Derivative Formulas

Activity 27

While the primary focus of this lab is to help you develop shortcut skills for finding derivative formulas, there are inevitable notational issues that must be addressed. It turns out that the latter issue is the one we are going to address first.

If $y = f(x)$, we say that ***the derivative of y with respect to x*** is equal to $f'(x)$. Symbolically we write: $\frac{dy}{dx} = f'(x)$

> While the symbol $\frac{dy}{dx}$ certainly ***looks*** like a fraction, ***it is not a fraction***. The symbol is Leibniz notation for the first derivative of y with respect to x. The short way of reading the symbol aloud is "d-y-d-x" (don't enunciate the dashes).

If $z = g(t)$, we say that the ***the derivative of z with respect to t*** is equal to $g'(t)$. Symbolically we write: $\frac{dz}{dt} = g'(t)$. (Read aloud as "d-z-d-t equals g-prime of t.")

Problem 27.1
Take the derivative of both sides of each equation with respect to the independent variable as indicated in the function notation. Write and say the derivative using Leibniz notation on the left side of the equal sign and function notation on the right side of the equal sign. Make sure that every one in your group says at least one of the derivative equations aloud using both the formal reading and informal reading of the Leibniz notation.

27.1.1 $y = k(t)$ **27.1.2** $V = f(r)$ **27.1.3** $T = g(P)$

Activity 28

$\frac{dy}{dx}$ is ***the name of a derivative*** in the same way that $f'(x)$ is the name of a derivative. We need a different symbol that tells us to ***take*** the derivative of a given expression (in the same way that we have symbols that tell us to take a square root, sine, or logarithm of an expression).

The symbol $\frac{d}{dx}$ is used to tell us to ***take the derivative with respect to x of something***. **The symbol itself is an incomplete phrase** in the same way that the symbol $\sqrt{\ }$ is an incomplete phrase; in both cases we need to indicate the object to be manipulated – what number or formula are we taking the square root of? … what number or formula are we differentiating?

One thing you can do to help you remember the difference between the symbols $\frac{dy}{dx}$ and $\frac{d}{dx}$ is to get in the habit of always writing grouping symbols after $\frac{d}{dx}$. In this way the symbols $\frac{d}{dx}\left(\sin(x)\right)$ mean "the derivative with respect to x of the sine of x." Similarly, the symbols $\frac{d}{dt}\left(t^2\right)$ mean "the derivative with respect to t of t-squared."

Problem 28.1

Write the Leibniz notation for each of the following expressions.

28.1.1 The derivative with respect to β of $\cos(\beta)$.

28.1.2 The derivative with respect to x of $\frac{dy}{dx}$.

28.1.3 The derivative with respect to t of $\ln(x)$. (Yes, we will do such things.)

28.1.4 The derivative of z with respect to x.

28.1.5 The derivative with respect to t of $g(8)$. (Yes, we will also do such things.)

Activity 29

The first differentiation rule we are going to explore is called ***<u>the power rule of differentiation</u>***.

Equation 29.1	$\frac{d}{dx}\left(x^n\right) = n\,x^{n-1}$ for $\text{n} \neq 0$

When n is a positive integer, it is fairly easy to establish this rule using Definition 19.1. The proof of the rule gets a little more complicated when n is negative, fractional, or irrational. For purposes of this lab, we are going to just accept the rule as valid.

This rule is one you just "do in your head" and then write down the result. Three examples of what you would be expected to write when differentiating power functions are shown below.

Given Function	$y = x^7$	$f(t) = \sqrt[3]{t^7}$	$z = \frac{1}{y^5}$
What you should ***think*** ***<u>or</u> write*** (***as necessary***)	$y = x^7$	$f(t) = t^{7/3}$	$z = y^{-5}$
What you should write	$\frac{dy}{dx} = 7\,x^6$	$f'(t) = \frac{7}{3}t^{4/3}$ $= \frac{7}{3}\sqrt[3]{t^4}$	$\frac{dz}{dy} = -5\,y^{-6}$ $= -\frac{5}{y^6}$

Notice that the type of notation used when naming the derivative is dictated by the manner in which the original function is expressed. For example, $y = x^7$ is telling us the relationship between two variables; in this situation we name the derivative using the notation $\frac{dy}{dx}$. On the other hand, function notation is being used to name the rule in $f(t) = \sqrt[3]{t^7}$; in this situation we name the derivative using the function notation $f'(t)$.

