# MTH 252 Lab Supplement

## Contents



## Antiderivatives

1. Find the general antiderivative of each function. Check your answer by taking the derivative of the second column.

| Function | Antiderivative |
|---|---|
| $f(x) = 4x^3$ | |
| $f(x) = x^5$ | |
| $f(x) = x^n$ | |
| $f(x) = \sqrt[3]{x^2} - \dfrac{2}{x^4}$ | |
| $f(x) = \dfrac{x^4 + 2\sqrt{x} - 3x}{x}$ | |
| $f(x) = \dfrac{1}{x}$ | |
| $f(x) = e^x$ | |

| | |
|---|---|
| $f(x)=\cos(x)$ | |
| $f(x)=\sin(x)$ | |
| $f(x)=\sec^2(x)$ | |
| $f(x)=\frac{1}{\sqrt{1-x^2}}$ | |
| $f(x)=-\frac{1}{\sqrt{1-x^2}}$ | |
| $f(x)=\frac{1}{1+x^2}$ | |

2. Find equations for $f'$ and $f$, given $f''(x)=x$, $f'(0)=1$, $f(0)=0$.

3. Find equations for $g'$ and $g$, given $g''(t)=2e^t+3\cos(t)$, $g'(0)=1$, $g(0)=3$.

4. A stone is dropped off a cliff. It hits the ground 6 seconds later. How high is the cliff? (Hint: Acceleration due to gravity is a constant -32 ft/sec$^2$.)

5. A car traveling 84 ft/s begins to decelerate at a constant rate of 14 ft/s$^2$. After how many seconds does the car come to a stop and how far will the car have traveled before stopping?

6. You are the question designer! Create a limit problem where you must correctly apply L'Hospital's Rule twice in order to evaluate the limit. The ground work is laid out for you below. Determine the functions $f'(x)$, $g'(x)$, $f(x)$, and $g(x)$ to result in the following limit. Don't forget about the criteria needed in order to apply L'Hospital's Rule.

$$\lim_{x\to 0}\frac{f(x)}{g(x)} \overset{\text{L'H}}{=} \lim_{x\to 0}\frac{f'(x)}{g'(x)} \quad \leftarrow \text{After 1}^{\text{st}} \text{ application of L'Hospital's Rule}$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0}\frac{f''(x)}{g''(x)} \quad \leftarrow \text{After 2}^{\text{nd}} \text{ application of L'Hospital's Rule}$$

$$= \lim_{x\to 0}\frac{e^x+\cos(x)}{12x-6} \quad \leftarrow \text{The resulting limit}$$

$$= \frac{e^0+\cos(0)}{12(0)-6} \quad \leftarrow \text{Evaluating the limit}$$

$$= \frac{1+1}{-6}$$

$$= -\frac{1}{3}$$

## Trigonometric Substitution

1. Evaluate the integral using the recommended trigonometric substitution:

a. $\displaystyle\int \frac{x}{\sqrt{x^2-1}}\,dx, \quad x=\sec\theta$ ← Check your answer using substitution!

b. $\displaystyle\int_0^5 \frac{1}{(25+x^2)^2}\,dx, \quad x=5\tan\theta$

c. $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx, \quad x=2\sin\theta$

2. Evaluate the integral using the necessary trigonometric substitution:

a. $\int_{-6}^{6} \sqrt{36 - x^2}\, dx$ ← Check your answer using geometry!

b. $\int \frac{1}{x\sqrt{x^2 - 16}}\, dx$

c. $\int \frac{1}{x^2\sqrt{1 - x^2}}\, dx$

d. $\int_{-2}^{2} \frac{1}{(4 + x^2)^{3/2}}\, dx$

## Approximate Integrals Technology Lab (Optional)

Use the link below to be taken to the pre-created GeoGebra applet:

https://www.geogebra.org/m/SKAvZYtn

1. Consider the integral $\int_{0}^{1} xe^{2x^2}\, dx$ . Using the GeoGebra applet, fill in the table with the given approximations using the sliders, as needed. Round to 8 decimal places.

| | Approximation | Over or under? |
|---|---|---|
| $M_6$ | | |
| $T_6$ | | |
| $M_{16}$ | | |
| $T_{16}$ | | |

Based on the shape of the curve, explain how you determined whether $M_N$ and $T_N$ were over- or under-approximations.

