Supplemental Solutions for the Critical Numbers and Graphing from Formulas Lab

Exercise 9.1

E9.1.1

$$\begin{aligned}\cosh^2(t)-\sinh^2(t)&=\left(\frac{e^t+e^{-t}}{2}\right)^2-\left(\frac{e^t-e^{-t}}{2}\right)^2\\&=\frac{e^{2t}+2+e^{-2t}}{4}-\frac{e^{2t}-2+e^{-2t}}{4}\\&=\frac{e^{2t}+2+e^{-2t}-e^{2t}+2-e^{-2t}}{4}\\&=\frac{4}{4}\\&=1\end{aligned}$$

E9.1.2 For $f(t)=\cosh(t)$ and $g(t)=\sinh(t)$:

$$\begin{aligned}f'(t)&=\frac{d}{dt}\left(\frac{1}{2}\left(e^t+e^{-t}\right)\right)\\&=\frac{1}{2}\left(e^t+e^{-t}\cdot-1\right)\\&=\frac{1}{2}\left(e^t-e^{-t}\right)\\&=\sinh(t)\end{aligned}$$

$$\begin{aligned}g'(t)&=\frac{d}{dt}\left(\frac{1}{2}\left(e^t-e^{-t}\right)\right)\\&=\frac{1}{2}\left(e^t-e^{-t}\cdot-1\right)\\&=\frac{1}{2}\left(e^t+e^{-t}\right)\\&=\cosh(t)\end{aligned}$$

It follows that $f''(t)=\cosh(t)$ and $g''(t)=\sinh(t)$

E9.1.3 The domains of both f and g are $(-\infty,\infty)$. f' and g' are both always defined, so the only issue is when $f'(t)=0$ and when $g'(t)=0$.

$$\begin{aligned}f'(t)=0\quad&\Rightarrow\quad\frac{e^t-e^{-t}}{2}=0\\&\Rightarrow\quad e^t=e^{-t}\\&\Rightarrow\quad e^t=\frac{1}{e^t}\\&\Rightarrow\quad\left(e^t\right)^2=1\\&\Rightarrow\quad e^t=\pm1\end{aligned}$$

e^t is never negative and $e^t=1$ when $t=0$. Therefore, the only critical number for $f(t)=\cosh(t)$ is 0.

$$\begin{aligned}g'(t)=0\quad&\Rightarrow\quad\frac{e^t+e^{-t}}{2}=0\\&\Rightarrow\quad e^t=-e^{-t}\end{aligned}$$

Since real number powers of e are always positive, e^t never equals $-e^{-t}$. Therefore, $g(t)=\sinh(t)$ has no critical numbers.

E9.1.4 **Table 9.1.1K**: Sign analysis for $\sinh(t)$

Interval	$\sinh(t)$
$(-\infty,0)$	negative
$(0,\infty)$	positive

Table 9.1.2K: Sign analysis for $\cosh(t)$

Interval	$\cosh(t)$
$(-\infty,\infty)$	positive

$f(t)=\sinh(t)$, $f'(t)=\cosh(t)$ and $f''(t)=\sinh(t)$. So based upon the signs shown in tables 9.1.1K and 9.1.2K we can conclude that $f(t)=\sinh(t)$ is always increasing, concave down on $(-\infty,0)$ and concave up on $(0,\infty)$. We should note that the point $(0,0)$ is an inflection point on f and that the slope of f at that point is $\cosh(0)$ which is 1.

$g(t)=\cosh(t)$, $g'(t)=\sinh(t)$, and $g''(t)=\cosh(t)$. So based upon the signs shown in tables 9.1.1K and 9.1.2K we can conclude that $g(t)=\cosh(t)$ is decreasing on $(-\infty,0)$, increasing on $(0,\infty)$, and always concave up. We should note that the point $(0,1)$ is a local minimum point on the graph of g and that g has a horizontal tangent line at that point.

