Implicit Differentiation

Activity 42

Some points that satisfy the equation $y^3 - 4y = x^2 - 1$ are graphed in Figure 42.1. Clearly this set of points does not constitute a function where y is a function of x; for example, there are three points that have an x-coordinate of 1.

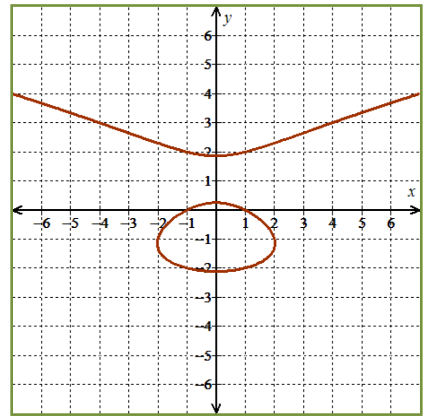

Figure 42.1: $y^3 - 4y = x^2 - 1$

Never-the-less, there is a unique tangent line to the curve at each point on the curve and so long as the tangent line is not vertical it has a unique slope. We still identify the value of this slope using the symbol $\frac{dy}{dx}$, so it would be helpful if we had a formula for $\frac{dy}{dx}$. If we could solve $y^3 - 4y = x^2 - 1$ for y, finding the formula for $\frac{dy}{dx}$ would be a snap. Hopefully you quickly see that such an approach is just not possible.

To get around this problem we are going to employ a technique called ***implicit differentiation***. We use this technique to find the formula for $\frac{dy}{dx}$ whenever the equation relating x and y is not explicitly solved for y. What we are going to do is treat y ***as if it were*** a function of x and set the derivatives of the two sides of the equation equal to one another. This is actually a reasonable thing to do because so long as we are at a point on the curve where the tangent line is not vertical, we could make y a function of x using appropriate restrictions on the domain and range.

Since we are treating y as a function of x, we need to make sure that we use the chain rule when differentiating terms like y^3. When u is a function of x, we know that $\frac{d}{dx}\left(u^3\right) = 3u^2\frac{d}{dx}(u)$. Since the name we give $\frac{d}{dx}(y)$ is $\frac{dy}{dx}$, it follows that when y is a function of x, $\frac{d}{dx}\left(y^3\right) = 3y^2\frac{dy}{dx}$. The derivation of $\frac{dy}{dx}$ for the equation $y^3 - 4y = x^2 - 1$ is shown in example 42.1.

Example 42.1

$$y^3 - 4y = x^2 - 1$$

$$\frac{d}{dx}\left(y^3 - 4y\right) = \frac{d}{dx}\left(x^2 - 1\right)$$

Begin by differentiating both sides of the equation with respect to x.

$$3y^2\frac{dy}{dx} - 4\frac{dy}{dx} = 2x$$

The chain rule only comes into play on the terms involving y.

$$\left(3y^2 - 4\right)\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{3y^2 - 4}$$

We now solve the equation for $\frac{dy}{dx}$.

At first it might be unsettling that the formula for $\frac{dy}{dx}$ contains both the variables x and y. However, if you think it through you should conclude that the formula ***must*** include the variable y; otherwise, how could the formula generate three different slopes at the points $(1,2)$, $(1,0)$, and $(1,-2)$? These slopes are given below. The reader should verify their values by drawing lines onto Figure 42.1 with the indicated slopes at the indicated points.

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{2(1)}{3(2)^2-4} = \frac{1}{4} \qquad \left.\frac{dy}{dx}\right|_{(1,0)} = \frac{2(1)}{3(0)^2-4} = -\frac{1}{2} \qquad \left.\frac{dy}{dx}\right|_{(1,-2)} = \frac{2(1)}{3(-2)^2-4} = \frac{1}{4}$$

Problem 42.1

Use the process of implicit differentiation to find a formula for $\frac{dy}{dx}$ for the curves generated by each of the following equations. ***Do not*** simplify the equations before taking the derivatives.

You will need to use the product rule for differentiation in problems 42.1.4-42.1.6.

42.1.1 $3x^4 = -6y^5$

42.1.2 $\sin(x) = \sin(y)$

42.1.3 $4y^2 - 2y = 4x^2 - 2x$

42.1.4 $x = ye^y$

42.1.5 $y = xe^y$

42.1.6 $xy = e^{xy-1}$

Problem 42.2

Several points that satisfy the equation $y = xe^y$ are graphed in Figure 42.2. Find the slope and equation of the tangent line to this curve at the origin. (Please note that you already found the formula for $\frac{dy}{dx}$ in problem 42.1.5.)

