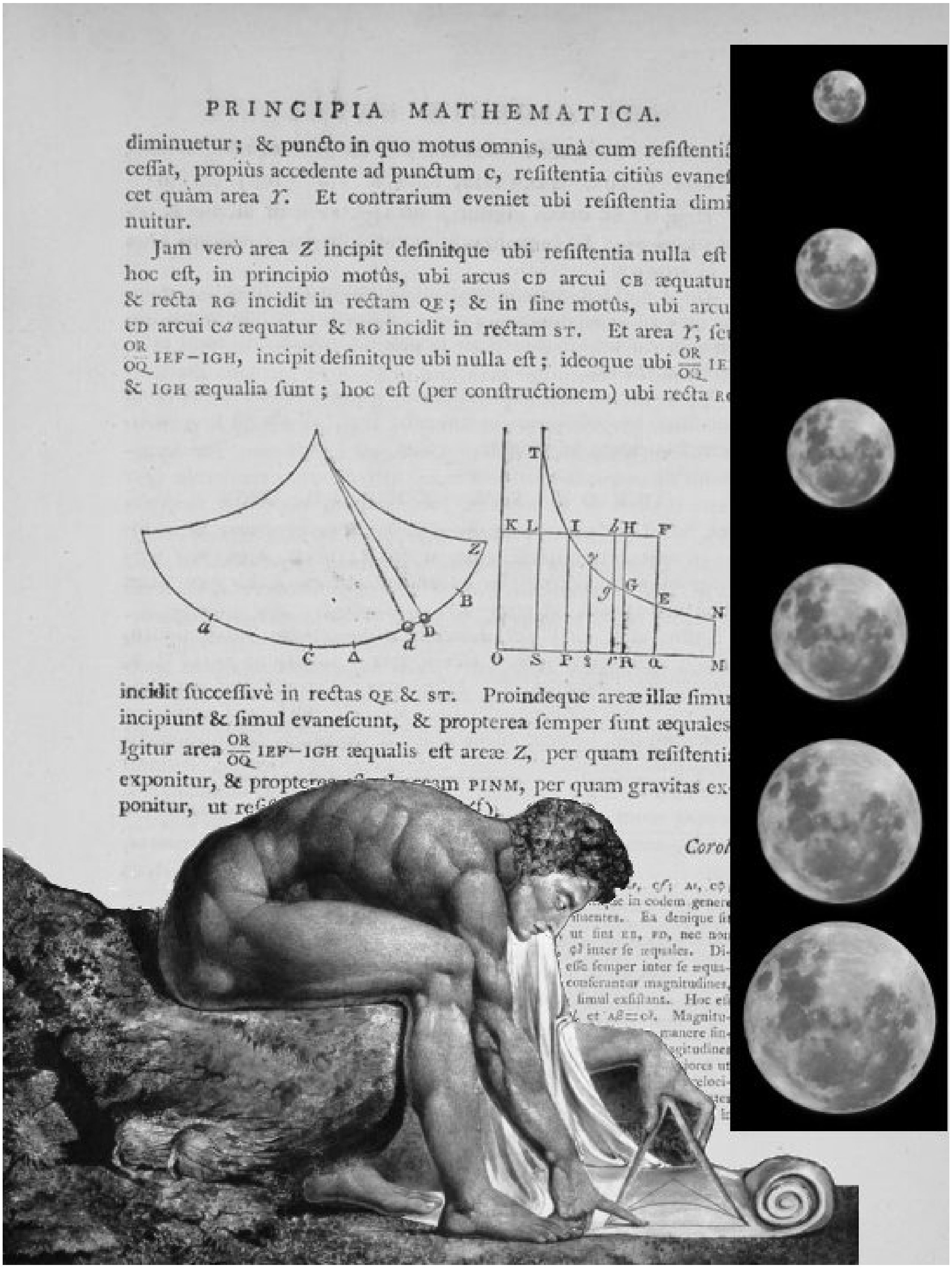

Portland Community College

MTH 251 Laboratory Manual

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Implicit Differentiation (Activities 42-43)

Related Rates (Activities 44-47)

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To the Student

MTH 251 is taught at Portland Community College using a lecture/lab format. The laboratory time is set aside for students to investigate the topics and practice the skills that are covered during their lecture periods.

The lab activities have been written under the presumption that students will be working in groups and will be actively discussing the examples and problems included in each activity. Many of the exercises and problems lend themselves quite naturally to discussion. Some of the more algebraic problems are not so much discussion problems as they are "practice and help" problems.

You do not need to fully understand an example before starting on the associated problems. The intent is that your understanding of the material will grow while you work on the problems.

When working through the lab activities the students in a given group should be working on the same activity at the same time. Sometimes this means an individual student will have to go a little more slowly than he or she may like and sometimes it means an individual student will need to move on to the next activity before he or she fully grasps the current activity.

Many instructors will want you to focus some of your energy on the way you write your mathematics. It is important that you do not rush through the activities. Write your solutions as if they are going to be graded; that way you will know during lab time if you understand the proper way to write and organize your work.

If your lab section meets more than once a week, ***you should not work on lab activities between lab sections that occur during the same week***. It is OK to work on lab activities outside of class once the entire classroom time allotted for that lab has passed.

There are not written solutions for the lab activity problems. Between your group mates, your instructor, and (if you have one) your lab assistant, you should know whether or not you have the correct answers and proper writing strategies for these problems.

Each lab has a section of supplementary exercises; these exercises are fully keyed. The supplementary exercises are not simply copies of the problems in the lab activities. While some questions will look familiar, many others will challenge you to apply the material covered in the lab to a new type of problem. These questions are meant to supplement your textbook homework, not replace your textbook homework.

The MTH 251 Laboratory Manual was written by Steve Simonds.

The cover art for the manual was designed by Phil Thurber.

The cover includes a page from Isaac Newton's ***Philosophiæ Naturalis Principia Mathematica***.

The cover includes William Blake's "**Newton**."

To the Instructor

This manual is significantly different from earlier versions of the lab manual. The topics have been arranged in a developmental order. Because of this, students who work each activity in the order they appear may not get to all of the topics covered in a particular week.

It is strongly recommended that the instructor pick and choose what they consider to be the most vital activities for a given week and that the instructor have the students work those activities first; for some activities you might also want to have the students only work selected problems in the activity. Students who complete the high priority activities and problems can then go back and work the activities that they initially skipped. There are also fully keyed problems in the supplementary exercises that the students could work on both during lab time and outside of class.

A suggested schedule for the labs is shown in Table P1. Again, the instructor should choose what they feel to be the most relevant activities and problems for a given week and have the students work those activities and problems first.

Table P1: Possible 10 week schedule for the labs. (Students should consult their syllabus for their schedule.)

Week	Labs (Lab Activities)	Supplementary Exercises
1	Rates of Change/Limits and Continuity (1-4)	E1 (all)
2	Limits and Continuity (5-16)	E2 (all)
3	Introduction to the First Derivative (17-20)	E3 (all)
4	Functions, Derivatives, and Antiderivatives (21-24)	E4.1-E4.5
5	Functions, Derivatives, and Antiderivatives (25-26)	E4.6-E4.10
6	Derivative Formulas (27-37)	E5 (all)
7	The Chain Rule (38-41)	E6 (all)
8	Implicit Differentiation/Related Rates (42-47)	E7 (all), E8 (all)
9 and 10	Critical Numbers and Graphing from Formulas (48-54)	E9 (all)

Rates of Change

Activity 1

Motion is frequently modeled using calculus. A building block for this application is the concept of ***average velocity***. Average velocity is defined to be net displacement divided by elapsed time. More precisely, if p is a position function for something moving along a numbered line, then we define the average velocity over the time interval $[t_0, t_1]$ to be:

Expression 1.1: $\dfrac{p(t_1) - p(t_0)}{t_1 - t_o}$

Problem 1.1

According to simplified Newtonian physics, if an object is dropped from a height of 200 m and allowed to freefall to the ground, then the height of the object (measure in m) is given by the position function $p(t) = 200 - 4.9t^2$ where t is the amount of time that has passed since the object was dropped (measured in s).

1.1.1 What, *including unit*, are the values of $p(t)$ three seconds and five seconds into the object's fall? Use these values when working problem 1.1.2.

1.1.2 Calculate $\dfrac{p(5\text{s}) - p(3\text{ s})}{5\text{ s} - 3\text{ s}}$; *include units **while making** the calculation*. What does the result tell you in the context of this problem?

1.1.3 Use Expression 1.1 to find a formula for the average velocity of this object over the general time interval $[t_0, t_1]$. The first couple of lines of this process are shown below. ***Copy these lines onto your paper*** and continue the simplification process.

$$\frac{p(t_1) - p(t_0)}{t_1 - t_0} = \frac{\left[200 - 4.9t_1^{\,2}\right] - \left[200 - 4.9t_0^{\,2}\right]}{t_1 - t_0}$$

$$= \frac{200 - 4.9t_1^{\,2} - 200 + 4.9t_0^{\,2}}{t_1 - t_0}$$

$$= \frac{-4.9t_1^{\,2} + 4.9t_0^{\,2}}{t_1 - t_0}$$

Hint

In the next step you should factor -4.9 from the numerator; the remaining factor will factor further.

1.1.4 Check the formula you derived in problem 1.1.3 using $t_0 = 3$ and $t_1 = 5$; that is, compare the value generated by the formula to that you found in problem 1.1.2.

1.1.5 Using the formula found in problem 1.1.3, replace t_0 with 3 but leave t_1 as a variable; simplify the result. Then copy Table 1.1 onto your paper and fill in the missing entries.

Table 1.1: $y = \dfrac{p(t_1) - p(3)}{t_1 - 3}$

t_1 (s)	y (m/s)
2.9	
2.99	
2.999	
3.001	
3.01	
3.1	

1.1.6 As the value of t_1 gets closer to 3, the values in the y column of Table 1.1 appear to be converging on a single number; what is this number and what do you think it tells you in the context of this problem?

Activity 2

One of the building blocks in differential calculus is ***the secant line to a curve***. It is very easy for a line to be considered a secant line to a curve; the only requirement that must be fulfilled is that the line intersects the curve in at least two points.

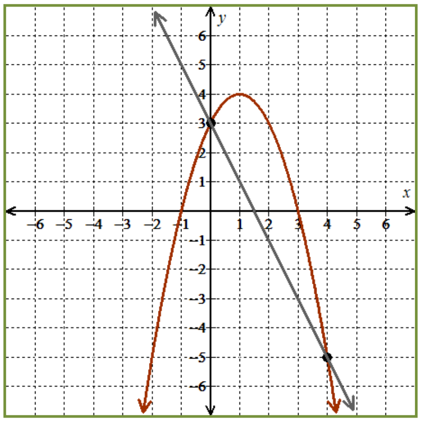


Figure 2.1: f

In Figure 2.1, a secant line to the curve $y = f(x)$ has been drawn through the points $(0,3)$ and $(4,-5)$. You should verify that the slope of this line is -2.

The formula for *f* is $f(x) = 3 + 2x - x^2$. We can use this formula to come up with a generalized formula for the slope of secant lines to this curve. Specifically, the slope of the line connecting the point $(x_0, f(x_0))$ to the point $(x_1, f(x_1))$ is derived in Example 2.1.

Example 2.1

$$
\begin{aligned}
m_{\sec} &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\
&= \frac{(3 + 2x_1 - x_1^2) - (3 + 2x_0 - x_0^2)}{x_1 - x_0} \\
&= \frac{3 + 2x_1 - x_1^2 - 3 - 2x_0 + x_0^2}{x_1 - x_0} \\
&= \frac{(2x_1 - 2x_0) - (x_1^2 - x_0^2)}{x_1 - x_0} \\
&= \frac{2(x_1 - x_0) - (x_1 + x_0)(x_1 - x_0)}{x_1 - x_0} \\
&= \frac{[2 - (x_1 + x_0)](x_1 - x_0)}{x_1 - x_0} \\
&= 2 - x_1 - x_0 \quad \text{for } x_1 \neq x_0
\end{aligned}
$$

This factoring technique is called ***factoring by grouping***.

We can check our formula using the line in Figure 2.1. If we let $x_0 = 0$ and $x_1 = 4$ then our simplified slope formula gives us:

$$
\begin{aligned}
2 - x_1 - x_0 &= 2 - 4 - 0 \\
&= -2 \quad \checkmark
\end{aligned}
$$

Problem 2.1

Let $g(x)=x^2-5$.

2.1.1 Following Example 2.1, find a formula for the slope of the secant line connecting the points $(x_0, g(x_0))$ and $(x_1, g(x_1))$. Please note that factoring by grouping will ***not*** be necessary when simplifying the expression,

2.1.2 Check your slope formula using the two points indicated in Figure 2.2. That is, use the graph to find the slope between the two points and then use your formula to find the slope; make sure that the two values agree!

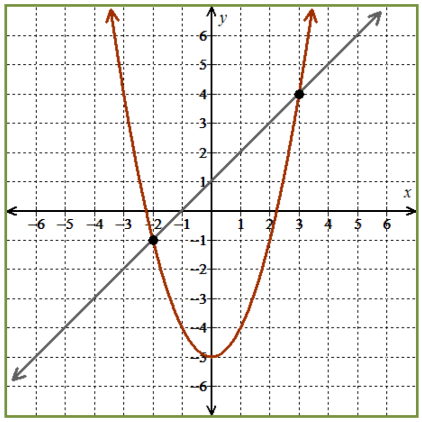


Figure 2.2: g

Activity 3

While it's easy to see that the formula $\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$ gives the slope of the line connecting two points on the function f, the resultant expression can at times be awkward to work with. We actually already saw that when we had to use slight-of-hand factoring in Example 2.1.

The algebra associated with secant lines (and average velocities) can sometimes be simplified if we designate the variable h to be the run between the two points (or the length of the time interval). With this designation we have $x_1-x_0=h$ which gives us $x_1=x_0+h$. Making these substitutions we get Equation 3.1. The expression on the right side of Equation 3.1 is called ***the difference quotient for f***.

Equation 3.1
$$\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{f(x_0+h)-f(x_0)}{h}$$

Let's revisit the function $f(x)=3+2x-x^2$ from Example 2.1. The difference quotient for this function is derived in Example 3.1.

Example 3.1
$$\begin{aligned}\frac{f(x_0+h)-f(x_0)}{h}&=\frac{\left[3+2(x_0+h)-(x_0+h)^2\right]-\left[3+2x_0-x_0{}^2\right]}{h}\\&=\frac{3+2x_0+2h-x_0{}^2-2x_0h-h^2-3-2x_0+x_0{}^2}{h}\\&=\frac{2h-2x_0h-h^2}{h}\\&=\frac{h(2-2x_0-h)}{h}\\&=2-2x_0-h\ \text{ for }\ h\neq 0\end{aligned}$$

Please notice that all of the terms without a factor of h subtracted to zero. Please notice, too, that we avoided all of the tricky factoring that appeared in Example 2.1!

For simplicity's sake, we generally drop the variable subscript when applying the difference quotient. So for future reference we will define the difference quotient as follows:

Definition 3.1

The ***difference quotient*** for the function $y = f(x)$ is the expression $\dfrac{f(x+h) - f(x)}{h}$.

Problem 3.1

Completely simplify the difference quotient for each of the following functions. Please note that the template for the difference quotient needs to be adapted to the function name and independent variable in each given equation. For example, the difference quotient for the function in problem 3.1.1 is $\dfrac{v(t+h) - v(t)}{h}$.

Please make sure that you lay out your work in a manner consistent with the way the work is shown in example 3.1 (excluding the subscripts, of course).

3.1.1 $v(t) = 2.5t^2 - 7.5t$ **3.1.2** $g(x) = 3 - 7x$ **3.1.3** $w(x) = \dfrac{3}{x+2}$

Problem 3.2

Suppose that an object is tossed into the air in such a way that the elevation of the object (measured in ft) is given by the function $s(t) = 40 + 40t - 16t^2$ where t is the amount of time that has passed since the object was tossed (measured in s).

3.2.1 Simplify the difference quotient for s.

3.2.2 Ignoring the unit, use the difference quotient to determine the average velocity over the interval $[1.6, 2.8]$. (**Hint:** Use $t = 1.6$ and $h = 1.2$. Make sure that you understand why!)

3.2.3 What, *including unit*, are the values of $s(1.6)$ and $s(2.8)$? Use these values when working problem 3.2.4.

3.2.4 Use the expression $\dfrac{s(2.8) - s(1.6)}{2.8 - 1.6}$ to verify the value you found in problem 3.2.2. *Include the unit **while making** this calculation*.

3.2.5 Ignoring the unit, use the difference quotient to determine the average velocity over the interval $[0.4, 2.4]$.

Problem 3.3

Moose and squirrel were having casual conversation when suddenly, without any apparent provocation, Boris Badenov launched anti-moose missile in their direction. Fortunately, squirrel had ability to fly as well as great knowledge of missile technology, and he was able to disarm missile well before it hit ground.

The elevation (ft) of the tip of the missile t seconds after it was launched is given by the function $h(t) = -16t^2 + 294.4t + 15$.

3.3.1 What, including unit, is the value of $h(12)$ and what does the value tell you about the flight of the missile?

3.3.2 What, including unit, is the value of $\frac{h(10\text{ s}) - h(0\text{ s})}{10\text{ s}}$ and what does this value tell you about the flight of the missile?

3.3.3 The velocity (ft/s) function for the missile is $v(t) = -32t + 294.4$. What, including unit, is the value of $\frac{v(10\text{ s}) - v(0\text{ s})}{10\text{ s}}$ and what does this value tell you about the flight of the missile?

Problem 3.4

Timmy lived a long life in the 19th century. When Timmy was seven he found a rock that weighed exactly half a stone. (Timmy lived in jolly old England, don't you know.) That rock sat on Timmy's window sill for the next 80 years and wouldn't you know the weight of that rock did not change even one smidge the entire time. In fact, the weight function for this rock was $w(t) = 0.5$ where $w(t)$ was the weight of the rock (stones) and t was the number of years that had passed since that day Timmy brought the rock home.

3.4.1 What was the average rate of change in the weight of the rock over the 80 years it sat on Timmy's window sill?

3.4.2 Ignoring the unit, simplify the expression $\frac{w(t_1) - w(t_0)}{t_1 - t_0}$. Does the result make sense in the context of this problem?

3.4.3 Showing each step in the process and ignoring the unit, simplify the difference quotient for w. Does the result make sense in the context of this problem?

Problem 3.5

Truth be told, there was one day in 1842 when Timmy's mischievous son Nigel took that rock outside and chucked it into the air. The velocity of the rock (ft/s) was given by $v(t) = 60 - 32t$ where t was the number of seconds that had passed since Nigel chucked the rock.

3.5.1 What, *including unit*, are the values of $v(0)$, $v(1)$, and $v(2)$ and what do these values tell you in the context of this problem? Don't just write that the values tell you the velocity at certain times; explain what the velocity values tell you about the motion of the rock.

3.5.2 Ignoring the unit, simplify the difference quotient for v.

3.5.3 What is the unit for the difference quotient for v? What does the value of the difference quotient (including unit) tell you in the context of this problem?

Problem 3.6

Suppose that a vat was undergoing a controlled drain and that the amount of fluid left in the vat (gal) was given by the formula $V(t) = 100 - 2t^{3/2}$ where t is the number of minutes that had passed since the draining process began.

3.6.1 What, *including unit*, is the value of $V(4)$ and what does that value tell you in the context of this problem?

3.6.2 Ignoring the unit, write down the formula you get for the difference quotient of V when $t = 4$. Copy Table 3.1 onto your paper and fill in the missing values. ***Look for a pattern in the output and write down enough digits for each entry so that the pattern is clearly illustrated***; the first two entries in the output column have been given to help you understand what is meant by this direction.

Table 3.1: $y = \dfrac{V(4+h) - V(4)}{h}$

h	y
–0.1	–5.962
–0.01	–5.9962
–0.001	
0.001	
0.01	
0.1	

3.6.3 What is the unit for the y values in Table 3.1? What do these values (with their unit) tell you in the context of this problem?

3.6.4 As the value of h gets closer to 0, the values in the y column of Table 3.1 appear to be converging to a single number; what is this number and what do you think that value (with its unit) tells you in the context of this problem?

Limits and Continuity

Activity 4

While working problem 3.6 you completed Table 4.1 (formerly Table 3.1). In the context of that problem the difference quotient being evaluated returned the average rate of change in the volume of fluid remaining in a vat between times $t=4$ and $t=4+h$. As the elapsed time closes in on 0 this average rate of change converges to -6. From that we deduce that the rate of change in the volume 4 minutes into the draining process must have been -6 gal/min.

The context for Problem 3.6

Suppose that a vat was undergoing a controlled drain and that the amount of fluid left in the vat (gal) was given by the formula $V(t)=100-2t^{3/2}$ where t is the number of minutes that had passed since the draining process began.

Table 4.1: $y=\dfrac{V(4+h)-V(4)}{h}$

h	y
–0.1	–5.962
–0.01	–5.9962
–0.001	–5.99962
0.001	–6.00037
0.01	–6.0037
0.1	–6.037

Please note that we could not deduce the rate of change 4 minutes into the process by replacing h with 0; in fact, there are at least two things preventing us from doing so. From a strictly mathematical perspective, we cannot replace h with 0 because that would lead to division by zero in the difference quotient. From a more physical perspective, replacing h with 0 would in essence stop the clock. If time is frozen, so is the amount of fluid in the vat and the entire concept of "rate of change" becomes moot.

It turns out that it is frequently more useful (not to mention interesting) to explore the ***trend*** in a function as the input variable ***approaches*** a number rather than the actual value of the function at that number. Mathematically we describe these trends using **limits**.

If we call the difference quotient in the heading for Table 4.1 $f(h)$, then we could describe the trend evidenced in the table by saying "the limit of $f(h)$ as h approaches zero is -6." Please note that as h changes value, the value of $f(h)$ changes, not the value of the limit. The limit value is a fixed number to which the value of $f(h)$ converges. Symbolically we write $\lim_{h\to 0} f(h)=-6$

Most of the time the value of a function at the number a and the limit of the function as x approaches a are in fact the same number. When this occurs we say that the function is ***continuous*** at a. However, to help you better understand the concept of limit we need to have you confront situations where the function value and limit value are not equal to one another. Graphs can be useful for helping distinguish function values from limit values, so that is the perspective you are going to use in the first couple of problems in this lab.

Problem 4.1

Several function values and limit values for the function in Figure 4.1 are given below. You and your group mates should take turns reading the equations aloud. Make sure that you read the symbols correctly, that's part of what you are learning! Also, discuss why the values are what they are and make sure that you get help from your instructor to clear up any confusion.

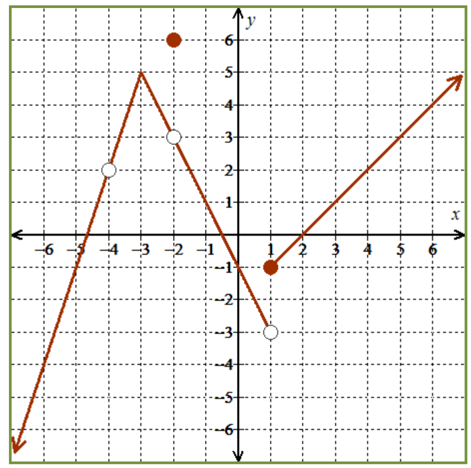


Figure 4.1: f

$f(-2)=6$ but $\lim\limits_{x\to -2} f(x)=3$

$f(-4)$ is undefined but $\lim\limits_{x\to -4} f(x)=2$

$f(1)=-1$ but $\lim\limits_{x\to 1} f(x)$ does not exist

$\lim\limits_{x\to 1^-} f(x)=-3$ but $\lim\limits_{x\to 1^+} f(x)=-1$

The limit of $f(x)$ as x approaches 1 from the left.

The limit of $f(x)$ as x approaches 1 from the right.

Problem 4.2

Copy each of the following expressions onto your paper and either state the value or state that the value is undefined or doesn't exist. Make sure that when discussing the values you use proper terminology. All expressions are in reference to the function g shown in Figure 4.2.

4.2.1	$g(5)$	**4.2.2**	$\lim\limits_{t\to 5} g(t)$	**4.2.3**	$g(3)$
4.2.4	$\lim\limits_{t\to 3^-} g(t)$	**4.2.5**	$\lim\limits_{t\to 3^+} g(t)$	**4.2.6**	$\lim\limits_{t\to 3} g(t)$
4.2.7	$g(2)$	**4.2.8**	$\lim\limits_{t\to 2} g(t)$	**4.2.9**	$g(-2)$
4.2.10	$\lim\limits_{t\to -2^-} g(t)$	**4.2.11**	$\lim\limits_{t\to -2^+} g(t)$	**4.2.12**	$\lim\limits_{t\to -2} g(t)$

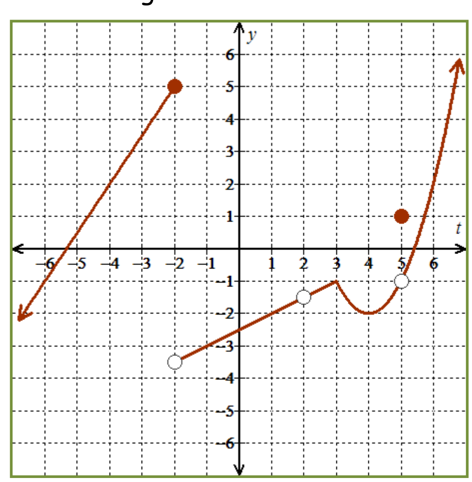


Figure 4.2: g

Problem 4.3

Values of the function $f(x)=\dfrac{3x^2-16x+5}{2x^2-13x+15}$ are shown in Table 4.2. Both of the questions below are in reference to this function.

4.3.1 What is the value of $f(5)$?

4.3.2 What is the value of $\lim\limits_{x\to 5}\dfrac{3x^2-16x+5}{2x^2-13x+15}$?

Table 4.2: $f(x)=\dfrac{3x^2-16x+5}{2x^2-13x+15}$

x	$f(x)$
4.99	2.0014
4.999	2.00014
4.9999	2.000014
5.0001	1.999986
5.001	1.99986
5.01	1.9986

Problem 4.4

Values of the function $p(t)=\sqrt{t-12}$ are shown in Table 4.3. Both of the questions below are in reference to this function.

4.4.1 What is the value of $p(21)$?

4.4.2 What is the value of $\lim\limits_{t\to 21}\sqrt{t-12}$?

Table 4.3: $p(t)=\sqrt{t-12}$

t	$p(t)$
20.9	2.98
20.99	2.998
20.999	2.9998
21.001	3.0002
21.01	3.002
21.1	3.02

Problem 4.5

Create tables similar to tables 4.2 and 4.3 from which you can deduce each of the following limit values. Make sure that you include table numbers, table captions, and meaningful column headings. Make sure that your input values follow patterns similar to those used in tables 4.2 and 4.3. Make sure that you round your output values in such a way that a clear and compelling pattern in the output is clearly demonstrated by your stated values. Make sure that you state the limit value!

4.5.1 $\lim\limits_{t\to 6}\dfrac{t^2-10t+24}{t-6}$ **4.5.2** $\lim\limits_{x\to -1^+}\dfrac{\sin(x+1)}{3x+3}$ **4.5.3** $\lim\limits_{h\to 0^-}\dfrac{h}{4-\sqrt{16-h}}$

Activity 5

When proving the value of a limit we frequently rely upon laws that are easy to prove using the technical definitions of limit. These laws can be found in Appendix C (pages C1 and C2). The first of these type laws are called replacement laws. Replacement laws allow us to replace limit expressions with the actual values of the limits.

Problem 5.1

The value of each of the following limits can be established using one of the replacement laws. Copy each limit expression onto your own paper, state the value of the limit (e.g. $\lim\limits_{x\to 9}5=5$), and state the replacement law (by number) that establishes the value of the limit.

5.1.1 $\lim\limits_{t\to\pi}t$ **5.1.2** $\lim\limits_{x\to 14}14$ **5.1.3** $\lim\limits_{x\to 14}x$

Problem 5.2

The algebraic limit laws allow us to replace limit expressions with equivalent limit expressions. When applying limit laws our first goal is to come up with an expression in which every limit in the expression can be replaced with its value based upon one of the replacement laws. This process is shown in example 5.1. Please note that all replacement laws are saved for the second to last step and that each replacement is explicitly shown. Please note also that each limit law used is referenced by number.

Example 5.1

$$\begin{aligned}
\lim_{x\to 7}\left(4x^2+3\right) &= \lim_{x\to 7}\left(4x^2\right)+\lim_{x\to 7}3 && \text{Limit Law A1}\\
&= 4\cdot\lim_{x\to 7}x^2+\lim_{x\to 7}3 && \text{Limit Law A3}\\
&= 4\cdot\left(\lim_{x\to 7}x\right)^2+\lim_{x\to 7}3 && \text{Limit Law A6}\\
&= 4\cdot 7^2+3 && \text{Limit Laws R1 and R2}\\
&= 199
\end{aligned}$$

Use the limit laws to establish the value of each of the following limits. Make sure that you use the step-by-step, vertical format shown in example 5.1. Make sure that you cite the limit laws used in each step. To help you get started, ***the steps necessary in <u>problem 5.2.1</u> are outlined below***.

To help you get started, ***the steps necessary in <u>problem 5.2.1</u> are outlined below***.

Step 1: Apply Law A6 **Step 2**: Apply Law A1 **Step 3**: Apply Law A3

Step 4: Apply Laws R1 and R2

5.2.1 $\lim_{t\to 4}\sqrt{6t+1}$ **5.2.2** $\lim_{y\to 7}\dfrac{y+3}{y-\sqrt{y+9}}$ **5.2.3** $\lim_{x\to \pi}\left(x\cos(x)\right)$

Activity 6

Many limits have the form $\frac{0}{0}$ which means the expressions in both the numerator and denominator limit to zero (e.g. $\lim_{x\to 3}\frac{2x-6}{x-3}$). The form $\frac{0}{0}$ is called ***<u>indeterminate</u>*** because we do not know the value of the limit (or even if it exists) so long as the limit has that form. When confronted with limits of form $\frac{0}{0}$ we must first manipulate the expression so that common factors causing the zeros in the numerator and denominator are isolated. Limit law A7 can then be used to justify eliminating the common factors and once they are gone we may proceed with the application of the remaining limit laws. Examples 6.1 and 6.2 illustrate this situation.

Example 6.1

$$\lim_{x\to 3}\frac{x^2-8x+15}{x-3}=\lim_{x\to 3}\frac{(x-5)(x-3)}{x-3}$$

$$=\lim_{x\to 3}(x-5) \qquad \text{Limit Law A7}$$

$$=\lim_{x\to 3}x-\lim_{x\to 3}5 \qquad \text{Limit Law A2}$$

$$=3-5 \qquad \text{Limit Laws R1 and R2}$$

$$=-2$$

This limit does not have indeterminate form, so we may apply the remaining limit laws.

Example 6.2

$$\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\sin^2(\theta)}=\lim_{\theta\to 0}\left(\frac{1-\cos(\theta)}{\sin^2(\theta)}\cdot\frac{1+\cos(\theta)}{1+\cos(\theta)}\right)$$

$$=\lim_{\theta\to 0}\frac{1-\cos^2(\theta)}{\sin^2(\theta)\cdot(1+\cos(\theta))}$$

$$=\lim_{\theta\to 0}\frac{\sin^2(\theta)}{\sin^2(\theta)\cdot(1+\cos(\theta))}$$

$$=\lim_{\theta\to 0}\frac{1}{1+\cos(\theta)} \qquad \text{Limit Law A7}$$

$$=\frac{\lim_{\theta\to 0}1}{\lim_{\theta\to 0}(1+\cos(\theta))} \qquad \text{Limit Law A5}$$

$$=\frac{\lim_{\theta\to 0}1}{\lim_{\theta\to 0}1+\lim_{\theta\to 0}\cos(\theta)} \qquad \text{Limit Law A1}$$

$$=\frac{\lim_{\theta\to 0}1}{\lim_{\theta\to 0}1+\cos\left(\lim_{\theta\to 0}\theta\right)} \qquad \text{Limit Law A6}$$

$$=\frac{1}{1+\cos(0)} \qquad \text{Limit Laws R1 and R2}$$

$$=\frac{1}{2}$$

This limit does not have indeterminate form, so we may apply the remaining limit laws.

As seen in example 6.2, trigonometric identities can come into play while trying to eliminate the form $\frac{0}{0}$. Elementary rules of logarithms can also play a role in this process. Before you begin evaluating limits whose initial form is $\frac{0}{0}$, you need to make sure that you recall some of these basic rules. That is the purpose of problem 6.1.

Problem 6.1

Complete each of the following identities (over the real numbers). Make sure that you check with your lecture instructor so that you know which of these identities you are expected to memorize.

The following identities are valid for all values of x and y.

$1-\cos^2(x)=$ ______________________ $\tan^2(x)+1=$ ______________________

$\sin(2x)=$ ______________________ $\tan(2x)=$ ______________________

$\sin(x+y)=$ ______________________ $\cos(x+y)=$ ______________________

$\sin\left(\frac{x}{2}\right)=$ ______________________ $\cos\left(\frac{x}{2}\right)=$ ______________________

There are three versions of the following identity; write them all.

$\cos(2x)=$ ______________________ $\cos(2x)=$ ______________________

$\cos(2x)=$ ______________________

The following identities are valid for all positive values of x and y and all values of n.

$\ln(xy)=$ ______________________ $\ln\left(\frac{x}{y}\right)=$ ______________________

$\ln(x^n)=$ ______________________ $\ln(e^n)=$ ______________________

Problem 6.2

Use the limit laws to establish the value of each of the following limits after first manipulating the expression so that it no longer has form $\frac{0}{0}$. Make sure that you use the step-by-step, vertical format shown in examples 6.1 and 6.2. Make sure that you cite each limit law used.

6.2.1 $\lim_{x \to -4} \frac{x+4}{2x^2+5x-12}$ **6.2.2** $\lim_{x \to 0} \frac{\sin(2x)}{\sin(x)}$ **6.2.3** $\lim_{\beta \to 0} \frac{\sin(\beta+\pi)}{\sin(\beta)}$

6.2.4 $\lim_{t \to 0} \frac{\cos(2t)-1}{\cos(t)-1}$ **6.2.5** $\lim_{x \to 1} \frac{4\ln(x)+2\ln(x^3)}{\ln(x)-\ln(\sqrt{x})}$ **6.2.6** $\lim_{w \to 9} \frac{9-w}{\sqrt{w}-3}$

Activity 7

We are frequently interested in a function's "end behavior;" that is, what is the behavior of the function as the input variable increases without bound or decreases without bound.

Many times a function will approach a horizontal asymptote as its end behavior. Assuming that the horizontal asymptote $y = L$ represents the end behavior of the function f both as x increases without bound and as x decreases without bound, we write $\lim_{x \to \infty} f(x) = L$ and $\lim_{x \to -\infty} f(x) = L$.

The formalistic way to read $\lim_{x \to \infty} f(x) = L$ is "the limit of $f(x)$ as x approaches infinity equals L."

When read that way, however, the words need to be taken ***anything** but literally*. In the first place, x isn't approaching anything! The entire point is that x is increasing without any bound on how large its value becomes. Secondly, there is no place on the real number line called "infinity;" infinity is not a number. Hence x certainly can't be approaching something that isn't even there!

Problem 7.1

For the function in Figure 7.1 (Appendix B, page B1) we could (correctly) write $\lim_{x \to \infty} f_1(x) = -2$ and $\lim_{x \to -\infty} f_1(x) = -2$. Go ahead and write (and say aloud) the analogous limits for the functions in figures 7.2-7.5 (pages B1 and B2).

Problem 7.2

Values of the function $f(x) = \frac{3x^2-16x+5}{2x^2-13x+15}$ are shown in Table 7.1. Both of the questions below are in reference to this function.

7.2.1 What is the value of $\lim_{x \to -\infty} f(x)$?

7.2.2 What is the horizontal asymptote for the graph of $y = f(x)$?

Table 7.1: $f(x) = \frac{3x^2-16x+5}{2x^2-13x+15}$

x	$f(x)$
−1,000	1.498
−10,000	1.4998
−100,000	1.49998
−1,000,000	1.499998

Problem 7.3

Jorge and Vanessa were in a heated discussion about horizontal asymptotes. Jorge said that functions never cross horizontal asymptotes. Vanessa said Jorge was nuts. Vanessa whipped out her trusty calculator and generated the values in Table 7.2 to prove her point.

Table 7.2: $g(t) = 1 + \frac{\sin(t)}{t}$

t	$g(t)$
10^3	1.0008
10^4	.99997
10^5	1.0000004
10^6	.9999997
10^7	1.00000004
10^8	1.000000009
10^9	1.0000000005
10^{10}	.99999999995

7.3.1 What is the value of $\lim_{t \to \infty} g(t)$?

7.3.2 What is the horizontal asymptote for the graph of $y = g(t)$?

7.3.3 Just how many times does the curve $y = g(t)$ cross its horizontal asymptote?

Activity 8

When using limit laws to establish limit values as $x \to \infty$ or $x \to -\infty$, limit laws A1-A6 and R2 are still in play (when applied in a valid manner), but limit law R1 cannot be applied. (The reason limit law R1 cannot be applied is discussed in detail in problem 11.4)

There is a new replacement law that can only be applied when $x \to \infty$ or $x \to -\infty$; this is replacement law R3. Replacement law R3 essentially says that if the value of a function is increasing without any bound on large it becomes or if the function is decreasing without any bound on how large its absolute value becomes, then the value of a constant divided by that function must be approaching zero. An analogy can be found in extremely poor party planning. Let's say that you plan to have a pizza party and you buy five pizzas. Suppose that as the hour of the party approaches more and more guests come in the door … in fact the guests never stop coming! Clearly as the number of guests continues to rise the amount of pizza each guest will receive quickly approaches zero (assuming the pizzas are equally divided among the guests).

Problem 8.1

Consider the function $f(x) = \frac{12}{x}$. Complete Table 8.1 ***without the use of your calculator***. What limit value and limit law are being illustrated in the table?

Table 8.1: $f(x) = \frac{12}{x}$

x	$f(x)$
1,000	
10,000	
100,000	
1,000,000	

Activity 9

Many limits have the form $\frac{\infty}{\infty}$ which we take to mean that the expressions in both the numerator and denominator are increasing or decreasing without bound. When confronted with a limit of type $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ or $\lim_{x\to-\infty}\frac{f(x)}{g(x)}$ that has the form $\frac{\infty}{\infty}$, we can frequently resolve the limit if we first divide the dominant factor of the dominant term of the denominator from both the numerator and the denominator. When we do this, we need to completely simplify each of the resultant fractions and make sure that the resultant limit exists before we start to apply limit laws. We then apply the algebraic limit laws until all of the resultant limits can be replaced using limit laws R2 and R3. This process is illustrated in example 9.1.

Example 9.1

$$\lim_{t\to\infty}\frac{3t^2+5t}{3-5t^2}=\lim_{t\to\infty}\left(\frac{3t^2+5t}{3-5t^2}\cdot\frac{1/t^2}{1/t^2}\right)$$

$$=\lim_{t\to\infty}\frac{3+\frac{5}{t}}{\frac{3}{t^2}-5}$$

The "form" of the limit is now $\frac{3+0}{0-5}$, so we can begin to apply the limit laws because the limits will all exist.

$$=\frac{\lim_{t\to\infty}\left(3+\frac{5}{t}\right)}{\lim_{t\to\infty}\left(\frac{3}{t^2}-5\right)} \quad \text{Limit Law A5}$$

$$=\frac{\lim_{t\to\infty}3+\lim_{t\to\infty}\frac{5}{t}}{\lim_{t\to\infty}\frac{3}{t^2}-\lim_{t\to\infty}5} \quad \text{Limit Laws A1 and A2}$$

$$=\frac{3+0}{0-5} \quad \text{Limit Laws R2 and R3}$$

$$=-\frac{3}{5}$$

Problem 9.1

Use the limit laws to establish the value of each limit after dividing the dominant term-factor in the denominator from both the numerator and denominator. Remember to simplify each resultant expression before you begin to apply the limit laws.

9.1.1 $\lim_{t\to-\infty}\frac{4t^2}{4t^2+t^3}$ **9.1.2** $\lim_{t\to\infty}\frac{6e^t+10e^{2t}}{2e^{2t}}$ **9.1.3** $\lim_{y\to\infty}\sqrt{\frac{4y+5}{5+9y}}$

Activity 10

Many limit values do not exist. Sometimes the non-existence is caused by the function value either increasing without bound or decreasing without bound. In these special cases we use the symbols ∞ and $-\infty$ to communicate the non-existence of the limits. Figures 10.1-10.3 can be used to illustrate some ways in which we communicate the ***non-existence*** of these type of limits.

In Figure 10.1 we could (correctly) write $\lim_{x\to 2} k(x)=\infty$, $\lim_{x\to 2^-} k(x)=\infty$, and $\lim_{x\to 2^+} k(x)=\infty$.

In Figure 10.2 we could (correctly) write $\lim_{t\to 4} w(t)=-\infty$, $\lim_{t\to 4^-} w(t)=-\infty$, and $\lim_{t\to 4^+} w(t)=-\infty$.

In Figure 10.3 we could (correctly) write $\lim_{x\to -3^-} T(x)=\infty$ and $\lim_{x\to -3^+} T(x)=-\infty$. There is no shorthand way of communicating the non-existence of the two sided limit $\lim_{x\to -3} T(x)$.

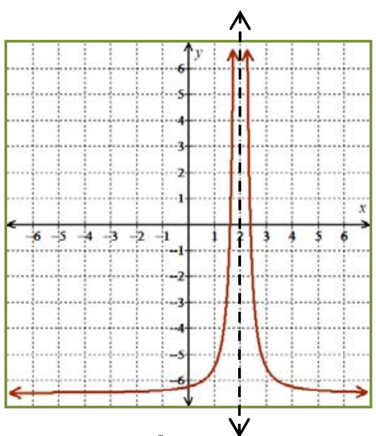


Figure 10.1: k

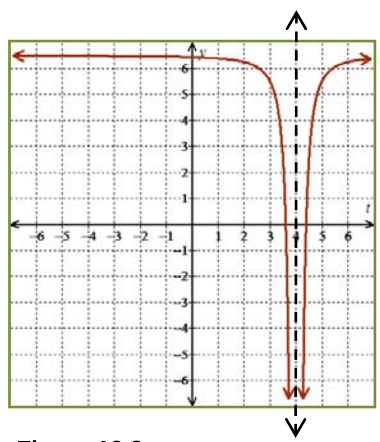


Figure 10.2: w

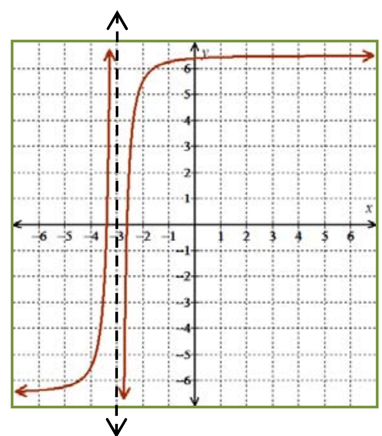


Figure 10.3: T

Problem 10.1

Draw onto Figure 10.4 a single function, f, that satisfies each of the following limit statements. Make sure that you draw the necessary asymptotes and that you label each asymptote with its equation.

- $\lim_{x\to 3^-} f(x)=-\infty$
- $\lim_{x\to 3^+} f(x)=\infty$
- $\lim_{x\to \infty} f(x)=0$
- $\lim_{x\to -\infty} f(x)=-2$

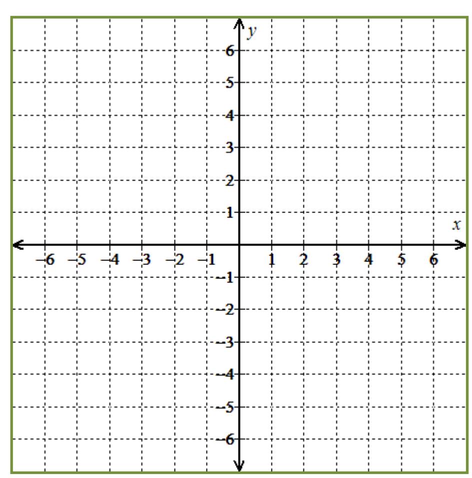


Figure 10.4: f

Activity 11

Whenever $\lim\limits_{x\to a} f(x)\neq 0$ but $\lim\limits_{x\to a} g(x)=0$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ ***does not exist*** because from either side of a the value of $\dfrac{f(x)}{g(x)}$ either increases without bound or decreasing without bound. In these situations the line $x=a$ is a vertical asymptote for the graph of $y=\dfrac{f(x)}{g(x)}$.

For example, the line $x=2$ is a vertical asymptote for the function $h(x)=\dfrac{x+5}{2-x}$. We say that $\lim\limits_{x\to 2}\dfrac{x+5}{2-x}$ has ***the form "not zero over zero."*** (Specifically, the form of $\lim\limits_{x\to 2}\dfrac{x+5}{2-x}$ is $\dfrac{7}{0}$.) Every limit with form "not zero over zero" ***does not exist***. However, we frequently can communicate the non-existence of the limit using an infinity symbol. In the case of $h(x)=\dfrac{x+5}{2-x}$ it's pretty easy to see that $h(1.99)$ is a positive number whereas $h(2.01)$ is a negative number. Consequently, we can infer that $\lim\limits_{x\to 2^-} h(x)=\infty$ and $\lim\limits_{x\to 2^+} h(x)=-\infty$. Remember, these equations are communicating that the limits ***do not exist*** as well as the reason for their non-existence. There is no short-hand way to communicate the non-existence of the two-sided limit $\lim\limits_{x\to 2} h(x)$.

Problem 11.1 Suppose that $g(t)=\dfrac{t+4}{t+3}$.

11.1.1 What is the vertical asymptote on the graph of $y=g(t)$.

11.1.2 Write an equality about $\lim\limits_{t\to -3^-} g(t)$.

11.1.3 Write an equality about $\lim\limits_{t\to -3^+} g(t)$.

11.1.4 Is it possible to write an equality about $\lim\limits_{t\to -3} g(t)$? If so, write it.

11.1.5 Which of the following limits exist? $\lim\limits_{t\to -3^-} g(t)$, $\lim\limits_{t\to -3^+} g(t)$, and $\lim\limits_{t\to -3} g(t)$

Problem 11.2 Suppose that $z(x)=\dfrac{7-3x^2}{(x-2)^2}$.

11.2.1 What is the vertical asymptote on the graph of $y=z(x)$.

11.2.2 Is it possible to write an equality about $\lim\limits_{x\to 2} z(x)$? If so, write it.

11.2.3 What is the horizontal asymptote on the graph of $y=z(x)$.

11.2.4 Which of the following limits exist? $\lim\limits_{x\to 2} z(x)$, $\lim\limits_{x\to \infty} z(x)$, and $\lim\limits_{x\to -\infty} z(x)$

Problem 11.3

Consider the function $f(x)=\dfrac{x+7}{x-8}$. Complete Table 11.1 ***without the use of your calculator***.

Use this as an opportunity to discuss why limits of form "not zero over zero" are "infinite limits." What limit equation is being illustrated in the table?

Table 11.1: $f(x)=\dfrac{x+7}{x-8}$

x	$x+7$	$x-8$	$f(x)$
8.1	15.1	.1	
8.01	15.01	.01	
8.001	15.001	.001	
8.0001	15.0001	.0001	

Problem 11.4

Hear me, and hear me loud … ∞ **does not exist**. This, in part, is why we cannot apply Limit Law R1 to an expression like $\lim\limits_{x\to\infty} x=\infty$. When we write, say, $\lim\limits_{x\to 7} x=7$, we are replacing the limit expression *with its **value** – that's what the replacement laws are all about*! When we write $\lim\limits_{x\to\infty} x=\infty$, we are not replacing the limit expression with a value! We are explicitly saying that ***the limit has no value*** (i.e. does not exist) as well as saying the reason the limit does not exist. The limit laws (R1-R3 and A1-A6) can only be applied when all of the limits in the equation exist. With this in mind, discuss and decide whether each of the following equations are true or false.

$$\lim_{x\to 0}\left(\frac{e^x}{e^x}\right)=\frac{\lim\limits_{x\to 0} e^x}{\lim\limits_{x\to 0} e^x}$$ **T or F**

$$\lim_{x\to -\infty}\left(\frac{e^{-x}}{e^{-x}}\right)=\frac{\lim\limits_{x\to -\infty} e^{-x}}{\lim\limits_{x\to -\infty} e^{-x}}$$ **T or F**

$$\lim_{x\to 1}\frac{e^x}{\ln(x)}=\frac{\lim\limits_{x\to 1} e^x}{\lim\limits_{x\to 1}\ln(x)}$$ **T or F**

$$\lim_{x\to 1}\frac{\ln(x)}{e^x}=\frac{\lim\limits_{x\to 1}\ln(x)}{\lim\limits_{x\to 1} e^x}$$ **T or F**

$$\lim_{x\to 0^+}\left(2\ln(x)\right)=2\lim_{x\to 0^+}\ln(x)$$ **T or F**

$$\lim_{\theta\to\infty}\frac{\sin(\theta)}{\sin(\theta)}=\frac{\lim\limits_{\theta\to\infty}\sin(\theta)}{\lim\limits_{\theta\to\infty}\sin(\theta)}$$ **T or F**

$$\lim_{x\to\infty}\left(e^x-\ln(x)\right)=\lim_{x\to\infty} e^x-\lim_{x\to\infty}\ln(x)$$ **T or F**

$$\lim_{x\to -\infty} e^{1/x}=e^{\lim\limits_{x\to -\infty}\frac{1}{x}}$$ **T or F**

Problem 11.5

Mindy tried to evaluate $\lim\limits_{x \to 6^+} \dfrac{4x-24}{x^2-12x+36}$ using the limit laws. Things went horribly wrong for Mindy (her work is shown below). Identify what is wrong in Mindy's work and discuss what a more reasonable approach might have been.

