## Limits and Continuity

**Activity 4**

While working problem 3.6 you completed Table 4.1 (formerly Table 3.1). In the context of that problem the difference quotient being evaluated returned the average rate of change in the volume of fluid remaining in a vat between times $t = 4$ and $t = 4 + h$. As the elapsed time closes in on $0$ this average rate of change converges to $-6$. From that we deduce that the rate of change in the volume 4 minutes into the draining process must have been $-6$ gal/min.

**The context for Problem 3.6**

Suppose that a vat was undergoing a controlled drain and that the amount of fluid left in the vat (gal) was given by the formula $V(t) = 100 - 2t^{3/2}$ where $t$ is the number of minutes that had passed since the draining process began.

**Table 4.1:** $y = \dfrac{V(4+h) - V(4)}{h}$

| $h$ | $y$ |
|------|------|
| $-0.1$ | $-5.962$ |
| $-0.01$ | $-5.9962$ |
| $-0.001$ | $-5.99962$ |
| $0.001$ | $-6.00037$ |
| $0.01$ | $-6.0037$ |
| $0.1$ | $-6.037$ |

Please note that we could not deduce the rate of change 4 minutes into the process by replacing $h$ with $0$; in fact, there are at least two things preventing us from doing so. From a strictly mathematical perspective, we cannot replace $h$ with $0$ because that would lead to division by zero in the difference quotient. From a more physical perspective, replacing $h$ with $0$ would in essence stop the clock. If time is frozen, so is the amount of fluid in the vat and the entire concept of "rate of change" becomes moot.

It turns out that it is frequently more useful (not to mention interesting) to explore the __*trend*__ in a function as the input variable __*approaches*__ a number rather than the actual value of the function at that number. Mathematically we describe these trends using __limits__.

If we call the difference quotient in the heading for Table 4.1 $f(h)$, then we could describe the trend evidenced in the table by saying "the limit of $f(h)$ as $h$ approaches zero is $-6$." Please note that as $h$ changes value, the value of $f(h)$ changes, not the value of the limit. The limit value is a fixed number to which the value of $f(h)$ converges. Symbolically we write $\lim_{h \to 0} f(h) = -6$

Most of the time the value of a function at the number $a$ and the limit of the function as $x$ approaches $a$ are in fact the same number. When this occurs we say that the function is __*continuous*__ at $a$. However, to help you better understand the concept of limit we need to have you confront situations where the function value and limit value are not equal to one another. Graphs can be useful for helping distinguish function values from limit values, so that is the perspective you are going to use in the first couple of problems in this lab.

**Problem 4.1**

Several function values and limit values for the function in Figure 4.1 are given below. You and your group mates should take turns reading the equations aloud. Make sure that you read the symbols correctly, that's part of what you are learning! Also, discuss why the values are what they are and make sure that you get help from your instructor to clear up any confusion.

**Figure 4.1:** $f$

$$f(-2) = 6 \text{ but } \lim_{x \to -2} f(x) = 3$$

$$f(-4) \text{ is undefined but } \lim_{x \to -4} f(x) = 2$$

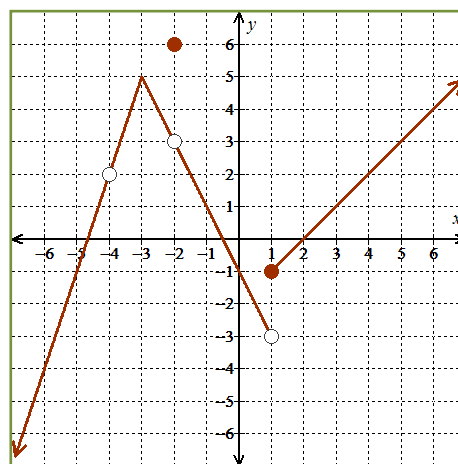$$f(1) = -1 \text{ but } \lim_{x \to 1} f(x) \text{ does not exist}$$

$$\lim_{x \to 1^-} f(x) = -3 \text{ but } \lim_{x \to 1^+} f(x) = -1$$

The limit of $f(x)$ as $x$ approaches 1 from the left.

The limit of $f(x)$ as $x$ approaches 1 from the right.

**Problem 4.2**

Copy each of the following expressions onto your paper and either state the value or state that the value is undefined or doesn't exist. Make sure that when discussing the values you use proper terminology. All expressions are in reference to the function $g$ shown in Figure 4.2.

**Figure 4.2:** $g$

**4.2.1**  $g(5)$       **4.2.2**  $\lim_{t \to 5} g(t)$       **4.2.3**  $g(3)$

**4.2.4**  $\lim_{t \to 3^-} g(t)$   **4.2.5**  $\lim_{t \to 3^+} g(t)$   **4.2.6**  $\lim_{t \to 3} g(t)$

**4.2.7**  $g(2)$       **4.2.8**  $\lim_{t \to 2} g(t)$       **4.2.9**  $g(-2)$

**4.2.10**  $\lim_{t \to -2^-} g(t)$   **4.2.11**  $\lim_{t \to -2^+} g(t)$   **4.2.12**  $\lim_{t \to -2} g(t)$
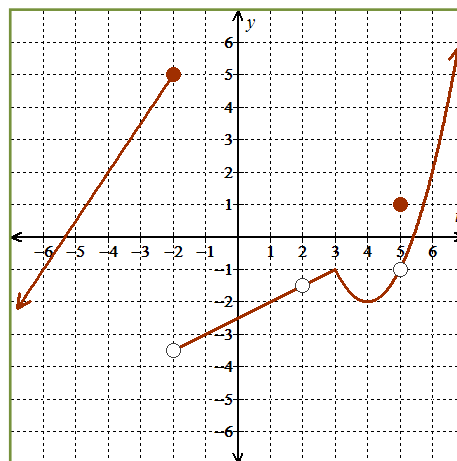
**Problem 4.3**

Values of the function $f(x) = \dfrac{3x^2 - 16x + 5}{2x^2 - 13x + 15}$ are shown

in Table 4.2. Both of the questions below are in reference to this function.

