Supplemental Solutions for the Related Rates Lab

Exercise 8.1

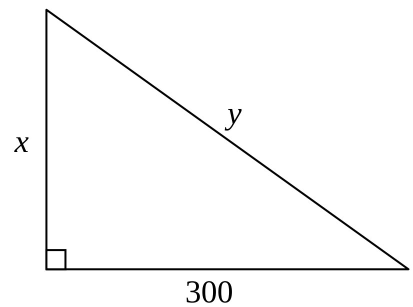


Figure E8.1K: Variable Diagram

Define x to be the elevation (ft) of the balloon t seconds after the balloon begins to rise and y to be the distance (ft) between the observer and the balloon at the same instant.

The relation equation is $x^2 + 300^2 = y^2$.

The rate equation is:

$$\frac{d}{dt}\left(x^2 + 90000\right) = \frac{d}{dt}\left(y^2\right) \Rightarrow 2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

When the elevation of the balloon is 400 feet:

$x = 400$, $y = 500$ (from the Pythagorean Theorem), and $\frac{dx}{dt} = 10$

Substituting these values into the rate equation we get:

$$2(400)(10) = 2(500)\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = 8$$

So the distance between the observer and the balloon is increasing at a rate of 8 ft/s at the instant the elevation of the balloon is 400 feet.

Exercise 8.2

Define x to be the vertical distance (ft) between the tip of the arm and the horizontal position of the arm t seconds after gate begins to close and θ to be the angle of elevation (rad) at the pivot point at the same instant in time.

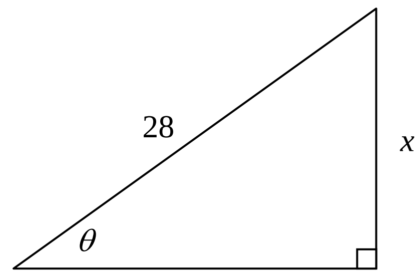


Figure E8.2K: Variable Diagram

The relation equation is $\sin(\theta) = \frac{x}{28}$.

The rate equation is: $\frac{d}{dt}(\sin(\theta)) = \frac{d}{dt}\left(\frac{x}{28}\right) \Rightarrow \cos(\theta)\frac{d\theta}{dt} = \frac{1}{28}\frac{dx}{dt}$

At the instant the angle at the pivot point is 30^{O}:

$$\theta = \frac{\pi}{6} \quad \text{and} \quad \frac{d\theta}{dt} = \left(-6\ \text{deg}/\text{s}\right)\left(\pi\ \text{rad}/180\ \text{deg}\right) = -\frac{\pi}{30}\ \text{rad}/\text{s}$$

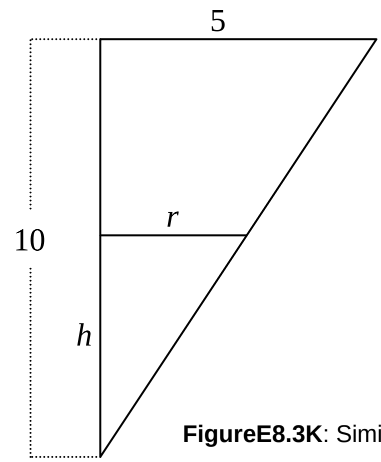
Substituting these values into the rate equation we have:

$$\cos\left(\frac{\pi}{6}\right)\left(-\frac{\pi}{30}\right) = \frac{1}{28}\frac{dx}{dt} \Rightarrow \frac{dx}{dt} \approx -2.54$$

So at the instant the angle of elevation of the arm is 30^{O}, the tip of the arm is approaching the ground at a rate of about 2.54 ft/s.

Exercise 8.3

Define V to be the volume of soda (cm^3) that remains in Jimbo's cup t seconds after he commences to sip and h to be the height of the soda in the cup (cm) at that very same instant.



FigureE8.3K: Similar Triangles

The volume formula for a right circular cone is $V = \frac{\pi}{3} r^2 h$ where V represents the volume of the cone, h represents the height of the cone, and r represents the radius at the top of the cone. Since neither of our defined variable is a radius, we need to purge that variable from our volume formula.

From the similar triangles shown in Figure E8.3K we have, $\frac{r}{h} = \frac{5}{10} \Rightarrow r = \frac{h}{2}$.

Substituting the expression $\frac{h}{2}$ for r in the volume formula we get our relation equation:

$$V = \frac{\pi}{3} r^2 h \Rightarrow V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h \Rightarrow V = \frac{\pi}{12} h^3$$

Our rate equation is: $\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi}{12} h^3\right) \Rightarrow \frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$

At the instant there are 100 cm^3 of soda remaining in the cup:

$$100 = \frac{\pi}{12} h^3 \Rightarrow h = \sqrt[3]{\frac{1200}{\pi}} \text{ ; also, } \frac{dV}{dt} = -0.25$$

Substituting these values into our rate equation we get:

$$-0.25 = \frac{\pi}{4}\left(\sqrt[3]{\frac{1200}{\pi}}\right)^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} \approx -0.006$$

So at the instant there are 100 cm^3 of soda remaining in the cup, the height of the soda in the cup is decreasing at a rate of about 0.006 cm/s. Somebody needs to tell Jimbo to put some juice into his sipping rate!

Exercise 8.4

Define V to be the volume of the snowball (cm^3) and A to be the surface area of the snowball (cm^2) t minutes after the snowball began to melt.

The volume and surface area formulas for a sphere in terms of the radius, r, of the sphere are, respectively, $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$. Solving the volume formula for r we have $r = \frac{3^{1/3}}{(4\pi)^{1/3}} V^{1/3}$ and substituting the resultant expression into the area formula we have our relation equation:

$$A = 4\pi r^2 \;\Rightarrow\; A = 4\pi\left(\frac{3^{1/3}}{(4\pi)^{1/3}} V^{1/3}\right)^2 \;\Rightarrow\; A = \sqrt[3]{36\pi}\, V^{2/3}$$

This gives us our rate equation:

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\sqrt[3]{36\pi}\, V^{2/3}\right) \;\Rightarrow\; \frac{dA}{dt} = \frac{2}{3}\sqrt[3]{36\pi}\, V^{-1/3}\frac{dV}{dt} \;\Rightarrow\; \frac{dA}{dt} = \frac{2}{3}\sqrt[3]{\frac{36\pi}{V}}\frac{dV}{dt}$$

When the radius of the snowball is 6 cm:

$$\begin{aligned} V &= \frac{4}{3}\pi(6)^3 \quad \text{and} \quad \frac{dV}{dt} = -25 \\ &= 288\pi \end{aligned}$$

Substituting these values into our rate equation we get:

$$\begin{aligned} \frac{dA}{dt} &= \frac{2}{3}\sqrt[3]{\frac{36\pi}{288\pi}}(-25) \\ &= -8\tfrac{1}{3} \end{aligned}$$

So at the instant the radius of the snowball is 6 cm, the surface area of the snowball is decreasing at the rate of $8\tfrac{1}{3}$ cm^2/minute.