The Chain Rule

Activity 38

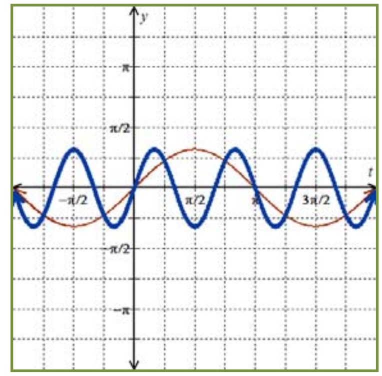

Figure 38.1: f and k

The functions $f(t)=\sin(t)$ and $k(t)=\sin(3t)$ are shown in Figure 38.1. Since $f'(t)=\cos(t)$, it is reasonable to speculate that $k'(t)=\cos(3t)$. But this would imply that $k'(0)=f'(0)=1$, and a quick glance of the two functions at 0 should convince you that this is not true; clearly $k'(0)>f'(0)$.

The function k moves through three periods for every one period generated by the function f. Since the amplitudes of the two functions are the same, the only way k can generate periods at a rate of 3:1 (compared to f) is if its rate of change is three times that of f. In fact, $k'(t)=3\cos(3t)$; please note that 3 is the first derivative of $3t$. This means that the formula for $k'(t)$ is the product of the rates of change of the outside function ($\frac{d}{du}(\sin(u))=\cos(u)$) and the inside function ($\frac{d}{dt}(3t)=3$).

k is an example of a composite function (as illustrated in Figure 38.2). If we define g by the rule $g(t)=3t$, then $k(t)=f(g(t))$.

$$t \xrightarrow{\quad g \quad} 3t \xrightarrow{\quad f \quad} \sin(3t)$$

Figure 38.2: $g(t)=3t$, $f(u)=\sin(u)$, and $k(t)=f(g(t))$

Taking the output from g and processing it through a second function, f, is the action that characterizes k as a composite function.

Note that $k'(0)=g'(0)\,f'(g(0))$. This last equation is an example of what we call ***the chain rule for differentiation***. Loosely, the chain rule tells us that when finding the rate of change for a composite function (at 0), we need to multiply the rate of change of the outside function, $f'(g(0))$, with the rate of change of the inside function, $g'(0)$. This is symbolized for general values of x in Equation 38.1 where u represents a function of x (e.g. $u=g(x)$).

Equation 38.1 $$\frac{d}{dx}(f(u))=f'(u)\cdot\frac{d}{dx}(u)$$

This rule is used to find derivative formulas in examples 38.1 and 38.2.

The factor of $\frac{d}{dx}\left(x^2\right)$ is called a ***chain rule factor***.

Example 38.1

Problem	Solution
Find $\frac{dy}{dx}$ if $y=\sin\left(x^2\right)$.	$\frac{dy}{dx}=\cos\left(x^2\right)\cdot\frac{d}{dx}\left(x^2\right)$ $=\cos\left(x^2\right)\cdot 2x$ $=2x\cos\left(x^2\right)$

The factor of $\frac{d}{dt}\left(\sec(t)\right)$ is called a ***chain rule factor***.

Example 38.2

Problem	Solution
Find $f'(t)$ if $f(t)=\sec^9(t)$.	$f(t)=\left[\sec(t)\right]^9$ $f'(t)=9\left[\sec(t)\right]^8\cdot\frac{d}{dt}\left(\sec(t)\right)$ $=9\sec^8(t)\cdot\sec(t)\tan(t)$ $=9\sec^9(t)\tan(t)$

Example 38.3

Problem	Solution
Find $\frac{dy}{dx}$ if $y=4^x$.	$\frac{dy}{dx}=\ln(4)\cdot 4^x$

Please note that **the chain rule was not applied** here because **the function being differentiated was not a composite function.**

While you ultimately want to perform the chain rule step in your head, your instructor may want you to illustrate the step while you are first practicing the rule. For this reason, the step will be explicitly shown in every example given in this lab.

Problem 38.1

Find the first derivative formula for each function. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative (e.g. $h'(t)$).

38.1.1	$h(t)=\cos\left(\sqrt{t}\right)$	**38.1.2**	$P=\sin\left(\theta^4\right)$	**38.1.3**	$w(\alpha)=\cot\left(\sqrt[3]{\alpha}\right)$
38.1.4	$z=7\left[\ln(t)\right]^3$	**38.1.5**	$z(\theta)=\sin^4(\theta)$	**38.1.6**	$P(\beta)=\tan^{-1}(\beta)$
38.1.7	$y=\left[\sin^{-1}(t)\right]^{17}$	**38.1.8**	$T=2^{\ln(x)}$	**38.1.9**	$y(x)=\sec^{-1}\left(e^x\right)$

Problem 38.2

A function, *f*, is shown in Figure 38.3. Answer each of the following questions in reference to this function.

