Rates of Change

Activity 1

Motion is frequently modeled using calculus. A building block for this application is the concept of ***average velocity***. Average velocity is defined to be net displacement divided by elapsed time. More precisely, if p is a position function for something moving along a numbered line, then we define the average velocity over the time interval $[t_0, t_1]$ to be:

Expression 1.1: $$\frac{p(t_1)-p(t_0)}{t_1-t_o}$$

Problem 1.1

According to simplified Newtonian physics, if an object is dropped from a height of 200 m and allowed to freefall to the ground, then the height of the object (measure in m) is given by the position function $p(t)=200-4.9t^2$ where t is the amount of time that has passed since the object was dropped (measured in s).

1.1.1 What, *including unit*, are the values of $p(t)$ three seconds and five seconds into the object's fall? Use these values when working problem 1.1.2.

1.1.2 Calculate $\dfrac{p(5\text{s})-p(3\text{s})}{5\text{s}-3\text{s}}$; *include units **while making** the calculation*. What does the result tell you in the context of this problem?

1.1.3 Use Expression 1.1 to find a formula for the average velocity of this object over the general time interval $[t_0, t_1]$. The first couple of lines of this process are shown below. ***Copy these lines onto your paper*** and continue the simplification process.

$$\begin{aligned}\frac{p(t_1)-p(t_0)}{t_1-t_0} &= \frac{\left[200-4.9t_1^{\,2}\right]-\left[200-4.9t_0^{\,2}\right]}{t_1-t_0}\\ &= \frac{200-4.9t_1^{\,2}-200+4.9t_0^{\,2}}{t_1-t_0}\\ &= \frac{-4.9t_1^{\,2}+4.9t_0^{\,2}}{t_1-t_0}\end{aligned}$$

Hint

In the next step you should factor -4.9 from the numerator; the remaining factor will factor further.

1.1.4 Check the formula you derived in problem 1.1.3 using $t_0=3$ and $t_1=5$; that is, compare the value generated by the formula to that you found in problem 1.1.2.

1.1.5 Using the formula found in problem 1.1.3, replace t_0 with 3 but leave t_1 as a variable; simplify the result. Then copy Table 1.1 onto your paper and fill in the missing entries.

Table 1.1: $y=\dfrac{p(t_1)-p(3)}{t_1-3}$

t_1 (s)	y (m/s)
2.9	
2.99	
2.999	
3.001	
3.01	
3.1	

1.1.6 As the value of t_1 gets closer to 3, the values in the y column of Table 1.1 appear to be converging on a single number; what is this number and what do you think it tells you in the context of this problem?

Activity 2

One of the building blocks in differential calculus is ***the secant line to a curve***. It is very easy for a line to be considered a secant line to a curve; the only requirement that must be fulfilled is that the line intersects the curve in at least two points.

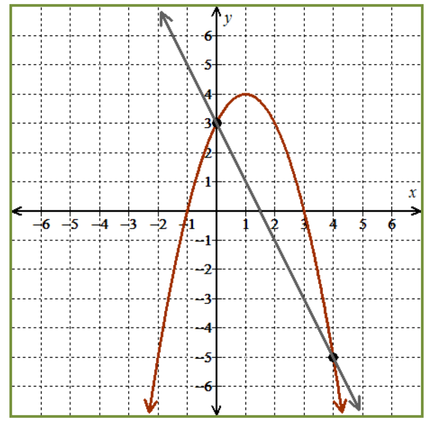

Figure 2.1: f

In Figure 2.1, a secant line to the curve $y = f(x)$ has been drawn through the points $(0,3)$ and $(4,-5)$. You should verify that the slope of this line is -2.

The formula for f is $f(x) = 3 + 2x - x^2$. We can use this formula to come up with a generalized formula for the slope of secant lines to this curve. Specifically, the slope of the line connecting the point $(x_0, f(x_0))$ to the point $(x_1, f(x_1))$ is derived in Example 2.1.

Example 2.1

$$
\begin{aligned}
m_{\sec} &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\
&= \frac{(3 + 2x_1 - x_1^2) - (3 + 2x_0 - x_0^2)}{x_1 - x_0} \\
&= \frac{3 + 2x_1 - x_1^2 - 3 - 2x_0 + x_0^2}{x_1 - x_0} \\
&= \frac{(2x_1 - 2x_0) - (x_1^2 - x_0^2)}{x_1 - x_0} \\
&= \frac{2(x_1 - x_0) - (x_1 + x_0)(x_1 - x_0)}{x_1 - x_0} \\
&= \frac{[2 - (x_1 + x_0)](x_1 - x_0)}{x_1 - x_0} \\
&= 2 - x_1 - x_0 \quad \text{for} \quad x_1 \neq x_0
\end{aligned}
$$

This factoring technique is called ***factoring by grouping***.