E9.1.5 Each of the hyperbolic limits are based upon these four basic limits: $\lim_{t\to\infty} e^t=\infty$, $\lim_{t\to-\infty} e^t=0$, $\lim_{t\to\infty} e^{-t}=0$, and $\lim_{t\to-\infty} e^{-t}=\infty$

$$\lim_{t\to-\infty}\cosh(t)=\lim_{t\to-\infty}\frac{e^t+e^{-t}}{2}$$
$$=\infty$$

$$\lim_{t\to-\infty}\sinh(t)=\lim_{t\to-\infty}\frac{e^t-e^{-t}}{2}$$
$$=-\infty$$

$$\lim_{t\to\infty}\cosh(t)=\lim_{t\to\infty}\frac{e^t+e^{-t}}{2}$$
$$=\infty$$

$$\lim_{t\to\infty}\sinh(t)=\lim_{t\to\infty}\frac{e^t-e^{-t}}{2}$$
$$=\infty$$

E9.1.6

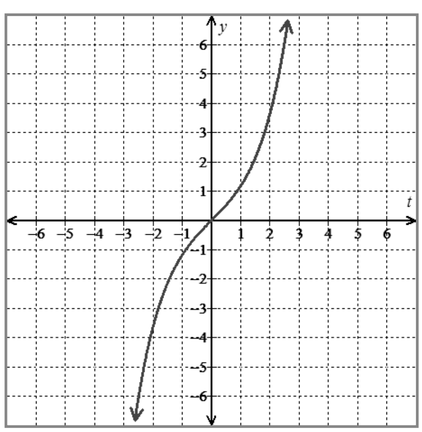

Figure 9.1.1K: $y=\sinh(t)$

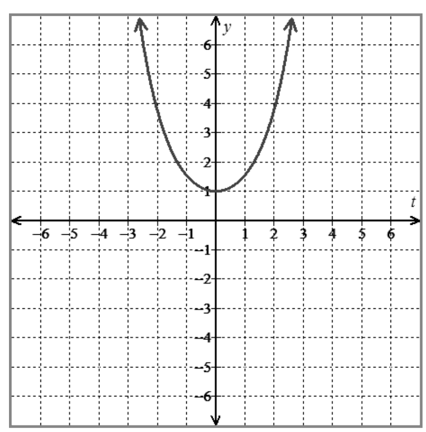

Figure 9.1.2K: $y=\cosh(t)$

E9.1.7

$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} = \frac{\frac{e^t - e^{-t}}{2}}{\frac{e^t + e^{-t}}{2}} = \frac{e^t - e^{-t}}{e^t + e^{-t}}$$

$$\operatorname{sech}(t) = \frac{1}{\cosh(t)} = \frac{2}{e^t + e^{-t}}$$

$$\operatorname{csch}(t) = \frac{1}{\sinh(t)} = \frac{2}{e^t - e^{-t}}$$

$$\coth(t) = \frac{1}{\tanh(t)} = \frac{e^t + e^{-t}}{e^t - e^{-t}}$$

E9.1.8 A good guess for $\frac{d}{dt}(\tanh(t))$ would be $\operatorname{sech}(t)$. Checking it out …

$$\begin{aligned}
\frac{d}{dt}(\tanh(t)) &= \frac{d}{dt}\left(\frac{e^t - e^{-t}}{e^t + e^{-t}}\right) \\
&= \frac{\left(e^t - e^{-t}\cdot -1\right)\left(e^t + e^{-t}\right) - \left(e^t - e^{-t}\right)\left(e^t + e^{-t}\cdot -1\right)}{\left(e^t + e^{-t}\right)^2} \\
&= \frac{\left(e^t + e^{-t}\right)\left(e^{-t} + e^{-t}\right) - \left(e^t - e^{-t}\right)\left(e^t - e^{-t}\right)}{\left(e^t + e^{-t}\right)^2} \\
&= \frac{e^{2t} + 1 + 1 + e^{-2t} - \left(e^{2t} - 1 - 1 + e^{-2t}\right)}{\left(e^t + e^{-t}\right)^2} \\
&= \frac{4}{\left(e^t + e^{-t}\right)^2} \\
&= \left[\frac{2}{e^t + e^{-t}}\right]^2 \\
&= \operatorname{sech}^2(t)
\end{aligned}$$

E9.1.9 $f'(t) = \left[\frac{2}{e^t + e^{-t}}\right]^2$ is always positive so f is always increasing.

