

Functions, Derivatives, and Antiderivatives

Activity 21

Functions, derivatives, and antiderivatives have many entangled properties. For example, over intervals where the first derivative of a function is always positive, we know that the function itself is always increasing. (Do you understand why?)

Many of these relationships can be expressed graphically. Consequently, it is imperative that you fully understand the meaning of some commonly used graphical expressions. These expressions are loosely defined in Table 21.1.

Table 21.1: Some Common Graphical Phrases

<p>The function is positive</p> <p>This means that the vertical-coordinate of the point on the function is positive. As such, a function is positive whenever it lies above the horizontal axis.</p>	<p>The function is negative</p> <p>This means that the vertical-coordinate of the point on the function is negative. As such, a function is negative whenever it lies below the horizontal axis.</p>
<p>The function is increasing</p> <p>This means that the vertical-coordinate of the function consistently increases as you move along the curve from left to right. Linear functions with positive slope are always increasing.</p>	<p>The function is decreasing</p> <p>This means that the vertical-coordinate of the function consistently decreases as you move along the curve from left to right. Linear functions with negative slope are always decreasing.</p>
<p>The function is concave up</p> <p>A function is concave up at a if the tangent line to the function at a lies below the curve. An upward opening parabola is everywhere concave up.</p>	<p>The function is concave down</p> <p>A function is concave down at a if the tangent line to the function at a lies above the curve. A downward opening parabola is everywhere concave down.</p>

Problem 21.1

Answer each of the following questions in reference to the function shown in Figure 21.1. Each answer is an interval (or intervals) along the x -axis. Use interval notation when expressing your answers. Make each interval as wide as possible; that is, do not break an interval into pieces if the interval does not need to be broken up. Assume that the slope of the function is constant on $(-\infty, -5)$, $(3, 4)$, and $(4, \infty)$.

- 21.1.1 Over what intervals is the function positive?
- 21.1.2 Over what intervals is the function negative?
- 21.1.3 Over what intervals is the function increasing?
- 21.1.4 Over what intervals is the function decreasing?
- 21.1.5 Over what intervals is the function concave up?
- 21.1.6 Over what intervals is the function concave down?
- 21.1.7 Over what intervals is the function linear?
- 21.1.8 Over what intervals is the function constant?

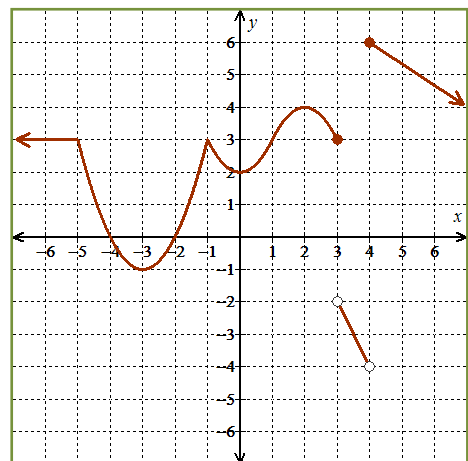


Figure 21.1: f

Activity 22

Much information about a function's first derivative can be gleaned simply by looking at a graph of the function. In fact, a person with good visual skills can "see" the graph of the derivative while looking at the graph of the function. This activity focuses on helping you develop that skill.

Problem 22.1

A parabolic function is shown in Figure 22.1. Each question in this problem is in reference to that function.

- 22.1.1 Several values of the function g' are given in Table 22.1. For each given value draw a nice long line segment at the corresponding point on g whose slope is equal to the value of g' . If we think of these line segments as actual lines, what do we call the lines?
- 22.1.2 What is the value of g' at 1? How do you know? Go ahead and enter that value into Table 22.1.
- 22.1.3 The function g is symmetric across the line $x = 1$; that is, if we move equal distance to the left and right from this line the corresponding y -coordinates on g are always equal. Notice that the slopes of the tangent lines are "equal but opposite" at points that are equally removed from the axis of symmetry; this is reflected in the values of $g'(-1)$ and $g'(3)$. Use the idea of "equal but opposite slope equidistance from the axis of symmetry" to complete Table 22.1.
- 22.1.4 Plot the points from Table 22.1 onto Figure 22.2 and connect the dots. Determine the formula for the resultant linear function.
- 22.1.5 The formula for g is $g(x) = -.5x^2 + x + 5.5$. Use Definition 19.1 to determine the formula for $g'(x)$.
- 22.1.6 The line you drew onto Figure 22.2 is not a tangent line to g . Just what exactly is this line?

