numbers we might see most often are temperatures. Most people on Earth use the Celsius scale; if you're not familiar with the Celsius temperature scale, think about these examples:

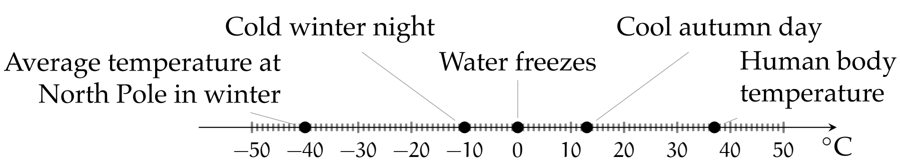


Figure A.1.2: Number line with interesting Celsius temperatures

Figure A.1.2 uses a **number line** to illustrate these positive and negative numbers. A number line is a useful device for visualizing how numbers relate to each other and combine with each other. Values to the right of 0 are called **positive** numbers and values to the left of 0 are called **negative numbers**.

Warning A.1.3 Subtraction Sign versus Negative Sign. Unfortunately, the symbol we use for subtraction looks just like the symbol we use for marking a negative number. It will help to identify when a "minus" sign means "subtract" or means "negative." The key is to see if there is a number to its left, not counting anything farther left than an open parenthesis. Here are some examples.

- -13 has one negative sign and no subtraction sign.
- $20 - 13$ has no negative signs and one subtraction sign.
- $-20 - 13$ has a negative sign and then a subtraction sign.
- $(-20)(-13)$ has two negative signs and no subtraction sign.

Checkpoint A.1.4 Identify "minus" signs.

In each expression, how many negative signs and subtraction signs are there?

a. $1 - 9$

has $\boxed{}$ negative signs and $\boxed{}$ subtraction signs.

b. $-12 + (-50)$

has $\boxed{}$ negative signs and $\boxed{}$ subtraction signs.

c. $\dfrac{-13 - (-15) - 17}{23 - 4}$

has $\boxed{}$ negative signs and $\boxed{}$ subtraction signs.

Explanation.

a. $1 - 9$ has zero negative signs and one subtraction sign.

b. $-12 + (-50)$ has two negative signs and zero subtraction signs.

c. $\dfrac{-13 - (-15) - 17}{23 - 4}$ has two negative signs and three subtraction signs.

A.1.2 Adding

An easy way to think about adding two numbers with the *same sign* is to simply (at first) ignore the signs, and add the numbers as if they were both positive. Then make sure your result is either positive or negative, depending on what the sign was of the two numbers you started with.

Example A.1.5 Add Two Negative Numbers. If you needed to add -18 and -7, note that both are negative. Maybe you have this expression in front of you:

$$-18 + -7$$

but that "plus minus" is awkward, and in this book you are more likely to have this expression:

$$-18 + (-7)$$

with extra parentheses. (How many subtraction signs do you see? How many negative signs?)

Since *both* our terms are *negative*, we can add 18 and 7 to get 25 and immediately realize that our final result should be negative. So our result is -25:

$$-18 + (-7) = -25$$

This approach works because adding numbers is like having two people tugging on a rope in one direction or the other, with strength indicated by each number. In Example A.1.5 we have two people pulling to the left, one with strength 18, the other with strength 7. Their forces combine to pull *left* with strength 25, giving us our total of -25, as illustrated in Figure A.1.6.

If we are adding two numbers that have *opposite* signs, then the two people tugging the rope are opposing each other. If either of them is using more strength, then the overall effect will be a net pull in that person's direction. And the overall pull on the rope will be the *difference* of the two strengths. This is illustrated in Figure A.1.7.

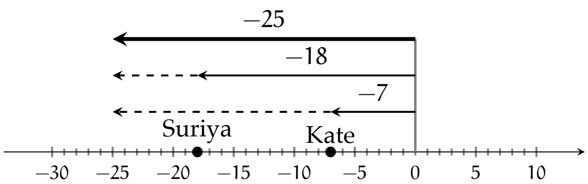


Figure A.1.6: Working together

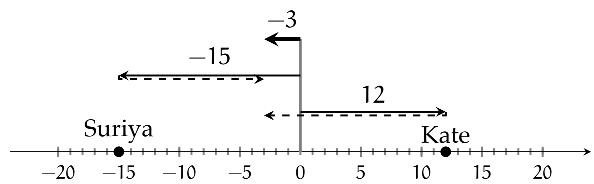


Figure A.1.7: Working in opposition

Example A.1.8 Adding One Number of Each Sign. Here are four examples of addition where one number is positive and the other is negative.

a. $-15 + 12$

We have one number of each sign, with sizes 15 and 12. Their difference is 3. But of the two numbers, the negative number dominated. So the result from adding these is -3.

b. $200 + (-100)$

We have one number of each sign, with sizes 200 and 100. Their difference is 100. But of the two numbers, the positive number dominated. So the result from adding these is 100.

c. $12.8 + (-20)$

We have one number of each sign, with sizes 12.8 and 20. Their difference is 7.2. But of the two numbers, the negative number dominated. So the result from adding these is -7.2.

d. $-87.3 + 87.3$

We have one number of each sign, both with size 87.3. The opposing forces cancel each other, leaving a result of 0.

Checkpoint A.1.9 Take a moment to practice adding when at least one negative number is involved. The expectation is that readers can make these calculations here without a calculator.

a. Add $-1 + 9$.

b. Add $-12 + (-98)$.

c. Add $100 + (-123)$.

d. Find the sum $-2.1 + (-2.1)$.

e. Find the sum $-34.67 + 81.53$.

Explanation.

a. The two numbers have opposite sign, so we can think to subtract $9 - 1 = 8$. Of the two numbers we added, the positive is larger, so we stick with postive 8 as the answer.

b. The two numbers are both negative, so we can add $12 + 98 = 110$, and take the negative of that as the answer: -110.

c. The two numbers have opposite sign, so we can think to subtract $123 - 100 = 23$. Of the two numbers we added, the negative is larger, so we take the negative of 23 as the answer. That is, the answer is -23.

d. The two numbers are both negative, so we can add $2.1 + 2.1 = 4.2$, and take the negative of that as the answer: -4.2.

e. The two numbers have opposite sign, so we can think to subtract $81.53 - 34.67 = 46.86$. Of the two numbers we added, the positive is larger, so we stick with postive 46.86 as the answer.

A.1.3 Subtracting

Perhaps you can handle a subtraction such as $18 - 5$, where a small positive number is subtracted from a larger number. There are other instances of subtraction that might leave you scratching your head. In such situations, we recommend that you view each subtraction as *adding* the opposite number.

	Original	Adding the Opposite
Subtracting a larger positive number:	$12 - 30$	$12 + (-30)$
Subtracting from a negative number:	$-8.1 - 17$	$-8.1 + (-17)$
Subtracting a negative number:	$42 - (-23)$	$42 + 23$

The benefit is that perhaps you already mastered addition with positive and negative numbers, and this strategy that you convert subtraction to addition means you don't have all that much more to learn. These examples might be computed as follows:

$$\begin{aligned} 12 - 30 &= 12 + (-30) \\ &= -18 \end{aligned} \qquad \begin{aligned} -8.1 - 17 &= -8.1 + (-17) \\ &= -25.1 \end{aligned} \qquad \begin{aligned} 42 - (-23) &= 42 + 23 \\ &= 65 \end{aligned}$$

Checkpoint A.1.10 Take a moment to practice subtracting when at least one negative number is involved. The expectation is that readers can make these calculations here without a calculator.

a. Subtract -1 from 9.

b. Subtract $32 - 50$.

c. Subtract $108 - (-108)$.

d. Find the difference $-5.9 - (-3.1)$.

e. Find the difference $-12.04 - 17.2$.

Explanation.

a. After writing this as $9 - (-1)$, we can rewrite it as $9 + 1$ and get 10.

b. Subtrcting in the oppsite order with the larger number first, $50 - 32 = 18$. But since we were asked to subtract the larger number *from* the smaller number, the answer is -18.

c. After writing this as $108 - (-108)$, we can rewrite it as $108 + 108$ and get 216.

d. After writing this as $-5.9-(-3.1)$, we can rewrite it as $-5.9+3.1$. Now it is the *sum* of two numbers of opposite sign, so we can subtract $5.9-3.1$ to get 2.8. But we were adding numbers where the negative number was larger, so the final answer should be -2.8.

e. Since we are subtracting a positive number from a negative number, the result should be an even more negative number. We can add $12.04+17.2$ to get 29.24, but our final answer should be the opposite, -29.24.

A.1.4 Multiplying

Making sense of multiplication of negative numbers isn't quite so straightforward, but it's possible. Should the product of 3 and -7 be a positive number or a negative number? Remembering that we can view multiplication as repeated addition, we can see this result on a number line:

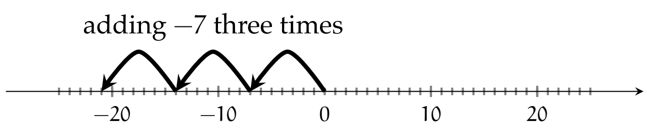


Figure A.1.11: Viewing $3\cdot(-7)$ as repeated addition

Figure A.1.11 illustrates that $3\cdot(-7)=-21$, and so it would seem that a positive number times a negative number will always give a negative result. (Note that it would not change things if the negative number came first in the product, since the order of multiplication doesn't affect the result.)

What about the product $-3\cdot(-7)$, where both factors are negative? Should the product be positive or negative? If $3\cdot(-7)$ can be seen as adding -7 three times as in Figure A.1.12, then it isn't too crazy to interpret $-3\cdot(-7)$ as *subtracting* -7 three times, as in Figure A.1.12.

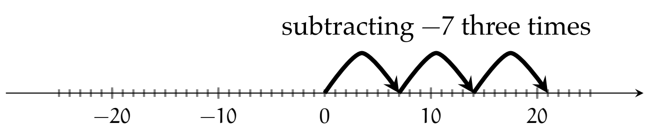


Figure A.1.12: Viewing $-3\cdot(-7)$ as repeated subtraction

This illustrates that $-3\cdot(-7)=21$, and it would seem that a negative number times a negative number always gives a positive result.

Positive and negative numbers are not the whole story. The number 0 is neither positive nor negative. What happens with multiplication by 0? You can choose to view $7\cdot 0$ as adding the number 0 seven times. And you can choose to view $0\cdot 7$ as adding the number 7 zero times. Either way, you really added nothing at all, which is the same as adding 0.

Fact A.1.13 Multiplication by 0. *Multiplying any number by 0 results in 0.*

Checkpoint A.1.14 Here are some practice exercises with multiplication and signed numbers. The expectation is that readers can make these calculations here without a calculator.

a. Multiply $-13\cdot 2$.

b. Find the product of 30 and -50.

c. Compute $-12(-7)$.

d. Find the product $-285(0)$.

Explanation.

a. Since $13 \cdot 2 = 26$, and we are multiplying numbers of opposite signs, the answer is negative: -26.

b. Since $30 \cdot 50 = 1500$, and we are multiplying numbers of opposite signs, the answer is negative: -1500.

c. Since $12 \cdot 7 = 84$, and we are multiplying numbers of the same sign, the answer is positive: 84.

d. Any number multiplied by 0 is 0.

A.1.5 Powers

For early sections of this book the only exponents you will see will be the **natural numbers** : $\{1, 2, 3, \ldots\}$. But negative numbers can and will arise as the *base* of a power.

An exponent is a shorthand for how many times to multiply by the base. For example,

$$(-2)^5 \text{ means } \overbrace{(-2) \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2)}^{\text{5 instances}}$$

Will the result here be positive or negative? Since we can view $(-2)^5$ as repeated multiplication, and we now understand that multiplying two negatives gives a positive result, this expression can be thought of this way:

$$\underbrace{\underbrace{(-2) \cdot (-2)}_{\text{positive}} \cdot \underbrace{(-2) \cdot (-2)}_{\text{positive}}}_{\text{positive}} \cdot (-2)$$

and that lone last negative number will be responsible for making the final product negative.

More generally, if the base of a power is negative, then whether or not the result is positive or negative depends on if the exponent is even or odd. It depends on whether or not the factors can all be paired up to "cancel" negative signs, or if there will be a lone factor left by itself.

Once you understand whether the result is positive or negative, for a moment you may forget about signs. Continuing the example, you may calculate that $2^5 = 32$, and then since we know $(-2)^5$ is negative, you can report

$$(-2)^5 = -32$$

Warning A.1.15 Negative Signs and Exponents. Expressions like -3^4 may not mean what you think they mean. What base do you see here? The correct answer is 3. The exponent 4 *only* applies to the 3, not to -3. So this expression, -3^4, is actually the same as $-\left(3^4\right)$, which is -81. Be careful not to treat -3^4 as having base -3. That would make it equivalent to $(-3)^4$, which is *positive* 81.

Checkpoint A.1.16 Here is some practice with natural exponents on negative bases. The expectation is that readers can make these calculations here without a calculator.

a. Compute $(-8)^2$.

b. Calculate the power $(-1)^{203}$.

c. Find $(-3)^3$.

d. Calculate -5^2.

Explanation.

a. Since 8^2 is 64 and we are raising a negative number to an *even* power, the answer is positive: 64.

b. Since 1^{203} is 1 and we are raising a negative number to an *odd* power, the answer is negative: -1.

c. Since 3^3 is 27 and we are raising a negative number to an *odd* power, the answer is negative: -27.

d. Careful: here we are raising *positive* 5 to the second power to get 25 and *then* negating the result: -25. Since we don't see "$(-5)^2$," the answer is not positive 25.

A.1.6 Summary

Addition Add two negative numbers: add their positive counterparts and make the result negative.

Add a positive with a negative: find their difference using subtraction, and keep the sign of the dominant number.

Subtraction Any subtraction can be converted to addition of the opposite number. For all but the most basic subtractions, this is a useful strategy.

Multiplication Multiply two negative numbers: multiply their positive counterparts and make the result positive.

Multiply a positive with a negative: multiply their positive counterparts and make the result negative.

Multiply any number by 0: the result will be 0.

Division (not discussed in this section) Division by some number is the same as multiplication by its reciprocal. So the multiplication rules can be adopted.

Division of 0 by any nonzero number always results in 0.

Division of any number by 0 is always undefined.

Powers Raise a negative number to an even power: raise the positive counterpart to that power.

Raise a negative number to an odd power: raise the positive counterpart to that power, then make the result negative.

Expressions like -2^4 mean $-\left(2^4\right)$, not $(-2)^4$.

A.1.7 Exercises

1. Add the following.

a. $-9+(-3)$

b. $-6+(-5)$

c. $-1+(-7)$

2. Add the following.

a. $-8+(-1)$

b. $-5+(-6)$

c. $-1+(-9)$

3. Add the following.

a. $5+(-9)$

b. $5+(-1)$

c. $8+(-8)$

4. Add the following.

a. $5+(-6)$

b. $8+(-2)$

c. $8+(-8)$

5. Add the following.

a. $-7+1$

b. $-3+10$

c. $-3+3$

6. Add the following.

a. $-9+2$

b. $-4+7$

c. $-3+3$

7. Add the following.

a. $-71+(-16)$

b. $-32+87$

c. $78+(-23)$

8. Add the following.

a. $-61+(-47)$

b. $-84+62$

c. $77+(-69)$

9. Subtract the following.

a. $3-7$

b. $8-2$

c. $5-19$

10. Subtract the following.

a. $4-10$

b. $5-1$

c. $5-14$

11. Subtract the following.

a. $-2-3$

b. $-8-4$

c. $-5-5$

12. Subtract the following.

a. $-1-2$

b. $-6-3$

c. $-5-5$

13. Subtract the following.

a. $-5-(-8)$

b. $-6-(-1)$

c. $-5-(-5)$

14. Subtract the following.

a. $-1-(-7)$

b. $-9-(-2)$

c. $-5-(-5)$

15. Perform the given addition and subtraction.

a. $-18-2+(-9)$

b. $7-(-13)+(-11)$

16. Perform the given addition and subtraction.

a. $-17-9+(-5)$

b. $4-(-14)+(-17)$

17. Perform the given addition and subtraction.

a. $-16-5+(-1)$

b. $1-(-14)+(-11)$

18. Perform the given addition and subtraction.

a. $-14-2+(-7)$

b. $9-(-14)+(-17)$

19. Multiply the following.

a. $(-9)\cdot(-1)$

b. $(-4)\cdot 5$

c. $7\cdot(-2)$

d. $(-6)\cdot 0$

20. Multiply the following.

a. $(-8)\cdot(-2)$

b. $(-7)\cdot 3$

c. $7\cdot(-5)$

d. $(-5)\cdot 0$

21. Multiply the following.

a. $(-1)\cdot(-6)\cdot(-3)$

b. $7\cdot(-7)\cdot(-1)$

c. $(-86)\cdot(-78)\cdot 0$

22. Multiply the following.

a. $(-1)\cdot(-4)\cdot(-5)$

b. $6\cdot(-7)\cdot(-4)$

c. $(-84)\cdot(-66)\cdot 0$

23. Multiply the following.

a. $(-1)(-2)(-2)(-2)$

b. $(3)(-3)(2)(-3)$

24. Multiply the following.

a. $(-1)(-1)(-3)(-1)$

b. $(1)(-3)(-3)(-1)$

25. Evaluate the following.

a. $\frac{-64}{-8}$

b. $\frac{54}{-6}$

c. $\frac{-63}{7}$

26. Evaluate the following.

a. $\frac{-28}{-7}$

b. $\frac{35}{-5}$

c. $\frac{-28}{7}$

27. Evaluate the following.

a. $\frac{-6}{-1}$

b. $\frac{10}{-1}$

c. $\frac{110}{-110}$

d. $\frac{-17}{-17}$

e. $\frac{11}{0}$

f. $\frac{0}{-3}$

28. Evaluate the following.

a. $\frac{-5}{-1}$

b. $\frac{8}{-1}$

c. $\frac{150}{-150}$

d. $\frac{-20}{-20}$

e. $\frac{10}{0}$

f. $\frac{0}{-8}$

29. Evaluate the following.

a. $(-9)^2$

b. -4^2

30. Evaluate the following.

a. $(-7)^2$

b. -6^2

31. Evaluate the following.

a. $(-3)^3$

b. -1^3

32. Evaluate the following.

a. $(-2)^3$

b. -4^3

33. Evaluate the following.

a. 4^2

b. 2^3

c. $(-4)^2$

d. $(-3)^3$

34. Evaluate the following.

a. 5^2

b. 4^3

c. $(-3)^2$

d. $(-5)^3$

35. Evaluate the following.

a. 1^{10}

b. $(-1)^{11}$

c. $(-1)^{12}$

d. 0^{20}

36. Evaluate the following.

a. 1^5

b. $(-1)^{13}$

c. $(-1)^{18}$

d. 0^{18}

Simplify without using a calculator.

