Supplemental Solutions for the Derivative Formulas Lab

Exercise 5.1

5.1.1

$$f(x)=\frac{1}{6}x^{6/11}$$
$$f'(x)=\frac{1}{11}x^{-5/11}$$
$$=\frac{1}{11\sqrt[11]{x^5}}$$

5.1.2

$$y=-5t^{-7}$$
$$\frac{dy}{dt}=35t^{-8}$$
$$=\frac{35}{t^8}$$

5.1.3

$$y(u)=12u^{1/3}$$
$$y'(u)=4u^{-2/3}$$
$$=\frac{4}{\sqrt[3]{u^2}}$$

5.1.4

$$z(\alpha)=e\alpha^{\pi}$$
$$z'(\alpha)=e\pi\alpha^{\pi-1}$$

5.1.5

$$z=8t^{-7/2}$$
$$\frac{dz}{dt}=-28t^{-9/2}$$
$$=-\frac{28}{\sqrt{t^9}}$$

5.1.6

$$T=4\cdot\frac{t^{7/3}}{t^2}$$
$$=4t^{1/3}$$
$$\frac{dT}{dt}=\frac{4}{3}t^{-2/3}$$
$$=\frac{4}{3\sqrt[3]{t^2}}$$

Exercise 5.2

5.2.1

$$y=5\sin(x)\cdot\cos(x)$$
$$\frac{dy}{dx}=\frac{d}{dx}\left(5\sin(x)\right)\cdot\cos(x)+5\sin(x)\cdot\frac{d}{dx}\left(\cos(x)\right)$$
$$=5\cos(x)\cdot\cos(x)+5\sin(x)\cdot\left(-\sin(x)\right)$$
$$=5\cos^2(x)-5\sin^2(x)$$

5.2.2

$$y=\frac{5}{7}t^2\cdot e^t$$
$$\frac{dy}{dt}=\frac{d}{dt}\left(\frac{5}{7}t^2\right)\cdot e^t+\frac{5}{7}t^2\cdot\frac{d}{dt}\left(e^t\right)$$
$$=\frac{10}{7}t\cdot e^t+\frac{5}{7}t^2e^t$$
$$=\frac{5te^t(2+t)}{7}$$

5.2.3

$$F(x)=4x\cdot\ln(x)$$
$$F'(x)=\frac{d}{dx}(4x)\cdot\ln(x)+4x\cdot\frac{d}{dx}\left(\ln(x)\right)$$
$$=4\cdot\ln(x)+4x\cdot\frac{1}{x}$$
$$=4\ln(x)+4$$

5.2.4 $z = x^2 \cdot \sin^{-1}(x)$

$$\frac{dz}{dx} = \frac{d}{dx}\left(x^2\right)\cdot \sin^{-1}(x) + x^2 \cdot \frac{d}{dx}\left(\sin^{-1}(x)\right)$$

$$= 2x \cdot \sin^{-1}(x) + x^2 \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= 2x\sin^{-1}(x) + \frac{x^2}{\sqrt{1-x^2}}$$

5.2.5 $T(t) = \left(1+t^2\right)\cdot \tan^{-1}(t)$

$$T'(t) = \frac{d}{dt}\left(1+t^2\right)\cdot \tan^{-1}(t) + \left(1+t^2\right)\cdot \frac{d}{dt}\left(\tan^{-1}(t)\right)$$

$$= 2t \cdot \tan^{-1}(t) + \left(1+t^2\right)\cdot \frac{1}{1+t^2}$$

$$= 2t\tan^{-1}(t) + 1$$

5.2.6 $T = \frac{1}{3}x^7 \cdot 7^x$

$$\frac{dT}{dx} = \frac{d}{dx}\left(\frac{1}{3}x^7\right)\cdot 7^x + \frac{1}{3}x^7 \cdot \frac{d}{dx}\left(7^x\right)$$

$$= \frac{7}{3}x^6 \cdot 7^x + \frac{1}{3}x^7 \cdot \ln(7)7^x$$

$$= \frac{x^6\,7^x\left(7+\ln(7)x\right)}{3}$$

Exercise 5.3

5.3.1 $q(\theta) = \frac{4e^\theta}{e^\theta + 1}$

$$q'(\theta) = \frac{\frac{d}{d\theta}\left(4e^\theta\right)\cdot\left(e^\theta+1\right) - 4e^\theta \cdot \frac{d}{d\theta}\left(e^\theta+1\right)}{\left(e^\theta+1\right)^2}$$

$$= \frac{4e^\theta\cdot\left(e^\theta+1\right) - 4e^\theta\cdot e^\theta}{\left(e^\theta+1\right)^2}$$