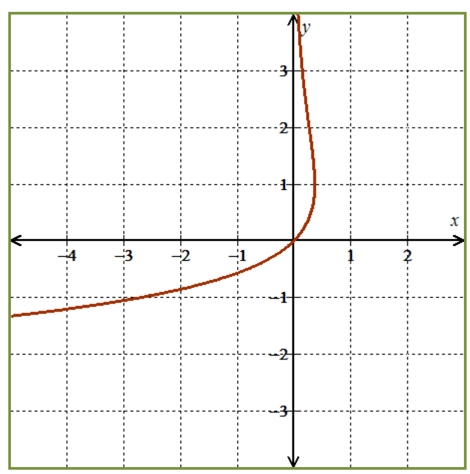

Figure 42.2: $y = xe^y$

Problem 42.3

Consider the set of points that satisfy the equation $xy = 4$.

42.3.1 Use implicit differentiation to find a formula for $\frac{dy}{dx}$.

42.3.2 Find a formula for $\frac{dy}{dx}$ after first solving the equation $xy = 4$ for y.

42.3.3 Show that the two formulas are in fact equivalent so long as $xy = 4$.

Problem 42.4

A set of points that satisfy the equation $x\cos(xy)=4-y$ is graphed in Figure 42.3. Find the slope and equation of the tangent line to this curve at the point $(0,4)$.

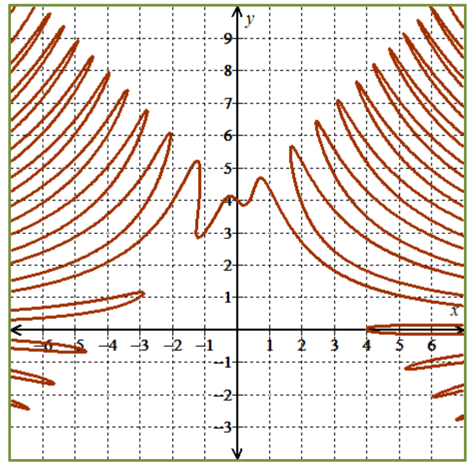

Figure 42.3: $x\cos(xy)=4-y$

Activity 43

Using the definition $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, it is easy to establish that $\frac{d}{dx}(x^2)=2x$. We can use this formula and implicit differentiation to find the formula for $\frac{d}{dx}(\sqrt{x})$.

If $y=\sqrt{x}$, then $y^2=x$ (and $y\geq 0$). Using implicit differentiation we have:

$$\begin{aligned} y^2 &= x \\ \frac{d}{dx}(y^2) &= \frac{d}{dx}(x) \\ 2y\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{2y} \end{aligned}$$

But $y=\sqrt{x}$, so we have:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2y} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

In a similar manner, we can use the fact that $\frac{d}{dx}(\sin(x))=\cos(x)$ to come up with a formula for $\frac{d}{dx}(\sin^{-1}(x))$.

If $y = \sin^{-1}(x)$, then $\sin(y) = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. This gives us:

$$\sin(y) = x$$
$$\frac{d}{dx}(\sin(y)) = \frac{d}{dx}(x)$$
$$\cos(y)\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2(y)}}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

This expression comes from the trigonometric identity $\sin^2(t) + \cos^2(t) = 1$. If you solve that equation for $\cos(t)$ you get $\cos(t) = \pm\sqrt{1-\sin^2(t)}$. Because the function $y = \sin^{-1}(x)$ never has negative slope we can discard the negative solution to the equation.

Here we use that fact that $\sin(y) = x$.

Problem 43.1

Use the fact that $\frac{d}{dx}(e^x) = e^x$ together with implicit differentiation to show that $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$.

Begin by using the fact that $y = \ln(x)$ implies that $e^y = x$ (and $y > 0$). Your first step is to differentiate both sides of the equation $e^y = x$ with respect to x.

Problem 43.2

Use the fact that $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ together with implicit differentiation to show that $\frac{d}{dx}(e^x) = e^x$.

Begin by using the fact that $y = e^x$ implies that $\ln(y) = x$. Your first step is to differentiate both sides of the equation $\ln(y) = x$ with respect to x.

Problem 43.3

Use the fact that $\frac{d}{dx}(\tan(x)) = \sec^2(x)$ together with implicit differentiation to show that $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$. Begin by using the fact that $y = \tan^{-1}(x)$ implies that $\tan(y) = x$ (and $-\frac{\pi}{2} < y < \frac{\pi}{2}$). Your first step is to differentiate both sides of the equation $\tan(y) = x$ with respect to x. Please note that you will need to use the Pythagorean identity that relates the tangent and secant functions while working this problem.