Supplemental Solutions for the Implicit Differentiation Lab

Exercise 7.1

$$
\begin{aligned}
x\sin(x\,y) &= y \\
\frac{d}{dx}\left[x\sin(x\,y)\right] &= \frac{d}{dx}(y) \\
\frac{d}{dx}(x)\cdot\sin(x\,y) + x\cdot\frac{d}{dx}\left(\sin(x\,y)\right) &= \frac{dy}{dx} \\
1\cdot\sin(x\,y) + x\cdot\cos(x\,y)\cdot\frac{d}{dx}(x\,y) &= \frac{dy}{dx} \\
\sin(x\,y) + x\cos(x\,y)\cdot\left[\frac{d}{dx}(x)\cdot y + x\cdot\frac{d}{dx}(y)\right] &= \frac{dy}{dx} \\
\sin(x\,y) + x\cos(x\,y)\cdot\left[1\cdot y + x\cdot\frac{dy}{dx}\right] &= \frac{dy}{dx} \\
\sin(x\,y) + x\,y\cos(x\,y) + x^2\cos(x\,y)\frac{dy}{dx} &= \frac{dy}{dx} \\
\sin(x\,y) + x\,y\cos(x\,y) &= \left[1 - x^2\cos(x\,y)\right]\frac{dy}{dx} \\
\frac{\sin(x\,y) + x\,y\cos(x\,y)}{1 - x^2\cos(x\,y)} &= \frac{dy}{dx}
\end{aligned}
$$

Since we're looking for points on the y-axis, we know that the y-coordinates are zero. A necessary element for the tangent line to be vertical is that the denominator of the derivative formula evaluates to zero (otherwise the tangent line would have a value for its slope). Putting this together gives us:

$$1 - x^2\cos(x\cdot 0) = 0 \;\Rightarrow\; 1 - x^2 = 0 \;\Rightarrow\; x = \pm 1$$

So the x-coordinates are 1 and -1.

Exercise 7.2

$$
\begin{aligned}
\ln\left(x^2\,y^2\right) &= x + y \\
\ln\left(x^2\right) + \ln\left(y^2\right) &= x + y \\
2\ln(x) + 2\ln(y) &= x + y \\
\frac{d}{dx}\left(2\ln(x) + 2\ln(y)\right) &= \frac{d}{dx}(x + y) \\
2\cdot\frac{1}{x} + 2\cdot\frac{1}{y}\frac{dy}{dx} &= 1 + \frac{dy}{dx} \\
\left(\frac{2}{y} - 1\right)\frac{dy}{dx} &= 1 - \frac{2}{x} \\
\frac{dy}{dx} &= \frac{1 - \frac{2}{x}}{\frac{2}{y} - 1} \\
\left.\frac{dy}{dx}\right|_{(1,-1)} &= \frac{1}{3}
\end{aligned}
$$

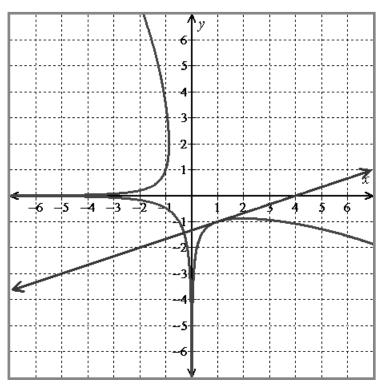

Figure E7.2K: $\ln\left(x^2\,y^2\right) = x + y$

The slope of the tangent line is $\frac{1}{3}$ and the line passes through the point $(1,-1)$, so the equation of the tangent line is $y = \frac{1}{3}x - \frac{4}{3}$.

Exercise 7.3

$$y = x^x \;\Rightarrow\; \ln(y) = \ln\left(x^x\right) \;\Rightarrow\; \ln(y) = x\cdot\ln(x)$$

$$\frac{d}{dx}\left(\ln(y)\right) = \frac{d}{dx}\left(x\cdot\ln(x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x)\cdot\ln(x) + x\cdot\frac{d}{dx}\left(\ln(x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = 1\cdot\ln(x) + x\cdot\frac{1}{x}$$