2. Confirm GeoGebra's result for $T_6$ by hand using the general formula for $T_N$ (the one with 1, 2, 2, … , 2, 2, 1). Be sure to state whether your result for $T_6$ matches the one that GeoGebra gave. Round your answer to 8 decimal places.

3. Determine the *exact* value of $\int_0^1 xe^{2x^2}\,dx$ using an integration technique we've learned in class. State the technique being used. Show all work with proper notation, and give both an exact value and an approximate value rounded to 8 decimal places. Is this close to the approximation from #2? What's the error?

4. Change the function and sliders in GeoGebra so that the applet approximates the integral $\int_{-1}^{1} x\sin(\pi x)\,dx$ . (To change the function, double-click in the Algebra window and edit. Use pi for π.) Fill in the table.

| | Approximation |
|---|---|
| $M_4$ | |
| $T_4$ | |
| $S_8$ | |
| $M_{16}$ | |
| $T_{16}$ | |
| $S_{32}$ | |

5. Confirm GeoGebra's result for $S_8$ by hand using the general formula for $S_N$ (the one with 1, 4, 2, 4, … , 4, 2, 4, 1). Be sure to state whether your result for $S_8$ matches the one that GeoGebra gave. Round your answer to 8 decimal places.

6. Determine the *exact* value of $\int_{-1}^{1} x\sin(\pi x)dx$ using an integration technique we've learned in class. State the technique being used. Show all work with proper notation, and give both an exact value and an approximate value, rounding to 8 decimal places. Is this close to the approximation from #5? What's the error?

## Error Bound Formulas

| | | |
|---|---|---|
| $Error(T_N) \le \dfrac{K_2(b-a)^3}{12N^2}$ | $Error(M_N) \le \dfrac{K_2(b-a)^3}{24N^2}$ | $Error(S_N) \le \dfrac{K_4(b-a)^5}{180N^4}$ |

1. Find the maximum possible error associated in using $T_{10}$ to approximate $\int_0^1 e^{-4x}dx$.

2. Find the value of N for which $S_N$ approximates $\int_0^{\pi/2} \sin(2x)dx$ with an error of at most 0.001.

3. Consider $\int_1^3 x\ln(x)dx$. Answer the following, rounding to 6 decimal places.

   a. Find the maximum possible error associated in using $S_4$ to approximate $\int_1^3 x\ln(x)dx$.

   b. Use $S_4$ to approximate $\int_1^3 x\ln(x)dx$.

   c. Evaluate $\int_1^3 x\ln(x)dx$. Give an exact and approximate value.

d. Determine the actual error when using $S_4$ to approximate $\int_1^3 x\ln(x)dx$.

## The Comparison Test for Improper Integrals

1. Use the Comparison Test to determine whether the following integrals converge or diverge. Your answer should be a sentence of how the Comparison Test was used and what your final conclusion is.

   a. $\int_1^{\infty} \frac{1}{\sqrt{x^4+1}}dx$

   b. $\int_2^{\infty} \frac{1}{\sqrt[3]{x^2-1}}dx$

   b.