37. $-6.53 + (-53.7)$

38. $-7.17 + (-33.3)$

39. $9.8 - 1.93$

40. $9.5 - 5.73$

41. $-4.43 + 5.2$

42. $-2.13 + 6.7$

43. $-4.83 - (-6.4)$

44. $-2.63 - (-7.1)$

45. $57 - 6.33$

46. $63 - 1.03$

47. $-70 + 4.73$

48. $-76 + 8.42$

49. It's given that $94 \cdot 32 = 3008$. Use this fact to calculate the following without using a calculator.
$9.4(-0.032) =$

50. It's given that $19 \cdot 79 = 1501$. Use this fact to calculate the following without using a calculator.
$1.9(-7.9) =$

51. It's given that $26 \cdot 16 = 416$. Use this fact to calculate the following without using a calculator.
$(-2.6)(-0.016)$

52. It's given that $33 \cdot 54 = 1782$. Use this fact to calculate the following without using a calculator.
$(-3.3)(-5.4)$

Applications

53. Consider the following situation in which you borrow money from your cousin:

- On June 1st, you borrowed 1200 dollars from your cousin.
- On July 1st, you borrowed 490 more dollars from your cousin.
- On August 1st, you paid back 690 dollars to your cousin.
- On September 1st, you borrowed another 820 dollars from your cousin.

How much money do you owe your cousin now?

54. Consider the following scenario in which you study your bank account.

- On Jan. 1, you had a balance of -310 dollars in your bank account.
- On Jan. 2, your bank charged 40 dollar overdraft fee.
- On Jan. 3, you deposited 870 dollars.
- On Jan. 10, you withdrew 750 dollars.

What is your balance on Jan. 11?

55. A mountain is 1100 feet *above* sea level. A trench is 360 feet *below* sea level. What is the difference in elevation between the mountain top and the bottom of the trench?

56. A mountain is 1200 feet *above* sea level. A trench is 420 feet *below* sea level. What is the difference in elevation between the mountain top and the bottom of the trench?

Challenge

57. Select the correct word to make each statement true.

a. A positive number minus a positive number is (☐ sometimes ☐ always ☐ never) negative.

b. A negative number plus a negative number is (☐ sometimes ☐ always ☐ never) negative.

c. A positive number minus a negative number is (☐ sometimes ☐ always ☐ never) positive.

d. A negative number multiplied by a negative number is (☐ sometimes ☐ always ☐ never) negative.

A.2 Fractions and Fraction Arithmetic

The word "fraction" comes from the Latin word *fractio*, which means "break into pieces." For thousands of years, cultures from all over the world have used fractions to understand parts of a whole.

A.2.1 Visualizing Fractions

Parts of a Whole. One approach to understanding fractions is to think of them as parts of a whole.

In Figure A.2.2, we see 1 whole divided into 7 parts. Since 3 parts are shaded, we have an illustration of the fraction $\frac{3}{7}$. The **denominator** 7 tells us how many parts to cut up the whole; since we have 7 parts, they're called "sevenths." The **numerator** 3 tells us how many sevenths to consider.

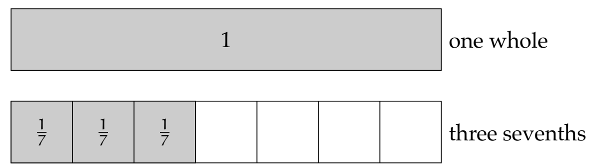


Figure A.2.2: Representing $\frac{3}{7}$ as parts of a whole.

Checkpoint A.2.3 A Fraction as Parts of a Whole. To visualize the fraction $\frac{14}{35}$, you might cut a rectangle into ___ equal parts, and then count up ___ of them.

Explanation. You could cut a rectangle into 35 equal pieces, and then 14 of them would represent $\frac{14}{35}$.

We can also locate fractions on number lines. When ticks are equally spread apart, as in Figure A.2.4, each tick represents a fraction.

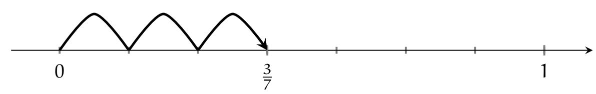
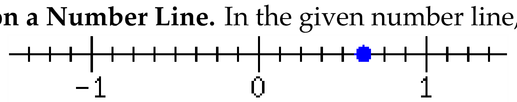


Figure A.2.4: Representing $\frac{3}{7}$ on a number line.

Checkpoint A.2.5 A Fraction on a Number Line. In the given number line, what fraction is marked?



Explanation. There are 8 subdivisions between 0 and 1, and the mark is at the fifth subdivision. So the mark is $\frac{5}{8}$ of the way from 0 to 1 and therefore represents the fraction $\frac{5}{8}$.

Division. Fractions can also be understood through division.

For example, we can view the fraction $\frac{3}{7}$ as 3 divided into 7 equal parts, as in Figure A.2.6. Just one of those parts represents $\frac{3}{7}$.

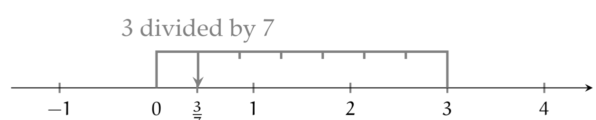


Figure A.2.6: Representing $\frac{3}{7}$ on a number line.

Checkpoint A.2.7 Seeing a Fraction as Division Arithmetic. The fraction $\frac{21}{40}$ can be thought of as dividing the whole number ___ into ___ equal-sized parts.

Explanation. Since $\frac{21}{40}$ means the same as $21 \div 40$, it can be thought of as dividing 21 into 40 equal parts.

A.2.2 Equivalent Fractions

It's common to have two fractions that represent the same amount. Consider $\frac{2}{5}$ and $\frac{6}{15}$ represented in various ways in Figures A.2.8–A.2.10.

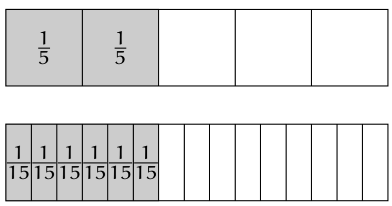


Figure A.2.8: $\frac{2}{5}$ and $\frac{6}{15}$ as equal parts of a whole

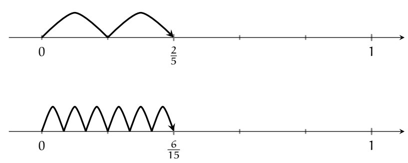


Figure A.2.9: $\frac{2}{5}$ and $\frac{6}{15}$ as equal on a number line

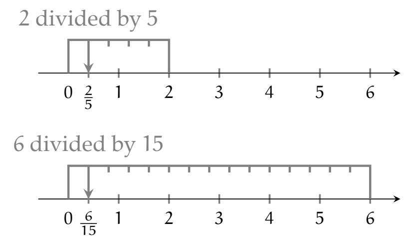


Figure A.2.10: $\frac{2}{5}$ and $\frac{6}{15}$ as equal results from division

Those two fractions, $\frac{2}{5}$ and $\frac{6}{15}$ are equal, as those figures demonstrate. In addition, both fractions are equal to 0.4 as a decimal. If we must work with this number, the fraction that uses smaller numbers, $\frac{2}{5}$, is preferable. Working with smaller numbers decreases the likelihood of making a human arithmetic error and it also increases the chances that you might make useful observations about the nature of that number.

So if you are handed a fraction like $\frac{6}{15}$, it is important to try to **reduce** it to "lowest terms." The most important skill you can have to help you do this is to know the multiplication table well. If you know it well, you know that $6 = 2 \cdot 3$ and $15 = 3 \cdot 5$, so you can break down the numerator and denominator that way. Both the numerator and denominator are divisible by 3, so they can be "factored out" and then as factors, cancel out.

$$\begin{aligned}\frac{6}{15} &= \frac{2 \cdot 3}{3 \cdot 5}\\ &= \frac{2 \cdot \cancel{3}}{\cancel{3} \cdot 5}\\ &= \frac{2 \cdot 1}{1 \cdot 5}\\ &= \frac{2}{5}\end{aligned}$$

Checkpoint A.2.11 Reduce these fractions into lowest terms.

a. $\dfrac{14}{42}$ b. $\dfrac{8}{30}$ c. $\dfrac{70}{90}$

Explanation.

a. With $\frac{14}{42}$, we have $\frac{2 \cdot 7}{2 \cdot 3 \cdot 7}$, which reduces to $\frac{1}{3}$.

b. With $\frac{8}{30}$, we have $\frac{2 \cdot 2 \cdot 2}{2 \cdot 3 \cdot 5}$, which reduces to $\frac{4}{15}$.

c. With $\frac{70}{90}$, we have $\frac{7 \cdot 10}{9 \cdot 10}$, which reduces to $\frac{7}{9}$.

Sometimes it is useful to do the opposite of reducing a fraction, and **build up** the fraction to use larger numbers.

Checkpoint A.2.12 Sayid scored $\frac{21}{25}$ on a recent exam. Build up this fraction so that the denominator is 100, so that Sayid can understand what percent score he earned.

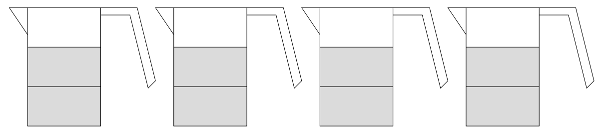
Explanation. To change the denominator from 25 to 100, it needs to be multiplied by 4. So we calculate

$$\begin{aligned}\frac{21}{25} &= \frac{21 \cdot 4}{25 \cdot 4}\\ &= \frac{84}{100}\end{aligned}$$

So the fraction $\frac{21}{25}$ is equivalent to $\frac{84}{100}$. (This means Sayid scored an 84%.)

A.2.3 Multiplying with Fractions

Example A.2.13 Suppose a recipe calls for $\frac{2}{3}$ cup of milk, but we'd like to quadruple the recipe (make it four times as big). We'll need four times as much milk, and one way to measure this out is to fill a measuring cup to $\frac{2}{3}$ full, four times:



When you count up the shaded thirds, there are eight of them. So multiplying $\frac{2}{3}$ by the whole number 4, the result is $\frac{8}{3}$. Mathematically:

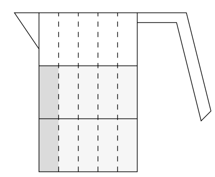
$$\begin{aligned} 4 \cdot \frac{2}{3} &= \frac{4 \cdot 2}{3} \\ &= \frac{8}{3} \end{aligned}$$

Fact A.2.14 Multiplying a Fraction and a Whole Number. *When you multiply a whole number by a fraction, you may just multiply the whole number by the numerator and leave the denominator alone. In other words, as long as* d *is not* 0*, then a whole number and a fraction multiply this way:*

$$a \cdot \frac{c}{d} = \frac{a \cdot c}{d}$$

Example A.2.15 We could also use multiplication to decrease amounts. Suppose we needed to cut the recipe down to just one fifth. Instead of *four* of the $\frac{2}{3}$ cup milk, we need *one fifth* of the $\frac{2}{3}$ cup milk. So instead of multiplying by 4, we multiply by $\frac{1}{5}$. But how much is $\frac{1}{5}$ of $\frac{2}{3}$ cup?

If we cut the measuring cup into five equal vertical strips along with the three equal horizontal strips, then in total there are $3 \cdot 5 = 15$ subdivisions of the cup. Two of those sections represent $\frac{1}{5}$ of the $\frac{2}{3}$ cup.



In the end, we have $\frac{2}{15}$ of a cup. The denominator 15 came from multiplying 5 and 3, the denominators of the fractions we had to multiply. The numerator 2 came from multiplying 1 and 2, the numerators of the fractions we had to multiply.

$$\begin{aligned} \frac{1}{5} \cdot \frac{2}{3} &= \frac{1 \cdot 2}{5 \cdot 3} \\ &= \frac{2}{15} \end{aligned}$$

Fact A.2.16 Multiplication with Fractions. *As long as* b *and* d *are not* 0*, then fractions multiply this way:*

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Checkpoint A.2.17 Simplify these fraction products.

a. $\frac{1}{3} \cdot \frac{10}{7}$

b. $\frac{12}{3} \cdot \frac{15}{3}$

c. $-\frac{14}{5} \cdot \frac{2}{3}$

d. $\frac{70}{27} \cdot \frac{12}{-20}$

Explanation.

a. Multiplying numerators gives 10, and multiplying denominators gives 21. The answer is $\frac{10}{21}$.

b. Before we multiply fractions, note that $\frac{12}{3}$ reduces to 4, and $\frac{15}{3}$ reduces to 5. So we just have $4 \cdot 5 = 20$.

c. Multiplying numerators gives 28, and multiplying denominators gives 15. The result should be negative, so the answer is $-\frac{28}{15}$.

d. Before we multiply fractions, note that $\frac{12}{-20}$ reduces to $\frac{-3}{5}$. So we have $\frac{70}{27} \cdot \frac{-3}{5}$. Both the numerator of the first fraction and denominator of the second fraction are divisible by 5, so it helps to reduce both fractions accordingly and get $\frac{14}{27} \cdot \frac{-3}{1}$. Both the denominator of the first fraction and numerator of the second fraction are divisible by 3, so it helps to reduce both fractions accordingly and get $\frac{14}{9} \cdot \frac{-1}{1}$. Now we are just multiplying $\frac{14}{9}$ by -1, so the result is $\frac{-14}{9}$.

A.2.4 Division with Fractions

How does division with fractions work? Are we able to compute/simplify each of these examples?

a. $3 \div \frac{2}{7}$

b. $\frac{18}{19} \div 5$

c. $\frac{14}{3} \div \frac{8}{9}$

d. $\frac{\frac{2}{5}}{\frac{5}{2}}$

We know that when we divide something by 2, this is the same as multiplying it by $\frac{1}{2}$. Conversely, dividing a number or expression by $\frac{1}{2}$ is the same as multiplying by $\frac{2}{1}$, or just 2. The more general property is that when we divide a number or expression by $\frac{a}{b}$, this is equivalent to multiplying by the reciprocal $\frac{b}{a}$.

Fact A.2.18 Division with Fractions. *As long as* b, c *and* d *are not* 0, *then division with fractions works this way:*

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

Example A.2.19 With our examples from the beginning of this subsection:

a.
$$\begin{aligned} 3 \div \frac{2}{7} &= 3 \cdot \frac{7}{2} \\ &= \frac{3}{1} \cdot \frac{7}{2} \\ &= \frac{21}{2} \end{aligned}$$

c.
$$\begin{aligned} \frac{14}{3} \div \frac{8}{9} &= \frac{14}{3} \cdot \frac{9}{8} \\ &= \frac{14}{1} \cdot \frac{3}{8} \\ &= \frac{7}{1} \cdot \frac{3}{4} \\ &= \frac{21}{4} \end{aligned}$$

b.
$$\begin{aligned} \frac{18}{19} \div 5 &= \frac{18}{19} \div \frac{5}{1} \\ &= \frac{18}{19} \cdot \frac{1}{5} \\ &= \frac{18}{95} \end{aligned}$$

d.
$$\begin{aligned} \frac{\frac{2}{5}}{\frac{5}{2}} &= \frac{2}{5} \div \frac{5}{2} \\ &= \frac{2}{5} \cdot \frac{2}{5} \\ &= \frac{4}{25} \end{aligned}$$

Checkpoint A.2.20 Simplify these fraction division expressions.

a. $\frac{1}{3} \div \frac{10}{7}$ b. $\frac{12}{5} \div 5$ c. $-14 \div \frac{3}{2}$ d. $\frac{70}{9} \div \frac{11}{-20}$

Explanation.

a.
$$\begin{aligned}\frac{1}{3} \div \frac{10}{7} &= \frac{1}{3} \cdot \frac{7}{10}\\ &= \frac{7}{30}\end{aligned}$$

c.
$$\begin{aligned}-14 \div \frac{3}{2} &= -14 \cdot \frac{2}{3}\\ &= -\frac{14}{1} \cdot \frac{2}{3}\\ &= -\frac{28}{3}\end{aligned}$$

b.
$$\begin{aligned}\frac{12}{5} \div 5 &= \frac{12}{5} \cdot \frac{1}{5}\\ &= \frac{12}{25}\end{aligned}$$

d.
$$\begin{aligned}\frac{70}{9} \div \frac{11}{-20} &= -\frac{70}{9} \cdot \frac{20}{11}\\ &= -\frac{1400}{99}\end{aligned}$$

A.2.5 Adding and Subtracting Fractions

With whole numbers and integers, operations of addition and subtraction are relatively straightforward. The situation is almost as straightforward with fractions *if the two fractions have the same denominator*. Consider

$$\frac{7}{2} + \frac{3}{2} = 7 \text{ halves} + 3 \text{ halves}$$

In the same way that 7 tacos and 3 tacos make 10 tacos, we have:

$$\begin{array}{ccccc} 7 \text{ halves} & + & 3 \text{ halves} & = & 10 \text{ halves}\\ \frac{7}{2} & + & \frac{3}{2} & = & \frac{10}{2}\\ & & & = & 5\end{array}$$

Fact A.2.21 Adding/Subtracting with Fractions Having the Same Denominator. *To add or subtract two fractions having the same denominator,* keep *that denominator, and add or subtract the numerators.*

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \qquad \frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$$

If it's possible, useful, or required of you, simplify the result by reducing to lowest terms.