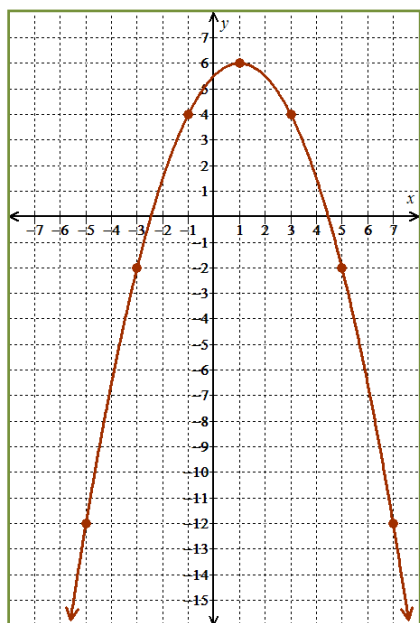


Figure 22.1: g

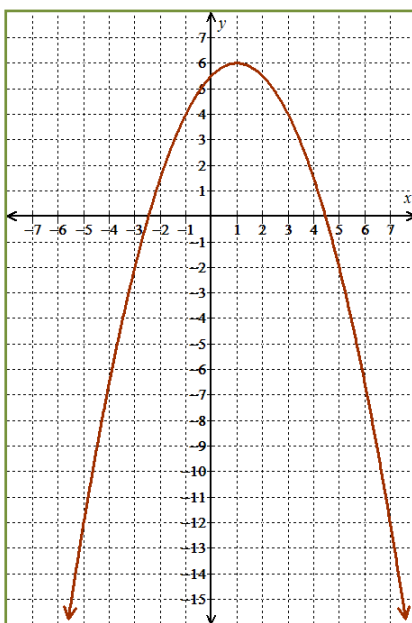


Figure 22.2: g

Table 22.1: $y = g'(x)$

x	y
-5	6
-3	
-1	2
1	
3	-2
5	-4
7	

Problem 22.2

A function f is shown in Figure 22.3 and the corresponding first derivative function f' is shown in Figure 22.4. Answer each of the following questions referencing these two functions.

- 22.2.1 Draw the tangent line to f at the three points indicated in Figure 22.3 after first using the graph of f' to determine the exact slope of the respective tangent lines.
- 22.2.2 Write a sentence relating the slope of the tangent line to f with the corresponding y -coordinate on f' .
- 22.2.3 Copy each of the following phrases onto your paper and supply the words or phrases that correctly complete each sentence.
- Over the interval where f' is always negative f is always _____.
 - Over the intervals where f' is always positive f is always _____.
 - Over the interval where f' is always increasing f is always _____.
 - Over the interval where f' is always decreasing f is always _____.

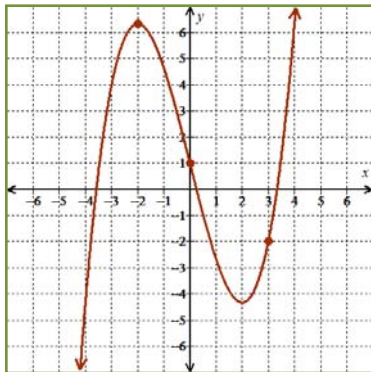


Figure 22.3: f

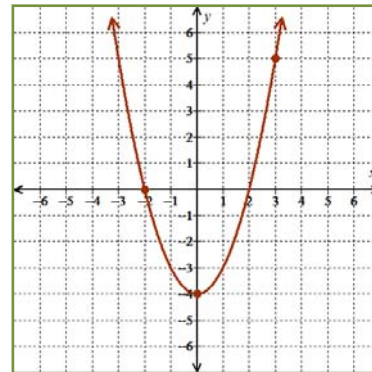


Figure 22.4: f'

Problem 22.3

In each of figures 22.5 and 22.6 a function (the thin curve) is given; both of these functions are symmetric about the y -axis. The first derivative of each function (the thick curves) have been drawn over the interval $(0, 7)$. Use the given portion of the first derivative together with the symmetry of the function to help you draw each first derivative over the interval $(-7, 0)$.

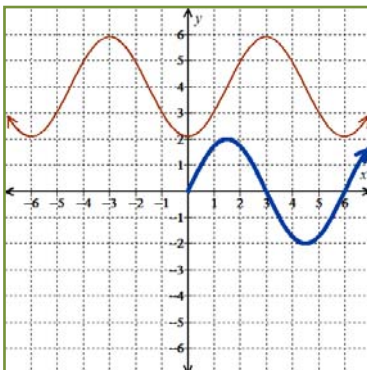


Figure 22.5: g and g'

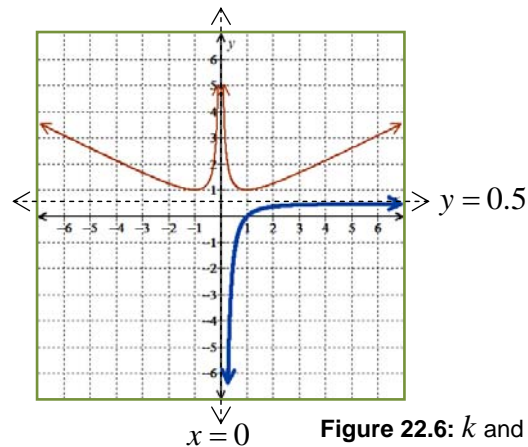


Figure 22.6: k and k'

Problem 22.4

A graph of the function $y = \frac{1}{x}$ is shown in Figure 22.7; call this function f .

- 22.4.1 Except at 0, there is something that is always true about the value of f' ; what is the common trait?
- 22.4.2 Use Definition 19.1 to find the formula for $f'(x)$.
- 22.4.3 Does the formula for $f'(x)$ support your answer to problem 22.4.1?
- 22.4.4 Use the formula for $f'(x)$ to determine the horizontal and vertical asymptotes for the graph of $y = f'(x)$.
- 22.4.5 Keeping it simple, draw onto Figure 22.8 a curve with the asymptotes found in problem 22.4.4 and the property determined in problem 22.4.1. Does the curve you drew have the properties you would expect to see in the first derivative of f ? For example, f is concave down over $(-\infty, 0)$ and concave up over $(0, \infty)$; what are the corresponding differences in the behavior of f' over those two intervals?