This "solution" is not correct!
Do not emulate Mindy's work!!

$$\lim_{x \to 6^+} \frac{4x-24}{x^2-12x+36} = \lim_{x \to 6^+} \frac{4(x-6)}{(x-6)^2}$$

$$= \lim_{x \to 6^+} \frac{4}{x-6}$$

$$= \frac{\lim\limits_{x \to 6^+} 4}{\lim\limits_{x \to 6^+} (x-6)} \qquad \text{Limit Law A5}$$

$$= \frac{\lim\limits_{x \to 6^+} 4}{\lim\limits_{x \to 6^+} x - \lim\limits_{x \to 6^+} 6} \qquad \text{Limit Law A2}$$

$$= \frac{4}{6-6} \qquad \text{Limit Laws R1 and R2}$$

$$= \frac{4}{0}$$

$$= \infty$$

Activity 12

Many statements we make about functions are only true over intervals where the function is ***continuous***. When we say a function is continuous over an interval, we basically mean that there are no breaks in the function over that interval; that is, there are no vertical asymptotes, holes, jumps, or gaps along that interval.

Definition 12.1

The function f is continuous at the number a if and only if $\lim\limits_{x \to a} f(x) = f(a)$.

There are three ways that the defining property can fail to be satisfied at a given value of a. To facilitate exploration of these three manner of failure, we can break the defining property into a spectrum of three properties.

i. $f(a)$ must be defined   *ii.* $\lim\limits_{x \to a} f(x)$ must exist   *iii.* $\lim\limits_{x \to a} f(x)$ must equal $f(a)$

Please note that if either property i or property ii fails to be satisfied at a given value of a, then property iii also fails to be satisfied at a.

Problem 12.1

State the values of t at which the function shown in Figure 12.1 is discontinuous. For each instance of discontinuity, state (by number) all of the sub-properties in Definition 12.1 that fail to be satisfied.

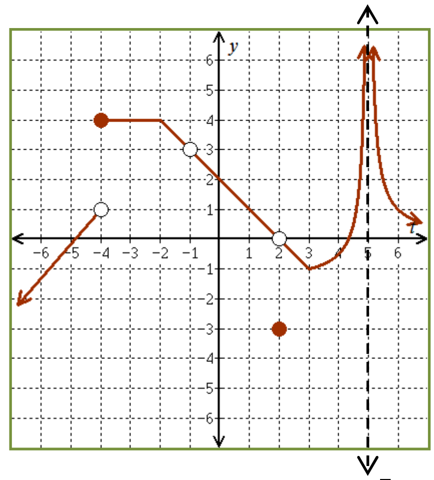


Figure 12.1: h

Activity 13

When a function has a discontinuity at a, the function is sometimes continuous from only the right or only the left at a. (Please note that when we say "the function is continuous at a" we mean that the function is continuous from **both** the right and left at a.)

Definition 13.1

The function f is continuous from the left at a if and only if $\lim_{x\to a^-} f(x) = f(a)$ and f is continuous from the right at a if and only if $\lim_{x\to a^+} f(x) = f(a)$.

Some discontinuities are classified as ***removable discontinuities***. Specifically, discontinuities that are holes or skips (holes with a secondary point) are called removable.

Definition 13.2

We say that f has a removable discontinuity at a if f is discontinuous at a but $\lim_{x\to a} f(x)$ exists.

Problem 13.1

Referring to the function h shown in Figure 13.1, state the values of t where the function is continuous from the right but not the left. Then state the values of t where the function is continuous from the left but not the right.

Problem 13.2

Referring again to the function h shown in Figure 13.1, state the values of t where the function has removable discontinuities.

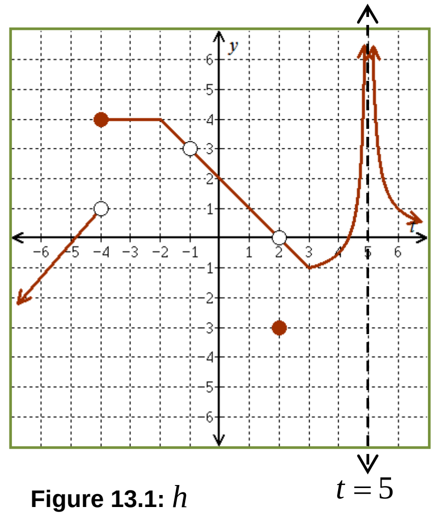


Figure 13.1: h

Activity 14

Now that we have a definition for continuity at a number, we can go ahead and define what we mean when we say a function is continuous over an interval.

Definition 14.1

The function f is continuous over an open interval if and only if it is continuous at each and every number on that interval.

The function is continuous over the closed interval $[a,b]$ if and only if it is continuous on (a,b), continuous from the right at a, and continuous from the left at b.

Similar definitions apply to half-open intervals.

Problem 14.1

Write a definition for continuity over the half-open interval $(a,b]$.

Problem 14.2

Referring to the function in Figure 14.1, decide whether each of the following statements are true or false.

14.2.1 h is continuous on $[-4,-1)$

14.2.2 h is continuous on $(-4,-1)$

14.2.3 h is continuous on $(-4,-1]$

14.2.4 h is continuous on $(-1,2]$

14.2.5 h is continuous on $(-1,2)$

14.2.6 h is continuous on $(-\infty,-4)$

14.2.7 h is continuous on $(-\infty,-4]$

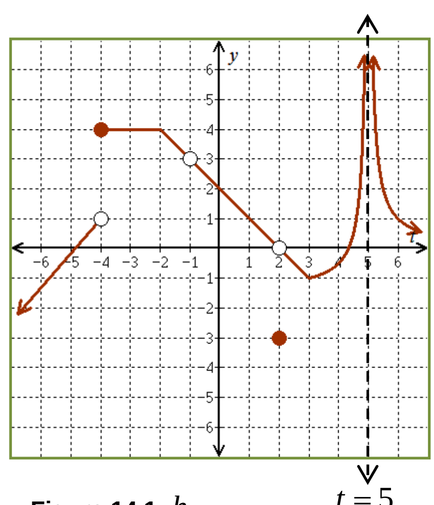


Figure 14.1: h

Problem 14.3

Several functions are described below. Your task is to draw each function on its provided axis system. Do not introduce any unnecessary discontinuities or intercepts that are not directly implied by the stated properties. Make sure that you draw all implied asymptotes and label them with their equations.

14.3.1 Draw onto Figure 14.2 a function that satisfies all of the following properties.

- $\lim_{x \to 4^-} f(x) = 2$
- $\lim_{x \to 4^+} f(x) = 5$
- $f(0) = 4$ and $f(4) = 5$
- $\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 4$

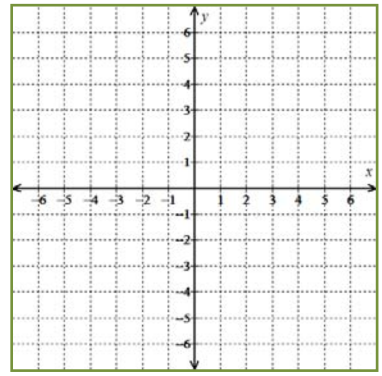


Figure 14.2: f

14.3.2 Draw onto Figure 14.3 a function that satisfies all of the following properties.

- $\lim_{x \to -2} g(x) = \infty$
- $\lim_{x \to -\infty} g(x) = \infty$
- $g(0) = 4$, $g(3) = -2$, and $g(6) = 0$
- g is continuous and has constant slope on $(0, \infty)$.

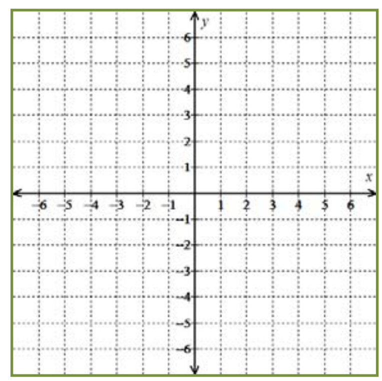


Figure 14.3: g

14.3.3 Draw onto Figure 14.4 a function that satisfies all of the following properties.

- The only discontinuities on m occur at -4 and 3
- m has no x-intercepts
- $m(-6) = 5$
- $\lim_{x \to -4^+} m(x) = -2$
- $\lim_{x \to 3} m(x) = -\infty$
- $\lim_{x \to \infty} m(x) = -\infty$
- m has a constant slope of -2 over $(-\infty, -4)$
- m is continuous over $[-4, 3)$

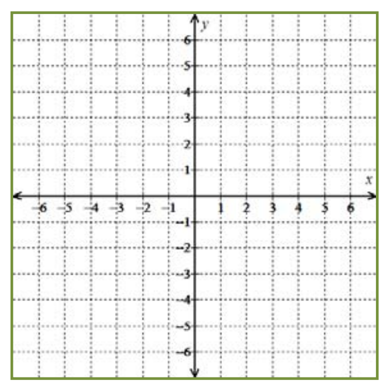


Figure 14.4: m

Activity 15

Discontinuities are a little more challenging to identify when working with formulas than when working with graphs. One reason for the added difficulty is that when working with a function formula you have to dig into your memory bank and retrieve fundamental properties about certain types of functions.

Problem 15.1

15.1.1 What would cause a discontinuity on a rational function (a polynomial divided by another polynomial)?

15.1.2 What is always true about the ***argument*** of the function, <u>u</u>, over intervals where the function $y = \ln(u)$ is continuous?

15.1.3 Name three values of θ where the function $y = \tan(\theta)$ is discontinuous.

15.1.4 What is the domain of the function $k(t) = \sqrt{t-4}$?

15.1.5 What is the domain of the function $g(t) = \sqrt[3]{t-4}$?

Activity 16

Piece-wise defined functions are functions where the formula used depends upon the value of the input. When looking for discontinuities on piece-wise defined functions, you need to investigate the behavior at values where the formula changes as well as values where the issues discussed in Activity 15 might pop up.

Problem 16.1

This question is all about the function *f* shown to the right. Answer question 16.1.1 at each of the values 1, 3, 4, 5, 7, and 8. At the values where the answer to question 16.1.1 is yes, go ahead and answer questions 16.1.2-16.1.4; skip questions 16.1.2-16.1.4 at the values where the answer to question 16.1.1 is no.

$$f(x) = \begin{cases} \dfrac{4}{5-x} & \text{if} \quad x < 1 \\ \dfrac{x-3}{x-3} & \text{if} \quad 1 < x < 4 \\ 2x+1 & \text{if} \quad 4 \le x \le 7 \\ \dfrac{15}{8-x} & \text{if} \quad x > 7 \end{cases}$$

16.1.1 Is *f* discontinuous at the given value?

16.1.2 Is *f* continuous only from the left at the given value?

16.1.3 Is *f* continuous only from the right at the given value?

16.1.4 Is the discontinuity removable?

Problem 16.2

Consider the function g shown to the right. The letter C represents the same real number in all three of the piece-wise formulas.

$$g(x)=\begin{cases} \dfrac{C}{x-17} & \text{if} \quad x<10 \\ C+3x & \text{if} \quad x=10 \\ 2C-4 & \text{if} \quad x>10 \end{cases}$$

16.2.1 Find the value for C that makes the function continuous on $(-\infty,10]$. Make sure that your reasoning is clear.

16.2.2 Is it possible to find a value for C that makes the function continuous over $(-\infty,\infty)$? Explain.

Problem 16.3

Consider the function f shown to the right. State the values of x where each of the following occur. If a stated property doesn't occur, make sure that you state that (as opposed to simply not responding to the question). No explanation necessary.

$$f(x)=\begin{cases} \dfrac{5}{x-10} & \text{if} \quad x\le 5 \\ \dfrac{5}{5x-30} & \text{if} \quad 5<x<7 \\ \dfrac{x-2}{12-x} & \text{if} \quad x>7 \end{cases}$$

16.3.1 At what values of x is f discontinuous?

16.3.2 At what values of x is f continuous from the left but not the right?

16.3.3 At what values of x is f continuous from the right but not the left?

16.3.4 At what values of x does f have removable discontinuities?

Introduction to the First Derivative

Activity 17

Most of the focus in the *Rates of Change* lab was on average rates of change. While the idea of rates of change at one specific instant was alluded to, we couldn't explore that idea formally because we hadn't yet talked about limits. Now that we have covered average rates of change and limits we can put those two ideas together to discuss rates of change at specific instances in time.

Suppose that an object is tossed into the air in such a way that the elevation of the object (measured in ft) is given by the function $s(t) = 40 + 40t - 16t^2$ where t is the amount of time that has passed since the object was tossed (measured in s). Let's determine the velocity of the object 2 seconds into this flight.

Recall that the difference quotient $\frac{s(2+h) - s(2)}{h}$ gives us the average velocity for the object between the times $t = 2$ and $t = 2 + h$. So long as h is positive, we can think of h as the length of the time interval. To infer the velocity exactly 2 seconds into the flight we need the time interval as close to 0 as possible; this is done using the appropriate limit in example 17.1.

Example 17.1

$$\begin{aligned}
v(2) &= \lim_{h \to 0} \frac{s(2+h) - s(2)}{h} \\
&= \lim_{h \to 0} \frac{\left[40 + 40(2+h) - 16(2+h)^2\right] - \left[40 + 40(2) - 16(2)^2\right]}{h} \\
&= \lim_{h \to 0} \frac{40 + 80 + 40h - 64 - 64h - 16h^2 - 56}{h} \\
&= \lim_{h \to 0} \frac{-24h - 16h^2}{h} \\
&= \lim_{h \to 0} \frac{h(-24 - 16h)}{h} \\
&= \lim_{h \to 0} (-24 - 16h) \\
&= -24 - 16 \cdot 0 \\
&= -24
\end{aligned}$$

From this we can infer that the velocity of the object 2 seconds into its flight is -24 ft/s. From that we know that the object is falling at a speed of 24 ft/s.

Problem 17.1

Suppose that an object is tossed into the air so that its elevation (measured in m) is given by the function $p(t) = 300 + 10t - 4.9t^2$ where t is the amount of time that has passed since the object was tossed (measured in s).

17.1.1 Evaluate $\lim_{h \to 0} \frac{p(4+h) - p(4)}{h}$ showing each step in the simplification process (as illustrated in example 17.1).

17.1.2 What is the unit for the value calculated in problem 17.1.1 and what does the value (including unit) tell you about the motion of the object?

17.1.3 Copy Table 17.1 onto your paper and compute and record the missing values. Do these values support your answer to problem 17.1.2?

Table 17.1: Average Velocities

t_1	$\frac{p(t_1) - p(4)}{t_1 - 4}$
3.9	
3.99	
3.999	
4.001	
4.01	
4.1	

Activity 18

In previous activities we saw that if p is a position function, then the difference quotient for p can be used to calculate average velocities and the expression $\lim_{h \to 0} \frac{p(t_0 + h) - p(t_0)}{h}$ calculates the instantaneous velocity at time t_0.

Graphically, the difference quotient of a function f can be used to calculate the slope of secant lines to f. What happens when we take the run of the secant line to zero? Basically, we are connecting two points on the line that are really, really, (***really***), close to one another. As mentioned above, sending h to zero turns an average velocity into an instantaneous velocity. Graphically, sending h to zero turns a secant line into ***a tangent line***.

The tangent line to the function $f(x) = \sqrt{x}$ at 4 is shown in Figure 18.1. A calculation of the slope of this line is shown in Example 18.1.

Example 18.1

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0}\frac{f(4+h)-f(4)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{4+h}-\sqrt{4}}{h} \\ &= \lim_{h\to 0}\left(\frac{\sqrt{4+h}-2}{h}\cdot\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right) \\ &= \lim_{h\to 0}\frac{4+h-4}{h\left(\sqrt{4+h}+2\right)} \\ &= \lim_{h\to 0}\frac{h}{h\left(\sqrt{4+h}+2\right)} \\ &= \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} \\ &= \frac{1}{\sqrt{4+0}+2} \\ &= \frac{1}{4} \end{aligned}$$

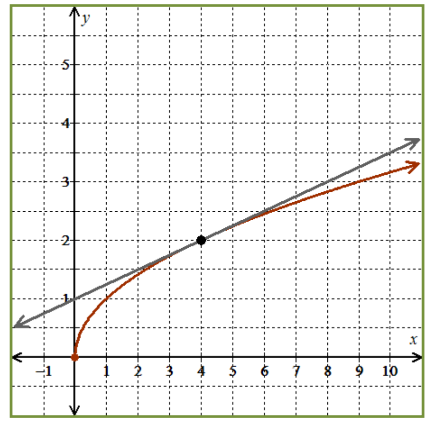


Figure 18.1: f

You should verify that the slope of the tangent line shown in Figure 18.1 is indeed $\frac{1}{4}$. You should also verify that the ***equation*** of the tangent line is $y=\frac{1}{4}x+1$.

Problem 18.1

Consider the function $g(x)=5-\sqrt{4-x}$.

18.1.1 Find the slope of the tangent line shown in Figure 18.2 using $m_{\tan}=\lim\limits_{h\to 0}\frac{g(3+h)-g(3)}{h}$. Show work consistent with that illustrated in example 18.1.

18.1.2 Use the line in Figure 18.2 to verify your answer to problem 18.1.1.

18.1.3 State the equation of the tangent line to g at 3.

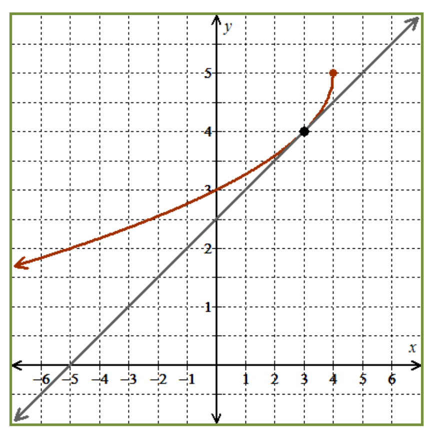


Figure 18.2: g

Activity 19

So far we've seen two applications of expressions of form $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$. It turns out that this expression is so important in mathematics, the sciences, economics, and many other fields that it deserves a name in and of its own right. We call the expression "***the first derivative of f at a***."

So far we've always fixed the value of a before making the calculation. There's no reason why we couldn't use a variable for a, make the calculation, and then replace the variable with specific values; in fact, it seems like this might be a better plan all around. This leads us to a definition of ***the first derivative function***.

Definition 19.1 – The First Derivative Function

If f is a function of x, then we define ***the first derivative function***, f', as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}.$$

The symbols $f'(x)$ are read aloud as "f prime of x" or "f prime at x."

As we've already seen, $f'(a)$ gives us the slope of the tangent line to f at a.

We've also seen that if s is a position function, then $s'(a)$ gives us the instantaneous velocity at a. It's not too much of a stretch to infer that the velocity function for s would be $v(t) = s'(t)$

Problem 19.1

A graph of the function $f(x) = \frac{3}{2-x}$ is shown in Figure 19.1 and the formula for $f'(x)$ is derived in Example 19.1.

19.1.1 Use the formula $f'(x) = \frac{3}{(2-x)^2}$ to calculate $f'(1)$ and $f'(5)$.

19.1.2 Draw onto Figure 19.1 a line through the point $(1,3)$ with a slope of $f'(1)$. Also draw a line though the point $(5,-1)$ with a slope of $f'(5)$. What are the names for the two lines you just drew? What are their equations?

19.1.3 Showing work consistent with that shown in example 19.1, find the formula for $g'(x)$ where $g(x) = \frac{5}{2x+1}$.

Example 19.1

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{3}{2-(x+h)}-\dfrac{3}{2-x}}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{3}{2-x-h}\cdot\dfrac{2-x}{2-x}-\dfrac{3}{2-x}\cdot\dfrac{2-x-h}{2-x-h}}{\dfrac{h}{1}}\\
&= \lim_{h\to 0}\left(\frac{6-3x-6+3x+3h}{(2-x-h)(2-x)}\cdot\frac{1}{h}\right)\\
&= \lim_{h\to 0}\frac{3h}{(2-x-h)(2-x)h}\\
&= \lim_{h\to 0}\frac{3}{(2-x-h)(2-x)}\\
&= \frac{3}{(2-x-0)(2-x)}\\
&= \frac{3}{(2-x)^2}
\end{aligned}$$

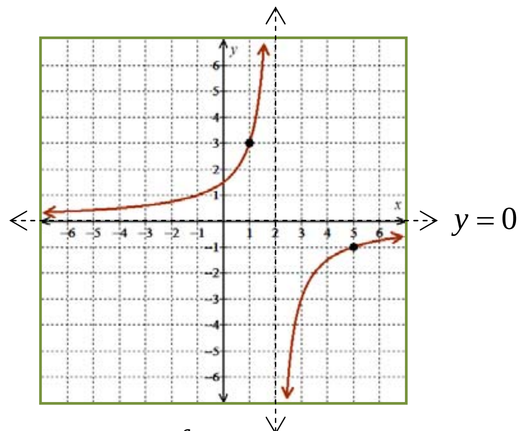


Figure 19.1: f

Problem 19.2

Suppose that the elevation of an object (measured in ft) is given by $s(t)=-16t^2+112t+5$ where t is the amount of time that has passed since the object was launched into the air (measured in s).

19.2.1 Use Equation 19.2 to find the formula for the velocity function associated with this motion. The first two lines of your presentation should be an exact copy of Equation 19.2.

Equation 19.2

$$\begin{aligned}
v(t) &= s'(t)\\
&= \lim_{h\to 0}\frac{s(t+h)-s(t)}{h}
\end{aligned}$$

19.2.2 Find the values of $v(2)$ and $v(5)$. What is the unit on each of these values? What do the values tell you about the motion of the object; don't just say "the velocity" – describe what is actually happening to the object 2 seconds and 5 seconds into its travel.

19.2.3 Use the velocity function to determine when the object reaches its maximum elevation. (Think about what must be true about the velocity at that instant.) Also, what is the common mathematical term for the point on the parabola $y = s(t)$ that occurs at that value of *t*?

19.2.4 Use Equation 19.3 to find the formula for $v'(t)$. The first line of your presentation should be an exact copy of Equation 19.3.

Equation 19.3	$v'(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$

19.2.5 What is the common name for the function $v'(t)$? Is its formula consistent with what you know about objects in freefall on Earth?

Problem 19.3

What is the constant slope of the function $w(x) = 12$? Verify this by using Definition 19.1 to find the formula for the function $w'(x)$.

Activity 20

We can think about the instantaneous velocity as being ***<u>the instantaneous rate of change</u>*** in position. In general, whenever you see the phrase "***<u>rate of change</u>***" you can assume that the rate of change at one instant is being discussed. When we want to discuss average rates of change over a time interval we always say "***<u>average rate of change</u>***."

In general, if *f* is any function, then **<u>*f*´(*a*) tells us the rate of change in *f* at *a*</u>**. Additionally, if *f* is an applied function with an input unit of i_{unit} and an output unit of f_{unit}, then the unit on $f'(a)$ is $\frac{f_{unit}}{i_{unit}}$. Please note that this unit loses all meaning if it is simplified in any way. Consequently, *we do not simplify derivative units in any way, shape, or form.*

For example, if $v(t)$ is the velocity of your car (measured in mi/hr) where *t* is the amount of time that has passed since you hit the road (measured in minutes), then the unit on $v'(t)$ is $\frac{\text{mi/hr}}{\text{min}}$.

Problem 20.1

Determine the unit for the first derivative function for each of the following functions. Remember, *we do not simplify derivative units in any way, shape, or form.*

20.1.1 $V(r)$ is the volume of a sphere (measured in ml) with radius r (measured in mm).

20.1.2 $A(x)$ is the area of a square (measured in ft^2) with sides of length x (measured in ft).

20.1.3 $V(t)$ is the volume of water in a bathtub (measured in gal) where t is the amount of time that has elapsed since the tub began to drain (measured in minutes).

20.1.4 $R(t)$ is the rate at which a bathtub is draining (measured in gal/min) where t is the amount of time that has elapsed since the tub began to drain (measured in minutes).

Problem 20.2

Akbar was given a formula for the function described in problem 20.1.3. Akbar did some calculations and decided that the value of $V'(20)$ was (without unit) 1.5. Nguyen took one look and Akbar's value and said "that's wrong." What is it about Akbar's value that caused Nguyen to dismiss it as wrong?

Problem 20.3

After a while Nguyen convinced Akbar that he was wrong, so Akbar set about doing the calculation over again. This time Akbar came up with a value of $-12,528$. Nguyen took one look at Akbar's value and declared "still wrong." What's the problem now?

Problem 20.4

Consider the function described in problem 20.1.4.

20.4.1 What would it mean if the value of $R(t)$ was zero for all $t > 0$.

20.4.2 What would it mean if the value of $R'(t)$ was zero for all $0 < t < 2.25$.

Functions, Derivatives, and Antiderivatives

Activity 21

Functions, derivatives, and antiderivatives have many entangled properties. For example, over intervals where the first derivative of a function is always positive, we know that the function itself is always increasing. (Do you understand why?)

Many of these relationships can be expressed graphically. Consequently, it is imperative that you fully understand the meaning of some commonly used graphical expressions. These expressions are loosely defined in Table 21.1.

Table 21.1: Some Common Graphical Phrases

The function is positive This means that the vertical-coordinate of the point on the function is positive. As such, a function is positive whenever it lies above the horizontal axis.	**The function is negative** This means that the vertical-coordinate of the point on the function is negative. As such, a function is negative whenever it lies below the horizontal axis.
The function is increasing This means that the vertical-coordinate of the function consistently increases as you move along the curve from left to right. Linear functions with positive slope are always increasing.	**The function is decreasing** This means that the vertical-coordinate of the function consistently decreases as you move along the curve from left to right. Linear functions with negative slope are always decreasing.
The function is concave up A function is concave up at a if the tangent line to the function at a lies below the curve. An upward opening parabola is everywhere concave up.	**The function is concave down** A function is concave down at a if the tangent line to the function at a lies above the curve. A downward opening parabola is everywhere concave down.

Problem 21.1

Answer each of the following questions in reference to the function shown in Figure 21.1. Each answer is an interval (or intervals) along the x-axis. Use interval notation when expressing your answers. Make each interval as wide as possible; that is, do not break an interval into pieces if the interval does not need to be broken up. Assume that the slope of the function is constant on $(-\infty,-5)$, $(3,4)$, and $(4,\infty)$.

21.1.1 Over what intervals is the function positive?

21.1.2 Over what intervals is the function negative?

21.1.3 Over what intervals is the function increasing?

21.1.4 Over what intervals is the function decreasing?

21.1.5 Over what intervals is the function concave up?

21.1.6 Over what intervals is the function concave down?

21.1.7 Over what intervals is the function linear?

21.1.8 Over what intervals is the function constant?

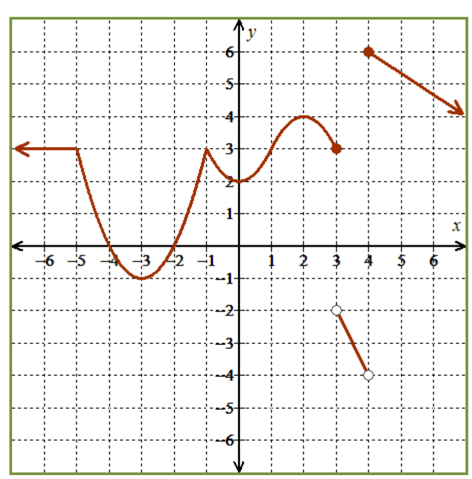


Figure 21.1: f

Activity 22

Much information about a function's first derivative can be gleaned simply by looking at a graph of the function. In fact, a person with good visual skills can "see" the graph of the derivative while looking at the graph of the function. This activity focuses on helping you develop that skill.

Problem 22.1

A parabolic function is shown in Figure 22.1. Each question in this problem is in reference to that function.

22.1.1 Several values of the function g' are given in Table 22.1. For each given value draw a nice long line segment at the corresponding point on g whose slope is equal to the value of g'. If we think of these line segments as actual lines, what do we call the lines?

22.1.2 What is the value of g' at 1? How do you know? Go ahead and enter that value into Table 22.1.

22.1.3 The function g is symmetric across the line $x=1$; that is, if we move equal distance to the left and right from this line the corresponding y-coordinates on g are always equal. Notice that the slopes of the tangent lines are "equal but opposite" at points that are equally removed from the axis of symmetry; this is reflected in the values of $g'(-1)$ and $g'(3)$. Use the idea of "equal but opposite slope equidistance from the axis of symmetry" to complete Table 22.1.

22.1.4 Plot the points from Table 22.1 onto Figure 22.2 and connect the dots. Determine the formula for the resultant linear function.

22.1.5 The formula for g is $g(x)=-.5x^2+x+5.5$. Use Definition 19.1 to determine the formula for $g'(x)$.

22.1.6 The line you drew onto Figure 22.2 is ***not*** a tangent line to g. Just what exactly is this line?

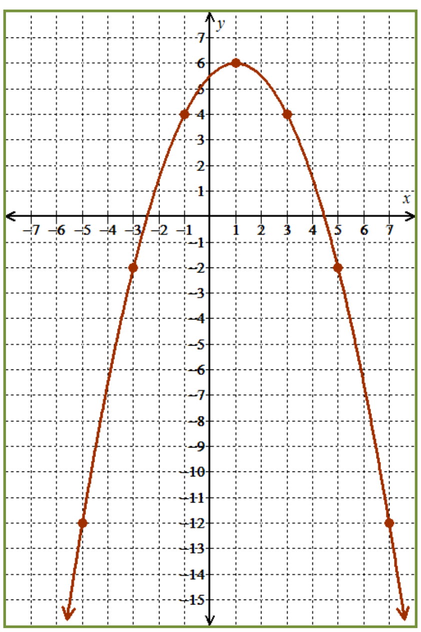


Figure 22.1: g

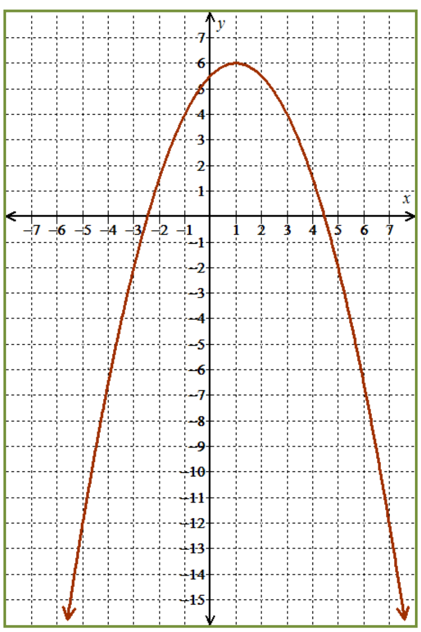


Figure 22.2: g

Table 22.1: $y=g'(x)$

x	y
-5	6
-3	
-1	2
1	
3	-2
5	-4
7	

Problem 22.2
A function f is shown in Figure 22.3 and the corresponding first derivative function f' is shown in Figure 22.4. Answer each of the following questions referencing these two functions.

22.2.1 Draw the tangent line to f at the three points indicated in Figure 22.3 *after first using the graph of* f' *to determine the exact slope of the respective tangent lines.*

22.2.2 Write a sentence relating the slope of the tangent line to f with the corresponding y-coordinate on f'.

22.2.3 Copy each of the following phrases onto your paper and supply the words or phrases that correctly complete each sentence.

- Over the interval where f' is always negative f is always __________.
- Over the intervals where f' is always positive f is always ____________.
- Over the interval where f' is always increasing f is always ___________.
- Over the interval where f' is always decreasing f is always ___________.

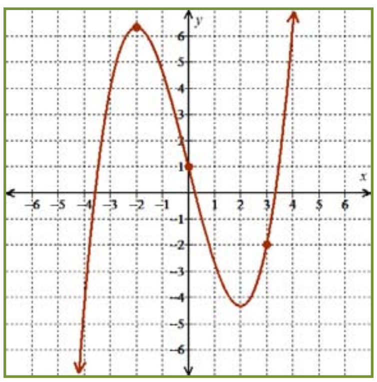


Figure 22.3: f

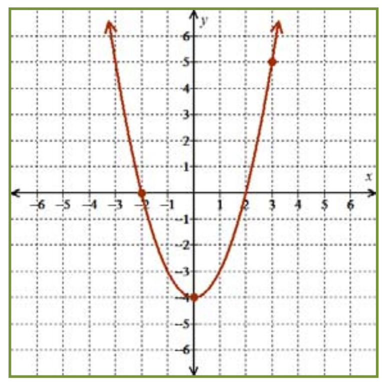


Figure 22.4: f'

Problem 22.3
In each of figures 22.5 and 22.6 a function (the thin curve) is given; both of these functions are symmetric about the y-axis. The first derivative of each function (the thick curves) have been drawn over the interval $(0,7)$. Use the given portion of the first derivative together with the symmetry of the function to help you draw each first derivative over the interval $(-7,0)$.

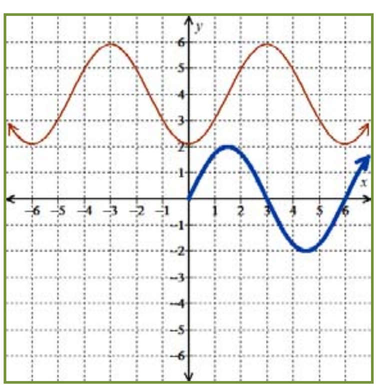


Figure 22.5: g and g'

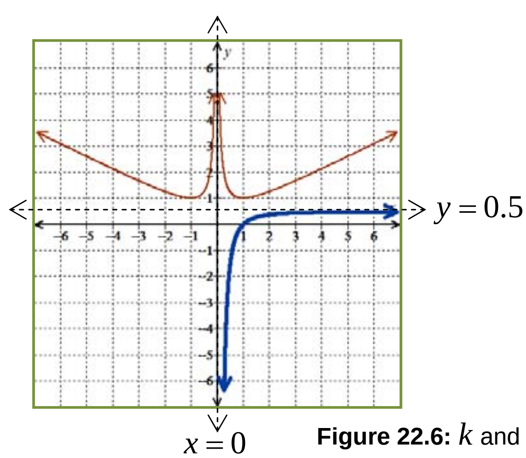


Figure 22.6: k and k'

Problem 22.4

A graph of the function $y = \frac{1}{x}$ is shown in Figure 22.7; call this function *f*.

22.4.1 Except at 0, there is something that is always true about the value of f'; what is the common trait?

22.4.2 Use Definition 19.1 to find the formula for $f'(x)$.

22.4.3 Does the formula for $f'(x)$ support your answer to problem 22.4.1?

22.4.4 Use the formula for $f'(x)$ to determine the horizontal and vertical asymptotes for the graph of $y = f'(x)$.

22.4.5 Keeping it simple, draw onto Figure 22.8 a curve with the asymptotes found in problem 22.4.4 and the property determined in problem 22.4.1. Does the curve you drew have the properties you would expect to see in the first derivative of *f*? For example, *f* is concave down over $(-\infty, 0)$ and concave up over $(0, \infty)$; what are the corresponding differences in the behavior of f' over those two intervals?

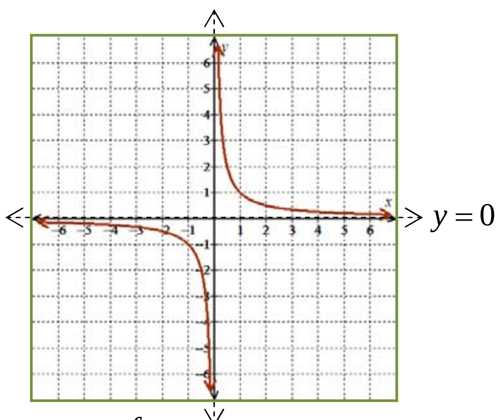


Figure 22.7: f

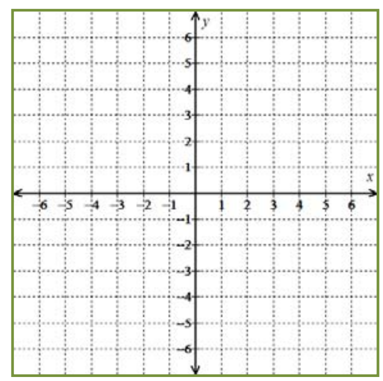


Figure 22.8: f'

Problem 22.5

A graph of the function g is shown in Figure 22.9. The absolute minimum value ever obtained by g' is -3. With that in mind, draw g' onto Figure 22.10. Make sure that you draw and label any and all necessary asymptotes. Make sure that your graph of g' adequately reflects the symmetry in the graph of g.

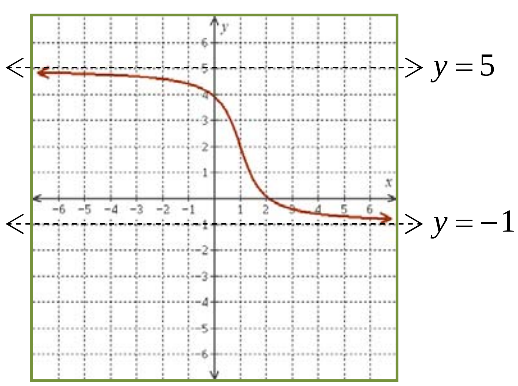


Figure 22.9: g

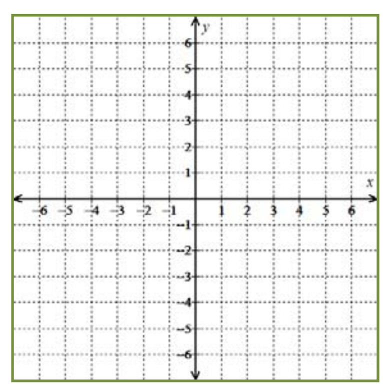


Figure 22.10: g'

Problem 22.6

A function, w, is shown in Figure 22.11. A larger version of Figure 22.11 is available in Appendix B (page B3). Answer each of the following questions in reference to this function.

22.6.1 An inflection point is a point where the function is continuous and the concavity of the function changes. The inflection points on w occur at 2, 3.25, and 6. With that in mind, over each interval stated in Table 22.3 exactly two of the words in Table 22.2 apply to w'. Complete Table 22.2 with the appropriate pairs of words.

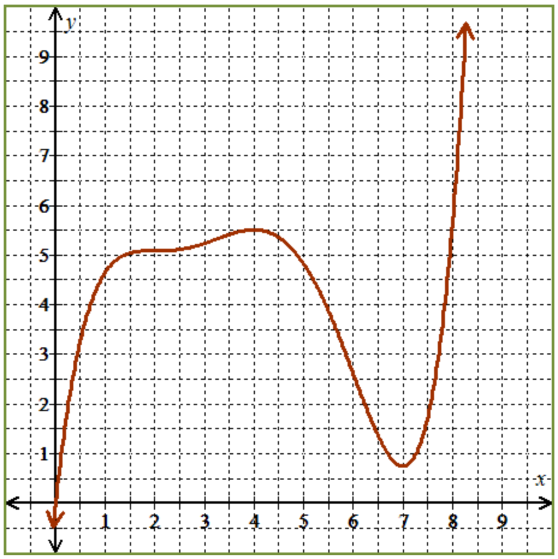


Figure 22.11: w

Table 22.2: Properties

positive
negative
increasing
decreasing

Table 22.3: Properties of w'

Interval	Properties
$(-\infty, 2)$	
$(2, 3.25)$	
$(3.25, 4)$	
$(4, 6)$	
$(6, 7)$	
$(7, \infty)$	

22.6.2 In Table 22.4, three possible values are given for w' at several values of x. In each case, one of the values is correct. Use tangent lines to w to determine each of the correct values. (This is where you probably want to use the graph on page B3.)

22.6.3 The value of w' is the same at 2, 4, and 7. What is this common value?

22.6.4 Put it all together and draw w' onto Figure 22.12.

Table 22.4: Choose the correct values for w'

x	Proposed values
0	$2/3$ or $8/3$ or $28/3$
1	$1/2$ or $3/2$ or $5/2$
3	$1/3$ or 1 or 3
5	$-1/2$ or -1 or $-3/2$
6	$-4/3$ or $-8/3$ or -4
8	1 or 6 or 12

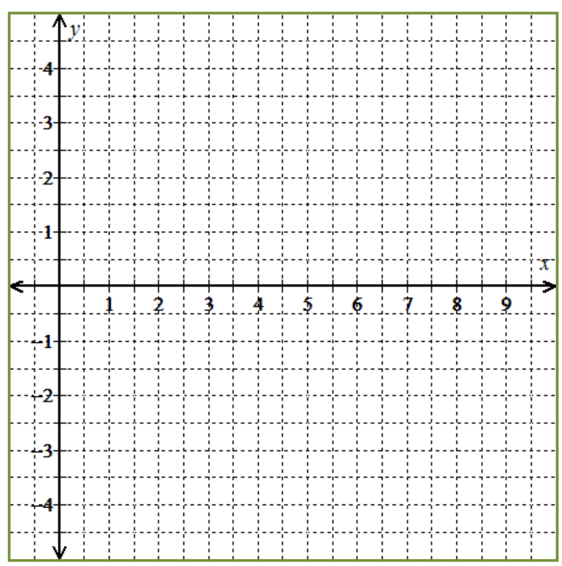


Figure 22.12: w'

Activity 23

A function is said to be ***nondifferentiable*** at any value its first derivative is undefined. There are three graphical behaviors that lead to non-differentiability.

- f is nondifferentiable at a if f is discontinuous at a.
- f is nondifferentiable at a if the slope of f is different from the left and right at a.
- f is nondifferentiable at a if f has a vertical tangent line at a.

Problem 23.1

Consider the function k shown in Figure 23.1.

23.1.1 There are four values where k is nondifferentiable; what are these values?

23.1.2 Draw k' onto Figure 23.2.

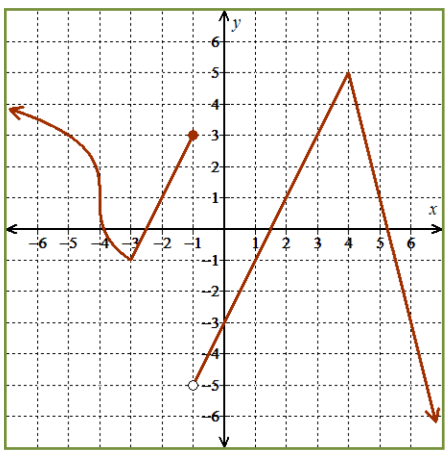


Figure 23.1: k

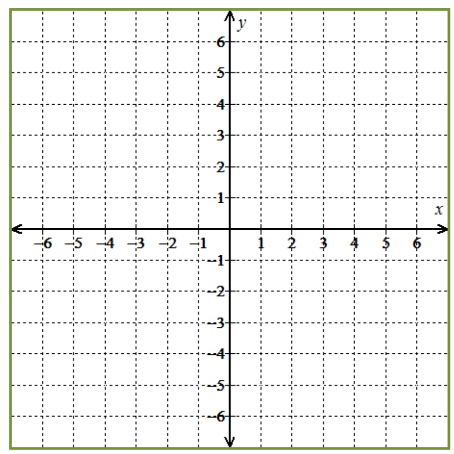


Figure 23.2: k'

Problem 23.2

Consider the function g shown in Figure 23.3.

23.2.1 g' has been drawn onto Figure 23.4 over the interval $(-5,-2.5)$. Use the piece-wise symmetry and periodic behavior of g to help you draw the remainder of g' over $(-7,7)$

23.2.2 What six syllable word applies to g at -5, 0, and 5?

23.2.3 What five syllable and six syllable words apply to g' at -5, 0, and 5?

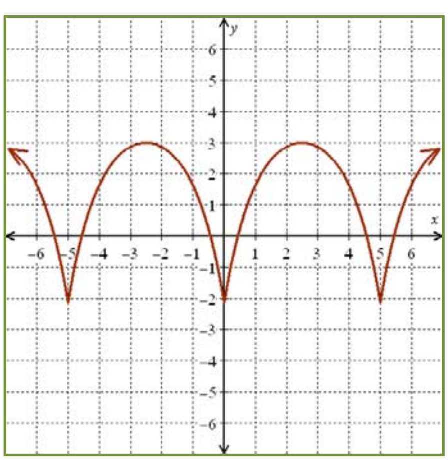


Figure 23.3: g

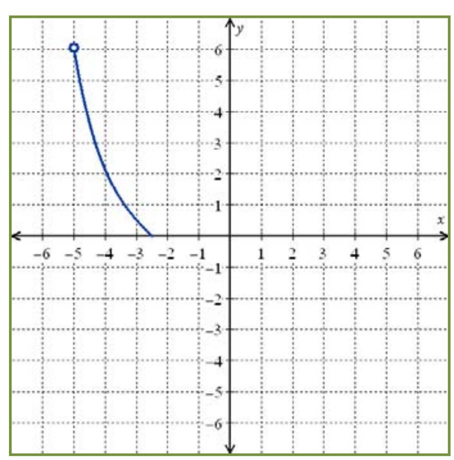


Figure 23.4: g'

Activity 24

Seeing as the first derivative of f is a function in its own right, f' must have its own first derivative. The first derivative of f' is the second derivative of f and is symbolized as f'' (f double-prime). Likewise, f''' (f triple-prime) is the first derivative of f'', the second derivative of f', and the third derivative of f.

Table 24.1: f' and f

When f' is …	f is …
Positive	Increasing
Negative	Decreasing
Constantly Zero	Constant
Increasing	Concave Up
Decreasing	Concave Down
Constant	Linear

Table 24.2: f'' and f'

When f'' is …	f' is …
Positive	Increasing
Negative	Decreasing
Constantly Zero	Constant
Increasing	Concave Up
Decreasing	Concave Down
Constant	Linear

Table 24.3: f''' and f''

When f''' is …	f'' is …
Positive	Increasing
Negative	Decreasing
Constantly Zero	Constant
Increasing	Concave Up
Decreasing	Concave Down
Constant	Linear

All of the graphical relationships you've established between f and f' work their way down the derivative chain; this is illustrated in tables 24.1, 24.2, and 24.3.

Problem 24.1

Extrapolating from tables 24.1 and 24.2, what must be true about f over intervals where f'' is, respectively, positive, negative, and constantly zero?

Problem 24.2

A function, g, and its first three derivatives are shown in figures 24.1-24.4, although not in that order. Determine which curve is which function (g, g', g'', and g''').

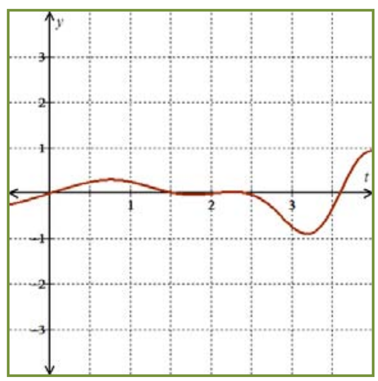


Figure 24.1

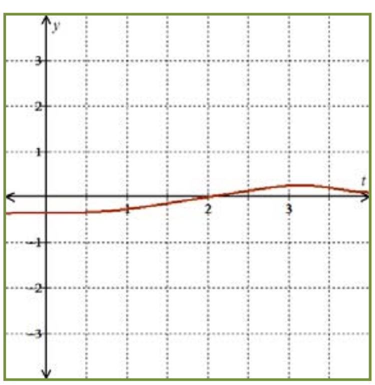


Figure 24.2

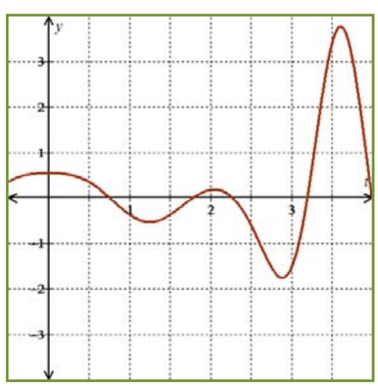


Figure 24.3

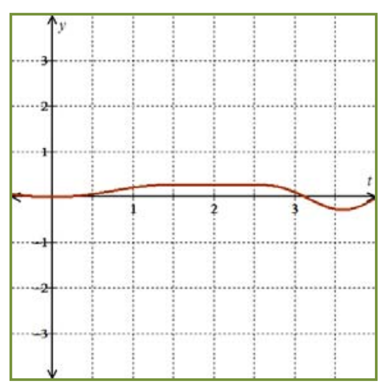


Figure 24.4

Problem 24.3

Three containers are shown in figures 24.5-24.7. Each of the following questions are in reference to these containers.