Checkpoint A.2.22 Add or subtract these fractions.

a. $\frac{1}{3} + \frac{10}{3}$ b. $\frac{13}{6} - \frac{5}{6}$

Explanation.

a. Since the denominators are both 3, we can add the numerators: $1 + 10 = 11$. The answer is $\frac{11}{3}$.

b. Since the denominators are both 6, we can subtract the numerators: $13 - 5 = 8$. The answer is $\frac{8}{6}$, but that reduces to $\frac{4}{3}$.

Whenever we'd like to combine fractional amounts that don't represent the same number of parts of a whole (that is, when the denominators are different), finding sums and differences is more complicated.

Example A.2.23 Quarters and Dimes. Find the sum $\frac{3}{4} + \frac{2}{10}$. Does this seem intimidating? Consider this:

- $\frac{1}{4}$ of a dollar is a quarter, and so $\frac{3}{4}$ of a dollar is 75 cents.
- $\frac{1}{10}$ of a dollar is a dime, and so $\frac{2}{10}$ of a dollar is 20 cents.

So if you know what to look for, the expression $\frac{3}{4} + \frac{2}{10}$ is like adding 75 cents and 20 cents, which gives you 95 cents. As a fraction of one dollar, that is $\frac{95}{100}$. So we can report

$$\frac{3}{4} + \frac{2}{10} = \frac{95}{100}.$$

(Although we should probably reduce that last fraction to $\frac{19}{20}$.)

This example was not something you can apply to other fraction addition situations, because the denominators here worked especially well with money amounts. But there is something we can learn here. The fraction $\frac{3}{4}$ was equivalent to $\frac{75}{100}$, and the other fraction $\frac{2}{10}$ was equivalent to $\frac{20}{100}$. These *equivalent* fractions have the same denominator and are therefore "easy" to add. What we saw happen was:

$$\begin{aligned}\frac{3}{4} + \frac{2}{10} &= \frac{75}{100} + \frac{20}{100}\\ &= \frac{95}{100}\end{aligned}$$

This realization gives us a strategy for adding (or subtracting) fractions.

Fact A.2.24 Adding/Subtracting Fractions with Different Denominators. *To add (or subtract) generic fractions together, use their denominators to find a* ***common denominator****. This means some whole number that is a whole multiple of both of the original denominators. Then rewrite the two fractions as* equivalent *fractions that use this common denominator. Write the result* keeping *that denominator and adding (or subtracting) the numerators. Reduce the fraction if that is useful or required.*

Example A.2.25 Let's add $\frac{2}{3} + \frac{2}{5}$. The denominators are 3 and 5, so the number 15 would be a good common denominator.

$$\begin{aligned}\frac{2}{3} + \frac{2}{5} &= \frac{2 \cdot 5}{3 \cdot 5} + \frac{2 \cdot 3}{5 \cdot 3}\\ &= \frac{10}{15} + \frac{6}{15}\\ &= \frac{16}{15}\end{aligned}$$

Checkpoint A.2.26 A chef had $\frac{2}{3}$ cups of flour and needed to use $\frac{1}{8}$ cup to thicken a sauce. How much flour is left? [ ]

Explanation. We need to compute $\frac{2}{3} - \frac{1}{8}$. The denominators are 3 and 8. One common denominator is 24, so we move to rewrite each fraction using 24 as the denominator:

$$\begin{aligned}\frac{2}{3} - \frac{1}{8} &= \frac{2 \cdot 8}{3 \cdot 8} - \frac{1 \cdot 3}{8 \cdot 3}\\ &= \frac{16}{24} - \frac{3}{24}\\ &= \frac{13}{24}\end{aligned}$$

The numerical result is $\frac{13}{24}$, but a pure number does not answer this question. The amount of flour remaining is $\frac{13}{24}$ *cups*.

A.2.6 Mixed Numbers and Improper Fractions

A simple recipe for bread contains only a few ingredients:

1 1/2	tablespoons yeast
1 1/2	tablespoons kosher salt
6 1/2	cups unbleached, all-purpose flour (more for dusting)

Each ingredient is listed as a **mixed number** that quickly communicates how many whole amounts and how many parts are needed. It's useful for quickly communicating a practical amount of something you are cooking with, measuring on a ruler, purchasing at the grocery store, etc. But it causes trouble in an algebra class. The number 1 1/2 means "one ***and*** one half." So really,

$$1\,\frac{1}{2} = 1 + \frac{1}{2}$$

The trouble is that with 1 1/2, you have two numbers written right next to each other. Normally with two math expressions written right next to each other, they should be *multiplied*, not *added*. But with a mixed number, they *should* be added.

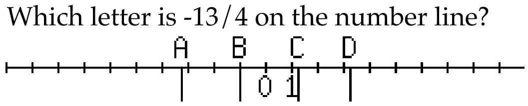
Fortunately we just reviewed how to add fractions. If we need to do any arithmetic with a mixed number like 1 1/2, we can treat it as $1 + \frac{1}{2}$ and simplify to get a "nice" fraction instead: $\frac{3}{2}$. A fraction like $\frac{3}{2}$ is called an **improper fraction** because it's actually larger than 1. And a "proper" fraction would be something small that is only *part* of a whole instead of *more* than a whole.

$$\begin{aligned}1\,\frac{1}{2} &= 1 + \frac{1}{2}\\ &= \frac{1}{1} + \frac{1}{2}\\ &= \frac{2}{2} + \frac{1}{2}\\ &= \frac{3}{2}\end{aligned}$$

A.2.7 Exercises

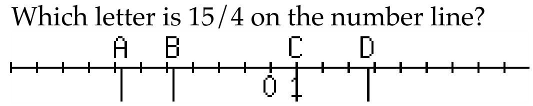
Review and Warmup

1. Which letter is -13/4 on the number line?



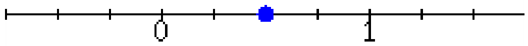
(☐ A ☐ B ☐ C ☐ D)

2. Which letter is 15/4 on the number line?

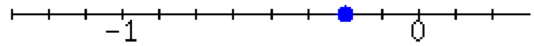


(☐ A ☐ B ☐ C ☐ D)

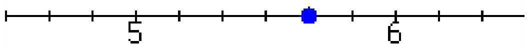
3. The dot in the graph can be represented by what fraction?



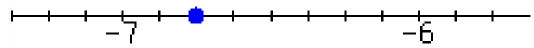
4. The dot in the graph can be represented by what fraction?



5. The dot in the graph can be represented by what fraction?



6. The dot in the graph can be represented by what fraction?



Reducing Fractions

7. Reduce the fraction $\frac{7}{56}$.

8. Reduce the fraction $\frac{20}{45}$.

9. Reduce the fraction $\frac{70}{84}$.

10. Reduce the fraction $\frac{14}{15}$.

11. Reduce the fraction $\frac{168}{105}$.

12. Reduce the fraction $\frac{30}{6}$.

Building Fractions

13. Find an equivalent fraction to $\frac{3}{4}$ with denominator 20.

14. Find an equivalent fraction to $\frac{3}{5}$ with denominator 10.

15. Find an equivalent fraction to $\frac{3}{11}$ with denominator 44.

16. Find an equivalent fraction to $\frac{11}{13}$ with denominator 26.

Multiplying/Dividing Fractions

17. Multiply: $\frac{4}{9} \cdot \frac{4}{9}$

18. Multiply: $\frac{5}{6} \cdot \frac{5}{8}$

19. Multiply: $\frac{15}{7} \cdot \frac{2}{5}$

20. Multiply: $\frac{3}{7} \cdot \frac{8}{15}$

21. Multiply: $8 \cdot \frac{1}{3}$

22. Multiply: $5 \cdot \frac{2}{9}$

23. Multiply: $-\frac{20}{11} \cdot \frac{7}{5}$

24. Multiply: $-\frac{8}{7} \cdot \frac{11}{26}$

25. Multiply: $28 \cdot \left(-\frac{6}{7}\right)$

26. Multiply: $10 \cdot \left(-\frac{7}{5}\right)$

27. Multiply: $\frac{6}{49} \cdot \frac{5}{4} \cdot \frac{7}{25}$

28. Multiply: $\frac{5}{49} \cdot \frac{2}{25} \cdot \frac{21}{4}$

29. Multiply: $\frac{35}{2} \cdot \frac{1}{25} \cdot 6$

30. Multiply: $\frac{21}{2} \cdot \frac{1}{49} \cdot 10$

31. Divide: $\frac{2}{5} \div \frac{7}{4}$

32. Divide: $\frac{3}{8} \div \frac{5}{3}$

33. Divide: $\frac{1}{12} \div \left(-\frac{10}{9}\right)$

34. Divide: $\frac{7}{20} \div \left(-\frac{9}{25}\right)$

35. Divide: $-\frac{16}{9} \div (-20)$

36. Divide: $-\frac{25}{6} \div (-20)$

37. Divide: $28 \div \frac{7}{5}$

38. Divide: $6 \div \frac{3}{2}$

39. Multiply: $2\frac{1}{10} \cdot 1\frac{4}{21}$

40. Multiply: $2\frac{2}{9} \cdot 1\frac{7}{8}$

Adding/Subtracting Fractions

41. Add: $\frac{3}{32} + \frac{1}{32}$

42. Add: $\frac{17}{32} + \frac{3}{32}$

43. Add: $\frac{4}{7} + \frac{15}{28}$

44. Add: $\frac{4}{9} + \frac{11}{27}$

45. Add: $\frac{4}{9} + \frac{13}{18}$

46. Add: $\frac{1}{10} + \frac{21}{40}$

47. Add: $\dfrac{2}{7}+\dfrac{3}{10}$

48. Add: $\dfrac{1}{6}+\dfrac{1}{7}$

49. Add: $\dfrac{1}{6}+\dfrac{1}{10}$

50. Add: $\dfrac{3}{10}+\dfrac{1}{6}$

51. Add: $\dfrac{5}{6}+\dfrac{9}{10}$

52. Add: $\dfrac{4}{5}+\dfrac{7}{10}$

53. Add: $-\dfrac{1}{11}+\dfrac{6}{11}$

54. Add: $-\dfrac{4}{13}+\dfrac{12}{13}$

55. Add: $-\dfrac{5}{7}+\dfrac{11}{21}$

56. Add: $-\dfrac{4}{7}+\dfrac{3}{14}$

57. Add: $-\dfrac{1}{6}+\dfrac{2}{9}$

58. Add: $-\dfrac{7}{8}+\dfrac{6}{7}$

59. Add: $-1+\dfrac{9}{10}$

60. Add: $1+\dfrac{4}{5}$

61. Add: $\dfrac{1}{10}+\dfrac{1}{6}+\dfrac{1}{5}$

62. Add: $\dfrac{3}{10}+\dfrac{2}{9}+\dfrac{1}{6}$

63. Add: $\dfrac{1}{5}+\dfrac{3}{10}+\dfrac{5}{6}$

64. Add: $\dfrac{7}{9}+\dfrac{3}{10}+\dfrac{1}{6}$

65. Subtract: $\dfrac{17}{12}-\dfrac{7}{12}$

66. Subtract: $\dfrac{29}{12}-\dfrac{19}{12}$

67. Subtract: $\dfrac{3}{7}-\dfrac{11}{14}$

68. Subtract: $\dfrac{5}{9}-\dfrac{5}{54}$

69. Subtract: $\dfrac{31}{35}-\dfrac{2}{7}$

70. Subtract: $\dfrac{44}{45}-\dfrac{4}{9}$

71. Subtract: $-\dfrac{1}{6}-\dfrac{9}{10}$

72. Subtract: $-\dfrac{3}{10}-\dfrac{5}{6}$

73. Subtract: $-\dfrac{3}{10}-\left(-\dfrac{5}{6}\right)$

74. Subtract: $-\dfrac{5}{6}-\left(-\dfrac{3}{10}\right)$

75. Subtract: $-4-\dfrac{18}{5}$

76. Subtract: $-2-\dfrac{3}{2}$

Applications

77. Kandace walked $\frac{2}{7}$ of a mile in the morning, and then walked $\frac{1}{8}$ of a mile in the afternoon. How far did Kandace walk altogether?

Kandace walked a total of $\boxed{}$ of a mile.

78. Tracei walked $\frac{1}{8}$ of a mile in the morning, and then walked $\frac{4}{11}$ of a mile in the afternoon. How far did Tracei walk altogether?

Tracei walked a total of $\boxed{}$ of a mile.

79. Jessica and Carl are sharing a pizza. Jessica ate $\frac{3}{8}$ of the pizza, and Carl ate $\frac{1}{6}$ of the pizza. How much of the pizza was eaten in total?

They ate $\boxed{}$ of the pizza.

80. A trail's total length is $\frac{32}{63}$ of a mile. It has two legs. The first leg is $\frac{2}{7}$ of a mile long. How long is the second leg?

The second leg is $\boxed{}$ of a mile in length.

81. A trail's total length is $\frac{14}{45}$ of a mile. It has two legs. The first leg is $\frac{1}{5}$ of a mile long. How long is the second leg?

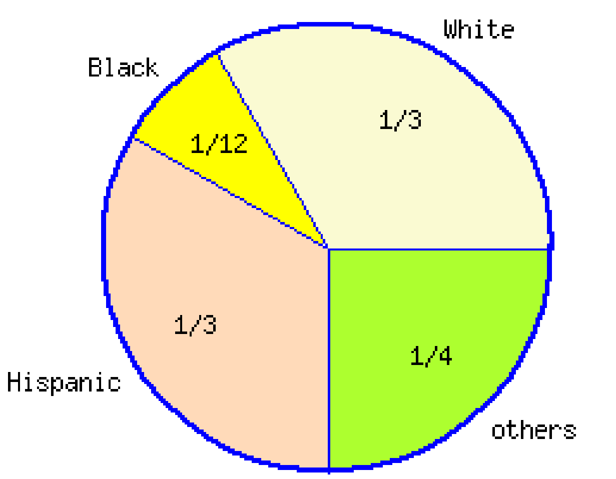
The second leg is $\boxed{}$ of a mile in length.

82. James is participating in a running event. In the first hour, he completed $\frac{2}{9}$ of the total distance. After another hour, in total he had completed $\frac{47}{90}$ of the total distance.

What fraction of the total distance did James complete during the second hour?

James completed [ ] of the distance during the second hour.

83. The pie chart represents a school's student population.



Together, white and black students make up [ ] of the school's population.

84. Each page of a book is $6\frac{3}{4}$ inches in height, and consists of a header (a top margin), a footer (a bottom margin), and the middle part (the body). The header is $\frac{5}{6}$ of an inch thick and the middle part is $5\frac{5}{6}$ inches from top to bottom.

What is the thickness of the footer?

The footer is [ ] of an inch thick.

85. Priscilla and Adrian are sharing a pizza. Priscilla ate $\frac{1}{6}$ of the pizza, and Adrian ate $\frac{1}{10}$ of the pizza. How much more pizza did Priscilla eat than Adrian?

Priscilla ate [ ] more of the pizza than Adrian ate.

86. Emiliano and Maria are sharing a pizza. Emiliano ate $\frac{2}{7}$ of the pizza, and Maria ate $\frac{1}{6}$ of the pizza. How much more pizza did Emiliano eat than Maria?

Emiliano ate [ ] more of the pizza than Maria ate.

87. A school had a fund-raising event. The revenue came from three resources: ticket sales, auction sales, and donations. Ticket sales account for $\frac{3}{4}$ of the total revenue; auction sales account for $\frac{1}{8}$ of the total revenue. What fraction of the revenue came from donations?

[ ] of the revenue came from donations.

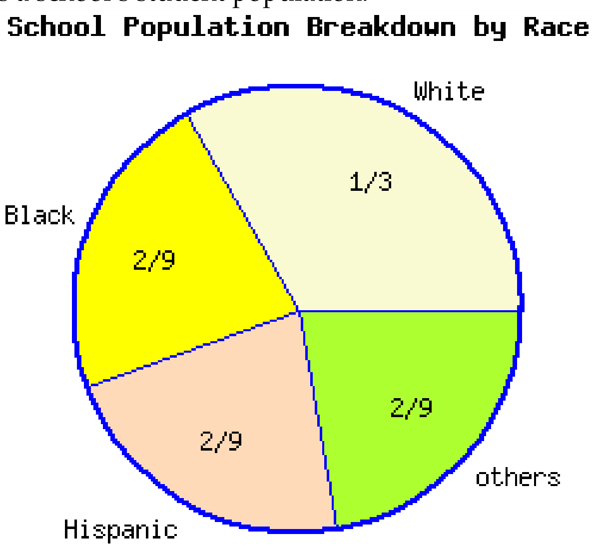
88. A few years back, a car was purchased for \$23,500. Today it is worth $\frac{1}{5}$ of its original value. What is the car's current value?

The car's current value is [ ].

89. A few years back, a car was purchased for \$20,000. Today it is worth $\frac{1}{5}$ of its original value. What is the car's current value?

The car's current value is [ ].

90. The pie chart represents a school's student population.



[ ] more of the school is white students than black students.

91. A town has 300 residents in total, of which $\frac{5}{6}$ are Asian Americans. How many Asian Americans reside in this town?

There are [ ] Asian Americans residing in this town.