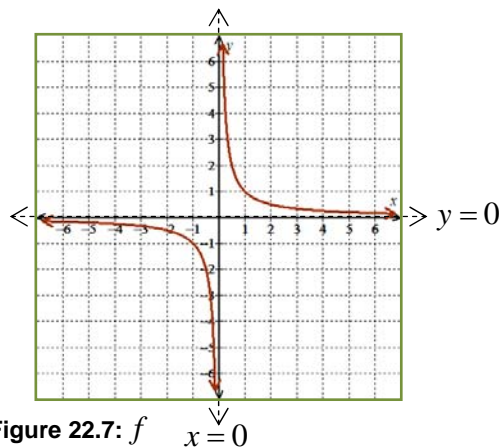


Figure 22.7: f

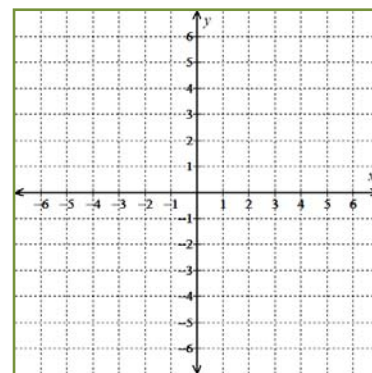


Figure 22.8: f'

Problem 22.5

A graph of the function g is shown in Figure 22.9. The absolute minimum value ever obtained by g' is -3 . With that in mind, draw g' onto Figure 22.10. Make sure that you draw and label any and all necessary asymptotes. Make sure that your graph of g' adequately reflects the symmetry in the graph of g .

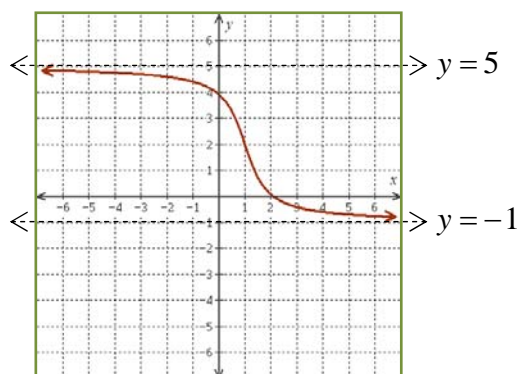


Figure 22.9: g

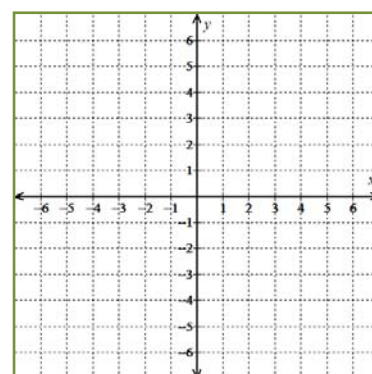


Figure 22.10: g'

Problem 22.6

A function, w , is shown in Figure 22.11. A larger version of Figure 22.11 is available in Appendix B (page B3). Answer each of the following questions in reference to this function.

- 22.6.1 An inflection point is a point where the function is continuous and the concavity of the function changes. The inflection points on w occur at 2, 3.25, and 6. With that in mind, over each interval stated in Table 22.3 exactly two of the words in Table 22.2 apply to w' . Complete Table 22.2 with the appropriate pairs of words.

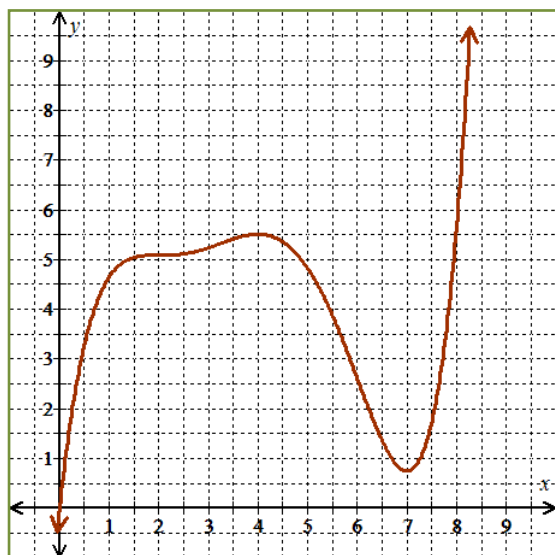


Figure 22.11: w

Table 22.2: Properties

positive
negative
increasing
decreasing

Table 22.3: Properties of w'

Interval	Properties
$(-\infty, 2)$	
$(2, 3.25)$	
$(3.25, 4)$	
$(4, 6)$	
$(6, 7)$	
$(7, \infty)$	

- 22.6.2 In Table 22.4, three possible values are given for w' at several values of x . In each case, one of the values is correct. Use tangent lines to w to determine each of the correct values. (This is where you probably want to use the graph on page B3.)
- 22.6.3 The value of w' is the same at 2, 4, and 7. What is this common value?
- 22.6.4 Put it all together and draw w' onto Figure 22.12.

