Critical Numbers and Graphing from Formulas

Activity 48

In the first activity of this lab you are going to discuss a few questions with your group mates that will hopefully motivate you for one of the topics covered in the lab.

Problem 48.1
Discuss how you could use the first derivative formula to help you determine the vertex of the parabola $y = -2x^2 + 18x - 7$ and then determine the vertex. Remember that the vertex is a point in the xy-plane and as such is identified using an ordered pair.

Problem 48.2
The curves in figures 48.1 and 48.2 were generated by two of the four functions given below. Use the given functions along with their first derivatives to determine which functions generated the curves. Please note that the y-scales have deliberately been omitted from the graphs and that different scales were used to generate the two graphs. Resist any temptation to use your calculator; use of your calculator totally obviates the point of the exercise.

Potential Functions:	$f_1(x) = \dfrac{1}{(x-2)^{10/7}} + C_1$	$f_2(x) = \dfrac{1}{(x-2)^{2/7}} + C_2$
C_1 - C_4 are unknown constants	$f_3(x) = (x-2)^{2/7} + C_3$	$f_4(x) = (x-2)^{10/7} + C_4$

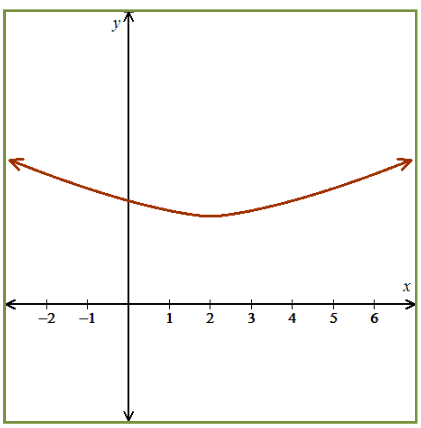

Figure 48.1: mystery curve 1

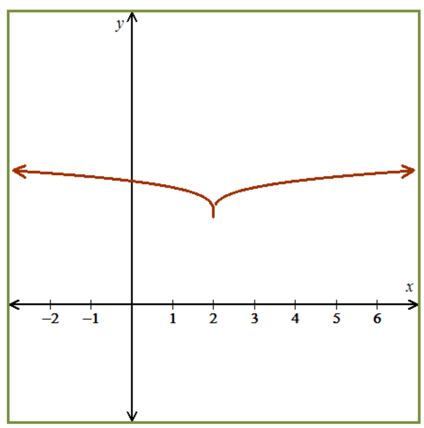

Figure 48.2: mystery curve 2

Activity 49

The vertex of the parabola in problem 48.1 is called a ***local maximum point*** and the points $(2, C_3)$ and $(2, C_4)$ in problem 48.2 are called ***local minimum points***. Collectively, these points are called ***local extreme points***.

While working Activity 48 you hopefully came to the conclusion that the local extreme points had certain characteristics in common. In the first place, they must occur at a number in the domain of the function (which eliminated f_1 and f_2 from contention in problem 48.2). Secondly, one of two things must be true about the first derivative when a function has a local extreme point; it either has a value of zero or it does not exist. This leads us to the definition of a ***critical number*** of a function.

> **Definition 49.1 – Critical Numbers**
>
> If f is a function, then we define ***the critical numbers of f*** as the numbers in the domain of f where the value of f' is either zero or does not exist.

Problem 49.1

The function g shown in Figure 49.1 has a vertical tangent line at -3. Veronica says that -3 is a critical number of g but Tito disagrees. Tito contends that -3 is not a critical number because g does not have a local extreme point at -3. Who's right?

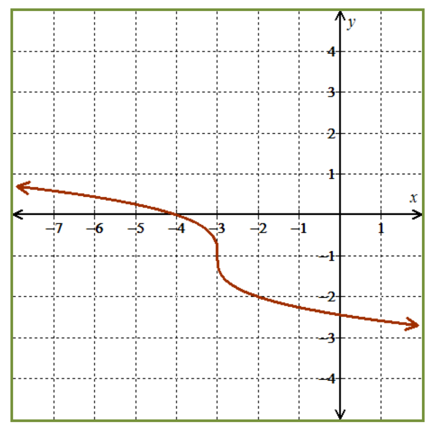

Figure 49.1: g

Problem 49.2

Answer each of the following questions (using complete sentences) in reference to the function f shown in Figure 49.2.