92. A company received a grant, and decided to spend $\frac{1}{8}$ of this grant in research and development next year. Out of the money set aside for research and development, $\frac{4}{9}$ will be used to buy new equipment. What fraction of the grant will be used to buy new equipment?

[ ] of the grant will be used to buy new equipment.

93. A food bank just received 14 kilograms of emergency food. Each family in need is to receive $\frac{2}{5}$ kilograms of food. How many families can be served with the 14 kilograms of food?

[ ] families can be served with the 14 kilograms of food.

94. A construction team maintains a 52-mile-long sewage pipe. Each day, the team can cover $\frac{4}{5}$ of a mile. How many days will it take the team to complete the maintenance of the entire sewage pipe?

It will take the team [ ] days to complete maintaining the entire sewage pipe.

95. A child is stacking up tiles. Each tile's height is $\frac{2}{3}$ of a centimeter. How many layers of tiles are needed to reach 12 centimeters in total height?

To reach the total height of 12 centimeters, [ ] layers of tiles are needed.

96. A restaurant made 300 cups of pudding for a festival.

Customers at the festival will be served $\frac{1}{5}$ of a cup of pudding per serving. How many customers can the restaurant serve at the festival with the 300 cups of pudding?

The restaurant can serve ______ customers at the festival with the 300 cups of pudding.

97. A 2×4 piece of lumber in your garage is $62\frac{11}{16}$ inches long. A second 2×4 is $55\frac{7}{16}$ inches long. If you lay them end to end, what will the total length be?

The total length will be ______ inches.

98. A 2×4 piece of lumber in your garage is $38\frac{11}{16}$ inches long. A second 2×4 is $44\frac{7}{8}$ inches long. If you lay them end to end, what will the total length be?

The total length will be ______ inches.

99. Each page of a book consists of a header, a footer and the middle part. The header is $\frac{1}{9}$ inches in height; the footer is $\frac{11}{18}$ inches in height; and the middle part is $3\frac{4}{9}$ inches in height.
What is the total height of each page in this book? Use mixed number in your answer if needed.

Each page in this book is ______ inches in height.

100. To pave the road on Ellis Street, the crew used $4\frac{5}{8}$ tons of cement on the first day, and used $4\frac{9}{10}$ tons on the second day. How many tons of cement were used in all?

______ tons of cement were used in all.

101. When driving on a high way, noticed a sign saying exit to Johnstown is $1\frac{3}{4}$ miles away, while exit to Jerrystown is $3\frac{1}{2}$ miles away. How far is Johnstown from Jerrystown?

Johnstown and Jerrystown are ______ miles apart.

102. A cake recipe needs $3\frac{1}{2}$ cups of flour. Using this recipe, to bake 9 cakes, how many cups of flour are needed?

To bake 9 cakes, ______ cups of flour are needed.

Sketching Fractions

103. Sketch a number line showing each fraction. (Be sure to carefully indicate the correct number of equal parts of the whole.)

(a) $\dfrac{2}{3}$ (b) $\dfrac{6}{8}$ (c) $\dfrac{5}{4}$ (d) $-\dfrac{4}{5}$

104. Sketch a number line showing each fraction. (Be sure to carefully indicate the correct number of equal parts of the whole.)

(a) $\dfrac{1}{6}$ (b) $\dfrac{3}{9}$ (c) $\dfrac{7}{6}$ (d) $-\dfrac{8}{5}$

105. Sketch a picture of the product $\frac{3}{5} \cdot \frac{1}{2}$, using a number line or rectangles.

106. Sketch a picture of the sum $\frac{2}{3} + \frac{1}{8}$, using a number line or rectangles.

Challenge

107. Given that $a \neq 0$, simplify $\dfrac{3}{a} + \dfrac{6}{a}$.

108. Given that $a \neq 0$, simplify $\dfrac{4}{a} + \dfrac{3}{2a}$.

109. Given that $a \neq 0$, simplify $\dfrac{5}{a} - \dfrac{9}{5a}$.

A.3 Absolute Value and Square Root

In this section, we will learn the basics of **absolute value** and **square root**. These are actions you can *do* to a given number, often changing the number into something else.

A.3.1 Introduction to Absolute Value

Definition A.3.2 The **absolute value** of a number is the distance between that number and 0 on a number line. For the absolute value of x, we write $|x|$. ◊

Let's look at $|2|$ and $|-2|$, the absolute value of 2 and the absolute value of -2.

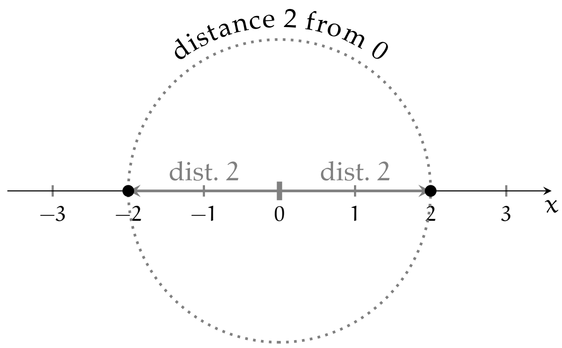


Figure A.3.3: $|2|$ and $|-2|$

Since the distance between 2 and 0 on the number line is 2 units, the absolute value of 2 is 2. We write $|2| = 2$.

Since the distance between -2 and 0 on the number line is also 2 units, the absolute value of -2 is also 2. We write $|-2| = 2$.

Fact A.3.4 Absolute Value. *Taking the absolute value of a number results in whatever the "positive version" of that number is. This is because the real meaning of absolute value is its* distance *from zero.*

Checkpoint A.3.5 Calculating Absolute Value. Try calculating some absolute values.

a. $|57|$

b. $|-43|$

c. $\left|\frac{2}{-5}\right|$

Explanation.

a. 57 is 57 units away from 0 on a number line, so $|57| = 57$. Another way to think about this is that the "positive version" of 57 is 57.

b. -43 is 43 units away from 0 on a number line, so $|-43| = 43$. Another way to think about this is that the "positive version" of -43 is 43.

c. $\frac{2}{-5}$ is $\frac{2}{5}$ units away from 0 on a number line, so $\left|\frac{2}{-5}\right| = \frac{2}{5}$. Another way to think about this is that the "positive version" of $\frac{2}{-5}$ is $\frac{2}{5}$.

Warning A.3.6 Absolute Value Does Not Exactly "Make Everything Positive". Students may see an expression like $|2-5|$ and incorrectly think it is OK to "make everything positive" and write $2+5$. This is incorrect since $|2-5|$ works out to be 3, not 7, as we are actually taking the absolute value of -3 (the equivalent number inside the absolute value).

A.3.2 Square Root Facts

If you have learned your basic multiplication table, you know:

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Figure A.3.7: Multiplication table with squares

The numbers along the diagonal are special; they are known as **perfect squares**. And for working with square roots, it will be helpful if you can memorize these first few perfect square numbers.

"Taking a square root" is the opposite action of squaring a number. For example, when you square 3, the result is 9. So when you take the square root of 9, the result is 3. Just knowing that 9 comes about as 3^2 lets us realize that 3 is the square root of 9. This is why memorizing the perfect squares from the multiplication table can be so helpful.

The notation we use for taking a square root is the **radical**, $\sqrt{}$. For example, "the square root of 9" is denoted $\sqrt{9}$. And now we know enough to be able to write $\sqrt{9} = 3$.

Tossing in a few extra special square roots, it's advisable to memorize the following:

$$\begin{array}{cccc} \sqrt{0}=0 & \sqrt{1}=1 & \sqrt{4}=2 & \sqrt{9}=3 \\ \sqrt{16}=4 & \sqrt{25}=5 & \sqrt{36}=6 & \sqrt{49}=7 \\ \sqrt{64}=8 & \sqrt{81}=9 & \sqrt{100}=10 & \sqrt{121}=11 \\ \sqrt{144}=12 & \sqrt{169}=13 & \sqrt{196}=14 & \sqrt{225}=15 \end{array}$$

A.3.3 Calculating Square Roots with a Calculator

Most square roots are actually numbers with decimal places that go on forever. Take $\sqrt{5}$ as an example:

$$\sqrt{4} = 2 \qquad \sqrt{5} = ? \qquad \sqrt{9} = 3$$

Since 5 is between 4 and 9, then $\sqrt{5}$ must be somewhere between 2 and 3. There are no whole numbers between 2 and 3, so $\sqrt{5}$ must be some number with decimal places. If the decimal places eventually stopped, then squaring it would give you another number with decimal places that stop further out. But squaring it gives you 5 with no decimal places. So the only possibility is that $\sqrt{5}$ is a decimal between 2 and 3 that goes on forever. With a calculator, we can see:

$$\sqrt{5} \approx 2.236$$

Actually the decimal will not terminate, and that is why we used the ≈ symbol instead of an equals sign. To get 2.236 we rounded down slightly from the true value of $\sqrt{5}$. With a calculator, we can check that $2.236^2 = 4.999696$, a little shy of 5.

A.3.4 Square Roots of Fractions

We can calculate the square root of some fractions by hand, such as $\sqrt{\frac{1}{4}}$. The idea is the same: can you think of a number that you would square to get $\frac{1}{4}$? Being familiar with fraction multiplication, we know that $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ and so $\sqrt{\frac{1}{4}} = \frac{1}{2}$.

Checkpoint A.3.8 Square Roots of Fractions. Try calculating some absolute values.

a. $\sqrt{\dfrac{1}{25}}$ b. $\sqrt{\dfrac{4}{9}}$ c. $\sqrt{\dfrac{81}{121}}$

Explanation.

a. Since $\sqrt{1} = 1$ and $\sqrt{25} = 5$, then $\sqrt{\dfrac{1}{25}} = \dfrac{1}{5}$.

b. Since $\sqrt{4} = 2$ and $\sqrt{9} = 3$, then $\sqrt{\dfrac{4}{9}} = \dfrac{2}{3}$.

c. Since $\sqrt{81} = 9$ and $\sqrt{121} = 11$, then $\sqrt{\dfrac{81}{121}} = \dfrac{9}{11}$.

A.3.5 Square Root of Negative Numbers

Can we find the square root of a negative number, such as $\sqrt{-25}$? That would mean that there is some number out there that multiplies by itself to make -25. Would $\sqrt{-25}$ be positive or negative? Either way, once you square it (multiply it by itself) the result would be positive. So it couldn't possibly square to -25. So there is no square root of -25 or of any negative number for that matter.

Imaginary Numbers. Mathematicians imagined a new type of number, neither positive nor negative, that would square to a negative result. But that is beyond the scope of this chapter.

If you are confronted with an expression like $\sqrt{-25}$, or any other square root of a negative number, you can state that "there is no real square root" or that the result "does not exist" (as a real number).

A.3.6 Exercises

Review and Warmup

1. Evaluate the expressions.

a. 1^2 b. 3^2 c. 5^2 d. 7^2 e. 9^2 f. 11^2

2. Evaluate the expressions.

a. 2^2 b. 4^2 c. 6^2 d. 8^2 e. 10^2 f. 12^2

Absolute Value Evaluate the following.

3. $|-10|$

4. $|-7|$

5. $|-48.89|$

6. $|-15.76|$

7. $\left|-\dfrac{3}{59}\right|$

8. $\left|-\dfrac{19}{11}\right|$

9.
a. $-|1-9|$
b. $|-1-9|$
c. $-2|9-1|$

10.
a. $-|4-6|$
b. $|-4-6|$
c. $-2|6-4|$

11. a. $|10|$
b. $|-10|$
c. $-|10|$
d. $-|-10|$

12. a. $|2|$
b. $|-2|$
c. $-|2|$
d. $-|-2|$

13. a. $|2|$
b. $|-3|$
c. $|0|$
d. $|14+(-8)|$
e. $|-8-(-1)|$

14. a. $|3|$
b. $|-7|$
c. $|0|$
d. $|18+(-1)|$
e. $|-8-(-3)|$

15. Which of the following are square numbers? There may be more than one correct answer.
☐ 101 ☐ 80 ☐ 56 ☐ 1 ☐ 16 ☐ 25

16. Which of the following are square numbers? There may be more than one correct answer.
☐ 36 ☐ 81 ☐ 55 ☐ 64 ☐ 50 ☐ 62

Square Roots Evaluate the following.

17. a. $\sqrt{64}$
b. $\sqrt{16}$
c. $\sqrt{1}$

18. a. $\sqrt{81}$
b. $\sqrt{144}$
c. $\sqrt{25}$

19. a. $\sqrt{\frac{100}{49}}$
b. $\sqrt{-\frac{121}{36}}$

20. a. $\sqrt{\frac{144}{121}}$
b. $\sqrt{-\frac{100}{81}}$

Evaluate the following.

21. Do not use a calculator.
a. $\sqrt{4}$
b. $\sqrt{0.04}$
c. $\sqrt{400}$

22. Do not use a calculator.
a. $\sqrt{9}$
b. $\sqrt{0.09}$
c. $\sqrt{900}$

23. Do not use a calculator.
a. $\sqrt{16}$
b. $\sqrt{1600}$
c. $\sqrt{160000}$

24. Do not use a calculator.
a. $\sqrt{25}$
b. $\sqrt{2500}$
c. $\sqrt{250000}$

25. Do not use a calculator.
a. $\sqrt{36}$
b. $\sqrt{0.36}$
c. $\sqrt{0.0036}$

26. Do not use a calculator.
a. $\sqrt{64}$
b. $\sqrt{0.64}$
c. $\sqrt{0.0064}$

Evaluate the following.

27. Use a calculator to approximate with a decimal.
$\sqrt{61}$

28. Use a calculator to approximate with a decimal.
$\sqrt{83}$

Evaluate the following.

29. $\sqrt{\frac{121}{144}}$

30. $\sqrt{\frac{1}{100}}$

31. $-\sqrt{9}$

32. $-\sqrt{16}$

33. $\sqrt{-25}$

34. $\sqrt{-49}$

35. $\sqrt{-\frac{49}{100}}$

36. $\sqrt{-\frac{64}{81}}$

37. $-\sqrt{\dfrac{81}{100}}$

38. $-\sqrt{\dfrac{121}{144}}$

39. a. $\sqrt{25}-\sqrt{9}$
b. $\sqrt{25-9}$

40. a. $\sqrt{25}-\sqrt{9}$
b. $\sqrt{25-9}$

41. $\dfrac{3}{\sqrt{100}}$

42. $\dfrac{3}{\sqrt{49}}$

A.4 Percentages

Percent-related problems arise in everyday life. This section reviews some basic calculations that can be made with percentages.

A.4.1 Converting Percents, Decimals, and English

In many situations when translating from English to math, the word "of" translates as multiplication. Also the word "is" (and many similar words related to "to be") translates to an equals sign. For example:

$$\text{One third of thirty is ten.}$$
$$\frac{1}{3} \cdot 30 = 10$$

Here is another example, this time involving a percentage. We know that "2 is 50% of 4," so we can say:

$$\begin{aligned} &2 \text{ is } 50\% \text{ of } 4 \\ &2 = 0.5 \cdot 4 \end{aligned}$$

Example A.4.2 Translate each statement involving percents below into an equation. Define any variables used. (Solving these equations is an exercise).

a. How much is 30% of \$24.00?

b. \$7.20 is what percent of \$24.00?

c. \$7.20 is 30% of how much money?

Explanation. Each question can be translated from English into a math equation by reading it slowly and looking for the right signals.

a. The word "is" means about the same thing as the equals sign. "How much" is a question phrase, and we can let x be the unknown amount (in dollars). The word "of" translates to multiplication, as discussed earlier. So we have:

$$\overset{\text{how much}}{x} \overset{\text{is}}{=} \overset{30\%}{0.30} \overset{\text{of}}{\cdot} \overset{\$24}{24}$$

b. Let P be the unknown value. We have:

$$\overset{\$7.20}{7.2} \overset{\text{is}}{=} \overset{\text{what percent}}{P} \overset{\text{of}}{\cdot} \overset{\$24}{24}$$

With this setup, P is going to be a decimal value (0.30) that you would translate into a percentage (30%).

c. Let x be the unknown amount (in dollars). We have:

$$\overset{\$7.20}{7.2} \overset{\text{is}}{=} \overset{30\%}{0.30} \overset{\text{of}}{\cdot} \overset{\text{how much}}{x}$$

Checkpoint A.4.3 Solve each equation from Example A.4.2.

a. How much is 30% of \$24.00?

$$x = 0.3 \cdot 24$$

x is ______.

b. \$7.20 is what percent of \$24.00?

$$7.2 = P \cdot 24$$

P is ______.

c. \$7.20 is 30% of how much money?

$$7.2 = 0.3 \cdot x$$

x is ______.

Explanation.

a.
$$\begin{aligned} x &= 0.3 \cdot 24 \\ x &= 8 \end{aligned}$$

b.
$$\begin{aligned} 7.2 &= P \cdot 24 \\ \frac{7.2}{24} &= \frac{P \cdot 24}{24} \\ 0.3 &= P \end{aligned}$$

c.
$$\begin{aligned} 7.2 &= 0.3 \cdot x \\ \frac{7.2}{0.3} &= \frac{0.3 \cdot x}{0.3} \\ 24 &= x \end{aligned}$$

A.4.2 Setting up and Solving Percent Equations

An important skill for solving percent-related problems is to boil down a complicated word problem into a simple form like “2 is 50% of 4.” Let’s look at some further examples.

Example A.4.4
In Fall 2016, Portland Community College had 89,900 enrolled students. According to Figure A.4.5, how many black students were enrolled at PCC in Fall 2016?

