

Related Rates

Activity 44

Consider the toy rocket shown in Figure 44.1. As the rocket's elevation (x , measured in feet) changes the angle of elevation from the ground to the base of the rocket (θ , measured in radians) also changes. Consequently, there is a relationship between the **rates** at which the elevation and angle of elevation **change**. This is an example of the topic we call **related rates**.

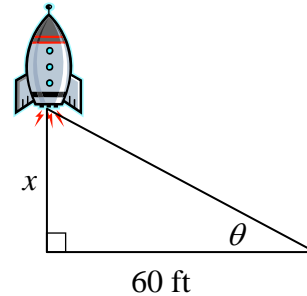


Figure 44.1: Johnny's Rocket

If we assume that the rocket flies straight upward (and subsequently falls straight downward) and we define x and θ as functions of time, t , where t is the amount of time that has passed (s) since the rocket was launched, then the rates of change in the elevation and angle of elevation are, respectively, $\frac{dx}{dt}$ (measure in ft/s) and $\frac{d\theta}{dt}$ (measured in rad/s).

Using simple right triangle trigonometry we determine the relation equation between x and θ (Equation 44.1). Since x and θ both vary as functions of t , we can differentiate both sides of Equation 44.1 with respect to t resulting in the rate equation (Equation 44.2)

$$\text{Equation 44.1: } \tan(\theta) = \frac{x}{60}$$

$$\text{Equation 44.2: } \sec^2(\theta) \frac{d\theta}{dt} = \frac{1}{60} \frac{dx}{dt}$$

Problem 44.1

Rates of change in the rocket's elevation are given for two elevations in Table 44.1.

- 44.1.1 Use Equation 44.1 to determine the values of θ at the given elevations. **Do not approximate these value ... state their exact values.**
- 44.1.2 Use Equation 44.2 to determine the corresponding values of $\frac{d\theta}{dt}$. Round each of these values to the nearest hundredth.
- 44.1.3 Write two complete sentences that fully communicate what is happening to the rocket's elevation and angle of elevation as implied by the values in Table 44.1. Each sentence should include the elevation of the rocket, a clear indication of whether the rocket is rising or falling, the speed at which the rocket is moving, a clear indication of whether the angle of elevation is increasing or decreasing, and the rate at which the angle is increasing or decreasing. All values should be stated using appropriate units.

Table 44.1: Tracking Johnny's Rocket

x (ft)	$\frac{dx}{dt}$ (ft/s)	θ (rad)	$\frac{d\theta}{dt}$ (rad/s)
$60\sqrt{3}$	6		
60	-38		

Activity 45

There is something quite different about the manner in which variables are defined while working related rates problems from the way in which you defined variables when working applied problems in earlier classes. In most applications, you define your variables to explicitly represent the quantity (or quantities) you are asked to find. When defining variables for a related rates problem, however, you define a variable from which you can *infer* the value you are trying to find and, more uniquely, you define variables for quantities whose values you are actually given!

Problem 45.1

At 2 PM one day, Muffin and Rex were sleeping atop one another. A loud noise startled the animals and Muffin began to run due north at a constant rate of 1.7 ft/s while Rex ran due east at a constant rate of 2.1 ft/s. The critters maintained these paths and rates for several seconds. Suppose that you wanted to calculate the rate at which the distance between the cat and dog was changing three seconds into their run.

- 45.1.1 A right triangle representing this problem has been drawn in Figure 45.1. The length of each side and the measure of each angle have (temporarily) been assigned variable names; these measurements are the *potential* variables for the problem. Circle the variable whose rate you are trying to determine and the two variables whose rate you are given. These three measurements are the actual variables for the problem.
- 45.1.2 One of the remaining pieces in Figure 45.1 does not change value as the animals run. Cross out the corresponding variable and replace it with its fixed value.
- 45.1.3 Cross out any potential variable not addressed in problems 45.1.1 and 45.1.2; although changing value, these pieces of the picture are irrelevant to the question at hand.
- 45.1.4 Copy the new and improved diagram (with the relevant variables and fixed value) onto your own paper; this is your working diagram for the problem. Explicitly define a time variable and then define your three length variables making sure that you establish their dependence upon time. The *rate* variables will emerge when you differentiate the length variables with respect to time.
- 45.1.5 Use your diagram to determine the relation equation and then differentiate both sides of that equation with respect to time. The resultant equation is your rate equation.
- 45.1.6 What are the values of the length variables three seconds into the animals' panicked runs? Which rate values in the rate equation do you know and what are their values?
- 45.1.7 Substitute the known values into the rate equation and solve for the unknown rate.
- 45.1.8 Write a contextual conclusion sentence that clearly indicates whether the distance between the cat and dog is increasing or decreasing and clearly communicates the rate at which this change is happening.