$$= \frac{4e^\theta\left(e^\theta+1-e^\theta\right)}{\left(e^\theta+1\right)^2}$$

$$= \frac{4e^\theta}{\left(e^\theta+1\right)^2}$$

5.3.2 $$u(x)=\frac{2\ln(x)}{x^4}$$

$$u'(x)=\frac{\frac{d}{dx}\left(2\ln(x)\right)\cdot x^4-2\ln(x)\cdot\frac{d}{dx}\left(x^4\right)}{\left(x^4\right)^2}$$

$$=\frac{2\cdot\frac{1}{x}\cdot x^4-2\ln(x)\cdot 4x^3}{x^8}$$

$$=\frac{2x^3-8x^3\ln(x)}{x^8}$$

$$=\frac{2x^3\left(1-4\ln(x)\right)}{x^8}$$

$$=\frac{2-8\ln(x)}{x^5}$$

5.3.3 $$F=\frac{t^{1/2}}{3t^2-5t^{3/2}}$$

$$=\frac{t^{1/2}}{t^{1/2}\left(3t^{3/2}-5t\right)}$$

$$=\frac{1}{3t^{3/2}-5t}$$

$$\frac{dF}{dt}=\frac{\frac{d}{dt}(1)\cdot\left(3t^{3/2}-5t\right)-1\cdot\frac{d}{dt}\left(3t^{3/2}-5t\right)}{\left(3t^{3/2}-5t\right)^2}$$

$$=\frac{0\cdot\left(3t^{3/2}-5t\right)-1\cdot\left(\frac{9}{2}t^{1/2}-5\right)}{\left(3t^{3/2}-5t\right)^2}$$

$$=\frac{5-\frac{9}{2}t^{1/2}}{\left(3t^{3/2}-5t\right)^2}\cdot\frac{2}{2}$$

$$=\frac{10-9\sqrt{t}}{2\left(3\sqrt{t^3}-5t\right)^2}$$

5.3.4 $F(x)=\dfrac{\tan(x)}{\tan^{-1}(x)}$

$$F'(x)=\frac{\frac{d}{dx}\left(\tan(x)\right)\cdot\tan^{-1}(x)-\tan(x)\cdot\frac{d}{dx}\left(\tan^{-1}(x)\right)}{\left(\tan^{-1}(x)\right)^2}$$

$$=\frac{\sec^2(x)\cdot\tan^{-1}(x)-\tan(x)\cdot\frac{1}{1+x^2}}{\left(\tan^{-1}(x)\right)^2}\cdot\frac{1+x^2}{1+x^2}$$

$$=\frac{\left(1+x^2\right)\sec^2(x)\tan^{-1}(x)-\tan(x)}{\left(1+x^2\right)\left(\tan^{-1}(x)\right)^2}$$

5.3.5 $P=\dfrac{\tan^{-1}(t)}{1+t^2}$

$$\frac{dP}{dt}=\frac{\frac{d}{dt}\left(\tan^{-1}(t)\right)\cdot\left(1+t^2\right)-\tan^{-1}(t)\cdot\frac{d}{dt}\left(1+t^2\right)}{\left(1+t^2\right)^2}$$

$$=\frac{\frac{1}{1+t^2}\cdot\left(1+t^2\right)-\tan^{-1}(t)\cdot 2t}{\left(1+t^2\right)^2}$$

$$=\frac{1-2t\tan^{-1}(t)}{\left(1+t^2\right)^2}$$

5.3.6 $y=\dfrac{4}{\sin(\beta)-2\cos(\beta)}$

$$\frac{dy}{d\beta}=\frac{\frac{d}{d\beta}(4)\cdot\left(\sin(\beta)-2\cos(\beta)\right)-4\cdot\frac{d}{d\beta}\left(\sin(\beta)-2\cos(\beta)\right)}{\left(\sin(\beta)-2\cos(\beta)\right)^2}$$

$$=\frac{0\cdot\left(\sin(\beta)-2\cos(\beta)\right)-4\cdot\left(\cos(\beta)-2\left(-\sin(\beta)\right)\right)}{\left(\sin(\beta)-2\cos(\beta)\right)^2}$$

$$=\frac{-4\left(\cos(\beta)+2\sin(\beta)\right)}{\left(\sin(\beta)-2\cos(\beta)\right)^2}$$