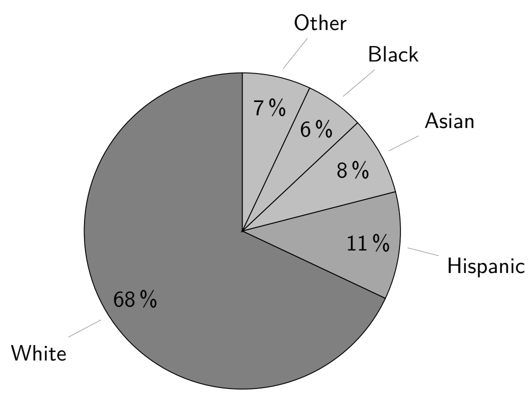


Figure A.4.5: Racial breakdown of PCC students in Fall 2016

Explanation. After reading this word problem and the chart, we can translate the problem into “what is 6% of 89,900?” Let x be the number of black students enrolled at PCC in Fall 2016. We can set up and solve the equation:

$$\begin{aligned} \overset{\text{what}}{x} &\overset{\text{is}}{=} \overset{6\%}{0.06} \overset{\text{of}}{\cdot} \overset{89,900}{89900} \\ x &= 5394 \end{aligned}$$

There was not much “solving” to do, since the variable we wanted to isolate was already isolated.

As of Fall 2016, Portland Community College had 5394 black students. Note: this is not likely to be perfectly accurate, because the numbers we started with (89,900 enrolled students and 6%) appear to be rounded.

Example A.4.6
The bar graph in Figure A.4.7 displays how many students are in each class at a local high school. According to the bar graph, what percentage of the school's student population is freshman?

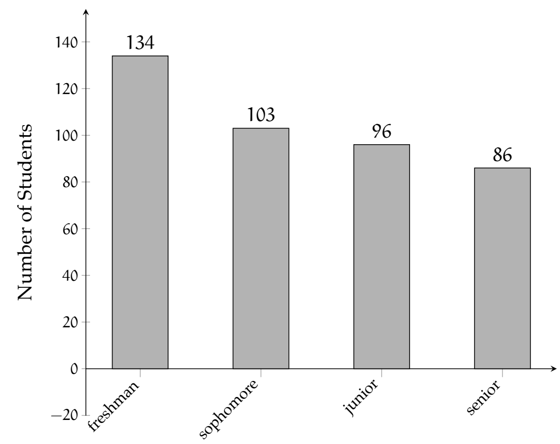


Figure A.4.7: Number of students at a high school by class

Explanation. The school's total number of students is:

$$134 + 103 + 96 + 86 = 419$$

With that calculated, we can translate the main question:

"What percentage of the school's student population is freshman?"

into:

"What percent of 419 is 134?"

Using P to represent the unknown quantity, we write and solve the equation:

$$\begin{aligned}
\underbrace{P}_{\text{what percent}} \underbrace{\cdot}_{\text{of}} \underbrace{419}_{419} \underbrace{=}_{\text{is}} \underbrace{134}_{134}&\\
\frac{P \cdot 419}{419} &= \frac{134}{419}\\
P &\approx 0.3198\\
P &\approx 31.98\%
\end{aligned}$$

Approximately 31.98% of the school's student population is freshman.

Remark A.4.8 When solving equations that do *not* have context we state the solution set. However, when solving an equation or inequality that arises in an application problem (such as the context of the high school in Example A.4.6), it makes more sense to summarize our result with a sentence, using the context of the application. This allows us to communicate the full result, including appropriate units.

Example A.4.9 Carlos just received his monthly paycheck. His gross pay (the amount before taxes and related things are deducted) was \$2,346.19, and his total tax and other deductions was \$350.21. The rest was deposited directly into his checking account. What percent of his gross pay went into his checking account?

Explanation. Train yourself to read the word problem and not try to pick out numbers to substitute into formulas. You may find it helps to read the problem over to yourself three or more times before you attempt to solve it. There are *three* dollar amounts to discuss in this problem, and many students fall into a trap of using the wrong values in the wrong places. There is the gross pay, the amount that was deducted, and the amount that was deposited. Only two of these have been explicitly written down. We need to use subtraction to find the dollar amount that was deposited:

$$2346.19 - 350.21 = 1995.98$$

Now, we can translate the main question:

"What percent of his gross pay went into his checking account?"

into:

"What percent of \$2346.19 is \$1995.98?"

Using P to represent the unknown quantity, we write and solve the equation:

$$\begin{aligned}
\overset{\text{what percent}}{\overset{|}{P}} \;\overset{\text{of}}{\overset{|}{\cdot}}\; \overset{\$2346.19}{\overset{|}{2346.19}} &\overset{\text{is}}{\overset{|}{=}} \overset{\$1995.98}{\overset{|}{1995.98}}\\
\frac{P \cdot 2346.19}{2346.19} &= \frac{1995.98}{2346.19}\\
P &\approx 0.8507\\
P &\approx 85.07\%
\end{aligned}$$

Approximately 85.07% of his gross pay went into his checking account.

Checkpoint A.4.10 Alexis sells cars for a living, and earns 28% of the dealership's sales profit as commission. In a certain month, she plans to earn \$2200 in commissions. How much total sales profit does she need to bring in for the dealership?

Alexis needs to bring in ____________ in sales profit.

Explanation. Be careful that you do not calculate 28% of \$2200. That might be what a student would do who doesn't thoroughly read the question. If you have ever trained yourself to quickly find numbers in word problems and substitute them into formulas, you must *unlearn* this. The issue is that \$2200 is not the dealership's sales profit, and if you mistakenly multiply $0.28 \cdot 2200 = 616$, then \$616 makes no sense as an answer to this question. How could Alexis bring in only \$616 of sales profit, and earn \$2200 in commission?

We can translate the problem into "\$2200 is 28% of what?" Letting x be the sales profit for the dealership (in dollars), we can write and solve the equation:

$$\begin{aligned}
\overset{\$2200}{2200} &\overset{\text{is}}{=} \overset{28\%}{0.28} \overset{\text{of}}{\cdot} \overset{\text{what}}{x}\\
\frac{2200}{0.28} &= \frac{0.28x}{0.28}\\
7857.14 &\approx x\\
x &\approx 7857.14
\end{aligned}$$

To earn \$2200 in commission, Alexis needs to bring in approximately \$7857.14 of sales profit for the dealership.

Example A.4.11
According to e-Literate, the average cost of a new college textbook has been increasing. Find the percentage of increase from 2009 to 2013.

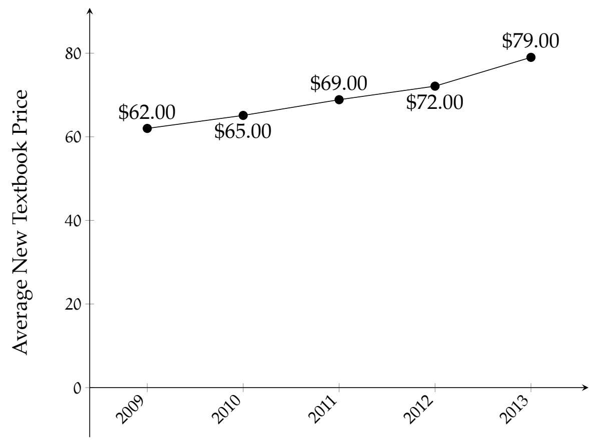


Figure A.4.12: Average New Textbook Price from 2009 to 2013

Explanation. The actual amount of increase from 2009 to 2013 was $79 - 62 = 17$, dollars. We need to answer the question "\$17 is what percent of \$62?" Note that we are comparing the 17 to 62, not to 79. In these situations where one amount is the earlier amount, the earlier original amount is the one that represents 100%. Let P represent the percent of increase. We can set up and solve the equation:

$$\begin{aligned}
\overset{\$17}{17} &\overset{\text{is}}{=} \overset{\text{what percent}}{P} \overset{\text{of}}{\cdot} \overset{\$62}{62}\\
17 &= 62P\\
\frac{17}{62} &= \frac{62P}{62}\\
0.2742 &\approx P
\end{aligned}$$

From 2009 to 2013, the average cost of a new textbook increased by approximately 27.42%.

Checkpoint A.4.13 Last month, a full tank of gas for a car you drive cost you \$40.00. You hear on the news that gas prices have risen by 12%. By how much, in dollars, has the cost of a full tank gone up?

A full tank of gas now costs ______ more than it did last month.

Explanation. Let x represent the amount of increase. We can set up and solve the equation:

$$
\begin{aligned}
\overset{\text{12\%}}{0.12} \overset{\text{of}}{\cdot} \overset{\text{old cost}}{40} &\overset{\text{is}}{=} \overset{\text{how much}}{x}\\
4.8 &= x
\end{aligned}
$$

A full tank now costs \$4.80 more than it did last month.

Example A.4.14 Enrollment at your neighborhood's elementary school two years ago was 417 children. After a 15% increase last year and a 15% decrease this year, what's the new enrollment?

Explanation. It is tempting to think that increasing by 15% and then decreasing by 15% would bring the enrollment right back to where it started. But the 15% decrease applies to the enrollment *after* it had already increased. So that 15% decrease is going to translate to *more* students lost than were gained.

Using 100% as corresponding to the enrollment from two years ago, the enrollment last year was $100\% + 15\% = 115\%$ of that. But then using 100% as corresponding to the enrollment from last year, the enrollment this year was $100\% - 15\% = 85\%$ of that. So we can set up and solve the equation

$$
\begin{aligned}
\overset{\text{this year's enrollment}}{x} &\overset{\text{is}}{=} \overset{\text{85\%}}{0.85} \overset{\text{of}}{\cdot} \overset{\text{115\%}}{1.15} \overset{\text{of}}{\cdot} \overset{\text{enrollment two years ago}}{417}\\
x &= 0.85 \cdot 1.15 \cdot 417\\
x &= 407.6175
\end{aligned}
$$

We would round and report that enrollment is now 408 students. (The percentage rise and fall of 15% were probably rounded in the first place, which is why we did not end up with a whole number.)

A.4.3 Exercises

Review and Warmup Write the following percentages as decimals.

1. a. 18% b. 53%

2. a. 19% b. 58%

3. a. 0.21 b. 0.65

4. a. 0.22 b. 0.62

5. a. 3% b. 30% c. 100% d. 300%

6. a. 4% b. 40% c. 100% d. 400%

Write the following decimals as percentages.

7. a. 0.05 b. 0.5 c. 5 d. 1

8. a. 0.06 b. 0.6 c. 6 d. 1

9. a. 7.45 b. 0.745 c. 0.0745

10. a. 8.19
b. 0.819
c. 0.0819

11. a. 973%
b. 97.3%
c. 9.73%

12. a. 147%
b. 14.7%
c. 1.47%

Basic Percentage Calculation

13. 2% of 220 is ______.

14. 7% of 320 is ______.

15. 40% of 420 is ______.

16. 20% of 520 is ______.

17. 670% of 620 is ______.

18. 430% of 710 is ______.

19. 97% of ______ is 785.7.

20. 66% of ______ is 600.6.

21. 4% of ______ is 4.8.

22. 9% of ______ is 19.8.

23. 390% of ______ is 1248.

24. 220% of ______ is 924.

Answer with a percent.

25. 450 is ______ of 900.

26. 354 is ______ of 590.

27. 64.4 is ______ of 28.

28. 223.6 is ______ of 86.

29. 19 is about ______ of 60.

30. 2 is about ______ of 17.

Applications

31. A town has 1500 registered residents. Among them, 38% were Democrats, 34% were Republicans. The rest were Independents. How many registered Independents live in this town?

There are ______ registered Independent residents in this town.

32. A town has 1900 registered residents. Among them, 35% were Democrats, 22% were Republicans. The rest were Independents. How many registered Independents live in this town?

There are ______ registered Independent residents in this town.

33. Dave is paying a dinner bill of $30.00. Dave plans to pay 11% in tips. How much tip will Dave pay?

Dave will pay ______ in tip.

34. Tracei is paying a dinner bill of $34.00. Tracei plans to pay 18% in tips. How much tip will Tracei pay?

Tracei will pay ______ in tip.

35. Michael is paying a dinner bill of $37.00. Michael plans to pay 14% in tips. How much in total (including bill and tip) will Michael pay?

Michael will pay ______ in total (including bill and tip).

36. Jessica is paying a dinner bill of $41.00. Jessica plans to pay 10% in tips. How much in total (including bill and tip) will Jessica pay?

Jessica will pay ______ in total (including bill and tip).

37. A watch's wholesale price was $440.00. The retailer marked up the price by 35%. What's the watch's new price (markup price)?

The watch's markup price is ______.

38. A watch's wholesale price was $480.00. The retailer marked up the price by 25%. What's the watch's new price (markup price)?

The watch's markup price is ______.

39. In the past few seasons' basketball games, Lindsay attempted 110 free throws, and made 22 of them. What percent of free throws did Lindsay make?

Lindsay made [ ] of free throws in the past few seasons.

40. In the past few seasons' basketball games, Douglas attempted 370 free throws, and made 74 of them. What percent of free throws did Douglas make?

Douglas made [ ] of free throws in the past few seasons.

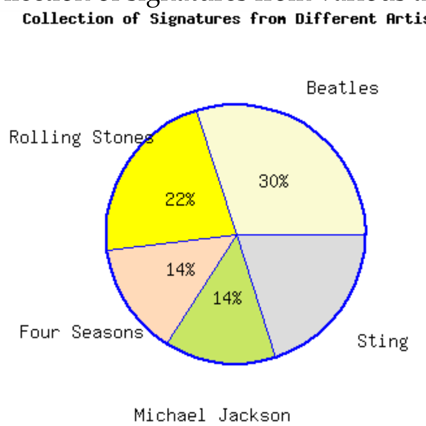
41. A painting is on sale at $425.00. Its original price was $500.00. What percentage is this off its original price?

The painting was [ ] off its original price.

42. A painting is on sale at $385.00. Its original price was $550.00. What percentage is this off its original price?

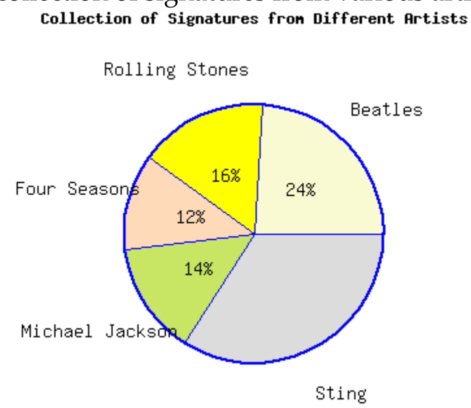
The painting was [ ] off its original price.

43. The pie chart represents a collector's collection of signatures from various artists.



If the collector has a total of 1050 signatures, there are [ ] signatures by Sting.

44. The pie chart represents a collector's collection of signatures from various artists.



If the collector has a total of 1250 signatures, there are [ ] signatures by Sting.

45. In the last election, 67% of a county's residents, or 24790 people, turned out to vote. How many residents live in this county?

This county has [ ] residents.

46. In the last election, 54% of a county's residents, or 22356 people, turned out to vote. How many residents live in this county?

This county has [ ] residents.

47. 58.14 grams of pure alcohol was used to produce a bottle of 15.3% alcohol solution. What is the weight of the solution in grams?

The alcohol solution weighs [ ].

48. 57.4 grams of pure alcohol was used to produce a bottle of 28.7% alcohol solution. What is the weight of the solution in grams?

The alcohol solution weighs [ ].

49. Ravi paid a dinner and left 16%, or $3.68, in tips. How much was the original bill (without counting the tip)?
The original bill (not including the tip) was [ ].

50. Laney paid a dinner and left 12%, or $3.24, in tips. How much was the original bill (without counting the tip)?
The original bill (not including the tip) was [ ].

51. Parnell sells cars for a living. Each month, he earns $1,300.00 of base pay, plus a certain percentage of commission from his sales.
One month, Parnell made $53,500.00 in sales, and earned a total of $4,472.55 in that month (including base pay and commission). What percent commission did Parnell earn?
Parnell earned ______ in commission.

52. Selena sells cars for a living. Each month, she earns $1,300.00 of base pay, plus a certain percentage of commission from her sales.
One month, Selena made $58,000.00 in sales, and earned a total of $3,904.20 in that month (including base pay and commission). What percent commission did Selena earn?
Selena earned ______ in commission.

53. The following is a nutrition fact label from a certain macaroni and cheese box.

Nutrition Facts	
Serving Size 1 cup	
Servings Per Container 2	
Amount Per Serving	
Calories 300	Calories from Fat 110
	% Daily Value
Total Fat 6.5 g	10%
Saturated Fat 2 g	15%
Trans Fat 2 g	
Cholesterol 30 mg	10%
Sodium 400 mg	20%
Total Carbohydrate 30 g	11%
Dietary Fiber 0 g	0%
Sugars 5 g	
Protein 5 g	
Vitamin A 2 mg	3%
Vitamin C 2 mg	2.5%
Calcium 2 mg	20%
Iron 3 mg	4%

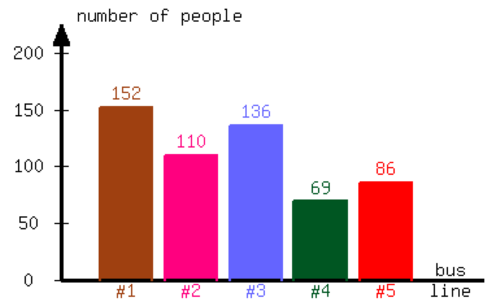
The highlighted row means each serving of macaroni and cheese in this box contains 6.5 g of fat, which is 10% of an average person's daily intake of fat. What's the recommended daily intake of fat for an average person?
The recommended daily intake of fat for an average person is ______.