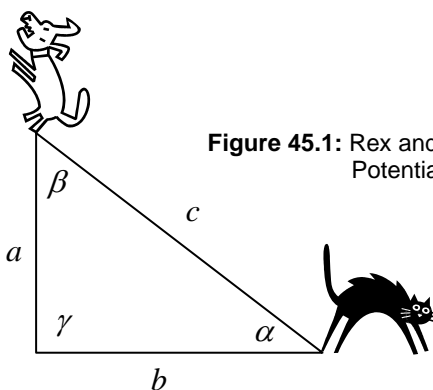


Figure 45.1: Rex and Muffin and the Potential Variables

Problem 45.2

Schuyler's clock is kaput; the minute hand functions as it should but the hour hand is stuck at 4. The minute hand on the clock is 30 cm long and the hour hand is 10 cm long. In this problem you are going to determine the rate of change between the tips of the hands every time the minute hand points directly at 12.

- 45.2.1 A triangle representing this problem has been drawn in Figure 45.3; the minute hand has deliberately been drawn so that it points at a number other than 12. This diagram represents the motion of the minute hand, not its position at one specific time. The hour hand, however, has been drawn in its fixed position.

The length of each side of the triangle and the measure of each angle of the triangle have (temporarily) been assigned variable names; these are the *potential* variables for the problem. Circle the variable whose rate you are trying to determine. One of the variables represents a piece whose rate of change is constant when the minute hand falls between 10 and 4 (in the clock-wise direction). Identify this piece and circle the corresponding variable. The two circled variables are the actual variables for the problem.

- 45.2.2 Two of the remaining pieces in Figure 45.3 do not change value as the time passes. Cross out the corresponding variables and replace them with their fixed values.
- 45.2.3 Cross out any potential variable not addressed in problems 45.2.1 and 45.2.2; although changing value, these pieces of the picture are irrelevant to the question at hand.
- 45.2.4 Copy the new and improved diagram (with the relevant variables and fixed values) onto your own paper; this is your working diagram for the problem. Explicitly define a time variable and then define your angular and length variables making sure that you establish their dependence upon time. Make sure that you use radians as the measure of the angular variable. The *rate* variables will emerge when you differentiate the angular and length variables with respect to time.
- 45.2.5 Use your diagram to determine the relation equation and then differentiate both sides of that equation with respect to time. The resultant equation is your rate equation. You might want to look up the law of cosines before writing down your relation equation.
- 45.2.6 What are the values of the variables when the minute hand points to 12? What is the known rate value in your rate equation? (Careful ... is this rate positive or negative?)
- 45.2.7 Substitute the known values into the rate equation and solve for the unknown rate. Round your solution to the nearest 10^{th} .
- 45.2.8 Write a contextual conclusion sentence that clearly indicates whether the distance between the tip of the clock hands is increasing or decreasing and clearly communicates the rate at which this change is happening.



Figure 45.2: Schuyler's clock

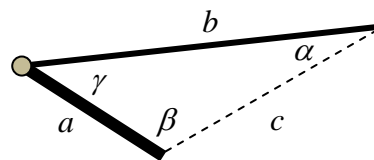


Figure 45.3: Potential Variables

Activity 46

Sometimes the relation equation is based upon a given or "known" formula.

Problem 46.1

Slushy is flowing out of the bottom of a cup at the constant rate of $0.1 \text{ cm}^3/\text{s}$. The cup is the shape of a right circular cone. The height of the cup is 12 cm and the cup (when full) holds a total of $36\pi \text{ cm}^3$ of slushy. Determine the rate at which the height of the remaining slushy changes at the instant there are exactly $8\pi \text{ cm}^3$ of slushy remaining in the cup.

- 46.1.1** The volume formula for a right circular cylinder is $V = \frac{\pi}{3} r^2 h$ where V is the volume of the cone, h is the height of the cone, and r is the radius at the top of the cone. We ultimately are going to define two of these variables in terms of the amount of slushy remaining in the cup. Which are the two variables relevant to the question at hand? That is, which quantity's rate of change are you given and which quantity's rate of change are you trying to determine?
- 46.1.2** Hopefully you determined that V and h are the relevant variables. This means that r needs to be eliminated from the volume formula. A cross section of the cup and slushy is shown in Figure 46.2.
- What is the radius at the top of the cup?
 - Use the concept of similar triangle to express r in terms of h . Substitute this expression into the volume formula and simplify. The resultant equation is the relation equation for the problem.
- 46.1.3** Explicitly define h and V (including units) in terms of the amount of slushy remaining in the cup. Make sure that you communicate that each variable is dependent upon time and that you explicitly define your time variable. The *rate* variables will emerge when you differentiate the height and volume variables with respect to time.
- 46.1.4** Go ahead and complete the problem in a manner consistent with that implied in problems 45.1 and 45.2.