Problem 29.1

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

29.1.1 $f(x) = x^{43}$ **29.1.2** $z = \frac{1}{t^7}$ **29.1.3** $P = \sqrt[5]{t^2}$ **29.1.4** $h(x) = \frac{1}{\sqrt{x}}$

Activity 30

The next rule you are going to practice is the ***constant factor rule of differentiation***. This is another rule you do in your head.

Equation 30.1	$\frac{d}{dx}\left(k\, f(x)\right) = k \cdot \frac{d}{dx}\left(f(x)\right)$ for $k \in \Re$

In the following examples and problems several derivative rules are used that are shown in Appendix C (pages C5 and C6). While working this lab you should refer to those rules; you may want to cover up the chain rule and implicit derivative columns this week! You should make a goal of having all of the basic formulas memorized within a week.

Given Function	$f(x) = 6x^9$	$f(\theta) = -2\cos(\theta)$	$z = 2\ln(x)$
What you should write	$f'(x) = 54x^8$	$f'(\theta) = 2\sin(\theta)$	$\frac{dz}{dx} = \frac{2}{x}$

Problem 30.1

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

30.1.1 $z = 7t^4$ **30.1.2** $P(x) = -7\sin(x)$ **30.1.3** $h(t) = \frac{1}{3}\ln(t)$

30.1.4 $z(x) = \pi\tan(x)$ **30.1.5** $P = \frac{-8}{t^4}$ **30.1.6** $T = 4\sqrt{t}$

Activity 31

When an expression is divided by the constant k, we can think of the expression as being multiplied by the fraction $\frac{1}{k}$. In this way, the constant factor rule of differentiation can be applied when a formula is multiplied ***or*** divided by a constant.

Given Function	$f(x)=\frac{x^4}{8}$	$y=\frac{3\tan(\alpha)}{2}$	$z=\frac{\ln(y)}{3}$
What you should ***think***	$f(x)=\frac{1}{8}x^4$	$y=\frac{3}{2}\tan(\alpha)$	$z=\frac{1}{3}\ln(y)$
What you should write	$f'(x)=\frac{1}{2}x^3$	$\frac{dy}{d\alpha}=\frac{3}{2}\sec^2(\alpha)$	$\frac{dz}{dy}=\frac{1}{3y}$

Problem 31.1

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

31.1.1 $z(t)=\frac{\sin^{-1}(t)}{6}$

31.1.2 $V(r)=\frac{\pi r^3}{3}$

31.1.3 $f(r)=\frac{G m_1 m_2}{r^2}$

G, m_1, and m_2 are constants.

Activity 32

When taking the derivative of two or more terms, you can take the derivatives term by term and insert plus or minus signs as appropriate. Collectively we call this ***<u>the sum and difference rules of differentiation</u>***.

Equation 32.1 $$\frac{d}{dx}\left(f(x)\pm g(x)\right)=\frac{d}{dx}\left(f(x)\right)\pm\frac{d}{dx}\left(g(x)\right)$$

In the following examples and problems we introduce linear and constant terms into the functions being differentiated.

Equation 32.2	$\frac{d}{dx}(k)=0 \text{ for } k\in\mathbb{R}$	**Equation 32.3**	$\frac{d}{dx}(kx)=k \text{ for } k\in\mathbb{R}$

Problem 32.1

Explain why both the constant rule (Equation 32.2) and linear rule (Equation 32.3) are "obvious."

Hint: Think about the graphs $y=k$ and $y=kx$. What does a first derivative tell you about a graph?

A couple of examples using the sum and difference rule are shown below.