54. The following is a nutrition fact label from a certain macaroni and cheese box.

Nutrition Facts	
Serving Size 1 cup	
Servings Per Container 2	
Amount Per Serving	
Calories 300	Calories from Fat 110
	% Daily Value
Total Fat 12.6 g	18%
Saturated Fat 2 g	15%
Trans Fat 2 g	
Cholesterol 30 mg	10%
Sodium 400 mg	20%
Total Carbohydrate 30 g	11%
Dietary Fiber 0 g	0%
Sugars 5 g	
Protein 5 g	
Vitamin A 2 mg	3%
Vitamin C 2 mg	2.5%
Calcium 2 mg	20%
Iron 3 mg	4%

The highlighted row means each serving of macaroni and cheese in this box contains 12.6 g of fat, which is 18% of an average person's daily intake of fat. What's the recommended daily intake of fat for an average person?
The recommended daily intake of fat for an average person is ______.

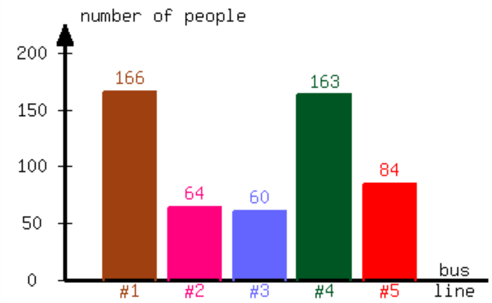
55. A community college conducted a survey about the number of students riding each bus line available. The following bar graph is the result of the survey.



What percent of students ride Bus #1?

Approximately [ ] of students ride Bus #1.

56. A community college conducted a survey about the number of students riding each bus line available. The following bar graph is the result of the survey.



What percent of students ride Bus #1?

Approximately [ ] of students ride Bus #1.

57. Donna earned \$75.75 of interest from a mutual fund, which was 0.75% of his total investment. How much money did Donna invest into this mutual fund?

Donna invested [ ] in this mutual fund.

58. Dave earned \$71.05 of interest from a mutual fund, which was 0.49% of his total investment. How much money did Dave invest into this mutual fund?

Dave invested [ ] in this mutual fund.

59. A town has 1900 registered residents. Among them, there are 589 Democrats and 513 Republicans. The rest are Independents. What percentage of registered voters in this town are Independents?

In this town, [ ] of all registered voters are Independents.

60. A town has 2300 registered residents. Among them, there are 851 Democrats and 805 Republicans. The rest are Independents. What percentage of registered voters in this town are Independents?

In this town, [ ] of all registered voters are Independents.

Percent Increase/Decrease

61. The population of cats in a shelter decreased from 100 to 80. What is the percentage decrease of the shelter's cat population?

The percentage decrease is [ ].

62. The population of cats in a shelter decreased from 120 to 114. What is the percentage decrease of the shelter's cat population?

The percentage decrease is [ ].

63. The population of cats in a shelter increased from 57 to 72. What is the percentage increase of the shelter's cat population?
The percentage increase is approximately [ ].

64. The population of cats in a shelter increased from 65 to 75. What is the percentage increase of the shelter's cat population?
The percentage increase is approximately [ ].

65. Last year, a small town's population was 770. This year, the population decreased to 769. What is the percentage decrease?
The percentage decrease of the town's population was approximately ______.

66. Last year, a small town's population was 800. This year, the population decreased to 797. What is the percentage decrease?
The percentage decrease of the town's population was approximately ______.

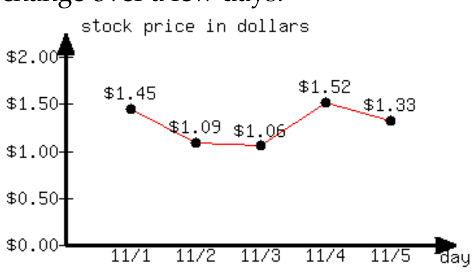
67. Your salary used to be \$36,000 per year. You had to take a 2% pay cut. After the cut, your salary was ______ per year.
Then, you earned a 2% raise. After the raise, your salary was ______ per year.

68. Your salary used to be \$49,000 per year. You had to take a 2% pay cut. After the cut, your salary was ______ per year.
Then, you earned a 2% raise. After the raise, your salary was ______ per year.

69. A house was bought two years ago at the price of \$340,000. Each year, the house's value decreased by 4%. What's the house's value this year?
The house's value this year is ______.

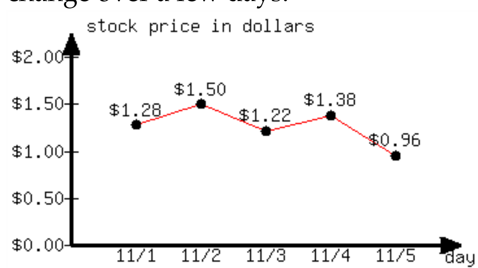
70. A house was bought two years ago at the price of \$200,000. Each year, the house's value decreased by 5%. What's the house's value this year?
The house's value this year is ______.

71. This line graph shows a certain stock's price change over a few days.



From 11/1 to 11/5, what is the stock price's percentage change?
From 11/1 to 11/5, the stock price's percentage change was approximately ______.

72. This line graph shows a certain stock's price change over a few days.



From 11/1 to 11/5, what is the stock price's percentage change?
From 11/1 to 11/5, the stock price's percentage change was approximately ______.

A.5 Order of Operations

Mathematical symbols are a means of communication, and it's important that when you write something, everyone else knows exactly what you intended. For example, if we say in English, "two times three squared," do we mean that:

- 2 is multiplied by 3, and then the result is squared?
- or that 2 is multiplied by the result of squaring 3?

English is allowed to have ambiguities like this. But mathematical language needs to be precise and mean the same thing to everyone reading it. For this reason, a standard **order of operations** has been adopted, which we review here.

A.5.1 Grouping Symbols

Consider the math expression $2 \cdot 3^2$. There are two mathematical operations here: a multiplication and an exponentiation. The result of this expression will change depending on which operation you decide to execute first: the multiplication or the exponentiation. If you multiply $2 \cdot 3$, and then square the result, you have 36. If you square 3, and then multiply 2 by the result, you have 18. If we want all people everywhere to interpret $2 \cdot 3^2$ in the same way, then only *one* of these can be correct.

The first tools that we have to tell readers what operations to execute first are grouping symbols, like parentheses and brackets. If you *intend* to execute the multiplication first, then writing

$$(2 \cdot 3)^2$$

clearly tells your reader to do that. And if you *intend* to execute the power first, then writing

$$2 \cdot \left(3^2\right)$$

clearly tells your reader to do that.

To visualize the difference between $2 \cdot \left(3^2\right)$ or $(2 \cdot 3)^2$, consider these garden plots:

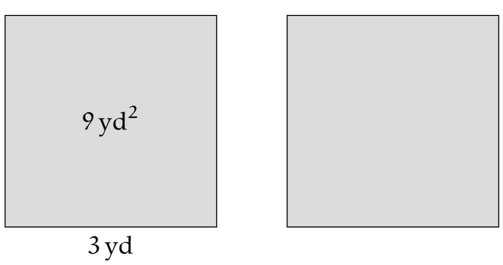


Figure A.5.2: 3 yd is squared, then doubled: $2 \cdot \left(3^2\right)$

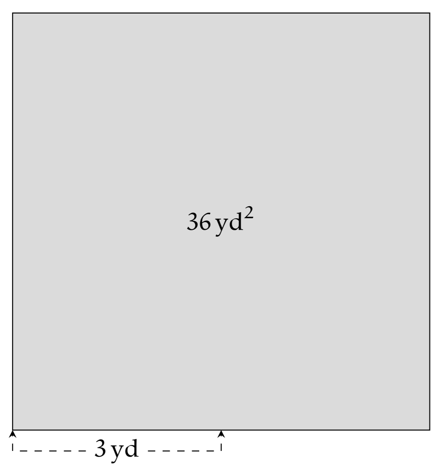


Figure A.5.3: 3 yd is doubled, then squared: $(2 \cdot 3)^2$

If we calculate 3^2, we have the area of one of the small square garden plots on the left. If we then double that, we have $2 \cdot \left(3^2\right)$, the area of the left garden plot.

But if we calculate $(2 \cdot 3)^2$, then first we are doubling 3. So we are calculating the area of a square garden plot whose sides are twice as long. We end up with the area of the garden plot on the right.

The point is that these amounts are different.

Checkpoint A.5.4 Calculate the value of $30 - ((2 + 3) \cdot 2)$, respecting the order that the grouping symbols are telling you to execute the arithmetic operations.

Explanation. The grouping symbols tell us what to work on first. In this exercise, we have grouping symbols within grouping symbols, so any operation in there (the addition) should be executed first:

$$\begin{aligned} 30 - ((2 + 3) \cdot 2) &= 30 - (5 \cdot 2) \\ &= 30 - 10 \\ &= 20 \end{aligned}$$

A.5.2 Beyond Grouping Symbols

If all math expressions used grouping symbols for each and every arithmetic operation, we wouldn't need to say anything more here. In fact, some computer systems work that way, *requiring* the use of grouping symbols all the time. But it is much more common to permit math expressions with no grouping symbols at all, like $5 + 3 \cdot 2$. Should the addition $5 + 3$ be executed first, or should the multiplication $3 \cdot 2$? We need what's known formally as the **order of operations** to tell us what to do.

The **order of operations** is nothing more than an agreement that we all have made to prioritize the arithmetic operations in a certain order.

List A.5.5: Order of Operations

(P)arentheses and other grouping symbols Grouping symbols should always direct you to the highest priority arithmetic first.

(E)xponentiation After grouping symbols, exponentiation has the highest priority. Execute any exponentiation before other arithmetic operations.

(M)ultiplication, (D)ivision, and Negation After all exponentiation has been executed, start executing multiplications, divisions, and negations. These things all have equal priority. If there are more than one of them in your expression, the highest priority is the one that is leftmost (which comes first as you read it).

(A)ddition and (S)ubtraction After all other arithmetic has been executed, these are all that is left. Addition and subtraction have equal priority. If there are more than one of them in your expression, the highest priority is the one that is leftmost (which comes first as you read it).

A common acronym to help you remember this order of operations is PEMDAS. There are a handful of mnemonic devices for remembering this ordering (such as Please Excuse My Dear Aunt Sally, People Eat More Donuts After School, etc.).

We'll start with a few examples that only invoke a few operations each.

Example A.5.6 Use the order of operations to simplify the following expressions.

a. $10 + 2 \cdot 3$. With this expression, we have the operations of addition and multiplication. The order of operations says the multiplication has higher priority, so execute that first:

$$\begin{aligned} 10 + 2 \cdot 3 &= 10 + \boxed{2 \cdot 3} \\ &= 10 + 6 \\ &= 16 \end{aligned}$$

b. $4 + 10 \div 2 - 1$. With this expression, we have addition, division, and subtraction. According to the order of operations, the first thing we need to do is divide. After that, we'll apply the addition and subtraction, working left to right:

$$\begin{aligned} 4 + 10 \div 2 - 1 &= 4 + \boxed{10 \div 2} - 1 \\ &= \boxed{4 + 5} - 1 \\ &= 9 - 1 \\ &= 8 \end{aligned}$$

c. $7 - 10 + 4$. This example *only* has subtraction and addition. While the acronym PEMDAS may mislead you to do addition before subtraction, remember that these operations have the same priority, and so we work left to right when executing them:

$$\begin{aligned} 7 - 10 + 4 &= \boxed{7 - 10} + 4 \\ &= -3 + 4 \\ &= 1 \end{aligned}$$

d. $20 \div 4 \cdot 7$. This expression has only division and multiplication. Again, remember that although PEMDAS shows "MD," the operations of multiplication and division have the same priority, so we'll apply them left to right:

$$\begin{aligned} 20 \div 4 \cdot 5 &= \boxed{20 \div 4} \cdot 5 \\ &= 5 \cdot 5 \\ &= 25 \end{aligned}$$

e. $(6 + 7)^2$. With this expression, we have addition inside a set of parentheses, and an exponent of 2 outside of that. We must compute the operation inside the parentheses first, and after that we'll apply the exponent:

$$\begin{aligned} (6 + 7)^2 &= (\boxed{6 + 7})^2 \\ &= 13^2 \\ &= 169 \end{aligned}$$

f. $4(2)^3$. This expression has multiplication and an exponent. There are parentheses too, but no operation inside them. Parentheses used in this manner make it clear that the 4 and 2 are separate numbers, not to be confused with 42. In other words, $4(2)^3$ and 42^3 mean very different things. Exponentiation has the higher priority, so we'll apply the exponent first, and then we'll multiply:

$$\begin{aligned} 4(2)^3 &= 4\boxed{(2)^3} \\ &= 4(8) \\ &= 32 \end{aligned}$$

Remark A.5.7 There are many different ways that we write multiplication. We can use the symbols $\cdot$, $\times$, and $*$ to denote multiplication. We can also use parentheses to denote multiplication, as we've seen in Example A.5.6, Item f. Once we start working with variables, there is even another way. No matter how multiplication is written, it does not change the priority that multiplication has in the order of operations.

Checkpoint A.5.8 Practice with order of operations. Simplify this expression one step at a time, using the order of operations.

$5 - 3(7 - 4)^2$

Explanation.

$$\begin{aligned} 5 - 3(\boxed{7-4})^2 &= 5 - 3\boxed{(3)^2} \\ &= 5 - \boxed{3(9)} \\ &= 5 - 27 \\ &= -22 \end{aligned}$$

A.5.3 Absolute Value Bars, Radicals, and Fraction Bars are Grouping Symbols

When we first discussed grouping symbols, we only mentioned parentheses and brackets. Each of the following operations has an *implied* grouping symbol aside from parentheses and brackets.

Absolute Value Bars The absolute value bars, as in $|2 - 5|$, group the expression inside it just like a set of parentheses would.

Radicals The same is true of the radical symbol—everything inside the radical is grouped, as with $\sqrt{12 - 3}$.

Fraction Bars With a horizontal division bar, the numerator is treated as one group and the denominator as another, as with $\frac{2+3}{5-2}$.

We don't *need* parentheses for these three things since the absolute value bars, radical, and horizontal division bar each denote this grouping on their own. As far as priority in the order of operations goes, it's important to remember that these work just like our most familiar grouping symbols, parentheses.

With absolute value bars and radicals, these grouping symbols also *do* something to what's inside (but only *after* the operations inside have been executed). For example, $|-2| = 2$, and $\sqrt{9} = 3$.

Example A.5.9 Use the order of operations to simplify the following expressions.

a. $4 - 3\,|5 - 7|$. For this expression, we'll treat the absolute value bars just like we treat parentheses. This implies we'll simplify what's inside the bars first, and then compute the absolute value. After that, we'll multiply and then finally subtract:

$$\begin{aligned}
4-3|5-7| &= 4-3\left|\boxed{5-7}\right| \\
&= 4-3\boxed{|-2|} \\
&= 4-\boxed{3(2)} \\
&= 4-6 \\
&= -2
\end{aligned}$$

We may not do $4-3=1$ first, because 3 is connected to the absolute value bars by multiplication (although implicitly), which has a higher order than subtraction.

b. $8-\sqrt{5^2-8\cdot 2}$. This expression has an expression inside the radical of $5^2-8\cdot 2$. We'll treat this radical like we would a set of parentheses, and simplify that internal expression first. We'll then apply the square root, and then our last step will be to subtract that expression from 8:

$$\begin{aligned}
8-\sqrt{5^2-8\cdot 2} &= 8-\sqrt{\boxed{5^2}-8\cdot 2} \\
&= 8-\sqrt{25-\boxed{8\cdot 2}} \\
&= 8-\sqrt{\boxed{25-16}} \\
&= 8-\boxed{\sqrt{9}} \\
&= 8-3 \\
&= 5
\end{aligned}$$

c. $\dfrac{2^4+3\cdot 6}{5-18\div 2}$. For this expression, the first thing we want to do is to recognize that the main fraction bar serves as a separator that groups the numerator and groups the denominator. Another way this expression could be written is $(2^4+3\cdot 6)\div(15-18\div 2)$. This implies we'll simplify the numerator and denominator separately according to the order of operations (since there are implicit parentheses around each of these). As a final step we'll simplify the resulting fraction (which is division).

$$\begin{aligned}
\frac{2^4+3\cdot 6}{5-18\div 2} &= \frac{\boxed{2^4}+3\cdot 6}{5-\boxed{18\div 2}} \\
&= \frac{16+\boxed{3\cdot 6}}{\boxed{5-9}} \\
&= \frac{\boxed{16+18}}{-4} \\
&= \frac{34}{-4} \\
&= -\frac{17}{2}
\end{aligned}$$

Checkpoint A.5.10 More Practice with Order of Operations. Use the order of operations to evaluate

$$\frac{6+3\left|9-10\right|}{\sqrt{3+18\div 3}}.$$

Explanation. We start by identifying the innermost, highest priority operations:

$$\begin{aligned}
\frac{6+3\left|9-10\right|}{\sqrt{3+18\div 3}} &= \frac{6+3\boxed{\left|\boxed{9-10}\right|}}{\sqrt{3+\boxed{18\div 3}}}\\
&= \frac{6+3\boxed{\left|-1\right|}}{\sqrt{\boxed{3+6}}}\\
&= \frac{6+\boxed{3(1)}}{\boxed{\sqrt{9}}}\\
&= \frac{\boxed{6+3}}{3}\\
&= \frac{9}{3} = 3
\end{aligned}$$

A.5.4 Negation and Distinguishing $(-a)^m$ from $-a^m$

We noted in the order of operations that using the negative sign to negate a number has the same priority as multiplication and division. To understand why this is, observe that $-1 \cdot 23 = -23$, just for one example. So negating 23 gives the same result as multiplying 23 by -1. For this reason, negation has the same priority in the order of operations as multiplication. This can be a source of misunderstandings.

How would you write a math expression that takes the number -4 and squares it?

$$-4^2? \qquad (-4)^2? \qquad \text{it doesn't matter?}$$

It *does* matter. The second option, $(-4)^2$ is squaring the number -4. The parentheses emphasize this.