Exercise 5.4

E5.4.1

$$\begin{aligned}\frac{d}{dx}\left(f(x)g(x)h(x)\right) &= \frac{d}{dx}\left(f(x)\right)\left[g(x)h(x)\right]+f(x)\frac{d}{dx}\left[g(x)h(x)\right]\\ &= f'(x)\left[g(x)h(x)\right]+f(x)\left[\frac{d}{dx}\left(g(x)\right)h(x)+g(x)\frac{d}{dx}\left(h(x)\right)\right]\\ &= f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x)\end{aligned}$$

E5.4.2

$$f(x)=x^2e^x\sin(x)\cos(x)$$

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left(x^2\right)e^x\sin(x)\cos(x)+x^2\frac{d}{dx}\left(e^x\right)\sin(x)\cos(x)+x^2e^x\frac{d}{dx}\left(\sin(x)\right)\cos(x)+x^2e^x\sin(x)\frac{d}{dx}\left(\cos(x)\right)\\ &= 2xe^x\sin(x)\cos(x)+x^2e^x\sin(x)\cos(x)+x^2e^x\cos(x)\cos(x)+x^2e^x\sin(x)\left(-\sin(x)\right)\\ &= xe^x\left(2\sin(x)\cos(x)+x\sin(x)\cos(x)+x\cos^2(x)-x\sin^2(x)\right)\end{aligned}$$

Exercise 5.5

E5.5.1

$$\begin{aligned}f(x) &= \frac{x^3+x^2}{x}\\ &= x^2+x;\ \ x\neq 0\\ f'(x) &= 2x+1;\ \ x\neq 0\end{aligned}$$

$f(5)=30$ and $f'(5)=11$ So the tangent line to f at 5 passes through the point $(5,30)$ and has a slope of 11. The equation of this line is $y=11x-25$.

E5.5.2

$$\begin{aligned}h(x) &= \frac{x}{1+x}\\ h'(x) &= \frac{\frac{d}{dx}(x)\cdot(1+x)-x\cdot\frac{d}{dx}(1+x)}{(1+x)^2}\\ &= \frac{1\cdot(1+x)-x\cdot 1}{(1+x)^2}\\ &= \frac{1}{(1+x)^2}\end{aligned}$$

$h(-2)=2$ and $h'(-2)=1$ So the tangent line to h at -2 passes through the point $(-2,2)$ and has a slope of 1. The equation of this line is $y=x+4$.

E5.5.3
$$K(x) = \frac{1+x}{2x+2}$$
$$= \frac{1+x}{2(x+1)}$$
$$= \frac{1}{2};\ h \neq -1$$

Since the graph of $y = K(x)$ is simply the horizontal line $y = 0.5$ with a hole at -1, the tangent line to K at 8 must be the line with equation $y = 0.5$.

E5.5.4
$$r(x) = 3x^{1/3} \cdot e^x$$
$$r'(x) = \frac{d}{dx}\left(3x^{1/3}\right) \cdot e^x + 3x^{1/3} \cdot \frac{d}{dx}\left(e^x\right)$$
$$= x^{-2/3} e^x + 3x^{1/3} e^x$$
$$= \frac{e^x}{x^{2/3}} + \frac{3x^{1/3} e^x}{1} \cdot \frac{x^{2/3}}{x^{2/3}}$$
$$= \frac{e^x (1+3x)}{\sqrt[3]{x^2}}$$

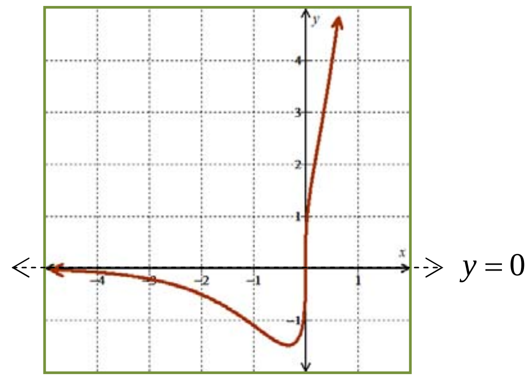

Figure E5.5.4K: $y = 3\sqrt[3]{x}\, e^x$

$r(0) = 0$ and $r'(0)$ has the form $\frac{1}{0}$. This form for $r'(0)$ implies that r has a vertical tangent line at 0. So the tangent to r at 0 is the line $x = 0$.

Exercise 5.6

$$y = 4\sqrt{t}\, t^5 = 4t^{11/2} \qquad \frac{dy}{dt} = 22t^{9/2} \qquad \frac{d^2y}{dt^2} = 99t^{7/2} = 99\sqrt{t^7}$$

Exercise 5.7

E5.7.1 f' is positive over $(-\infty, -8)$ and $(2, \infty)$; f' is negative over $(-8, 2)$.