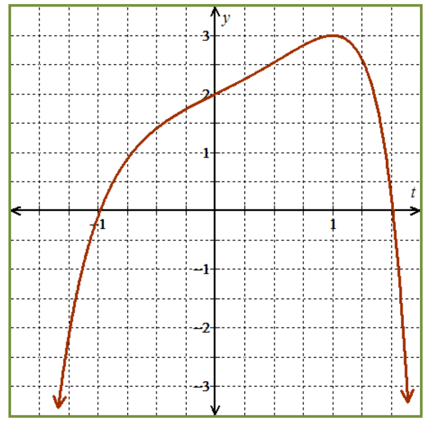

Figure 38.3: f

38.2.1 Use the graph to rank the following in decreasing order: $f(1)$, $f'(1)$, $f''(1)$, $f'(0)$, and $f'(-1)$.

38.2.2 The formula for *f* is $f(t) = te^t - e^{t^2} + 3$. Find the formulas for f' and f'' and use the formulas to verify your answer to problem 38.2.1.

38.2.3 Find the equation of the tangent line to *f* at 0.

38.2.4 Find the equation of the tangent line to f' at 0.

38.2.5 There is an antiderivative of *f* that passes through the point $(0,7)$. Find the equation of the tangent line to this antiderivative at 0.

Activity 39

When finding derivatives of complex formulas you need to apply the rules for differentiation in the reverse of order of operations. For example, when finding $\frac{d}{dx}\left(\sin\left(xe^x\right)\right)$ the first rule you need to apply is the derivative formula for $\sin(u)$ but when finding $\frac{d}{dx}\left(x\sin\left(e^x\right)\right)$ the first rule that needs to be applied is the product rule.

Problem 39.1

Find the first derivative formula for each function. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative (e.g. $f'(x)$).

39.1.1 $f(x) = \sin\left(xe^x\right)$

39.1.2 $g(x) = x\sin\left(e^x\right)$

39.1.3 $y = \dfrac{\tan(\ln(x))}{x}$

39.1.4 $z = 5t + \dfrac{\cos^2\left(t^2\right)}{3}$

39.1.5 $f(y) = \sin\left(\dfrac{\ln(y)}{y}\right)$

39.1.6 $G = x\sin^{-1}\left(x\ln(x)\right)$

Activity 40

As always, you want to simplify an expression before jumping in to take its derivative. Never-the-less, it can build confidence to see that the rules work even if you don't simplify first.

Problem 40.1

Consider the functions $f(x)=\sqrt{x^2}$ and $g(x)=\left(\sqrt{x}\right)^2$.

40.1.1 Assuming that x is not negative, how does each of these formulas simplify? Use the simplified formula to find the formulas for $f'(x)$ and $g'(x)$.

40.1.2 Use the chain rule (without first simplifying) to find the formulas for $f'(x)$ and $g'(x)$; simplify each result (assuming that x is positive).

40.1.3 Are f and g the same function? Explain why or why not.

Problem 40.2

So long as x falls on the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $\tan^{-1}\left(\tan(x)\right)=x$. Use the chain rule to find $\frac{d}{dx}\left[\tan^{-1}\left(\tan(x)\right)\right]$ and show that it simplifies as it should.

Problem 40.3

How does the formula $g(t)=\ln\left(e^{5t}\right)$ simplify and what does this tell you about the formula for $g'(t)$? After answering those questions use the chain rule to find the formula for $g'(t)$ and show that it simplifies as it should.

Problem 40.4

Consider the function $g(t)=\ln\left(\frac{5}{t^3\sec(t)}\right)$.

40.4.1 Find the formula for $g'(t)$ without first simplifying the formula for $g(t)$.

40.4.2 Use the quotient, product, and power rules ***of logarithms*** to expand the formula for $g(t)$ into three logarithmic terms. Then find $g'(t)$ by taking the derivative of the expanded version of g.

40.4.3 Show that the two resultant formulas are in fact the same. Also, reflect upon which process of differentiation was less work and easier to "clean up."