Given Function	$y = 4\sqrt[5]{t^6} - \dfrac{1}{6\sqrt{t}} + 8t$	$P(\gamma) = \dfrac{\sin(\gamma) - \cos(\gamma)}{2} + 4$
What you should ***think*** ***<u>or</u>*** ***write*** (***as necessary***)	$y = 4t^{6/5} - \dfrac{1}{6}t^{-1/2} + 8t$	$P(\gamma) = \dfrac{1}{2}\sin(\gamma) - \dfrac{1}{2}\cos(\gamma) + 4$
What you should definitely write	$\dfrac{dy}{dt} = \dfrac{24}{5}t^{1/5} + \dfrac{1}{12}t^{-3/2} + 8$ $= \dfrac{24\sqrt[5]{t}}{5} + \dfrac{1}{12\sqrt{t^3}} + 8$	$P'(\gamma) = \dfrac{1}{2}\cos(\gamma) + \dfrac{1}{2}\sin(\gamma)$

Problem 32.2

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

32.2.1 $T = \sin(t) - 2\cos(t) + 3$

32.2.2 $k(\theta) = \dfrac{4\sec(\theta) - 3\csc(\theta)}{4}$

32.2.3 $r(x) = \dfrac{x}{5} + 7$

32.2.4 $r = \dfrac{2}{3\sqrt[3]{x}} - \dfrac{\ln(x)}{9} + \ln(2)$

Activity 33

The next rule we are going to explore is called ***<u>the product rule of differentiation</u>***. We use this rule when there are ***two or more <u>variable</u> factors*** in the expression we are differentiating. (Remember, we already have the constant factor rule to deal with two factors when one of the two factors is a constant.)

Equation 33.1
$$\frac{d}{dx}\big(f(x)g(x)\big) = \frac{d}{dx}\big(f(x)\big)\cdot g(x) + f(x)\cdot\frac{d}{dx}\big(g(x)\big)$$

Intuitively, what is happening in this rule is that we are alternately treating one factor as a constant (the one not being differentiated) and the other factor as a variable function (the one that is being differentiated). We then add these two rates of change together.

Ultimately, you want to perform this rule in your head just like all of the other rules. Your instructor, however, may initially want you to show steps; under that presumption, steps are going to be shown in each and every example of this lab when the product rule is applied.
Two simple examples of the product rule are shown at the top of the next page.

Given Function	$y(x)=x^2\sin(x)$	$P=e^t\cos(t)$
Derivative	$y'(x)=\frac{d}{dx}(x^2)\cdot\sin(x)+x^2\cdot\frac{d}{dx}(\sin(x))$ $=2x\sin(x)+x^2\cos(x)$	$\frac{dP}{dt}=\frac{d}{dt}(e^t)\cdot\cos(t)+e^t\cdot\frac{d}{dt}(\cos(t))$ $=e^t\cos(t)-e^t\sin(t)$

A decision you'll need to make is how to handle a constant factor in a term that requires the product rule. Two options for taking the derivative of $f(x)=5x^2\ln(x)$ are shown below.

Option A	Option B
$f'(x)=5\left[\frac{d}{dx}(x^2)\cdot\ln(x)+x^2\cdot\frac{d}{dx}(\ln(x))\right]$ $=5\left[2x\ln(x)+x^2\cdot\frac{1}{x}\right]$ $=10x\ln(x)+5x$	$f'(x)=\frac{d}{dx}(5x^2)\cdot\ln(x)+5x^2\cdot\frac{d}{dx}(\ln(x))$ $=10x\ln(x)+5x^2\cdot\frac{1}{x}$ $=10x\ln(x)+5x$

In Option B we are treating the factor of 5 as a part of the first variable factor. In doing so, the factor of 5 distributes itself. This is the preferred treatment of the author, so this is what you will see illustrated in this lab.

Problem 33.1

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

33.1.1 $T(t)=2\sec(t)\tan(t)$

33.1.2 $k=\frac{e^t\sqrt{t}}{2}$

33.1.3 $y=4x\ln(x)+3^x-x^3$

33.1.4 $f(x)=\cot(x)\cot(x)-1$

Problem 33.2

Find each of the following derivatives ***without first simplifying the formula***; that is, go ahead and use the product rule on the expression as written. Simplify each resultant derivative formula. For each derivative, ***check*** your answer by simplifying the original expression and then taking the derivative of that simplified expression.