**4.3.1** What is the value of $f(5)$?

**4.3.2** What is the value of $\displaystyle\lim_{x \to 5} \dfrac{3x^2 - 16x + 5}{2x^2 - 13x + 15}$?

**Table 4.2:** $f(x) = \dfrac{3x^2 - 16x + 5}{2x^2 - 13x + 15}$

| $x$ | $f(x)$ |
|---|---|
| 4.99 | 2.0014 |
| 4.999 | 2.00014 |
| 4.9999 | 2.000014 |
| 5.0001 | 1.999986 |
| 5.001 | 1.99986 |
| 5.01 | 1.9986 |

**Problem 4.4**

Values of the function $p(t) = \sqrt{t - 12}$ are shown in Table 4.3. Both of the questions below are in reference to this function.

**4.4.1** What is the value of $p(21)$?

**4.4.2** What is the value of $\displaystyle\lim_{t \to 21} \sqrt{t - 12}$?

**Table 4.3:** $p(t) = \sqrt{t - 12}$

| $t$ | $p(t)$ |
|---|---|
| 20.9 | 2.98 |
| 20.99 | 2.998 |
| 20.999 | 2.9998 |
| 21.001 | 3.0002 |
| 21.01 | 3.002 |
| 21.1 | 3.02 |

**Problem 4.5**

Create tables similar to tables 4.2 and 4.3 from which you can deduce each of the following limit values. Make sure that you include table numbers, table captions, and meaningful column headings. Make sure that your input values follow patterns similar to those used in tables 4.2 and 4.3. Make sure that you round your output values in such a way that a clear and compelling pattern in the output is clearly demonstrated by your stated values. Make sure that you state the limit value!

**4.5.1** $\displaystyle\lim_{t \to 6} \dfrac{t^2 - 10t + 24}{t - 6}$

**4.5.2** $\displaystyle\lim_{x \to -1^+} \dfrac{\sin(x + 1)}{3x + 3}$

**4.5.3** $\displaystyle\lim_{h \to 0^-} \dfrac{h}{4 - \sqrt{16 - h}}$

**Activity 5**

When proving the value of a limit we frequently rely upon laws that are easy to prove using the technical definitions of limit. These laws can be found in Appendix C (pages C1 and C2). The first of these type laws are called replacement laws. Replacement laws allow us to replace limit expressions with the actual values of the limits.

**Problem 5.1**

The value of each of the following limits can be established using one of the replacement laws. Copy each limit expression onto your own paper, state the value of the limit (e.g. $\displaystyle\lim_{x \to 9} 5 = 5$), and state the replacement law (by number) that establishes the value of the limit.

**5.1.1** $\displaystyle\lim_{t \to \pi} t$

**5.1.2** $\displaystyle\lim_{x \to 14} 14$

**5.1.3** $\displaystyle\lim_{x \to 14} x$

**Problem 5.2**

The algebraic limit laws allow us to replace limit expressions with equivalent limit expressions. When applying limit laws our first goal is to come up with an expression in which every limit in the expression can be replaced with its value based upon one of the replacement laws. This process is shown in example 5.1. Please note that all replacement laws are saved for the second to last step and that each replacement is explicitly shown. Please note also that each limit law used is referenced by number.

**Example 5.1**

$$\lim_{x \to 7}\left(4x^2 + 3\right) = \lim_{x \to 7}\left(4x^2\right) + \lim_{x \to 7} 3 \quad \text{Limit Law A1}$$

$$= 4 \cdot \lim_{x \to 7} x^2 + \lim_{x \to 7} 3 \quad \text{Limit Law A3}$$

$$= 4 \cdot \left(\lim_{x \to 7} x\right)^2 + \lim_{x \to 7} 3 \quad \text{Limit Law A6}$$

$$= 4 \cdot 7^2 + 3 \quad \text{Limit Laws R1 and R2}$$

$$= 199$$

Use the limit laws to establish the value of each of the following limits. Make sure that you use the step-by-step, vertical format shown in example 5.1. Make sure that you cite the limit laws used in each step. To help you get started, **the steps necessary in _problem 5.2.1_ are outlined below**.

> To help you get started, **the steps necessary in _problem 5.2.1_ are outlined below**.
>
> **Step 1**: Apply Law A6    **Step 2**: Apply Law A1    **Step 3**: Apply Law A3
>
> **Step 4**: Apply Laws R1 and R2

**5.2.1** $\displaystyle \lim_{t \to 4} \sqrt{6t + 1}$

**5.2.2** $\displaystyle \lim_{y \to 7} \frac{y + 3}{y - \sqrt{y + 9}}$

**5.2.3** $\displaystyle \lim_{x \to \pi} \left(x \cos(x)\right)$

**Activity 6**

Many limits have the form $\dfrac{0}{0}$ which means the expressions in both the numerator and denominator limit to zero (e.g. $\displaystyle \lim_{x \to 3} \frac{2x - 6}{x - 3}$). The form $\dfrac{0}{0}$ is called **_indeterminate_** because we do not know the value of the limit (or even if it exists) so long as the limit has that form. When confronted with limits of form $\dfrac{0}{0}$ we must first manipulate the expression so that common factors causing the zeros in the numerator and denominator are isolated. Limit law A7 can then be used to justify eliminating the common factors and once they are gone we may proceed with the application of the remaining limit laws. Examples 6.1 and 6.2 illustrate this situation.