Activity 41

So far we have worked with the chain rule as expressed using function notation. In some applications it is easier to think of the chain rule using Leibniz notation. Consider the following example

During the 1990s, the amount of electricity used per day in Etown increased as a function of population at the rate of 18 kW/person. On July 1, 1997, the population of Etown was 100,000 and the population was decreasing at a rate of 6 people/day. In Equation 41.1 (page 61) we use these values to determine the rate at which electrical usage was changing (*with respect to **time***) in Etown on 7/1/1997. Please note that in this extremely simplified example we are ignoring all factors that contribute to citywide electrical usage other than population (such as temperature).

Equation 41.1 $\left(18\dfrac{\text{kW}}{\text{person}}\right)\left(-6\dfrac{\text{people}}{\text{day}}\right)=-128\dfrac{\text{kW}}{\text{day}}$

Let's define $g(t)$ as the population of Etown t years after January 1, 1990 and $f(u)$ as the daily amount of electricity used in Etown when the population was u. From the given information, $g(7.5)=100{,}000$, $g'(7.5)=-6$, and $f'(u)=18$ for all values of u. If we let $y=f(u)$ where $u=g(t)$, then we have (from Equation 41.1):

Equation 41.2 $\left(\left.\dfrac{dy}{du}\right|_{u=100{,}000}\right)\left(\left.\dfrac{du}{dt}\right|_{t=7.5}\right)=\left.\dfrac{dy}{dt}\right|_{t=7.5}$

You should note that Equation 41.2 is an application of the chain rule expressed in Leibniz notation; specifically, the expression on the left side of the equal sign represents $f'\left(g(7.5)\right)g'(7.5)$.

In general, we can express the chain rule as shown in Equation 41.3.

Equation 41.3 $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$

Problem 41.1

Suppose that Carla is jogging in her sweet new running shoes. Suppose further that $r=f(t)$ is Carla's pace (mi/hr) t hours after 1 pm and $y=h(r)$ is Carla's heart rate (bpm) when she jogs at a rate of r mi/hr. In this context we can assume that all of Carla's motion was in one direction, so the words speed and velocity are completely interchangeable.

41.1.1 What is the meaning of $f(.75)=7$?

41.1.2 What is the meaning of $h(7)=125$?

41.1.3 What is the meaning of $(h\circ f)(.75)=125$?

41.1.4 What is the meaning of $\left.\dfrac{dr}{dt}\right|_{t=.75}=-0.00003$?

41.1.5 What is the meaning of $\left.\dfrac{dy}{dr}\right|_{r=7}=8$?

41.1.6 Assuming that all of the previous values are for real, what is the value of $\left.\dfrac{dy}{dt}\right|_{t=.75}$ and what does this value tell you about Carla?

Problem 41.2

Portions of SW 35th Avenue are extremely hilly. Suppose that you are riding your bike along SW 35th Ave from Vermont Street to Capitol Highway. Let $u = d(t)$ be the distance you have travelled (ft) where t is the number of seconds that have passed since you began your journey. Suppose that $y = e(u)$ is the elevation (m) of SW 35th Ave where u is the distance (ft) from Vermont St headed towards Capital Highway.

41.2.1 What, including units, would be the meanings of $d(25) = 300$, $e(300) = 140$, and $(e \circ d)(25) = 140$?

41.2.2 What, including units, would be the meanings of $\left.\frac{du}{dt}\right|_{t=25} = 14$ and $\left.\frac{dy}{du}\right|_{u=300} = -0.1$

41.2.3 Suppose that the values stated in problem 41.1.2 are accurate. What, including unit, is the value of $\left.\frac{dy}{dt}\right|_{t=25}$? What does this value tell you in the context of this problem?

Problem 41.3

According to Hooke's Law, the force (lb), F, required to hold a spring in place when its displacement from the natural length of the spring is x (ft), is given by the formula $F = k\,x$ where k is called the spring constant. The value of k varies from spring to spring.

Suppose that it requires 120 lb of force to hold a given spring 1.5 ft beyond its natural length.

41.2.1 Find the spring constant for this spring. Include units when substituting the values for F and x into Hooke's Law so that you know the unit on k.

41.2.2 What, including unit, is the constant value of $\frac{dF}{dx}$?

41.2.3 Suppose that the spring is stretched at a constant rate of .032 ft/s. If we define t to be the amount of time (s) that passes since the stretching begins, what, including unit, is the constant value of $\frac{dx}{dt}$?

41.2.4 Use the chain rule to find the constant value (including unit) of $\frac{dF}{dt}$. What is the contextual significance of this value?