Figure 46.1: Slushy cups

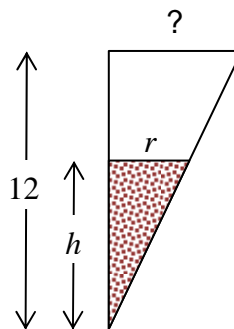


Figure 46.2: Slushy cross-section

Problem 46.2

In the classic TV series *Batman*, story lines almost always lasted for two episodes. At the end of the first episode, Batman and Robin were invariably caught in some diabolical trap intended to end their lives. In one episode, Egghead decided to both roast and crush the dynamic duo. He placed the caped crusaders in a room whose temperature steadily rose as two of the walls closed in on the men in tights.

According to the ideal gas law, the pressure (P), temperature (T), and volume (V) inside the room were related by the equation $\frac{PV}{T} = k$ where k is a constant specific to the types and amounts of gas that were in the room.

When the heroes were placed in the room, the room was $2\text{ m} \times 2\text{ m} \times 2\text{ m}$, the temperature in the room was 20°C , and the air pressure in the room was 100 kPa. Once Egghead activated the trap, two of the walls moved toward each other at the constant rate of .1 m/min (in total) and the temperature in the room rose at a constant rate of $2^\circ\text{C}/\text{min}$. Since the volume and temperature both changed at constant rates, it might seem intuitive to you that the pressure also changed at a constant rate. Let's see if that is in fact the case.

- 46.2.1 Use the initial conditions in the room to determine the value of k (without unit) and rewrite the ideal gas law using this value for k .
- 46.2.2 Explicitly define P , T , and V (including units) in terms of the amount of time that had passed since the trap was set in motion.
- 46.2.3 Your rate equation comes from differentiating the ideal gas law equation with respect to time. There is a simple algebraic adjustment you can make to the equation that will greatly simplify this task. Go ahead and adjust the equation and then find your rate equation.
- 46.2.4 Determine the values of P , T , and V at both five minutes and eight minutes into the motion of the trap. State the known rate values (remembering to think about the sign on each rate).

You might want to think twice about the value of $\frac{dV}{dt}$. Use all of these values along with the rate equation to determine the value of $\frac{dP}{dt}$ at both five minutes and eight minutes into the motion of the trap.



Figure 46.3: Egghead



Figure 46.4: The Dynamic Duo

Activity 47

Solve each of the following problems using procedures similar to those suggested in the previous problems.

Problem 47.1

It was a dark night and 5.5 ft tall Bahram was walking towards a street lamp whose light was perched 40 ft into the air. As he walked, the light caused a shadow to fall behind Bahram. When Bahram was 80 feet from the base of the lamp he was walking at a pace of 2 ft/s. At what rate was the length of Bahram's shadow changing at that instant? Make sure that each stated and calculated rate value has the correct sign. Make sure that your conclusion sentence clearly communicates whether the length of the shadow was increasing or decreasing at the indicated time.

Problem 47.2

The gravitational force, F , between two objects in space with masses m_1 and m_2 is given by the formula $F = \frac{G m_1 m_2}{r^2}$ where r is the distance between the objects' centers of mass and G is the universal gravitational constant.

Two pieces of space junk, one with mass 500 kg and the other with mass 3000 kg, were drifting directly toward one another. When the objects' centers of mass were 250 km apart the lighter piece was moving at a rate of 0.5 km/hour and the heavier piece was moving at a rate of 0.9 km/hour. Leaving G as a constant, determine the rate at which the gravitational force between the two objects was changing at that instant. Make sure that each stated and calculated rate value has the correct sign. Make sure that your conclusion sentence clearly communicates whether the force was increasing or decreasing at that given time. The unit for F is Newtons (N).

Problem 47.3

An eighteen inch pendulum sitting atop a table is in the downward part of its motion. The pivot point for the pendulum is 30 inches above the table top. When the pendulum is 45° away from vertical, the angle formed at the pivot is decreasing at the rate of $25^\circ/\text{s}$. At what rate is the end of the pendulum approaching the table top at this instant?

Hint: Draw a right triangle with one acute angle vertex at the pivot point and the other acute angle vertex at the end of the pendulum. The rate of change of one piece of this triangle is the same as the rate you are trying to find.

Make sure that each stated and calculated rate value has the correct sign. Make sure that you use appropriate units when defining your variables.