$f(0) = 0$ and $f'(0) = 1$

E9.1.10

$$\lim_{t\to-\infty}\tanh(t)=\lim_{t\to-\infty}\frac{e^{t}-e^{-t}}{e^{t}+e^{-t}}$$
$$=\lim_{t\to-\infty}\left(\frac{e^{t}-e^{-t}}{e^{t}+e^{-t}}\cdot\frac{e^{t}}{e^{t}}\right)$$
$$==\lim_{t\to-\infty}\frac{e^{2t}-1}{e^{2t}+1}$$
$$=\frac{0-1}{0+1}$$
$$=-1$$

$$\lim_{t\to\infty}\tanh(t)=\lim_{t\to\infty}\frac{e^{t}-e^{-t}}{e^{t}+e^{-t}}$$
$$=\lim_{t\to\infty}\left(\frac{e^{t}-\frac{1}{e^{t}}}{e^{t}+\frac{1}{e^{t}}}\cdot\frac{\frac{1}{e^{t}}}{\frac{1}{e^{t}}}\right)$$
$$=\lim_{t\to\infty}\left(\frac{1-\frac{1}{e^{2t}}}{1+\frac{1}{e^{2t}}}\right)$$
$$=\frac{1-0}{1+0}$$
$$=1$$

E9.1.11

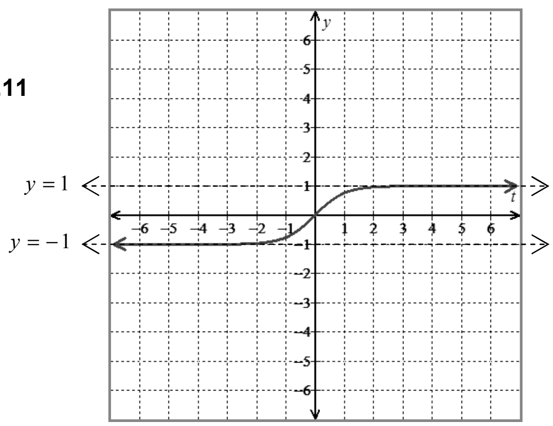

Figure 9.1.3K: $y=\tanh(t)$

Exercise 9.2

$$k'(t)=\frac{8}{3}t^{5/3}-\frac{512}{3}t^{-1/3}$$
$$=\frac{8t^{5/3}}{3}-\frac{512}{3t^{1/3}}$$
$$=\frac{8t^{5/3}}{3}\cdot\frac{t^{1/3}}{t^{1/3}}-\frac{512}{3t^{1/3}}$$
$$=\frac{8t^{2}-512}{3t^{1/3}}$$
$$=\frac{8\left(t^{2}-64\right)}{3t^{1/3}}$$
$$=\frac{8(t-8)(t+8)}{3t^{1/3}}$$

The domain of k is $(-\infty,\infty)$. $k'(t)=0$ at 8 and -8. Over the domain of k, k' is undefined at 0. So the critical numbers of k are 8, -8, and 0.

Table 9.2.1K: Behavior of k based upon sign analysis for k'

Interval	k'	k
$(-\infty,-8)$	negative	decreasing
$(-8,0)$	positive	increasing
$(0,8)$	negative	decreasing
$(8,\infty)$	positive	increasing

The local minimum points on k are $(-8,-768)$ and $(8,768)$. The only local maximum point on k is $(0,0)$.