24.3.1 Suppose that water is being poured into each of the containers at a constant rate. Let h_5, h_6, and h_7 be the heights (measured in cm) of the liquid in containers 24.5-24.7, respectively, t seconds after the water began to fill the containers. What would you expect the sign to be on the second derivative functions h_5'', h_6'', h_7'' while the containers are being filled? (Hint: Think about the shape of the curves $y = h_5(t)$, $y = h_6(t)$, and $y = h_7(t)$.)

24.3.2 Suppose that water is being drained from each of the containers at a constant rate. Let h_5, h_6, and h_7 be the heights (measured in cm) of the liquid remaining in the containers t seconds after the water began to drain. What would you expect the sign to be on the second derivative functions h_5'', h_6'', h_7'' while the containers are being drained?

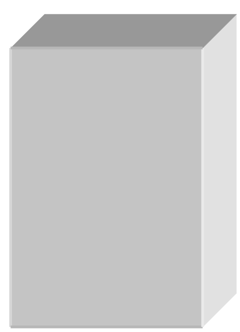


Figure 24.5

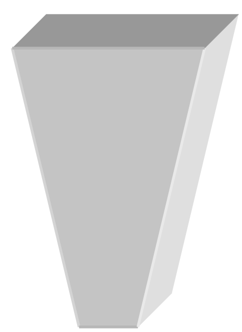


Figure 24.6

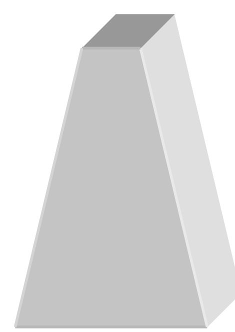


Figure 24.7

Problem 24.4

During the recession of 2008-2009, the total number of employed Americans decreased every month. One month a talking head on the television made the observation that "at least the second derivative was positive this month." Why was it a good thing that the second derivative was positive?

Problem 24.5

During the early 1980s the problem was inflation. Every month the average price for a gallon of milk was higher than the month before. Was it a good thing when the second derivative of this function was positive? Explain.

Activity 25

The derivative continuum can be expressed backwards as well as forwards. When you move from function to function in the reverse direction the resultant functions are called antiderivatives and the process is called antidifferentiation. These relationships are shown in figures 25.1 and 25.2.

$$f \xrightarrow{\text{differentiate}} f' \xrightarrow{\text{differentiate}} f'' \xrightarrow{\text{differentiate}} f''' \xrightarrow{\text{differentiate}} \ldots$$

Figure 25.1: Differentiating

$$\ldots \xrightarrow{\text{antidifferentiate}} f''' \xrightarrow{\text{antidifferentiate}} f'' \xrightarrow{\text{antidifferentiate}} f' \xrightarrow{\text{antidifferentiate}} f \xrightarrow{\text{antidifferentiate}} F$$

Figure 25.2: Antidifferentiating

There are (at least) two important differences between the differentiation chain and the antidifferentiation chain (besides their reversed order).

- When you differentiate, the resultant function is unique. When you antidifferentiate, you do not get a unique function - you get a family of functions; specifically, you get a set of parallel curves.
- We introduce a new function in the antidifferentiation chain. We say that F is an antiderivative of f. This is where we stop in that direction; we do not have a variable name for an antiderivative of F.

Since F is considered an antiderivative of f, it must be the case that f is the first derivative of F. Hence we can add F to our derivative chain resulting in Figure 25.3

$$F \xrightarrow{\text{differentiate}} f \xrightarrow{\text{differentiate}} f' \xrightarrow{\text{differentiate}} f'' \xrightarrow{\text{differentiate}} f''' \xrightarrow{\text{differentiate}} \ldots$$

Figure 25.3: Differentiating

Problem 25.1

Each of the linear functions in Figure 25.4 have the same first derivative function.

25.1.1 Draw this common first derivative function onto Figure 25.4 and label it g.

25.1.2 Each of the given lines Figure 25.4 is called what in relation to g?

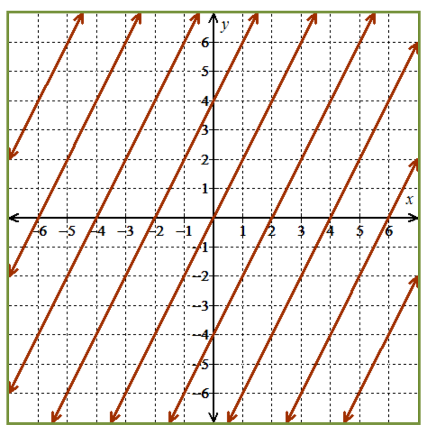


Figure 25.4

Problem 25.2

The function f is shown in Figure 25.5. Reference this function in the following questions.

25.2.1 At what values of x is f nondifferentiable?

25.2.2 At what values of x are antiderivatives of f nondifferentiable?

25.2.3 Draw onto Figure 25.6 the ***continuous*** antiderivative of f that passes through the point $(-3,1)$. Please note that every antiderivative of f increases exactly one unit over the interval $(-3,-2)$.

25.2.4 Because f is not continuous, there are other antiderivatives of f that pass through the point $(-3,1)$. Specifically, antiderivatives of f may or may not be continuous at -1. Draw onto figures 25.7 and 25.8 different antiderivatives of f that pass through the point $(-3,1)$.

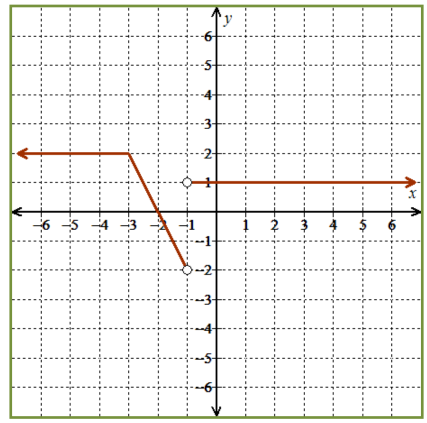


Figure 25.5: f

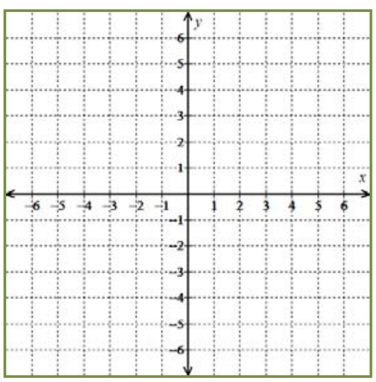


Figure 25.6

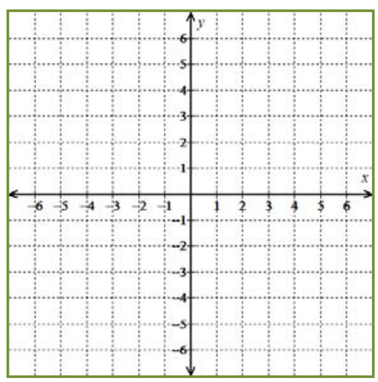


Figure 25.7

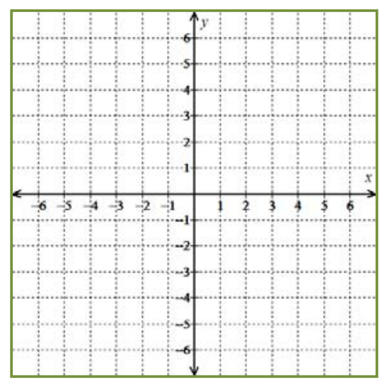


Figure 25.8

Problem 25.3

The function $y=\sin(x)$ is an example of a periodic function. Specifically, the function has a period of 2π because over any interval of length 2π the behavior of the function is exactly the same as it was the previous interval of length 2π. A little more precisely, $\sin(x+2\pi)=\sin(x)$ regardless of the value of x.

Jasmine was thinking and told her lab assistant that derivatives and antiderivatives of periodic functions must also be periodic. Jasmine's lab assistant told her that she was half right. Which half did Jasmine have correct? Also, draw a function that illustrates that the other half of Jasmine's statement is not correct.

Problem 25.4

Consider the function g shown in Figure 25.9.

25.4.1 Let G be an antiderivative of g. Suppose that G is continuous on $[-6,6]$, $G(-6)=-3$, and that the greatest value G ever achieves is 6. Draw G onto Figure 25.10.

25.4.2 At what values of t is G nondifferentiable?

25.4.3 At what values of t is g nondifferentiable?

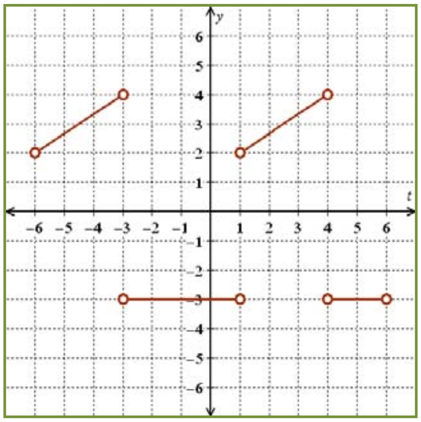


Figure 25.9 g

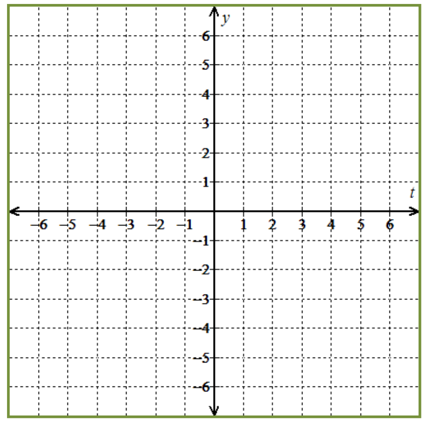


Figure 25.10 G

Problem 25.5

Answer the following question in reference to a ***continuous*** function g whose first derivative is shown in Figure 25.11. You do not need to state how you made your determination; just state the interval(s) or values of x that satisfy the stated property.

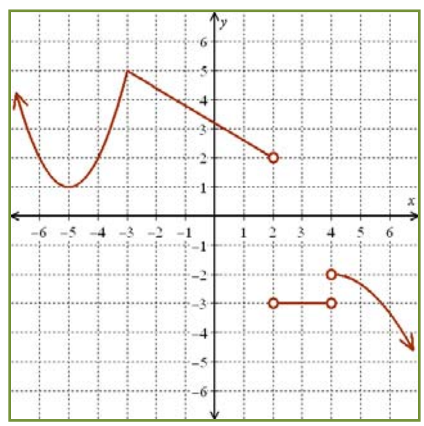


Figure 25.11 g'

Note: The correct answer to one or more of these questions may be "There is no way of knowing."

25.5.1 Over these intervals, g'' is positive and increasing.

25.5.2 At these values of x, g is nondifferentiable.

25.5.3 Over these intervals, g is never negative.

25.5.4 At these values of x, every antiderivative of g is nondifferentiable.

25.5.5 Over these intervals, the value of g'' is constant.

25.5.6 Over these intervals, g is linear.

25.5.7 Over these intervals, antiderivatives of g are linear.

25.5.8 Over these intervals, g''' is never negative.

Activity 26

When given a function formula, we often find the first and second derivative formulas to determine behaviors of the given function. In a later lab we will use the first and second derivative formulas to help us graph a function given the formula for the function. One thing we do with the derivative formulas is determine where they are positive, negative, zero, and undefined. This helps us determine where the given function is increasing, decreasing, concave up, concave down, and linear.

Problem 26.1

26.1.1 Draw onto Figure 26.1 a ***continuous*** function f that has a horizontal tangent line at the point $(-1,2)$ along with the properties stated in Table 26.1.

26.1.2 Given the conditions stated in problem 26.1.1, does it have to be the case that $f''(-1)=0$?

26.1.3 Is f increasing at -1? How do you know?

26.1.4 Can a continuous, everywhere differentiable function satisfy the properties stated in Table 26.1 and not have a slope of zero at -1? Draw a picture that supports your answer.

Table 26.1: Signs on f' and f''

Interval	f'	f''
$(-\infty,-1)$	Positive	Negative
$(-1,\infty)$	Positive	Positive

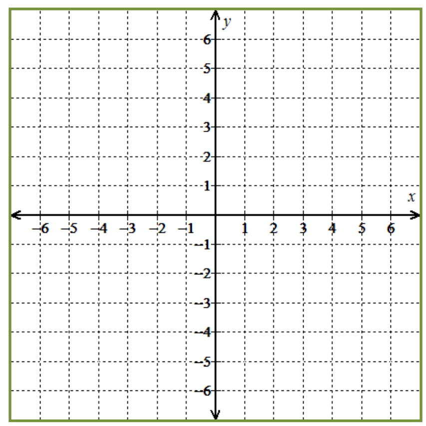


Figure 26.1 f

Problem 26.2

26.2.1 Draw onto Figure 26.2 a ***continuous*** function g that has a vertical tangent line at the point $(-1,2)$ along with the properties stated in Table 26.2.

26.2.2 Given the conditions stated in problem 26.2.1, does it have to be the case that $g''(-1)$ is undefined?

26.2.3 Is g increasing at -1? How do you know?

26.2.4 Can a continuous, everywhere differentiable function satisfy the properties stated in Table 26.2 and not have a vertical tangent line at -1? Draw a picture that supports your answer.

Table 26.2: Signs on g' and g''

Interval	g'	g''
$(-\infty,-1)$	Positive	Positive
$(-1,\infty)$	Positive	Negative

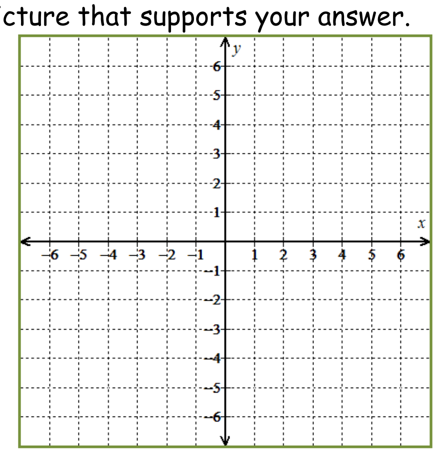


Figure 26.2 g

Problem 26.3

26.3.1 Draw onto Figure 26.3 a ***continuous*** function k that passes through the point $(-1,2)$ and also satisfies the properties stated in Table 26.3.

26.3.2 At what values of x is k nondifferentiable?

Table 26.3: Signs on k' and k''

Interval	k'	k''
$(-\infty,-1)$	Negative	Positive
$(-1,\infty)$	Positive	Negative

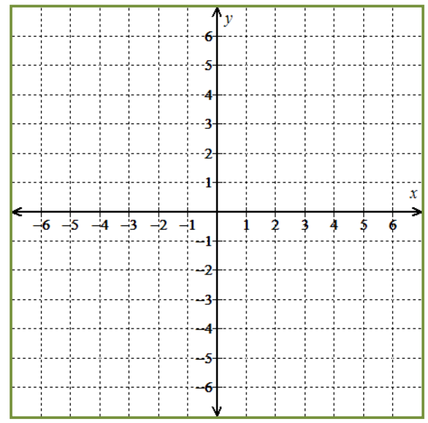


Figure 26.3 k

Problem 26.4

You should know by now that the first derivative continually increases over intervals where the second derivative is constantly positive and that the first derivative continually decreases over intervals where the second derivative is constantly negative. A person might infer from this that a function changes more and more quickly over intervals where the second derivative is constantly positive and that a function changes more and more slowly over intervals where the second derivative is constantly negative. We are going to explore that idea in this problem.

26.4.1 Suppose that $V(t)$ is the volume of water in an ice cube (ml) where t is the amount of time that has passed since noon (measure in minutes). Suppose that $V'(6)=0\ ^{\text{ml}}/_{\text{min}}$ and that $V''(t)$ has a constant value of $-.3\,\frac{^{\text{ml}}/_{\text{min}}}{\text{min}}$ over the interval $[6,11]$. What is the value of $V'(11)$? When is V changing more quickly, at 12:06 pm or at 12:11 pm?

26.4.2 Referring to the function V in problem 26.4.1, what would the shape of V be over the interval $[6,11]$? (Choose from options a – d.)

26.4.3 Again referring to options a-d, which functions are changing more and more rapidly from left to right and which functions are changing more and more slowly from left to right? Which functions have positive second derivatives and which functions have negative second derivatives? Do the functions with positive second derivative values both change more and more quickly from left to right?

26.4.4 Consider the signs on both the first and second derivatives in options a-d. Is there something that the two functions that change more and more quickly have in common that is different in the functions that change more and more slowly?

Problem 26.5

Resolve each of the following disputes.

26.5.1 One day Sara and Jermaine were working on an assignment. One question asked them to draw a function over the domain $(-2,\infty)$ with the properties that the function is always increasing and always concave down. Sara insisted that the curve must have a vertical asymptote at -2 and Jermaine insisted that the function must have a horizontal asymptote somewhere. Were either of these students correct?.

26.5.2 The next question Sara and Jermaine encountered described the same function with the added condition that the function is never positive. Sara and Jermaine made the same contentions about asymptotes. Is one of them now correct?

26.5.3 At another table Pedro and Yoshi were asked to draw a continuous curve that, among other properties, was never concave up. Pedro said "OK, so the curve is always concave down" to which Yoshi replied "Pedro, you need to open your mind to other possibilities." Who's right?

26.5.4 In the next problem Pedro and Yoshi were asked to draw a function that is everywhere continuous and that is concave down at every value of x **except** 3. Yoshi declared "impossible" and Pedro responded "have some faith, Yosh-man." Pedro then began to draw. Is it possible that Pedro came up with such a function?

Problem 26.6

Determine the correct answer to each of the following questions. Pictures of the situation may help you determine the correct answers.

26.6.1 Which of the following propositions is true? If a given proposition is not true, draw a graph that illustrates its untruth.

26.6.1a If the graph of f has a vertical asymptote, then the graph of f' must also have a vertical asymptote.

26.6.1b If the graph of f' has a vertical asymptote, then the graph of f must also have a vertical asymptote.

26.6.2 Suppose that the function f is everywhere continuous and concave down. Suppose further that $f(7)=5$ and $f'(7)=3$. Which of the following is true?

a. $f(9)<11$ b. $f(9)=11$ c. $f(9)>11$

d. There is not enough information to determine the relationship between $f(9)$ and 11.

Derivative Formulas

Activity 27

While the primary focus of this lab is to help you develop shortcut skills for finding derivative formulas, there are inevitable notational issues that must be addressed. It turns out that the latter issue is the one we are going to address first.

If $y = f(x)$, we say that ***<u>the derivative of y with respect to x</u>*** is equal to $f'(x)$. Symbolically we write: $\frac{dy}{dx} = f'(x)$

> While the symbol $\frac{dy}{dx}$ certainly ***looks*** like a fraction, ***<u>it is not a fraction</u>***. The symbol is Leibniz notation for the first derivative of y with respect to x. The short way of reading the symbol aloud is "d-y-d-x" (don't enunciate the dashes).

If $z = g(t)$, we say that the ***<u>the derivative of z with respect to t</u>*** is equal to $g'(t)$. Symbolically we write: $\frac{dz}{dt} = g'(t)$. (Read aloud as "d-z-d-t equals g-prime of t.")

Problem 27.1
Take the derivative of both sides of each equation with respect to the independent variable as indicated in the function notation. Write and say the derivative using Leibniz notation on the left side of the equal sign and function notation on the right side of the equal sign. Make sure that every one in your group says at least one of the derivative equations aloud using both the formal reading and informal reading of the Leibniz notation.

27.1.1 $y = k(t)$ **27.1.2** $V = f(r)$ **27.1.3** $T = g(P)$

Activity 28

$\frac{dy}{dx}$ is ***the name of a derivative*** in the same way that $f'(x)$ is the name of a derivative. We need a different symbol that tells us to ***<u>take</u>*** the derivative of a given expression (in the same way that we have symbols that tell us to take a square root, sine, or logarithm of an expression).

The symbol $\frac{d}{dx}$ is used to tell us to ***take the derivative with respect to x of something***. **<u>The symbol itself is an incomplete phrase</u>** in the same way that the symbol $\sqrt{}$ is an incomplete phrase; in both cases we need to indicate the object to be manipulated – what number or formula are we taking the square root of? … what number or formula are we differentiating?

One thing you can do to help you remember the difference between the symbols $\frac{dy}{dx}$ and $\frac{d}{dx}$ is to get in the habit of always writing grouping symbols after $\frac{d}{dx}$. In this way the symbols $\frac{d}{dx}(\sin(x))$ mean "the derivative with respect to x of the sine of x." Similarly, the symbols $\frac{d}{dt}(t^2)$ mean "the derivative with respect to t of t-squared."

Problem 28.1

Write the Leibniz notation for each of the following expressions.

28.1.1 The derivative with respect to β of $\cos(\beta)$.

28.1.2 The derivative with respect to x of $\frac{dy}{dx}$.

28.1.3 The derivative with respect to t of $\ln(x)$. (Yes, we will do such things.)

28.1.4 The derivative of z with respect to x.

28.1.5 The derivative with respect to t of $g(8)$. (Yes, we will also do such things.)

Activity 29

The first differentiation rule we are going to explore is called ***the power rule of differentiation***.

Equation 29.1	$\frac{d}{dx}(x^n) = n\,x^{n-1}$ for $n \neq 0$

When n is a positive integer, it is fairly easy to establish this rule using Definition 19.1. The proof of the rule gets a little more complicated when n is negative, fractional, or irrational. For purposes of this lab, we are going to just accept the rule as valid.

This rule is one you just "do in your head" and then write down the result. Three examples of what you would be expected to write when differentiating power functions are shown below.

Given Function	$y = x^7$	$f(t) = \sqrt[3]{t^7}$	$z = \frac{1}{y^5}$
What you should ***think*** ***or*** ***write*** (***as necessary***)	$y = x^7$	$f(t) = t^{7/3}$	$z = y^{-5}$
What you should write	$\frac{dy}{dx} = 7x^6$	$f'(t) = \frac{7}{3}t^{4/3}$ $= \frac{7}{3}\sqrt[3]{t^4}$	$\frac{dz}{dy} = -5y^{-6}$ $= -\frac{5}{y^6}$

Notice that the type of notation used when naming the derivative is dictated by the manner in which the original function is expressed. For example, $y = x^7$ is telling us the relationship between two variables; in this situation we name the derivative using the notation $\frac{dy}{dx}$. On the other hand, function notation is being used to name the rule in $f(t) = \sqrt[3]{t^7}$; in this situation we name the derivative using the function notation $f'(t)$.

Problem 29.1

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

29.1.1 $f(x) = x^{43}$ **29.1.2** $z = \frac{1}{t^7}$ **29.1.3** $P = \sqrt[5]{t^2}$ **29.1.4** $h(x) = \frac{1}{\sqrt{x}}$

Activity 30

The next rule you are going to practice is the ***constant factor rule of differentiation***. This is another rule you do in your head.

Equation 30.1	$\frac{d}{dx}\left(k\, f(x)\right) = k \cdot \frac{d}{dx}\left(f(x)\right)$ for $k \in \Re$

In the following examples and problems several derivative rules are used that are shown in Appendix C (pages C5 and C6). While working this lab you should refer to those rules; you may want to cover up the chain rule and implicit derivative columns this week! You should make a goal of having all of the basic formulas memorized within a week.

Given Function	$f(x) = 6x^9$	$f(\theta) = -2\cos(\theta)$	$z = 2\ln(x)$
What you should write	$f'(x) = 54x^8$	$f'(\theta) = 2\sin(\theta)$	$\frac{dz}{dx} = \frac{2}{x}$

Problem 30.1

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

30.1.1 $z = 7t^4$ **30.1.2** $P(x) = -7\sin(x)$ **30.1.3** $h(t) = \frac{1}{3}\ln(t)$

30.1.4 $z(x) = \pi\tan(x)$ **30.1.5** $P = \frac{-8}{t^4}$ **30.1.6** $T = 4\sqrt{t}$

Activity 31

When an expression is divided by the constant k, we can think of the expression as being multiplied by the fraction $\frac{1}{k}$. In this way, the constant factor rule of differentiation can be applied when a formula is multiplied ***or*** divided by a constant.

Given Function	$f(x)=\frac{x^4}{8}$	$y=\frac{3\tan(\alpha)}{2}$	$z=\frac{\ln(y)}{3}$
What you should ***think***	$f(x)=\frac{1}{8}x^4$	$y=\frac{3}{2}\tan(\alpha)$	$z=\frac{1}{3}\ln(y)$
What you should write	$f'(x)=\frac{1}{2}x^3$	$\frac{dy}{d\alpha}=\frac{3}{2}\sec^2(\alpha)$	$\frac{dz}{dy}=\frac{1}{3y}$

Problem 31.1

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

31.1.1 $z(t)=\frac{\sin^{-1}(t)}{6}$

31.1.2 $V(r)=\frac{\pi r^3}{3}$

31.1.3 $f(r)=\frac{G m_1 m_2}{r^2}$

G, m_1, and m_2 are constants.

Activity 32

When taking the derivative of two or more terms, you can take the derivatives term by term and insert plus or minus signs as appropriate. Collectively we call this ***<u>the sum and difference rules of differentiation</u>***.

Equation 32.1	$\frac{d}{dx}\big(f(x)\pm g(x)\big)=\frac{d}{dx}\big(f(x)\big)\pm\frac{d}{dx}\big(g(x)\big)$

In the following examples and problems we introduce linear and constant terms into the functions being differentiated.

Equation 32.2	$\frac{d}{dx}(k)=0 \text{ for } k\in\mathbb{R}$	**Equation 32.3**	$\frac{d}{dx}(kx)=k \text{ for } k\in\mathbb{R}$

Problem 32.1

Explain why both the constant rule (Equation 32.2) and linear rule (Equation 32.3) are "obvious."

Hint: Think about the graphs $y=k$ and $y=kx$. What does a first derivative tell you about a graph?

A couple of examples using the sum and difference rule are shown below.

Given Function	$y = 4\sqrt[5]{t^6} - \dfrac{1}{6\sqrt{t}} + 8t$	$P(\gamma) = \dfrac{\sin(\gamma) - \cos(\gamma)}{2} + 4$
What you should ***think or write*** **(*as necessary*)**	$y = 4t^{6/5} - \dfrac{1}{6}t^{-1/2} + 8t$	$P(\gamma) = \dfrac{1}{2}\sin(\gamma) - \dfrac{1}{2}\cos(\gamma) + 4$
What you should definitely write	$\dfrac{dy}{dt} = \dfrac{24}{5}t^{1/5} + \dfrac{1}{12}t^{-3/2} + 8$ $= \dfrac{24\sqrt[5]{t}}{5} + \dfrac{1}{12\sqrt{t^3}} + 8$	$P'(\gamma) = \dfrac{1}{2}\cos(\gamma) + \dfrac{1}{2}\sin(\gamma)$

Problem 32.2

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

32.2.1 $T = \sin(t) - 2\cos(t) + 3$

32.2.2 $k(\theta) = \dfrac{4\sec(\theta) - 3\csc(\theta)}{4}$

32.2.3 $r(x) = \dfrac{x}{5} + 7$

32.2.4 $r = \dfrac{2}{3\sqrt[3]{x}} - \dfrac{\ln(x)}{9} + \ln(2)$

Activity 33

The next rule we are going to explore is called ***the product rule of differentiation***. We use this rule when there are ***two or more variable factors*** in the expression we are differentiating. (Remember, we already have the constant factor rule to deal with two factors when one of the two factors is a constant.)

Equation 33.1 $$\frac{d}{dx}\left(f(x)g(x)\right) = \frac{d}{dx}\left(f(x)\right)\cdot g(x) + f(x)\cdot\frac{d}{dx}\left(g(x)\right)$$

Intuitively, what is happening in this rule is that we are alternately treating one factor as a constant (the one not being differentiated) and the other factor as a variable function (the one that is being differentiated). We then add these two rates of change together.

Ultimately, you want to perform this rule in your head just like all of the other rules. Your instructor, however, may initially want you to show steps; under that presumption, steps are going to be shown in each and every example of this lab when the product rule is applied.
Two simple examples of the product rule are shown at the top of the next page.

Given Function	$y(x)=x^2\sin(x)$	$P=e^t\cos(t)$
Derivative	$y'(x)=\frac{d}{dx}(x^2)\cdot\sin(x)+x^2\cdot\frac{d}{dx}(\sin(x))$ $=2x\sin(x)+x^2\cos(x)$	$\frac{dP}{dt}=\frac{d}{dt}(e^t)\cdot\cos(t)+e^t\cdot\frac{d}{dt}(\cos(t))$ $=e^t\cos(t)-e^t\sin(t)$

A decision you'll need to make is how to handle a constant factor in a term that requires the product rule. Two options for taking the derivative of $f(x)=5x^2\ln(x)$ are shown below.

Option A	Option B
$f'(x)=5\left[\frac{d}{dx}(x^2)\cdot\ln(x)+x^2\cdot\frac{d}{dx}(\ln(x))\right]$ $=5\left[2x\ln(x)+x^2\cdot\frac{1}{x}\right]$ $=10x\ln(x)+5x$	$f'(x)=\frac{d}{dx}(5x^2)\cdot\ln(x)+5x^2\cdot\frac{d}{dx}(\ln(x))$ $=10x\ln(x)+5x^2\cdot\frac{1}{x}$ $=10x\ln(x)+5x$

In Option B we are treating the factor of 5 as a part of the first variable factor. In doing so, the factor of 5 distributes itself. This is the preferred treatment of the author, so this is what you will see illustrated in this lab.

Problem 33.1

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

33.1.1 $T(t)=2\sec(t)\tan(t)$

33.1.2 $k=\frac{e^t\sqrt{t}}{2}$

33.1.3 $y=4x\ln(x)+3^x-x^3$

33.1.4 $f(x)=\cot(x)\cot(x)-1$

Problem 33.2

Find each of the following derivatives ***without first simplifying the formula***; that is, go ahead and use the product rule on the expression as written. Simplify each resultant derivative formula. For each derivative, ***check*** your answer by simplifying the original expression and then taking the derivative of that simplified expression.

33.2.1 $\frac{d}{dx}\left(x^4x^7\right)$

33.2.2 $\frac{d}{dx}\left(x\cdot x^{10}\right)$

33.2.3 $\frac{d}{dx}\left(\sqrt{x}\cdot\sqrt{x^{21}}\right)$

Activity 34

The next rule we are going to explore is called ***the quotient rule of differentiation***. We **never** use this rule unless there is ***a variable factor in the denominator of the expression*** we are differentiating. (Remember, we already have the constant factor rule to deal with constant factors in the denominator.)

Equation 34.1 $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{d}{dx}\left(f(x)\right)\cdot g(x)-f(x)\cdot\frac{d}{dx}\left(g(x)\right)}{\left[g(x)\right]^2};\ g(x)\neq 0$$

Like all of the other rules, you ultimately want to perform the quotient rule in your head. Your instructor, however, may initially want you to show steps when applying the quotient rule; under that presumption, steps are going to be shown in each and every example of this lab when the quotient rule is applied. Two simple examples of the quotient rule are shown below.

Given Function	$y=\frac{4x^3}{\ln(x)}$	$V=\frac{\csc(t)}{\tan(t)}$
Derivative	$\frac{dy}{dx}=\frac{\frac{d}{dx}(4x^3)\cdot\ln(x)-4x^3\cdot\frac{d}{dx}(\ln(x))}{\left[\ln(x)\right]^2}$ $=\frac{12x^2\ln(x)-4x^3\cdot\frac{1}{x}}{\left[\ln(x)\right]^2}$ $=\frac{12x^2\ln(x)-4x^2}{\left[\ln(x)\right]^2}$	$\frac{dV}{dt}=\frac{\frac{d}{dt}(\csc(t))\cdot\tan(t)-\csc(t)\cdot\frac{d}{dt}(\tan(t))}{\left[\tan(t)\right]^2}$ $=\frac{-\csc(t)\cot(t)\cdot\tan(t)-\csc(t)\cdot\sec^2(t)}{\tan^2(t)}$ $=\frac{-\csc(t)\left(1+\sec^2(t)\right)}{\tan^2(t)}$

Problem 34.1

Find the first derivative formula for each of the following functions. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative.

34.1.1 $g(x)=\frac{4\ln(x)}{x}$

34.1.2 $j(y)=\frac{\sqrt[3]{y^5}}{\cos(y)}$

34.1.3 $y=\frac{\sin(x)}{4\sec(x)}$

34.1.4 $f(t)=\frac{t^2}{e^t}$

Problem 34.2

Find each of the following derivatives ***without first simplifying the formula***; that is, go ahead and use the quotient rule on the expression as written. For each derivative, ***check*** your answer by simplifying the original expression and then taking the derivative of that simplified expression.

34.2.1 $\frac{d}{dt}\left(\frac{\sin(t)}{\sin(t)}\right)$

34.2.2 $\frac{d}{dx}\left(\frac{x^6}{x^2}\right)$

34.2.3 $\frac{d}{dx}\left(\frac{10}{2x}\right)$

Activity 35

In problems 33.2 and 34.2 you applied the product and quotient rules to expressions where the derivative could have been found much more quickly had you simplified the expression before taking the derivative. For example, while you can find the correct derivative formula using the quotient rule when working problem 34.2.3, the derivative can be found much more quickly if you simplify the expression ***before*** applying the rules of differentiation. This is illustrated in Example 35.1.

Example 35.1

$$\begin{aligned}\frac{d}{dx}\left(\frac{10}{2x}\right) &= \frac{d}{dx}\left(5x^{-1}\right)\\ &= -5x^{-2}\\ &= -\frac{5}{x^2}\end{aligned}$$

Part of learning to take derivatives is learning to make good choices about the methodology to employ when taking derivatives. In examples 35.2 and 35.3, the need to use the product rule or quotient rule is obviated by first simplifying the expression being differentiated.

Example 35.2

Problem	**Solution**
Find $\frac{dy}{dx}$ if $y = \sec(x)\cos(x)$.	$y = \sec(x)\cos(x)$ $= \frac{1}{\cos(x)} \cdot \cos(x)$ $= 1$ $\frac{dy}{dx} = 0$

Example 35.3

Problem	**Solution**
Find $f'(t)$ if $f(t) = \frac{4t^5 - 3t^3}{2t^2}$.	$f(t) = \frac{4t^5 - 3t^3}{2t^2}$ $= \frac{4t^5}{2t^2} - \frac{3t^3}{2t^2}$ $= 2t^3 - \frac{3}{2}t$ $f'(t) = 6t^2 - \frac{3}{2}$

Problem 35.1

Find the derivative with respect to x for each of the following functions after first completely simplifying the formula being differentiated. In each case you should ***not*** use either the product rule or the quotient rule while finding the derivative formula.

35.1.1 $y = \dfrac{4x^{12} - 5x^4 + 3x^2}{x^4}$

35.1.2 $g(x) = \dfrac{-4\sin(x)}{\cos(x)}$

35.1.3 $h(x) = \dfrac{4 - x^6}{3x^{-2}}$

35.1.4 $z(x) = \sin^2(x) + \cos^2(x)$

35.1.5 $z = (x+4)(x-4)$

35.1.6 $T(x) = \dfrac{\ln(x)}{\ln(x^2)}$

Activity 36

Sometimes both the product rule and quotient rule need to be applied when finding a derivative formula.

Problem 36.1

Consider the functions $f(x) = x^2 \cdot \dfrac{\sin(x)}{e^x}$ and $g(x) = \dfrac{x^2\sin(x)}{e^x}$.

36.1.1 Discuss why f and g are in fact two representations of the same function.

36.1.2 Find $f'(x)$ by first applying the product rule and then applying the quotient rule (where necessary).

36.1.3 Find $g'(x)$ by first applying the quotient rule and then applying the product rule (where necessary).

36.1.4 Rigorously establish that the formulas for $f'(x)$ and $g'(x)$ are indeed the same.

Activity 37

Derivative formulas can give us much information about the behavior of a function. For example, the derivative formula for $f(x) = x^2$ is $f'(x) = 2x$. Clearly f' is negative when x is negative and f' is positive when x is positive. This tells us that f is decreasing when x is negative and that f is increasing when x is positive. This matches the behavior of the parabola $y = x^2$.

Problem 37.1

Answer each of the following questions about applied functions.

37.1.1 The amount of time (seconds), T, required for a pendulum to complete one period is a function of the pendulum's length (meters), L. Specifically, $T = 2\pi\sqrt{\frac{L}{g}}$ where g is the acceleration constant for Earth (roughly 9.8 m/s^2).

37.1.1.a Find $\frac{dT}{dL}$ after first rewriting the formula for T as a constant times $\sqrt{L}$.

37.1.1.b The sign on $\frac{dT}{dL}$ is the same regardless of the value of L. What is this sign and what does it tell you about the relative periods of two pendulums with different lengths?

37.1.2 The gravitational force (Newtons) between two objects of masses m_1 and m_2 (kg) is a function of the distance (meters) between the objects' centers of mass, r. Specifically, $F(r) = \frac{G\,m_1\,m_2}{r^2}$ where G is the universal gravitational constant (which is approximately $6.7\times10^{-11}\ \mathrm{n\cdot m/kg^2}$).

37.1.2.a Leaving G, m_1, and m_2 as constants, find $F'(r)$ after first rewriting the formula for F as a constant times a power of r.

37.1.2.b The sign on $F'(r)$ is the same regardless of the value of F. What is this sign and what does it tell you about the effect on the gravitational force between two objects when the distance between the objects is changed?

37.1.2.c Leaving G, m_1, and m_2 as constants, evaluate $F'(1.00\times10^{12})$, $F'(1.01\times10^{12})$, $F'(1.02\times10^{12})$.

37.1.2.d Calculate $\frac{F(1.02\times10^{12}) - F(1.00\times10^{12})}{1.02\times10^{12} - 1.00\times10^{12}}$. Which of the quantities found in part c comes closest to this value? Draw a sketch of F and discuss why this result makes sense.

The Chain Rule

Activity 38

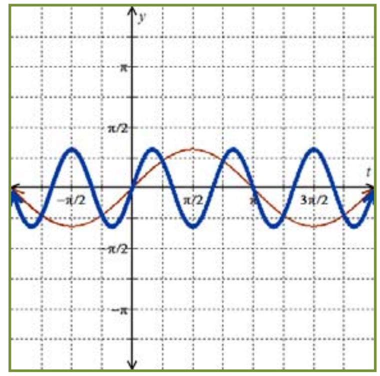


Figure 38.1: f and k

The functions $f(t)=\sin(t)$ and $k(t)=\sin(3t)$ are shown in Figure 38.1. Since $f'(t)=\cos(t)$, it is reasonable to speculate that $k'(t)=\cos(3t)$. But this would imply that $k'(0)=f'(0)=1$, and a quick glance of the two functions at 0 should convince you that this is not true; clearly $k'(0)>f'(0)$.

The function k moves through three periods for every one period generated by the function f. Since the amplitudes of the two functions are the same, the only way k can generate periods at a rate of 3:1 (compared to f) is if its rate of change is three times that of f. In fact, $k'(t)=3\cos(3t)$; please note that 3 is the first derivative of $3t$. This means that the formula for $k'(t)$ is the product of the rates of change of the outside function ($\frac{d}{du}(\sin(u))=\cos(u)$) and the inside function ($\frac{d}{dt}(3t)=3$).

k is an example of a composite function (as illustrated in Figure 38.2). If we define g by the rule $g(t)=3t$, then $k(t)=f(g(t))$.

$$t \xrightarrow{\quad g \quad} 3t \xrightarrow{\quad f \quad} \sin(3t)$$

Figure 38.2: $g(t)=3t$, $f(u)=\sin(u)$, and $k(t)=f(g(t))$

Taking the output from g and processing it through a second function, f, is the action that characterizes k as a composite function.

Note that $k'(0)=g'(0)\,f'(g(0))$. This last equation is an example of what we call ***the chain rule for differentiation***. Loosely, the chain rule tells us that when finding the rate of change for a composite function (at 0), we need to multiply the rate of change of the outside function, $f'(g(0))$, with the rate of change of the inside function, $g'(0)$. This is symbolized for general values of x in Equation 38.1 where u represents a function of x (e.g. $u=g(x)$).

Equation 38.1 $$\frac{d}{dx}(f(u))=f'(u)\cdot\frac{d}{dx}(u)$$

This rule is used to find derivative formulas in examples 38.1 and 38.2.

Example 38.1

Problem	Solution
Find $\frac{dy}{dx}$ if $y=\sin\left(x^2\right)$.	$\frac{dy}{dx}=\cos\left(x^2\right)\cdot\frac{d}{dx}\left(x^2\right)$ $=\cos\left(x^2\right)\cdot 2x$ $=2x\cos\left(x^2\right)$

The factor of $\frac{d}{dx}\left(x^2\right)$ is called a ***chain rule factor***.

Example 38.2

Problem	Solution
Find $f'(t)$ if $f(t)=\sec^9(t)$.	$f(t)=\left[\sec(t)\right]^9$ $f'(t)=9\left[\sec(t)\right]^8\cdot\frac{d}{dt}\left(\sec(t)\right)$ $=9\sec^8(t)\cdot\sec(t)\tan(t)$ $=9\sec^9(t)\tan(t)$

The factor of $\frac{d}{dt}\left(\sec(t)\right)$ is called a ***chain rule factor***.

Example 38.3

Problem	Solution
Find $\frac{dy}{dx}$ if $y=4^x$.	$\frac{dy}{dx}=\ln(4)\cdot 4^x$

Please note that **the chain rule was not applied** here because **the function being differentiated was not a composite function.**

While you ultimately want to perform the chain rule step in your head, your instructor may want you to illustrate the step while you are first practicing the rule. For this reason, the step will be explicitly shown in every example given in this lab.

Problem 38.1

Find the first derivative formula for each function. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative (e.g. $h'(t)$).

38.1.1 $h(t)=\cos\left(\sqrt{t}\right)$ **38.1.2** $P=\sin\left(\theta^4\right)$ **38.1.3** $w(\alpha)=\cot\left(\sqrt[3]{\alpha}\right)$

38.1.4 $z=7\left[\ln(t)\right]^3$ **38.1.5** $z(\theta)=\sin^4(\theta)$ **38.1.6** $P(\beta)=\tan^{-1}(\beta)$

38.1.7 $y=\left[\sin^{-1}(t)\right]^{17}$ **38.1.8** $T=2^{\ln(x)}$ **38.1.9** $y(x)=\sec^{-1}\left(e^x\right)$

Problem 38.2

A function, f, is shown in Figure 38.3. Answer each of the following questions in reference to this function.

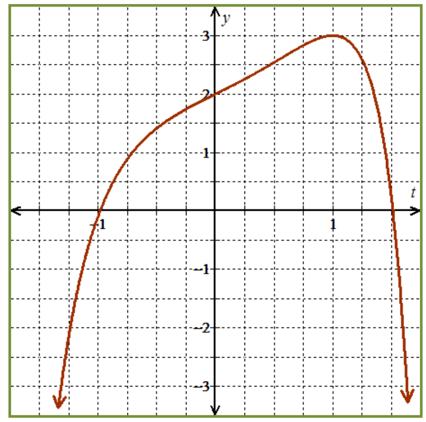


Figure 38.3: f

38.2.1 Use the graph to rank the following in decreasing order: $f(1)$, $f'(1)$, $f''(1)$, $f'(0)$, and $f'(-1)$.

38.2.2 The formula for f is $f(t) = te^t - e^{t^2} + 3$. Find the formulas for f' and f'' and use the formulas to verify your answer to problem 38.2.1.

38.2.3 Find the equation of the tangent line to f at 0.

38.2.4 Find the equation of the tangent line to f' at 0.

38.2.5 There is an antiderivative of f that passes through the point $(0,7)$. Find the equation of the tangent line to this antiderivative at 0.

Activity 39

When finding derivatives of complex formulas you need to apply the rules for differentiation in the reverse of order of operations. For example, when finding $\frac{d}{dx}\left(\sin\left(xe^x\right)\right)$ the first rule you need to apply is the derivative formula for $\sin(u)$ but when finding $\frac{d}{dx}\left(x\sin\left(e^x\right)\right)$ the first rule that needs to be applied is the product rule.

Problem 39.1

Find the first derivative formula for each function. In each case take the derivative with respect to the independent variable as implied by the expression on the right side of the equal sign. Make sure that you use the appropriate name for each derivative (e.g. $f'(x)$).

39.1.1 $f(x) = \sin\left(xe^x\right)$

39.1.2 $g(x) = x\sin\left(e^x\right)$

39.1.3 $y = \dfrac{\tan(\ln(x))}{x}$

39.1.4 $z = 5t + \dfrac{\cos^2\left(t^2\right)}{3}$

39.1.5 $f(y) = \sin\left(\dfrac{\ln(y)}{y}\right)$

39.1.6 $G = x\sin^{-1}\left(x\ln(x)\right)$

Activity 40

As always, you want to simplify an expression before jumping in to take its derivative. Never-the-less, it can build confidence to see that the rules work even if you don't simplify first.

Problem 40.1

Consider the functions $f(x)=\sqrt{x^2}$ and $g(x)=\left(\sqrt{x}\right)^2$.

40.1.1 Assuming that x is not negative, how does each of these formulas simplify? Use the simplified formula to find the formulas for $f'(x)$ and $g'(x)$.

40.1.2 Use the chain rule (without first simplifying) to find the formulas for $f'(x)$ and $g'(x)$; simplify each result (assuming that x is positive).

40.1.3 Are f and g the same function? Explain why or why not.

Problem 40.2

So long as x falls on the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $\tan^{-1}\left(\tan(x)\right)=x$. Use the chain rule to find $\frac{d}{dx}\left[\tan^{-1}\left(\tan(x)\right)\right]$ and show that it simplifies as it should.

Problem 40.3

How does the formula $g(t)=\ln\left(e^{5t}\right)$ simplify and what does this tell you about the formula for $g'(t)$? After answering those questions use the chain rule to find the formula for $g'(t)$ and show that it simplifies as it should.

Problem 40.4

Consider the function $g(t)=\ln\left(\frac{5}{t^3\sec(t)}\right)$.

40.4.1 Find the formula for $g'(t)$ without first simplifying the formula for $g(t)$.

40.4.2 Use the quotient, product, and power rules ***of logarithms*** to expand the formula for $g(t)$ into three logarithmic terms. Then find $g'(t)$ by taking the derivative of the expanded version of g.

40.4.3 Show that the two resultant formulas are in fact the same. Also, reflect upon which process of differentiation was less work and easier to "clean up."

Activity 41

So far we have worked with the chain rule as expressed using function notation. In some applications it is easier to think of the chain rule using Leibniz notation. Consider the following example

During the 1990s, the amount of electricity used per day in Etown increased as a function of population at the rate of 18 kW/person. On July 1, 1997, the population of Etown was 100,000 and the population was decreasing at a rate of 6 people/day. In Equation 41.1 (page 61) we use these values to determine the rate at which electrical usage was changing (*with respect to* ***time***) in Etown on 7/1/1997. Please note that in this extremely simplified example we are ignoring all factors that contribute to citywide electrical usage other than population (such as temperature).

Equation 41.1	$\left(18\dfrac{\text{kW}}{\text{person}}\right)\left(-6\,\dfrac{\text{people}}{\text{day}}\right)=-128\dfrac{\text{kW}}{\text{day}}$

Let's define $g(t)$ as the population of Etown t years after January 1, 1990 and $f(u)$ as the daily amount of electricity used in Etown when the population was u. From the given information, $g(7.5)=100{,}000$, $g'(7.5)=-6$, and $f'(u)=18$ for all values of u. If we let $y=f(u)$ where $u=g(t)$, then we have (from Equation 41.1):

Equation 41.2	$\left(\left.\dfrac{dy}{du}\right\|_{u=100,000}\right)\left(\left.\dfrac{du}{dt}\right\|_{t=7.5}\right)=\left.\dfrac{dy}{dt}\right\|_{t=7.5}$

You should note that Equation 41.2 is an application of the chain rule expressed in Leibniz notation; specifically, the expression on the left side of the equal sign represents $f'\big(g(7.5)\big)g'(7.5)$.

In general, we can express the chain rule as shown in Equation 41.3.

Equation 41.3	$\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$

Problem 41.1

Suppose that Carla is jogging in her sweet new running shoes. Suppose further that $r=f(t)$ is Carla's pace (mi/hr) t hours after 1 pm and $y=h(r)$ is Carla's heart rate (bpm) when she jogs at a rate of r mi/hr. In this context we can assume that all of Carla's motion was in one direction, so the words speed and velocity are completely interchangeable.

41.1.1 What is the meaning of $f(.75)=7$?

41.1.2 What is the meaning of $h(7)=125$?

41.1.3 What is the meaning of $(h\circ f)(.75)=125$?

41.1.4 What is the meaning of $\left.\dfrac{dr}{dt}\right|_{t=.75}=-0.00003$?

41.1.5 What is the meaning of $\left.\dfrac{dy}{dr}\right|_{r=7}=8$?

41.1.6 Assuming that all of the previous values are for real, what is the value of $\left.\dfrac{dy}{dt}\right|_{t=.75}$ and what does this value tell you about Carla?

Problem 41.2

Portions of SW 35th Avenue are extremely hilly. Suppose that you are riding your bike along SW 35th Ave from Vermont Street to Capitol Highway. Let $u = d(t)$ be the distance you have travelled (ft) where t is the number of seconds that have passed since you began your journey. Suppose that $y = e(u)$ is the elevation (m) of SW 35th Ave where u is the distance (ft) from Vermont St headed towards Capital Highway.