We can check our formula using the line in Figure 2.1. If we let $x_0 = 0$ and $x_1 = 4$ then our simplified slope formula gives us:

$$
\begin{aligned}
2 - x_1 - x_0 &= 2 - 4 - 0 \\
&= -2 \quad ✓
\end{aligned}
$$

Problem 2.1

Let $g(x) = x^2 - 5$.

2.1.1 Following Example 2.1, find a formula for the slope of the secant line connecting the points $(x_0, g(x_0))$ and $(x_1, g(x_1))$. Please note that factoring by grouping will ***not*** be necessary when simplifying the expression.

2.1.2 Check your slope formula using the two points indicated in Figure 2.2. That is, use the graph to find the slope between the two points and then use your formula to find the slope; make sure that the two values agree!

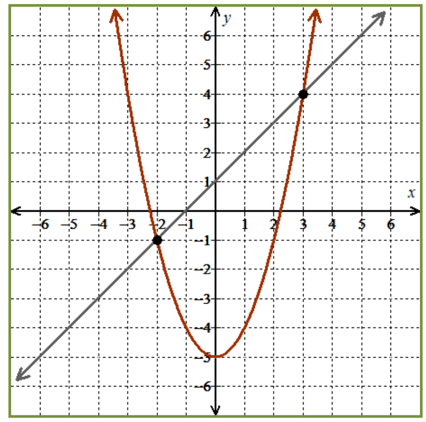

Figure 2.2: g

Activity 3

While it's easy to see that the formula $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ gives the slope of the line connecting two points on the function f, the resultant expression can at times be awkward to work with. We actually already saw that when we had to use slight-of-hand factoring in Example 2.1.

The algebra associated with secant lines (and average velocities) can sometimes be simplified if we designate the variable h to be the run between the two points (or the length of the time interval). With this designation we have $x_1 - x_0 = h$ which gives us $x_1 = x_0 + h$. Making these substitutions we get Equation 3.1. The expression on the right side of Equation 3.1 is called ***the difference quotient for f***.

Equation 3.1 $$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}$$

Let's revisit the function $f(x) = 3 + 2x - x^2$ from Example 2.1. The difference quotient for this function is derived in Example 3.1.

Example 3.1

$$\begin{aligned}\frac{f(x_0 + h) - f(x_0)}{h} &= \frac{\left[3 + 2(x_0 + h) - (x_0 + h)^2\right] - \left[3 + 2x_0 - x_0^{\,2}\right]}{h} \\ &= \frac{3 + 2x_0 + 2h - x_0^{\,2} - 2x_0 h - h^2 - 3 - 2x_0 + x_0^{\,2}}{h} \\ &= \frac{2h - 2x_0 h - h^2}{h} \\ &= \frac{h(2 - 2x_0 - h)}{h} \\ &= 2 - 2x_0 - h \quad \text{for } h \neq 0\end{aligned}$$

Please notice that all of the terms without a factor of h subtracted to zero. Please notice, too, that we avoided all of the tricky factoring that appeared in Example 2.1!

For simplicity's sake, we generally drop the variable subscript when applying the difference quotient. So for future reference we will define the difference quotient as follows:

Definition 3.1

The ***difference quotient*** for the function $y = f(x)$ is the expression $\dfrac{f(x+h) - f(x)}{h}$.

Problem 3.1

Completely simplify the difference quotient for each of the following functions. Please note that the template for the difference quotient needs to be adapted to the function name and independent variable in each given equation. For example, the difference quotient for the function in problem 3.1.1 is $\dfrac{v(t+h) - v(t)}{h}$.

Please make sure that you lay out your work in a manner consistent with the way the work is shown in example 3.1 (excluding the subscripts, of course).

3.1.1 $v(t) = 2.5t^2 - 7.5t$ **3.1.2** $g(x) = 3 - 7x$ **3.1.3** $w(x) = \dfrac{3}{x+2}$

Problem 3.2

Suppose that an object is tossed into the air in such a way that the elevation of the object (measured in ft) is given by the function $s(t) = 40 + 40t - 16t^2$ where t is the amount of time that has passed since the object was tossed (measured in s).