Exercise 9.3

$$f'(x) = 2\cos(x)\cdot -\sin(x) + \cos(x)$$
$$= \cos(x)(1 - 2\sin(x))$$

The domain of f has been restricted to $[0, 2\pi]$. Over $[0, 2\pi]$, $\cos(x) = 0$ at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ and $\sin(x) = \frac{1}{2}$ at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. So over the restricted domain $f'(x) = 0$ at $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{\pi}{6}$, and $\frac{5\pi}{6}$. f' is never undefined. So the critical numbers of f are $\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{5\pi}{6}$, and $\frac{3\pi}{2}$.

Table 9.3.1K: Behavior of f based upon sign analysis for f'

Interval	f'	f
$\left(0, \frac{\pi}{6}\right)$	positive	increasing
$\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$	negative	decreasing
$\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$	positive	increasing
$\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$	negative	decreasing
$\left(\frac{3\pi}{2}, 2\pi\right)$	positive	increasing

Over $[0, 2\pi]$ the local minimum points on f are $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{3\pi}{2}, -1\right)$ and the local maximum points are $\left(\frac{\pi}{6}, \frac{5}{4}\right)$ and $\left(\frac{5\pi}{6}, \frac{5}{4}\right)$.

Exercise 9.4

$$g'(t) = \frac{(1)(t^3) - (t+9)(3t^2)}{(t^3)^2}$$
$$= \frac{t^3 - 3t^3 - 27t^2}{t^6}$$
$$= \frac{-2t^3 - 27t^2}{t^6}$$
$$= \frac{t^2(-2t - 27)}{t^6}$$
$$= \frac{-2t - 27}{t^4}$$

$$g''(t) = \frac{(-2)(t^4) - (-2t - 27)(4t^3)}{(t^4)^2}$$
$$= \frac{-2t^4 + 8t^4 + 108t^3}{t^8}$$
$$= \frac{6t^4 + 108t^3}{t^8}$$
$$= \frac{2t^3(3t + 54)}{t^8}$$
$$= \frac{2\cdot 3(t + 18)}{t^5}$$
$$= \frac{6(t + 18)}{t^5}$$

$g''(t)=0$ at -18 and $g''(t)$ is undefined at 0.

Table 9.4.1K: Behavior of g based upon sign analysis for g''

Interval	g''	g
$(-\infty,-18)$	positive	concave up
$(-18,0)$	negative	concave down
$(0,\infty)$	positive	concave up

The only inflection point on g is $\left(-18,\dfrac{1}{648}\right)$; g has a vertical asymptote at 0.

Exercise 9.5

E9.5.2

$$\begin{aligned} f'(x) &= \frac{(1)(x+2)^2-(x-3)(2(x+2))}{\left[(x+2)^2\right]^2} \\ &= \frac{(x+2)\left[(x+2)-(x-3)\cdot 2\right]}{(x+2)^4} \\ &= \frac{\left[x+2-2x+6\right]}{(x+2)^3} \\ &= \frac{-x+8}{(x+2)^3} \end{aligned}$$

$$\begin{aligned} f''(x) &= \frac{(-1)(x+2)^3-(-x+8)\left(3(x+2)^2\right)}{\left[(x+2)^3\right]^2} \\ &= \frac{(x+2)^2\left[-1\cdot(x+2)-(-x+8)\cdot 3\right]}{(x+2)^6} \\ &= \frac{\left[-x-2+3x-24\right]}{(x+2)^4} \\ &= \frac{2(x-13)}{(x+2)^4} \end{aligned}$$

Table 9.5.1K:
Behavior of f based upon sign analysis for f'

Interval	f'	f
$(-\infty,-2)$	negative	decreasing
$(-2,8)$	positive	increasing
$(8,\infty)$	negative	decreasing

Table 9.5.2K:
Behavior of f based upon sign analysis for f''

Interval	f''	f
$(-\infty,-2)$	negative	concave down
$(-2,13)$	negative	concave down
$(13,\infty)$	positive	concave up