Table 22.4: Choose the correct values for w'

x	Proposed values
0	$\frac{2}{3}$ or $\frac{8}{3}$ or $\frac{28}{3}$
1	$\frac{1}{2}$ or $\frac{3}{2}$ or $\frac{5}{2}$
3	$\frac{1}{3}$ or 1 or 3
5	$-\frac{1}{2}$ or -1 or $-\frac{3}{2}$
6	$-\frac{4}{3}$ or $-\frac{8}{3}$ or -4
8	1 or 6 or 12

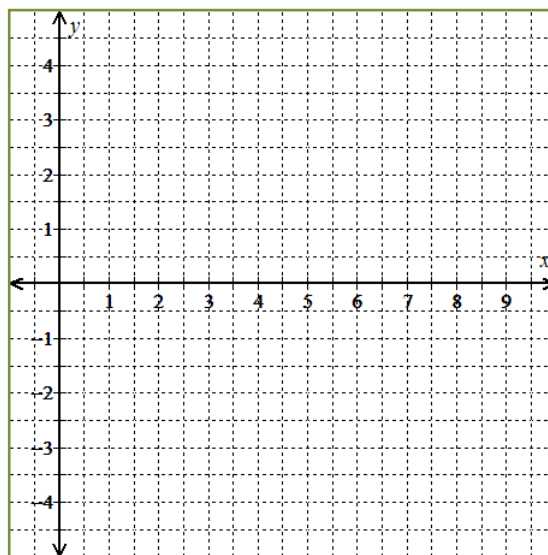


Figure 22.12: w'

Activity 23

A function is said to be **nondifferentiable** at any value its first derivative is undefined. There are three graphical behaviors that lead to non-differentiability.

- f is nondifferentiable at a if f is discontinuous at a .
- f is nondifferentiable at a if the slope of f is different from the left and right at a .
- f is nondifferentiable at a if f has a vertical tangent line at a .

Problem 23.1

Consider the function k shown in Figure 23.1.

23.1.1 There are four values where k is nondifferentiable; what are these values?

23.1.2 Draw k' onto Figure 23.2.

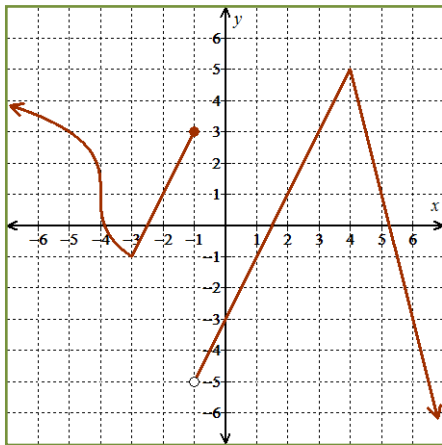


Figure 23.1: k

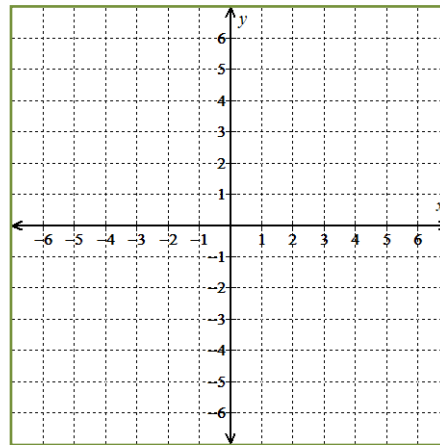


Figure 23.2: k'

Problem 23.2

Consider the function g shown in Figure 23.3.

23.2.1 g' has been drawn onto Figure 23.4 over the interval $(-5, -2.5)$. Use the piece-wise symmetry and periodic behavior of g to help you draw the remainder of g' over $(-7, 7)$

23.2.2 What six syllable word applies to g at $-5, 0,$ and 5 ?

23.2.3 What five syllable and six syllable words apply to g' at $-5, 0,$ and 5 ?

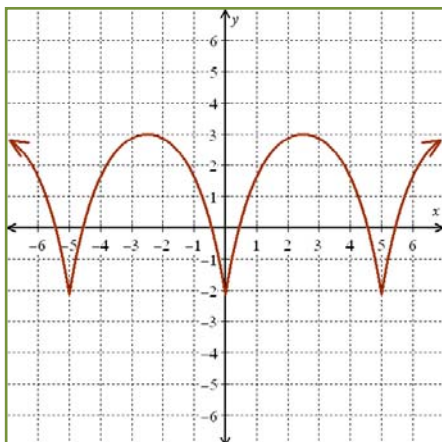


Figure 23.3: g

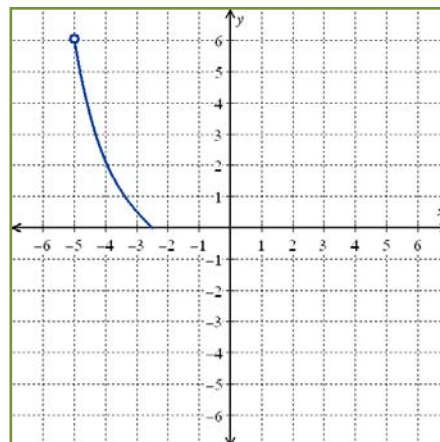


Figure 23.4: g'

Activity 24

Seeing as the first derivative of f is a function in its own right, f' must have its own first derivative. The first derivative of f' is the second derivative of f and is symbolized as f'' (f double-prime). Likewise, f''' (f triple-prime) is the first derivative of f'' , the second derivative of f' , and the third derivative of f .