41.2.1 What, including units, would be the meanings of $d(25) = 300$, $e(300) = 140$, and $(e \circ d)(25) = 140$?

41.2.2 What, including units, would be the meanings of $\left.\frac{du}{dt}\right|_{t=25} = 14$ and $\left.\frac{dy}{du}\right|_{u=300} = -0.1$

41.2.3 Suppose that the values stated in problem 41.1.2 are accurate. What, including unit, is the value of $\left.\frac{dy}{dt}\right|_{t=25}$? What does this value tell you in the context of this problem?

Problem 41.3

According to Hooke's Law, the force (lb), F, required to hold a spring in place when its displacement from the natural length of the spring is x (ft), is given by the formula $F = kx$ where k is called the spring constant. The value of k varies from spring to spring.

Suppose that it requires 120 lb of force to hold a given spring 1.5 ft beyond its natural length.

41.3.1 Find the spring constant for this spring. Include units when substituting the values for F and x into Hooke's Law so that you know the unit on k.

41.3.2 What, including unit, is the constant value of $\frac{dF}{dx}$?

41.3.3 Suppose that the spring is stretched at a constant rate of .032 ft/s. If we define t to be the amount of time (s) that passes since the stretching begins, what, including unit, is the constant value of $\frac{dx}{dt}$?

41.3.4 Use the chain rule to find the constant value (including unit) of $\frac{dF}{dt}$. What is the contextual significance of this value?

Implicit Differentiation

Activity 42

Some points that satisfy the equation $y^3 - 4y = x^2 - 1$ are graphed in Figure 42.1. Clearly this set of points does not constitute a function where y is a function of x; for example, there are three points that have an x-coordinate of 1.

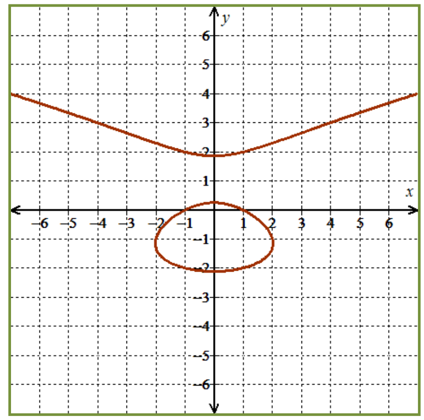


Figure 42.1: $y^3 - 4y = x^2 - 1$

Never-the-less, there is a unique tangent line to the curve at each point on the curve and so long as the tangent line is not vertical it has a unique slope. We still identify the value of this slope using the symbol $\frac{dy}{dx}$, so it would be helpful if we had a formula for $\frac{dy}{dx}$. If we could solve $y^3 - 4y = x^2 - 1$ for y, finding the formula for $\frac{dy}{dx}$ would be a snap. Hopefully you quickly see that such an approach is just not possible.

To get around this problem we are going to employ a technique called ***implicit differentiation***. We use this technique to find the formula for $\frac{dy}{dx}$ whenever the equation relating x and y is not explicitly solved for y. What we are going to do is treat y ***as if it were*** a function of x and set the derivatives of the two sides of the equation equal to one another. This is actually a reasonable thing to do because so long as we are at a point on the curve where the tangent line is not vertical, we could make y a function of x using appropriate restrictions on the domain and range.

Since we are treating y as a function of x, we need to make sure that we use the chain rule when differentiating terms like y^3. When u is a function of x, we know that $\frac{d}{dx}\left(u^3\right) = 3u^2 \frac{d}{dx}(u)$.

Since the name we give $\frac{d}{dx}(y)$ is $\frac{dy}{dx}$, it follows that when y is a function of x, $\frac{d}{dx}\left(y^3\right) = 3y^2 \frac{dy}{dx}$.

The derivation of $\frac{dy}{dx}$ for the equation $y^3 - 4y = x^2 - 1$ is shown in example 42.1.

Example 42.1

$$y^3 - 4y = x^2 - 1$$

$$\frac{d}{dx}\left(y^3 - 4y\right) = \frac{d}{dx}\left(x^2 - 1\right)$$

Begin by differentiating both sides of the equation with respect to x.

$$3y^2 \frac{dy}{dx} - 4\frac{dy}{dx} = 2x$$

The chain rule only comes into play on the terms involving y.

$$\left(3y^2 - 4\right)\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{3y^2 - 4}$$

We now solve the equation for $\frac{dy}{dx}$.

At first it might be unsettling that the formula for $\frac{dy}{dx}$ contains both the variables x and y. However, if you think it through you should conclude that the formula ***must*** include the variable y; otherwise, how could the formula generate three different slopes at the points $(1,2)$, $(1,0)$, and $(1,-2)$? These slopes are given below. The reader should verify their values by drawing lines onto Figure 42.1 with the indicated slopes at the indicated points.

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{2(1)}{3(2)^2-4} = \frac{1}{4} \qquad \left.\frac{dy}{dx}\right|_{(1,0)} = \frac{2(1)}{3(0)^2-4} = -\frac{1}{2} \qquad \left.\frac{dy}{dx}\right|_{(1,-2)} = \frac{2(1)}{3(-2)^2-4} = \frac{1}{4}$$

Problem 42.1

Use the process of implicit differentiation to find a formula for $\frac{dy}{dx}$ for the curves generated by each of the following equations. ***Do not*** simplify the equations before taking the derivatives.

You will need to use the product rule for differentiation in problems 42.1.4-42.1.6.

42.1.1 $3x^4 = -6y^5$

42.1.2 $\sin(x) = \sin(y)$

42.1.3 $4y^2 - 2y = 4x^2 - 2x$

42.1.4 $x = ye^y$

42.1.5 $y = xe^y$

42.1.6 $xy = e^{xy-1}$

Problem 42.2

Several points that satisfy the equation $y = xe^y$ are graphed in Figure 42.2. Find the slope and equation of the tangent line to this curve at the origin. (Please note that you already found the formula for $\frac{dy}{dx}$ in problem 42.1.5.)

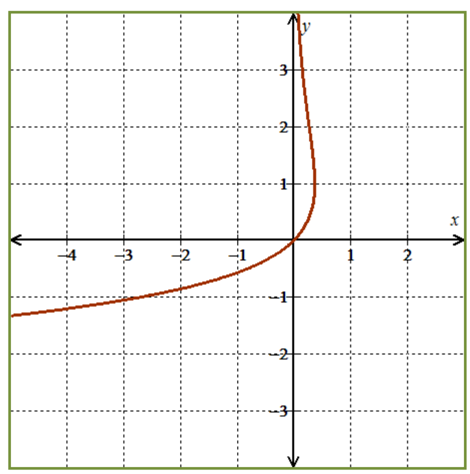


Figure 42.2: $y = xe^y$

Problem 42.3

Consider the set of points that satisfy the equation $xy = 4$.

42.3.1 Use implicit differentiation to find a formula for $\frac{dy}{dx}$.

42.3.2 Find a formula for $\frac{dy}{dx}$ after first solving the equation $xy = 4$ for y.

42.3.3 Show that the two formulas are in fact equivalent so long as $xy = 4$.

Problem 42.4

A set of points that satisfy the equation $x\cos(xy) = 4 - y$ is graphed in Figure 42.3. Find the slope and equation of the tangent line to this curve at the point $(0,4)$.

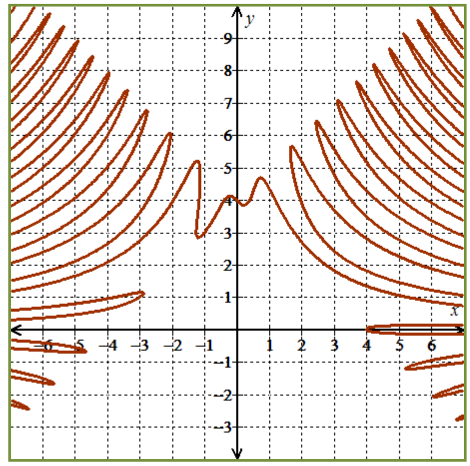


Figure 42.3: $x\cos(xy) = 4 - y$

Activity 43

Using the definition $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, it is easy to establish that $\frac{d}{dx}(x^2) = 2x$. We can use this formula and implicit differentiation to find the formula for $\frac{d}{dx}(\sqrt{x})$.

If $y = \sqrt{x}$, then $y^2 = x$ (and $y \geq 0$). Using implicit differentiation we have:

$$\begin{aligned} y^2 &= x \\ \frac{d}{dx}(y^2) &= \frac{d}{dx}(x) \\ 2y\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{2y} \end{aligned}$$

But $y = \sqrt{x}$, so we have:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2y} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

In a similar manner, we can use the fact that $\frac{d}{dx}(\sin(x)) = \cos(x)$ to come up with a formula for $\frac{d}{dx}(\sin^{-1}(x))$.

If $y = \sin^{-1}(x)$, then $\sin(y) = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. This gives us:

$$\sin(y) = x$$
$$\frac{d}{dx}(\sin(y)) = \frac{d}{dx}(x)$$
$$\cos(y)\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2(y)}}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

This expression comes from the trigonometric identity $\sin^2(t) + \cos^2(t) = 1$. If you solve that equation for $\cos(t)$ you get $\cos(t) = \pm\sqrt{1-\sin^2(t)}$. Because the function $y = \sin^{-1}(x)$ never has negative slope we can discard the negative solution to the equation.

Here we use that fact that $\sin(y) = x$.

Problem 43.1

Use the fact that $\frac{d}{dx}(e^x) = e^x$ together with implicit differentiation to show that $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$.

Begin by using the fact that $y = \ln(x)$ implies that $e^y = x$ (and $y > 0$). Your first step is to differentiate both sides of the equation $e^y = x$ with respect to x.

Problem 43.2

Use the fact that $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ together with implicit differentiation to show that $\frac{d}{dx}(e^x) = e^x$.

Begin by using the fact that $y = e^x$ implies that $\ln(y) = x$. Your first step is to differentiate both sides of the equation $\ln(y) = x$ with respect to x.

Problem 43.3

Use the fact that $\frac{d}{dx}(\tan(x)) = \sec^2(x)$ together with implicit differentiation to show that $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$. Begin by using the fact that $y = \tan^{-1}(x)$ implies that $\tan(y) = x$ (and $-\frac{\pi}{2} < y < \frac{\pi}{2}$). Your first step is to differentiate both sides of the equation $\tan(y) = x$ with respect to x. Please note that you will need to use the Pythagorean identity that relates the tangent and secant functions while working this problem.

Related Rates

Activity 44

Consider the toy rocket shown in Figure 44.1. As the rocket's elevation (x, measured in feet) changes the angle of elevation from the ground to the base of the rocket (θ, measured in radians) also changes. Consequently, there is a relationship between the ***rates*** at which the elevation and angle of elevation ***change***. This is an example of the topic we call ***related rates***.

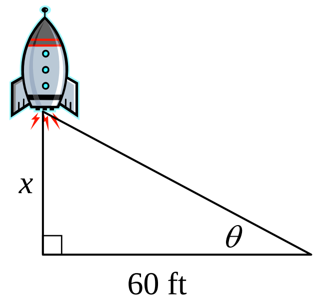


Figure 44.1: Johnny's Rocket

If we assume that the rocket flies straight upward (and subsequently falls straight downward) and we define x and θ as functions of time, t, where t is the amount of time that has passed (s) since the rocket was launched, then the rates of change in the elevation and angle of elevation are, respectively, $\frac{dx}{dt}$ (measure in ft/s) and $\frac{d\theta}{dt}$ (measured in rad/s).

Using simple right triangle trigonometry we determine the relation equation between x and θ (Equation 44.1). Since x and θ both vary as functions of t, we can differentiate both sides of Equation 44.1 with respect to t resulting in the rate equation (Equation 44.2)

Equation 44.1: $\tan(\theta) = \frac{x}{60}$ **Equation 44.2:** $\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{60}\frac{dx}{dt}$

Problem 44.1

Rates of change in the rocket's elevation are given for two elevations in Table 44.1.

44.1.1 Use Equation 44.1 to determine the values of θ at the given elevations. ***Do not approximate these value … state their exact values***.

44.1.2 Use Equation 44.2 to determine the corresponding values of $\frac{d\theta}{dt}$. Round each of these values to the nearest hundredth.

44.1.3 Write two complete sentences that fully communicate what is happening to the rocket's elevation and angle of elevation as implied by the values in Table 44.1. Each sentence should include the elevation of the rocket, a clear indication of whether the rocket is rising or falling, the speed at which the rocket is moving, a clear indication of whether the angle of elevation is increasing or decreasing, and the rate at which the angle is increasing or decreasing. All values should be stated using appropriate units.

Table 44.1: Tracking Johnny's Rocket

x (ft)	$\frac{dx}{dt}$ (ft/s)	θ (rad)	$\frac{d\theta}{dt}$ (rad/s)
$60\sqrt{3}$	6		
60	-38		

Activity 45
There is something quite different about the manner in which variables are defined while working related rates problems from the way in which you defined variables when working applied problems in earlier classes. In most applications, you define your variables to explicitly represent the quantity (or quantities) you are asked to find. When defining variables for a related rates problem, however, you define a variable from which you can ***infer*** the value you are trying to find and, more uniquely, you define variables for quantities whose values you are actually given!

Problem 45.1
At 2 PM one day, Muffin and Rex were sleeping atop one another. A loud noise startled the animals and Muffin began to run due north at a constant rate of 1.7 ft/s while Rex ran due east at a constant rate of 2.1 ft/s. The critters maintained these paths and rates for several seconds. Suppose that you wanted to calculate the rate at which the distance between the cat and dog was changing three seconds into their run.

45.1.1 A right triangle representing this problem has been drawn in Figure 45.1. The length of each side and the measure of each angle have (temporarily) been assigned variable names; these measurements are the ***potential*** variables for the problem. Circle the variable whose rate you are trying to determine and the two variables whose rate you are given. These three measurements are the actual variables for the problem.

45.1.2 One of the remaining pieces in Figure 45.1 does not change value as the animals run. Cross out the corresponding variable and replace it with its fixed value.

45.1.3 Cross out any potential variable not addressed in problems 45.1.1 and 45.1.2; although changing value, these pieces of the picture are irrelevant to the question at hand.

45.1.4 Copy the new and improved diagram (with the relevant variables and fixed value) onto your own paper; this is your working diagram for the problem. Explicitly define a time variable and then define your three length variables making sure that you establish their dependence upon time. The ***rate*** variables will emerge when you differentiate the length variables with respect to time.

45.1.5 Use your diagram to determine the relation equation and then differentiate both sides of that equation with respect to time. The resultant equation is your rate equation.

45.1.6 What are the values of the length variables three seconds into the animals' panicked runs? Which rate values in the rate equation do you know and what are their values?

45.1.7 Substitute the known values into the rate equation and solve for the unknown rate.

45.1.8 Write a contextual conclusion sentence that clearly indicates whether the distance between the cat and dog is increasing or decreasing and clearly communicates the rate at which this change is happening.

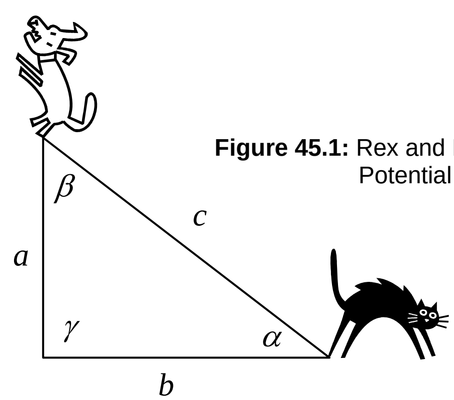


Figure 45.1: Rex and Muffin and the Potential Variables

Problem 45.2

Schuyler's clock is kaput; the minute hand functions as it should but the hour hand is stuck at 4. The minute hand on the clock is 30 cm long and the hour hand is 10 cm long. In this problem you are going to determine the rate of change between the tips of the hands every time the minute hand points directly at 12.

45.2.1 A triangle representing this problem has been drawn in Figure 45.3; the minute hand has deliberately been drawn so that it points at a number other than 12. This diagram represents the motion of the minute hand, not its position at one specific time. The hour hand, however, has been drawn is in its fixed position.

The length of each side of the triangle and the measure of each angle of the triangle have (temporarily) been assigned variable names; these are the ***potential*** variables for the problem. Circle the variable whose rate you are trying to determine. One of the variables represents a piece whose rate of change is constant when the minute hand falls between 10 and 4 (in the clock-wise direction). Identify this piece and circle the corresponding variable. The two circled variables are the actual variables for the problem.

45.2.2 Two of the remaining pieces in Figure 45.3 do not change value as the time passes. Cross out the corresponding variables and replace them with their fixed values.

45.2.3 Cross out any potential variable not addressed in problems 45.2.1 and 45.2.2; although changing value, these pieces of the picture are irrelevant to the question at hand.

45.2.4 Copy the new and improved diagram (with the relevant variables and fixed values) onto your own paper; this is your working diagram for the problem. Explicitly define a time variable and then define your angular and length variables making sure that you establish their dependence upon time. Make sure that you use radians as the measure of the angular variable. The ***rate*** variables will emerge when you differentiate the angular and length variables with respect to time.

45.2.5 Use your diagram to determine the relation equation and then differentiate both sides of that equation with respect to time. The resultant equation is your rate equation. You might want to look up the law of cosines before writing down your relation equation.

45.2.6 What are the values of the variables when the minute hand points to 12? What is the known rate value in your rate equation? (Careful … is this rate positive or negative?)

45.2.7 Substitute the known values into the rate equation and solve for the unknown rate. Round your solution to the nearest 10^{th}.

45.2.8 Write a contextual conclusion sentence that clearly indicates whether the distance between the tip of the clock hands is increasing or decreasing and clearly communicates the rate at which this change is happening.

Figure 45.2: Schuyler's clock

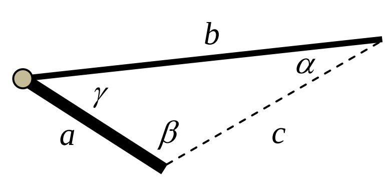


Figure 45.3: Potential Variables

Activity 46

Sometimes the relation equation is based upon a given or "known" formula.

Problem 46.1

Slushy is flowing out of the bottom of a cup at the constant rate of 0.1 cm^3/s. The cup is the shape of a right circular cone. The height of the cup is 12 cm and the cup (when full) holds a total of $36\pi\ \text{cm}^3$ of slushy. Determine the rate at which the height of the remaining slushy changes at the instant there are exactly $8\pi\ \text{cm}^3$ of slushy remaining in the cup.

46.1.1 The volume formula for a right circular cylinder is $V = \frac{\pi}{3} r^2 h$ where V is the volume of the cone, h is the height of the cone, and r is the radius at the top of the cone. We ultimately are going to define two of these variables in terms of the amount of slushy remaining in the cup. Which are the two variables relevant to the question at hand? That is, which quantity's rate of change are you given and which quantity's rate of change are you trying to determine?

46.1.2 Hopefully you determined that V and h are the relevant variables. This means that r needs to be eliminated from the volume formula. A cross section of the cup and slushy is shown in Figure 46.2.

- What is the radius at the top of the cup?
- Use the concept of similar triangle to express r in terms of h. Substitute this expression into the volume formula and simplify. The resultant equation is the relation equation for the problem.

46.1.3 Explicitly define h and V (including units) in terms of the amount of slushy remaining in the cup. Make sure that you communicate that each variable is dependent upon time and that you explicitly define your time variable. The ***rate*** variables will emerge when you differentiate the height and volume variables with respect to time.

46.1.4 Go ahead and complete the problem in a manner consistent with that implied in problems 45.1 and 45.2.

Figure 46.1: Slushy cups

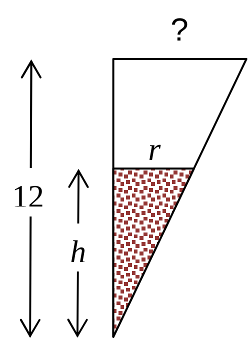


Figure 46.2: Slushy cross-section

Problem 46.2

In the classic TV series Batman, story lines almost always lasted for two episodes. At the end of the first episode, Batman and Robin were invariably caught in some diabolical trap intended to end their lives. In one episode, Egghead decided to both roast and crush the dynamic duo. He placed the caped crusaders in a room whose temperature steadily rose as two of the walls closed in on the men in tights.

According to the ideal gas law, the pressure (P), temperature (T), and volume (V) inside the room were related by the equation $\frac{PV}{T} = k$ where k is a constant specific to the types and amounts of gas that were in the room.

When the heroes were placed in the room, the room was 2 m $\times$ 2 m $\times$ 2 m, the temperature in the room was 20°C, and the air pressure in the room was 100 kPa. Once Egghead activated the trap, two of the walls moved toward each other at the constant rate of .1 m/min (in total) and the temperature in the room rose at a constant rate of 2°C/min. Since the volume and temperature both changed at constant rates, it might seem intuitive to you that the pressure also changed at a constant rate. Let's see if that is in fact the case.

46.2.1 Use the initial conditions in the room to determine the value of k (without unit) and rewrite the ideal gas law using this value for k.

46.2.2 Explicitly define P, T, and V (including units) in terms of the amount of time that had passed since the trap was set in motion.

46.2.3 Your rate equation comes from differentiating the ideal gas law equation with respect to time. There is a simple algebraic adjustment you can make to the equation that will greatly simplify this task. Go ahead and adjust the equation and then find your rate equation.

46.2.4 Determine the values of P, T, and V at both five minutes and eight minutes into the motion of the trap. State the known rate vales (remembering to think about the sign on each rate). You might want to think twice about the value of $\frac{dV}{dt}$. Use all of these values along with the rate equation to determine the value of $\frac{dP}{dt}$ at both five minutes and eight minutes into the motion of the trap.

Figure 46.3: Egghead

Figure 46.4: The Dynamic Duo

Activity 47

Solve each of the following problems using procedures similar to those suggested in the previous problems.

Problem 47.1

It was a dark night and 5.5 ft tall Bahram was walking towards a street lamp whose light was perched 40 ft into the air. As he walked, the light caused a shadow to fall behind Bahram. When Bahram was 80 feet from the base of the lamp he was walking at a pace of 2 ft/s. At what rate was the length of Bahram's shadow changing at that instant? Make sure that each stated and calculated rate value has the correct sign. Make sure that your conclusion sentence clearly communicates whether the length of the shadow was increasing or decreasing at the indicated time.

Problem 47.2

The gravitational force, F, between two objects in space with masses m_1 and m_2 is given by the formula $F = \dfrac{G m_1 m_2}{r^2}$ where r is the distance between the objects' centers of mass and G is the universal gravitational constant.

Two pieces of space junk, one with mass 500 kg and the other with mass 3000 kg, were drifting directly toward one another. When the objects' centers of mass were 250 km apart the lighter piece was moving at a rate of 0.5 km/hour and the heavier piece was moving at a rate of 0.9 km/hour. Leaving G as a constant, determine the rate at which the gravitational force between the two objects was changing at that instant. Make sure that each stated and calculated rate value has the correct sign. Make sure that your conclusion sentence clearly communicates whether the force was increasing or decreasing at that given time. The unit for F is Newtons (N).

Problem 47.3

An eighteen inch pendulum sitting atop a table is in the downward part of its motion. The pivot point for the pendulum is 30 inches above the table top. When the pendulum is 45° away from vertical, the angle formed at the pivot is decreasing at the rate of 25°/s. At what rate is the end of the pendulum approaching the table top at this instant?

Hint: Draw a right triangle with one acute angle vertex at the pivot point and the other acute angle vertex at the end of the pendulum. The rate of change of one piece of this triangle is the same as the rate you are trying to find.

Make sure that each stated and calculated rate value has the correct sign. Make sure that you use appropriate units when defining your variables.

Critical Numbers and Graphing from Formulas

Activity 48

In the first activity of this lab you are going to discuss a few questions with your group mates that will hopefully motivate you for one of the topics covered in the lab.

Problem 48.1
Discuss how you could use the first derivative formula to help you determine the vertex of the parabola $y = -2x^2 + 18x - 7$ and then determine the vertex. Remember that the vertex is a point in the xy-plane and as such is identified using an ordered pair.

Problem 48.2
The curves in figures 48.1 and 48.2 were generated by two of the four functions given below. Use the given functions along with their first derivatives to determine which functions generated the curves. Please note that the y-scales have deliberately been omitted from the graphs and that different scales were used to generate the two graphs. Resist any temptation to use your calculator; use of your calculator totally obviates the point of the exercise.

Potential Functions:	$f_1(x) = \dfrac{1}{(x-2)^{10/7}} + C_1$	$f_2(x) = \dfrac{1}{(x-2)^{2/7}} + C_2$
C_1 - C_4 are unknown constants	$f_3(x) = (x-2)^{2/7} + C_3$	$f_4(x) = (x-2)^{10/7} + C_4$

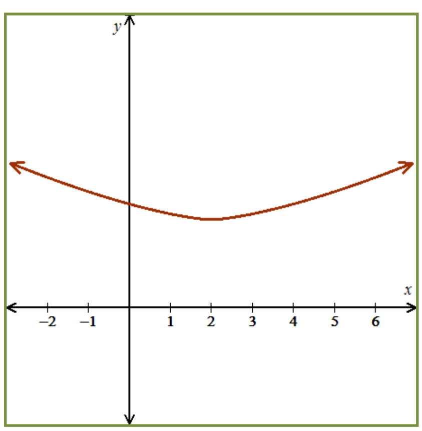


Figure 48.1: mystery curve 1

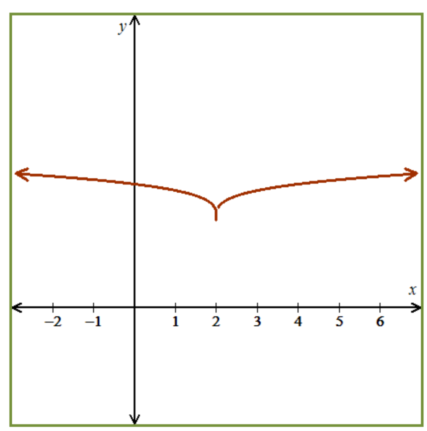


Figure 48.2: mystery curve 2

Activity 49

The vertex of the parabola in problem 48.1 is called a ***local maximum point*** and the points $(2, C_3)$ and $(2, C_4)$ in problem 48.2 are called ***local minimum points***. Collectively, these points are called ***local extreme points***.

While working Activity 48 you hopefully came to the conclusion that the local extreme points had certain characteristics in common. In the first place, they must occur at a number in the domain of the function (which eliminated f_1 and f_2 from contention in problem 48.2). Secondly, one of two things must be true about the first derivative when a function has a local extreme point; it either has a value of zero or it does not exist. This leads us to the definition of a ***critical number*** of a function.

Definition 49.1 – Critical Numbers

If f is a function, then we define ***the critical numbers of f*** as the numbers in the domain of f where the value of f' is either zero or does not exist.

Problem 49.1

The function g shown in Figure 49.1 has a vertical tangent line at -3. Veronica says that -3 is a critical number of g but Tito disagrees. Tito contends that -3 is not a critical number because g does not have a local extreme point at -3. Who's right?

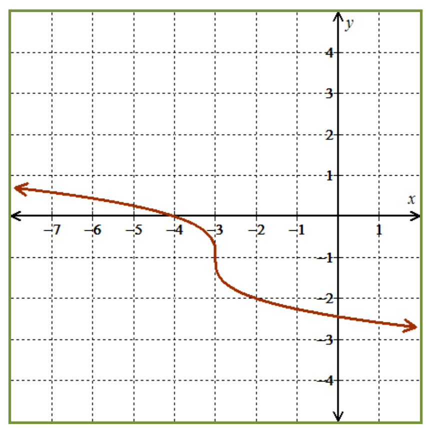


Figure 49.1: g

Problem 49.2

Answer each of the following questions (using complete sentences) in reference to the function f shown in Figure 49.2.

49.2.1 What are the critical numbers of f?

49.2.2 What are the local extreme points on f? Classify the points as local minimums or local maximums and remember that points on the plane are represented by ordered pairs.

49.2.3 What is the absolute maximum value of f over the interval $(-7, 7)$. Please note that the function value is the value of the y-coordinate at the point on the curve and as such is a number.

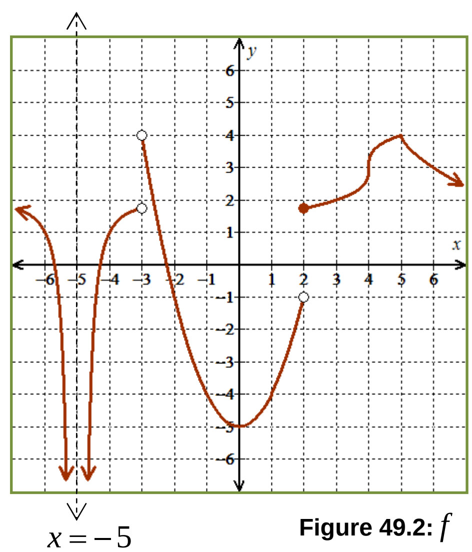


Figure 49.2: f

Problem 49.3
Decide whether each of the following statements is true or false.

49.3.1 A function always has a local extreme point at each of its critical numbers.

49.3.2 If the point $(t_1, h(t_1))$ is a local minimum point on h, then t_1 must be a critical number of h.

49.3.3 If $g'(2.7) = 0$, then g must have a local extreme point at 2.7.

49.3.4 If $g'(2.7) = 0$, then 2.7 must be a critical number of g.

49.3.5 If $g'(9)$ does not exist, then 9 must be a critical number of g.

Activity 50

When finding critical numbers based upon a function formula, there are three issues that need to be considered; the domain of the function, the zeros of the first derivative, and the numbers in the domain of the function where the first derivative is undefined. When writing a formal analysis of this process each of these questions must be explicitly addressed. The following outline shows the work you need to show when you are asked to write a formal determination of critical numbers based upon a function formula.

A process for formal determination of critical numbers

Please note that you should present your work in narrative form using complete sentences that establish the significance of all stated intervals and values. For example, for the function $g(t) = \sqrt{t-7}$ the first sentence you should write is "The domain of g is $[7, \infty)$."

1. Using interval notation, state the domain of the function.
2. Differentiate the function and completely simplify the resultant formula. Simplification includes:
    a. Manipulating all negative exponents into positive exponents.
    b. If any rational expression occurs in the formula, all terms must be written in rational form and a common denominator must be established for the terms. The final expression should be a single rational expression.
    c. The final expression must be completely factored.
3. State the values where the first derivative is equal to zero; show any non-trivial work necessary in making this determination. Make sure that you write a sentence addressing this issue even if the first derivative has no zeros.
4. State the values **<u>in the domain of the function</u>** where the first derivative does not exist; show any non-trivial work necessary in making this determination. Make sure that you write a sentence addressing this issue even if no such numbers exist.
5. State the critical numbers of the function. Make sure that you write this conclusion even if the function has no critical numbers.

Because the domain of a function is such an important issue when determining critical numbers, there are some things you should keep in mind when determining domains.

- Division by zero is never a good thing.
- Over the real numbers, you **cannot** take **even** roots of negative numbers.
- Over the real numbers, you **can** take **odd** roots of negative numbers.
- $\sqrt[n]{0}=0$ for any positive integer n.
- Over the real numbers, you can only take logarithms of positive numbers.
- The sine and cosine function are defined at all real numbers. You should check with your lecture instructor to see if you are expected to know the domains of the other four trigonometric functions and/or the inverse trigonometric functions.

Problem 50.1

Formally establish the critical numbers for each of the following functions following the procedure outlined on page 75.

50.1.1 $f(x)=x^2-9x+4$

50.1.2 $g(t)=7t^3+39t^2-24t+4$

50.1.3 $p(t)=(t+8)^{2/3}$

50.1.4 $z(x)=x\ln(x)$

50.1.5 $y(\theta)=e^{\cos(\theta)}$

50.1.6 $T(t)=\sqrt{t-4}\sqrt{16-t}$

Problem 50.2

The first derivative of the function $m(x)=\dfrac{\sqrt{x-5}}{x-7}$ is $m'(x)=\dfrac{3-x}{2\sqrt{x-5}(x-7)^2}$.

50.2.1 Roland says that 5 is a critical number of m but Yuna disagrees. Who is correct and why?

50.2.2 Roland says that 7 is a critical number of m but Yuna disagrees. Who is correct and why?

50.2.3 Roland says that 3 is a critical number of m but Yuna disagrees. Who is correct and why?

Activity 51

Once you have determined the critical numbers of a function, the next thing you might want to determine is the behavior of the function at each of its critical numbers. One way you could do that involves a sign table for the first derivative of the function

Problem 51.1

The first derivative of the function $f(x)=x^3-21x^2+135x-24$ is $f'(x)=3(x-5)(x-9)$. The critical numbers of f are trivially shown to be 5 and 9.

Copy Table 51.1 onto your paper and fill in the missing information. Then state the local minimum and maximum points on f. Specifically address both minimum and maximum points even if one and/or the other does not exist. Remember that points on the plane are represented by ordered pairs. **Make sure that you state points on f and not f'!**

Table 51.1: $f'(x) = 3(x-5)(x-9)$

Interval	Sign of f'	Behavior of f
$(-\infty, 5)$		
$(5, 9)$		
$(9, \infty)$		

Problem 51.2

The first derivative of the function $g(t) = \dfrac{\sqrt{t-4}}{(t-1)^2}$ is $g'(t) = \dfrac{-3(t-5)}{2(t-1)^3\sqrt{t-4}}$.

51.2.1 State the critical numbers of g; you **do not** need to show a formal determination of the critical numbers. You **do** need to write a complete sentence.

51.2.2 Copy Table 51.2 onto your paper and fill in the missing information.

Table 51.2: $g'(t) = \dfrac{-3(t-5)}{2(t-1)^3\sqrt{t-4}}$

Interval	Sign of g'	Behavior of g
$(4, 5)$		
$(5, \infty)$		

51.2.3 Why did we not include any part of the interval $(-\infty, 4)$ in Table 51.2?

51.2.4 Formally we say that $g(t_0)$ is a local minimum value of g if there exists an open interval centered at t_0 over which $g(t_0) < g(t)$ for every value of t on that interval (other than t_0, of course). Since g is not defined to the left of 4, it is impossible for this definition to be satisfied at 4; hence g does not have a local minimum value at 4.

State the local minimum and maximum points on g. Specifically address both minimum and maximum points even if one and/or the other does not exist.

51.2.5 Write a formal definition for a local maximum point on g.

Problem 51.3

The first derivative of the function $k(x)=\frac{\sqrt[3]{(x-2)^2}}{x-1}$ is $k'(x)=\frac{4-x}{3(x-1)^2\sqrt[3]{x-2}}$.

51.3.1 State the critical numbers of k; you **do not** need to show a formal determination of the critical numbers. You **do** need to write a complete sentence.

51.3.2 Copy Table 51.3 onto your paper and fill in the missing information. Then state the local minimum and maximum points on k. Specifically address both minimum and maximum points even if one and/or the other does not exist.

Table 51.3: $k'(x)=\frac{4-x}{3(x-1)^2\sqrt[3]{x-2}}$

Interval	Sign of k'	Behavior of k
$(-\infty,1)$		
$(1,2)$		
$(2,4)$		
$(4,\infty)$		

51.3.3 The number 1 was included as an endpoint in Table 51.3 even though 1 is not a critical number of k. Why did we have to include the intervals $(-\infty,1)$ and $(1,2)$ in the table as opposed to just using the single interval $(-\infty,2)$?

Problem 51.4

Perform each of the following for the functions in problems 51.4.1-51.4.3.

- Formally establish the critical numbers of the function.
- Create a table similar to tables 51.1-51.3. Number the tables, in order, 51.4-51.6. Don't forget to include table headings and column headings.
- State the local minimum points and local maximum points on the function. Make sure that you explicitly address both types of points even if there are none of one type and/or the other.

51.4.1 $k(x)=x^3+9x^2-10$

51.4.2 $g(t)=(t+2)^3(t-6)$

51.4.3 $F(x)=\frac{x^2}{\ln(x)}$

Problem 51.5

Consider a function f whose first derivative is $f'(x)=(x-9)^4$.

51.5.1 Is 9 definitely a critical number of f? Explain why or why not.

51.5.2 Other than at 9, what is always the sign of $f'(x)$? What does this sign tell you about the function f?

51.5.3 What type of point does f have at 9? (Hint, draw a freehand sketch of the curve.)

51.5.4 How could the second derivative of f be used to confirm your conclusion in problem 51.5.3? Go ahead and do it.

Activity 52

When searching for inflection points on a function, you can narrow your search by identifying numbers where the function is continuous (from both directions) and the second derivative is either zero or undefined. (By definition an inflection point cannot occur at a number where the function is not continuous from both directions.) You can then build a sign table for the second derivative that implies the concavity of the given function.

When performing this analysis, you need to simplify the second derivative formula in the same way you simplify the first derivative formula when looking for critical numbers and local extreme points.

Problem 52.1

Identify the inflection points for the function shown in Figure 49.2 (page 74).

Problem 52.2

The first two derivatives of the function $y(x)=\dfrac{(x+2)^2}{(x+3)^3}$ are $y'(x)=\dfrac{-x(x+2)}{(x+3)^4}$ and

$$y''(x)=\frac{2\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)}{(x+3)^5}.$$

52.2.1 Yolanda was given this information and asked to find the inflection points on y. The first thing Yolanda wrote was "The critical numbers of y are $\sqrt{3}$ and $-\sqrt{3}$." Explain to Yolanda why this is not true

52.2.2 What are the critical numbers of y and in what way are they important when asked to identify the inflection points on y?

52.2.3 Copy Table 52.1 (page 80) onto your paper and fill in the missing information.

52.2.4 State the inflection points on y; you may round the dependent coordinate of each point to the nearest hundredth.

52.2.5 The function y has a vertical asymptote at -3. Given that fact, it was impossible that y would have an inflection point at -3. Why, then, did we never-the-less break the interval $\left(-\infty,-\sqrt{3}\right)$ at -3 when creating our concavity table?

Table 52.1: $y''(x)=\dfrac{2\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)}{(x+3)^5}$

Interval	Sign of y''	Behavior of y
$(-\infty,-3)$		
$\left(-3,-\sqrt{3}\right)$		
$\left(-\sqrt{3},\sqrt{3}\right)$		
$\left(\sqrt{3},\infty\right)$		

Problem 52.3

Perform each of the following for the functions in problems 52.3.1-52.3.3.

- State the domain of the function.
- Find, and completely simplify, the formula for the second derivative of the function. It is ***not*** necessary to simplify the formula for the first derivative of the function.
- State the values in the domain of the function where the second derivative is either zero or does not exist.
- Create a table similar to Table 52.1. Number the tables, in order, 52.2-52.4. Don't forget to include table headings and column headings.
- State the inflection points on the function. Make sure that you explicitly address this question even if there are no inflection points.

52.3.1 $f(x)=x^4-12x^3+54x^2-10x+6$ **52.3.2** $g(x)=(x-2)^2e^x$

52.3.3 $G(x)=\sqrt{x^3}+6\sqrt{x}$

Problem 52.4

The second derivative of the function $w(t)=t^{1.5}-9t^{0.5}$ is $w''(t)=\dfrac{3(t+3)}{4t^{1.5}}$ yet w has no inflection points. Why is that?

Activity 53

We are frequently interested in a function's "end behavior;" that is, what is the behavior of the function as the input variable increases without bound or decreases without bound.

Many times a function will approach a horizontal asymptote as its end behavior. Assuming that the horizontal asymptote $y = L$ represents the end behavior of the function f both as x increases without bound and as x decreases without bound, we write $\lim_{x \to \infty} f(x) = L$ and $\lim_{x \to -\infty} f(x) = L$.

While working the *Limits and Continuity* lab you investigated strategies for formally establishing limit values as $x \to \infty$ or $x \to -\infty$. In this activity you are going to investigate a more informal strategy for determining these type limits.

Consider $\lim_{x \to \infty} \frac{4x-2}{3+20x}$. When the value of x is really large, we say that the term $4x$ dominates the numerator of the expression $\frac{4x-2}{3+20x}$ and the term $20x$ dominates the denominator. We actually call those terms ***the dominant terms*** of the numerator and denominator. The dominant terms are significant because when the value of x is really large, the other terms in the expression contribute almost nothing to the value of the expression. That is, for really large values of x:

$$\begin{aligned}\frac{4x-2}{3+20x} &\approx \frac{4x}{20x} \\ &= \frac{1}{5}\end{aligned}$$

For example, even if x has the paltry value of $1{,}000$,

$$\begin{aligned}\frac{4x-2}{3+20x} &= \frac{3998}{20003} \\ &\approx \frac{4000}{20000} \\ &= \frac{1}{5}\end{aligned}$$

This tells us that $\lim_{x \to \infty} \frac{4x-2}{3+20x} = \frac{1}{5}$ and that $y = \frac{1}{5}$ is a horizontal asymptote for the graph of $y = \frac{4x-2}{3+20x}$.

Problem 53.1

The formulas used to graph figures 7.1-7.5 (pages B1 and B2) are given below. Focusing first on the dominant terms of the expressions, match the formulas with the functions (f_1 through f_5).

53.1.1 $y = \frac{3x+6}{x-2}$

53.1.2 $y = \frac{16+4x}{6+x}$

53.1.3 $y = \frac{6x^2-6x-36}{36-3x-3x^2}$

53.1.4 $y = \frac{-2x+8}{x^2-100}$

53.1.5 $y = \frac{15}{x-5}$

Problem 53.2

Use the concept of dominant terms to informally determine the value of each of the following limits.

53.2.1 $\lim_{x \to -\infty} \frac{4 + x - 7x^3}{14x^3 + x^2 + 2}$

53.2.2 $\lim_{t \to -\infty} \frac{4t^2 + 1}{4t^3 - 1}$

53.2.3 $\lim_{\gamma \to \infty} \frac{8}{2\gamma^3}$

53.2.4 $\lim_{x \to \infty} \frac{(3x+1)(6x-2)}{(4+x)(1-2x)}$

53.2.5 $\lim_{t \to \infty} \frac{4e^t - 8e^{-t}}{e^t + e^{-t}}$

53.2.6 $\lim_{t \to -\infty} \frac{4e^t - 8e^{-t}}{e^t + e^{-t}}$

Activity 54

Let's put it all together and produce some graphs.

Problem 54.1

Consider the function $f(x) = \frac{8x^2 - 8}{(2x-4)^2}$.

54.1.1 Evaluate each of the following limits: $\lim_{x \to \infty} f(x)$, $\lim_{x \to -\infty} f(x)$, $\lim_{x \to 2^-} f(x)$, and $\lim_{x \to 2^+} f(x)$.

54.1.2 What are the horizontal and vertical asymptotes for a graph of *f* ?

54.1.3 What are the horizontal and vertical intercepts for a graph of *f* ?

54.1.4 Use the formulas $f'(x) = \frac{4(1-2x)}{(x-2)^3}$ and $f''(x) = \frac{4(4x+1)}{(x-2)^4}$ to help you accomplish each of the following.

- State the critical numbers of *f*.
- Create well-documented increasing/decreasing and concavity tables for *f*.
- State the local minimum, local maximum, and inflection points on *f*. Make sure that you explicitly address all three types of points whether they exist or not.

54.1.5 Graph $y = f(x)$ onto Figure 54.1. Make sure that you choose a scale that allows you to clearly illustrate each of the features found in problems 54.1.1-54.1.4. Make sure that all axes and asymptotes are well labeled and also write the coordinates of each local extreme point and inflection point next to the point on the graph.

54.1.6 Check your graph using a graphing calculator.

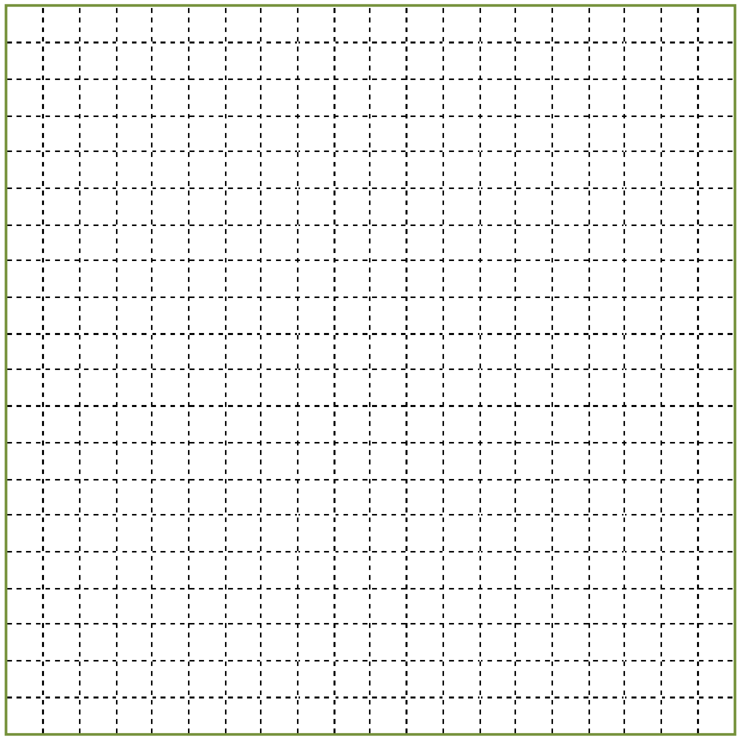


Figure 54.1: f

Problem 54.2

Consider the function $g(t)=\dfrac{1}{\left(e^t+4\right)^2}$.

54.2.1 Evaluate each of the following limits: $\lim\limits_{t\to\infty} g(t)$ and $\lim\limits_{t\to-\infty} g(t)$.

54.2.2 What are the horizontal and vertical asymptotes for a graph of g?

54.2.3 What are the horizontal and vertical intercepts for a graph of g ?

54.2.4 Use the formulas $g'(t)=\dfrac{-2e^t}{\left(e^t+4\right)^3}$ and $g''(t)=\dfrac{4\left(e^t-2\right)e^t}{\left(e^t+4\right)^2}$ to help you accomplish each of the following.

- State the critical numbers of g.
- Create well-documented increasing/decreasing and concavity tables for g.
- State the local minimum, local maximum, and inflection points on g. Make sure that you explicitly address all three types of points whether they exist or not.

54.2.5 Graph $y = g(t)$ onto Figure 54.2. Make sure that you choose a scale that allows you to clearly illustrate each of the features found in problems 54.2.1-54.2.4. Make sure that all axes and asymptotes are well labeled and also write the coordinates of each local extreme point and inflection point next to the point on the graph.

54.2.6 Check your graph using a graphing calculator.

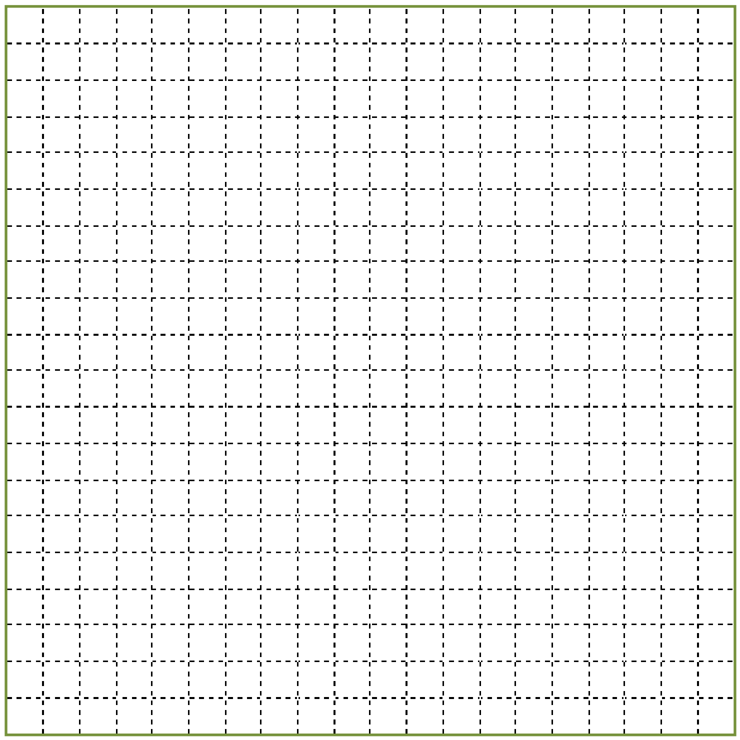


Figure 54.2: g

Supplemental Exercises for the Rates of Change Lab

Exercise 1.1

The function z shown in Figure E1.1 was generated by the formula $y = 2 + 4x - x^2$.

E1.1.1 Simplify the difference quotient for z.

E1.1.2 Use the graph to find the slope of the secant line to z between the points where $x = -1$ and $x = 2$. Check your simplified difference quotient for z by using it to find the slope of the same secant line.

E1.1.3 Replace x with 4 in your difference quotient formula and simplify the result. Then copy Table E1.1 onto your paper and fill in the missing values.

E1.1.4 As the value of h gets closer to 0, the values in the y column of Table 1.1 appear to be converging on a single number; what is this number?

E1.1.5 The value found in problem 1.1.4 is called ***the slope of the <u>tangent line</u> to z at 4***. Draw onto Figure 1 the line that passes through the point $(4, 2)$ with this slope. The line you just drew is called ***<u>the tangent line</u> to z at 4***.