E5.7.2 f' is increasing over and $(-3, \infty)$; f' is decreasing over $(-\infty, -3)$.

E5.7.3 f'' is positive over $(-3, \infty)$; f'' is negative over $(-\infty, -3)$.

E5.7.4
$$f'(x) = x^2 + 6x - 16$$
$$f''(x) = 2x + 6$$

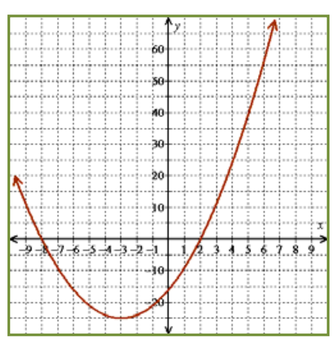

Figure E5.7.4Ka: $y = f'(x)$

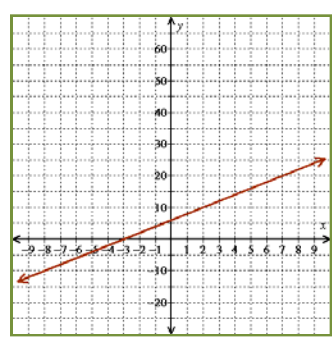

Figure E5.7.4Kb: $y = f''(x)$

Exercise 5.8

E5.8.1 Over the interval $\left[0,2\pi\right]$, $g'(t)=0$ at $\frac{\pi}{2},\frac{7\pi}{6},\frac{3\pi}{2}$, and $\frac{11\pi}{6}$.

E5.8.2

$$\begin{aligned} g(t) &= 3\sin(t)+3\sin(t)\cdot\sin(t)-5 \\ g'(t) &= 3\cos(t)+\frac{d}{dt}\left(3\sin(t)\right)\cdot\sin(t)+3\sin(t)\cdot\frac{d}{dt}\left(\sin(t)\right)+0 \\ &= 3\cos(t)+3\cos(t)\cdot\sin(t)+3\sin(t)\cdot\cos(t) \\ &= 3\cos(t)+6\cos(t)\sin(t) \\ &= 3\cos(t)\left[1+2\sin(t)\right] \end{aligned}$$

So $g'(t)=0$ where $\cos(t)=0$ or $\sin(t)=-\frac{1}{2}$. Over $\left[0,2\pi\right]$, $\cos(t)=0$ at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Over $\left[0,2\pi\right]$, $\sin(t)=-\frac{1}{2}$ at $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. This comports with our answer to problem E5.8.1

Exercise 5.9

$$\begin{aligned} f(x) &= \sin(2x) \\ &= 2\sin(x)\cdot\cos(x) \\ f'(x) &= \frac{d}{dx}\left(2\sin(x)\right)\cdot\cos(x)+2\sin(x)\cdot\frac{d}{dx}\left(\cos(x)\right) \\ &= 2\cos(x)\cdot\cos(x)+2\sin(x)\cdot\left(-\sin(x)\right) \\ &= 2\cos(x)\cdot\cos(x)-2\sin(x)\cdot\sin(x) \\ f''(x) &= \left[\frac{d}{dx}\left(2\cos(x)\right)\cdot\cos(x)+2\cos(x)\cdot\frac{d}{dx}\left(\cos(x)\right)\right]-\left[\frac{d}{dx}\left(2\sin(x)\right)\cdot\sin(x)+2\sin(x)\cdot\frac{d}{dx}\left(\sin(x)\right)\right] \\ &= \left[\left(-2\sin(x)\right)\cdot\cos(x)+2\cos(x)\left(-\sin(x)\right)\right]-\left[2\cos(x)\sin(x)+2\sin(x)\cos(x)\right] \\ &= -8\sin(x)\cos(x) \end{aligned}$$

$f'(\pi)=2$ and $f''(\pi)=0$ So the tangent line to f' at π passes through the point $(\pi,2)$ and has a slope of 0. The equation of this line is $y=2$.

Exercise 5.10

E5.10.1

$$\begin{aligned} h(x) &= f(x)g(x) \\ h'(x) &= \frac{d}{dx}\left(f(x)\right)g(x)+f(x)\frac{d}{dx}\left(g(x)\right) \\ &= f'(x)g(x)+f(x)g'(x) \\ h'(4) &= f'(4)g(4)+f(4)g'(4) \\ &= (39)(289)+(46)(293) \\ &= 24,749 \end{aligned}$$