Table 24.1: f' and f

When f' is ...	f is ...
Positive	Increasing
Negative	Decreasing
Constantly Zero	Constant
Increasing	Concave Up
Decreasing	Concave Down
Constant	Linear

All of the graphical relationships you've established between f and f' work their way down the derivative chain; this is illustrated in tables 24.1, 24.2, and 24.3.

Problem 24.1

Extrapolating from tables 24.1 and 24.2, what must be true about f over intervals where f'' is, respectively, positive, negative, and constantly zero?

Table 24.2: f'' and f'

When f'' is ...	f' is ...
Positive	Increasing
Negative	Decreasing
Constantly Zero	Constant
Increasing	Concave Up
Decreasing	Concave Down
Constant	Linear

Problem 24.2

A function, g , and its first three derivatives are shown in figures 24.1-24.4, although not in that order. Determine which curve is which function (g , g' , g'' , and g''').

Table 24.3: f''' and f''

When f''' is ...	f'' is ...
Positive	Increasing
Negative	Decreasing
Constantly Zero	Constant
Increasing	Concave Up
Decreasing	Concave Down
Constant	Linear

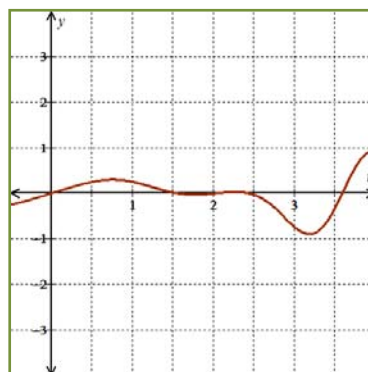


Figure 24.1

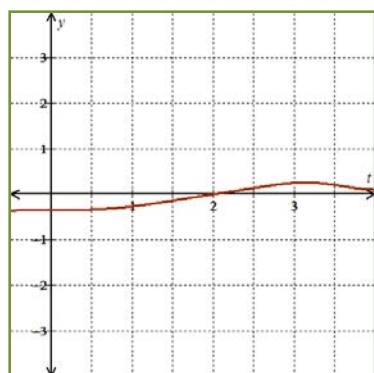


Figure 24.2

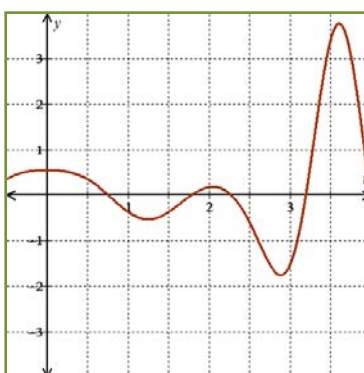


Figure 24.3

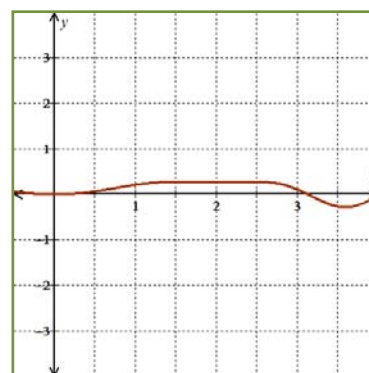


Figure 24.4

Problem 24.3

Three containers are shown in figures 24.5-24.7. Each of the following questions are in reference to these containers.

- 24.3.1** Suppose that water is being poured into each of the containers at a constant rate. Let h_5 , h_6 , and h_7 be the heights (measured in cm) of the liquid in containers 24.5-24.7, respectively, t seconds after the water began to fill the containers. What would you expect the sign to be on the second derivative functions h_5'' , h_6'' , h_7'' while the containers are being filled? (Hint: Think about the shape of the curves $y = h_5(t)$, $y = h_6(t)$, and $y = h_7(t)$.)
- 24.3.2** Suppose that water is being drained from each of the containers at a constant rate. Let h_5 , h_6 , and h_7 be the heights (measured in cm) of the liquid remaining in the containers t seconds after the water began to drain. What would you expect the sign to be on the second derivative functions h_5'' , h_6'' , h_7'' while the containers are being drained?

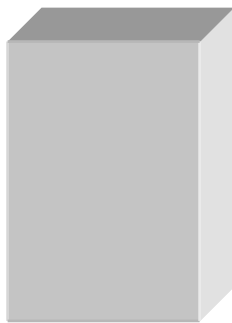


Figure 24.5

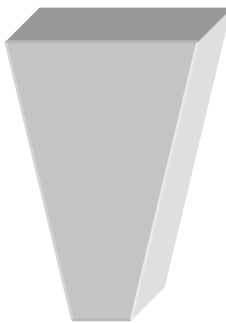


Figure 24.6

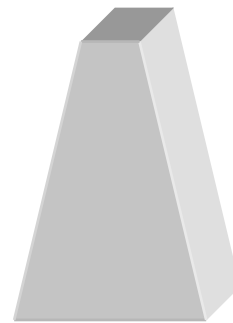


Figure 24.7

Problem 24.4

During the recession of 2008-2009, the total number of employed Americans decreased every month. One month a talking head on the television made the observation that "at least the second derivative was positive this month." Why was it a good thing that the second derivative was positive?

Problem 24.5

During the early 1980s the problem was inflation. Every month the average price for a gallon of milk was higher than the month before. Was it a good thing when the second derivative of this function was positive? Explain.