Table E1.1: $y = \dfrac{z(4+h) - z(4)}{h}$

h	y
–0.1	–3.9
–0.01	–3.99
–0.001	–3.999
0.001	–4.001
0.01	–4.01
0.1	–4.01

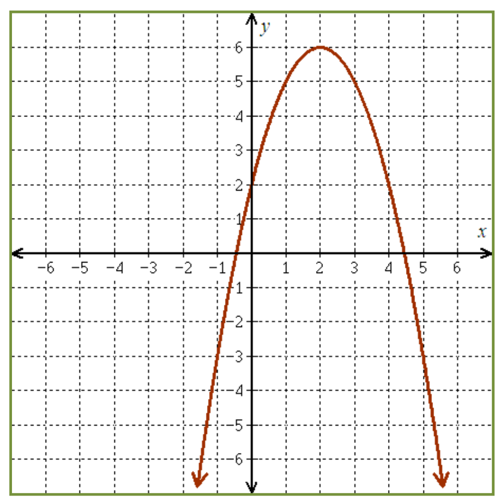


Figure E1.1: $y = 2 + 4x - x^2$

Exercise 1.2

Find the difference quotient for each function showing all relevant steps in an organized manner.

E1.2.1 $f(x) = 3 - 7x$

E1.2.2 $g(x) = \dfrac{7}{x + 4}$

E1.2.3 $z(x) = \pi$

E1.2.4 $s(t) = t^3 - t - 9$

E1.2.5 $k(t) = \dfrac{(t - 8)^2}{t}$

Exercise 1.3

Suppose that an object is tossed into the air in such a way that the elevation of the object (measured in ft) is given by the function $s(t) = 150 + 60t - 16t^2$ where t is the amount of time that has passed since the object was tossed (measure in seconds).

E1.3.1 Find the difference quotient for s.

E1.3.2 Use the difference quotient to determine the average velocity of the object over the interval $[4, 4.2]$ and then verify the value by calculating $\dfrac{s(4.2)-s(4)}{4.2-4}$.

Exercise 1.4

Several applied functions are given below. In each case, find the indicated quantity (including unit) and interpret the value in the context of the application.

E1.4.1 The velocity of a roller coaster (in ft/s) is given by $v(t) = -100\sin\left(\dfrac{\pi t}{15}\right)$ where t is the amount of time (s) that has passed since the coaster topped the first hill. Find and interpret $\dfrac{v(7.5)-v(0)}{7.5\text{s}-0\text{s}}$.

E1.4.2 The elevation of a ping pong ball relative to the table top (in m) is given by the function $h(t) = 1.1\left|\cos\left(\dfrac{2\pi t}{3}\right)\right|$ where t is the amount of time (s) that has passed since the ball went into play. Find and interpret $\dfrac{h(3)-h(1.5)}{3\text{s}-1.5\text{s}}$.

E1.4.3 The period of a pendulum (s) is given by $P(x) = \dfrac{6}{x+1}$ where x is the number of swings the pendulum has made. Find and interpret $\dfrac{P(29)-P(1)}{29\,\text{swing}-1\,\text{swing}}$.

E1.4.4 The acceleration of a rocket (mph/s) is given by $h(t) = .02 + .13t$ where t is the amount of time (s) that has passed since lift-off. Find and interpret $\dfrac{h(120)-h(60)}{120\text{s}-60\text{s}}$.

Supplemental Exercises for the Limits and Continuity Lab

Exercise 2.1

For each of tables E2.2-E2.5, state the limit suggested by the values in the table and state whether or not the limit exists. The correct answer for Table E2.1 has been given to help you understand the directions to the question.

Table E2.1: $y = g(t)$

t	y	Limit	Limit exist?
$-10{,}000$	99.97	$\lim_{t \to -\infty} g(t) = \infty$	No
$-100{,}000$	999.997		
$-1{,}000{,}000$	9999.9997		

Table E2.2: $y = f(x)$

x	y	Limit	Limit exist?
51,000	-3.2×10^{-5}		
51,0000	-3.02×10^{-7}		
5,100,000	-3.002×10^{-9}		

Table E2.3: $y = z(t)$

t	y	Limit	Limit exist?
.33	.66		
.333	.666		
.3333	.6666		

Table E2.4: $y = g(\theta)$

θ	y	Limit	Limit exist?
$-.9$	2,999,990		
$-.99$	2,999,999		
$-.999$	2,999,999.9		

Table E2.5: $y = T(t)$

t	y	Limit	Limit exist?
.778	29,990		
.7778	299,999		
.77778	2,999,999.9		

Exercise 2.2

Sketch onto Figure E2.1 a function, f, with each of the following properties. Make sure that your graph includes all of the relevant features addressed in lab.

- The only discontinuities on f occur at -4 and 3
- f has no x-intercepts
- f is continuous from the right at -4
- $\lim_{x\to -4^-} f(x) = 1$ and $\lim_{x\to -4^+} f(x) = -2$
- $\lim_{x\to 3} f(x) = -\infty$
- $\lim_{x\to \infty} f(x) = -\infty$
- f has a constant slope of -2 over $(-\infty, -4)$

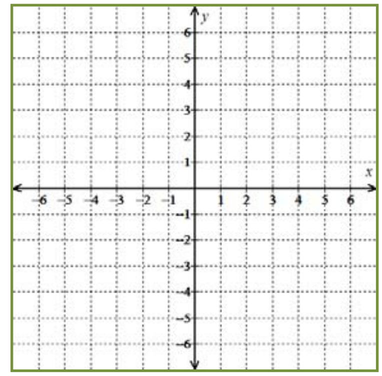


Figure E2.1: f

Exercise 2.3

Sketch onto Figure E2.2 a function, f, that satisfies each of the properties stated below. Assume that there are no intercepts or discontinuities other than those directly implied by the given properties. Make sure that your graph includes all of the relevant features addressed in lab.

- $\lim_{x\to -\infty} f(x) = \lim_{x\to \infty} f(x) = 3$
- $\lim_{x\to -4^+} f(x) = -2$ and $\lim_{x\to -4^-} f(x) = 5$
- $\lim_{x\to 3^-} f(x) = -\infty$ and $\lim_{x\to 3^+} f(x) = \infty$
- $f(-2) = 0$, $f(0) = -1$, and $f(-4) = 5$

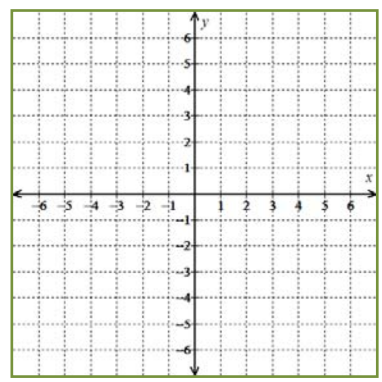


Figure E2.2 f

Exercise 2.4

Determine all of the values of x where the function f (given below) has discontinuities. At each value where f has a discontinuity, determine if f is continuous from either the right or left at x and also state whether or not the discontinuity is removable.

$$f(x) = \begin{cases} \dfrac{2\pi}{x} & \text{if} \quad x \le 3 \\ \sin\left(\dfrac{2\pi}{x}\right) & \text{if} \quad 3 < x < 4 \\ \dfrac{\sin(x)}{\sin(x)} & \text{if} \quad 4 < x \le 7 \\ 3 - \dfrac{2x-8}{x-4} & \text{if} \quad x > 7 \end{cases}$$

Exercise 2.5

Determine the value of k that makes the function $g(t)=\begin{cases} t^2+kt-k & \text{if} \quad t\ge 3 \\ t^2-4k & \text{if} \quad t<3 \end{cases}$ continuous over $(-\infty,\infty)$.

Exercise 2.6

Determine the appropriate symbol to write after an equal sign following each of the given limits. In each case, the appropriate symbol is either a real number, ∞, or $-\infty$. Also, state whether or not each limit exists and if the limit exists prove its existence (and value) by applying the appropriate limit laws. The Rational Limit Form table in Appendix C (Pages C3 and C4) summarizes strategies to be employed based upon the initial form of the limit.

E2.6.1 $\lim\limits_{x\to 4^-}\left(5-\dfrac{1}{x-4}\right)$

E2.6.2 $\lim\limits_{x\to\infty}\dfrac{e^{2/x}}{e^{1/x}}$

E2.6.3 $\lim\limits_{x\to 2^+}\dfrac{x^2-4}{x^2+4}$

E2.6.4 $\lim\limits_{x\to 2^+}\dfrac{x^2-4}{x^2-4x+4}$

E2.6.5 $\lim\limits_{x\to\infty}\dfrac{\ln(x)+\ln\left(x^6\right)}{7\ln\left(x^2\right)}$

E2.6.6 $\lim\limits_{x\to-\infty}\dfrac{3x^3+2x}{3x-2x^3}$

E2.6.7 $\lim\limits_{x\to\infty}\sin\left(\dfrac{\pi e^{3x}}{2e^x+4e^{3x}}\right)$

E2.6.8 $\lim\limits_{x\to\infty}\dfrac{\ln\left(1/x\right)}{\ln\left(x/x\right)}$

E2.6.9 $\lim\limits_{x\to 5}\sqrt{\dfrac{x^2-12x+35}{5-x}}$

E2.6.10 $\lim\limits_{h\to 0}\dfrac{4(3+h)^2-5(3+h)-21}{h}$

E2.6.11 $\lim\limits_{h\to 0}\dfrac{5h^2+3}{2-3h^2}$

E2.6.12 $\lim\limits_{h\to 0}\dfrac{\sqrt{9-h}-3}{h}$

E2.6.13 $\lim\limits_{\theta\to\frac{\pi}{2}}\dfrac{\sin\left(\theta+\frac{\pi}{2}\right)}{\sin(2\theta+\pi)}$

E2.6.14 $\lim\limits_{x\to 0^+}\dfrac{\ln\left(x^e\right)}{\ln\left(e^x\right)}$

Exercise 2.7

Draw sketches of the functions $y = e^x$, $y = e^{-x}$, $y = \ln(x)$, and $y = 1/x$. Note the coordinates of any and all intercepts. This is something you should be able to do from memory/intuition. If you cannot already do so, spend some time reviewing whatever you need to review so that you can do so.

Once you have the graphs drawn, fill in each of the blanks below and also circle whether or not each limit exists. The appropriate symbol for some of the blanks is either ∞ or $-\infty$

		Does limit exist?			Does limit exist?
$\lim\limits_{x\to\infty} e^x =$	________	yes or no	$\lim\limits_{x\to-\infty} e^x =$	________	yes or no
$\lim\limits_{x\to 0} e^x =$	________	yes or no	$\lim\limits_{x\to\infty} e^{-x} =$	________	yes or no
$\lim\limits_{x\to-\infty} e^{-x} =$	________	yes or no	$\lim\limits_{x\to 0} e^{-x} =$	________	yes or no
$\lim\limits_{x\to\infty} \ln(x) =$	________	yes or no	$\lim\limits_{x\to 1} \ln(x) =$	________	yes or no
$\lim\limits_{x\to 0^+} \ln(x) =$	________	yes or no	$\lim\limits_{x\to\infty} \frac{1}{x} =$	________	yes or no
$\lim\limits_{x\to-\infty} \frac{1}{x} =$	________	yes or no	$\lim\limits_{x\to 0^+} \frac{1}{x} =$	________	yes or no
$\lim\limits_{x\to 0^-} \frac{1}{x} =$	________	yes or no	$\lim\limits_{x\to\infty} e^{1/x} =$	________	yes or no
$\lim\limits_{x\to\infty} \frac{1}{e^x} =$	________	yes or no	$\lim\limits_{x\to-\infty} \frac{1}{e^x} =$	________	yes or no
$\lim\limits_{x\to\infty} \frac{1}{e^{-x}} =$	________	yes or no	$\lim\limits_{x\to-\infty} \frac{1}{e^{-x}} =$	________	yes or no

Supplemental Exercises for the Introduction to the First Derivative Lab

Exercise 3.1

Find the first derivative formula for each of the following functions twice: first by evaluating $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ and then by evaluating $\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$.

E3.1.1 $f(x) = x^2$ **E3.1.2** $f(x) = \sqrt{x}$ **E3.1.3** $f(x) = 7$

Exercise 3.2

It can be shown that $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$. Use these limits to help you to establish the first derivative formula for $\sin(x)$.

Hint: Begin with $\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$ and use the sum formula for $\sin(x+h)$.

Exercise 3.3

Suppose that an object is tossed into the air in such a way that the elevation of the object (measured in ft) is given by the function $s(t) = 150 + 60t - 16t^2$ where t is the amount of time that has passed since the object was tossed (measure in seconds).

E3.3.1 Find the velocity function for this motion and use that function to determine the velocity of the object 4.1 s into its motion.

E3.3.2 Find the acceleration function for this motion and use that function to determine the acceleration of the object 4.1 s into its motion.

Exercise 3.4

Determine the unit on the first derivative function for each of the following functions. Remember, *we do not simplify derivative units in any way, shape, or form.*

E3.4.1 $R(p)$ is Carl's heart rate (beats/min) when he jogs at a rate of p (measured in ft/min).

E3.4.2 $F(v)$ is the fuel consumption rate (gal/mi) of Hanh's pick-up when she drives it on level ground at a constant speed of v (measured in mi/hr).

E3.4.3 $v(t)$ is the velocity of the space shuttle (mi/hr) where t is the amount of time that has passed since lift-off (measured in seconds).

E3.4.4 $h(t)$ is the elevation of the space shuttle (mi) where t is the amount of time that has passed since lift-off (measured in seconds).

Exercise 3.5

Referring to the functions in Exercise 3.4, write sentences that explain the meaning of each of the following function values.

E3.5.1 $R(300) = 84$ **E3.5.2** $R'(300) = .02$ **E3.5.3** $F(50) = .03$

E3.5.4 $F'(50) = -.0006$ **E3.5.5** $v(20) = 266$ **E3.5.6** $v'(20) = 18.9$

E3.5.7 $h(20) = .7$ **E3.5.8** $h'(20) = .074$

Exercise 3.6

It can be shown that the derivative formula for the function $f(x) = \ln(x) + 2x$ is $f'(x) = \dfrac{2x+1}{x}$.

E3.6.1 Determine the equation of the tangent line to f at 1.

E3.6.2 A graph of f is shown in Figure E3.1; axis scales have deliberately been omitted from the graph. The graph shows that f quickly resembles a line. In a detailed sketch of f we would reflect this apparent linear behavior by adding a skew asymptote. What is the slope of this skew asymptote?

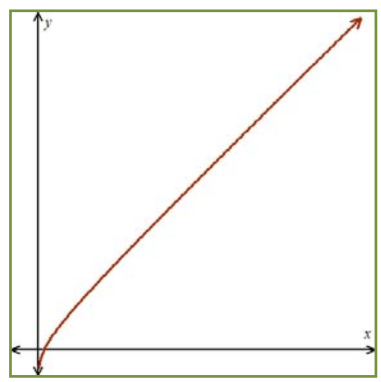


Figure E3.1: $f(x) = \ln(x) + 2x$

Supplemental Exercises for the Functions, Derivatives, and Antiderivatives Lab

Exercise 4.1

Sketch the first derivatives of the functions shown in figures E4.1a, E4.2a, and E4.3a onto, respectively, figures E4.1b, E4.2b, and E4.3b.

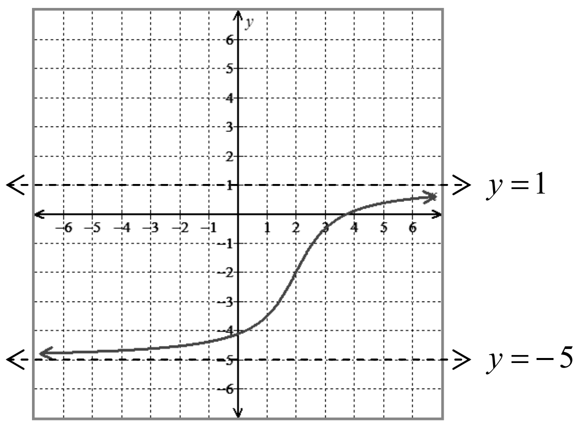


Figure E4.1a

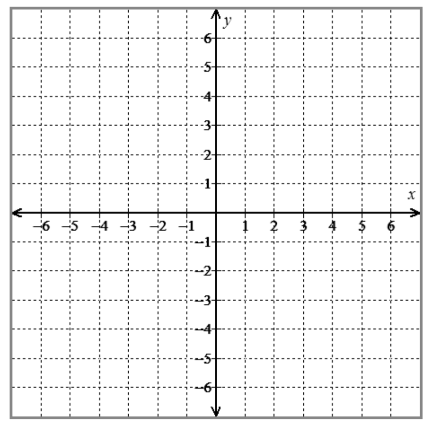


Figure E4.1b

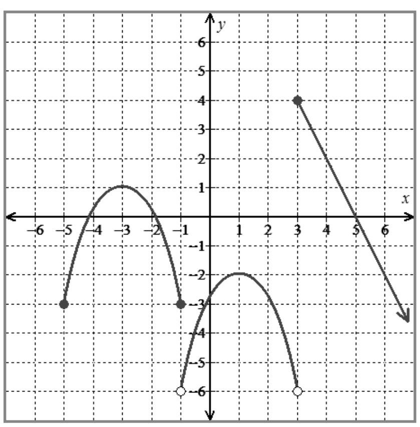


Figure E4.2a

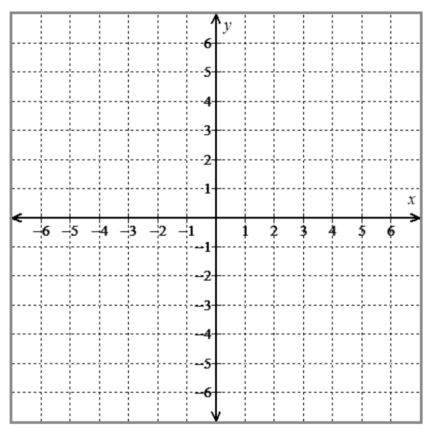


Figure E4.2b

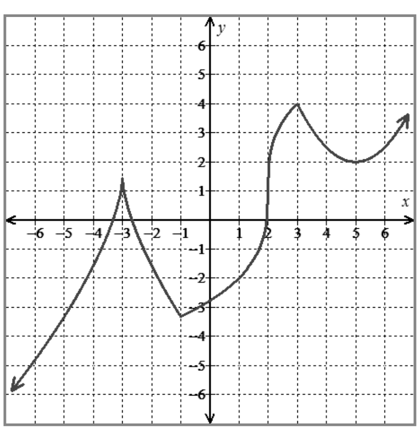


Figure E4.3c

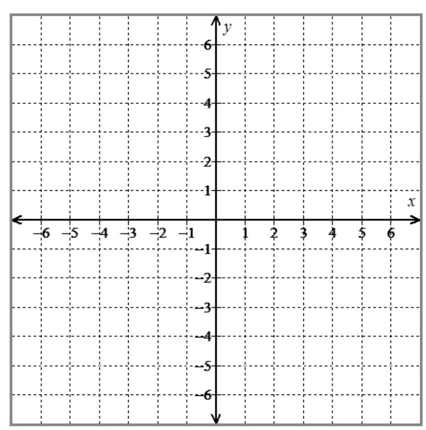


Figure E4.3b

Exercise 4.2

At the bottom of the page seven curves are identified as Figures A-G. For each statement below, identify the figure letters that go along with the given statement. Assume each statement is being made only in reference to the portion of the curves shown in each of the figures. Assume in all cases that the axes have the traditional positive/negative orientation. The first question has been answered for you to help you get started.

<u>C, D, and E</u> **E4.2.1** Each of these functions is always increasing.

__________ **E4.2.2** The first derivative of each of these functions is always increasing.

__________ **E4.2.3** The second derivative of each of these functions is always negative.

__________ **E4.2.4** The first derivative of each of these functions is always positive.

__________ **E4.2.5** Any antiderivative of each of these functions is always concave up.

__________ **E4.2.6** The second derivative of each of these functions is always equal to zero.

__________ **E4.2.7** Any antiderivative of each of these functions graphs to a line.

__________ **E4.2.8** The first derivative of each of these functions is a constant function.

__________ **E4.2.9** The second derivative of each of these functions is never positive.

__________ **E4.2.10** The first derivative of each of these functions **must** be linear.

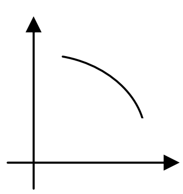


Figure A

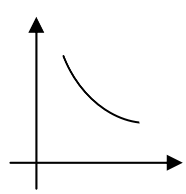


Figure B

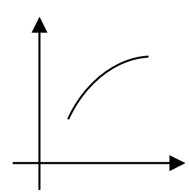


Figure C

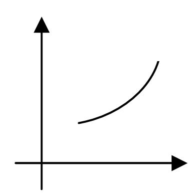


Figure D

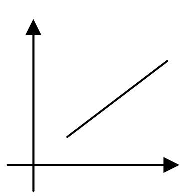


Figure E

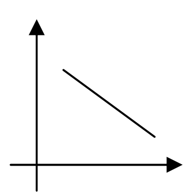


Figure F

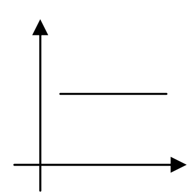


Figure G

Exercise 4.3

As water drains from a tank the drainage rate continually decreases over time. Suppose that a tank that initially holds 360 gal of water drains in exactly 6 minutes. Answer each of the following questions about this problem situation.

E4.3.1 Suppose that $V(t)$ is the amount of water (gal) left in the tank where t is the amount of time that has elapsed (s) since the drainage began. What are the units on $V'(45)$ and $V''(45)$? Which of the following is the most realistic value for $V'(45)$?

a 0.8 **b** 1.2 **c** -0.8 **d** -1.2

E4.3.2 Suppose that $R(t)$ is the water's flow rate (gal/s) where t is the amount of time that has elapsed (s) since the drainage began. What are the units on $R'(45)$ and $R''(45)$? Which of the following is the most realistic value for $R'(45)$?

a 1.0 **b** 0.001 **c** -1.0 **d** -0.001

E4.3.3 Suppose that $V(t)$ is the amount of water (gal) left in the tank where t is the amount of time that has elapsed (s) since the drainage began. Which of the following is the most realistic value for $V''(45)$?

a 1.0 **b** 0.001 **c** -1.0 **d** -0.001

Exercise 4.4

For each statement (E4.4.1-E4.4.5), decide which of the following is true.

i. The statement is true regardless of the specific function f.
ii. The statement is true for some functions and false for other functions.
iii. The statement is false regardless of the specific function f.

E4.4.1 If a function f is increasing over the entire interval $(-3,7)$, then $f'(0)>0$.

E4.4.2 If a function f is increasing over the entire interval $(-3,7)$, then $f'(4)=0$.

E4.4.3 If a function f is concave down over the entire interval $(-3,7)$, then $f'(-2)<0$.

E4.4.4 If the slope of f is increasing over the entire interval $(-3,7)$, then $f'(0)>0$.

E4.4.5 If a function f has a local maximum at 3, then $f'(3)=0$.

Exercise 4.5

For a certain function g, $g'(t) = -8$ at all values of t on $(-\infty, -3)$ and $g'(t) = -2$ at all values of t on $(-3, \infty)$.

E4.5.1 Janice thinks that g must be discontinuous at -3 but Lisa disagrees. Who's right?

E4.5.2 Lisa thinks that the graph of g'' is the line $y = 0$ but Janice disagrees. Who's right?

Exercise 4.6

Each statement below is in reference to the function shown in Figure E4.4. Decide whether each statement is true or false. A caption has been omitted from Figure E4.4 because the identity of the curve changes from question to question.

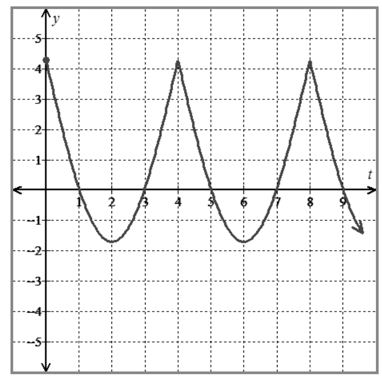


Figure E4.4

E4.6.1 If the given function is f', then $f(3) > f(2)$.

E4.6.2 If the given function is f, then $f'(3) > f'(2)$.

E4.6.3 If the given function is f, then $f'(1) > f''(1)$.

E4.6.4 If the given function is f', then the tangent line to f at 6 is horizontal.

E4.6.5 If the given function is f', then f'' is always increasing.

E4.6.6 If the given function is f', then f is nondifferentiable at 4.

E4.6.7 If the given function is f', then f' is nondifferentiable at 4.

E4.6.8 The first derivative of the given function is periodic (starting at 0).

E4.6.9 Antiderivatives of the given function are periodic (starting at 0).

E4.6.10 If the given function was measuring the rate at which the volume of air in your lungs was changing t seconds after you were frightened, then there was the same amount of air in your lungs 9 seconds after the fright as was there 7 seconds after the fright.

E4.6.11 If the given function is measuring the position of a weight attached to a spring relative to a table edge (with positive and negative positions corresponding, respectively, to the weight being above and below the edge of the table), then the weight is in the same position 9 seconds after it begins to bob as it is 7 seconds after the bobbing commenced (assuming that t represents the number of seconds that pass after the weight begins to bob).

E4.6.12 If the given function is measuring the velocity of a weight attached to a spring relative to a table edge (with positive and negative positions corresponding, respectively, to the weight being above and below the edge of the table), then the spring was moving downward over the interval $(1,3)$ (assuming that t represents the number of seconds that pass after the weight begins to bob).

Exercise 4.7

Complete as many cells in Table E4.1 as possible so that when read left to right the relationships between the derivatives, function, and antiderivative are always true. Please note that several cells will remain blank.

Table E4.1

f''	f'	f	F
		Positive	
		Negative	
		Constantly Zero	
		Increasing	
		Decreasing	
		Constant	
		Concave Up	
		Concave Down	
		Linear	

Exercise 4.8

Answer each question about the function f whose first derivative is shown in Figure E4.5. The correct answer to one or more of the questions may be "there's no way of knowing."

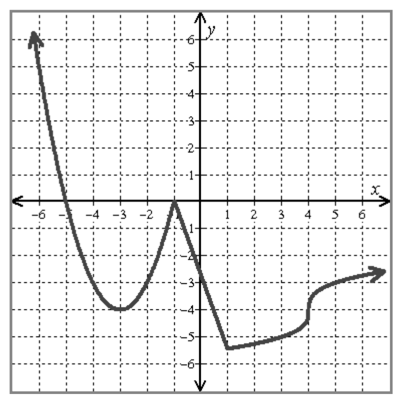


Figure E4.5: f'

E4.8.1 Where, over $(-6,6)$, is f' nondifferentiable?

E4.8.2 Where, over $(-6,6)$, are antiderivatives of f nondifferentiable?

E4.8.3 Where, over $(-6,6)$, is f decreasing?

E4.8.4 Where, over $(-6,6)$, is f concave down?

E4.8.5 Where, over $(-6,6)$, is f'' decreasing?

E4.8.6 Where, over $(-6,6)$, is f'' positive?

E4.8.7 Where, over $(-6,6)$, are antiderivatives of f concave up?

E4.8.8 Where, over $(-6,6)$, are antiderivatives of f increasing?

E4.8.9 Where, over $(-6,6)$, does f have its maximum value?

E4.8.10 Suppose that $f(-3)=14$. What, then, is the equation of the tangent line to f at -3?

Exercise 4.9

A certain antiderivative, F, of the function f shown in Figure E4.6a passes through the points $(-3,6)$ and $(3,-2)$. Draw this antiderivative (over the interval $(-6,6)$) onto Figure E4.6b.

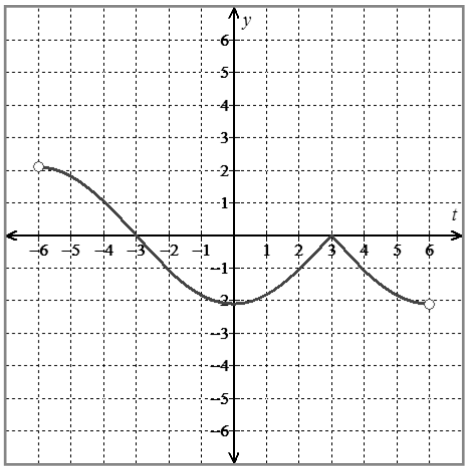


Figure E4.6a: f

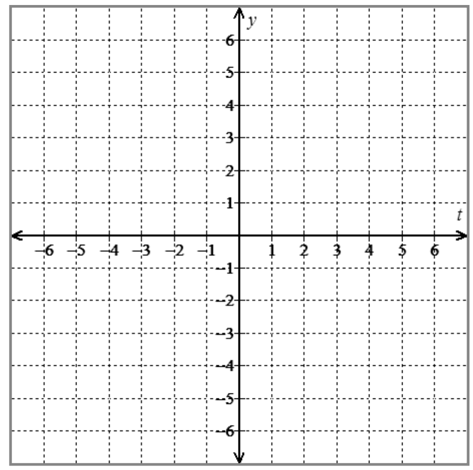


Figure E4.6b: F

Exercise 4.10

Draw onto Figure E4.7 the function f given the following properties of f.

- $f(0)=f(4)=3$
- The only discontinuity on f occurs at the vertical asymptote $x=2$.
- $f'(-1)=0$
- $f'(x)>0$ on $(-\infty,-1)$, $(-1,2)$, and $(4,\infty)$
- $f'(x)<0$ on $(2,4)$
- $f''(x)>0$ on $(-1,2)$ and $(2,4)$
- $f''(x)<0$ on $(-\infty,-1)$
- $f''(x)=0$ on $(4,\infty)$
- $\lim_{x\to 4^-} f'(x)=-1$ and $\lim_{x\to 4^+} f'(x)=1$

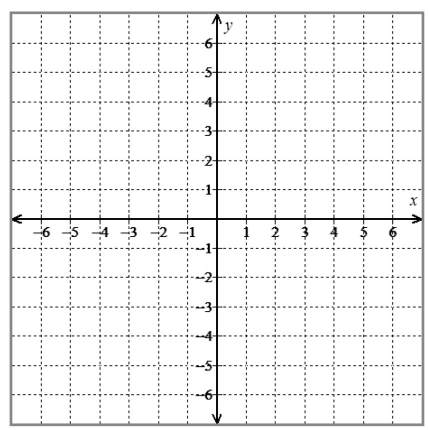


Figure E4.7 f

Supplemental Exercises for the Derivative Formulas Lab

Exercise 5.1

Apply the constant factor and power rules of differentiation to find the first derivative of each of the following functions. In each case take the derivative with respect to the independent variable suggested by the expression on the right side of the equal sign. Make sure that you assign the proper name to the derivative function. Please note that there is no need to employ either the quotient rule or the product rule to find any of these derivative formulas.

E5.1.1 $f(x)=\dfrac{\sqrt[11]{x^6}}{6}$ **E5.1.2** $y=-\dfrac{5}{t^7}$ **E5.1.3** $y(u)=12\sqrt[3]{u}$

E5.1.4 $z(\alpha)=e\alpha^{\pi}$ **E5.1.5** $z=\dfrac{8}{\sqrt{t^7}}$ **E5.1.6** $T=\dfrac{4\sqrt[3]{t^7}}{t^2}$

Exercise 5.2

Apply the constant factor and product rules of differentiation to find the first derivative of each of the following functions. Write out the Leibniz notation for the product rule as applied to the given function. In each case take the derivative with respect to the independent variable suggested by the expression on the right side of the equal sign. Make sure that you assign the proper name to the derivative function.

E5.2.1 $y=5\sin(x)\cos(x)$ **E5.2.2** $y=\dfrac{5t^2e^t}{7}$ **E5.2.3** $F(x)=4x\ln(x)$

E5.2.4 $z=x^2\sin^{-1}(x)$ **E5.2.5** $T(t)=\left(1+t^2\right)\tan^{-1}(t)$ **E5.2.6** $T=\dfrac{x^7\cdot 7^x}{3}$

Exercise 5.3

Apply the constant factor and quotient rules of differentiation to find the first derivative of each of the following functions. Write out the Leibniz notation for the quotient rule as applied to the given function. In each case take the derivative with respect to the independent variable suggested by the expression on the right side of the equal sign. Make sure that you assign the proper name to the derivative function.

E5.3.1 $q(\theta)=\dfrac{4e^{\theta}}{e^{\theta}+1}$ **E5.3.2** $u(x)=2\dfrac{\ln(x)}{x^4}$ **E5.3.3** $F=\dfrac{\sqrt{t}}{3t^2-5\sqrt{t^3}}$

E5.3.4 $F(x)=\dfrac{\tan(x)}{\tan^{-1}(x)}$ **E5.3.5** $p=\dfrac{\tan^{-1}(t)}{1+t^2}$ **E5.3.6** $y=\dfrac{4}{\sin(\beta)-2\cos(\beta)}$

Exercise 5.4

Multiple applications of the product rule are required when finding the derivative formula of the product of three of more functions. After simplification, however, the predictable pattern shown below emerges.

$$\frac{d}{dx}(f\,g)=\frac{d}{dx}(f)\,g+f\,\frac{d}{dx}(g)$$

$$\frac{d}{dx}(f\,g\,h)=\frac{d}{dx}(f)\,g\,h+f\,\frac{d}{dx}(g)\,h+f\,g\,\frac{d}{dx}(h)$$

$$\frac{d}{dx}(f\,g\,h\,k)=\frac{d}{dx}(f)\,g\,h\,k+f\,\frac{d}{dx}(g)\,h\,k+f\,g\,\frac{d}{dx}(h)\,k+f\,g\,h\,\frac{d}{dx}(k)$$

E5.4.1 Apply the product rule twice to the expression $\frac{d}{dx}\big(f(x)g(x)h(x)\big)$ to verify the formula stated above for the product of three functions. The first application of the product rule is shown below to help you get you started.

$$\frac{d}{dx}\big(f(x)g(x)h(x)\big)=\frac{d}{dx}\big(f(x)\big)\big[g(x)h(x)\big]+f(x)\frac{d}{dx}\big[g(x)h(x)\big]$$

E5.4.2 Apply the formula shown for the product of four functions to determine the derivative with respect to x of the function $f(x)=x^2\,e^x\sin(x)\cos(x)$

Exercise 5.5

Find the first derivative with respect to x for each of the following functions and for each function find the equation of the tangent line at the stated value of x. Make sure that you use appropriate techniques of differentiation.

E5.5.1 $f(x)=\dfrac{x^3+x^2}{x}$; tangent line at 5

E5.5.2 $h(x)=\dfrac{x}{1+x}$; tangent line at -2

E5.5.3 $K(x)=\dfrac{1+x}{2x+2}$; tangent line at 8

E5.5.4 $r(x)=3\sqrt[3]{x}\,e^x$; tangent line at 0

Exercise 5.6

Find the second derivative of y with respect to t if $y=4\sqrt{t}\,t^5$. Make sure that you use appropriate techniques of differentiation and that you use proper notation on both sides of the equal sign.

Exercise 5.7

A function f is shown in Figure E5.1. Each question below is in reference to this function. Please note that the numeric endpoints of each interval answer are all integers.

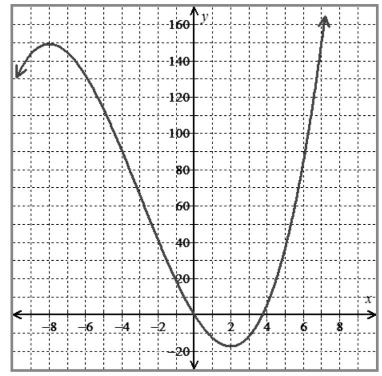


Figure E5.1

E5.7.1 Over what intervals is f' positive and over what intervals is f' negative?

E5.7.2 Over what intervals is f' increasing and over what intervals is f' decreasing?

E5.7.3 Over what intervals is f'' positive and over what intervals is f'' negative?

E5.7.4 The formula for f is $f(x) = \dfrac{x^3}{3} + 3x^2 - 16x$. Find the formulas for f' and f'', and then graph the derivatives and verify your answers to parts a-c.

Exercise 5.8

A function g is shown in Figure E5.2. Each question below is in reference to this function.

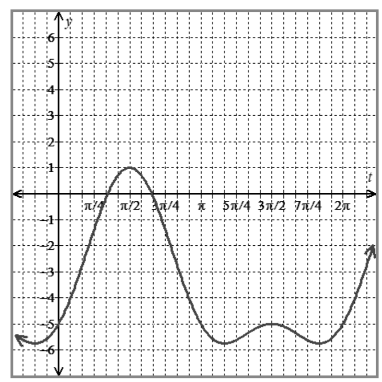


Figure E5.2

E5.8.1 Where over the interval $[0, 2\pi]$ does $g'(t) = 0$? Please note that the t-scale in Figure E5.2 is $\dfrac{\pi}{12}$.

E5.8.2 The formula for g is $g(t) = 3\sin(t) + 3\sin^2(t) - 5$. Find the formula for g' and use that formula to solve the equation $g'(t) = 0$ over the interval $[0, 2\pi]$. Compare your answers to those found in part a.

Hint: $\sin^2(t) = \sin(t)\sin(t)$

Exercise 5.9

Find the equation of the tangent line to f' at π if $f(x) = \sin(2x)$.

Hint: Begin by applying a double angle identity. Also, make sure that you read the question carefully.

Exercise 5.10

In Table E5.1 several function and derivative values are given for the functions f and g. This entire problem is based upon these two functions.

Suppose that you were asked to find the value of $h'(2)$ where $h(x)=f(x)g(x)$. The first thing you would need to do is find a formula for $h'(x)$. You would then replace x with 2, substitute the appropriate function and derivative values, and simplify. This is illustrated in example E5.1.

Example E5.1

$$h'(x)=\frac{d}{dx}(f(x))g(x)+f(x)\frac{d}{dx}(g(x))$$
$$=f'(x)g(x)+f(x)g'(x)$$

$$h'(2)=f'(2)g(2)+f(2)g'(2)$$
$$=(7)(11)+(4)(37)$$
$$=225$$

Following the process outlined in example E5.1, find each of the following.

E5.10.1 Find $h'(4)$ where $h(x)=f(x)g(x)$.

E5.10.2 Find $h''(2)$ where $h(x)=f(x)g(x)$.

E5.10.3 Find $k'(3)$ where $k(x)=\dfrac{g(x)}{f(x)}$.

E5.10.4 $p'(4)$ where $p(x)=6\sqrt{x}\,f(x)$.

E5.10.5 $r'(1)$ where $r(x)=[g(x)]^2$.

E5.10.6 $s'(2)$ where $s(x)=x\,f(x)g(x)$.

E5.10.7 $F'(4)$ where $F(x)=\sqrt{x}\,g(4)$.

E5.10.8 $T''(0)$ where $T(x)=\dfrac{f(x)}{e^x}$.

Table E5.1

x	$f(x)$	$f'(x)$	$f''(x)$	$g(x)$	$g'(x)$	$g''(x)$
0	2	−1	−2	−3	−3	−2
1	1	0	4	−5	2	16
2	4	7	10	11	37	58
3	15	20	16	87	126	124
4	46	39	22	289	293	214

Supplemental Exercises for the Chain Rule Lab

Exercise 6.1

The functions $f(x)=e^{3x}$, $g(x)=\left(e^{x}\right)^{3}$, and $h(x)=\left(e^{3}\right)^{x}$ are equivalent. Find the first derivative formulas for each of the functions (without altering their given forms) and then explicitly establish that the derivative formulas are the same.

Exercise 6.2

Consider the function $k(\theta)=\sin^{-1}(\sin(\theta))$.

E6.2.1 Over the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $k(\theta)=\theta$. What does this tell you about the formula for $k'(\theta)$ over $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$?

E6.2.2 Use the chain rule and an appropriate trigonometric identity to verify your answer to question E6.2.1. Please note that $\cos(\theta)>0$ over $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

E6.2.3 What is the constant value of $k'(\theta)$ over $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$. Hint: What is the sign on $\cos(\theta)$ over that interval?

E6.2.4 What is the value of $k'\left(\frac{\pi}{2}\right)$?

Exercise 6.3

Consider the function f shown in Figure E6.1.

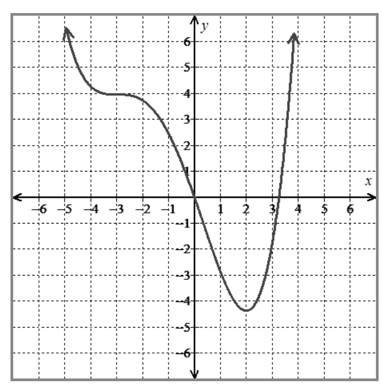


Figure E6.1: f

E6.3.1 Suppose that $g(x)=\left[f(x)\right]^{4}$. Over what intervals is g' positive?

E6.3.2 Suppose that $r(x)=e^{f(x)}$. Over what intervals is r' positive?

E6.3.3 Suppose that $w(x)=e^{f(-x)}$. Over what intervals is w' positive?

E6.3.4 Suppose that $h(x)=\frac{1}{f(x)}$. Is h nondifferentiable at -3?

(Please note that f has a horizontal tangent line at -3.)

Exercise 6.4

Decide whether or not it is necessary to use the chain rule when finding the derivative with respect to x of each of the following functions.

E6.4.1 $f(x)=\ln(x+1)$

E6.4.2 $f(x)=\dfrac{2}{x^5}$

E6.4.3 $f(x)=\cos(\pi)$

E6.4.4 $f(x)=\cos(x)$

E6.4.5 $f(x)=\cos(\pi x)$

E6.4.6 $f(x)=\dfrac{\cos(\pi x)}{\pi}$

Exercise 6.5

Find the first derivative with respect to x of each of the following functions. In all cases, look for appropriate simplifications before taking the derivative. Please note that some of the functions will be simpler to differentiate if you first use the rules of logarithms to expand and simplify the logarithmic expression.

E6.5.1 $f(x)=\tan^{-1}\left(\sqrt{x}\right)$

E6.5.2 $f(x)=e^{e^{\sin(x)}}$

E6.5.3 $f(x)=\sin^{-1}\left(\cos(x)\right)$

E6.5.4 $f(x)=\tan\left(x\sec(x)\right)$

E6.5.5 $f(x)=\tan(x)\sec\left(\sec(x)\right)$

E6.5.6 $f(x)=\sqrt[3]{\left(\sin\left(x^2\right)\right)^2}$

E6.5.7 $f(x)=4x\sin^2(x)$

E6.5.8 $f(x)=\ln\left(x\ln(x)\right)$

E6.5.9 $f(x)=\ln\left(\dfrac{5}{xe^x}\right)$

E6.5.10 $f(x)=2\ln\left(\sqrt[3]{x\tan^2(x)}\right)$

E6.5.11 $f(x)=\ln\left(\dfrac{e^{x+2}}{\sqrt{x+2}}\right)$

E6.5.12 $f(x)=\ln\left(x^e+e\right)$

E6.5.13 $f(x)=\sec^4\left(e^x\right)$

E6.5.14 $f(x)=\sec^{-1}\left(e^x\right)$

E6.5.15 $f(x)=\csc\left(\dfrac{1}{\sqrt{x}}\right)$

E6.5.16 $f(x)=\dfrac{1}{\csc\left(\sqrt{x}\right)}$

E6.5.17 $f(x)=\dfrac{\tan^{-1}(2x)}{2}$

E6.5.18 $f(x)=x^3\sin\left(\dfrac{x}{3}\right)$

E6.5.19 $f(x)=\dfrac{4}{\sqrt{\dfrac{3}{x^7}}}$

E6.5.20 $f(x)=\dfrac{e^{xe^x}}{x}$

E6.5.21 $f(x)=xe^{xe^2}$

E6.5.22 $f(x)=\dfrac{\sin^5(x)-\sqrt{\sin(x)}}{\sin(x)}$

E6.5.23 $f(x)=4x\sin(x)\cos\left(x^2\right)$

E6.5.24 $f(x)=\sin\left(x\cos^2(x)\right)$

Supplemental Exercises for the Implicit Differentiation Lab

Exercise 7.1

The curve $x\sin(xy) = y$ is shown in Figure E7.1. Find a formula for $\frac{dy}{dx}$ and use that formula to determine the x-coordinate at each of the two points the curve crosses the x-axis. (Note: The tangent line to the curve is vertical at each of these points.) Scales have deliberately been omitted in Figure E7.1.

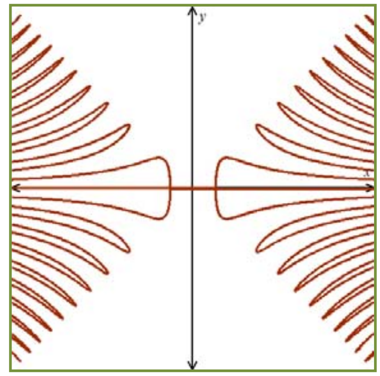


Figure E7.1: $x\sin(xy) = y$

Exercise 7.2

Solutions to the equation $\ln\left(x^2 y^2\right) = x + y$ are graphed in Figure E7.2. Determine the equation of the tangent line to this curve at the point $(1, -1)$.

Hint: It is easier to differentiate if you first use rules of logarithms to completely expand the logarithmic expression.

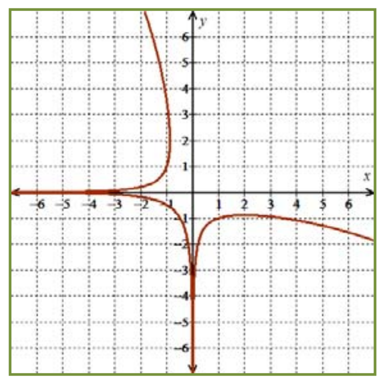


Figure E7.2: $\ln\left(x^2 y^2\right) = x + y$

Exercise 7.3

You have formulas that allow you to differentiate x^2, 2^x, and 2^2. You don't, however, have a formula to differentiate x^x. In this exercise you are going to use a process called **<u>logarithmic differentiation</u>** to determine the derivative formula for the function $y = x^x$. Example E7.1 (page A22) shows this process for a different function.

The function $y = x^x$ is only defined for positive values of x (which in turn means y is also positive), so we can say that $\ln(y) = \ln\left(x^x\right)$. What you need to do is use implicit differentiation to find a formula for $\frac{dy}{dx}$ ***<u>after first applying the power rule of logarithms to the logarithmic expression on the right side of the equal sign</u>***. Once you have your formula for $\frac{dy}{dx}$, substitute x^x for y. Voila! You will have the derivative formula for x^x. So go ahead and do it.

Exercise 7.4

In the olden days (pre-symbolic calculators) we would use the process of logarithmic differentiation to find derivative formulas for complicated functions. The reason this process is "simpler" than straight forward differentiation is that we can obviate the need for the product and quotient rules if we completely expand the logarithmic expression before taking the derivative.

Use the process of logarithmic differentiation to find a first derivative formula for each of the following functions. The process of logarithmic differentiation is illustrated in .

7.4.1 $y = \dfrac{x\sin(x)}{\sqrt{x-1}}$ **7.4.2** $y = \dfrac{e^{2x}}{\sin^4(x)\sqrt[4]{x^5}}$ **7.4.3** $y = \dfrac{\ln(4x^3)}{x^5\ln(x)}$

Example E7.1

$$y = \frac{x^{e^x}}{4x+1}$$

$$\ln(y) = \ln\left(\frac{x^{e^x}}{4x+1}\right)$$

Set the natural logarithms of the two expressions equal to one another.

$$\ln(y) = \ln\left(x^{e^x}\right) - \ln(4x+1)$$

$$\ln(y) = e^x\ln(x) - \ln(4x+1)$$

Completely expand the logarithmic expression on the right side of the equal sign.

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left(e^x\ln(x) - \ln(4x+1)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left(e^x\right)\cdot\ln(x) + e^x\cdot\frac{d}{dx}(\ln(x)) - \frac{1}{4x+1}\cdot\frac{d}{dx}(4x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = e^x\ln(x) + \frac{e^x}{x} - \frac{4}{4x+1}$$

$$\frac{dy}{dx} = y\cdot\left(e^x\ln(x) + \frac{e^x}{x} - \frac{4}{4x+1}\right)$$

Solve for $\frac{dy}{dx}$ after going through the process of implicit differentiation.

$$\frac{dy}{dx} = \frac{x^{e^x}}{4x+1}\cdot\left(e^x\ln(x) + \frac{e^x}{x} - \frac{4}{4x+1}\right)$$

Replace y with its original formula in the formula for $\frac{dy}{dx}$.

Supplemental Exercises for the Related Rates Lab

Exercise 8.1

A weather balloon is rising vertically at the rate of 10 ft/s. An observer is standing on the ground 300 ft horizontally from the point where the balloon was released. At what rate is the distance between the observer and the balloon changing when the balloon is 400 ft height?

Exercise 8.2

Imagine a railroad-crossing gate. For purposes of this problem we are going to treat the arm of the gate as a line that pivots on the vertical pole via a round gear whose center is exactly 4 feet off the ground. The distance between the tip of the arm and the center of the gear is exactly 28 ft. When the arm is being lowered the angle of elevation of the arm decreases at a constant rate of 6^{o}/sec. Find the rate at which the tip of the arm approaches the ground (vertically) at the instant the angle of elevation of the arm is 30^{o}.