$$\lim_{x \to 3} \frac{x^2 - 8x + 15}{x - 3} = \lim_{x \to 3} \frac{(x - 5)(x - 3)}{x - 3}$$

$$= \lim_{x \to 3} (x - 5) \qquad \text{Limit Law A7}$$

**Example 6.1**

$$= \lim_{x \to 3} x - \lim_{x \to 3} 5 \qquad \text{Limit Law A2}$$

This limit does not have indeterminate form, so we may apply the remaining limit laws.

$$= 3 - 5 \qquad \text{Limit Laws R1 and R2}$$

$$= -2$$

$$\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\sin^2(\theta)} = \lim_{\theta \to 0} \left( \frac{1 - \cos(\theta)}{\sin^2(\theta)} \cdot \frac{1 + \cos(\theta)}{1 + \cos(\theta)} \right)$$

$$= \lim_{\theta \to 0} \frac{1 - \cos^2(\theta)}{\sin^2(\theta) \cdot (1 + \cos(\theta))}$$

This limit does not have indeterminate form, so we may apply the remaining limit laws.

$$= \lim_{\theta \to 0} \frac{\sin^2(\theta)}{\sin^2(\theta) \cdot (1 + \cos(\theta))}$$

$$= \lim_{\theta \to 0} \frac{1}{1 + \cos(\theta)} \qquad \text{Limit Law A7}$$

**Example 6.2**

$$= \frac{\lim_{\theta \to 0} 1}{\lim_{\theta \to 0} (1 + \cos(\theta))} \qquad \text{Limit Law A5}$$

$$= \frac{\lim_{\theta \to 0} 1}{\lim_{\theta \to 0} 1 + \lim_{\theta \to 0} \cos(\theta)} \qquad \text{Limit Law A1}$$

$$= \frac{\lim_{\theta \to 0} 1}{\lim_{\theta \to 0} 1 + \cos\left( \lim_{\theta \to 0} \theta \right)} \qquad \text{Limit Law A6}$$

$$= \frac{1}{1 + \cos(0)} \qquad \text{Limit Laws R1 and R2}$$

$$= \frac{1}{2}$$

As seen in example 6.2, trigonometric identities can come into play while trying to eliminate the form $\frac{0}{0}$. Elementary rules of logarithms can also play a role in this process. Before you begin evaluating limits whose initial form is $\frac{0}{0}$, you need to make sure that you recall some of these basic rules. That is the purpose of problem 6.1.

**Problem 6.1**

Complete each of the following identities (over the real numbers).  Make sure that you check with your lecture instructor so that you know which of these identities you are expected to memorize.

The following identities are valid for all values of $x$ and $y$.

$1 - \cos^2(x) =$ _____     $\tan^2(x) + 1 =$ _____

$\sin(2x) =$ _____     $\tan(2x) =$ _____

$\sin(x + y) =$ _____     $\cos(x + y) =$ _____

$\sin\left(\dfrac{x}{2}\right) =$ _____     $\cos\left(\dfrac{x}{2}\right) =$ _____

There are three versions of the following identity; write them all.

$\cos(2x) =$ _____     $\cos(2x) =$ _____

$\cos(2x) =$ _____

The following identities are valid for all positive values of $x$ and $y$ and all values of $n$.

$\ln(xy) =$ _____     $\ln\left(\dfrac{x}{y}\right) =$ _____

$\ln(x^n) =$ _____     $\ln(e^n) =$ _____

**Problem 6.2**

Use the limit laws to establish the value of each of the following limits after first manipulating the expression so that it no longer has form $\dfrac{0}{0}$. Make sure that you use the step-by-step, vertical format shown in examples 6.1 and 6.2. Make sure that you cite each limit law used.

**6.2.1** $\quad \displaystyle\lim_{x \to -4} \frac{x+4}{2x^2+5x-12}$

**6.2.2** $\quad \displaystyle\lim_{x \to 0} \frac{\sin(2x)}{\sin(x)}$

**6.2.3** $\quad \displaystyle\lim_{\beta \to 0} \frac{\sin(\beta+\pi)}{\sin(\beta)}$

**6.2.4** $\quad \displaystyle\lim_{t \to 0} \frac{\cos(2t)-1}{\cos(t)-1}$

**6.2.5** $\quad \displaystyle\lim_{x \to 1} \frac{4\ln(x)+2\ln(x^3)}{\ln(x)-\ln(\sqrt{x})}$

**6.2.6** $\quad \displaystyle\lim_{w \to 9} \frac{9-w}{\sqrt{w}-3}$

**Activity 7**

We are frequently interested in a function's "end behavior;" that is, what is the behavior of the function as the input variable increases without bound or decreases without bound.

Many times a function will approach a horizontal asymptote as its end behavior. Assuming that the horizontal asymptote $y = L$ represents the end behavior of the function $f$ both as $x$ increases without bound and as $x$ decreases without bound, we write $\displaystyle\lim_{x \to \infty} f(x) = L$ and $\displaystyle\lim_{x \to -\infty} f(x) = L$.

The formalistic way to read $\displaystyle\lim_{x \to \infty} f(x) = L$ is "the limit of $f(x)$ as $x$ approaches infinity equals $L$."

When read that way, however, the words need to be taken **_anything but literally_**. In the first place, $x$ isn't approaching anything! The entire point is that $x$ is increasing without any bound on how large its value becomes. Secondly, there is no place on the real number line called "infinity;" infinity is not a number. Hence $x$ certainly can't be approaching something that isn't even there!

**Problem 7.1**

For the function in Figure 7.1 (Appendix B, page B1) we could (correctly) write $\displaystyle\lim_{x \to \infty} f_1(x) = -2$ and $\displaystyle\lim_{x \to -\infty} f_1(x) = -2$. Go ahead and write (and say aloud) the analogous limits for the functions in figures 7.2-7.5 (pages B1 and B2).

**Problem 7.2**

Values of the function $f(x) = \dfrac{3x^2-16x+5}{2x^2-13x+15}$ are shown in Table 7.1. Both of the questions below are in reference to this function.