3.2.1 Simplify the difference quotient for s.

3.2.2 Ignoring the unit, use the difference quotient to determine the average velocity over the interval $[1.6, 2.8]$. (**Hint:** Use $t = 1.6$ and $h = 1.2$. Make sure that you understand why!)

3.2.3 What, *including unit*, are the values of $s(1.6)$ and $s(2.8)$? Use these values when working problem 3.2.4.

3.2.4 Use the expression $\dfrac{s(2.8) - s(1.6)}{2.8 - 1.6}$ to verify the value you found in problem 3.2.2. *Include the unit* ***while making*** *this calculation*.

3.2.5 Ignoring the unit, use the difference quotient to determine the average velocity over the interval $[0.4, 2.4]$.

Problem 3.3

Moose and squirrel were having casual conversation when suddenly, without any apparent provocation, Boris Badenov launched anti-moose missile in their direction. Fortunately, squirrel had ability to fly as well as great knowledge of missile technology, and he was able to disarm missile well before it hit ground.

The elevation (ft) of the tip of the missile t seconds after it was launched is given by the function $h(t) = -16t^2 + 294.4t + 15$.

3.3.1 What, including unit, is the value of $h(12)$ and what does the value tell you about the flight of the missile?

3.3.2 What, including unit, is the value of $\dfrac{h(10\text{ s}) - h(0\text{ s})}{10\text{ s}}$ and what does this value tell you about the flight of the missile?

3.3.3 The velocity (ft/s) function for the missile is $v(t) = -32t + 294.4$. What, including unit, is the value of $\dfrac{v(10\text{ s}) - v(0\text{ s})}{10\text{ s}}$ and what does this value tell you about the flight of the missile?

Problem 3.4

Timmy lived a long life in the 19th century. When Timmy was seven he found a rock that weighed exactly half a stone. (Timmy lived in jolly old England, don't you know.) That rock sat on Timmy's window sill for the next 80 years and wouldn't you know the weight of that rock did not change even one smidge the entire time. In fact, the weight function for this rock was $w(t) = 0.5$ where $w(t)$ was the weight of the rock (stones) and t was the number of years that had passed since that day Timmy brought the rock home.

3.4.1 What was the average rate of change in the weight of the rock over the 80 years it sat on Timmy's window sill?

3.4.2 Ignoring the unit, simplify the expression $\dfrac{w(t_1) - w(t_0)}{t_1 - t_0}$. Does the result make sense in the context of this problem?

3.4.3 Showing each step in the process and ignoring the unit, simplify the difference quotient for w. Does the result make sense in the context of this problem?

Problem 3.5

Truth be told, there was one day in 1842 when Timmy's mischievous son Nigel took that rock outside and chucked it into the air. The velocity of the rock (ft/s) was given by $v(t)=60-32t$ where t was the number of seconds that had passed since Nigel chucked the rock.

3.5.1 What, *including unit*, are the values of $v(0)$, $v(1)$, and $v(2)$ and what do these values tell you in the context of this problem? Don't just write that the values tell you the velocity at certain times; explain what the velocity values tell you about the motion of the rock.

3.5.2 Ignoring the unit, simplify the difference quotient for v.

3.5.3 What is the unit for the difference quotient for v? What does the value of the difference quotient (including unit) tell you in the context of this problem?

Problem 3.6

Suppose that a vat was undergoing a controlled drain and that the amount of fluid left in the vat (gal) was given by the formula $V(t)=100-2t^{3/2}$ where t is the number of minutes that had passed since the draining process began.

3.6.1 What, *including unit*, is the value of $V(4)$ and what does that value tell you in the context of this problem?

3.6.2 Ignoring the unit, write down the formula you get for the difference quotient of V when $t=4$. Copy Table 3.1 onto your paper and fill in the missing values. ***Look for a pattern in the output and write down enough digits for each entry so that the pattern is clearly illustrated***; the first two entries in the output column have been given to help you understand what is meant by this direction.

Table 3.1: $y=\dfrac{V(4+h)-V(4)}{h}$

h	y
–0.1	–5.962
–0.01	–5.9962
–0.001	
0.001	
0.01	
0.1	

3.6.3 What is the unit for the y values in Table 3.1? What do these values (with their unit) tell you in the context of this problem?

3.6.4 As the value of h gets closer to 0, the values in the y column of Table 3.1 appear to be converging to a single number; what is this number and what do you think that value (with its unit) tells you in the context of this problem?