Exercise 8.3

Jimbo is drinking soda from a conical cup; the radius of the cup at its top is 5 cm and the height of the cup is 10 cm. Jimbo is drinking through a straw at a constant rate of 0.25 cm^3/s. Assuming that the cup remains vertical whilst Jimbo drinks, find the rate of change in the height of the liquid when there are exactly 100 cm^3 of soda left in the cup

Exercise 8.4

A certain snowball maintains a perfectly spherical shape as it melts; the snowball melts at a constant rate of 25 cm^3/min. Determine the rate at which the surface area of the snowball changes at the instant the radius of the snowball has a radius of 6 cm.

Hint: What quantity is measured using cubic centimeters? That quantity and the surface area are the variables for the problem.

Supplemental Exercises for the Critical Numbers and Graphing from Formulas Lab

Exercise 9.1

The sine and cosine function are called circular functions because for any given value of t the point $(\cos(t), \sin(t))$ lies on the circle with equation $x^2 + y^2 = 1$.

There are analogous functions called hyperbolic sine and hyperbolic cosine. As you might suspect, these functions generate points that lie on a hyperbola; specifically, for all values of t the point $(\cosh(t), \sinh(t))$ lies on the hyperbola $x^2 - y^2 = 1$. It turns out that there are alternate formulas for the hyperbolic functions. Specifically:

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \text{ and } \sinh(t) = \frac{e^t - e^{-t}}{2}$$

E9.1.1 Use the exponential formulas to verify the identity $\cosh^2(t) - \sinh^2(t) = 1$.

E9.1.2 Use the exponential formulas to determine the first derivatives (with respect to t) of $\cosh(t)$ and $\sinh(t)$. Determine, by inference, the second derivative formulas for each of these functions.

E9.1.3 Determine the critical numbers of $\cosh(t)$ and $\sinh(t)$.

E9.1.4 Determine the intervals over which $\cosh(t)$ and $\sinh(t)$ are increasing/decreasing and over which they are concave up/concave down.

E9.1.5 Use the exponential formulas to determine each of the following limits: $\lim\limits_{t \to -\infty} \cosh(t)$, $\lim\limits_{t \to \infty} \cosh(t)$, $\lim\limits_{t \to -\infty} \sinh(t)$, and $\lim\limits_{t \to \infty} \sinh(t)$.

E9.1.6 Use the information you determined in problems E9.1.3-E9.1.5 to help you draw freehand sketches of $y = \cosh(t)$ and $y = \sinh(t)$.

E9.1.7 There are four more hyperbolic functions that correspond to the four additional circular functions; e.g. $\tanh(t) = \dfrac{\sinh(t)}{\cosh(t)}$. Find the exponential formulas for these four functions.

E9.1.8 What would you guess to be the first derivative of $\tanh(t)$? Take the derivative of the exponential formula for $\tanh(t)$ to verify your suspicion.

E9.1.9 Let $f(t) = \tanh(t)$. What is the sign on $f'(t)$ at all values of t? What does this tell you about the function f? What are the values of $f(0)$ and $f'(0)$?

E9.1.10 Use the exponential formula to determine $\lim\limits_{t \to -\infty} \tanh(t)$ and $\lim\limits_{t \to \infty} \tanh(t)$.

E9.1.11 Use the information you determined in problems E9.1.9-E9.1.10 to help you draw a freehand sketch of $y = \tanh(t)$.

Exercise 9.2

Consider the function $k(t)=t^{8/3}-256t^{2/3}$.

E9.2.1 What are the critical numbers of k? Remember to show all relevant work! Remember that your formula for $k'(t)$ needs to be a single, completely factored, fraction!

E9.2.2 Create an increasing/decreasing table for k.

E9.2.3 State each local minimum point and local maximum point on k.

Exercise 9.3

Consider the function $f(x)=\cos^2(x)+\sin(x)$ over the restricted domain $[0,2\pi]$.

E9.3.1 What are the critical numbers of f? Remember to show all relevant work! Remember that your formula for $f'(x)$ needs to be a single, completely factored, fraction!

E9.3.2 Create an increasing/decreasing table for f.

E9.3.3 State each local minimum point and local maximum point on f.

Exercise 9.4

Consider $g(t)=\dfrac{t+9}{t^3}$. Find (and completely simplify) $g''(t)$ and state all numbers where $g''(t)$ is zero or undefined. Then construct a concavity table for g and state all of the inflection points on g.

Exercise 9.5

For each of the following functions build increasing/decreasing tables and concavity tables and then state all local minimum points, local maximum points, and inflection points on the function. Also determine and state all horizontal asymptotes and vertical asymptotes for the function. Finally, draw a detailed sketch of the function.

E9.5.1 $f(x)=\dfrac{x-3}{(x+2)^2}$ **E9.5.2** $g(x)=x^{2/3}(x+5)$ **E9.5.3** $k(x)=\dfrac{(x-4)^2}{x+3}$

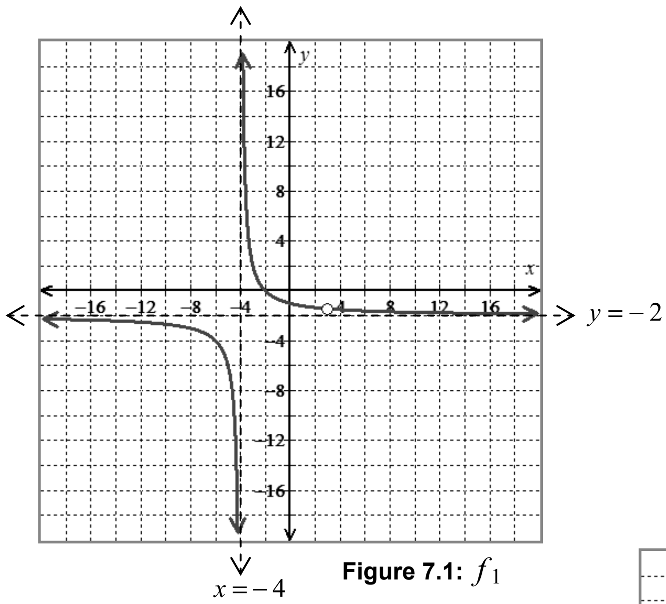


Figure 7.1: f_1

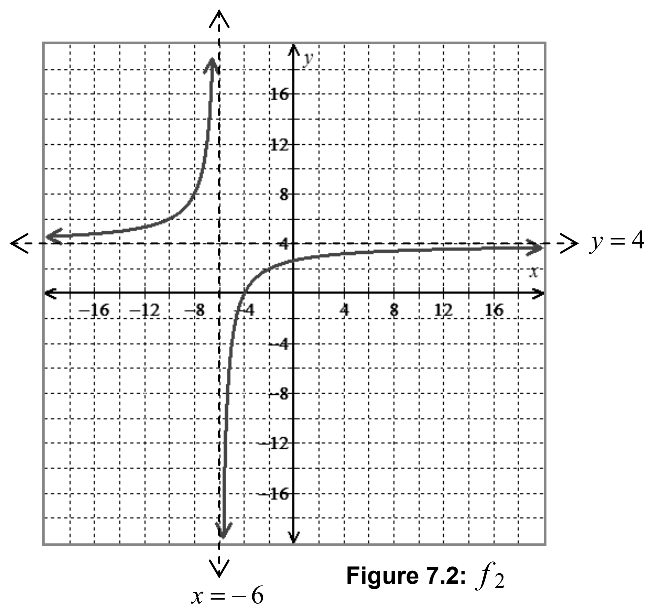


Figure 7.2: f_2

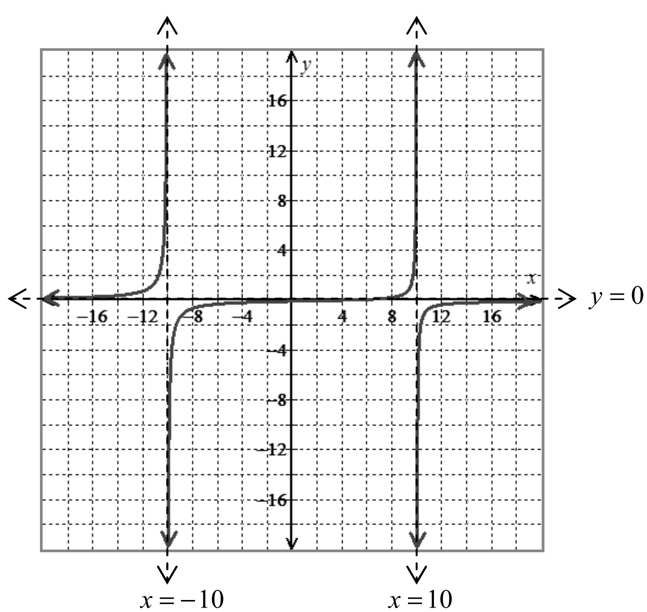


Figure 7.3: f_3

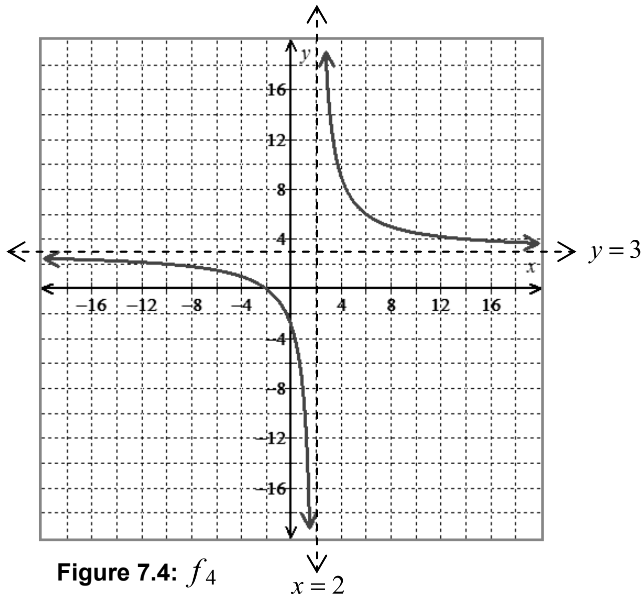


Figure 7.4: f_4

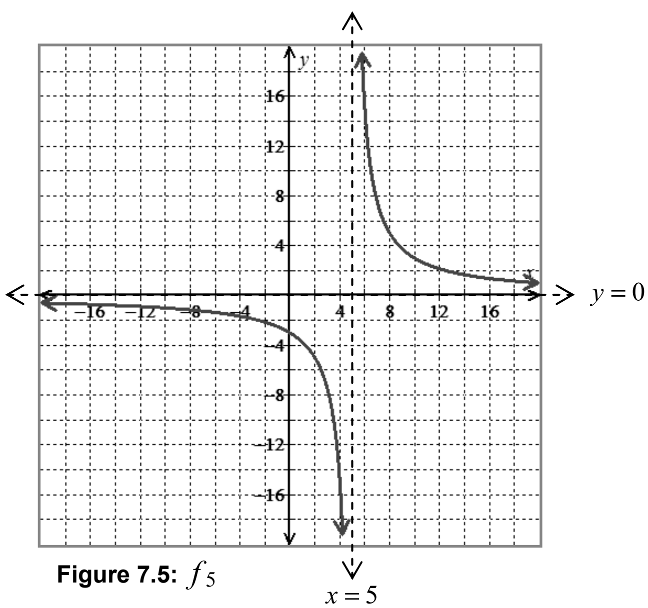


Figure 7.5: f_5

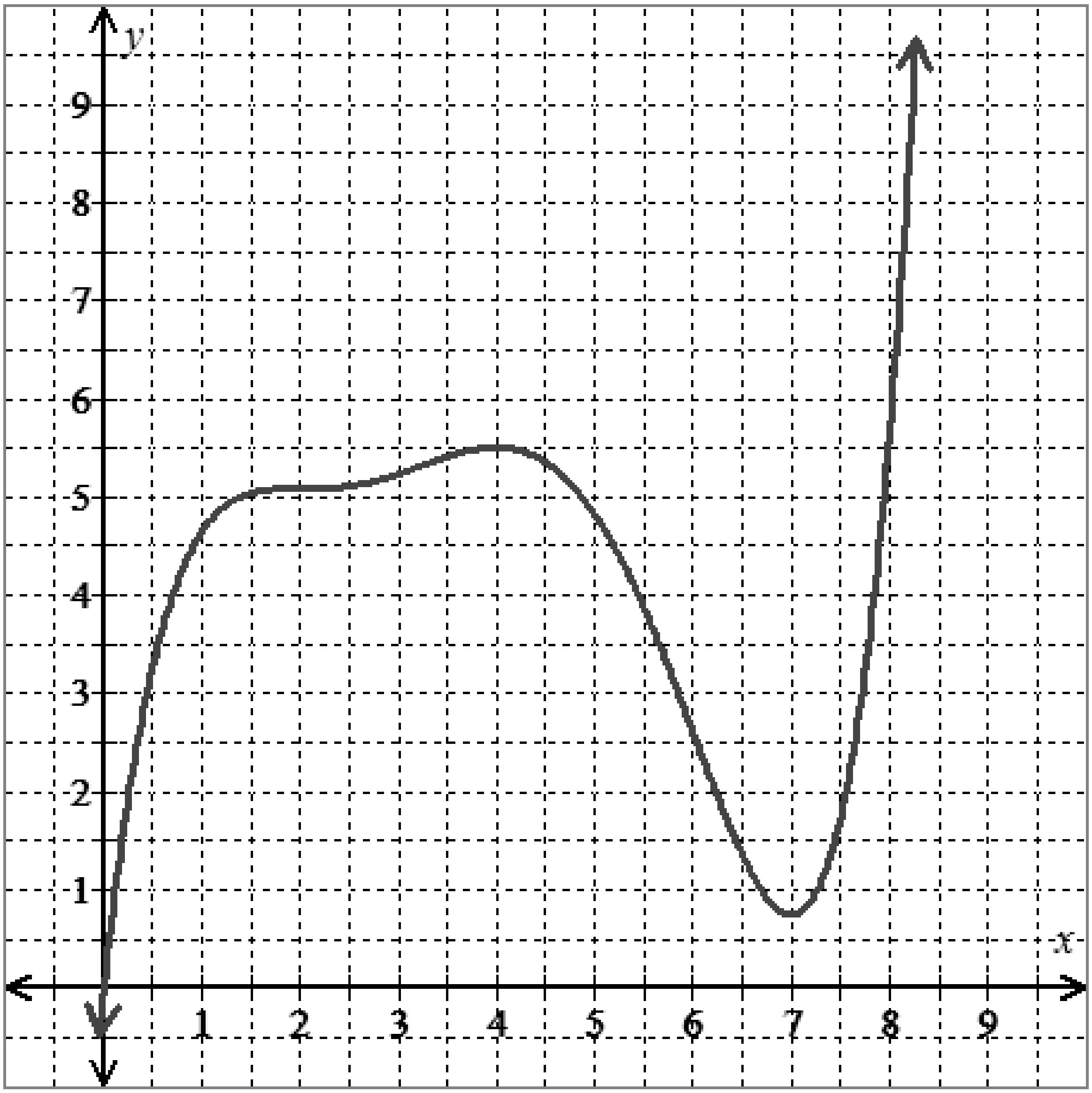


Figure 22.11: *w*

Replacement Limit Laws

The following laws allow you to replace a limit expression with an actual value. For example, the expression $\lim_{t \to 5} t$ can be replaced with the number 5, the expression $\lim_{x \to 7^-} 6$ can be replaced with the number 6, and the expression $\lim_{z \to \infty} \frac{12}{e^z}$ can be replaced with the number 0.

In all cases both a and C represent real numbers.

Limit Law R1

$$\lim_{x \to a} x = \lim_{x \to a^-} x = \lim_{x \to a^+} x = a$$

Limit Law R2

$$\lim_{x \to a} C = \lim_{x \to a^-} C = \lim_{x \to a^+} C = C$$

$$\lim_{x \to \infty} C = C \text{ and } \lim_{x \to -\infty} C = C$$

Limit Law R3

$$\lim_{x \to \infty} f(x) = \infty \Rightarrow \lim_{x \to \infty} \frac{C}{f(x)} = 0 \qquad \lim_{x \to \infty} f(x) = -\infty \Rightarrow \lim_{x \to \infty} \frac{C}{f(x)} = 0$$

$$\lim_{x \to -\infty} f(x) = \infty \Rightarrow \lim_{x \to -\infty} \frac{C}{f(x)} = 0 \qquad \lim_{x \to -\infty} f(x) = -\infty \Rightarrow \lim_{x \to -\infty} \frac{C}{f(x)} = 0$$

Infinite Limit Laws: Limits as the output increases or decreases without bound

Many limit values do not exist. Sometimes the non-existence is caused by the function value either increasing without bound or decreasing without bound. In these special cases we use the symbols ∞ and $-\infty$ to communicate the non-existence of the limits.

Some examples of these type of ***non-existent*** limits follow.

$$\lim_{x \to \infty} x^n = \infty \text{ (if } n \text{ is a positive real number)}$$

$$\lim_{x \to -\infty} x^n = \infty \text{ (if } n \text{ is an even positive integer)}$$

$$\lim_{x \to -\infty} x^n = -\infty \text{ (if } n \text{ is an odd positive integer)}$$

$$\lim_{x \to \infty} e^x = \infty, \ \lim_{x \to -\infty} e^{-x} = \infty, \ \lim_{x \to \infty} \ln(x) = \infty, \text{ and } \lim_{x \to 0^+} \ln(x) = -\infty$$

Algebraic Limit Laws

The following laws allow you to replace a limit expression with equivalent limit expressions. For example, the expression $\lim_{x\to 5}\left(2+x^2\right)$ can be replaced with the expression $\lim_{x\to 5} 2 + \lim_{x\to 5} x^2$.

Limit laws A1-A6 are valid if and only if *every limit in the equation exists*.

In all cases a can represent a real number, a real number from the left, a real number from the right, the symbol ∞, or the symbol $-\infty$.

Limit Law A1: $\lim_{x\to a}\left(f(x)+g(x)\right)=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$

Limit Law A2: $\lim_{x\to a}\left(f(x)-g(x)\right)=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$

Limit Law A3: $\lim_{x\to a}\left(C\,f(x)\right)=C\lim_{x\to a} f(x)$ where C is any real number.

Limit Law A4: $\lim_{x\to a}\left(f(x)\cdot g(x)\right)=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$

Limit Law A5: $\lim_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ ***if and only if*** $\lim_{x\to a} g(x)\neq 0$

Limit Law A6: $\lim_{x\to a} f\left(g(x)\right)=f\left(\lim_{x\to a} g(x)\right)$ (when f is continuous at $\lim_{x\to a} g(x)$)

Limit Law A7: If there exists an open interval centered at a over which $f(x)=g(x)$ for $x\neq a$, then $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)$ (provided that both limits exist)

Additionally, if there exists an open interval centered at a over which $f(x)=g(x)$ for $x\neq a$, then $\lim_{x\to a} f(x)=\infty$ if and only if $\lim_{x\to a} g(x)=\infty$ and, similarly, $\lim_{x\to a} f(x)=-\infty$ if and only if $\lim_{x\to a} g(x)=-\infty$.

Rational Limit Forms

Form	Example	Course of Action
$\dfrac{\text{real number}}{\text{non-zero real number}}$	$\lim\limits_{x\to 3}\dfrac{2x+16}{5x-4}$ (the form is $\dfrac{22}{11}$)	The value of the limit is the number to which the limit form simplifies; for example, $\lim\limits_{x\to 3}\dfrac{2x+16}{5x-4}=2$. You can immediately begin applying limit laws if you are "proving" the limit value.
$\dfrac{\text{non-zero real number}}{\text{zero}}$	$\lim\limits_{x\to 7^-}\dfrac{x+7}{7-x}$ (the form is $\dfrac{14}{0}$)	The limit value doesn't exist. The expression whose limit is being found is either increasing without bound or decreasing without bound (or possibly both if you have a two-sided limit). You may be able to communicate the non-existence of the limit using ∞ or $-\infty$. For example, if the value of x is a little less than 7, the value of $\dfrac{x+7}{7-x}$ is positive. Hence you could write $\lim\limits_{x\to 7^-}\dfrac{x+7}{7-x}=\infty$.
$\dfrac{\text{zero}}{\text{zero}}$	$\lim\limits_{x\to -2}\dfrac{x^2-4}{x^2+3x+2}$ (the form is $\dfrac{0}{0}$)	This is an ***indeterminate form limit***. You do not know the value of the limit (or even if it exists) nor can you begin to apply limit laws. You need to manipulate the expression whose limit is being found until the resultant limit no longer has indeterminate form. For example: $\lim\limits_{x\to -2}\dfrac{x^2-4}{x^2+3x+2}=\lim\limits_{x\to -2}\dfrac{(x-2)(x+2)}{(x+1)(x+2)}=\lim\limits_{x\to -2}\dfrac{(x-2)}{(x+1)}$ (The limit form is now $\dfrac{-4}{-1}$; you may begin applying the limit laws.)
$\dfrac{\text{real number}}{\text{infinity}}$	$\lim\limits_{x\to 0^+}\dfrac{1-4e^x}{\ln(x)}$ (the form is $\dfrac{-3}{-\infty}$)	The limit value is zero and this is justified by logic similar to that used to justify Limit Law R3.

Rational Limit Forms

Form	Example	Course of Action
$\dfrac{\text{infinity}}{\text{real number}}$	$\lim\limits_{x \to 0^+} \dfrac{\ln(x)}{1-4e^x}$ (the form is $\dfrac{-\infty}{-3}$)	The limit value doesn't exist. The expression whose limit is being found is either increasing without bound or decreasing without bound (or possibly both if you have a two-sided limit). You may be able to communicate the non-existence of the limit using ∞ or $-\infty$. For example, you can write $\lim\limits_{x \to 0^+} \dfrac{\ln(x)}{1-4e^x} = \infty$.
$\dfrac{\text{infinity}}{\text{infinity}}$	$\lim\limits_{x \to \infty} \dfrac{3+e^x}{1+3e^x}$	This is an ***indeterminate form limit***. You do not know the value of the limit (or even if it exists) nor can you begin to apply limit laws. You need to manipulate the expression whose limit is being found until the resultant limit no longer has indeterminate form. For example: $$\lim_{x \to \infty} \frac{3+e^x}{1+3e^x} = \lim_{x \to \infty} \left(\frac{3+e^x}{1+3e^x} \cdot \frac{1/e^x}{1/e^x} \right) = \lim_{x \to \infty} \frac{\frac{3}{e^x}+1}{\frac{1}{e^x}+3}$$ (The limit form is now $\frac{1}{3}$; you may begin applying the limit laws.)

There are other indeterminate limit forms, although the two mentioned in Table 1 are the only two *rational* indeterminate forms. The other indeterminate forms are discussed in MTH 252.

Derivative Formulas: k, a, and n represent constants; u and y represent functions of x

Basic Formulas	Chain Rule Format	Implicit Derivative Format
$\frac{d}{dx}(k)=0$		
$\frac{d}{dx}(x^n)=n\,x^{n-1}$	$\frac{d}{dx}(u^n)=n\,u^{n-1}\cdot\frac{d}{dx}(u)$	$\frac{d}{dx}(y^n)=n\,y^{n-1}\frac{dy}{dx}$
$\frac{d}{dx}(\sqrt{x})=\frac{1}{2\sqrt{x}}$	$\frac{d}{dx}(\sqrt{u})=\frac{1}{2\sqrt{u}}\cdot\frac{d}{dx}(u)$	$\frac{d}{dx}(\sqrt{y})=\frac{1}{2\sqrt{y}}\frac{dy}{dx}$
$\frac{d}{dx}(\sin(x))=\cos(x)$	$\frac{d}{dx}(\sin(u))=\cos(u)\cdot\frac{d}{dx}(u)$	$\frac{d}{dx}(\sin(y))=\cos(y)\frac{dy}{dx}$
$\frac{d}{dx}(\cos(x))=-\sin(x)$	$\frac{d}{dx}(\cos(u))=-\sin(u)\cdot\frac{d}{dx}(u)$	$\frac{d}{dx}(\cos(y))=-\sin(y)\frac{dy}{dx}$
$\frac{d}{dx}(\tan(x))=\sec^2(x)$	$\frac{d}{dx}(\tan(u))=\sec^2(u)\cdot\frac{d}{dx}(u)$	$\frac{d}{dx}(\tan(y))=\sec^2(y)\frac{dy}{dx}$
$\frac{d}{dx}(\sec(x))=\sec(x)\tan(x)$	$\frac{d}{dx}(\sec(u))=\sec(u)\tan(u)\cdot\frac{d}{dx}(u)$	$\frac{d}{dx}(\sec(y))=\sec(y)\tan(y)\frac{dy}{dx}$
$\frac{d}{dx}(\cot(x))=-\csc^2(x)$	$\frac{d}{dx}(\cot(u))=-\csc^2(u)\cdot\frac{d}{dx}(u)$	$\frac{d}{dx}(\cot(y))=-\csc^2(y)\frac{dy}{dx}$
$\frac{d}{dx}(\csc(x))=-\csc(x)\cot(x)$	$\frac{d}{dx}(\csc(u))=-\csc(u)\cot(u)\cdot\frac{d}{dx}(u)$	$\frac{d}{dx}(\csc(y))=-\csc(y)\cot(y)\frac{dy}{dx}$
$\frac{d}{dx}(\tan^{-1}(x))=\frac{1}{1+x^2}$	$\frac{d}{dx}(\tan^{-1}(u))=\frac{1}{1+u^2}\cdot\frac{d}{dx}(u)$	$\frac{d}{dx}(\tan^{-1}(y))=\frac{1}{1+y^2}\frac{dy}{dx}$
$\frac{d}{dx}(\sin^{-1}(x))=\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\sin^{-1}(u))=\frac{1}{\sqrt{1-u^2}}\cdot\frac{d}{dx}(u)$	$\frac{d}{dx}(\sin^{-1}(y))=\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}$
$\frac{d}{dx}(\sec^{-1}(x))=\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}(\sec^{-1}(u))=\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\cdot\frac{d}{dx}(u)$	$\frac{d}{dx}(\sec^{-1}(y))=\frac{1}{\lvert y\rvert\sqrt{y^2-1}}\frac{dy}{dx}$
$\frac{d}{dx}(e^x)=e^x$	$\frac{d}{dx}(e^u)=e^u\cdot\frac{d}{dx}(u)$	$\frac{d}{dx}(e^y)=e^y\frac{dy}{dx}$
$\frac{d}{dx}(a^x)=\ln(a)\cdot a^x$	$\frac{d}{dx}(a^u)=\ln(a)\cdot a^u\cdot\frac{d}{dx}(u)$	$\frac{d}{dx}(a^y)=\ln(a)\cdot a^y\frac{dy}{dx}$
$\frac{d}{dx}(\ln(x))=\frac{1}{x}$	$\frac{d}{dx}(\ln(u))=\frac{1}{u}\cdot\frac{d}{dx}(u)$	$\frac{d}{dx}(\ln(y))=\frac{1}{y}\frac{dy}{dx}$

Constant Factor Rule of Differentiation: k represents a constant

$\frac{d}{dx}(k\,f(x))=k\frac{d}{dx}(f(x))$ **Alternate Form:** If $y=f(x)$, then $\frac{d}{dx}(k\,y)=k\frac{dy}{dx}$

Sum/Difference Rule of Differentiation

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx}\left(f(x)\right) \pm \frac{d}{dx}\left(g(x)\right)$$

Product Rule of Differentiation

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \frac{d}{dx}\left(f(x)\right)\cdot g(x) + f(x)\cdot\frac{d}{dx}\left(g(x)\right)$$

Quotient Rule of Differentiation

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}\left(f(x)\right)\cdot g(x) - f(x)\cdot\frac{d}{dx}\left(g(x)\right)}{\left[g(x)\right]^2};\ g(x)\neq 0$$

Chain Rule of Differentiation

$$\frac{d}{dx}\left(f\left(g(x)\right)\right) = f'\left(g(x)\right)\cdot g'(x)$$

Alternate Form: If $u = g(x)$, then $\frac{d}{dx}\left(f(u)\right) = f'(u)\cdot\frac{d}{dx}(u)$

Alternate Form: If $y = f(u)$ where $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

Some Useful Rules of Algebra

For positive integers m and n: $\sqrt[n]{x^m} = x^{m/n}$

For real numbers k and n: $\frac{k}{x^n} = k\,x^{-n};\ x\neq 0$ and $\frac{f(x)}{k} = \frac{1}{k}f(x);\ k\neq 0$

For positive real numbers A and B and all real numbers n:

$\ln(AB) = \ln(A) + \ln(B)$, $\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)$, and $\ln\left(A^n\right) = n\cdot\ln(A)$

Supplemental Solutions for the Rates of Change Lab

Exercise 1.1

E1.1.1 The difference quotient for z is:

$$\begin{aligned}\frac{z(x+h)-z(x)}{h} &= \frac{\left[2+4(x+h)-(x+h)^2\right]-\left[2+4x-x^2\right]}{h}\\ &= \frac{2+4x+4h-x^2-2xh-h^2-2-4x+x^2}{h}\\ &= \frac{4h-2xh-h^2}{h}\\ &= \frac{h(4-2x-h)}{h}\\ &= 4-2x-h;\ \text{for } h\neq 0\end{aligned}$$

E1.1.2 The rise between the two points is 9 and the run is 3, so the slope between the two points is given by $\frac{9}{3}=3$.

Using the difference quotient, if we let $x=-1$ and $h=3$ we get:

$$\begin{aligned}4-2x-h &= 4-2(-1)-3\\ &= 3 \quad \checkmark\end{aligned}$$

E1.1.3 $$\begin{aligned}\frac{z(4+h)-z(4)}{h} &= 4-2(4)-h\\ &= -4-h\end{aligned}$$

Table E1.1K: $y=\frac{z(4+h)-z(4)}{h}$

h	y
–0.1	–3.9
–0.01	–3.99
–0.001	–3.999
0.001	–4.001
0.01	–4.01
0.1	–4.01

E1.1.4 The values are converging on -4.

E1.1.5

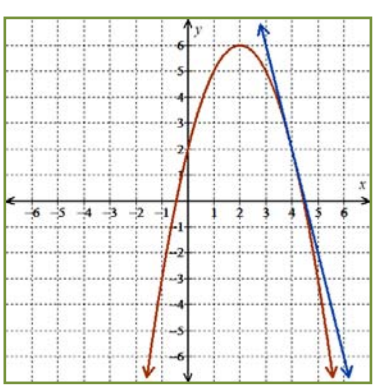


Figure E1.1K: $y=2+4x-x^2$

Exercise 1.2

E1.2.1
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\left[3-7(x+h)\right]-\left[3-7x\right]}{h}\\ &= \frac{3-7x-7h-3+7x}{h}\\ &= \frac{-7h}{h}\\ &= -7;\ \text{for } h\neq 0\end{aligned}$$

E1.2.2
$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{\frac{7}{x+h+4}-\frac{7}{x+4}}{h}\\ &= \frac{\frac{7}{x+h+4}\cdot\frac{x+4}{x+4}-\frac{7}{x+4}\cdot\frac{x+h+4}{x+h+4}}{\frac{h}{1}}\\ &= \frac{7x+28-7x-7h-28}{(x+h+4)(x+4)}\cdot\frac{1}{h}\\ &= \frac{-7h}{(x+h+4)(x+4)h}\\ &= \frac{-7}{(x+4)(x+h+4)};\ \text{for } h\neq 0\end{aligned}$$

E1.2.3
$$\begin{aligned}\frac{z(x+h)-z(x)}{h} &= \frac{\pi-\pi}{h}\\ &= \frac{0}{h}\\ &= 0;\ \text{for } h\neq 0\end{aligned}$$

E1.2.4
$$\begin{aligned}\frac{s(t+h)-s(t)}{h} &= \frac{\left[(t+h)^3-(t+h)-9\right]-\left[t^3-t-9\right]}{h}\\ &= \frac{t^3+3t^2h+3th^2+h^3-t-h-9-t^3+t+9}{h}\\ &= \frac{3t^2h+3th^2+h^3-h}{h}\\ &= \frac{\left(3t^2+3th+h^2-1\right)h}{h}\\ &= 3t^2+3th+h^2-1;\ \text{for } h\neq 0\end{aligned}$$

E1.2.5

$$\frac{k(t+h)-k(t)}{h}=\frac{\frac{(t+h-8)^2}{t+h}-\frac{(t-8)^2}{t}}{h}$$

$$=\frac{\frac{t^2+th-8t+th+h^2-8h-8t-8h+64}{t+h}-\frac{t^2-8t-8t+64}{t}}{h}$$

$$=\frac{\frac{t^2+2th-16t+h^2-16h+64}{t+h}\cdot\frac{t}{t}-\frac{t^2-16t+64}{t}\cdot\frac{t+h}{t+h}}{\frac{h}{1}}$$

$$=\frac{t^3+2t^2h-16t^2+th^2-16th+64t-\left[t^3+t^2h-16t^2-16th+64t+64h\right]}{t(t+h)}\cdot\frac{1}{h}$$

$$=\frac{t^3+2t^2h-16t^2+th^2-16th+64t-t^3-t^2h+16t^2+16th-64t-64h}{t(t+h)}\cdot\frac{1}{h}$$

$$=\frac{t^2h+th^2-64h}{t(t+h)h}$$

$$=\frac{(t^2+th-64)h}{t(t+h)h}$$

$$=\frac{t^2+th-64}{t(t+h)};\ \text{for }\ h\neq 0$$

Exercise 1.3

E1.3.1

$$\frac{s(t+h)-s(t)}{h}=\frac{\left[150+60(t+h)-16(t+h)^2\right]-\left[150+60t-16t^2\right]}{h}$$

$$=\frac{150+60t+60h-16t^2-32th-16h^2-150-60t+16t^2}{h}$$

$$=\frac{60h-32th-16h^2}{h}$$

$$=\frac{(60-32t-16h)h}{h}$$

$$=60-32t-16h\quad\text{for } h\neq 0$$

E1.3.2 Letting $t=4$ and $h=0.2$ we have (from the difference quotient):

$$60-32t-16h=60-32(4)-16(0.2)$$
$$=-71.2$$

Checking …

$$\frac{s(4.2)-s(4)}{4.2-4}=\frac{119.76-134}{.2}\quad\text{(Phew!)}$$
$$=-71.2$$

E1.4.1 $\dfrac{v(7.5)-v(0)}{7.5\,\text{s}-0\,\text{s}}=-13\tfrac{1}{3}\ \dfrac{\text{ft/s}}{\text{s}}$

This value tells us that during the first 7.5 seconds of descent, the average rate of change in the coaster's velocity is $-13\tfrac{1}{3}\ \dfrac{\text{ft/s}}{\text{s}}$. In other words, ***on average***, with each passing second the velocity is $13\tfrac{1}{3}\ \text{ft/s}$ less than it was the preceding second. We could also say that the average acceleration experienced by the coaster over the first 7.5 seconds of descent is $-13\tfrac{1}{3}\ \dfrac{\text{ft/s}}{\text{s}}$.

E1.4.2 $\dfrac{h(3)-h(1.5)}{3\,\text{s}-1.5\,\text{s}}=0\ \text{m/s}$

This value tells us that the average velocity experienced by the ball between the 1.5 second of play and the third second of play is $0\ \text{m/s}$. Please note that this does not imply that the ball doesn't move; it simply means that the ball is at the same elevation at the two times.

E1.4.3 $\dfrac{P(29)-P(1)}{29\,\text{swing}-1\,\text{swing}}=-0.1\ \text{s}/\text{swing}$

This value tells us that between the first swing and the 29th swing the average rate of change in the pendulum's period is $-0.1\ \text{s}/\text{swing}$. In other words, ***on average***, with each passing swing the period decreases by a tenth of a second.

E1.4.4 $\dfrac{h(120)-h(60)}{120\,\text{s}-60\,\text{s}}=.13\,\dfrac{\text{mph/s}}{\text{s}}$

This value tells us that during the second minute of flight, the average rate of change in the rocket's acceleration is $.13\,\dfrac{\text{mph/s}}{\text{s}}$. In other words, ***on average***, with each passing second the acceleration is $.13\ \text{mph/s}$ more than it was the preceding second. You might have noticed that since the acceleration function is linear, the rate of change in the acceleration is constant. That is, with each passing second the acceleration ***actually is*** $.13\ \text{mph/s}$ more than it was the preceding second.

Solutions to the Supplemental Exercises for the Limits and Continuity Lab

Exercise 2.1

Table	Limit	Existence?	Comments
E2.2	$\lim\limits_{x \to \infty} f(x) = 0$	Yes	The relevant pattern in the y column is the powers of 10. The first nonzero digit is moving farther and farther to the right of the decimal point.
E2.3	$\lim\limits_{t \to \frac{1}{3}^-} z(t) = \frac{2}{3}$	Yes	You should definitely recognize decimals approaching common fractions. Good catch if you noted the t was approaching $\frac{1}{3}$ only from the left.
E2.4	$\lim\limits_{\theta \to -1^+} g(\theta) = 3,000,000$	Yes	The last entry in the output column is the same in these two tables. Hopefully you recognized that you need to look at the pattern in the output, not just the last entry. Recognizing that t approaches $\frac{7}{9}$ probably requires some guessing and checking. Stay focused when determining "from the left" or "from the right;" that can get tricky … especially when the numbers are negative.
E2.5	$\lim\limits_{t \to \frac{7}{9}^+} T(t) = \infty$	No	

Exercise 2.2

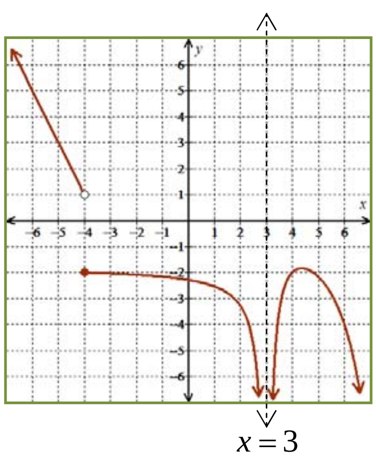


Figure E2.1K: f

Exercise 2.3

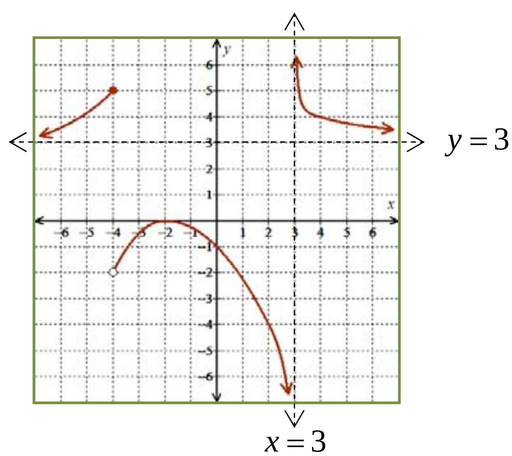


Figure E2.2K: f

Exercise 2.4

f is discontinuous from both directions at 0 and the discontinuity is not removable.

f is discontinuous at 3 although it is continuous from the left at 3. The discontinuity is not removable. (The problem is that the limits are different from the left and right of 3.)

f is discontinuous from both directions at 4 but the discontinuity is removable ($\lim_{x\to 4} f(x)=1$).

f is discontinuous from both directions at 2π but the discontinuity is removable ($\lim_{x\to 2\pi} f(x)=1$).

Exercise 2.5

Because the two formulas are polynomials, the only number where continuity is at issue is 3. The top formula is used at 3, so $g(3)$ is defined regardless of the value of k. Basically all we need to do is ensure that the limits from the left and the right of 3 are both equal to $g(3)$. We have:

$$g(3)=9+2k\text{ , } \lim_{t\to 3^+} g(t)=9+2k\text{ , and } \lim_{t\to 3^-} g(t)=9-4k$$

So g will be continuous at 3 (and, consequently, over $(-\infty,\infty)$) if and only if $9+2k=9-4k$. This gives us $k=0$. Doh! Turns out the g isn't piece-wise at all, it's simply the parabolic function $g(t)=t^2$.

Exercise 2.6

E2.6.1 $\lim_{x\to 4^-}\left(5-\frac{1}{x-4}\right)=\infty$ This limit does **not** exit.

E2.6.2

$$\lim_{x\to\infty}\frac{e^{2/x}}{e^{1/x}}=\lim_{x\to\infty}e^{1/x}$$

$$=e^{\lim_{x\to\infty}\frac{1}{x}} \quad \text{LL A6}$$

$$=e^0 \quad \text{LL R3}$$

$$=1$$

This limit **does** exist.

E2.6.3

$$\lim_{x\to 2^+}\frac{x^2-4}{x^2+4}=\frac{\lim_{x\to 2^+}(x^2-4)}{\lim_{x\to 2^+}(x^2+4)} \quad \text{LL A5}$$

$$=\frac{\lim_{x\to 2^+}x^2-\lim_{x\to 2^+}4}{\lim_{x\to 2^+}x^2+\lim_{x\to 2^+}4} \quad \text{LL A1 and A2}$$

$$=\frac{\left(\lim_{x\to 2^+}x\right)^2-\lim_{x\to 2^+}4}{\left(\lim_{x\to 2^+}x\right)^2+\lim_{x\to 2^+}4} \quad \text{LL A6}$$

$$=\frac{2^2-4}{2^2+4} \quad \text{LL R1 and R2}$$

$$=0$$

This limit **does** exist.

E2.6.4

$$\lim_{x\to 2^+}\frac{x^2-4}{x^2-4x+4}=\lim_{x\to 2^+}\frac{(x+2)(x-2)}{(x-2)(x-2)}$$

$$=\lim_{x\to 2^+}\frac{(x+2)}{(x-2)} \qquad \text{LL A7}$$

$$=\infty$$

We can't apply limit laws A1-A6 yet because the limit has the indeterminate form $\frac{0}{0}$.

This limit does **not** exist.

E2.6.5

$$\lim_{x\to\infty}\frac{\ln(x)+\ln\left(x^6\right)}{7\ln\left(x^2\right)}=\lim_{x\to\infty}\frac{\ln(x)+6\ln(x)}{7\cdot 2\ln(x)}$$

$$=\lim_{x\to\infty}\frac{7\ln(x)}{14\ln(x)}$$

$$=\lim_{x\to\infty}\frac{1}{2} \qquad \text{LL A7}$$

$$=\frac{1}{2} \qquad \text{LL R2}$$

We can't apply limit laws A1-A6 yet because the limit has the indeterminate form $\frac{\infty}{\infty}$.

This limit **does** exist.

E2.6.6

$$\lim_{x\to-\infty}\frac{3x^3+2x}{3x-2x^3}=\lim_{x\to-\infty}\left(\frac{3x^3+2x}{3x-2x^3}\cdot\frac{1/x^3}{1/x^3}\right)$$

$$=\lim_{x\to-\infty}\frac{3+\frac{2}{x^2}}{\frac{3}{x^2}-2}$$

$$=\frac{\lim\limits_{x\to-\infty}\left(3+\frac{2}{x^2}\right)}{\lim\limits_{x\to-\infty}\left(\frac{3}{x^2}-2\right)} \qquad \text{LL A5}$$

$$=\frac{\lim\limits_{x\to-\infty}3+\lim\limits_{x\to-\infty}\frac{2}{x^x}}{\lim\limits_{x\to-\infty}\frac{3}{x^2}-\lim\limits_{x\to-\infty}2} \qquad \text{LL A1 and A2}$$

$$=\frac{3+0}{0-2} \qquad \text{LL R2 and R3}$$

$$=-\frac{3}{2}$$

We can't apply limit laws A1-A6 yet because the limit has the indeterminate form $\frac{\infty}{\infty}$.

This limit **does** exist.

E2.6.7

$$\lim_{x\to\infty}\sin\left(\frac{\pi e^{3x}}{2e^x+4e^{3x}}\right)=\lim_{x\to\infty}\sin\left(\frac{\pi e^{3x}}{2e^x+4e^{3x}}\cdot\frac{\frac{1}{e^{3x}}}{\frac{1}{e^{3x}}}\right)$$

$$=\lim_{x\to\infty}\sin\left(\frac{\pi}{\frac{2}{e^{2x}}+4}\right)$$

$$=\sin\left(\lim_{x\to\infty}\frac{\pi}{\frac{2}{e^{2x}}+4}\right) \qquad \text{LL A6}$$

$$=\sin\left(\frac{\lim\limits_{x\to\infty}\pi}{\lim\limits_{x\to\infty}\left(\frac{2}{e^{2x}}+4\right)}\right) \qquad \text{LL A5}$$

$$=\sin\left(\frac{\lim\limits_{x\to\infty}\pi}{\lim\limits_{x\to\infty}\frac{2}{e^{2x}}+\lim\limits_{x\to\infty}4}\right) \qquad \text{LL A1}$$

$$=\sin\left(\frac{\pi}{0+4}\right) \qquad \text{LL R2 and R3}$$

$$=1/\sqrt{2}$$

We can't apply limit laws A1-A6 yet because the limit has the indeterminate form $\frac{\infty}{\infty}$.

This limit **does** exist.

E2.6.8 $\lim_{x\to\infty}\frac{\ln\left(\frac{1}{x}\right)}{\ln\left(\frac{x}{x}\right)}$ does **not** exist because the domain of the function $y=\frac{\ln\left(\frac{1}{x}\right)}{\ln\left(\frac{x}{x}\right)}$ is $\varnothing$.

E2.6.9

$$\lim_{x\to 5}\sqrt{\frac{x^2-12x+35}{5-x}}=\lim_{x\to 5}\sqrt{\frac{(x-7)(x-5)}{5-x}}$$

$=\lim_{x\to 5}\sqrt{-1\cdot(x-7)}$ LL A7

$=\lim_{x\to 5}\sqrt{7-x}$

$=\sqrt{\lim_{x\to 5}(7-x)}$ LL A6

$=\sqrt{\lim_{x\to 5}7-\lim_{x\to 5}x}$ LL A2

$=\sqrt{7-5}$ LL R1 and R2

$=\sqrt{2}$

We can't apply limit laws A1-A6 yet because the inside limit has the indeterminate form $\frac{0}{0}$.

This limit **does** exist.

E2.6.10

$$\lim_{h\to 0}\frac{4(3+h)^2-5(3+h)-21}{h}=\lim_{h\to 0}\frac{4\left(9+6h+h^2\right)-15-5h-21}{h}$$

$=\lim_{h\to 0}\frac{36+24h+4h^2-15-5h-21}{h}$

$=\lim_{h\to 0}\frac{19h+4h^2}{h}$

$=\lim_{h\to 0}\frac{h(19+4h)}{h}$

$=\lim_{h\to 0}(19+4h)$ LL A7

$=\lim_{h\to 0}19+\lim_{h\to 0}(4h)$ LL A1

$=\lim_{h\to 0}19+4\cdot\lim_{h\to 0}h$ LL A3

$=19+4(0)$ LL R1 and R2

$=19$

We can't apply limit laws A1-A6 yet because the limit has the indeterminate form $\frac{0}{0}$.

This limit **does** exist.

E2.6.11

$$\lim_{h\to 0}\frac{5h^2+3}{2-3h^2}=\frac{\lim_{h\to 0}\left(5h^2+3\right)}{\lim_{h\to 0}\left(2-3h^2\right)}$$ LL A5

$=\frac{\lim_{h\to 0}\left(5h^2\right)+\lim_{h\to 0}3}{\lim_{h\to 0}2-\lim_{h\to 0}\left(3h^2\right)}$ LL A1 and A2

$=\frac{5\cdot\lim_{h\to 0}h^2+\lim_{h\to 0}3}{\lim_{h\to 0}2-3\cdot\lim_{h\to 0}h^2}$ LL A3

$=\frac{5\left(\lim_{h\to 0}h\right)^2+\lim_{h\to 0}3}{\lim_{h\to 0}2-3\left(\lim_{h\to 0}h\right)^2}$ LL A6

$=\frac{5\cdot 0^2+3}{2-3\cdot 0^2}$ LL R1 and R2

$=\frac{3}{2}$

This limit **does** exist.

