Supplemental Solutions for the Introduction to the First Derivative Lab

Exercise 3.1

E3.1.1

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} \frac{(2x+h)h}{h} \\ &= \lim_{h \to 0} (2x+h) \\ &= 2x + 0 \\ &= 2x \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \\ &= \lim_{t \to x} \frac{t^2 - x^2}{t - x} \\ &= \lim_{t \to x} \frac{(t+x)(t-x)}{t - x} \\ &= \lim_{t \to x} (t+x) \\ &= x + x \\ &= 2x \end{aligned}$$

E3.1.2

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h \to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + x\right)} \\ &= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x+0} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \\ &= \lim_{t \to x} \frac{\sqrt{t} - \sqrt{x}}{t - x} \\ &= \lim_{t \to x} \frac{\sqrt{t} - \sqrt{x}}{\left(\sqrt{t} + \sqrt{x}\right)\left(\sqrt{t} - \sqrt{x}\right)} \\ &= \lim_{t \to x} \frac{1}{\sqrt{t} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

E3.1.3

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{7 - 7}{h} \\ &= \lim_{h \to 0} \frac{0}{h} \\ &= \lim_{h \to 0} 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \\ &= \lim_{t \to x} \frac{7 - 7}{t - x} \\ &= \lim_{t \to x} \frac{0}{t - x} \\ &= \lim_{t \to x} 0 \\ &= 0 \end{aligned}$$

This step is not "optional." On the preceding line the limit has the indeterminate form $\frac{0}{0}$.

Exercise 3.2

If $f(x) = \sin(x)$, then:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \lim_{h \to 0} \left[\frac{\sin(x)\left[\cos(h) - 1\right]}{h} + \frac{\cos(x)\sin(h)}{h} \right] \\
&= \sin(x) \cdot \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \cdot \lim_{h \to 0} \frac{\sin(h)}{h} \\
&= \sin(x) \cdot 0 + \cos(x) \cdot 1 \\
&= \cos(x)
\end{aligned}$$

Here we've applied limit laws A1 and A5. Remember, from this limit symbol's perspective the variable is *h*; so the factors of $\sin(x)$ and $\cos(x)$ are constants from the perspective of the limit symbol.

Here we've used the two given limit values.

Exercise 3.3

E3.3.1 The velocity function is:

$$\begin{aligned}
v(t) &= \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} \\
&= \lim_{h \to 0} \frac{\left[150 + 60(t+h) - 16(t+h)^2\right] - \left[150 + 60t - 16t^2\right]}{h} \\
&= \lim_{h \to 0} \frac{150 + 60t + 60h - 16t^2 - 32th - 16h^2 - 150 - 60t + 16t^2}{h} \\
&= \lim_{h \to 0} \frac{60h - 32th - 16h^2}{h} \\
&= \lim_{h \to 0} \frac{(60 - 32t - 16h)h}{h} \\
&= \lim_{h \to 0} (60 - 32t - 16h) \\
&= 60 - 32t - 16 \cdot 0 \\
&= 60 - 32t
\end{aligned}$$

The velocity of the object 4.1 s into its motion is (including unit):

$$\begin{aligned}
v(4.1) &= \left(60 - 32(4.1)\right) \text{ft}/\text{s} \\
&= -71.2 \text{ ft}/\text{s}
\end{aligned}$$

E3.3.2 The acceleration function is:

$$\begin{aligned} a(t) &= \lim_{h \to 0} \frac{v(t+h) - v(t)}{h} \\ &= \lim_{h \to 0} \frac{[60 - 32(t+h)] - [60 - 32t]}{h} \\ &= \lim_{h \to 0} \frac{60 - 32t - 32h - 60 + 32t}{h} \\ &= \lim_{h \to 0} \frac{-32h}{h} \\ &= \lim_{h \to 0} (-32) \\ &= -32 \end{aligned}$$

The acceleration of the object 4.1 s into its motion is (including unit):

$$a(4.1) = -32 \, \frac{\text{ft}/\text{s}}{\text{s}}$$

Exercise 3.4

E3.4.1 The unit for R' is $\frac{\text{beats}/\text{min}}{\text{ft}/\text{min}}$.

E3.4.2 The unit for F' is $\frac{\text{gal}/\text{mi}}{\text{mi}/\text{hr}}$.

E3.4.3 The unit for v' is $\frac{\text{mi}/\text{hr}}{\text{s}}$.

E3.4.4 The unit for h' is mi/s.

Exercise 3.5

E3.5.1 When Carl jogs at a pace of 300 ft/min his heart rate is 84 beats/min.

E3.5.2 At the pace of 300 ft/min, Carl's heart rate changes relative to his pace at a rate of $0.02 \, \frac{\text{beats}/\text{min}}{\text{ft}/\text{min}}$. So, if Carl picks up his pace from 300 ft/min to 301 ft/min we'd expect his heart rate to increase to about 84.02 beats/min. Conversely, if Carl decreases his pace from 300 ft/min to 299 ft/min we'd expect his heart rate to decrease to about 83.98 beats/min.

E3.5.3 When Hanh drives her pick-up on level ground at a constant speed of 50 mph, the truck burns fuel at the rate of 0.03 gal/mi.

E3.5.4 (On level ground), at the speed of 50 mph, the rate at which Hanh's truck burns fuel changes relative to the speed at a rate of $-.0006\frac{\text{gal}/\text{mi}}{\text{mi}/\text{hr}}$. So, if Hanh increased her speed from 50 mph to 51 mph, we'd expect the fuel consumption rate for her truck to decrease to about .0294 gal/mi. Conversely, if Hanh decreased her speed from 50 mph to 49 mph we'd expect to fuel consumption rate for her truck to increase to about .0306 gal/mi.

E3.5.5 Twenty seconds after lift-off the space shuttle is cruising at the rate of 266 mi/hr.

E3.5.6 Twenty seconds after lift-off the velocity of the space shuttle is increasing at the rate $18.9\frac{\text{mi}/\text{hr}}{\text{s}}$. So, we'd expect that 19 seconds into lift-off the velocity was ***about*** 247.1 mi/hr and 21 seconds into lift-off the velocity will be ***about*** 284.9 mi/hr.

E3.5.7 Twenty seconds after lift-off the space shuttle is at an elevation of 0.7 miles.

E3.5.8 Twenty seconds after lift-off the space shuttle's elevation is increasing at a rate of 0.074 mi/s. So, we'd expect that 19 seconds into lift-off the elevation was ***about*** 0.626 miles and 21 seconds into lift-off the elevation will be ***about*** 0.774 mile.

Exercise 3.6

E3.6.1 The point on *f* when $x=1$ is $(1,2)$. The value of $f'(1)$ tells us the slope of the tangent line to *f* through $(1,2)$; this value is 3. Consequently, the equation of the tangent line to *f* at 1 is $y=3x-1$.

E3.6.2 The long term slope of *f* is given by:

$$\begin{aligned}\lim_{x\to\infty} f'(x) &= \lim_{x\to\infty}\frac{2x+1}{x}\\ &= 2\end{aligned}$$

So the slope of the skew asymptote is 3.