49.2.1 What are the critical numbers of f?

49.2.2 What are the local extreme points on f? Classify the points as local minimums or local maximums and remember that points on the plane are represented by ordered pairs.

49.2.3 What is the absolute maximum value of f over the interval $(-7, 7)$. Please note that the function value is the value of the y-coordinate at the point on the curve and as such is a number.

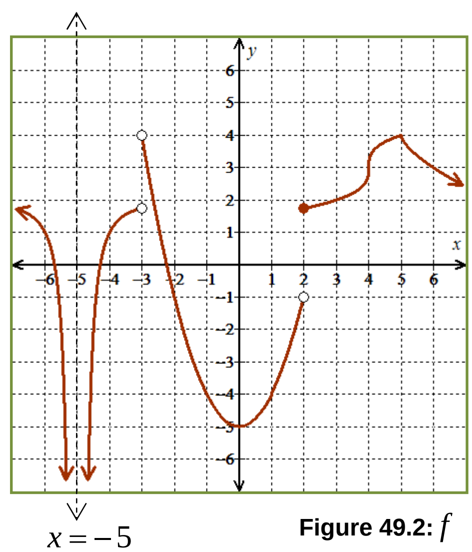

Figure 49.2: f

Problem 49.3
Decide whether each of the following statements is true or false.

49.3.1 A function always has a local extreme point at each of its critical numbers.

49.3.2 If the point $(t_1, h(t_1))$ is a local minimum point on h, then t_1 must be a critical number of h.

49.3.3 If $g'(2.7)=0$, then g must have a local extreme point at 2.7.

49.3.4 If $g'(2.7)=0$, then 2.7 must be a critical number of g.

49.3.5 If $g'(9)$ does not exist, then 9 must be a critical number of g.

Activity 50

When finding critical numbers based upon a function formula, there are three issues that need to be considered; the domain of the function, the zeros of the first derivative, and the numbers in the domain of the function where the first derivative is undefined. When writing a formal analysis of this process each of these questions must be explicitly addressed. The following outline shows the work you need to show when you are asked to write a formal determination of critical numbers based upon a function formula.

A process for formal determination of critical numbers

Please note that you should present your work in narrative form using complete sentences that establish the significance of all stated intervals and values. For example, for the function $g(t)=\sqrt{t-7}$ the first sentence you should write is "The domain of g is $[7,\infty)$."

1. Using interval notation, state the domain of the function.
2. Differentiate the function and completely simplify the resultant formula. Simplification includes:
   a. Manipulating all negative exponents into positive exponents.
   b. If any rational expression occurs in the formula, all terms must be written in rational form and a common denominator must be established for the terms. The final expression should be a single rational expression.
   c. The final expression must be completely factored.
3. State the values where the first derivative is equal to zero; show any non-trivial work necessary in making this determination. Make sure that you write a sentence addressing this issue even if the first derivative has no zeros.
4. State the values **in the domain of the function** where the first derivative does not exist; show any non-trivial work necessary in making this determination. Make sure that you write a sentence addressing this issue even if no such numbers exist.
5. State the critical numbers of the function. Make sure that you write this conclusion even if the function has no critical numbers.

Because the domain of a function is such an important issue when determining critical numbers, there are some things you should keep in mind when determining domains.

- Division by zero is never a good thing.
- Over the real numbers, you **cannot** take **even** roots of negative numbers.
- Over the real numbers, you **can** take **odd** roots of negative numbers.
- $\sqrt[n]{0}=0$ for any positive integer n.
- Over the real numbers, you can only take logarithms of positive numbers.
- The sine and cosine function are defined at all real numbers. You should check with your lecture instructor to see if you are expected to know the domains of the other four trigonometric functions and/or the inverse trigonometric functions.

Problem 50.1

Formally establish the critical numbers for each of the following functions following the procedure outlined on page 75.

50.1.1 $f(x)=x^2-9x+4$

50.1.2 $g(t)=7t^3+39t^2-24t+4$

50.1.3 $p(t)=(t+8)^{2/3}$

50.1.4 $z(x)=x\ln(x)$

50.1.5 $y(\theta)=e^{\cos(\theta)}$

50.1.6 $T(t)=\sqrt{t-4}\sqrt{16-t}$

Problem 50.2

The first derivative of the function $m(x)=\dfrac{\sqrt{x-5}}{x-7}$ is $m'(x)=\dfrac{3-x}{2\sqrt{x-5}(x-7)^2}$.