**7.2.1** What is the value of $\displaystyle\lim_{x \to -\infty} f(x)$?

**7.2.2** What is the horizontal asymptote for the graph of $y = f(x)$?

**Table 7.1:** $f(x) = \dfrac{3x^2-16x+5}{2x^2-13x+15}$

| $x$ | $f(x)$ |
|---|---|
| $-1{,}000$ | $1.498$ |
| $-10{,}000$ | $1.4998$ |
| $-100{,}000$ | $1.49998$ |
| $-1{,}000{,}000$ | $1.499998$ |

**Problem 7.3**

Jorge and Vanessa were in a heated discussion about horizontal asymptotes. Jorge said that functions never cross horizontal asymptotes. Vanessa said Jorge was nuts. Vanessa whipped out her trusty calculator and generated the values in Table 7.2 to prove her point.

**7.3.1** What is the value of $\lim\limits_{t \to \infty} g(t)$?

**7.3.2** What is the horizontal asymptote for the graph of $y = g(t)$?

**7.3.3** Just how many times does the curve $y = g(t)$ cross its horizontal asymptote?

**Table 7.2:** $g(t) = 1 + \dfrac{\sin(t)}{t}$

| $t$ | $g(t)$ |
|-----|--------|
| $10^3$ | 1.0008 |
| $10^4$ | .99997 |
| $10^5$ | 1.0000004 |
| $10^6$ | .9999997 |
| $10^7$ | 1.00000004 |
| $10^8$ | 1.000000009 |
| $10^9$ | 1.0000000005 |
| $10^{10}$ | .99999999995 |

**Activity 8**

When using limit laws to establish limit values as $x \to \infty$ or $x \to -\infty$, limit laws A1-A6 and R2 are still in play (when applied in a valid manner), but limit law R1 cannot be applied. (The reason limit law R1 cannot be applied is discussed in detail in problem 11.4)

There is a new replacement law that can only be applied when $x \to \infty$ or $x \to -\infty$; this is replacement law R3. Replacement law R3 essentially says that if the value of a function is increasing without any bound on large it becomes or if the function is decreasing without any bound on how large its absolute value becomes, then the value of a constant divided by that function must be approaching zero. An analogy can be found in extremely poor party planning. Let's say that you plan to have a pizza party and you buy five pizzas. Suppose that as the hour of the party approaches more and more guests come in the door … in fact the guests never stop coming! Clearly as the number of guests continues to rise the amount of pizza each guest will receive quickly approaches zero (assuming the pizzas are equally divided among the guests).

**Problem 8.1**

Consider the function $f(x) = \dfrac{12}{x}$. Complete Table 8.1 **_without the use of your calculator_**. What limit value and limit law are being illustrated in the table?

**Table 8.1:** $f(x) = \dfrac{12}{x}$

| $x$ | $f(x)$ |
|-----|--------|
| 1,000 | |
| 10,000 | |
| 100,000 | |
| 1,000,000 | |

**Activity 9**

Many limits have the form $\dfrac{\infty}{\infty}$ which we take to mean that the expressions in both the numerator

and denominator are increasing or decreasing without bound. When confronted with a limit of type

$\displaystyle\lim_{x\to\infty}\dfrac{f(x)}{g(x)}$ or $\displaystyle\lim_{x\to-\infty}\dfrac{f(x)}{g(x)}$ that has the form $\dfrac{\infty}{\infty}$, we can frequently resolve the limit if we first

divide the dominant factor of the dominant term of the denominator from both the numerator and the denominator. When we do this, we need to completely simplify each of the resultant fractions and make sure that the resultant limit exists before we start to apply limit laws. We then apply the algebraic limit laws until all of the resultant limits can be replaced using limit laws R2 and R3. This process is illustrated in example 9.1.

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**Example 9.1**

$$\lim_{t\to\infty}\frac{3t^2+5t}{3-5t^2}=\lim_{t\to\infty}\left(\frac{3t^2+5t}{3-5t^2}\cdot\frac{1/t^2}{1/t^2}\right)$$

> The "form" of the limit is now $\dfrac{3+0}{0-5}$, so we can begin to apply the limit laws because the limits will all exist.

$$=\lim_{t\to\infty}\frac{3+\dfrac{5}{t}}{\dfrac{3}{t^2}-5}$$

$$=\frac{\displaystyle\lim_{t\to\infty}\left(3+\frac{5}{t}\right)}{\displaystyle\lim_{t\to\infty}\left(\frac{3}{t^2}-5\right)} \qquad \text{Limit Law A5}$$

$$=\frac{\displaystyle\lim_{t\to\infty}3+\lim_{t\to\infty}\frac{5}{t}}{\displaystyle\lim_{t\to\infty}\frac{3}{t^2}-\lim_{t\to\infty}5} \qquad \text{Limit Laws A1 and A2}$$

$$=\frac{3+0}{0-5} \qquad \text{Limit Laws R2 and R3}$$

$$=-\frac{3}{5}$$

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**Problem 9.1**

Use the limit laws to establish the value of each limit after dividing the dominant term-factor in the denominator from both the numerator and denominator. Remember to simplify each resultant expression before you begin to apply the limit laws.

**9.1.1** $\displaystyle\lim_{t\to-\infty}\frac{4t^2}{4t^2+t^3}$     **9.1.2** $\displaystyle\lim_{t\to\infty}\frac{6e^t+10e^{2t}}{2e^{2t}}$     **9.1.3** $\displaystyle\lim_{y\to\infty}\sqrt{\frac{4y+5}{5+9y}}$

**Activity 10**

Many limit values do not exist. Sometimes the non-existence is caused by the function value either increasing without bound or decreasing without bound. In these special cases we use the symbols $\infty$ and $-\infty$ to communicate the non-existence of the limits. Figures 10.1-10.3 can be used to illustrate some ways in which we communicate the **_non-existence_** of these type of limits.