Activity 25

The derivative continuum can be expressed backwards as well as forwards. When you move from function to function in the reverse direction the resultant functions are called antiderivatives and the process is called antidifferentiation. These relationships are shown in figures 25.1 and 25.2.

$$f \xrightarrow{\text{differentiate}} f' \xrightarrow{\text{differentiate}} f'' \xrightarrow{\text{differentiate}} f''' \xrightarrow{\text{differentiate}} \dots$$

Figure 25.1: Differentiating

$$\dots \xrightarrow{\text{antidifferentiate}} f''' \xrightarrow{\text{antidifferentiate}} f'' \xrightarrow{\text{antidifferentiate}} f' \xrightarrow{\text{antidifferentiate}} f \xrightarrow{\text{antidifferentiate}} F$$

Figure 25.2: Antidifferentiating

There are (at least) two important differences between the differentiation chain and the antidifferentiation chain (besides their reversed order).

- When you differentiate, the resultant function is unique. When you antidifferentiate, you do not get a unique function - you get a family of functions; specifically, you get a set of parallel curves.
- We introduce a new function in the antidifferentiation chain. We say that F is an antiderivative of f . This is where we stop in that direction; we do not have a variable name for an antiderivative of F .

Since F is considered an antiderivative of f , it must be the case that f is the first derivative of F . Hence we can add F to our derivative chain resulting in Figure 25.3

$$F \xrightarrow{\text{differentiate}} f \xrightarrow{\text{differentiate}} f' \xrightarrow{\text{differentiate}} f'' \xrightarrow{\text{differentiate}} f''' \xrightarrow{\text{differentiate}} \dots$$

Figure 25.3: Differentiating

Problem 25.1

Each of the linear functions in Figure 25.4 have the same first derivative function.

25.1.1 Draw this common first derivative function onto Figure 25.4 and label it g .

25.1.2 Each of the given lines Figure 25.4 is called what in relation to g ?

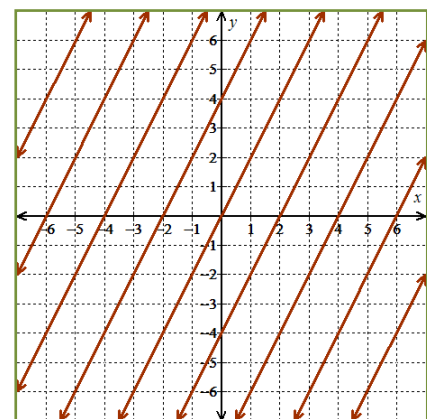


Figure 25.4

Problem 25.2

The function f is shown in Figure 25.5. Reference this function in the following questions.

- 25.2.1 At what values of x is f nondifferentiable?
- 25.2.2 At what values of x are antiderivatives of f nondifferentiable?
- 25.2.3 Draw onto Figure 25.6 the continuous antiderivative of f that passes through the point $(-3,1)$. Please note that every antiderivative of f increases exactly one unit over the interval $(-3, -2)$.

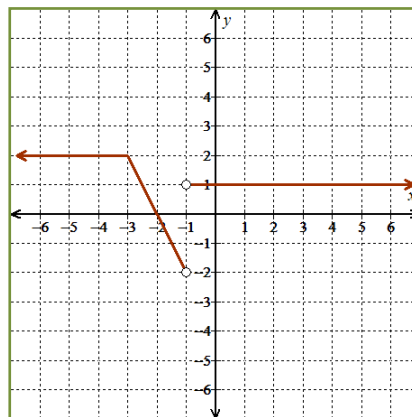


Figure 25.5: f

- 25.2.4 Because f is not continuous, there are other antiderivatives of f that pass through the point $(-3,1)$. Specifically, antiderivatives of f may or may not be continuous at -1 . Draw onto figures 25.7 and 25.8 different antiderivatives of f that pass through the point $(-3,1)$.

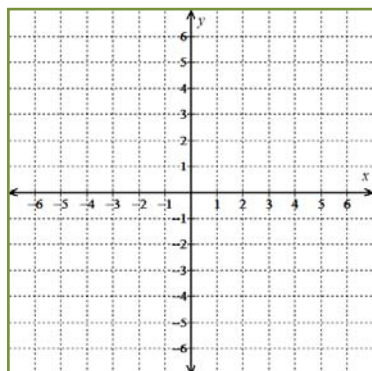


Figure 25.6

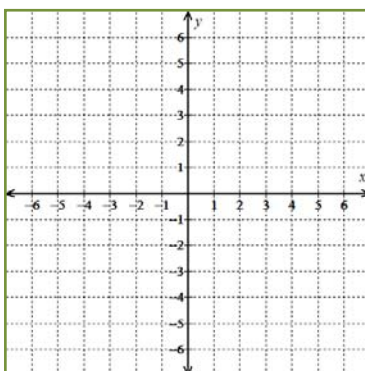


Figure 25.7

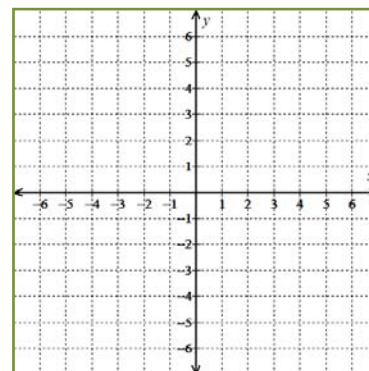


Figure 25.8

Problem 25.3

The function $y = \sin(x)$ is an example of a periodic function. Specifically, the function has a period of 2π because over any interval of length 2π the behavior of the function is exactly the same as it was the previous interval of length 2π . A little more precisely, $\sin(x + 2\pi) = \sin(x)$ regardless of the value of x .