33.2.1 $\frac{d}{dx}\left(x^4x^7\right)$

33.2.2 $\frac{d}{dx}\left(x\cdot x^{10}\right)$

33.2.3 $\frac{d}{dx}\left(\sqrt{x}\cdot\sqrt{x^{21}}\right)$

Activity 34

The next rule we are going to explore is called ***the quotient rule of differentiation***. We **never** use this rule unless there is ***a variable factor in the denominator of the expression*** we are differentiating. (Remember, we already have the constant factor rule to deal with constant factors in the denominator.)

Equation 34.1	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{d}{dx}(f(x))\cdot g(x)-f(x)\cdot\frac{d}{dx}(g(x))}{\left[g(x)\right]^2};\; g(x)\neq 0$

Like all of the other rules, you ultimately want to perform the quotient rule in your head. Your instructor, however, may initially want you to show steps when applying the quotient rule; under that presumption, steps are going to be shown in each and every example of this lab when the quotient rule is applied. Two simple examples of the quotient rule are shown below.

Given Function	$y=\frac{4x^3}{\ln(x)}$	$V=\frac{\csc(t)}{\tan(t)}$
Derivative	$\frac{dy}{dx}=\frac{\frac{d}{dx}(4x^3)\cdot\ln(x)-4x^3\cdot\frac{d}{dx}(\ln(x))}{\left[\ln(x)\right]^2}$ $=\frac{12x^2\ln(x)-4x^3\cdot\frac{1}{x}}{\left[\ln(x)\right]^2}$ $=\frac{12x^2\ln(x)-4x^2}{\left[\ln(x)\right]^2}$	$\frac{dV}{dt}=\frac{\frac{d}{dt}(\csc(t))\cdot\tan(t)-\csc(t)\cdot\frac{d}{dt}(\tan(t))}{\left[\tan(t)\right]^2}$ $=\frac{-\csc(t)\cot(t)\cdot\tan(t)-\csc(t)\cdot\sec^2(t)}{\tan^2(t)}$ $=\frac{-\csc(t)\left(1+\sec^2(t)\right)}{\tan^2(t)}$

Problem 34.1

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

34.1.1 $g(x)=\frac{4\ln(x)}{x}$

34.1.2 $j(y)=\frac{\sqrt[3]{y^5}}{\cos(y)}$

34.1.3 $y=\frac{\sin(x)}{4\sec(x)}$

34.1.4 $f(t)=\frac{t^2}{e^t}$

Problem 34.2

Find each of the following derivatives ***without first simplifying the formula***; that is, go ahead and use the quotient rule on the expression as written. For each derivative, ***check*** your answer by simplifying the original expression and then taking the derivative of that simplified expression.

34.2.1 $\frac{d}{dt}\left(\frac{\sin(t)}{\sin(t)}\right)$

34.2.2 $\frac{d}{dx}\left(\frac{x^6}{x^2}\right)$

34.2.3 $\frac{d}{dx}\left(\frac{10}{2x}\right)$

Activity 35

In problems 33.2 and 34.2 you applied the product and quotient rules to expressions where the derivative could have been found much more quickly had you simplified the expression before taking the derivative. For example, while you can find the correct derivative formula using the quotient rule when working problem 34.2.3, the derivative can be found much more quickly if you simplify the expression ***before*** applying the rules of differentiation. This is illustrated in Example 35.1.

Example 35.1

$$\frac{d}{dx}\left(\frac{10}{2x}\right)=\frac{d}{dx}\left(5x^{-1}\right)$$
$$=-5x^{-2}$$
$$=-\frac{5}{x^2}$$

Part of learning to take derivatives is learning to make good choices about the methodology to employ when taking derivatives. In examples 35.2 and 35.3, the need to use the product rule or quotient rule is obviated by first simplifying the expression being differentiated.