In Figure 10.1 we could (correctly) write $\lim\limits_{x \to 2} k(x) = \infty$, $\lim\limits_{x \to 2^-} k(x) = \infty$, and $\lim\limits_{x \to 2^+} k(x) = \infty$.

In Figure 10.2 we could (correctly) write $\lim\limits_{t \to 4} w(t) = -\infty$, $\lim\limits_{t \to 4^-} w(t) = -\infty$, and $\lim\limits_{t \to 4^+} w(t) = -\infty$.

In Figure 10.3 we could (correctly) write $\lim\limits_{x \to -3^-} T(x) = \infty$ and $\lim\limits_{x \to -3^+} T(x) = -\infty$. There is no shorthand way of communicating the non-existence of the two sided limit $\lim\limits_{x \to -3} T(x)$.



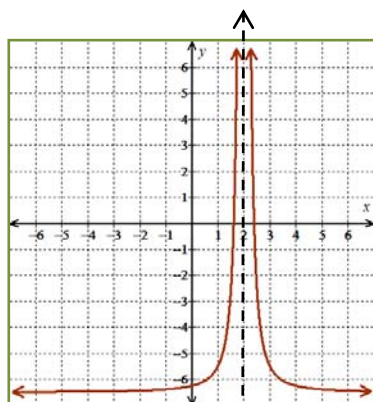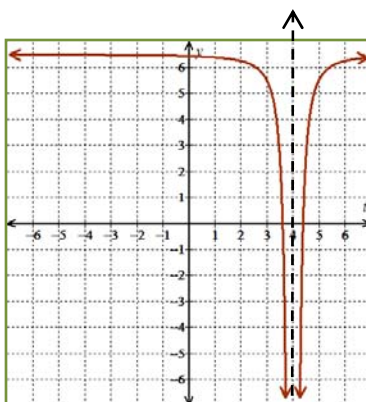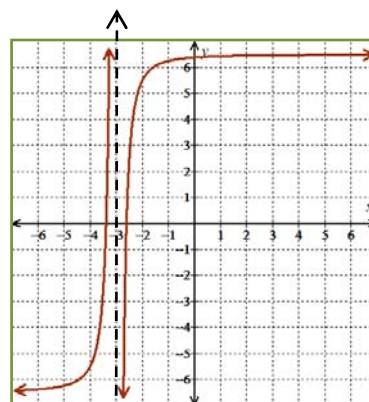**Figure 10.1:** $k$    $x = 2$      **Figure 10.2:** $w$    $t = 4$      $x = -3$    **Figure 10.3:** $T$

**Problem 10.1**

Draw onto Figure 10.4 a single function, $f$, that satisfies each of the following limit statements. Make sure that you draw the necessary asymptotes and that you label each asymptote with its equation.

- $\lim\limits_{x \to 3^-} f(x) = -\infty$

- $\lim\limits_{x \to 3^+} f(x) = \infty$

- $\lim\limits_{x \to \infty} f(x) = 0$

- $\lim\limits_{x \to -\infty} f(x) = -2$
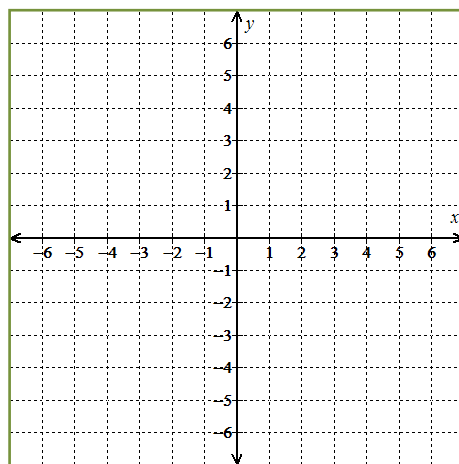


**Figure 10.4:** $f$

**Activity 11**

Whenever $\lim\limits_{x \to a} f(x) \neq 0$ but $\lim\limits_{x \to a} g(x) = 0$, then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ ***does not exist*** because from either

side of $a$ the value of $\dfrac{f(x)}{g(x)}$ either increases without bound or decreasing without bound. In

these situations the line $x = a$ is a vertical asymptote for the graph of $y = \dfrac{f(x)}{g(x)}$.

For example, the line $x = 2$ is a vertical asymptote for the function $h(x) = \dfrac{x+5}{2-x}$. We say that

$\lim\limits_{x \to 2} \dfrac{x+5}{2-x}$ has ***the form "not zero over zero."*** (Specifically, the form of $\lim\limits_{x \to 2} \dfrac{x+5}{2-x}$ is $\dfrac{7}{0}$.) Every

limit with form "not zero over zero" ***does not exist***. However, we frequently can communicate the

non-existence of the limit using an infinity symbol. In the case of $h(x) = \dfrac{x+5}{2-x}$ it's pretty easy to

see that $h(1.99)$ is a positive number whereas $h(2.01)$ is a negative number. Consequently, we can

infer that $\lim\limits_{x \to 2^-} h(x) = \infty$ and $\lim\limits_{x \to 2^+} h(x) = -\infty$. Remember, these equations are communicating

that the limits ***do not exist*** as well as the reason for their non-existence. There is no short-hand

way to communicate the non-existence of the two-sided limit $\lim\limits_{x \to 2} h(x)$.