50.2.1 Roland says that 5 is a critical number of m but Yuna disagrees. Who is correct and why?

50.2.2 Roland says that 7 is a critical number of m but Yuna disagrees. Who is correct and why?

50.2.3 Roland says that 3 is a critical number of m but Yuna disagrees. Who is correct and why?

Activity 51

Once you have determined the critical numbers of a function, the next thing you might want to determine is the behavior of the function at each of its critical numbers. One way you could do that involves a sign table for the first derivative of the function

Problem 51.1

The first derivative of the function $f(x)=x^3-21x^2+135x-24$ is $f'(x)=3(x-5)(x-9)$. The critical numbers of *f* are trivially shown to be 5 and 9.

Copy Table 51.1 onto your paper and fill in the missing information. Then state the local minimum and maximum points on *f*. Specifically address both minimum and maximum points even if one and/or the other does not exist. Remember that points on the plane are represented by ordered pairs. **Make sure that you state points on *f* and not f'!**

Table 51.1: $f'(x)=3(x-5)(x-9)$

Interval	Sign of f'	Behavior of f
$(-\infty,5)$		
$(5,9)$		
$(9,\infty)$		

Problem 51.2

The first derivative of the function $g(t)=\dfrac{\sqrt{t-4}}{(t-1)^2}$ is $g'(t)=\dfrac{-3(t-5)}{2(t-1)^3\sqrt{t-4}}$.

51.2.1 State the critical numbers of g; you **do not** need to show a formal determination of the critical numbers. You **do** need to write a complete sentence.

51.2.2 Copy Table 51.2 onto your paper and fill in the missing information.

Table 51.2: $g'(t)=\dfrac{-3(t-5)}{2(t-1)^3\sqrt{t-4}}$

Interval	Sign of g'	Behavior of g
$(4,5)$		
$(5,\infty)$		

51.2.3 Why did we not include any part of the interval $(-\infty,4)$ in Table 51.2?

51.2.4 Formally we say that $g(t_0)$ is a local minimum value of g if there exists an open interval centered at t_0 over which $g(t_0)<g(t)$ for every value of t on that interval (other than t_0, of course). Since g is not defined to the left of 4, it is impossible for this definition to be satisfied at 4; hence g does not have a local minimum value at 4.

State the local minimum and maximum points on g. Specifically address both minimum and maximum points even if one and/or the other does not exist.

51.2.5 Write a formal definition for a local maximum point on g.

Problem 51.3

The first derivative of the function $k(x)=\dfrac{\sqrt[3]{(x-2)^2}}{x-1}$ is $k'(x)=\dfrac{4-x}{3(x-1)^2\sqrt[3]{x-2}}$.

51.3.1 State the critical numbers of k; you **do not** need to show a formal determination of the critical numbers. You **do** need to write a complete sentence.

51.3.2 Copy Table 51.3 onto your paper and fill in the missing information. Then state the local minimum and maximum points on k. Specifically address both minimum and maximum points even if one and/or the other does not exist.

Table 51.3: $k'(x)=\dfrac{4-x}{3(x-1)^2\sqrt[3]{x-2}}$

Interval	Sign of k'	Behavior of k
$(-\infty,1)$		
$(1,2)$		
$(2,4)$		
$(4,\infty)$		

51.3.3 The number 1 was included as an endpoint in Table 51.3 even though 1 is not a critical number of k. Why did we have to include the intervals $(-\infty,1)$ and $(1,2)$ in the table as opposed to just using the single interval $(-\infty,2)$?

Problem 51.4

Perform each of the following for the functions in problems 51.4.1-51.4.3.

- Formally establish the critical numbers of the function.
- Create a table similar to tables 51.1-51.3. Number the tables, in order, 51.4-51.6. Don't forget to include table headings and column headings.
- State the local minimum points and local maximum points on the function. Make sure that you explicitly address both types of points even if there are none of one type and/or the other.

51.4.1 $k(x)=x^3+9x^2-10$

51.4.2 $g(t)=(t+2)^3(t-6)$

51.4.3 $F(x)=\dfrac{x^2}{\ln(x)}$

Problem 51.5
Consider a function f whose first derivative is $f'(x)=(x-9)^4$.