Example 35.2

Problem	Solution
Find $\frac{dy}{dx}$ if $y=\sec(x)\cos(x)$.	$y=\sec(x)\cos(x)$ $=\frac{1}{\cos(x)}\cdot\cos(x)$ $=1$ $\frac{dy}{dx}=0$

Example 35.3

Problem	Solution
Find $f'(t)$ if $f(t)=\frac{4t^5-3t^3}{2t^2}$.	$f(t)=\frac{4t^5-3t^3}{2t^2}$ $=\frac{4t^5}{2t^2}-\frac{3t^3}{2t^2}$ $=2t^3-\frac{3}{2}t$ $f'(t)=6t^2-\frac{3}{2}$

Problem 35.1

Find the derivative with respect to x for each of the following functions after first completely simplifying the formula being differentiated. In each case you should ***not*** use either the product rule or the quotient rule while finding the derivative formula.

35.1.1 $y=\dfrac{4x^{12}-5x^4+3x^2}{x^4}$ **35.1.2** $g(x)=\dfrac{-4\sin(x)}{\cos(x)}$ **35.1.3** $h(x)=\dfrac{4-x^6}{3x^{-2}}$

35.1.4 $z(x)=\sin^2(x)+\cos^2(x)$ **35.1.5** $z=(x+4)(x-4)$ **35.1.6** $T(x)=\dfrac{\ln(x)}{\ln(x^2)}$

Activity 36

Sometimes both the product rule and quotient rule need to be applied when finding a derivative formula.

Problem 36.1

Consider the functions $f(x)=x^2\cdot\dfrac{\sin(x)}{e^x}$ and $g(x)=\dfrac{x^2\sin(x)}{e^x}$.

36.1.1 Discuss why f and g are in fact two representations of the same function.

36.1.2 Find $f'(x)$ by first applying the product rule and then applying the quotient rule (where necessary).

36.1.3 Find $g'(x)$ by first applying the quotient rule and then applying the product rule (where necessary).

36.1.4 Rigorously establish that the formulas for $f'(x)$ and $g'(x)$ are indeed the same.

Activity 37

Derivative formulas can give us much information about the behavior of a function. For example, the derivative formula for $f(x)=x^2$ is $f'(x)=2x$. Clearly f' is negative when x is negative and f' is positive when x is positive. This tells us that f is decreasing when x is negative and that f is increasing when x is positive. This matches the behavior of the parabola $y=x^2$.

Problem 37.1

Answer each of the following questions about applied functions.

37.1.1 The amount of time (seconds), T, required for a pendulum to complete one period is a function of the pendulum's length (meters), L. Specifically, $T = 2\pi\sqrt{\frac{L}{g}}$ where g is the acceleration constant for Earth (roughly 9.8 m/s^2).

37.1.1.a Find $\frac{dT}{dL}$ after first rewriting the formula for T as a constant times $\sqrt{L}$.

37.1.1.b The sign on $\frac{dT}{dL}$ is the same regardless of the value of L. What is this sign and what does it tell you about the relative periods of two pendulums with different lengths?

37.1.2 The gravitational force (Newtons) between two objects of masses m_1 and m_2 (kg) is a function of the distance (meters) between the objects' centers of mass, r. Specifically, $F(r) = \frac{G m_1 m_2}{r^2}$ where G is the universal gravitational constant (which is approximately $6.7\times10^{-11}\ \mathrm{n\cdot m/kg^2}$).

37.1.2.a Leaving G, m_1, and m_2 as constants, find $F'(r)$ after first rewriting the formula for F as a constant times a power of r.

37.1.2.b The sign on $F'(r)$ is the same regardless of the value of F. What is this sign and what does it tell you about the effect on the gravitational force between two objects when the distance between the objects is changed?

37.1.2.c Leaving G, m_1, and m_2 as constants, evaluate $F'(1.00\times10^{12})$, $F'(1.01\times10^{12})$, $F'(1.02\times10^{12})$.

37.1.2.d Calculate $\frac{F(1.02\times10^{12}) - F(1.00\times10^{12})}{1.02\times10^{12} - 1.00\times10^{12}}$. Which of the quantities found in part c comes closest to this value? Draw a sketch of F and discuss why this result makes sense.