**Problem 11.1** Suppose that $g(t) = \dfrac{t+4}{t+3}$.

**11.1.1** What is the vertical asymptote on the graph of $y = g(t)$.

**11.1.2** Write an equality about $\lim\limits_{t \to -3^-} g(t)$.

**11.1.3** Write an equality about $\lim\limits_{t \to -3^+} g(t)$.

**11.1.4** Is it possible to write an equality about $\lim\limits_{t \to -3} g(t)$? If so, write it.

**11.1.5** Which of the following limits exist? $\lim\limits_{t \to -3^-} g(t)$, $\lim\limits_{t \to -3^+} g(t)$, and $\lim\limits_{t \to -3} g(t)$

**Problem 11.2** Suppose that $z(x) = \dfrac{7 - 3x^2}{(x-2)^2}$.

**11.2.1** What is the vertical asymptote on the graph of $y = z(x)$.

**11.2.2** Is it possible to write an equality about $\lim\limits_{x \to 2} z(x)$? If so, write it.

**11.2.3** What is the horizontal asymptote on the graph of $y = z(x)$.

**11.2.4** Which of the following limits exist? $\lim\limits_{x \to 2} z(x)$, $\lim\limits_{x \to \infty} z(x)$, and $\lim\limits_{x \to -\infty} z(x)$

**Problem 11.3**

Consider the function $f(x) = \dfrac{x+7}{x-8}$. Complete Table 11.1 **_without the use of your calculator_**.

Use this as an opportunity to discuss why limits of form "not zero over zero" are "infinite limits." What limit equation is being illustrated in the table?

**Table 11.1:** $f(x) = \dfrac{x+7}{x-8}$

| $x$ | $x+7$ | $x-8$ | $f(x)$ |
|------|--------|--------|--------|
| 8.1 | 15.1 | .1 | |
| 8.01 | 15.01 | .01 | |
| 8.001 | 15.001 | .001 | |
| 8.0001 | 15.0001 | .0001 | |

**Problem 11.4**

Hear me, and hear me loud … ∞ **does not exist**. This, in part, is why we cannot apply Limit Law R1 to an expression like $\lim\limits_{x\to\infty} x = \infty$. When we write, say, $\lim\limits_{x\to 7} x = 7$, we are replacing the limit expression *with its **value** – that's what the replacement laws are all about*! When we write $\lim\limits_{x\to\infty} x = \infty$, we are not replacing the limit expression with a value! We are explicitly saying that **_the limit has no value_** (i.e. does not exist) as well as saying the reason the limit does not exist. The limit laws (R1-R3 and A1-A6) can only be applied when all of the limits in the equation exist. With this in mind, discuss and decide whether each of the following equations are true or false.

$$\lim_{x\to 0}\left(\frac{e^x}{e^x}\right) = \frac{\lim\limits_{x\to 0} e^x}{\lim\limits_{x\to 0} e^x} \quad \textbf{T or F}$$
$$\lim_{x\to -\infty}\left(\frac{e^{-x}}{e^{-x}}\right) = \frac{\lim\limits_{x\to -\infty} e^{-x}}{\lim\limits_{x\to -\infty} e^{-x}} \quad \textbf{T or F}$$

$$\lim_{x\to 1}\frac{e^x}{\ln(x)} = \frac{\lim\limits_{x\to 1} e^x}{\lim\limits_{x\to 1}\ln(x)} \quad \textbf{T or F}$$
$$\lim_{x\to 1}\frac{\ln(x)}{e^x} = \frac{\lim\limits_{x\to 1}\ln(x)}{\lim\limits_{x\to 1} e^x} \quad \textbf{T or F}$$

$$\lim_{x\to 0^+}\left(2\ln(x)\right) = 2\lim_{x\to 0^+}\ln(x) \quad \textbf{T or F}$$
$$\lim_{\theta\to\infty}\frac{\sin(\theta)}{\sin(\theta)} = \frac{\lim\limits_{\theta\to\infty}\sin(\theta)}{\lim\limits_{\theta\to\infty}\sin(\theta)} \quad \textbf{T or F}$$

$$\lim_{x\to\infty}\left(e^x - \ln(x)\right) = \lim_{x\to\infty} e^x - \lim_{x\to\infty}\ln(x) \quad \textbf{T or F}$$
$$\lim_{x\to -\infty} e^{1/x} = e^{\lim\limits_{x\to -\infty}\frac{1}{x}} \quad \textbf{T or F}$$

**Problem 11.5**

Mindy tried to evaluate $\lim\limits_{x \to 6^+} \dfrac{4x - 24}{x^2 - 12x + 36}$ using the limit laws. Things went horribly wrong for

Mindy (her work is shown below). Identify what is wrong in Mindy's work and discuss what a more reasonable approach might have been.

$$\lim_{x \to 6^+} \frac{4x - 24}{x^2 - 12x + 36} = \lim_{x \to 6^+} \frac{4(x-6)}{(x-6)^2}$$

$$= \lim_{x \to 6^+} \frac{4}{x-6}$$

$$= \frac{\lim\limits_{x \to 6^+} 4}{\lim\limits_{x \to 6^+} (x-6)} \qquad \text{Limit Law A5}$$

$$= \frac{\lim\limits_{x \to 6^+} 4}{\lim\limits_{x \to 6^+} x - \lim\limits_{x \to 6^+} 6} \qquad \text{Limit Law A2}$$

$$= \frac{4}{6-6} \qquad \text{Limit Laws R1 and R2}$$

$$= \frac{4}{0}$$

$$= \infty$$

**This "solution" is not correct!**
**Do not emulate Mindy's work!!**

**Activity 12**

Many statements we make about functions are only true over intervals where the function is _continuous_. When we say a function is continuous over an interval, we basically mean that there are no breaks in the function over that interval; that is, there are no vertical asymptotes, holes, jumps, or gaps along that interval.

**Definition 12.1**

The function $f$ is continuous at the number $a$ if and only if $\lim\limits_{x \to a} f(x) = f(a)$.

There are three ways that the defining property can fail to be satisfied at a given value of $a$. To facilitate exploration of these three manner of failure, we can break the defining property into a spectrum of three properties.