51.5.1 Is 9 definitely a critical number of f? Explain why or why not.

51.5.2 Other than at 9, what is always the sign of $f'(x)$? What does this sign tell you about the function f?

51.5.3 What type of point does f have at 9? (Hint, draw a freehand sketch of the curve.)

51.5.4 How could the second derivative of f be used to confirm your conclusion in problem 51.5.3? Go ahead and do it.

Activity 52

When searching for inflection points on a function, you can narrow your search by identifying numbers where the function is continuous (from both directions) and the second derivative is either zero or undefined. (By definition an inflection point cannot occur at a number where the function is not continuous from both directions.) You can then build a sign table for the second derivative that implies the concavity of the given function.

When performing this analysis, you need to simplify the second derivative formula in the same way you simplify the first derivative formula when looking for critical numbers and local extreme points.

Problem 52.1
Identify the inflection points for the function shown in Figure 49.2 (page 74).

Problem 52.2

The first two derivatives of the function $y(x)=\dfrac{(x+2)^2}{(x+3)^3}$ are $y'(x)=\dfrac{-x(x+2)}{(x+3)^4}$ and

$$y''(x)=\frac{2\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)}{(x+3)^5}.$$

52.2.1 Yolanda was given this information and asked to find the inflection points on y. The first thing Yolanda wrote was "The critical numbers of y are $\sqrt{3}$ and $-\sqrt{3}$." Explain to Yolanda why this is not true

52.2.2 What are the critical numbers of y and in what way are they important when asked to identify the inflection points on y?

52.2.3 Copy Table 52.1 (page 80) onto your paper and fill in the missing information.

52.2.4 State the inflection points on y; you may round the dependent coordinate of each point to the nearest hundredth.

52.2.5 The function y has a vertical asymptote at -3. Given that fact, it was impossible that y would have an inflection point at -3. Why, then, did we never-the-less break the interval $\left(-\infty,-\sqrt{3}\right)$ at -3 when creating our concavity table?

Table 52.1: $y''(x)=\dfrac{2\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)}{(x+3)^5}$

Interval	Sign of y''	Behavior of y
$(-\infty,-3)$		
$\left(-3,-\sqrt{3}\right)$		
$\left(-\sqrt{3},\sqrt{3}\right)$		
$\left(\sqrt{3},\infty\right)$		

Problem 52.3

Perform each of the following for the functions in problems 52.3.1-52.3.3.

- State the domain of the function.
- Find, and completely simplify, the formula for the second derivative of the function. It is ***not*** necessary to simplify the formula for the first derivative of the function.
- State the values in the domain of the function where the second derivative is either zero or does not exist.
- Create a table similar to Table 52.1. Number the tables, in order, 52.2-52.4. Don't forget to include table headings and column headings.
- State the inflection points on the function. Make sure that you explicitly address this question even if there are no inflection points.

52.3.1 $f(x)=x^4-12x^3+54x^2-10x+6$ **52.3.2** $g(x)=(x-2)^2e^x$

52.3.3 $G(x)=\sqrt{x^3}+6\sqrt{x}$

Problem 52.4

The second derivative of the function $w(t)=t^{1.5}-9t^{0.5}$ is $w''(t)=\dfrac{3(t+3)}{4t^{1.5}}$ yet w has no inflection points. Why is that?

Activity 53

We are frequently interested in a function's "end behavior;" that is, what is the behavior of the function as the input variable increases without bound or decreases without bound.

Many times a function will approach a horizontal asymptote as its end behavior. Assuming that the horizontal asymptote $y = L$ represents the end behavior of the function f both as x increases without bound and as x decreases without bound, we write $\lim_{x \to \infty} f(x) = L$ and $\lim_{x \to -\infty} f(x) = L$.

While working the *Limits and Continuity* lab you investigated strategies for formally establishing limit values as $x \to \infty$ or $x \to -\infty$. In this activity you are going to investigate a more informal strategy for determining these type limits.