   c. $\int_5^{\infty} \frac{1}{\sqrt{x^2-4x-2}}dx$

   d. $\int_1^{\infty} \frac{1}{x^{1/3}+x^3}dx$

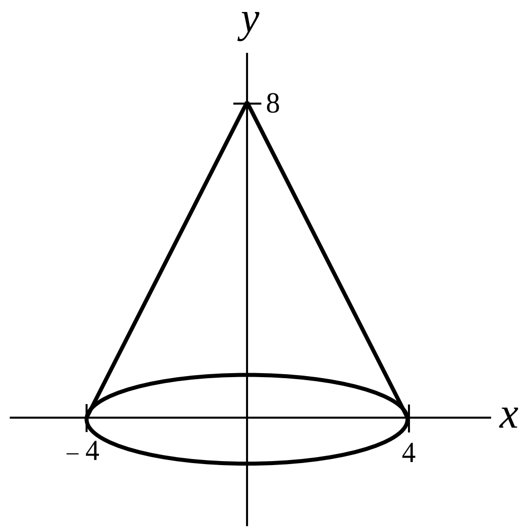
2. Determine whether the improper integral $\int_0^{\infty} \frac{x}{(x^2+1)^3}dx$ converges or diverges by using the Comparison Test. If it converges, evaluate it.

3. Determine whether the improper integral $\int_0^{\infty} \frac{1}{1+x^2}dx$ converges or diverges by using the Comparison Test. If it converges, evaluate it.

4. Determine whether the improper integral $\int_1^{\infty} \frac{1}{x^2+3x+2}dx$ converges or diverges by using the Comparison Test. If it converges, evaluate it.

## Cross-Sectional Volumes

1. Consider a cone with base of radius 4 and height 8 (see figure).




   a. What shape are the cross-sections perpendicular to the *y*-axis? Are the widths of the cross-sections $\Delta x$ or $\Delta y$?

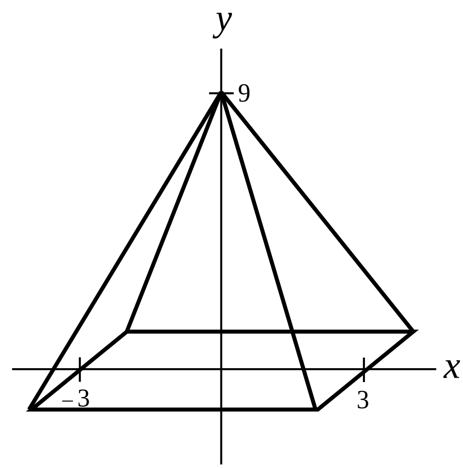
   b. Draw the cross-section at the following *y*-values and determine the radius and area.

| ***y*-value** | **Radius** | **Area** |
|---|---|---|
| $y = 0$ | | |
| $y = 4$ | | |
| $y = 8$ | | |

   c. Notice in part b that the radii are given by the *x*-coordinate (verify this in the table above). Find an equation that relates *x* and *y* so that we can get the radii in terms of *y* instead of *x*. Use this to get the area of the cross-sections in terms of *y*. Why do we need to do this?

   d. Integrate the area formula found in part c to find the volume of the cone. What should the bounds of integration be? Check your answer using the formula $V = \frac{1}{3}\pi r^2 h$.

2. Consider a pyramid with square base of dimension 6 by 6 and height of 9 (see figure).



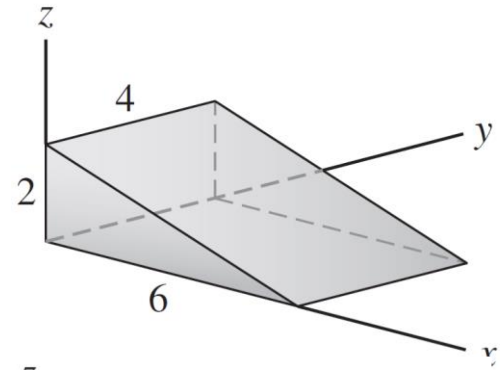

   a. What shape are the cross-sections perpendicular to the $y$-axis? Are the widths of the cross-sections $\Delta x$ or $\Delta y$?

   b. Draw an arbitrary cross-section and determine the formula for its area. (Find an equation that relates $x$ and $y$ and use as necessary in creating your formula.) Should this area formula be in terms of $x$ or $y$?