   *i.* $f(a)$ must be defined    *ii.* $\lim\limits_{x \to a} f(x)$ must exist    *iii.* $\lim\limits_{x \to a} f(x)$ must equal $f(a)$

Please note that if either property *i* or property *ii* fails to be satisfied at a given value of $a$, then property *iii* also fails to be satisfied at $a$.

**Problem 12.1**

State the values of $t$ at which the function shown in Figure 12.1 is discontinuous. For each instance of discontinuity, state (by number) all of the sub-properties in Definition 12.1 that fail to be satisfied.
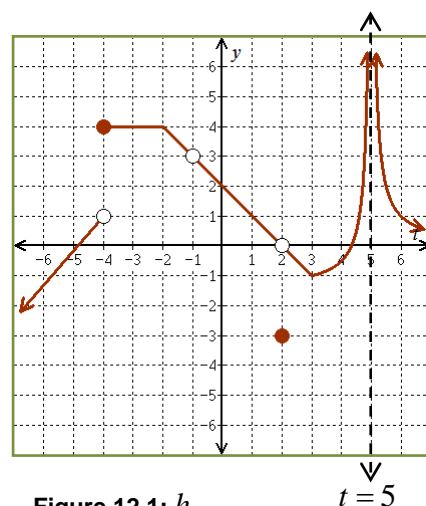


**Figure 12.1:** $h$

$t = 5$

**Activity 13**

When a function has a discontinuity at $a$, the function is sometimes continuous from only the right or only the left at $a$. (Please note that when we say "the function is continuous at $a$" we mean that the function is continuous from **both** the right and left at $a$.)

---

**Definition 13.1**

The function $f$ is continuous from the left at $a$ if and only if $\lim_{x \to a^-} f(x) = f(a)$ and $f$ is

continuous from the right at $a$ if and only if $\lim_{x \to a^+} f(x) = f(a)$.

---

Some discontinuities are classified as ***removable discontinuities***. Specifically, discontinuities that are holes or skips (holes with a secondary point) are called removable.

---

**Definition 13.2**

We say that $f$ has a removable discontinuity at $a$ if $f$ is discontinuous at $a$ but

$\lim_{x \to a} f(x)$ exists.

---

## Problem 13.1

Referring to the function $h$ shown in Figure 13.1, state the values of $t$ where the function is continuous from the right but not the left. Then state the values of $t$ where the function is continuous from the left but not the right.

## Problem 13.2

Referring again to the function $h$ shown in Figure 13.1, state the values of $t$ where the function has removable discontinuities.



**Figure 13.1:** $h$

$t = 5$
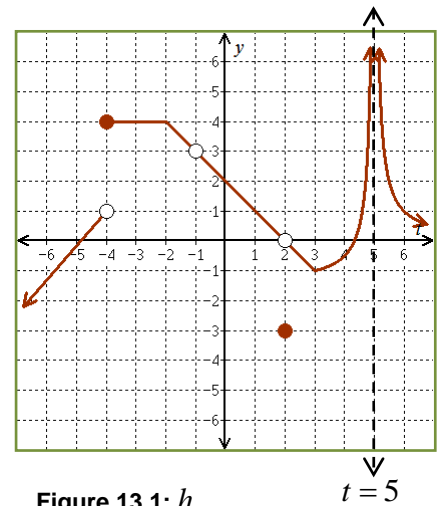
## Activity 14

Now that we have a definition for continuity at a number, we can go ahead and define what we mean when we say a function is continuous over an interval.

---

**Definition 14.1**

The function $f$ is continuous over an open interval if and only if it is continuous at each and every number on that interval.

The function is continuous over the closed interval $[a,b]$ if and only if it is continuous on $(a,b)$, continuous from the right at $a$, and continuous from the left at $b$.

Similar definitions apply to half-open intervals.

---

## Problem 14.1
Write a definition for continuity over the half-open interval $(a,b]$.

## Problem 14.2
Referring to the function in Figure 14.1, decide whether each of the following statements are true or false.

**14.2.1** $h$ is continuous on $[-4,-1)$
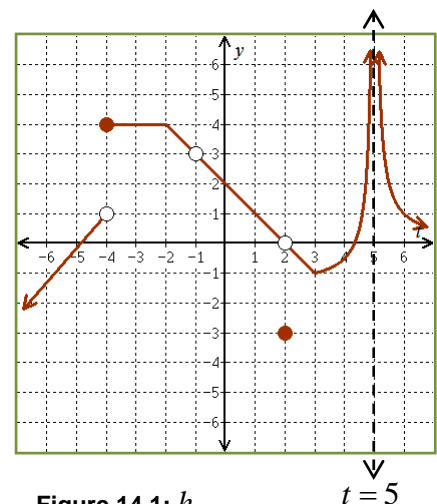
**14.2.2** $h$ is continuous on $(-4,-1)$

**14.2.3** $h$ is continuous on $(-4,-1]$

**14.2.4** $h$ is continuous on $(-1,2]$

**14.2.5** $h$ is continuous on $(-1,2)$

**14.2.6** $h$ is continuous on $(-\infty,-4)$

**14.2.7** $h$ is continuous on $(-\infty,-4]$



**Figure 14.1:** $h$

$t = 5$

**Problem 14.3**
Several functions are described below. Your task is to draw each function on its provided axis system. Do not introduce any unnecessary discontinuities or intercepts that are not directly implied by the stated properties. Make sure that you draw all implied asymptotes and label them with their equations.