We should begin by noting that $x=-2$ is a vertical asymptote for the graph of f. Based upon Table 9.5.1K we can conclude that f has no local minimum points and that the only local maximum point on f is $\left(8,\dfrac{1}{20}\right)$; we should note that the tangent line to f at 8 is horizontal. Based upon Table 9.5.2K we can conclude that the only inflection point on f is $\left(13,\dfrac{2}{45}\right)$. Finally, we can determine the horizontal asymptote(s) for f by looking at $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x-3}{(x+2)^2}$$
$$= \lim_{x\to\infty} \frac{x-3}{x^2+4x+4}$$
$$= \lim_{x\to\infty} \left(\frac{x-3}{x^2+4x+4} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \right)$$
$$= \lim_{x\to\infty} \left(\frac{\frac{1}{x} - \frac{3}{x^2}}{1 + \frac{4}{x} + \frac{4}{x^2}} \right)$$
$$= \frac{0-0}{1+0+0}$$
$$= 0$$

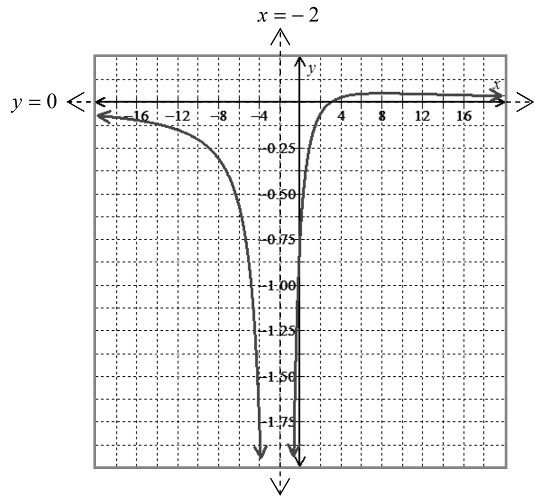

Figure 9.5.1K: $y = \frac{x-3}{(x+2)^2}$

Similarly, $\lim_{x\to-\infty} f(x) = 0$, so the only horizontal asymptote for the graph of f is $y = 0$.

E9.5.2

$$g(x) = x^{5/3} + 5x^{2/3}$$
$$g'(x) = \frac{5}{3}x^{2/3} + \frac{10}{3}x^{-1/3}$$
$$= \frac{5x^{2/3}}{3} + \frac{10}{3x^{1/3}}$$
$$= \frac{5x^{2/3}}{3} \cdot \frac{x^{1/3}}{x^{1/3}} + \frac{10}{3x^{1/3}}$$
$$= \frac{5x+10}{3x^{1/3}}$$
$$= \frac{5(x+2)}{3x^{1/3}}$$

$$g'(x) = \frac{5}{3}x^{2/3} + \frac{10}{3}x^{-1/3}$$
$$g''(x) = \frac{10}{9}x^{-1/3} - \frac{10}{9}x^{-4/3}$$
$$= \frac{10}{9x^{1/3}} - \frac{10}{9x^{4/3}}$$
$$= \frac{10}{9x^{1/3}} \cdot \frac{x}{x} - \frac{10}{9x^{4/3}}$$
$$= \frac{10(x-1)}{9x^{4/3}}$$

Table 9.5.3K:
Behavior of g based upon sign analysis for g'

Interval	g'	g
$(-\infty, -2)$	positive	increasing
$(-2, 0)$	negative	decreasing
$(0, \infty)$	positive	increasing

Table 9.5.4K:
Behavior of f based upon sign analysis for g''

Interval	g''	g
$(-\infty, 0)$	negative	concave down
$(0, 1)$	negative	concave down
$(1, \infty)$	positive	concave up

We should begin by noting g is everywhere continuous. Based upon Table 9.5.3K we can conclude that the only local minimum point on g is $(0,0)$ and that the only local maximum point on g is $\left(-2,3\sqrt[3]{4}\right)$. We should also note that g is nondifferentiable at $(0,0)$ and that g has a horizontal tangent line at $\left(-2,3\sqrt[3]{4}\right)$. Based upon Table 9.5.4K we can conclude that the only inflection point on g f is $(1,6)$. We should note that the slope of g at $(1,6)$ is given by $g'(1)$ which is $\frac{5}{\sqrt[3]{3}}$ (about 3 and a half). Finally we need to note that the graph of g has no asymptotes.