$$\frac{dy}{dx} = y\left(\ln(x)+1\right)$$

$$\frac{dy}{dx} = x^x\left(\ln(x)+1\right)$$

Exercise 7.4

7.4.1

$$y = \frac{x\sin(x)}{\sqrt{x-1}}$$

$$\ln(y) = \ln\left(\frac{x\sin(x)}{\sqrt{x-1}}\right)$$

$$\ln(y) = \ln\left(x\sin(x)\right) - \ln\left[(x-1)^{1/2}\right]$$

$$\ln(y) = \ln(x) + \ln\left(\sin(x)\right) - \frac{1}{2}\ln(x-1)$$

$$\frac{d}{dx}\left(\ln(y)\right) = \frac{d}{dx}\left[\ln(x) + \ln\left(\sin(x)\right) - \frac{1}{2}\ln(x-1)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\sin(x)}\cdot\frac{d}{dx}\left(\sin(x)\right) - \frac{1}{2}\cdot\frac{1}{x-1}\cdot\frac{d}{dx}(x-1)$$

$$\frac{dy}{dx} = y\left[\frac{1}{x} + \frac{1}{\sin(x)}\cdot\cos(x) - \frac{1}{2}\cdot\frac{1}{x-1}\cdot 1\right]$$

$$\frac{dy}{dx} = \frac{x\sin(x)}{\sqrt{x-1}}\left[\frac{1}{x} + \cot(x) - \frac{1}{2(x-1)}\right]$$

7.4.2

$$y=\frac{e^{2x}}{\sin^4(x)\sqrt[4]{x^5}}$$

$$\ln(y)=\ln\left(\frac{e^{2x}}{\sin^4(x)\sqrt[4]{x^5}}\right)$$

$$\ln(y)=\ln\left(e^{2x}\right)-\ln\left(\sin^4(x)\sqrt[4]{x^5}\right)$$

$$\ln(y)=2x-\left[\ln\left((\sin(x))^4\right)+\ln\left(x^{5/4}\right)\right]$$

$$\ln(y)=2x-4\ln(\sin(x))-\frac{5}{4}\ln(x)$$

$$\frac{d}{dx}(\ln(y))=\frac{d}{dx}\left[2x-4\ln(\sin(x))-\frac{5}{4}\ln(x)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=2-4\cdot\frac{1}{\sin(x)}\cdot\frac{d}{dx}(\sin(x))-\frac{5}{4}\cdot\frac{1}{x}$$

$$\frac{dy}{dx}=y\left[2-4\cdot\frac{1}{\sin(x)}\cdot\cos(x)-\frac{5}{4x}\right]$$

$$\frac{dy}{dx}=\frac{e^{2x}}{\sin^4(x)\sqrt[4]{x^5}}\left[2-4\cot(x)-\frac{5}{4x}\right]$$

7.4.3

$$y=\frac{\ln\left(4x^3\right)}{x^5\ln(x)}$$

$$\ln(y)=\ln\left(\frac{\ln\left(4x^3\right)}{x^5\ln(x)}\right)$$

$$\ln(y)=\ln\left(\ln\left(4x^3\right)\right)-\ln\left(x^5\ln(x)\right)$$

$$\ln(y)=\ln\left(\ln\left(4x^3\right)\right)-\left[\ln\left(x^5\right)+\ln(\ln(x))\right]$$

$$\ln(y)=\ln\left(\ln\left(4x^3\right)\right)-5\ln(x)-\ln(\ln(x))$$

$$\frac{d}{dx}(\ln(y))=\frac{d}{dx}\left[\ln\left(\ln\left(4x^3\right)\right)-5\ln(x)-\ln(\ln(x))\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{\ln\left(4x^3\right)}\cdot\frac{d}{dx}\left(\ln\left(4x^3\right)\right)-5\cdot\frac{1}{x}-\frac{1}{\ln(x)}\cdot\frac{d}{dx}(\ln(x))$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{\ln\left(4x^3\right)}\cdot\frac{1}{4x^3}\cdot\frac{d}{dx}\left(4x^3\right)-\frac{5}{x}-\frac{1}{\ln(x)}\cdot\frac{1}{x}$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{\ln\left(4x^3\right)}\cdot\frac{1}{4x^3}\cdot 12x^2-\frac{5}{x}-\frac{1}{x\ln(x)}$$

$$\frac{dy}{dx}=y\left[\frac{3}{x\ln\left(4x^3\right)}-\frac{5}{x}-\frac{1}{x\ln(x)}\right]$$

$$\frac{dy}{dx}=\frac{\ln\left(4x^3\right)}{x^5\ln(x)}\left[\frac{3}{x\ln\left(4x^3\right)}-\frac{5}{x}-\frac{1}{x\ln(x)}\right]$$