E2.6.12

$$\lim_{h\to 0}\frac{\sqrt{9-h}-3}{h}=\lim_{h\to 0}\left(\frac{\sqrt{9-h}-3}{h}\cdot\frac{\sqrt{9-h}+3}{\sqrt{9-h}+3}\right)$$

$$=\lim_{h\to 0}\frac{(9-h)-9}{h\left(\sqrt{9-h}+3\right)}$$

$$=\lim_{h\to 0}\frac{-h}{h\left(\sqrt{9-h}+3\right)}$$

We can't apply limit laws A1-A6 yet because the limit has the indeterminate form $\frac{0}{0}$.

$$=\lim_{h\to 0}\frac{-1}{\left(\sqrt{9-h}+3\right)}$$ LL A7

$$=\frac{\lim_{h\to 0}(-1)}{\lim_{h\to 0}\left(\sqrt{9-h}+3\right)}$$ LL A5

$$=\frac{\lim_{h\to 0}(-1)}{\lim_{h\to 0}\sqrt{9-h}+\lim_{h\to 0}3}$$ LL A1

$$=\frac{\lim_{h\to 0}(-1)}{\sqrt{\lim_{h\to 0}(9-h)}+\lim_{h\to 0}3}$$ LL A6

This limit **does** exist.

$$=\frac{\lim_{h\to 0}(-1)}{\sqrt{\lim_{h\to 0}9-\lim_{h\to 0}h}+\lim_{h\to 0}3}$$ LL A2

$$=\frac{-1}{\sqrt{9-0}+3}$$ LL R1 and R2

$$=-\frac{1}{6}$$

E2.6.13

$$\lim_{\theta\to\frac{\pi}{2}}\frac{\sin\left(\theta+\frac{\pi}{2}\right)}{\sin(2\theta+\pi)}=\lim_{\theta\to\frac{\pi}{2}}\frac{\sin(\theta)\cos\left(\frac{\pi}{2}\right)+\cos(\theta)\sin\left(\frac{\pi}{2}\right)}{\sin(2\theta)\cos(\pi)+\cos(2\theta)\sin(\pi)}$$

$$=\lim_{\theta\to\frac{\pi}{2}}\frac{\cos(\theta)}{-\sin(2\theta)}$$

$$=\lim_{\theta\to\frac{\pi}{2}}\frac{\cos(\theta)}{-2\sin(\theta)\cos(\theta)}$$

We can't apply limit laws A1-A6 yet because the limit has the indeterminate form $\frac{0}{0}$.

$$=\lim_{\theta\to\frac{\pi}{2}}\frac{1}{-2\sin(\theta)}$$ LL A7

$$=\frac{\lim_{\theta\to\frac{\pi}{2}}1}{\lim_{\theta\to\frac{\pi}{2}}\left(-2\sin(\theta)\right)}$$ LL A5

$$=\frac{\lim_{\theta\to\frac{\pi}{2}}1}{-2\lim_{\theta\to\frac{\pi}{2}}\sin(\theta)}$$ LL A3

This limit **does** exist.

$$=\frac{\lim_{\theta\to\frac{\pi}{2}}1}{-2\sin\left(\lim_{\theta\to\frac{\pi}{2}}\theta\right)}$$ LL A6

$$=\frac{1}{-2\sin\left(\frac{\pi}{2}\right)}$$ LL R1 and R2

$$=-\frac{1}{2}$$

E2.6.14 $\lim\limits_{x \to 0^+} \frac{\ln\left(x^e\right)}{\ln\left(e^x\right)} = -\infty$ This limit does **not** exist.

Exercise 2.7

	Does limit exist?		Does limit exist?
$\lim\limits_{x \to \infty} e^x = \infty$	No	$\lim\limits_{x \to -\infty} e^x = 0$	Yes
$\lim\limits_{x \to 0} e^x = 1$	Yes	$\lim\limits_{x \to \infty} e^{-x} = 0$	Yes
$\lim\limits_{x \to -\infty} e^{-x} = \infty$	No	$\lim\limits_{x \to 0} e^{-x} = 1$	Yes
$\lim\limits_{x \to \infty} \ln(x) = \infty$	No	$\lim\limits_{x \to 1} \ln(x) = 0$	Yes
$\lim\limits_{x \to 0^+} \ln(x) = -\infty$	No	$\lim\limits_{x \to \infty} \frac{1}{x} = 0$	Yes
$\lim\limits_{x \to -\infty} \frac{1}{x} = 0$	Yes	$\lim\limits_{x \to 0^+} \frac{1}{x} = \infty$	No
$\lim\limits_{x \to 0^-} \frac{1}{x} = -\infty$	No	$\lim\limits_{x \to \infty} e^{1/x} = 1$	Yes
$\lim\limits_{x \to \infty} \frac{1}{e^x} = 0$	Yes	$\lim\limits_{x \to -\infty} \frac{1}{e^x} = \infty$	No
$\lim\limits_{x \to \infty} \frac{1}{e^{-x}} = \infty$	No	$\lim\limits_{x \to -\infty} \frac{1}{e^{-x}} = 0$	Yes

Supplemental Solutions for the Introduction to the First Derivative Lab

Exercise 3.1

E3.1.1

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}\\
&= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}\\
&= \lim_{h\to 0}\frac{2xh+h^2}{h}\\
&= \lim_{h\to 0}\frac{(2x+h)h}{h}\\
&= \lim_{h\to 0}(2x+h)\\
&= 2x+0\\
&= 2x
\end{aligned}
$$

$$
\begin{aligned}
f'(x) &= \lim_{t\to x}\frac{f(t)-f(x)}{t-x}\\
&= \lim_{t\to x}\frac{t^2-x^2}{t-x}\\
&= \lim_{t\to x}\frac{(t+x)(t-x)}{t-x}\\
&= \lim_{t\to x}(t+x)\\
&= x+x\\
&= 2x
\end{aligned}
$$

E3.1.2

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)\\
&= \lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+x\right)}\\
&= \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}\\
&= \frac{1}{\sqrt{x+0}+\sqrt{x}}\\
&= \frac{1}{2\sqrt{x}}
\end{aligned}
$$

$$
\begin{aligned}
f'(x) &= \lim_{t\to x}\frac{f(t)-f(x)}{t-x}\\
&= \lim_{t\to x}\frac{\sqrt{t}-\sqrt{x}}{t-x}\\
&= \lim_{t\to x}\frac{\sqrt{t}-\sqrt{x}}{\left(\sqrt{t}+\sqrt{x}\right)\left(\sqrt{t}-\sqrt{x}\right)}\\
&= \lim_{t\to x}\frac{1}{\sqrt{t}+\sqrt{x}}\\
&= \frac{1}{\sqrt{x}+\sqrt{x}}\\
&= \frac{1}{2\sqrt{x}}
\end{aligned}
$$

E3.1.3

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{7-7}{h}\\
&= \lim_{h\to 0}\frac{0}{h}\\
&= \lim_{h\to 0}0\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
f'(x) &= \lim_{t\to x}\frac{f(t)-f(x)}{t-x}\\
&= \lim_{t\to x}\frac{7-7}{t-x}\\
&= \lim_{t\to x}\frac{0}{t-x}\\
&= \lim_{t\to x}0\\
&= 0
\end{aligned}
$$

This step is not "optional." On the preceding line the limit has the indeterminate form $\frac{0}{0}$.

Exercise 3.2

If $f(x)=\sin(x)$, then:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\ &= \lim_{h\to 0}\left[\frac{\sin(x)\left[\cos(h)-1\right]}{h}+\frac{\cos(x)\sin(h)}{h}\right] \\ &= \sin(x)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} \\ &= \sin(x)\cdot 0+\cos(x)\cdot 1 \\ &= \cos(x) \end{aligned}$$

Here we've applied limit laws A1 and A5. Remember, from this limit symbol's perspective the variable is *h*; so the factors of $\sin(x)$ and $\cos(x)$ are constants from the perspective of the limit symbol.

Here we've used the two given limit values.

Exercise 3.3

E3.3.1 The velocity function is:

$$\begin{aligned} v(t) &= \lim_{h\to 0}\frac{s(t+h)-s(t)}{h} \\ &= \lim_{h\to 0}\frac{\left[150+60(t+h)-16(t+h)^2\right]-\left[150+60t-16t^2\right]}{h} \\ &= \lim_{h\to 0}\frac{150+60t+60h-16t^2-32th-16h^2-150-60t+16t^2}{h} \\ &= \lim_{h\to 0}\frac{60h-32th-16h^2}{h} \\ &= \lim_{h\to 0}\frac{(60-32t-16h)h}{h} \\ &= \lim_{h\to 0}(60-32t-16h) \\ &= 60-32t-16\cdot 0 \\ &= 60-32t \end{aligned}$$

The velocity of the object 4.1 s into its motion is (including unit):

$$\begin{aligned} v(4.1) &= (60-32(4.1))\ {}^{\text{ft}}\!/_{\text{s}} \\ &= -71.2\ {}^{\text{ft}}\!/_{\text{s}} \end{aligned}$$

E3.3.2 The acceleration function is:

$$\begin{aligned} a(t) &= \lim_{h\to 0} \frac{v(t+h)-v(t)}{h} \\ &= \lim_{h\to 0} \frac{[60-32(t+h)]-[60-32t]}{h} \\ &= \lim_{h\to 0} \frac{60-32t-32h-60+32t}{h} \\ &= \lim_{h\to 0} \frac{-32h}{h} \\ &= \lim_{h\to 0}(-32) \\ &= -32 \end{aligned}$$

The acceleration of the object 4.1 s into its motion is (including unit):

$$a(4.1) = -32\,\frac{\text{ft/s}}{\text{s}}$$

Exercise 3.4

E3.4.1 The unit for R' is $\dfrac{\text{beats/min}}{\text{ft/min}}$.

E3.4.2 The unit for F' is $\dfrac{\text{gal/mi}}{\text{mi/hr}}$.

E3.4.3 The unit for v' is $\dfrac{\text{mi/hr}}{\text{s}}$.

E3.4.4 The unit for h' is mi/s.

Exercise 3.5

E3.5.1 When Carl jogs at a pace of 300 ft/min his heart rate is 84 beats/min.

E3.5.2 At the pace of 300 ft/min, Carl's heart rate changes relative to his pace at a rate of $0.02\,\dfrac{\text{beats/min}}{\text{ft/min}}$. So, if Carl picks up his pace from 300 ft/min to 301 ft/min we'd expect his heart rate to increase to about 84.02 beats/min. Conversely, if Carl decreases his pace from 300 ft/min to 299 ft/min we'd expect his heart rate to decrease to about 83.98 beats/min.

E3.5.3 When Hanh drives her pick-up on level ground at a constant speed of 50 mph, the truck burns fuel at the rate of 0.03 gal/mi.

E3.5.4 (On level ground), at the speed of 50 mph, the rate at which Hanh's truck burns fuel changes relative to the speed at a rate of $-.0006 \frac{\text{gal}/\text{mi}}{\text{mi}/\text{hr}}$. So, if Hanh increased her speed from 50 mph to 51 mph, we'd expect the fuel consumption rate for her truck to decrease to about .0294 gal/mi. Conversely, if Hanh decreased her speed from 50 mph to 49 mph we'd expect to fuel consumption rate for her truck to increase to about .0306 gal/mi.

E3.5.5 Twenty seconds after lift-off the space shuttle is cruising at the rate of 266 mi/hr.

E3.5.6 Twenty seconds after lift-off the velocity of the space shuttle is increasing at the rate $18.9 \frac{\text{mi}/\text{hr}}{\text{s}}$. So, we'd expect that 19 seconds into lift-off the velocity was ***about*** 247.1 mi/hr and 21 seconds into lift-off the velocity will be ***about*** 284.9 mi/hr.

E3.5.7 Twenty seconds after lift-off the space shuttle is at an elevation of 0.7 miles.

E3.5.8 Twenty seconds after lift-off the space shuttle's elevation is increasing at a rate of 0.074 mi/s. So, we'd expect that 19 seconds into lift-off the elevation was ***about*** 0.626 miles and 21 seconds into lift-off the elevation will be ***about*** 0.774 mile.

Exercise 3.6

E3.6.1 The point on f when $x=1$ is $(1,2)$. The value of $f'(1)$ tells us the slope of the tangent line to f through $(1,2)$; this value is 3. Consequently, the equation of the tangent line to f at 1 is $y=3x-1$.

E3.6.2 The long term slope of f is given by:

$$\begin{aligned}\lim_{x\to\infty} f'(x) &= \lim_{x\to\infty} \frac{2x+1}{x} \\ &= 2\end{aligned}$$

So the slope of the skew asymptote is 3.

Supplemental Solutions for the Functions, Derivatives, and Antiderivatives Lab

Exercise 4.1

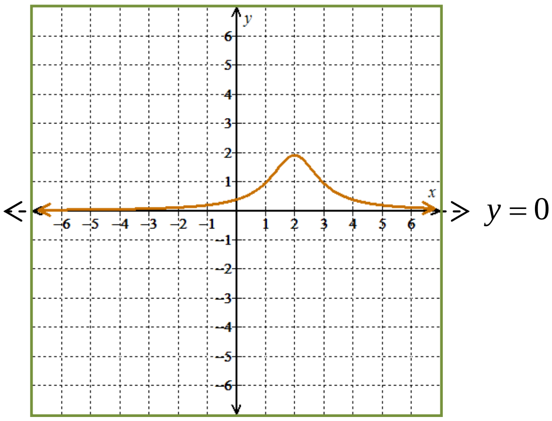


Figure E4.1bK

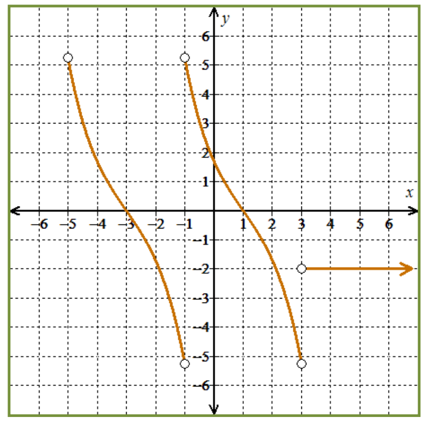


Figure E4.2bK

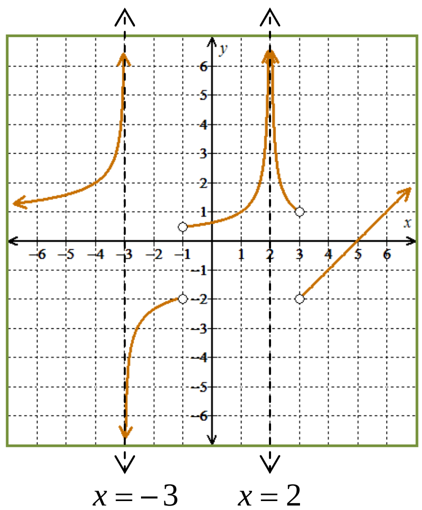


Figure E4.3bK

Exercise 4.2

E4.2.1	C, D, E	**E4.2.2**	B. D
E4.2.3	A, C	**E4.2.4**	C, D, E
E4.2.5	C, D, E	**E4.2.6**	E, F, G
E4.2.7	G	**E4.2.8**	E, F, G
E4.2.9	A, C, E, F, G	**E4.2.10**	E, F, G

Exercise 4.3

Initial observations … The average drainage rate is 60 gal/min which is equivalent to 1 gal/s. Initially the tank drains faster than this rate and towards the end of the drainage the tank is draining slower than this rate.

E4.3.1 The unit for $V'(45)$ is gal/s and the unit for $V''(45)$ is $\frac{\text{gal}/\text{s}}{\text{s}}$.

Since the amount of water left in the tank is decreasing, the value of $V'(45)$ cannot be positive. Since 45 seconds is early in the drainage process the drainage rate is almost certainly greater than the average rate of 1 gal/s. Hence the most realistic value for $V'(45)$ is -1.2.

E4.3.2 The unit for $R'(45)$ is $\frac{\text{gal}/\text{s}}{\text{s}}$ and the unit for $R''(45)$ is $\frac{\text{gal}/\text{s}}{\text{s}}\Big/\text{s}$.

Since the flow rate decreases over time, the value of $R'(45)$ cannot be positive. Since the average flow rate over the 360 seconds is 1 gal/s, it is not believable that 45 seconds into the process the flow rate was decreasing at a rate 1 gal/s/s. Hence the most realistic value for $R'(45)$ is -0.001.

E4.3.3 This question is similar to question E4.5.2 with one key difference; because V is defined as the amount of water *left* in the tank, V' has negative values. Since the value of V' is getting closer to zero, V' is increasing and, consequently, V'' cannot be negative. So the most realistic value for $V''(45)$ is 0.001.

Exercise 4.4

The answer to each question is ii.

Exercise 4.5

E4.5.1 Lisa is right. Maybe g is discontinuous at -3, but maybe it simply is pointy at -4.

E4.5.2 Janice is right although Lisa is pretty darn close to being right. The graph of g'' is ***basically*** the line $y=0$, but because g' is nondifferentiable at -3, g'' has a hole at -4.

Exercise 4.6

E4.6.1 False: $f'(t)<0$ on $(2,3) \Rightarrow f$ decreasing on that interval

E4.6.2 True: f concave up on $(2,3) \Rightarrow f'$ increasing on that interval

E4.6.3 False: $f'(1)<0$ while $f''(1)\geq 0$

E4.6.4 False: $f'(6)\approx -1.7$, so the tangent line to f at 6 is a decreasing line.

E4.6.5 False: f'' is periodic; it would be increasing over each period, however (e.g. $(0,4),(4,8),\ldots$

E4.6.6 False: $f'(4) \approx 4.2$; if f were nondifferentiable at 4 then f' would have no value at 4.

E4.6.7 True

E4.6.8 True

E4.6.9 False: Although the "shape" if the antiderivatives is periodic, the antiderivatives clearly end up higher then they started after each interval of length 4.

E4.6.10 False: The amount of air in your lungs was increasing the entire two seconds.

E4.6.11 True

E4.6.12 True: Negative velocity corresponds to downward motion (because the "position" is decreasing when the velocity is negative).

Exercise 4.7

Table E4.1K

f''	f'	f	F
		Positive	Increasing
		Negative	Decreasing
		Constantly Zero	Constant
	Positive	Increasing	Concave Up
	Negative	Decreasing	Concave Down
	Constantly Zero	Constant	Linear
Positive	Increasing	Concave Up	
Negative	Decreasing	Concave Down	
Constantly Zero	Constant	Linear	

Exercise 4.8

E4.8.1 -1, 1, and 4

E4.8.2 nowhere

E4.8.3 $(-5,6)$

E4.8.4 $(-6,-3)$ and $(-1,1)$

E4.8.5 $(4,6)$

E4.8.6 $(-3,-1)$, $(1,4)$, and $(4,6)$

E4.8.7 $(-6,-5)$

E4.8.8 there's no way to tell

E4.8.9 -5

E4.8.10 $y=-4x+2$

Exercise 4.9

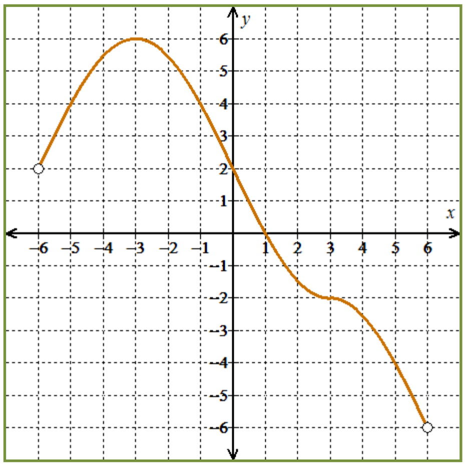


Figure E4.6bK: F

Exercise 4.10

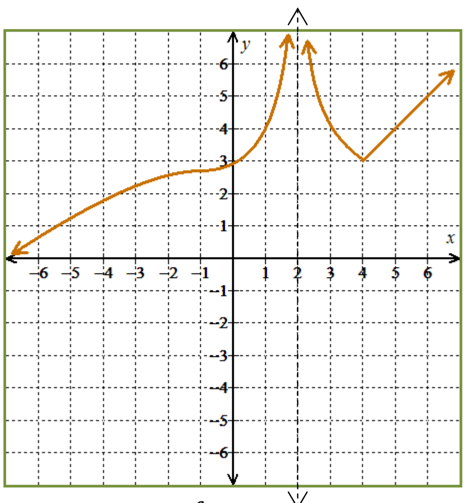


Figure E4.7K f $x = 2$

Supplemental Solutions for the Derivative Formulas Lab

Exercise 5.1

5.1.1
$$f(x)=\frac{1}{6}x^{6/11}$$
$$f'(x)=\frac{1}{11}x^{-5/11}$$
$$=\frac{1}{11\sqrt[11]{x^5}}$$

5.1.2
$$y=-5t^{-7}$$
$$\frac{dy}{dt}=35t^{-8}$$
$$=\frac{35}{t^8}$$

5.1.3
$$y(u)=12u^{1/3}$$
$$y'(u)=4u^{-2/3}$$
$$=\frac{4}{\sqrt[3]{u^2}}$$

5.1.4
$$z(\alpha)=e\alpha^{\pi}$$
$$z'(\alpha)=e\pi\,\alpha^{\pi-1}$$

5.1.5
$$z=8t^{-7/2}$$
$$\frac{dz}{dt}=-28t^{-9/2}$$
$$=-\frac{28}{\sqrt{t^9}}$$

5.1.6
$$T=4\cdot\frac{t^{7/3}}{t^2}$$
$$=4t^{1/3}$$
$$\frac{dT}{dt}=\frac{4}{3}t^{-2/3}$$
$$=\frac{4}{3\sqrt[3]{t^2}}$$

Exercise 5.2

5.2.1
$$y=5\sin(x)\cdot\cos(x)$$
$$\frac{dy}{dx}=\frac{d}{dx}\big(5\sin(x)\big)\cdot\cos(x)+5\sin(x)\cdot\frac{d}{dx}\big(\cos(x)\big)$$
$$=5\cos(x)\cdot\cos(x)+5\sin(x)\cdot\big(-\sin(x)\big)$$
$$=5\cos^2(x)-5\sin^2(x)$$

5.2.2
$$y=\frac{5}{7}t^2\cdot e^t$$
$$\frac{dy}{dt}=\frac{d}{dt}\left(\frac{5}{7}t^2\right)\cdot e^t+\frac{5}{7}t^2\cdot\frac{d}{dt}\left(e^t\right)$$
$$=\frac{10}{7}t\cdot e^t+\frac{5}{7}t^2\,e^t$$
$$=\frac{5t\,e^t(2+t)}{7}$$

5.2.3
$$F(x)=4x\cdot\ln(x)$$
$$F'(x)=\frac{d}{dx}(4x)\cdot\ln(x)+4x\cdot\frac{d}{dx}\big(\ln(x)\big)$$
$$=4\cdot\ln(x)+4x\cdot\frac{1}{x}$$
$$=4\ln(x)+4$$

5.2.4 $z = x^2 \cdot \sin^{-1}(x)$

$$\frac{dz}{dx} = \frac{d}{dx}\left(x^2\right)\cdot\sin^{-1}(x) + x^2\cdot\frac{d}{dx}\left(\sin^{-1}(x)\right)$$

$$= 2x\cdot\sin^{-1}(x) + x^2\cdot\frac{1}{\sqrt{1-x^2}}$$

$$= 2x\sin^{-1}(x) + \frac{x^2}{\sqrt{1-x^2}}$$

5.2.5 $T(t) = \left(1+t^2\right)\cdot\tan^{-1}(t)$

$$T'(t) = \frac{d}{dt}\left(1+t^2\right)\cdot\tan^{-1}(t) + \left(1+t^2\right)\cdot\frac{d}{dt}\left(\tan^{-1}(t)\right)$$

$$= 2t\cdot\tan^{-1}(t) + \left(1+t^2\right)\cdot\frac{1}{1+t^2}$$

$$= 2t\tan^{-1}(t) + 1$$

5.2.6 $T = \frac{1}{3}x^7\cdot 7^x$

$$\frac{dT}{dx} = \frac{d}{dx}\left(\frac{1}{3}x^7\right)\cdot 7^x + \frac{1}{3}x^7\cdot\frac{d}{dx}\left(7^x\right)$$

$$= \frac{7}{3}x^6\cdot 7^x + \frac{1}{3}x^7\cdot\ln(7)7^x$$

$$= \frac{x^6\,7^x\left(7+\ln(7)x\right)}{3}$$

Exercise 5.3

5.3.1 $q(\theta) = \frac{4e^\theta}{e^\theta+1}$

$$q'(\theta) = \frac{\frac{d}{d\theta}\left(4e^\theta\right)\cdot\left(e^\theta+1\right) - 4e^\theta\cdot\frac{d}{d\theta}\left(e^\theta+1\right)}{\left(e^\theta+1\right)^2}$$

$$= \frac{4e^\theta\cdot\left(e^\theta+1\right) - 4e^\theta\cdot e^\theta}{\left(e^\theta+1\right)^2}$$

$$= \frac{4e^\theta\left(e^\theta+1-e^\theta\right)}{\left(e^\theta+1\right)^2}$$

$$= \frac{4e^\theta}{\left(e^\theta+1\right)^2}$$

5.3.2 $u(x) = \dfrac{2\ln(x)}{x^4}$

$$u'(x) = \frac{\frac{d}{dx}\left(2\ln(x)\right)\cdot x^4 - 2\ln(x)\cdot\frac{d}{dx}\left(x^4\right)}{\left(x^4\right)^2}$$

$$= \frac{2\cdot\frac{1}{x}\cdot x^4 - 2\ln(x)\cdot 4x^3}{x^8}$$

$$= \frac{2x^3 - 8x^3\ln(x)}{x^8}$$

$$= \frac{2x^3\left(1-4\ln(x)\right)}{x^8}$$

$$= \frac{2-8\ln(x)}{x^5}$$

5.3.3 $F = \dfrac{t^{1/2}}{3t^2 - 5t^{3/2}}$

$$= \frac{t^{1/2}}{t^{1/2}\left(3t^{3/2} - 5t\right)}$$

$$= \frac{1}{3t^{3/2} - 5t}$$

$$\frac{dF}{dt} = \frac{\frac{d}{dt}(1)\cdot\left(3t^{3/2} - 5t\right) - 1\cdot\frac{d}{dt}\left(3t^{3/2} - 5t\right)}{\left(3t^{3/2} - 5t\right)^2}$$

$$= \frac{0\cdot\left(3t^{3/2} - 5t\right) - 1\cdot\left(\frac{9}{2}t^{1/2} - 5\right)}{\left(3t^{3/2} - 5t\right)^2}$$

$$= \frac{5 - \frac{9}{2}t^{1/2}}{\left(3t^{3/2} - 5t\right)^2}\cdot\frac{2}{2}$$

$$= \frac{10 - 9\sqrt{t}}{2\left(3\sqrt{t^3} - 5t\right)^2}$$

5.3.4 $F(x)=\dfrac{\tan(x)}{\tan^{-1}(x)}$

$$F'(x)=\frac{\frac{d}{dx}\left(\tan(x)\right)\cdot\tan^{-1}(x)-\tan(x)\cdot\frac{d}{dx}\left(\tan^{-1}(x)\right)}{\left(\tan^{-1}(x)\right)^2}$$

$$=\frac{\sec^2(x)\cdot\tan^{-1}(x)-\tan(x)\cdot\frac{1}{1+x^2}}{\left(\tan^{-1}(x)\right)^2}\cdot\frac{1+x^2}{1+x^2}$$

$$=\frac{\left(1+x^2\right)\sec^2(x)\tan^{-1}(x)-\tan(x)}{\left(1+x^2\right)\left(\tan^{-1}(x)\right)^2}$$

5.3.5 $P=\dfrac{\tan^{-1}(t)}{1+t^2}$

$$\frac{dP}{dt}=\frac{\frac{d}{dt}\left(\tan^{-1}(t)\right)\cdot\left(1+t^2\right)-\tan^{-1}(t)\cdot\frac{d}{dt}\left(1+t^2\right)}{\left(1+t^2\right)^2}$$

$$=\frac{\frac{1}{1+t^2}\cdot\left(1+t^2\right)-\tan^{-1}(t)\cdot 2t}{\left(1+t^2\right)^2}$$

$$=\frac{1-2t\tan^{-1}(t)}{\left(1+t^2\right)^2}$$

5.3.6 $y=\dfrac{4}{\sin(\beta)-2\cos(\beta)}$

$$\frac{dy}{d\beta}=\frac{\frac{d}{d\beta}(4)\cdot\left(\sin(\beta)-2\cos(\beta)\right)-4\cdot\frac{d}{d\beta}\left(\sin(\beta)-2\cos(\beta)\right)}{\left(\sin(\beta)-2\cos(\beta)\right)^2}$$

$$=\frac{0\cdot\left(\sin(\beta)-2\cos(\beta)\right)-4\cdot\left(\cos(\beta)-2\left(-\sin(\beta)\right)\right)}{\left(\sin(\beta)-2\cos(\beta)\right)^2}$$

$$=\frac{-4\left(\cos(\beta)+2\sin(\beta)\right)}{\left(\sin(\beta)-2\cos(\beta)\right)^2}$$

Exercise 5.4

E5.4.1

$$\begin{aligned}\frac{d}{dx}\big(f(x)g(x)h(x)\big) &= \frac{d}{dx}\big(f(x)\big)\big[g(x)h(x)\big] + f(x)\frac{d}{dx}\big[g(x)h(x)\big] \\ &= f'(x)\big[g(x)h(x)\big] + f(x)\left[\frac{d}{dx}\big(g(x)\big)h(x) + g(x)\frac{d}{dx}\big(h(x)\big)\right] \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)\end{aligned}$$

E5.4.2

$$f(x) = x^2 e^x \sin(x)\cos(x)$$

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left(x^2\right)e^x \sin(x)\cos(x) + x^2 \frac{d}{dx}\left(e^x\right)\sin(x)\cos(x) + x^2 e^x \frac{d}{dx}\big(\sin(x)\big)\cos(x) + x^2 e^x \sin(x)\frac{d}{dx}\big(\cos(x)\big) \\ &= 2xe^x \sin(x)\cos(x) + x^2 e^x \sin(x)\cos(x) + x^2 e^x \cos(x)\cos(x) + x^2 e^x \sin(x)\big(-\sin(x)\big) \\ &= xe^x\left(2\sin(x)\cos(x) + x\sin(x)\cos(x) + x\cos^2(x) - x\sin^2(x)\right)\end{aligned}$$

Exercise 5.5

E5.5.1

$$\begin{aligned}f(x) &= \frac{x^3 + x^2}{x} \\ &= x^2 + x;\ \ x \neq 0 \\ f'(x) &= 2x + 1;\ \ x \neq 0\end{aligned}$$

$f(5) = 30$ and $f'(5) = 11$ So the tangent line to f at 5 passes through the point $(5, 30)$ and has a slope of 11. The equation of this line is $y = 11x - 25$.

E5.5.2

$$\begin{aligned}h(x) &= \frac{x}{1+x} \\ h'(x) &= \frac{\frac{d}{dx}(x)\cdot(1+x) - x\cdot\frac{d}{dx}(1+x)}{(1+x)^2} \\ &= \frac{1\cdot(1+x) - x\cdot 1}{(1+x)^2} \\ &= \frac{1}{(1+x)^2}\end{aligned}$$

$h(-2) = 2$ and $h'(-2) = 1$ So the tangent line to h at -2 passes through the point $(-2, 2)$ and has a slope of 1. The equation of this line is $y = x + 4$.

E5.5.3 $$K(x)=\frac{1+x}{2x+2}$$
$$=\frac{1+x}{2(x+1)}$$
$$=\frac{1}{2};\ h\neq -1$$

Since the graph of $y=K(x)$ is simply the horizontal line $y=0.5$ with a hole at -1, the tangent line to K at 8 must be the line with equation $y=0.5$.

E5.5.4 $$r(x)=3x^{1/3}\cdot e^{x}$$
$$r'(x)=\frac{d}{dx}\left(3x^{1/3}\right)\cdot e^{x}+3x^{1/3}\cdot\frac{d}{dx}\left(e^{x}\right)$$
$$=x^{-2/3}e^{x}+3x^{1/3}e^{x}$$
$$=\frac{e^{x}}{x^{2/3}}+\frac{3x^{1/3}e^{x}}{1}\cdot\frac{x^{2/3}}{x^{2/3}}$$
$$=\frac{e^{x}(1+3x)}{\sqrt[3]{x^{2}}}$$

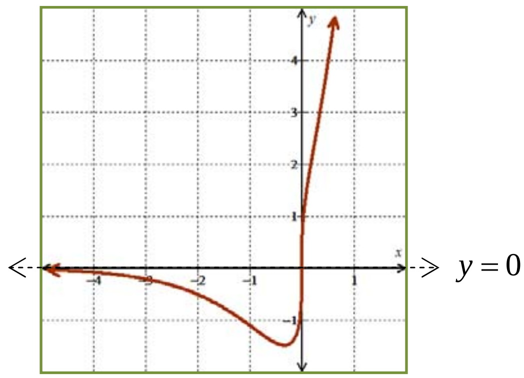


Figure E5.5.4K: $y=3\sqrt[3]{x}\,e^{x}$

$r(0)=0$ and $r'(0)$ has the form $\frac{1}{0}$. This form for $r'(0)$ implies that r has a vertical tangent line at 0. So the tangent to r at 0 is the line $x=0$.

Exercise 5.6 $y=4\sqrt{t}\,t^{5}$ $=4t^{11/2}$ $\qquad \frac{dy}{dt}=22t^{9/2}$ $\qquad \frac{d^{2}y}{dt^{2}}=99t^{7/2}$ $=99\sqrt{t^{7}}$

Exercise 5.7

E5.7.1 f' is positive over $(-\infty,-8)$ and $(2,\infty)$; f' is negative over $(-8,2)$.

E5.7.2 f' is increasing over and $(-3,\infty)$; f' is decreasing over $(-\infty,-3)$.

E5.7.3 f'' is positive over $(-3,\infty)$; f'' is negative over $(-\infty,-3)$.

E5.7.4 $$f'(x)=x^{2}+6x-16$$
$$f''(x)=2x+6$$

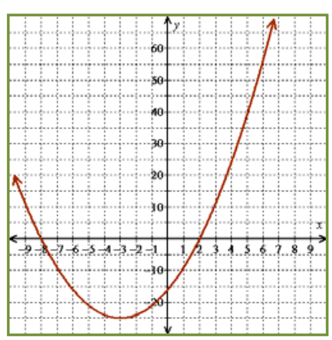


Figure E5.7.4Ka: $y=f'(x)$

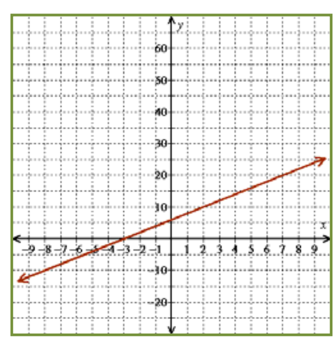


Figure E5.7.4Kb: $y=f''(x)$

Exercise 5.8

E5.8.1 Over the interval $[0,2\pi]$, $g'(t)=0$ at $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$, and $\frac{11\pi}{6}$.

E5.8.2
$$\begin{aligned} g(t) &= 3\sin(t)+3\sin(t)\cdot\sin(t)-5 \\ g'(t) &= 3\cos(t)+\frac{d}{dt}(3\sin(t))\cdot\sin(t)+3\sin(t)\cdot\frac{d}{dt}(\sin(t))+0 \\ &= 3\cos(t)+3\cos(t)\cdot\sin(t)+3\sin(t)\cdot\cos(t) \\ &= 3\cos(t)+6\cos(t)\sin(t) \\ &= 3\cos(t)[1+2\sin(t)] \end{aligned}$$

So $g'(t)=0$ where $\cos(t)=0$ or $\sin(t)=-\frac{1}{2}$. Over $[0,2\pi]$, $\cos(t)=0$ at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Over $[0,2\pi]$, $\sin(t)=-\frac{1}{2}$ at $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. This comports with our answer to problem E5.8.1

Exercise 5.9

$$\begin{aligned} f(x) &= \sin(2x) \\ &= 2\sin(x)\cdot\cos(x) \\ f'(x) &= \frac{d}{dx}(2\sin(x))\cdot\cos(x)+2\sin(x)\cdot\frac{d}{dx}(\cos(x)) \\ &= 2\cos(x)\cdot\cos(x)+2\sin(x)\cdot(-\sin(x)) \\ &= 2\cos(x)\cdot\cos(x)-2\sin(x)\cdot\sin(x) \\ f''(x) &= \left[\frac{d}{dx}(2\cos(x))\cdot\cos(x)+2\cos(x)\cdot\frac{d}{dx}(\cos(x))\right]-\left[\frac{d}{dx}(2\sin(x))\cdot\sin(x)+2\sin(x)\cdot\frac{d}{dx}(\sin(x))\right] \\ &= \left[(-2\sin(x))\cdot\cos(x)+2\cos(x)(-\sin(x))\right]-\left[2\cos(x)\sin(x)+2\sin(x)\cos(x)\right] \\ &= -8\sin(x)\cos(x) \end{aligned}$$

$f'(\pi)=2$ and $f''(\pi)=0$ So the tangent line to f' at π passes through the point $(\pi,2)$ and has a slope of 0. The equation of this line is $y=2$.

Exercise 5.10

E5.10.1
$$\begin{aligned} h(x) &= f(x)g(x) \\ h'(x) &= \frac{d}{dx}(f(x))g(x)+f(x)\frac{d}{dx}(g(x)) \\ &= f'(x)g(x)+f(x)g'(x) \\ h'(4) &= f'(4)g(4)+f(4)g'(4) \\ &= (39)(289)+(46)(293) \\ &= 24{,}749 \end{aligned}$$

E5.10.2 $h'(x) == f'(x)g(x) + f(x)g'(x)$

$$h''(x) = \left[\frac{d}{dx}\left(f'(x)\right)g(x) + f'(x)\frac{d}{dx}\left(g(x)\right)\right] + \left[\frac{d}{dx}\left(f(x)\right)g'(x) + f(x)\frac{d}{dx}\left(g'(x)\right)\right]$$

$$= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$$

$$h''(2) = f''(2)g(2) + 2f'(2)g'(2) + f(2)g''(2)$$

$$= (10)(11) + 2(7)(37) + (4)(58)$$

$$= 860$$

E5.10.3 $k(x) = \dfrac{g(x)}{f(x)}$

$$k'(x) = \frac{\frac{d}{dx}\left(g(x)\right)f(x) - g(x)\frac{d}{dx}\left(f(x)\right)}{\left[f(x)\right]^2}$$

$$= \frac{g'(x)f(x) - g(x)f'(x)}{\left[f(x)\right]^2}$$

$$k'(3) = \frac{g'(3)f(3) - g(3)f'(3)}{\left[f(3)\right]^2}$$

$$= \frac{(126)(15) - (87)(20)}{15^2}$$

$$= \frac{2}{3}$$

E5.10.4 $p(x) = 6\sqrt{x}\,f(x)$

$$p'(x) = \frac{d}{dx}\left(6\sqrt{x}\right)f(x) + 6\sqrt{x}\frac{d}{dx}\left(f(x)\right)$$

$$= 6\cdot\frac{1}{2\sqrt{x}}f(x) + 6\sqrt{x}\,f'(x)$$

$$= \frac{3f(x)}{\sqrt{x}} + 6\sqrt{x}\,f'(x)$$

$$p'(4) = \frac{3f(4)}{\sqrt{4}} + 6\sqrt{4}\,f'(4)$$

$$= \frac{3(46)}{2} + 6(2)(39)$$

$$= 537$$

E5.10.5

$$
\begin{aligned}
r(x) &= \left[g(x)\right]^2 \\
&= g(x)\cdot g(x) \\
r'(x) &= \frac{d}{dx}\left(g(x)\right)g(x) + g(x)\frac{d}{dx}\left(g(x)\right) \\
&= 2\,g(x)\,g'(x) \\
r'(1) &= 2\,g(1)\,g'(1) \\
&= 2(-5)(2) \\
&= -20
\end{aligned}
$$

E5.10.6

$$
\begin{aligned}
s(x) &= x\,f(x)\,g(x) \\
s'(x) &= \frac{d}{dx}(x)\,f(x)g(x) + x\frac{d}{dx}\left(f(x)\right)g(x) + x\,f(x)\frac{d}{dx}\left(g(x)\right) \\
&= f(x)g(x) + x\,f'(x)g(x) + x\,f(x)g'(x) \\
s'(2) &= f(2)g(2) + 2\,f'(2)g(2) + 2\,f(2)g'(2) \\
&= (4)(11) + 2(7)(11) + 2(4)(37) \\
&= 494
\end{aligned}
$$

E5.10.7

$$
\begin{aligned}
F(x) &= \sqrt{x}\,g(4) \\
&= 289\sqrt{x} \\
F'(x) &= \frac{289}{2\sqrt{x}}
\end{aligned}
\qquad
\begin{aligned}
F'(4) &= \frac{289}{2\sqrt{4}} \\
&= \frac{289}{4}
\end{aligned}
$$

E5.10.8

$$
\begin{aligned}
T(x) &= \frac{f(x)}{e^x} \\
T'(x) &= \frac{\frac{d}{dx}\left(f(x)\right)\cdot e^x - f(x)\cdot\frac{d}{dx}\left(e^x\right)}{\left(e^x\right)^2} \\
&= \frac{f'(x)e^x - f(x)e^x}{\left(e^x\right)^2} \\
&= \frac{\cancel{e^x}\left(f'(x) - f(x)\right)}{\cancel{e^x}\cdot e^x} \\
&= \frac{f'(x) - f(x)}{e^x}
\end{aligned}
$$

(Continued on next page.)

$$
\begin{aligned}
T''(x) &= \frac{\frac{d}{dx}\left(f'(x)-f(x)\right)\cdot e^x - \left(f'(x)-f(x)\right)\cdot\frac{d}{dx}\left(e^x\right)}{\left(e^x\right)^2} \\
&= \frac{\left(f''(x)-f'(x)\right)\cdot e^x - \left(f'(x)-f(x)\right)\cdot e^x}{\left(e^x\right)^2} \\
&= \frac{\cancel{e^x}\left(f''(x)-f'(x)-f'(x)+f(x)\right)}{\cancel{e^x}\cdot e^x} \\
&= \frac{f''(x)-2f'(x)+f(x)}{e^x} \\
T''(0) &= \frac{f''(0)-2f'(0)+f(0)}{e^0} \\
&= \frac{-2-2(-1)+2}{1} \\
&= 2
\end{aligned}
$$

Supplemental Solutions for the Chain Rule Lab

Exercise 6.1

$$f'(x) = e^{3x} \cdot \frac{d}{dx}(3x) = e^{3x} \cdot 3 = 3e^{3x}$$

$$g'(x) = 3\left(e^x\right)^2 \cdot \frac{d}{dx}\left(e^x\right) = 3\left(e^x\right)^2 \cdot e^x = 3\left(e^x\right)^3 = 3e^{3x}$$

$$h'(x) = \ln\left(e^3\right) \cdot \left(e^3\right)^x = 3e^{3x}$$

Exercise 6.2

E6.2.1 $k'(\theta) = 1$ over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

E6.2.2 Over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:

$$\begin{aligned} k'(\theta) &= \frac{1}{\sqrt{1-\sin^2(\theta)}} \cdot \frac{d}{d\theta}\left(\sin(\theta)\right) \\ &= \frac{1}{\sqrt{\cos^2(\theta)}} \cdot \cos(\theta) && \text{(I used the identity } \cos^2(\theta)+\sin^2(\theta)=1 \text{ here.)} \\ &= \frac{1}{\cos(\theta)} \cdot \cos(\theta) && \text{(As noted, } \cos(\theta) > 0 \text{ over } \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right).) \\ &= 1 \end{aligned}$$

E6.2.3 Over $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $\cos(\theta)$ is constantly negative. By definition, $\sqrt{u}$ is never negative. So over $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$:

$$\begin{aligned} k'(\theta) &== \frac{1}{\sqrt{\cos^2(\theta)}} \cdot \cos(\theta) \\ &= \frac{1}{-\cos(\theta)} \cdot \cos(\theta) \\ &= -1 \end{aligned}$$

E6.2.4 $k'\left(\frac{\pi}{2}\right)$ is undefined. The function k is nondifferentiable at $\frac{\pi}{2}$ because it transitions from a slope of 1 to a slope of -1 at that value.

Exercise 6.3

E6.3.1 $$g'(x) = 4\left[f(x)\right]^3 \cdot \frac{d}{dx}\left(f(x)\right)$$
$$= 4\left[f(x)\right]^3 f'(x)$$

g' is positive anywhere f and f' share the same sign. This occurs over $(0,2)$ and $(3.2,\infty)$. (The left endpoint of the second interval is an approximation.)

E6.3.2 $$r'(x) = e^{f(x)} \cdot \frac{d}{dx}\left(f(x)\right)$$
$$= e^{f(x)} f'(x)$$

Since the factor of $e^{f(x)}$ is always positive, $r'(x)$ is positive anywhere $f'(x)$ is positive. This occurs over $(2,\infty)$.

E6.3.3 $$w'(x) = e^{f(-x)} \cdot \frac{d}{dx}\left(f(-x)\right)$$
$$= e^{f(-x)} f'(-x) \cdot \frac{d}{dx}(-x)$$
$$= e^{f(-x)} f'(-x) \cdot (-1)$$
$$= -e^{f(-x)} f'(-x)$$

Since the factor of $-e^{f(-x)}$ is always negative, $w'(x)$ is positive anywhere $f'(-x)$ is also negative. Because the graph of $y = f'(x)$ is negative on the intervals $(-\infty,-3)$ and $(-3,2)$, $f'(-x)$ is negative on the intervals $(-2,3)$ and $(3,\infty)$. Ergo, $w'(x)$ is positive over $(-2,3)$ and $(3,\infty)$. (Major high five if you figured that out on your own!)

E6.3.4 $$h(x) = \left[f(x)\right]^{-1}$$
$$h'(x) = -\left[f(x)\right]^{-2} \cdot \frac{d}{dx}\left(f(x)\right)$$
$$= -\frac{f'(x)}{\left[f(x)\right]^2}$$

$$h'(-3) = -\frac{f'(-3)}{\left[f(-3)\right]^2}$$
$$= -\frac{0}{4^2}$$
$$= 0$$

So, the function h is ***not*** nondifferentiable at -3.