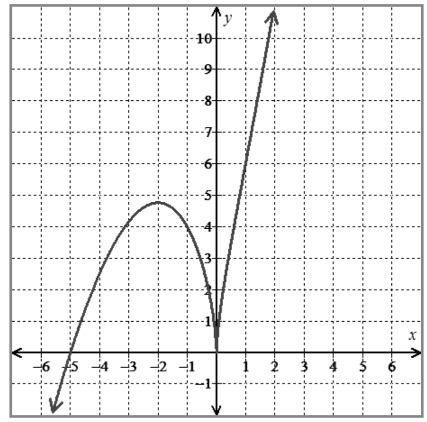

Figure 9.5.1K: $y = x^{2/3}(x+5)$

E9.5.3

$$k'(x) = \frac{2(x-4)(x+3)-(x-4)^2(1)}{(x+3)^2} = \frac{(x-4)\left[2(x+3)-(x-4)\right]}{(x+3)^2} = \frac{(x-4)(x+10)}{(x+3)^2}$$

Note that $k'(x) = \frac{x^2+6x-40}{(x+3)^2}$

$$k''(x) = \frac{(2x+6)(x+3)^2-\left(x^2+6x-40\right)\left(2(x+3)\right)}{\left[(x+3)^2\right]^2} = \frac{(x+3)\left[(2x+6)(x+3)-\left(x^2+6x-40\right)\cdot 2\right]}{(x+3)^4} = \frac{\left[2x^2+6x+6x+18-2x^2-12x+80\right]}{(x+3)^3} = \frac{98}{(x+3)^3}$$

Table 9.5.5K:
Behavior of k based upon sign analysis for k'

Interval	k'	k
$(-\infty,-10)$	positive	increasing
$(-10,-3)$	negative	decreasing
$(-3,4)$	negative	decreasing
$(4,\infty)$	positive	increasing

Table 9.5.6K:
Behavior of f based upon sign analysis for k''

Interval	k''	k
$(-\infty,-3)$	negative	concave down
$(-3,\infty)$	positive	concave up

We should begin by noting that $x=-3$ is a vertical asymptote for the graph of k. Based upon Based upon Table 9.5.5K we can conclude that the only local minimum point on k is $(4,0)$ and that the only local maximum point on k is $(-10,-28)$; we should note that k has horizontal tangent lines at both of its relative extreme points. Based upon Table 9.5.6K we can conclude that k has no inflection points (because there is a vertical tangent line at the only value of x where k changes concavity). Finally, we can determine the horizontal asymptote(s) for k by looking at $\lim\limits_{x \to -\infty} k(x)$ and $\lim\limits_{x \to \infty} k(x)$.

Because the degree of the numerator is one more than the degree of the denominator in the formula for k, neither $\lim\limits_{x \to -\infty} k(x)$ nor $\lim\limits_{x \to \infty} k(x)$ exists. Hence the graph of k has no horizontal asymptotes.

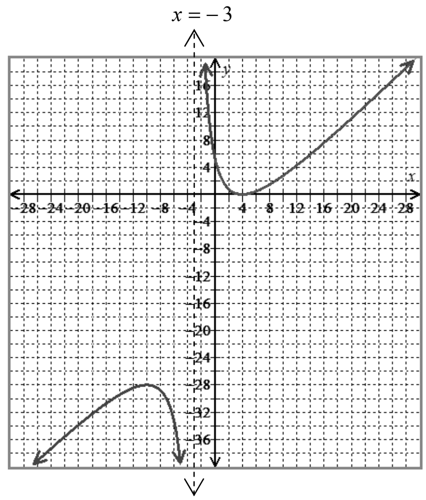

Figure 9.5.1K: $y = \dfrac{x-3}{(x+2)^2}$