Jasmine was thinking and told her lab assistant that derivatives and antiderivatives of periodic functions must also be periodic. Jasmine's lab assistant told her that she was half right. Which half did Jasmine have correct? Also, draw a function that illustrates that the other half of Jasmine's statement is not correct.

Problem 25.4

Consider the function g shown in Figure 25.9.

25.4.1 Let G be an antiderivative of g . Suppose that G is continuous on $[-6, 6]$, $G(-6) = -3$, and that the greatest value G ever achieves is 6. Draw G onto Figure 25.10.

25.4.2 At what values of t is G nondifferentiable?

25.4.3 At what values of t is g nondifferentiable?

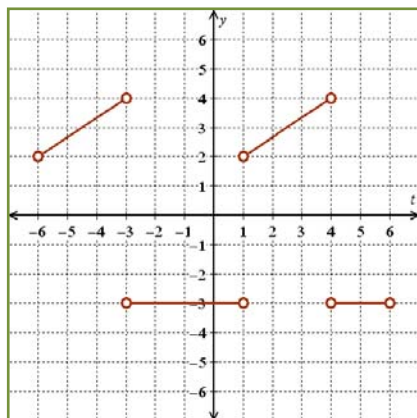


Figure 25.9 g

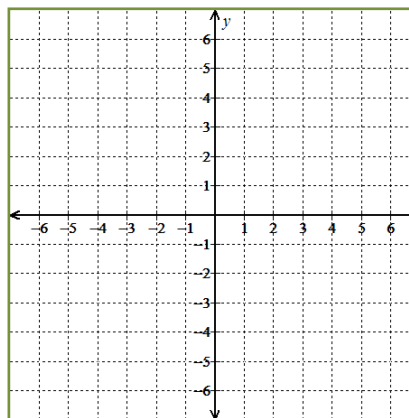


Figure 25.10 G

Problem 25.5

Answer the following question in reference to a continuous function g whose first derivative is shown in Figure 25.11. You do not need to state how you made your determination; just state the interval(s) or values of x that satisfy the stated property.

Note: The correct answer to one or more of these questions may be "There is no way of knowing."

25.5.1 Over these intervals, g'' is positive and increasing.

25.5.2 At these values of x , g is nondifferentiable.

25.5.3 Over these intervals, g is never negative.

25.5.4 At these values of x , every antiderivative of g is nondifferentiable.

25.5.5 Over these intervals, the value of g'' is constant.

25.5.6 Over these intervals, g is linear.

25.5.7 Over these intervals, antiderivatives of g are linear.

25.5.8 Over these intervals, g''' is never negative.

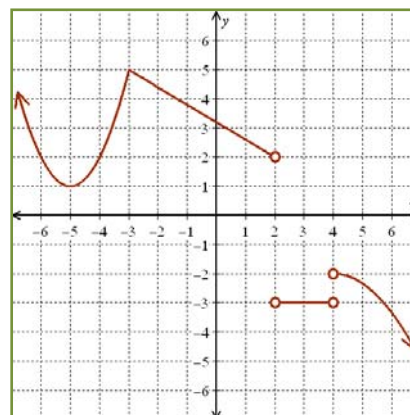


Figure 25.11 g'

Activity 26

When given a function formula, we often find the first and second derivative formulas to determine behaviors of the given function. In a later lab we will use the first and second derivative formulas to help us graph a function given the formula for the function. One thing we do with the derivative formulas is determine where they are positive, negative, zero, and undefined. This helps us determine where the given function is increasing, decreasing, concave up, concave down, and linear.

Problem 26.1

26.1.1 Draw onto Figure 26.1 a continuous function f that has a horizontal tangent line at the point $(-1, 2)$ along with the properties stated in Table 26.1.

26.1.2 Given the conditions stated in problem 26.1.1, does it have to be the case that $f''(-1) = 0$?

26.1.3 Is f increasing at -1 ? How do you know?

26.1.4 Can a continuous, everywhere differentiable function satisfy the properties stated in Table 26.1 and not have a slope of zero at -1 ? Draw a picture that supports your answer.

Table 26.1: Signs on f' and f''

Interval	f'	f''
$(-\infty, -1)$	Positive	Negative
$(-1, \infty)$	Positive	Positive

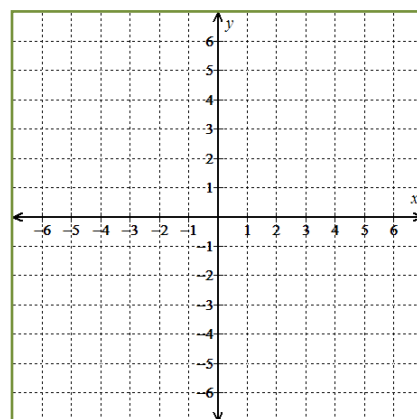


Figure 26.1 f

Problem 26.2

26.2.1 Draw onto Figure 26.2 a continuous function g that has a vertical tangent line at the point $(-1, 2)$ along with the properties stated in Table 26.2.