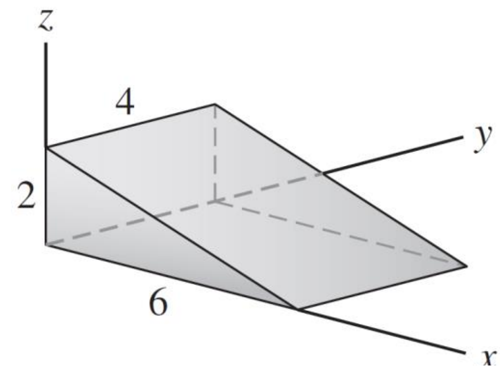
   c. Integrate the area formula found in part c with respect to the appropriate variable (so either $dx$ or $dy$) to find the volume of the pyramid. What should the bounds of integration be? Check your answer using the formula $V = \frac{1}{3}b^2h$.

3. Calculate the volume of the ramp shown by integrating the area of the cross-sections perpendicular to each axis. Draw an arbitrary cross-section for each and state what the shape is. You should get the same answer.

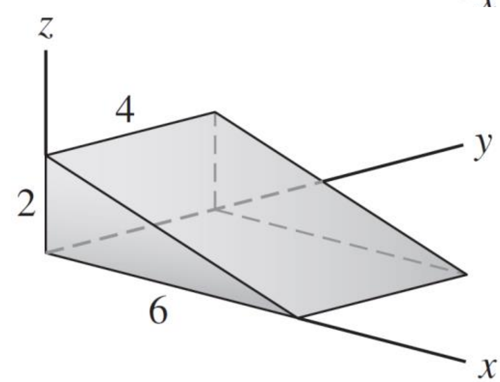
   a. $x$-axis




   b. $y$-axis




   c. $z$-axis




   d. Which approach from part (a)—(c) was easiest? Reflect on why.

   e. Check your answer using the formula $V = \frac{1}{2}lwh$.

## Method of Cylindrical Shells

Students can use the Method of Cylindrical Shells in many of the Active Calculus Section 6.2 questions. They can do some of the exercises twice: once using disks/washers and again using cylindrical shells.

Problems that lend themselves well to the Method of Cylindrical Shells:

Activity 6.2.2 e

Activity 6.2.3 a, b, d, e

Activity 6.2.4 c, d

Exercise 6.2.5 #2, 6, 7d, 8d, 8f, 9c

## Mean Value Theorem for Integrals

**The Mean Value Theorem for Integrals**: If *f* is continuous on [*a*, *b*], then there exists a number *c* in [*a*, *b*] such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

That is,

$$\int_a^b f(x)dx = f(c)(b-a)$$

1. For each of the following:

   - Find the average value $f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$ on the given interval.
   - Find *c* such that $f(c) = f_{ave}$.
   - Sketch the graph of *f* and a rectangle whose area is the same as the area under the graph of *f* on the given interval.

a. $g(x) = 3x^2 + 3$ on [-1, 3]

b. $g(x) = \frac{1}{x}$ on [1, $e$]

c. $f(x) = \cos(2x)$ on $\left[0, \frac{\pi}{4}\right]$

d. $g(t) = \frac{1}{1+t^2}$ on [-1, 1]

2. A ball is thrown vertically upwards from ground level with an initial velocity of 96 ft/sec. It's height, $h(t)$ in feet, as a function of time, $t$ in seconds, is given by $h(t) = 96t - 16t^2$.

   a. Find the average height of the ball during the time period that it's in the air.

   b. Determine when the ball's height is equal to its average height.

   c. Find the average velocity of the ball during the time period that it's in the air.

   d. Determine when the ball's velocity is equal to its average velocity.

   e. Find the average acceleration of the ball during the time period that it's in the air.

   f. Determine when the ball's acceleration is equal to its average acceleration.