E5.10.2

$$h'(x) = = f'(x)g(x) + f(x)g'(x)$$

$$h''(x) = \left[\frac{d}{dx}\left(f'(x)\right)g(x) + f'(x)\frac{d}{dx}\left(g(x)\right)\right] + \left[\frac{d}{dx}\left(f(x)\right)g'(x) + f(x)\frac{d}{dx}\left(g'(x)\right)\right]$$

$$= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$$

$$h''(2) = f''(2)g(2) + 2f'(2)g'(2) + f(2)g''(2)$$

$$= (10)(11) + 2(7)(37) + (4)(58)$$

$$= 860$$

E5.10.3

$$k(x) = \frac{g(x)}{f(x)}$$

$$k'(x) = \frac{\frac{d}{dx}\left(g(x)\right)f(x) - g(x)\frac{d}{dx}\left(f(x)\right)}{\left[f(x)\right]^2}$$

$$= \frac{g'(x)f(x) - g(x)f'(x)}{\left[f(x)\right]^2}$$

$$k'(3) = \frac{g'(3)f(3) - g(3)f'(3)}{\left[f(3)\right]^2}$$

$$= \frac{(126)(15) - (87)(20)}{15^2}$$

$$= \frac{2}{3}$$

E5.10.4

$$p(x) = 6\sqrt{x}\, f(x)$$

$$p'(x) = \frac{d}{dx}\left(6\sqrt{x}\right)f(x) + 6\sqrt{x}\frac{d}{dx}\left(f(x)\right)$$

$$= 6 \cdot \frac{1}{2\sqrt{x}} f(x) + 6\sqrt{x}\, f'(x)$$

$$= \frac{3f(x)}{\sqrt{x}} + 6\sqrt{x}\, f'(x)$$

$$p'(4) = \frac{3f(4)}{\sqrt{4}} + 6\sqrt{4}\, f'(4)$$

$$= \frac{3(46)}{2} + 6(2)(39)$$

$$= 537$$

E5.10.5

$$r(x) = \left[g(x)\right]^2$$
$$= g(x) \cdot g(x)$$
$$r'(x) = \frac{d}{dx}\big(g(x)\big)g(x) + g(x)\frac{d}{dx}\big(g(x)\big)$$
$$= 2\,g(x)g'(x)$$
$$r'(1) = 2\,g(1)g'(1)$$
$$= 2(-5)(2)$$
$$= -20$$

E5.10.6

$$s(x) = x\,f(x)g(x)$$
$$s'(x) = \frac{d}{dx}(x)\,f(x)g(x) + x\frac{d}{dx}\big(f(x)\big)g(x) + x\,f(x)\frac{d}{dx}\big(g(x)\big)$$
$$= f(x)g(x) + x\,f'(x)g(x) + x\,f(x)g'(x)$$
$$s'(2) = f(2)g(2) + 2\,f'(2)g(2) + 2\,f(2)g'(2)$$
$$= (4)(11) + 2(7)(11) + 2(4)(37)$$
$$= 494$$

E5.10.7

$$F(x) = \sqrt{x}\,g(4)$$
$$= 289\sqrt{x}$$
$$F'(x) = \frac{289}{2\sqrt{x}}$$

$$F'(4) = \frac{289}{2\sqrt{4}}$$
$$= \frac{289}{4}$$

E5.10.8

$$T(x) = \frac{f(x)}{e^x}$$
$$T'(x) = \frac{\frac{d}{dx}\big(f(x)\big)\cdot e^x - f(x)\cdot\frac{d}{dx}\big(e^x\big)}{\left(e^x\right)^2}$$
$$= \frac{f'(x)e^x - f(x)e^x}{\left(e^x\right)^2}$$
$$= \frac{\cancel{e^x}\big(f'(x) - f(x)\big)}{\cancel{e^x}\cdot e^x}$$
$$= \frac{f'(x) - f(x)}{e^x}$$

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$$
\begin{aligned}
T''(x) &= \frac{\frac{d}{dx}\left(f'(x) - f(x)\right)\cdot e^x - \left(f'(x) - f(x)\right)\cdot \frac{d}{dx}\left(e^x\right)}{\left(e^x\right)^2} \\
&= \frac{\left(f''(x) - f'(x)\right)\cdot e^x - \left(f'(x) - f(x)\right)\cdot e^x}{\left(e^x\right)^2} \\
&= \frac{\cancel{e^x}\left(f''(x) - f'(x) - f'(x) + f(x)\right)}{\cancel{e^x}\cdot e^x} \\
&= \frac{f''(x) - 2f'(x) + f(x)}{e^x} \\
T''(0) &= \frac{f''(0) - 2f'(0) + f(0)}{e^0} \\
&= \frac{-2 - 2(-1) + 2}{1} \\
&= 2
\end{aligned}
$$