26.2.2 Given the conditions stated in problem 26.2.1, does it have to be the case that $g''(-1)$ is undefined?

26.2.3 Is g increasing at -1 ? How do you know?

26.2.4 Can a continuous, everywhere differentiable function satisfy the properties stated in Table 26.2 and not have a vertical tangent line at -1 ? Draw a picture that supports your answer.

Table 26.2: Signs on g' and g''

Interval	g'	g''
$(-\infty, -1)$	Positive	Positive
$(-1, \infty)$	Positive	Negative

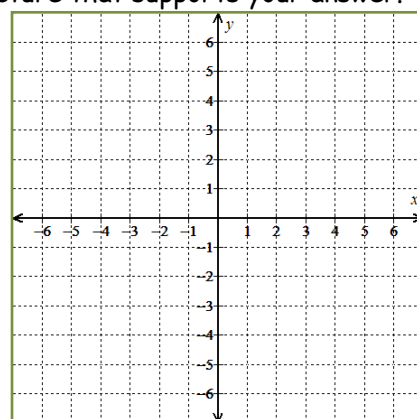


Figure 26.2 g

Problem 26.3

26.3.1 Draw onto Figure 26.3 a continuous function k that passes through the point $(-1, 2)$ and also satisfies the properties stated in Table 26.3.

26.3.2 At what values of x is k nondifferentiable?

Interval	k'	k''
$(-\infty, -1)$	Negative	Positive
$(-1, \infty)$	Positive	Negative

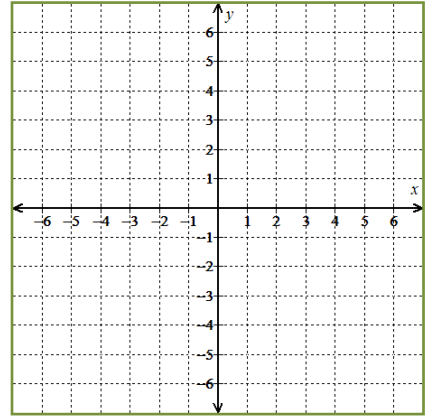


Figure 26.3 k

Problem 26.4

You should know by now that the first derivative continually increases over intervals where the second derivative is constantly positive and that the first derivative continually decreases over intervals where the second derivative is constantly negative. A person might infer from this that a function changes more and more quickly over intervals where the second derivative is constantly positive and that a function changes more and more slowly over intervals where the second derivative is constantly negative. We are going to explore that idea in this problem.

26.4.1 Suppose that $V(t)$ is the volume of water in an ice cube (ml) where t is the amount of time that has passed since noon (measure in minutes). Suppose that $V'(6) = 0 \frac{\text{ml}}{\text{min}}$ and that

$V''(t)$ has a constant value of $-.3 \frac{\text{ml}}{\text{min}}$ over the interval $[6, 11]$. What is the value of

$V'(11)$? When is V changing more quickly, at 12:06 pm or at 12:11 pm?

26.4.2 Referring to the function V in problem 26.4.1, what would the shape of V be over the interval $[6, 11]$? (Choose from options a - d.)



26.4.3 Again referring to options a-d, which functions are changing more and more rapidly from left to right and which functions are changing more and more slowly from left to right? Which functions have positive second derivatives and which functions have negative second derivatives? Do the functions with positive second derivative values both change more and more quickly from left to right?

- 26.4.4 Consider the signs on both the first and second derivatives in options a-d. Is there something that the two functions that change more and more quickly have in common that is different in the functions that change more and more slowly?

Problem 26.5

Resolve each of the following disputes.

- 26.5.1 One day Sara and Jermaine were working on an assignment. One question asked them to draw a function over the domain $(-2, \infty)$ with the properties that the function is always increasing and always concave down. Sara insisted that the curve must have a vertical asymptote at -2 and Jermaine insisted that the function must have a horizontal asymptote somewhere. Were either of these students correct?
- 26.5.2 The next question Sara and Jermaine encountered described the same function with the added condition that the function is never positive. Sara and Jermaine made the same contentions about asymptotes. Is one of them now correct?
- 26.5.3 At another table Pedro and Yoshi were asked to draw a continuous curve that, among other properties, was never concave up. Pedro said "OK, so the curve is always concave down" to which Yoshi replied "Pedro, you need to open your mind to other possibilities." Who's right?
- 26.5.4 In the next problem Pedro and Yoshi were asked to draw a function that is everywhere continuous and that is concave down at every value of x **except** 3. Yoshi declared "impossible" and Pedro responded "have some faith, Yosh-man." Pedro then began to draw. Is it possible that Pedro came up with such a function?

Problem 26.6

Determine the correct answer to each of the following questions. Pictures of the situation may help you determine the correct answers.

- 26.6.1 Which of the following propositions is true? If a given proposition is not true, draw a graph that illustrates its untruth.
- 26.6.1a If the graph of f has a vertical asymptote, then the graph of f' must also have a vertical asymptote.
- 26.6.1b If the graph of f' has a vertical asymptote, then the graph of f must also have a vertical asymptote.
- 26.6.2 Suppose that the function f is everywhere continuous and concave down. Suppose further that $f(7) = 5$ and $f'(7) = 3$. Which of the following is true?
- a. $f(9) < 11$ b. $f(9) = 11$ c. $f(9) > 11$
- d. There is not enough information to determine the relationship between $f(9)$ and 11.