Consider $\lim_{x \to \infty} \frac{4x-2}{3+20x}$. When the value of x is really large, we say that the term $4x$ dominates the numerator of the expression $\frac{4x-2}{3+20x}$ and the term $20x$ dominates the denominator. We actually call those terms ***the dominant terms*** of the numerator and denominator. The dominant terms are significant because when the value of x is really large, the other terms in the expression contribute almost nothing to the value of the expression. That is, for really large values of x:

$$\frac{4x-2}{3+20x} \approx \frac{4x}{20x} = \frac{1}{5}$$

For example, even if x has the paltry value of $1{,}000$,

$$\frac{4x-2}{3+20x} = \frac{3998}{20003} \approx \frac{4000}{20000} = \frac{1}{5}$$

This tells us that $\lim_{x \to \infty} \frac{4x-2}{3+20x} = \frac{1}{5}$ and that $y = \frac{1}{5}$ is a horizontal asymptote for the graph of $y = \frac{4x-2}{3+20x}$.

Problem 53.1

The formulas used to graph figures 7.1-7.5 (pages B1 and B2) are given below. Focusing first on the dominant terms of the expressions, match the formulas with the functions (f_1 through f_5).

53.1.1 $y = \frac{3x+6}{x-2}$

53.1.2 $y = \frac{16+4x}{6+x}$

53.1.3 $y = \frac{6x^2-6x-36}{36-3x-3x^2}$

53.1.4 $y = \frac{-2x+8}{x^2-100}$

53.1.5 $y = \frac{15}{x-5}$

Problem 53.2

Use the concept of dominant terms to informally determine the value of each of the following limits.

53.2.1 $\lim\limits_{x \to -\infty} \dfrac{4 + x - 7x^3}{14x^3 + x^2 + 2}$

53.2.2 $\lim\limits_{t \to -\infty} \dfrac{4t^2 + 1}{4t^3 - 1}$

53.2.3 $\lim\limits_{\gamma \to \infty} \dfrac{8}{2\gamma^3}$

53.2.4 $\lim\limits_{x \to \infty} \dfrac{(3x+1)(6x-2)}{(4+x)(1-2x)}$

53.2.5 $\lim\limits_{t \to \infty} \dfrac{4e^t - 8e^{-t}}{e^t + e^{-t}}$

53.2.6 $\lim\limits_{t \to -\infty} \dfrac{4e^t - 8e^{-t}}{e^t + e^{-t}}$

Activity 54

Let's put it all together and produce some graphs.

Problem 54.1

Consider the function $f(x) = \dfrac{8x^2 - 8}{(2x-4)^2}$.

54.1.1 Evaluate each of the following limits: $\lim\limits_{x \to \infty} f(x)$, $\lim\limits_{x \to -\infty} f(x)$, $\lim\limits_{x \to 2^-} f(x)$, and $\lim\limits_{x \to 2^+} f(x)$.

54.1.2 What are the horizontal and vertical asymptotes for a graph of *f*?

54.1.3 What are the horizontal and vertical intercepts for a graph of *f*?

54.1.4 Use the formulas $f'(x) = \dfrac{4(1-2x)}{(x-2)^3}$ and $f''(x) = \dfrac{4(4x+1)}{(x-2)^4}$ to help you accomplish each of the following.

- State the critical numbers of *f*.
- Create well-documented increasing/decreasing and concavity tables for *f*.
- State the local minimum, local maximum, and inflection points on *f*. Make sure that you explicitly address all three types of points whether they exist or not.

54.1.5 Graph $y = f(x)$ onto Figure 54.1. Make sure that you choose a scale that allows you to clearly illustrate each of the features found in problems 54.1.1-54.1.4. Make sure that all axes and asymptotes are well labeled and also write the coordinates of each local extreme point and inflection point next to the point on the graph.