Exercise 6.4

E6.4.1 yes **E6.4.2** no **E6.4.3** no

E6.4.4 no **E6.4.5** yes **E6.4.6** yes

Exercise 6.5

E6.5.1

$$f(x)=\tan^{-1}\left(\sqrt{x}\right)$$

$$f'(x)=\frac{1}{1+\left(\sqrt{x}\right)^2}\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=\frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2(1+x)\sqrt{x}}$$

E6.5.2

$$f(x)=e^{e^{\sin(x)}}$$

$$f'(x)=e^{e^{\sin(x)}}\cdot\frac{d}{dx}\left(e^{\sin(x)}\right)$$

$$=e^{e^{\sin(x)}}\cdot e^{\sin(x)}\cdot\frac{d}{dx}\left(\sin(x)\right)$$

$$=e^{e^{\sin(x)}}\cdot e^{\sin(x)}\cdot\cos(x)$$

$$=\cos(x)e^{\sin(x)+e^{\sin(x)}}$$

E6.5.3

$$f(x)=\sin^{-1}\left(\cos(x)\right)$$

$$f'(x)=\frac{1}{\sqrt{1-\cos^2(x)}}\cdot\frac{d}{dx}\left(\cos(x)\right)$$

$$=\frac{1}{\sqrt{\sin^2(x)}}\cdot\left(-\sin(x)\right)$$

$$=-\frac{\sin(x)}{\left|\sin(x)\right|}$$

E6.5.4

$$f(x)=\tan\left(x\sec(x)\right)$$

$$f'(x)=\sec^2\left(x\sec(x)\right)\cdot\frac{d}{dx}\left(x\sec(x)\right)$$

$$=\sec^2\left(x\sec(x)\right)\cdot\left(\frac{d}{dx}(x)\cdot\sec(x)+x\cdot\frac{d}{dx}\left(\sec(x)\right)\right)$$

$$=\sec^2\left(x\sec(x)\right)\cdot\left(1\cdot\sec(x)+x\cdot\sec(x)\tan(x)\right)$$

$$=\sec(x)\sec^2\left(x\sec(x)\right)\left(1+x\tan(x)\right)$$

E6.5.5 $f(x)=\tan(x)\sec(\sec(x))$

$$f'(x)=\frac{d}{dx}(\tan(x))\cdot\sec(\sec(x))+\tan(x)\cdot\frac{d}{dx}(\sec(\sec(x)))$$
$$=\sec^2(x)\sec(\sec(x))+\tan(x)\sec(\sec(x))\tan(\sec(x))\cdot\frac{d}{dx}(\sec(x))$$
$$=\sec^2(x)\sec(\sec(x))+\tan(x)\sec(\sec(x))\tan(\sec(x))\sec(x)\tan(x)$$
$$=\sec(x)\sec(\sec(x))\left[\sec(x)+\tan^2(x)\tan(\sec(x))\right]$$

E6.5.6 $f(x)=\sqrt[3]{\left(\sin(x^2)\right)^2}$

$$=\left(\sin(x^2)\right)^{2/3}$$
$$f'(x)=\frac{2}{3}\left(\sin(x^2)\right)^{-1/3}\cdot\frac{d}{dx}\left(\sin(x^2)\right)$$
$$=\frac{2}{3}\left(\sin(x^2)\right)^{-1/3}\cdot\cos(x^2)\cdot\frac{d}{dx}(x^2)$$
$$=\frac{2}{3}\left(\sin(x^2)\right)^{-1/3}\cos(x^2)\cdot 2x$$
$$=\frac{4x\cos(x^2)}{3\sqrt[3]{\sin(x^2)}}$$

E6.5.7 $f(x)=4x\sin^2(x)$

$$f'(x)=\frac{d}{dx}(4x)\cdot\sin^2(x)+4x\cdot\frac{d}{dx}\left((\sin(x))^2\right)$$
$$=4\cdot\sin^2(x)+4x\cdot 2\sin(x)\cdot\frac{d}{dx}(\sin(x))$$
$$=4\sin^2(x)+8x\sin(x)\cos(x)$$
$$=4\sin(x)(\sin(x)+2x\cos(x))$$

E6.5.8 $f(x)=\ln(x\ln(x))$

$$=\ln(x)+\ln(\ln(x))$$

$$f'(x)=\frac{1}{x}+\frac{1}{\ln(x)}\cdot\frac{d}{dx}(\ln(x))$$
$$=\frac{1}{x}+\frac{1}{\ln(x)}\cdot\frac{1}{x}$$
$$=\frac{\ln(x)+1}{x\ln(x)}$$

E6.5.9 $f(x)=\ln\left(\frac{5}{xe^x}\right)$

$$=\ln(5)-\ln(xe^x)$$
$$=\ln(5)-\left[\ln(x)+\ln(e^x)\right]$$
$$=\ln(5)-\ln(x)-x$$

$$f'(x)=0-\frac{1}{x}-1$$
$$=-\frac{1+x}{x}$$

E6.5.10

$$\begin{aligned} f(x) &= 2\ln\left(\sqrt[3]{x\tan^2(x)}\right) \\ &= 2\ln\left(\left(x\tan^2(x)\right)^{1/3}\right) \\ &= 2\cdot\frac{1}{3}\ln\left(x\tan^2(x)\right) \\ &= \frac{2}{3}\left[\ln(x)+\ln\left(\tan^2(x)\right)\right] \\ &= \frac{2}{3}\left[\ln(x)+2\ln\left(\tan(x)\right)\right] \end{aligned}$$

$$\begin{aligned} f'(x) &= \frac{2}{3}\left[\frac{1}{x}+2\cdot\frac{1}{\tan(x)}\cdot\frac{d}{dx}\left(\tan(x)\right)\right] \\ &= \frac{2}{3}\left[\frac{1}{x}+\frac{2}{\tan(x)}\cdot\sec^2(x)\right] \\ &= \frac{2\left(\tan(x)+2x\sec^2(x)\right)}{3x\tan(x)} \end{aligned}$$

E6.5.11

$$\begin{aligned} f(x) &= \ln\left(\frac{e^{x+2}}{\sqrt{x+2}}\right) \\ &= \ln\left(e^{x+2}\right)-\ln\left((x+2)^{1/2}\right) \\ &= x+2-\frac{1}{2}\ln(x+2) \end{aligned}$$

$$\begin{aligned} f'(x) &= 1+0-\frac{1}{2}\cdot\frac{1}{x+2}\cdot\frac{d}{dx}(x+2) \\ &= 1-\frac{1}{2}\cdot\frac{1}{x+2}\cdot 1 \\ &= \frac{2x+4-1}{2(x+2)} \\ &= \frac{2x+3}{2(x+2)} \end{aligned}$$

E6.5.12

$$\begin{aligned} f(x) &= \ln\left(x^e+e\right) \\ f'(x) &= \frac{1}{x^e+e}\cdot\frac{d}{dx}\left(x^e+e\right) \\ &= \frac{1}{x^e+e}\cdot\left(e\,x^{e-1}+0\right) \\ &= \frac{e\,x^{e-1}}{x^e+e} \end{aligned}$$

E6.5.13

$$\begin{aligned} f(x) &= \sec^4\left(e^x\right) \\ &= \left[\sec\left(e^x\right)\right]^4 \\ f'(x) &= 4\left[\sec\left(e^x\right)\right]^3\cdot\frac{d}{dx}\left(\sec\left(e^x\right)\right) \\ &= 4\sec^3\left(e^x\right)\cdot\sec\left(e^x\right)\tan\left(e^x\right)\cdot\frac{d}{dx}\left(e^x\right) \\ &= 4\sec^4\left(e^x\right)\tan\left(e^x\right)e^x \end{aligned}$$

E6.5.14

$$\begin{aligned} f(x) &= \sec^{-1}\left(e^x\right) \\ f'(x) &= \frac{1}{\left|e^x\right|\sqrt{\left(e^x\right)^2-1}}\frac{d}{dx}\left(e^x\right) \\ &= \frac{1}{\cancel{e^x}\sqrt{e^{2x}-1}}\cdot\cancel{e^x} \\ &= \frac{1}{\sqrt{e^{2x}-1}} \end{aligned}$$

E6.5.15

$$\begin{aligned} f(x) &= \csc\left(\frac{1}{\sqrt{x}}\right) \\ &= \csc\left(x^{-1/2}\right) \\ f'(x) &= -\csc\left(x^{-1/2}\right)\cot\left(x^{-1/2}\right)\cdot\frac{d}{dx}\left(x^{-1/2}\right) \\ &= -\csc\left(x^{-1/2}\right)\cot\left(x^{-1/2}\right)\cdot-\frac{1}{2}x^{-3/2} \\ &= \frac{\csc\left(\frac{1}{\sqrt{x}}\right)\cot\left(\frac{1}{\sqrt{x}}\right)}{2\sqrt{x^3}} \end{aligned}$$

E6.5.16

$$\begin{aligned} f(x) &= \frac{1}{\csc\left(\sqrt{x}\right)} \\ &= \sin\left(\sqrt{x}\right) \\ f'(x) &= \cos\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right) \\ &= \cos\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}} \\ &= \frac{\cos\left(\sqrt{x}\right)}{2\sqrt{x}} \end{aligned}$$

E6.5.17

$$\begin{aligned} f(x) &= \frac{\tan^{-1}(2x)}{2} \\ &= \frac{1}{2}\tan^{-1}(2x) \\ f'(x) &= \frac{1}{2}\cdot\frac{1}{1+(2x)^2}\cdot\frac{d}{dx}(2x) \\ &= \frac{1}{2}\cdot\frac{1}{1+4x^2}\cdot 2 \\ &= \frac{1}{1+4x^2} \end{aligned}$$

E6.5.18

$$\begin{aligned} f(x) &= x^3\sin\left(\frac{x}{3}\right) \\ f'(x) &= \frac{d}{dx}\left(x^3\right)\cdot\sin\left(\frac{x}{3}\right)+x^3\cdot\frac{d}{dx}\left(\sin\left(\frac{x}{3}\right)\right) \\ &= 3x^2\cdot\sin\left(\frac{x}{3}\right)+x^3\cdot\cos\left(\frac{x}{3}\right)\cdot\frac{d}{dx}\left(\frac{x}{3}\right) \\ &= 3x^2\sin\left(\frac{x}{3}\right)+x^3\cos\left(\frac{x}{3}\right)\cdot\frac{1}{3} \\ &= \frac{x^2\left(9\sin\left(\frac{x}{3}\right)+x\cos\left(\frac{x}{3}\right)\right)}{3} \end{aligned}$$

E6.5.19

$$\begin{aligned} f(x) &= \frac{4}{\sqrt{\frac{3}{x^7}}} \\ &= \frac{4}{\sqrt{3}}x^{7/2} \\ f'(x) &= \frac{4}{\sqrt{3}}\cdot\frac{7}{2}x^{5/2} \\ &= \frac{14\sqrt{x^5}}{\sqrt{3}} \end{aligned}$$

E6.5.20

$$\begin{aligned} f(x) &= \frac{e^{xe^x}}{x} \\ f'(x) &= \frac{\frac{d}{dx}\left(e^{xe^x}\right)\cdot x - e^{xe^x}\ \frac{d}{dx}(x)}{x^2} \\ &= \frac{e^{xe^x}\cdot\frac{d}{dx}\left(xe^x\right)\cdot x - e^{xe^x}\cdot 1}{x^2} \\ &= \frac{e^{xe^x}\cdot\left(\frac{d}{dx}(x)\cdot e^x + x\cdot\frac{d}{dx}\left(e^x\right)\right)\cdot x - e^{xe^x}}{x^2} \\ &= \frac{e^{xe^x}\left(1\cdot e^x + x\cdot e^x\right)\cdot x - e^{xe^x}}{x^2} \\ &= \frac{e^{xe^x}\left(xe^x + x^2e^x - 1\right)}{x^2} \end{aligned}$$

E6.5.21 $f(x) = x e^{xe^2}$

$$f'(x) = \frac{d}{dx}(x) \cdot e^{xe^2} + x \cdot \frac{d}{dx}\left(e^{xe^2}\right)$$
$$= 1 \cdot e^{xe^2} + x \cdot e^{xe^2} \cdot \frac{d}{dx}\left(xe^2\right)$$
$$= e^{xe^2} + x e^{xe^2} \cdot e^2$$
$$= e^{xe^2}\left(1 + xe^2\right)$$

E6.5.22 $f(x) = \dfrac{\sin^5(x) - \sqrt{\sin(x)}}{\sin(x)}$

$$= \frac{(\sin(x))^5}{\sin(x)} - \frac{\sqrt{\sin(x)}}{\sin(x)}$$
$$= (\sin(x))^4 - (\sin(x))^{-1/2}$$

$$f'(x) = 4(\sin(x))^3 \cdot \frac{d}{dx}(\sin(x)) + \frac{1}{2}(\sin(x))^{-3/2} \cdot \frac{d}{dx}(\sin(x))$$
$$= 4\sin^3(x)\cos(x) + \frac{1}{2} \cdot \frac{1}{\sqrt{\sin^3(x)}} \cdot \cos(x)$$
$$= \frac{\cos(x)\left[8\sqrt{\sin^9(x)} + 1\right]}{2\sqrt{\sin^3(x)}}$$

E6.5.23 $f(x) = 4x\sin(x)\cos\left(x^2\right)$

$$f'(x) = \frac{d}{dx}(4x) \cdot \sin(x)\cos\left(x^2\right) + 4x \cdot \frac{d}{dx}(\sin(x)) \cdot \cos\left(x^2\right) + 4x\sin(x) \cdot \frac{d}{dx}\left(\cos\left(x^2\right)\right)$$
$$= 4 \cdot \sin(x)\cos\left(x^2\right) + 4x \cdot \cos(x) \cdot \cos\left(x^2\right) + 4x\sin(x) \cdot \left(-\sin\left(x^2\right)\right) \cdot \frac{d}{dx}\left(x^2\right)$$
$$= 4\sin(x)\cos\left(x^2\right) + 4x\cos(x)\cos\left(x^2\right) - 4x\sin(x)\sin\left(x^2\right) \cdot 2x$$
$$= 4\sin(x)\cos\left(x^2\right) + 4x\cos(x)\cos\left(x^2\right) - 8x^2\sin(x)\sin\left(x^2\right)$$

E6.5.24 $f(x) = \sin\left(x\cos^2(x)\right)$

$$f'(x) = \cos\left(x\cos^2(x)\right) \cdot \frac{d}{dx}\left(x\cos^2(x)\right)$$
$$= \cos\left(x\cos^2(x)\right) \cdot \left[\frac{d}{dx}(x) \cdot \cos^2(x) + x \cdot \frac{d}{dx}\left((\cos(x))^2\right)\right]$$
$$= \cos\left(x\cos^2(x)\right) \cdot \left[1 \cdot \cos^2(x) + x \cdot 2\cos(x) \cdot \frac{d}{dx}(\cos(x))\right]$$
$$= \cos\left(x\cos^2(x)\right) \cdot \left[\cos^2(x) + 2x\cos(x) \cdot (-\sin(x))\right]$$
$$= \cos(x)\cos\left(x\cos^2(x)\right) \cdot \left[\cos(x) - 2x\sin(x)\right]$$

Supplemental Solutions for the Implicit Differentiation Lab

Exercise 7.1

$$x\sin(x\,y)=y$$
$$\frac{d}{dx}\left[x\sin(x\,y)\right]=\frac{d}{dx}(y)$$
$$\frac{d}{dx}(x)\cdot\sin(x\,y)+x\cdot\frac{d}{dx}\left(\sin(x\,y)\right)=\frac{dy}{dx}$$
$$1\cdot\sin(x\,y)+x\cdot\cos(x\,y)\cdot\frac{d}{dx}(x\,y)=\frac{dy}{dx}$$
$$\sin(x\,y)+x\cos(x\,y)\cdot\left[\frac{d}{dx}(x)\cdot y+x\cdot\frac{d}{dx}(y)\right]=\frac{dy}{dx}$$
$$\sin(x\,y)+x\cos(x\,y)\cdot\left[1\cdot y+x\cdot\frac{dy}{dx}\right]=\frac{dy}{dx}$$
$$\sin(x\,y)+x\,y\cos(x\,y)+x^2\cos(x\,y)\frac{dy}{dx}=\frac{dy}{dx}$$
$$\sin(x\,y)+x\,y\cos(x\,y)=\left[1-x^2\cos(x\,y)\right]\frac{dy}{dx}$$
$$\frac{\sin(x\,y)+x\,y\cos(x\,y)}{1-x^2\cos(x\,y)}=\frac{dy}{dx}$$

Since we're looking for points on the y-axis, we know that the y-coordinates are zero. A necessary element for the tangent line to be vertical is that the denominator of the derivative formula evaluates to zero (otherwise the tangent line would have a value for its slope). Putting this together gives us:

$$1-x^2\cos(x\cdot 0)=0\ \Rightarrow\ 1-x^2=0\ \Rightarrow\ x=\pm 1$$

So the x-coordinates are 1 and -1.

Exercise 7.2

$$\ln\left(x^2\,y^2\right)=x+y$$
$$\ln\left(x^2\right)+\ln\left(y^2\right)=x+y$$
$$2\ln(x)+2\ln(y)=x+y$$
$$\frac{d}{dx}\left(2\ln(x)+2\ln(y)\right)=\frac{d}{dx}(x+y)$$
$$2\cdot\frac{1}{x}+2\cdot\frac{1}{y}\frac{dy}{dx}=1+\frac{dy}{dx}$$
$$\left(\frac{2}{y}-1\right)\frac{dy}{dx}=1-\frac{2}{x}$$
$$\frac{dy}{dx}=\frac{1-\frac{2}{x}}{\frac{2}{y}-1}$$
$$\left.\frac{dy}{dx}\right|_{(1,-1)}=\frac{1}{3}$$

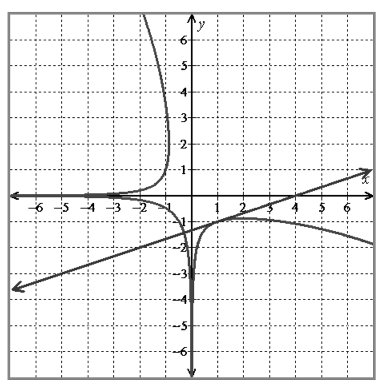


Figure E7.2K: $\ln\left(x^2\,y^2\right)=x+y$

The slope of the tangent line is $\frac{1}{3}$ and the line passes through the point $(1,-1)$, so the equation of the tangent line is $y=\frac{1}{3}x-\frac{4}{3}$.

Exercise 7.3

$$y = x^x \;\Rightarrow\; \ln(y) = \ln\left(x^x\right) \;\Rightarrow\; \ln(y) = x \cdot \ln(x)$$

$$\frac{d}{dx}\left(\ln(y)\right) = \frac{d}{dx}\left(x \cdot \ln(x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x) \cdot \ln(x) + x \cdot \frac{d}{dx}\left(\ln(x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln(x) + x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y\left(\ln(x) + 1\right)$$

$$\frac{dy}{dx} = x^x\left(\ln(x) + 1\right)$$

Exercise 7.4

7.4.1

$$y = \frac{x\sin(x)}{\sqrt{x-1}}$$

$$\ln(y) = \ln\left(\frac{x\sin(x)}{\sqrt{x-1}}\right)$$

$$\ln(y) = \ln\left(x\sin(x)\right) - \ln\left[(x-1)^{1/2}\right]$$

$$\ln(y) = \ln(x) + \ln\left(\sin(x)\right) - \frac{1}{2}\ln(x-1)$$

$$\frac{d}{dx}\left(\ln(y)\right) = \frac{d}{dx}\left[\ln(x) + \ln\left(\sin(x)\right) - \frac{1}{2}\ln(x-1)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\sin(x)} \cdot \frac{d}{dx}\left(\sin(x)\right) - \frac{1}{2} \cdot \frac{1}{x-1} \cdot \frac{d}{dx}(x-1)$$

$$\frac{dy}{dx} = y\left[\frac{1}{x} + \frac{1}{\sin(x)} \cdot \cos(x) - \frac{1}{2} \cdot \frac{1}{x-1} \cdot 1\right]$$

$$\frac{dy}{dx} = \frac{x\sin(x)}{\sqrt{x-1}}\left[\frac{1}{x} + \cot(x) - \frac{1}{2(x-1)}\right]$$

7.4.2

$$y=\frac{e^{2x}}{\sin^4(x)\sqrt[4]{x^5}}$$

$$\ln(y)=\ln\left(\frac{e^{2x}}{\sin^4(x)\sqrt[4]{x^5}}\right)$$

$$\ln(y)=\ln\left(e^{2x}\right)-\ln\left(\sin^4(x)\sqrt[4]{x^5}\right)$$

$$\ln(y)=2x-\left[\ln\left((\sin(x))^4\right)+\ln\left(x^{5/4}\right)\right]$$

$$\ln(y)=2x-4\ln(\sin(x))-\frac{5}{4}\ln(x)$$

$$\frac{d}{dx}(\ln(y))=\frac{d}{dx}\left[2x-4\ln(\sin(x))-\frac{5}{4}\ln(x)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=2-4\cdot\frac{1}{\sin(x)}\cdot\frac{d}{dx}(\sin(x))-\frac{5}{4}\cdot\frac{1}{x}$$

$$\frac{dy}{dx}=y\left[2-4\cdot\frac{1}{\sin(x)}\cdot\cos(x)-\frac{5}{4x}\right]$$

$$\frac{dy}{dx}=\frac{e^{2x}}{\sin^4(x)\sqrt[4]{x^5}}\left[2-4\cot(x)-\frac{5}{4x}\right]$$

7.4.3

$$y=\frac{\ln\left(4x^3\right)}{x^5\ln(x)}$$

$$\ln(y)=\ln\left(\frac{\ln\left(4x^3\right)}{x^5\ln(x)}\right)$$

$$\ln(y)=\ln\left(\ln\left(4x^3\right)\right)-\ln\left(x^5\ln(x)\right)$$

$$\ln(y)=\ln\left(\ln\left(4x^3\right)\right)-\left[\ln\left(x^5\right)+\ln(\ln(x))\right]$$

$$\ln(y)=\ln\left(\ln\left(4x^3\right)\right)-5\ln(x)-\ln(\ln(x))$$

$$\frac{d}{dx}(\ln(y))=\frac{d}{dx}\left[\ln\left(\ln\left(4x^3\right)\right)-5\ln(x)-\ln(\ln(x))\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{\ln\left(4x^3\right)}\cdot\frac{d}{dx}\left(\ln\left(4x^3\right)\right)-5\cdot\frac{1}{x}-\frac{1}{\ln(x)}\cdot\frac{d}{dx}(\ln(x))$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{\ln\left(4x^3\right)}\cdot\frac{1}{4x^3}\cdot\frac{d}{dx}\left(4x^3\right)-\frac{5}{x}-\frac{1}{\ln(x)}\cdot\frac{1}{x}$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{\ln\left(4x^3\right)}\cdot\frac{1}{4x^3}\cdot12x^2-\frac{5}{x}-\frac{1}{x\ln(x)}$$

$$\frac{dy}{dx}=y\left[\frac{3}{x\ln\left(4x^3\right)}-\frac{5}{x}-\frac{1}{x\ln(x)}\right]$$

$$\frac{dy}{dx}=\frac{\ln\left(4x^3\right)}{x^5\ln(x)}\left[\frac{3}{x\ln\left(4x^3\right)}-\frac{5}{x}-\frac{1}{x\ln(x)}\right]$$

Supplemental Solutions for the Related Rates Lab

Exercise 8.1

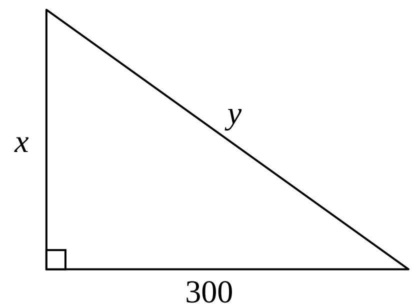


Figure E8.1K: Variable Diagram

Define x to be the elevation (ft) of the balloon t seconds after the balloon begins to rise and y to be the distance (ft) between the observer and the balloon at the same instant.

The relation equation is $x^2 + 300^2 = y^2$.

The rate equation is:

$$\frac{d}{dt}\left(x^2 + 90000\right) = \frac{d}{dt}\left(y^2\right) \;\Rightarrow\; 2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

When the elevation of the balloon is 400 feet:

$x = 400$, $y = 500$ (from the Pythagorean Theorem), and $\frac{dx}{dt} = 10$

Substituting these values into the rate equation we get:

$$2(400)(10) = 2(500)\frac{dy}{dt} \;\Rightarrow\; \frac{dy}{dt} = 8$$

So the distance between the observer and the balloon is increasing at a rate of 8 ft/s at the instant the elevation of the balloon is 400 feet.

Exercise 8.2

Define x to be the vertical distance (ft) between the tip of the arm and the horizontal position of the arm t seconds after gate begins to close and θ to be the angle of elevation (rad) at the pivot point at the same instant in time.

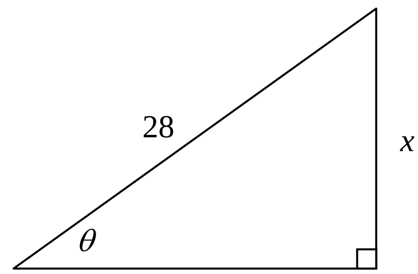


Figure E8.2K: Variable Diagram

The relation equation is $\sin(\theta) = \frac{x}{28}$.

The rate equation is: $\frac{d}{dt}\left(\sin(\theta)\right) = \frac{d}{dt}\left(\frac{x}{28}\right) \Rightarrow \cos(\theta)\frac{d\theta}{dt} = \frac{1}{28}\frac{dx}{dt}$

At the instant the angle at the pivot point is 30^{O}:

$$\theta = \frac{\pi}{6} \quad \text{and} \quad \frac{d\theta}{dt} = \left(-6\ ^{\text{deg}}\!/_{\text{s}}\right)\left(^{\pi\ \text{rad}}\!/_{180\ \text{deg}}\right)$$
$$= -\frac{\pi}{30}\ ^{\text{rad}}\!/_{\text{s}}$$

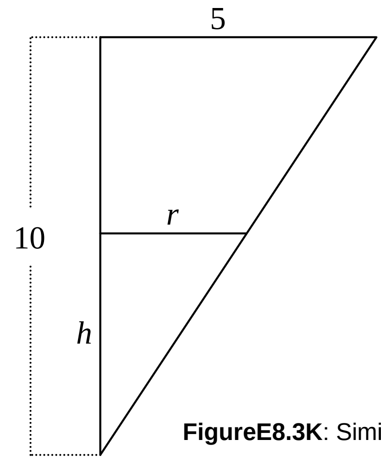
Substituting these values into the rate equation we have:

$$\cos\left(\frac{\pi}{6}\right)\left(-\frac{\pi}{30}\right) = \frac{1}{28}\frac{dx}{dt} \Rightarrow \frac{dx}{dt} \approx -2.54$$

So at the instant the angle of elevation of the arm is 30^{O}, the tip of the arm is approaching the ground at a rate of about 2.54 ft/s.

Exercise 8.3

Define V to be the volume of soda (cm^3) that remains in Jimbo's cup t seconds after he commences to sip and h to be the height of the soda in the cup (cm) at that very same instant.



FigureE8.3K: Similar Triangles

The volume formula for a right circular cone is $V = \frac{\pi}{3} r^2 h$ where V represents the volume of the cone, h represents the height of the cone, and r represents the radius at the top of the cone. Since neither of our defined variable is a radius, we need to purge that variable from our volume formula.

From the similar triangles shown in Figure E8.3K we have, $\frac{r}{h} = \frac{5}{10} \Rightarrow r = \frac{h}{2}$.

Substituting the expression $\frac{h}{2}$ for r in the volume formula we get our relation equation:

$$V = \frac{\pi}{3} r^2 h \Rightarrow V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h \Rightarrow V = \frac{\pi}{12} h^3$$

Our rate equation is: $\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi}{12} h^3\right) \Rightarrow \frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$

At the instant there are 100 cm^3 of soda remaining in the cup:

$$100 = \frac{\pi}{12} h^3 \Rightarrow h = \sqrt[3]{\frac{1200}{\pi}} \text{ ; also, } \frac{dV}{dt} = -0.25$$

Substituting these values into our rate equation we get:

$$-0.25 = \frac{\pi}{4}\left(\sqrt[3]{\frac{1200}{\pi}}\right)^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} \approx -0.006$$

So at the instant there are 100 cm^3 of soda remaining in the cup, the height of the soda in the cup is decreasing at a rate of about 0.006 cm/s. Somebody needs to tell Jimbo to put some juice into his sipping rate!

Exercise 8.4

Define V to be the volume of the snowball (cm^3) and A to be the surface area of the snowball (cm^2) t minutes after the snowball began to melt.

The volume and surface area formulas for a sphere in terms of the radius, r, of the sphere are, respectively, $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$. Solving the volume formula for r we have $r = \frac{3^{1/3}}{(4\pi)^{1/3}} V^{1/3}$ and substituting the resultant expression into the area formula we have our relation equation:

$$A = 4\pi r^2 \;\Rightarrow\; A = 4\pi\left(\frac{3^{1/3}}{(4\pi)^{1/3}} V^{1/3}\right)^2 \;\Rightarrow\; A = \sqrt[3]{36\pi}\, V^{2/3}$$

This gives us our rate equation:

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\sqrt[3]{36\pi}\, V^{2/3}\right) \;\Rightarrow\; \frac{dA}{dt} = \frac{2}{3}\sqrt[3]{36\pi}\, V^{-1/3}\frac{dV}{dt} \;\Rightarrow\; \frac{dA}{dt} = \frac{2}{3}\sqrt[3]{\frac{36\pi}{V}}\frac{dV}{dt}$$

When the radius of the snowball is 6 cm:

$$V = \frac{4}{3}\pi(6)^3 = 288\pi \quad \text{and} \quad \frac{dV}{dt} = -25$$

Substituting these values into our rate equation we get:

$$\frac{dA}{dt} = \frac{2}{3}\sqrt[3]{\frac{36\pi}{288\pi}}(-25) = -8\tfrac{1}{3}$$

So at the instant the radius of the snowball is 6 cm, the surface area of the snowball is decreasing at the rate of $8\tfrac{1}{3}$ cm^2/minute.

Supplemental Solutions for the Critical Numbers and Graphing from Formulas Lab

Exercise 9.1

E9.1.1

$$\begin{aligned}\cosh^2(t)-\sinh^2(t)&=\left(\frac{e^t+e^{-t}}{2}\right)^2-\left(\frac{e^t-e^{-t}}{2}\right)^2\\&=\frac{e^{2t}+2+e^{-2t}}{4}-\frac{e^{2t}-2+e^{-2t}}{4}\\&=\frac{e^{2t}+2+e^{-2t}-e^{2t}+2-e^{-2t}}{4}\\&=\frac{4}{4}\\&=1\end{aligned}$$

E9.1.2 For $f(t)=\cosh(t)$ and $g(t)=\sinh(t)$:

$$\begin{aligned}f'(t)&=\frac{d}{dt}\left(\frac{1}{2}\left(e^t+e^{-t}\right)\right)\\&=\frac{1}{2}\left(e^t+e^{-t}\cdot -1\right)\\&=\frac{1}{2}\left(e^t-e^{-t}\right)\\&=\sinh(t)\end{aligned}$$

$$\begin{aligned}g'(t)&=\frac{d}{dt}\left(\frac{1}{2}\left(e^t-e^{-t}\right)\right)\\&=\frac{1}{2}\left(e^t-e^{-t}\cdot -1\right)\\&=\frac{1}{2}\left(e^t+e^{-t}\right)\\&=\cosh(t)\end{aligned}$$

It follows that $f''(t)=\cosh(t)$ and $g''(t)=\sinh(t)$

E9.1.3 The domains of both f and g are $(-\infty,\infty)$. f' and g' are both always defined, so the only issue is when $f'(t)=0$ and when $g'(t)=0$.

$$\begin{aligned}f'(t)=0 \quad&\Rightarrow\quad \frac{e^t-e^{-t}}{2}=0\\&\Rightarrow\quad e^t=e^{-t}\\&\Rightarrow\quad e^t=\frac{1}{e^t}\\&\Rightarrow\quad \left(e^t\right)^2=1\\&\Rightarrow\quad e^t=\pm 1\end{aligned}$$

e^t is never negative and $e^t=1$ when $t=0$. Therefore, the only critical number for $f(t)=\cosh(t)$ is 0.

$$\begin{aligned}g'(t)=0 \quad&\Rightarrow\quad \frac{e^t+e^{-t}}{2}=0\\&\Rightarrow\quad e^t=-e^{-t}\end{aligned}$$

Since real number powers of e are always positive, e^t never equals $-e^{-t}$. Therefore, $g(t)=\sinh(t)$ has no critical numbers.

E9.1.4 **Table 9.1.1K**: Sign analysis for $\sinh(t)$

Interval	$\sinh(t)$
$(-\infty,0)$	negative
$(0,\infty)$	positive

Table 9.1.2K: Sign analysis for $\cosh(t)$

Interval	$\cosh(t)$
$(-\infty,\infty)$	positive

$f(t)=\sinh(t)$, $f'(t)=\cosh(t)$ and $f''(t)=\sinh(t)$. So based upon the signs shown in tables 9.1.1K and 9.1.2K we can conclude that $f(t)=\sinh(t)$ is always increasing, concave down on $(-\infty,0)$ and concave up on $(0,\infty)$. We should note that the point $(0,0)$ is an inflection point on f and that the slope of f at that point is $\cosh(0)$ which is 1.

$g(t)=\cosh(t)$, $g'(t)=\sinh(t)$, and $g''(t)=\cosh(t)$. So based upon the signs shown in tables 9.1.1K and 9.1.2K we can conclude that $g(t)=\cosh(t)$ is decreasing on $(-\infty,0)$, increasing on $(0,\infty)$, and always concave up. We should note that the point $(0,1)$ is a local minimum point on the graph of g and that g has a horizontal tangent line at that point.

E9.1.5 Each of the hyperbolic limits are based upon these four basic limits: $\lim\limits_{t\to\infty} e^t=\infty$, $\lim\limits_{t\to-\infty} e^t=0$, $\lim\limits_{t\to\infty} e^{-t}=0$, and $\lim\limits_{t\to-\infty} e^{-t}=\infty$

$$\lim_{t\to-\infty}\cosh(t)=\lim_{t\to-\infty}\frac{e^t+e^{-t}}{2}$$
$$=\infty$$

$$\lim_{t\to-\infty}\sinh(t)=\lim_{t\to-\infty}\frac{e^t-e^{-t}}{2}$$
$$=-\infty$$

$$\lim_{t\to\infty}\cosh(t)=\lim_{t\to\infty}\frac{e^t+e^{-t}}{2}$$
$$=\infty$$

$$\lim_{t\to\infty}\sinh(t)=\lim_{t\to\infty}\frac{e^t-e^{-t}}{2}$$
$$=\infty$$

E9.1.6

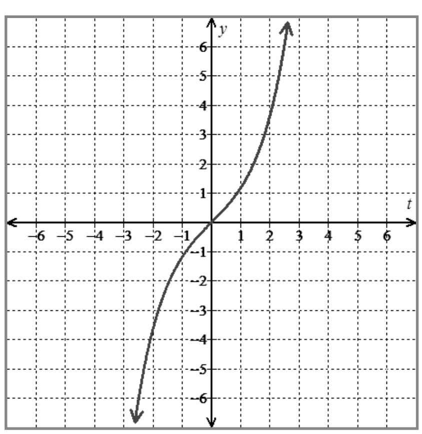


Figure 9.1.1K: $y=\sinh(t)$

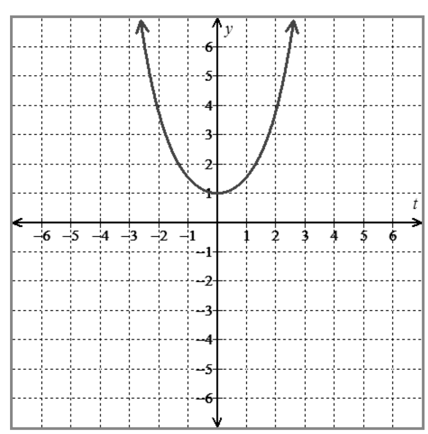


Figure 9.1.2K: $y=\cosh(t)$

E9.1.7 $\tanh(t) = \dfrac{\sinh(t)}{\cosh(t)} = \dfrac{\frac{e^t - e^{-t}}{2}}{\frac{e^t + e^{-t}}{2}} = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$ $\quad \operatorname{sech}(t) = \dfrac{1}{\cosh(t)} = \dfrac{2}{e^t + e^{-t}}$ $\quad \operatorname{csch}(t) = \dfrac{1}{\sinh(t)} = \dfrac{2}{e^t - e^{-t}}$ $\quad \coth(t) = \dfrac{1}{\tanh(t)} = \dfrac{e^t + e^{-t}}{e^t - e^{-t}}$

E9.1.8 A good guess for $\dfrac{d}{dt}(\tanh(t))$ would be $\operatorname{sech}(t)$. Checking it out …

$$\begin{aligned}
\frac{d}{dt}(\tanh(t)) &= \frac{d}{dt}\left(\frac{e^t - e^{-t}}{e^t + e^{-t}}\right) \\
&= \frac{\left(e^t - e^{-t}\cdot -1\right)\left(e^t + e^{-t}\right) - \left(e^t - e^{-t}\right)\left(e^t + e^{-t}\cdot -1\right)}{\left(e^t + e^{-t}\right)^2} \\
&= \frac{\left(e^t + e^{-t}\right)\left(e^{-t} + e^{-t}\right) - \left(e^t - e^{-t}\right)\left(e^t - e^{-t}\right)}{\left(e^t + e^{-t}\right)^2} \\
&= \frac{e^{2t} + 1 + 1 + e^{-2t} - \left(e^{2t} - 1 - 1 + e^{-2t}\right)}{\left(e^t + e^{-t}\right)^2} \\
&= \frac{4}{\left(e^t + e^{-t}\right)^2} \\
&= \left[\frac{2}{e^t + e^{-t}}\right]^2 \\
&= \operatorname{sech}^2(t)
\end{aligned}$$

E9.1.9 $f'(t) = \left[\dfrac{2}{e^t + e^{-t}}\right]^2$ is always positive so f is always increasing.

$f(0) = 0$ and $f'(0) = 1$

E9.1.10

$$\lim_{t\to-\infty}\tanh(t)=\lim_{t\to-\infty}\frac{e^t-e^{-t}}{e^t+e^{-t}}$$
$$=\lim_{t\to-\infty}\left(\frac{e^t-e^{-t}}{e^t+e^{-t}}\cdot\frac{e^t}{e^t}\right)$$
$$==\lim_{t\to-\infty}\frac{e^{2t}-1}{e^{2t}+1}$$
$$=\frac{0-1}{0+1}$$
$$=-1$$

$$\lim_{t\to\infty}\tanh(t)=\lim_{t\to\infty}\frac{e^t-e^{-t}}{e^t+e^{-t}}$$
$$=\lim_{t\to\infty}\left(\frac{e^t-\frac{1}{e^t}}{e^t+\frac{1}{e^t}}\cdot\frac{\frac{1}{e^t}}{\frac{1}{e^t}}\right)$$
$$=\lim_{t\to\infty}\left(\frac{1-\frac{1}{e^{2t}}}{1+\frac{1}{e^{2t}}}\right)$$
$$=\frac{1-0}{1+0}$$
$$=1$$

E9.1.11

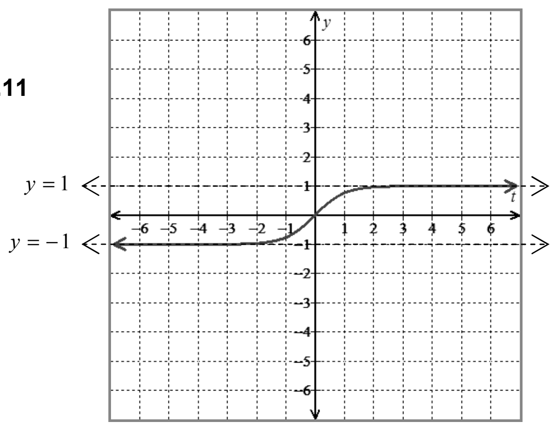


Figure 9.1.3K: $y=\tanh(t)$

Exercise 9.2

$$k'(t)=\frac{8}{3}t^{5/3}-\frac{512}{3}t^{-1/3}$$
$$=\frac{8t^{5/3}}{3}-\frac{512}{3t^{1/3}}$$
$$=\frac{8t^{5/3}}{3}\cdot\frac{t^{1/3}}{t^{1/3}}-\frac{512}{3t^{1/3}}$$
$$=\frac{8t^2-512}{3t^{1/3}}$$
$$=\frac{8(t^2-64)}{3t^{1/3}}$$
$$=\frac{8(t-8)(t+8)}{3t^{1/3}}$$

The domain of k is $(-\infty,\infty)$. $k'(t)=0$ at 8 and -8. Over the domain of k, k' is undefined at 0. So the critical numbers of k are 8, -8, and 0.

Table 9.2.1K: Behavior of k based upon sign analysis for k'

Interval	k'	k
$(-\infty,-8)$	negative	decreasing
$(-8,0)$	positive	increasing
$(0,8)$	negative	decreasing
$(8,\infty)$	positive	increasing

The local minimum points on k are $(-8,-768)$ and $(8,768)$. The only local maximum point on k is $(0,0)$.

Exercise 9.3

$$\begin{aligned} f'(x) &= 2\cos(x)\cdot -\sin(x) + \cos(x) \\ &= \cos(x)\left(1 - 2\sin(x)\right) \end{aligned}$$

The domain of f has been restricted to $[0,2\pi]$. Over $[0,2\pi]$, $\cos(x) = 0$ at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ and $\sin(x) = \frac{1}{2}$ at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. So over the restricted domain $f'(x) = 0$ at $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{\pi}{6}$, and $\frac{5\pi}{6}$. f' is never undefined. So the critical numbers of f are $\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{5\pi}{6}$, and $\frac{3\pi}{2}$.

Table 9.3.1K: Behavior of f based upon sign analysis for f'

Interval	f'	f
$\left(0,\frac{\pi}{6}\right)$	positive	increasing
$\left(\frac{\pi}{6},\frac{\pi}{2}\right)$	negative	decreasing
$\left(\frac{\pi}{2},\frac{5\pi}{6}\right)$	positive	increasing
$\left(\frac{5\pi}{6},\frac{3\pi}{2}\right)$	negative	decreasing
$\left(\frac{3\pi}{2},2\pi\right)$	positive	increasing

Over $[0,2\pi]$ the local minimum points on f are $\left(\frac{\pi}{2},1\right)$ and $\left(\frac{3\pi}{2},-1\right)$ and the local maximum points are $\left(\frac{\pi}{6},\frac{5}{4}\right)$ and $\left(\frac{5\pi}{6},\frac{5}{4}\right)$.

Exercise 9.4

$$\begin{aligned} g'(t) &= \frac{(1)(t^3) - (t+9)(3t^2)}{(t^3)^2} \\ &= \frac{t^3 - 3t^3 - 27t^2}{t^6} \\ &= \frac{-2t^3 - 27t^2}{t^6} \\ &= \frac{t^2(-2t-27)}{t^6} \\ &= \frac{-2t-27}{t^4} \end{aligned}$$

$$\begin{aligned} g''(t) &= \frac{(-2)(t^4) - (-2t-27)(4t^3)}{(t^4)^2} \\ &= \frac{-2t^4 + 8t^4 + 108t^3}{t^8} \\ &= \frac{6t^4 + 108t^3}{t^8} \\ &= \frac{2t^3(3t+54)}{t^8} \\ &= \frac{2\cdot 3(t+18)}{t^5} \\ &= \frac{6(t+18)}{t^5} \end{aligned}$$

$g''(t)=0$ at -18 and $g''(t)$ is undefined at 0.

Table 9.4.1K: Behavior of g based upon sign analysis for g''

Interval	g''	g
$(-\infty,-18)$	positive	concave up
$(-18,0)$	negative	concave down
$(0,\infty)$	positive	concave up

The only inflection point on g is $\left(-18,\dfrac{1}{648}\right)$; g has a vertical asymptote at 0.

Exercise 9.5

E9.5.2

$$\begin{aligned} f'(x) &= \frac{(1)(x+2)^2-(x-3)(2(x+2))}{\left[(x+2)^2\right]^2} \\ &= \frac{(x+2)\left[(x+2)-(x-3)\cdot 2\right]}{(x+2)^4} \\ &= \frac{[x+2-2x+6]}{(x+2)^3} \\ &= \frac{-x+8}{(x+2)^3} \end{aligned}$$

$$\begin{aligned} f''(x) &= \frac{(-1)(x+2)^3-(-x+8)\left(3(x+2)^2\right)}{\left[(x+2)^3\right]^2} \\ &= \frac{(x+2)^2\left[-1\cdot(x+2)-(-x+8)\cdot 3\right]}{(x+2)^6} \\ &= \frac{[-x-2+3x-24]}{(x+2)^4} \\ &= \frac{2(x-13)}{(x+2)^4} \end{aligned}$$

Table 9.5.1K:
Behavior of f based upon sign analysis for f'

Interval	f'	f
$(-\infty,-2)$	negative	decreasing
$(-2,8)$	positive	increasing
$(8,\infty)$	negative	decreasing

Table 9.5.2K:
Behavior of f based upon sign analysis for f''

Interval	f''	f
$(-\infty,-2)$	negative	concave down
$(-2,13)$	negative	concave down
$(13,\infty)$	positive	concave up

We should begin by noting that $x=-2$ is a vertical asymptote for the graph of f. Based upon Table 9.5.1K we can conclude that f has no local minimum points and that the only local maximum point on f is $\left(8,\dfrac{1}{20}\right)$; we should note that the tangent line to f at 8 is horizontal. Based upon Table 9.5.2K we can conclude that the only inflection point on f is $\left(13,\dfrac{2}{45}\right)$. Finally, we can determine the horizontal asymptote(s) for f by looking at $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x-3}{(x+2)^2}$$
$$= \lim_{x\to\infty} \frac{x-3}{x^2+4x+4}$$
$$= \lim_{x\to\infty} \left(\frac{x-3}{x^2+4x+4} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \right)$$
$$= \lim_{x\to\infty} \left(\frac{\frac{1}{x} - \frac{3}{x^2}}{1+\frac{4}{x}+\frac{4}{x^2}} \right)$$
$$= \frac{0-0}{1+0+0}$$
$$= 0$$

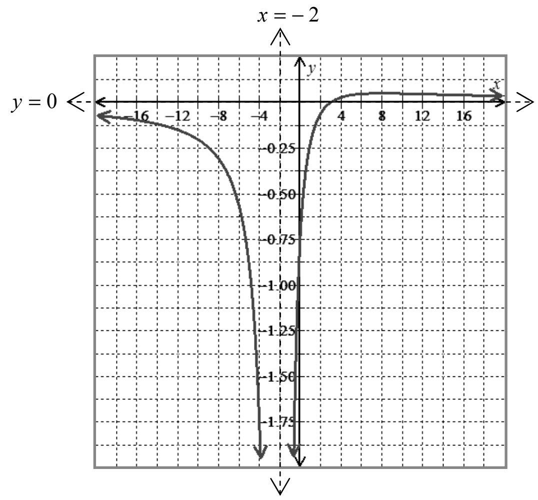


Figure 9.5.1K: $y = \dfrac{x-3}{(x+2)^2}$

Similarly, $\lim_{x\to-\infty} f(x) = 0$, so the only horizontal asymptote for the graph of f is $y = 0$.

E9.5.2

$$g(x) = x^{5/3} + 5x^{2/3}$$
$$g'(x) = \frac{5}{3}x^{2/3} + \frac{10}{3}x^{-1/3}$$
$$= \frac{5x^{2/3}}{3} + \frac{10}{3x^{1/3}}$$
$$= \frac{5x^{2/3}}{3} \cdot \frac{x^{1/3}}{x^{1/3}} + \frac{10}{3x^{1/3}}$$
$$= \frac{5x+10}{3x^{1/3}}$$
$$= \frac{5(x+2)}{3x^{1/3}}$$

$$g'(x) = \frac{5}{3}x^{2/3} + \frac{10}{3}x^{-1/3}$$
$$g''(x) = \frac{10}{9}x^{-1/3} - \frac{10}{9}x^{-4/3}$$
$$= \frac{10}{9x^{1/3}} - \frac{10}{9x^{4/3}}$$
$$= \frac{10}{9x^{1/3}} \cdot \frac{x}{x} - \frac{10}{9x^{4/3}}$$
$$= \frac{10(x-1)}{9x^{4/3}}$$

Table 9.5.3K:
Behavior of g based upon sign analysis for g'

Interval	g'	g
$(-\infty,-2)$	positive	increasing
$(-2,0)$	negative	decreasing
$(0,\infty)$	positive	increasing

Table 9.5.4K:
Behavior of f based upon sign analysis for g''

Interval	g''	g
$(-\infty,0)$	negative	concave down
$(0,1)$	negative	concave down
$(1,\infty)$	positive	concave up

We should begin by noting g is everywhere continuous. Based upon Table 9.5.3K we can conclude that the only local minimum point on g is $(0,0)$ and that the only local maximum point on g is $\left(-2,3\sqrt[3]{4}\right)$. We should also note that g is nondifferentiable at $(0,0)$ and that g has a horizontal tangent line at $\left(-2,3\sqrt[3]{4}\right)$. Based upon Table 9.5.4K we can conclude that the only inflection point on g f is $(1,6)$. We should note that the slope of g at $(1,6)$ is given by $g'(1)$ which is $\frac{5}{\sqrt[3]{3}}$ (about 3 and a half). Finally we need to note that the graph of g has no asymptotes.

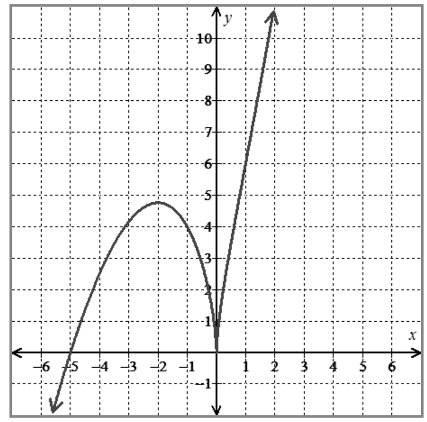


Figure 9.5.1K: $y = x^{2/3}(x+5)$

E9.5.3

$$k'(x) = \frac{2(x-4)(x+3)-(x-4)^2(1)}{(x+3)^2} = \frac{(x-4)\left[2(x+3)-(x-4)\right]}{(x+3)^2} = \frac{(x-4)(x+10)}{(x+3)^2}$$

Note that $k'(x) = \frac{x^2+6x-40}{(x+3)^2}$

$$k''(x) = \frac{(2x+6)(x+3)^2-\left(x^2+6x-40\right)\left(2(x+3)\right)}{\left[(x+3)^2\right]^2} = \frac{(x+3)\left[(2x+6)(x+3)-\left(x^2+6x-40\right)\cdot 2\right]}{(x+3)^4} = \frac{\left[2x^2+6x+6x+18-2x^2-12x+80\right]}{(x+3)^3} = \frac{98}{(x+3)^3}$$

Table 9.5.5K:
Behavior of k based upon sign analysis for k'

Interval	k'	k
$(-\infty,-10)$	positive	increasing
$(-10,-3)$	negative	decreasing
$(-3,4)$	negative	decreasing
$(4,\infty)$	positive	increasing

Table 9.5.6K:
Behavior of f based upon sign analysis for k''

Interval	k''	k
$(-\infty,-3)$	negative	concave down
$(-3,\infty)$	positive	concave up

We should begin by noting that $x=-3$ is a vertical asymptote for the graph of k. Based upon Based upon Table 9.5.5K we can conclude that the only local minimum point on k is $(4,0)$ and that the only local maximum point on k is $(-10,-28)$; we should note that k has horizontal tangent lines at both of its relative extreme points. Based upon Table 9.5.6K we can conclude that k has no inflection points (because there is a vertical tangent line at the only value of x where k changes concavity). Finally, we can determine the horizontal asymptote(s) for k by looking at $\lim_{x \to -\infty} k(x)$ and $\lim_{x \to \infty} k(x)$.

Because the degree of the numerator is one more than the degree of the denominator in the formula for k, neither $\lim_{x \to -\infty} k(x)$ nor $\lim_{x \to \infty} k(x)$ exists. Hence the graph of k has no horizontal asymptotes.

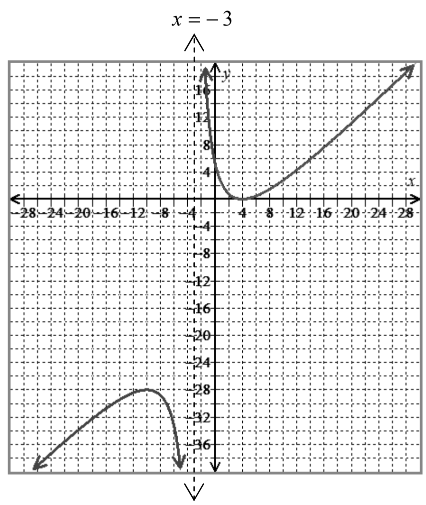


Figure 9.5.1K: $y = \dfrac{x-3}{(x+2)^2}$