**14.3.1** Draw onto Figure 14.2 a function that satisfies all of the following properties.

- $\lim\limits_{x \to 4^-} f(x) = 2$
- $\lim\limits_{x \to 4^+} f(x) = 5$
- $f(0) = 4$ and $f(4) = 5$
- $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to \infty} f(x) = 4$
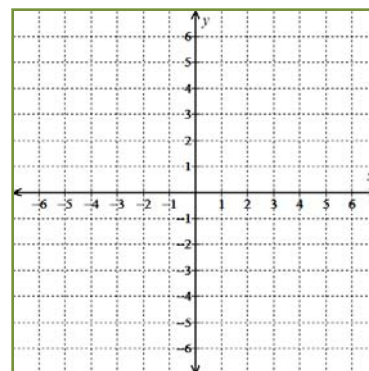
**Figure 14.2:** $f$

**14.3.2** Draw onto Figure 14.3 a function that satisfies all of the following properties.

- $\lim\limits_{x \to -2} g(x) = \infty$
- $\lim\limits_{x \to -\infty} g(x) = \infty$
- $g(0) = 4$, $g(3) = -2$, and $g(6) = 0$
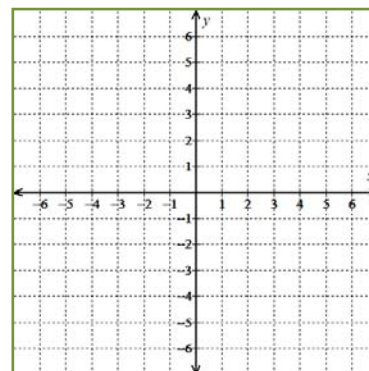- $g$ is continuous and has constant slope on $(0, \infty)$.

**Figure 14.3:** $g$

**14.3.3** Draw onto Figure 14.4 a function that satisfies all of the following properties.

- The only discontinuities on $m$ occur at $-4$ and $3$
- $m$ has no $x$-intercepts
- $m(-6) = 5$
- $\lim\limits_{x \to -4^+} m(x) = -2$
- $\lim\limits_{x \to 3} m(x) = -\infty$
- $\lim\limits_{x \to \infty} m(x) = -\infty$
- $m$ has a constant slope of $-2$ over $(-\infty, -4)$
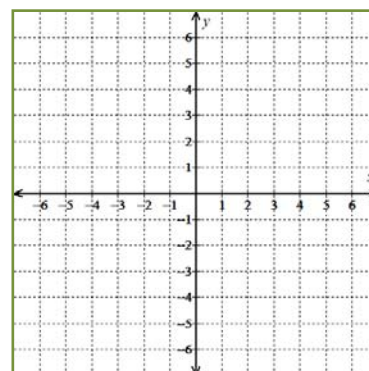- $m$ is continuous over $[-4, 3)$

**Figure 14.4:** $m$

**Activity 15**

Discontinuities are a little more challenging to identify when working with formulas than when working with graphs. One reason for the added difficulty is that when working with a function formula you have to dig into your memory bank and retrieve fundamental properties about certain types of functions.

**Problem 15.1**

**15.1.1**  What would cause a discontinuity on a rational function (a polynomial divided by another polynomial)?

**15.1.2**  What is always true about the *argument* of the function, $u$, over intervals where the function $y = \ln(u)$ is continuous?

**15.1.3**  Name three values of $\theta$ where the function $y = \tan(\theta)$ is discontinuous.

**15.1.4**  What is the domain of the function $k(t) = \sqrt{t-4}$ ?

**15.1.5**  What is the domain of the function $g(t) = \sqrt[3]{t-4}$ ?

**Activity 16**

Piece-wise defined functions are functions where the formula used depends upon the value of the input.  When looking for discontinuities on piece-wise defined functions, you need to investigate the behavior at values where the formula changes as well as values where the issues discussed in Activity 15 might pop up.

**Problem 16.1**

This question is all about the function $f$ shown to the right. Answer question 16.1.1 at each of the values $1, 3, 4, 5, 7,$ and $8$. At the values where the answer to question 16.1.1 is yes, go ahead and answer questions 16.1.2-16.1.4; skip questions 16.1.2-16.1.4 at the values where the answer to question 16.1.1 is no.

$$f(x) = \begin{cases} \dfrac{4}{5-x} & \text{if} \quad x < 1 \\[2mm] \dfrac{x-3}{x-3} & \text{if} \quad 1 < x < 4 \\[2mm] 2x+1 & \text{if} \quad 4 \le x \le 7 \\[2mm] \dfrac{15}{8-x} & \text{if} \quad x > 7 \end{cases}$$

**16.1.1**  Is $f$ discontinuous at the given value?

**16.1.2**  Is $f$ continuous only from the left at the given value?

**16.1.3**  Is $f$ continuous only from the right at the given value?

**16.1.4**  Is the discontinuity removable?

**Problem 16.2**

Consider the function $g$ shown to the right. The letter $C$ represents the same real number in all three of the piece-wise formulas.

$$g(x) = \begin{cases} \dfrac{C}{x-17} & \text{if} \quad x < 10 \\[2ex] C + 3x & \text{if} \quad x = 10 \\[2ex] 2C - 4 & \text{if} \quad x > 10 \end{cases}$$

**16.2.1** Find the value for $C$ that makes the function continuous on $(-\infty, 10]$. Make sure that your reasoning is clear.

**16.2.2** Is it possible to find a value for $C$ that makes the function continuous over $(-\infty, \infty)$? Explain.

**Problem 16.3**

Consider the function $f$ shown to the right. State the values of $x$ where each of the following occur. If a stated property doesn't occur, make sure that you state that (as opposed to simply not responding to the question). No explanation necessary.

$$f(x) = \begin{cases} \dfrac{5}{x-10} & \text{if} \quad x \leq 5 \\[2ex] \dfrac{5}{5x-30} & \text{if} \quad 5 < x < 7 \\[2ex] \dfrac{x-2}{12-x} & \text{if} \quad x > 7 \end{cases}$$

**16.3.1** At what values of $x$ is $f$ discontinuous?

**16.3.2** At what values of $x$ is $f$ continuous from the left but not the right?

**16.3.3** At what values of $x$ is $f$ continuous from the right but not the left?

**16.3.4** At what values of $x$ does $f$ have removable discontinuities?