But the expression -4^2 is different. There are two actions in this expression: a negation and and exponentiation. According to the order of operations, the exponentiation has higher priority than the negation, so the exponent of 2 in -4^2 applies to the 4 *before* the negative sign (multiplication by -1) is taken into account.

$$\begin{aligned}
-4^2 &= -\boxed{4^2}\\
&= -16
\end{aligned}$$

and this is not the same as $(-4)^2$, which is *positive* 16.

Warning A.5.11 Negative Numbers Raised to Powers. You may find yourself needing to raise a negative number to a power, and using a calculator to do the work for you. If you do not understand the issue described here, then you may get incorrect results.

- For example, entering `-4^2` into a calculator will result in -16, the negative of 4^2.
- But entering `(-4)^2` into a calculator will result in 16, the square of -4.

Go ahead and try entering these into your own calculator.

Checkpoint A.5.12 Negating and Raising to Powers. Compute the following:

a. $-3^4 = \boxed{\qquad\qquad}$ and $(-3)^4 = \boxed{\qquad\qquad}$

b. $-4^3 = \boxed{\qquad\qquad}$ and $(-4)^3 = \boxed{\qquad\qquad}$

c. $-1.1^2 = \boxed{\qquad\qquad}$ and $(-1.1)^2 = \boxed{\qquad\qquad}$

Explanation. In each part, the first expression asks you to exponentiate and then negate the result. The second expression has a negative number raised to a power. So the answers are:

a. $-3^4 = -81$ and $(-3)^4 = 81$

b. $-4^3 = -64$ and $(-4)^3 = -64$

c. $-1.1^2 = -1.21$ and $(-1.1)^2 = 1.21$

Remark A.5.13 You might observe in the previous example that there is no difference between -4^3 and $(-4)^3$. It's true that the results are the same, -64, but the two expressions still do say different things. With -4^3, you raise to a power first, then negate. With $(-4)^3$, you negate first, then raise to a power.

As was discussed in Subsection A.1.5, if the base of a power is negative, then whether or not the result is positive or negative depends on if the exponent is even or odd. It depends on whether or not the factors can all be paired up to "cancel" negative signs, or if there will be a lone factor left by itself.

A.5.5 More Examples

Here are some example exercises that involve applying the order of operations to more complicated expressions. Try these exercises and read the steps given in each solution.

Example A.5.14 Simplify $10 - 4(5 - 7)^3$.

Explanation. For the expression $10-4(5-7)^3$, we have simplify what's inside parentheses first, then we'll apply the exponent, then multiply, and finally subtract:

$$\begin{aligned}10 - 4(5-7)^3 &= 10 - 4(\boxed{5-7})^3\\ &= 10 - 4\boxed{(-2)^3}\\ &= 10 - \boxed{4(-8)}\\ &= 10 - (-32)\\ &= 10 + 32\\ &= 42\end{aligned}$$

Checkpoint A.5.15 Simplify $24 \div (15 \div 3 + 1) + 2$.

Explanation. With the expression $24 \div (15 \div 3 + 1) + 2$, we'll simplify what's inside the parentheses according to the order of operations, and then take 24 divided by that expression as our last step:

$$\begin{aligned} 24 \div (15 \div 3 + 1) + 2 &= 24 \div (\boxed{15 \div 3} + 1) + 2 \\ &= 24 \div (\boxed{5 + 1}) + 2 \\ &= \boxed{24 \div 6} + 2 \\ &= 4 + 2 \\ &= 6 \end{aligned}$$

Example A.5.16 Simplify $6 - (-8)^2 \div 4 + 1$.

Explanation. To simplify $6 - (-8)^2 \div 4 + 1$, we'll first apply the exponent of 2 to -8, making sure to recall that $(-8)^2 = 64$. After this, we'll apply division. As a final step, we'll be have subtraction and addition, which we'll apply working left-to-right:

$$\begin{aligned} 6 - (-8)^2 \div 4 + 1 &= 6 - \boxed{(-8)^2} \div 4 + 1 \\ &= 6 - \boxed{(64) \div 4} + 1 \\ &= \boxed{6 - 16} + 1 \\ &= -10 + 1 \\ &= -9 \end{aligned}$$

Checkpoint A.5.17 Simplify $(20 - 4^2) \div (4 - 6)^3$.

Explanation. The expression $(20 - 4^2) \div (4 - 6)^3$ has two sets of parentheses, so our first step will be to simplify what's inside each of those first according to the order of operations. Once we've done that, we'll apply the exponent and then finally divide:

$$\begin{aligned} (20 - 4^2) \div (4 - 6)^3 &= (20 - \boxed{4^2}) \div (4 - 6)^3 \\ &= (\boxed{20 - 16}) \div (4 - 6)^3 \\ &= 4 \div (\boxed{4 - 6})^3 \\ &= 4 \div \boxed{(-2)^3} \\ &= 4 \div (-8) \\ &= \frac{4}{-8} \\ &= \frac{1}{-2} \\ &= -\frac{1}{2} \end{aligned}$$

Checkpoint A.5.18 Simplify $\dfrac{2\,|9 - 15| + 1}{\sqrt{(-5)^2 + 12^2}}$.

Explanation. To simplify this expression, the first thing we want to recognize is the role of the main fraction bar, which groups the numerator and denominator. This implies we'll simplify the numerator and denominator separately according to the order of operations, and then reduce the fraction that results:

$$\begin{aligned}\frac{2\,|9-15|+1}{\sqrt{(-5)^2+12^2}} &= \frac{2\left|\boxed{9-15}\right|+1}{\sqrt{\boxed{(-5)^2}+12^2}}\\ &= \frac{2\boxed{|-6|}+1}{\sqrt{25+\boxed{12^2}}}\\ &= \frac{\boxed{2(6)}+1}{\sqrt{\boxed{25+144}}}\\ &= \frac{\boxed{12+1}}{\boxed{\sqrt{169}}}\\ &= \frac{13}{13}\\ &= 1\end{aligned}$$

A.5.6 Exercises

Review and Warmup

1. Multiply the following.

a. $(-9)\cdot(-1)$

b. $(-7)\cdot 3$

c. $9\cdot(-4)$

d. $(-8)\cdot 0$

2. Multiply the following.

a. $(-9)\cdot(-2)$

b. $(-5)\cdot 7$

c. $9\cdot(-7)$

d. $(-7)\cdot 0$

3. Multiply the following.

a. $(-1)\cdot(-6)\cdot(-2)$

b. $5\cdot(-6)\cdot(-3)$

c. $(-90)\cdot(-68)\cdot 0$

4. Multiply the following.

a. $(-1)\cdot(-4)\cdot(-4)$

b. $4\cdot(-6)\cdot(-5)$

c. $(-88)\cdot(-56)\cdot 0$

5. a. Compute -8^2.

b. Calculate the power -3^2.

c. Find -6^2.

d. Calculate $(-1)^{19}$.

6. a. Compute -1^{40}.

b. Calculate the power $(-2)^4$.

c. Find -4^4.

d. Calculate -8^2.

Order of Operations Skills Evaluate the following.

7. $6+7(3)$

8. $2+4(4)$

9. $5(3+3)$

10. $4(4+5)$

11. $(3\cdot 2)^2$

12. $(4\cdot 4)^2$

13. $5\cdot 3^2$

14. $5\cdot 2^3$

15. $(12-2)\cdot 5$

16. $(8-2)\cdot 3$

17. $14-3\cdot 2$

18. $17-3\cdot 5$

19. $6+4\cdot 8$
20. $7+2\cdot 6$
21. $4-5\cdot 8$
22. $5-3\cdot 10$
23. $5-2(-7)$
24. $1-5(-9)$
25. $-[1-(2-5)^2]$
26. $-[10-(2-7)^2]$
27. $7-5[10-(1+4\cdot 3)]$
28. $7-3[4-(8+4\cdot 5)]$
29. $6+3(59-2\cdot 3^3)$
30. $6+2(110-4\cdot 3^3)$
31. $-5[4-(5-2\cdot 5)^2]$
32. $-5[10-(3-4\cdot 2)^2]$
33. $42-3[4^2-(5-1)]$
34. $74-2[6^2-(4-2)]$
35. $(9-2)^2+3(9-2^2)$
36. $(13-3)^2+4(13-3^2)$
37. $(4\cdot 3)^2-4\cdot 3^2$
38. $(4\cdot 5)^2-4\cdot 5^2$
39. $8\cdot 4^2-125\div 5^2\cdot 4+5$
40. $9\cdot 2^2-36\div 3^2\cdot 4+9$
41. $5(9-4)^2-5(9-4^2)$
42. $6(6-2)^2-6(6-2^2)$
43. $\dfrac{5+1}{3-2}$
44. $\dfrac{4+6}{7-2}$
45. $\dfrac{8^2-3^2}{3+8}$
46. $\dfrac{8^2-2^2}{7+3}$
47. $\dfrac{27-(-4)^3}{3-10}$
48. $\dfrac{27-(-2)^3}{7-12}$
49. $\dfrac{(-2)\cdot(-9)-(-10)\cdot 9}{(-6)^2+(-38)}$
50. $\dfrac{(-2)\cdot(-3)-(-6)\cdot 7}{(-6)^2+(-38)}$
51. $-|1-5|$
52. $-|1-2|$
53. $2-7|4-9|+3$
54. $3-6|1-6|+3$
55. $-5^2-|8\cdot(-5)|$
56. $-4^2-|6\cdot(-9)|$
57. $8-4\left|-5+(4-5)^3\right|$
58. $9-8\left|-1+(3-5)^3\right|$
59. $\dfrac{\left|27+(-4)^3\right|}{-1}$
60. $\dfrac{\left|1+(-2)^3\right|}{-1}$
61. $\left|\dfrac{1+(-4)^3}{-3}\right|$
62. $\left|\dfrac{1+(-4)^3}{-3}\right|$
63. $\dfrac{-3|5-12|}{6-(-3)^2}$
64. $\dfrac{-3|1-7|}{7-(-1)^2}$
65. $\dfrac{8}{7}+8\cdot\dfrac{3}{7}$
66. $\dfrac{8}{3}+2\cdot\dfrac{2}{3}$
67. $\left(\dfrac{9}{10}-\dfrac{7}{50}\right)-2\left(\dfrac{7}{50}-\dfrac{9}{10}\right)$
68. $\left(\dfrac{3}{2}-\dfrac{1}{8}\right)-2\left(\dfrac{1}{8}-\dfrac{3}{2}\right)$
69. $\left|\dfrac{3}{2}-\dfrac{5}{6}\right|-2\left|\dfrac{5}{6}-\dfrac{3}{2}\right|$
70. $\left|\dfrac{3}{8}-\dfrac{9}{16}\right|-2\left|\dfrac{9}{16}-\dfrac{3}{8}\right|$
71. $\dfrac{1}{5}+4\left(\dfrac{3}{5}\right)^2$
72. $\dfrac{3}{5}+8\left(\dfrac{2}{5}\right)^2$
73. $\dfrac{3}{4}+\dfrac{1}{3}\div\dfrac{1}{3}-\dfrac{4}{5}$
74. $\dfrac{4}{3}+\dfrac{3}{5}\div\dfrac{1}{5}-\dfrac{5}{2}$
75. $4\sqrt{94-93}$
76. $5\sqrt{36+28}$
77. $5\sqrt{-8+8\cdot 3}$
78. $2\sqrt{73+3\cdot 9}$
79. $7-2\sqrt{61-25}$
80. $4-3\sqrt{2+7}$
81. $\sqrt{36}-3\sqrt{-9+130}$
82. $\sqrt{49}-5\sqrt{6+58}$
83. $\sqrt{33+4^2}$
84. $\sqrt{-36+10^2}$
85. $\sqrt{8^2+6^2}$
86. $\sqrt{9^2+12^2}$
87. $\dfrac{\sqrt{81}+3}{\sqrt{81}-3}$
88. $\dfrac{\sqrt{4}+6}{\sqrt{4}-6}$
89. $\dfrac{\sqrt{-8+6\cdot 4}+|-16-7|}{-67-(-4)^3}$
90. $\dfrac{\sqrt{1+8\cdot 3}+|-18-7|}{-2-(-2)^3}$
91. $4[17-5(3+8)]$
92. $4[15-3(1+8)]$
93. $-8^2-9[5-(6-4^3)]$
94. $-10^2-5[9-(4-4^3)]$

Challenge

95. In this challenge, your job is to create expressions, using addition, subtraction, multiplication, and parentheses. You may use the numbers, $1, 2, 3$, and 4 in your expression, using each number only once. For example, you could make the expression: $1 + 2 \cdot 3 - 4$.

a. The greatest value that it is possible to create under these conditions is $\boxed{}$.

b. The least value that it is possible to create under these conditions is $\boxed{}$.

A.6 Set Notation and Types of Numbers

When we talk about *how many* or *how much* of something we have, it often makes sense to use different types of numbers. For example, if we are counting dogs in a shelter, the possibilities are only $0, 1, 2, \ldots$. (It would be difficult to have $\frac{1}{2}$ of a dog.) On the other hand if you were weighing a dog in pounds, it doesn't make sense to only allow yourself to work with whole numbers. The dog might weigh something like 28.35 pounds. These examples highlight how certain kinds of numbers are appropriate for certain situations. We'll classify various types of numbers in this section.

A.6.1 Set Notation

What is the mathematical difference between these three "lists?"

$$28, 31, 30 \qquad \{28, 31, 30\} \qquad (28, 31, 30)$$

To a mathematician, the last one, $(28, 31, 30)$ is an *ordered* triple. What matters is not merely the three numbers, but *also* the order in which they come. The ordered triple $(28, 31, 30)$ is not the same as $(30, 31, 28)$; they have the same numbers in them, but the order has changed. For some context, February has 28 days; *then* March has 31 days; *then* April has 30 days. The order of the three numbers is meaningful in that context.

With curly braces and $\{28, 31, 30\}$, a mathematician sees a collection of numbers and does not particularly care in which order they are written. Such a collection is called a **set**. All that matters is that these numbers are part of a collection. They've been *written* in some particular order because that's necessary to write them down. But you might as well have put the three numbers in a bag and shaken up the bag. For some context, maybe your favorite three NBA players have jersey numbers 30, 31, and 28, and you like them all equally well. It doesn't really matter what order you use to list them.

So we can say:

$$\{28, 31, 30\} = \{30, 31, 28\} \qquad\qquad (28, 31, 30) \neq (30, 31, 28)$$

What about just writing $28, 31, 30$? This list of three numbers is ambiguous. Without the curly braces or parentheses, it's unclear to a reader if the order is important. **Set notation** is the use of curly braces to surround a list/collection of numbers, and we will use set notation frequently in this section.

Checkpoint A.6.2 Set Notation. Practice using (and not using) set notation.

According to Google, the three most common error codes from visiting a web site are `403`, `404`, and `500`.

a. Without knowing which error code is most common, express this set mathematically.

b. Error code `500` is the most common. Error code `403` is the least common of these three. And that leaves `404` in the middle. Express the error codes in a mathematical way that appreciates how frequently they happen, from most often to least often.

Explanation.

a. Since we only have to describe a collection of three numbers and their order doesn't matter, we can write {403,404,500}.

b. Now we must describe the same three numbers and we want readers to know that the order we are writing the numbers matters. We can write (500,404,403).

A.6.2 Different Number Sets

In the introduction, we mentioned how different sets of numbers are appropriate for different situations. Here are the basic sets of numbers that are used in basic algebra.

Natural Numbers When we count, we begin: $1, 2, 3, \ldots$ and continue on in that pattern. These numbers are known as **natural numbers**.

$\mathbb{N} = \{1, 2, 3, \ldots\}$

Whole Numbers If we include zero, then we have the set of **whole numbers**.

$\{0, 1, 2, 3, \ldots\}$ has no standard symbol, but some options are $\mathbb{N}_0$, $\mathbb{N} \cup \{0\}$, and $\mathbb{Z}_{\geq 0}$.

Integers If we include the negatives of whole numbers, then we have the set of **integers**.

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

A $\mathbb{Z}$ is used because one word in German for "numbers" is "Zahlen."

Rational Numbers A **rational number** is any number that *can* be written as a fraction of integers, where the denominator is nonzero. Alternatively, a **rational number** is any number that *can* be written with a decimal that terminates or that repeats.

$\mathbb{Q} = \left\{0, 1, -1, 2, \frac{1}{2}, -\frac{1}{2}, -2, 3, \frac{1}{3}, -\frac{1}{3}, -3, \frac{3}{2}, \frac{2}{3} \ldots\right\}$

$\mathbb{Q} = \left\{0, 1, -1, 2, 0.5, -0.5, -2, 3, 0.\overline{3}, -0.\overline{3}, -3, 1.5, 0.\overline{6} \ldots\right\}$

A $\mathbb{Q}$ is used because fractions are *q*uotients of integers.

Irrational Numbers Any number that *cannot* be written as a fraction of integers belongs to the set of **irrational numbers**. Another way to say this is that any number whose decimal places goes on forever without repeating is an **irrational number**. Some examples include $\pi \approx 3.1415926\ldots$, $\sqrt{15} \approx 3.87298\ldots$, $e \approx 2.71828\ldots$

There is no standard symbol for the set of irrational numbers.

Real Numbers Any number that can be marked somewhere on a number line is a **real number**. Real numbers might be the only numbers you are familiar with. For a number to *not* be real, you have to start considering things called *complex numbers*, which are not our concern right now.

The set of real numbers can be denoted with $\mathbb{R}$ for short.