54.1.6 Check your graph using a graphing calculator.

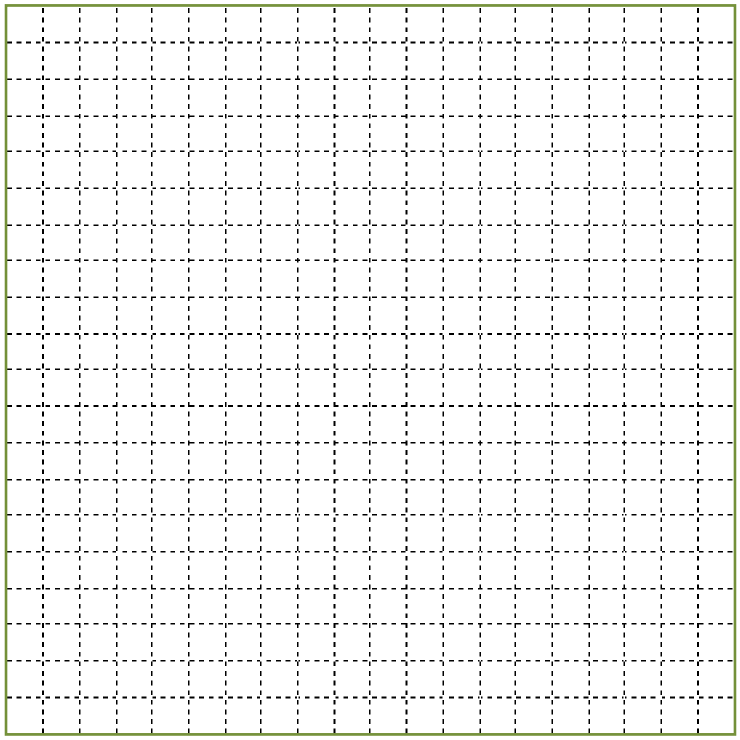

Figure 54.1: f

Problem 54.2

Consider the function $g(t)=\dfrac{1}{\left(e^t+4\right)^2}$.

54.2.1 Evaluate each of the following limits: $\lim\limits_{t\to\infty} g(t)$ and $\lim\limits_{t\to-\infty} g(t)$.

54.2.2 What are the horizontal and vertical asymptotes for a graph of g?

54.2.3 What are the horizontal and vertical intercepts for a graph of g ?

54.2.4 Use the formulas $g'(t)=\dfrac{-2e^t}{\left(e^t+4\right)^3}$ and $g''(t)=\dfrac{4\left(e^t-2\right)e^t}{\left(e^t+4\right)^2}$ to help you accomplish each of the following.

- State the critical numbers of g.
- Create well-documented increasing/decreasing and concavity tables for g.
- State the local minimum, local maximum, and inflection points on g. Make sure that you explicitly address all three types of points whether they exist or not.

54.2.5 Graph $y = g(t)$ onto Figure 54.2. Make sure that you choose a scale that allows you to clearly illustrate each of the features found in problems 54.2.1-54.2.4. Make sure that all axes and asymptotes are well labeled and also write the coordinates of each local extreme point and inflection point next to the point on the graph.

54.2.6 Check your graph using a graphing calculator.

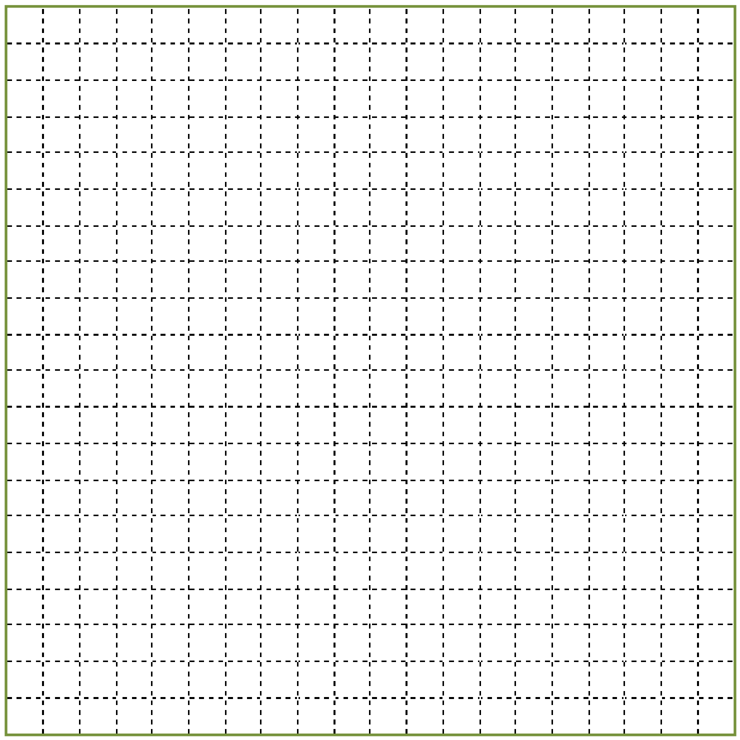

Figure 54.2: g