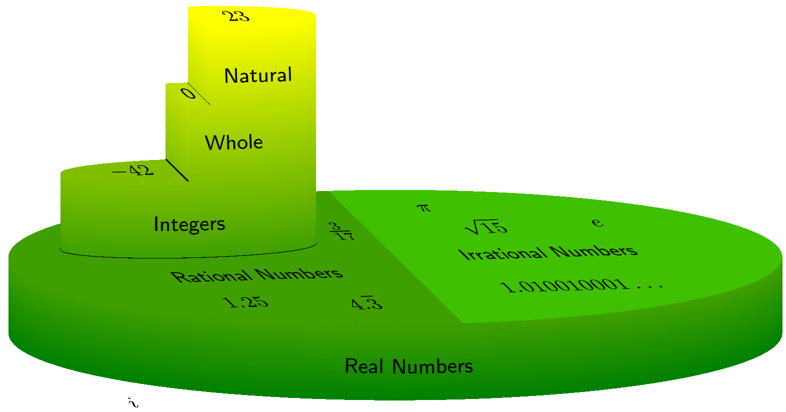


Figure A.6.3: Types of Numbers

Warning A.6.4 Rational Numbers in Other Forms. Any number that *can* be written as a ratio of integers is rational, even if it's not written that way at first. For example, these numbers might not look rational to you at first glance: -4, $\sqrt{9}$, 0π, and $\sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2}$. But they are all rational, because they can respectively be written as $\frac{-4}{1}$, $\frac{3}{1}$, $\frac{0}{1}$, and $\frac{1}{1}$.

Example A.6.5 Determine If Numbers Are This Type or That Type. Determine which numbers from the set $\left\{-102, -7.25, 0, \frac{\pi}{4}, 2, \frac{10}{3}, \sqrt{19}, \sqrt{25}, 10.\overline{7}\right\}$ are natural numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers.

Explanation. All of these numbers are real numbers, because all of these numbers can be positioned on the real number line.

Each real number is either rational or irrational, and not both. -102, -7.25, 0, and 2 are rational because we can see directly that their decimal expressions terminate. $10.\overline{7}$ is also rational, because its decimal expression repeats. $\frac{10}{3}$ is rational because it is a ratio of integers. And last but not least, $\sqrt{25}$ is rational, because that's the same thing as 5.

This leaves only $\frac{\pi}{4}$ and $\sqrt{19}$ as irrational numbers. Their decimal expressions go on forever without entering a repetitive cycle.

Only -102, 0, 2, and $\sqrt{25}$ (which is really 5) are integers.

Of these, only 0, 2, and $\sqrt{25}$ are whole numbers, because whole numbers excludes the negative integers.

Of these, only 2 and $\sqrt{25}$ are natural numbers, because the natural numbers exclude 0.

Checkpoint A.6.6

a. Give an example of a whole number that is not an integer.

b. Give an example of an integer that is not a whole number.

c. Give an example of a rational number that is not an integer.

d. Give an example of a irrational number.

e. Give an example of a irrational number that is also an integer.

Explanation.

a. Since all whole numbers belong to integers, we cannot write any whole number which is not an integer.

Type `DNE` (does not exist) for this question.

b. Any negative integer, like -1, is not a whole number, but is an integer.

c. Any terminating decimal, like 1.2, is a rational number, but is not an integer.

d. π is the easiest number to remember as an irrational number. Another constant worth knowing is $e \approx 2.718$. Finally, the square root of most integers are irrational, like $\sqrt{2}$ and $\sqrt{3}$.

e. All irrational numbers are non-repeating and non-terminating decimals. No irrational numbers are integers.

Checkpoint A.6.7 In the introduction, we mentioned that the different types of numbers are appropriate in different situation. Which number set do you think is most appropriate in each of the following situations?

a. The number of people in a math class that play the ukulele.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

b. The hypotenuse's length in a given right triangle.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

c. The proportion of people in a math class that have a cat.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

d. The number of people in the room with you who have the same birthday as you.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

e. The total revenue (in dollars) generated for ticket sales at a Timbers soccer game.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

Explanation.

a. The number of people who play the ukulele could be $0, 1, 2, \ldots$, so the whole numbers are the appropriate set.

b. A hypotenuse's length could be 1, 1.2, $\sqrt{2}$ (which is irrational), or any other positive number. So the real numbers are the appropriate set.

c. This proportion will be a ratio of integers, as both the total number of people in the class and the number of people who have a cat are integers. So the rational numbers are the appropriate set.

d. We know that the number of people must be a counting number, and since *you* are in the room with yourself, there is at least one person in that room with your birthday. So the natural numbers are the appropriate set.

e. The total revenue will be some number of dollars and cents, such as \$631,897.15, which is a terminating decimal and thus a rational number. So the rational numbers are the appropriate set.

A.6.3 Converting Repeating Decimals to Fractions

We have learned that a terminating decimal number is a rational number. It's easy to convert a terminating decimal number into a fraction of integers: you just need to multiply and divide by one of the numbers in the set $\{10, 100, 1000, \ldots\}$. For example, when we say the number 0.123 out loud, we say "one hundred and twenty-three thousandths." While that's a lot to say, it makes it obvious that this number can be written as a ratio:

$$0.123 = \frac{123}{1000}.$$

Similarly,

$$21.28 = \frac{2128}{100} = \frac{532 \cdot 4}{25 \cdot 4} = \frac{532}{25},$$

demonstrating how *any* terminating decimal can be written as a fraction.

Repeating decimals can also be written as a fraction. To understand how, use a calculator to find the decimal for, say, $\frac{73}{99}$ and $\frac{189}{999}$ You will find that

$$\frac{73}{99} = 0.73737373\ldots = 0.\overline{73} \qquad \frac{189}{999} = 0.189189189\ldots = 0.\overline{189}.$$

The pattern is that dividing a number by a number from $\{9, 99, 999, \ldots\}$ with the same number of digits will create a repeating decimal that starts as "0." and then repeats the numerator. We can use this observation to reverse engineer some fractions from repeating decimals.

Checkpoint A.6.8

a. Write the rational number $0.772772772\ldots$ as a fraction.

b. Write the rational number $0.69696969\ldots$ as a fraction.

Explanation.

a. The *three*-digit number 772 repeats after the decimal. So we will make use of the *three*-digit denominator 999. And we have $\frac{772}{999}$.

b. The *two*-digit number 69 repeats after the decimal. So we will make use of the *two*-digit denominator 99. And we have $\frac{69}{99}$. But this fraction can be reduced to $\frac{23}{33}$.

Converting a repeating decimal to a fraction is not always quite this straightforward. There are complications if the number takes a few digits before it begins repeating. For your interest, here is one example on how to do that.

Example A.6.9 Can we convert the repeating decimal $9.134343434\ldots = 9.1\overline{34}$ to a fraction? The trick is to separate its terminating part from its repeating part, like this:

$$9.1 + 0.034343434\ldots.$$

Now note that the terminating part is $\frac{91}{10}$, and the repeating part is almost like our earlier examples, except it has an extra 0 right after the decimal. So we have:

$$\frac{91}{10} + \frac{1}{10} \cdot 0.34343434\ldots.$$

With what we learned in the earlier examples and basic fraction arithmetic, we can continue:

$$9.134343434\ldots = \frac{91}{10} + \frac{1}{10} \cdot 0.34343434\ldots$$

$$\begin{aligned}
&= \frac{91}{10} + \frac{1}{10} \cdot \frac{34}{99} \\
&= \frac{91}{10} + \frac{34}{990} \\
&= \frac{91 \cdot 99}{10 \cdot 99} + \frac{34}{990} \\
&= \frac{9009}{990} + \frac{34}{990} = \frac{9043}{990}
\end{aligned}$$

Check that this is right by entering $\frac{9043}{990}$ into a calculator and seeing if it returns the decimal we started with, 9.134343434 . . ..

A.6.4 Exercises

Review and Warmup Write the decimal number as a fraction.

1. 0.55 **2.** 0.65 **3.** 7.65 **4.** 8.35 **5.** 0.992 **6.** 0.164

Write the fraction as a decimal number.

7. a. $\frac{1}{5}$
b. $\frac{18}{25}$

8. a. $\frac{2}{5}$
b. $\frac{21}{25}$

Write the mixed number as a decimal number.

9. a. $6\frac{5}{8} =$ ☐
b. $7\frac{9}{16} =$ ☐

10. a. $3\frac{5}{16} =$ ☐
b. $9\frac{3}{5} =$ ☐

Set Notation

11. There are two numbers that you can square to get 36. Express this collection of two numbers using set notation.

12. There are four positive, even, one-digit numbers. Express this collection of four numbers using set notation.

13. There are six two-digit perfect square numbers. Express this collection of six numbers using set notation.

14. There is a set of three small positive integers where you can square all three numbers, then add the results, and get 77. Express this collection of three numbers using set notation.

Types of Numbers Which of the following are whole numbers? There may be more than one correct answer.

15.
☐ -3 ☐ $\sqrt{4}$ ☐ $\frac{8}{29}$
☐ $-9.101001000100001\ldots$ ☐ $2.2\overline{39}$
☐ 48 ☐ $\sqrt{6}$ ☐ -61751

16.
☐ 3.419 ☐ -8 ☐ $\sqrt{6}$
☐ 60991 ☐ $6.8\overline{16}$ ☐ -52858
☐ 14 ☐ π

Which of the following are integers? There may be more than one correct answer.

17.

☐ 79 ☐ -71143 ☐ $-5.101001000100001\ldots$ ☐ $1.8\overline{37}$ ☐ -4 ☐ $\frac{7}{73}$ ☐ $\sqrt{10}$ ☐ 2738

18.

☐ -8 ☐ -35072 ☐ π ☐ $-\frac{10}{79}$ ☐ 44 ☐ $3.101001000100001\ldots$ ☐ 0 ☐ -3.877

Which of the following are rational numbers? There may be more than one correct answer.

19.

☐ -3 ☐ π ☐ 10 ☐ $\sqrt{10}$ ☐ $\frac{7}{82}$ ☐ $\sqrt{25}$ ☐ -26179 ☐ -6.897

20.

☐ π ☐ $\sqrt{11}$ ☐ -8 ☐ $\frac{6}{73}$ ☐ 74 ☐ 27783 ☐ -17285 ☐ $6.5\overline{40}$

Which of the following are irrational numbers? There may be more than one correct answer.

21.

☐ -7.955 ☐ $\frac{4}{77}$ ☐ -3 ☐ 0 ☐ $\sqrt{13}$ ☐ -8392 ☐ 40 ☐ π

22.

☐ -8 ☐ -9.315 ☐ π ☐ $\sqrt{49}$ ☐ 5 ☐ $\sqrt{11}$ ☐ -99499 ☐ $8.2\overline{60}$

Which of the following are real numbers? There may be more than one correct answer.

23.

☐ 52827 ☐ $\sqrt{6}$ ☐ 70 ☐ π ☐ -1.861 ☐ -90606 ☐ $-5.3\overline{95}$ ☐ -3

24.

☐ 0 ☐ -5.939 ☐ $7.2\overline{96}$ ☐ 36 ☐ $\sqrt{2}$ ☐ $7.101001000100001\ldots$ ☐ -86767 ☐ -8

Determine the validity of each statement by selecting True or False.

25.

(a) The number $\sqrt{3}$ is rational

(b) The number $\frac{-\sqrt{11}}{37\sqrt{11}}$ is irrational

(c) The number 0 is a natural number

(d) The number 0 is an integer

(e) The number 5 is an integer, but not a whole number

26.

(a) The number $\sqrt{4}$ is a real number, but not a rational number

(b) The number 0.78314404004000400004... is rational

(c) The number $\sqrt{2^2}$ is a real number, but not a rational number

(d) The number 0.900900900900900... is rational

(e) The number $\frac{23}{17}$ is rational, but not a natural number

27. In each situation, which number set do you think is most appropriate?

a. The number of dogs a student has owned throughout their lifetime.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

b. The difference between the projected annual expenditures and the actual annual expenditures for a given company.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

c. The length around swimming pool in the shape of a half circle with radius 10 ft.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

d. The proportion of students at a college who own a car.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

e. The width of a sheet of paper, in inches.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

f. The number of people eating in a non-empty restaurant.

This number is best considered as a (☐ natural number ☐ whole number ☐ integer ☐ rational number ☐ irrational number ☐ real number) .

28.

a. Give an example of a whole number that is not an integer.

b. Give an example of an integer that is not a whole number.

c. Give an example of a rational number that is not an integer.

d. Give an example of a irrational number.

e. Give an example of a irrational number that is also an integer.

Writing Decimals as Fractions

29. Write the rational number 6.16 as a fraction.

30. Write the rational number 74.973 as a fraction.

31. Write the rational number $0.\overline{79} = 0.7979\ldots$ as a fraction.

32. Write the rational number $0.\overline{936} = 0.936936\ldots$ as a fraction.

33. Write the rational number $7.9\overline{13} = 7.91313\ldots$ as a fraction.

34. Write the rational number $2.6\overline{218} = 2.6218218\ldots$ as a fraction.

Challenge

35. Imagine making up a number with the following pattern. After the decimal point, write the natural numbers 1, 2, 3, 4, 5, etc. The decimal digits will extend forever with this pattern: $0.12345\ldots$.Is the number a rational number or an irrational number? (☐ rational ☐ irrational)

Appendix B

Unit Conversions

Units of Length in the US/Imperial System	Units of Length in the Metric System	System to System Length Conversions
1 foot (ft) $= 12$ inches (in)	1 meter (m) $= 1000$ millimeters (mm)	1 inch (in) $= 2.54$ centimeters (cm)
1 yard (yd) $= 3$ feet (ft)	1 meter (m) $= 100$ centimeters (cm)	1 meter (m) ≈ 3.281 feet (ft)
1 yard (yd) $= 36$ inches (in)	1 meter (m) $= 10$ decimeters (dm)	1 meter (m) ≈ 1.094 yard (yd)
1 mile (mi) $= 5280$ feet (ft)	1 dekameter (dam) $= 10$ meters (m)	1 mile (mi) ≈ 1.609 kilometer (km)
	1 hectometer (hm) $= 100$ meters (m)	
	1 kilometer (km) $= 1000$ meters (m)	

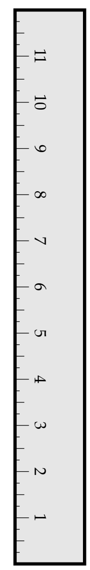
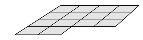
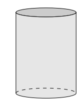


Table B.0.1: Length Unit Conversion Factors



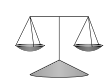
Units of Area in the US/Imperial System	Units of Area in the Metric System	System to System Area Conversions
1 acre $= 43560$ square feet (ft^2)	1 hectare (ha) $= 10000$ square meters (m^2)	1 hectare (ha) ≈ 2.471 acres
640 acres $= 1$ square mile (mi^2)	100 hectares (ha) $= 1$ square kilometer (km^2)	

Table B.0.2: Area Unit Conversion Factors



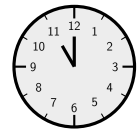
Units of Volume in the US/Imperial System	Units of Volume in the Metric System	System to System Volume Conversions
1 tablespoon (tbsp) $= 3$ teaspoon (tsp)	1 cubic centimeter (cc) $= 1$ cubic centimeter (cm^3)	1 cubic inch (in^3) ≈ 16.39 milliliters (mL)
1 fluid ounce (fl oz) $= 2$ tablespoons (tbsp)	1 milliliter (mL) $= 1$ cubic centimeter (cm^3)	1 fluid ounce (fl oz) ≈ 29.57 milliliters (mL)
1 cup (c) $= 8$ fluid ounces (fl oz)	1 liter (L) $= 1000$ milliliters (mL)	1 liter (L) ≈ 1.057 quarts (qt)
1 pint (pt) $= 2$ cups (c)	1 liter (L) $= 1000$ cubic centimeters (cm^3)	1 gallon (gal) ≈ 3.785 liters (L)
1 quart (qt) $= 2$ pints (pt)		
1 gallon (gal) $= 4$ quarts (qt)		
1 gallon (gal) $= 231$ cubic inches (in^3)		

Table B.0.3: Volume Unit Conversion Factors



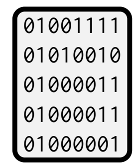
Units of Mass/Weight in the US/Imperial System	Units of Mass/Weight in the Metric System	System to System Mass/Weight Conversions
1 pound (lb) $= 16$ ounces (oz)	1 gram (g) $= 1000$ milligrams (mg)	1 ounce (oz) ≈ 28.35 grams (g)
1 ton (T) $= 2000$ pounds (lb)	1 gram (g) $= 1000$ kilograms (kg)	1 kilogram (kg) ≈ 2.205 pounds (lb)
	1 metric ton (t) $= 1000$ kilograms (kg)	

Table B.0.4: Weight/Mass Unit Conversion Factors



Precise Units of Time	Imprecise Units of Time	Units of Time in the Metric System
1 week (wk) $= 7$ days (d)	1 year (yr) ≈ 12 months (mo)	1 second (s) $= 1000$ milliseconds (ms)
1 day (d) $= 24$ hours (h)	1 year (yr) ≈ 52 weeks (wk)	1 second (s) $= 10^6$ microseconds (μs)
1 hour (h) $= 60$ minutes (min)	1 year (yr) ≈ 365 days (d)	1 second (s) $= 10^9$ nanoseconds (ns)
1 minute (min) $= 60$ seconds (s)	1 month (mo) ≈ 30 days (d)	

Table B.0.5: Time Unit Conversion Factors



1 byte (B) $= 8$ bits (b)	1 kilobit (kb) $= 1024$ bits (b)
1 kilobyte (kB) $= 1024$ bytes (B)	1 megabit (Mb) $= 1024$ kilobits (kb)
1 megabyte (MB) $= 1024$ kilobytes (kB)	
1 gigabyte (GB) $= 1024$ megabytes (MB)	
1 terabyte (TB) $= 1024$ gigabytes (GB)	

Table B.0.6: Computer Storage/Memory Conversion Factors

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