

## Section 1.1 Introduction to Algebra: Variables and Mathematical Models

A **variable** is a letter or symbol that represents a variety of different numbers.

If we let  $t$  represent the number of hours a woman has been driving, then  $55 \cdot t$ , usually written  $55t$ , represents the number of miles she has driven if she has been driving at a speed of 55 mph. We can **evaluate** this expression for different times. Notice that  $55t$  combines the number 55 and the variable  $t$  using the operation of multiplication. A combination of variables and numbers using the operations of addition, subtraction, multiplication, or division, as well as powers or roots, is called an **algebraic expression**.

$x+3$	$y-5$	$7w$	$\frac{t}{9}$	$2h+1$	$\sqrt{k}+8$
The variable $x$ increased by 3	The variable $y$ decreased by 5	7 times the variable $w$	The variable $t$ divided by 9	1 more than twice the variable $h$	8 more than the square root of the variable $k$

We can replace a variable in an algebraic expression by a number. This is called **substituting** the number for the variable. We are **evaluating the expression**. Many algebraic expressions contain more than one operation. The order in which we add, subtract, multiply, and divide is important. This concept is formally introduced in section 1.8. For now, follow this order:

1. Perform all operations within grouping symbols, such as parentheses.
2. Do all multiplications in the order in which they occur from left to right.
3. Do all additions and subtractions in the order in which they occur from left to right.

**Example:** Evaluate each algebraic expression for  $x=3$ .

a.  $2+5x$     b.  $4(x+5)$

$$\begin{aligned} \text{a. } 2+5x &= 2+5 \cdot 3 \\ &= 2+15 \\ &= 17 \end{aligned}$$

$$\begin{aligned} \text{b. } 4(x+5) &= 4(3+5) \\ &= 4(8) \\ &= 32 \end{aligned}$$

→ The horizontal fraction bar is a grouping symbol.

**Example:** Evaluate each algebraic expression for  $x=4$  and  $y=2$ .

a.  $3x-5y$

$$3x-5y = 3(4) - 5(2)$$

$$= 12 - 10$$

$$= 2$$

b.  $\frac{2x+3y+6}{x-y}$

$$= \frac{2(4)+3(2)+6}{4-2}$$

$$= \frac{8+6+6}{2}$$

$$= \frac{20}{2}$$

$$= 10$$

Key words for addition, subtraction, multiplication, and division				
Operation	Addition (+)	Subtraction (-)	Multiplication (·)	Division (÷)
Key words	plus	minus	times	divide
	sum	difference	product	quotient
	more than	less than	twice	ratio
	increased by	decreased by	multiplied by	divided by

**Example:** Write each English phrase as an algebraic expression. Let  $N$  represent the number.

*Define a variable*

a. the sum of a number and 3

$$N+3$$

b. nine less than a number

$$N-9$$

c. twice a number, decreased by 5

$$2N-5$$

d. the product of 7 and a number

$$7N$$

e. two more than the quotient of a number and 13

$$\frac{N}{13} + 2$$

**Example:** Write each English phrase as an algebraic expression. Let  $n$  represent the number. Pay close attention to the order when translating phrases involving subtraction.

- a. a number decreased by 9

$$n - 9$$

- b. a number subtracted from 9

$$9 - n$$

- c. nine less than a number

$$n - 9$$

- d. nine less a number

$$9 - n$$

An **equation** is a statement that two algebraic expressions are equal. An **equation always contains the equality symbol =**.

$$3x - 5 = 12 \quad 4a + 3 = 2a - 5 \quad 5(z + 1) = 3(z - 4)$$

**Solutions of an equation** are values of the variable that make the equation a true statement. To determine whether a number is a solution, substitute that number for the variable and evaluate each side of the equation. If the values on both sides of the equation are the same, the number is a solution.

**Example:** Determine whether the given number is a solution of the equation.

a.  $3x - 5 = 12; 6$

$$\begin{aligned} 3(6) - 5 &\stackrel{?}{=} 12 \\ 18 - 5 &\stackrel{?}{=} 12 \\ 13 &\neq 12 \end{aligned}$$

6 is not a solution.

$$\begin{aligned} 3x - 5 &= 3(6) - 5 \\ &= 18 - 5 \\ &= 13 \\ 12 &\neq 13 \end{aligned}$$

6 is not a solution

b.  $3(z + 1) = 4(z - 2); 11$

$$\begin{aligned} 3(11 + 1) &\stackrel{?}{=} 4(11 - 2) \\ 3(12) &\stackrel{?}{=} 4(9) \\ 36 &= 36 \end{aligned}$$

11 is a solution.

Alternate method

$$\begin{aligned} 3(z + 1) &= 3(11 + 1) \\ &= 3(12) \\ &= 36 \end{aligned}$$

$$\begin{aligned} 4(z - 2) &= 4(11 - 2) \\ &= 4(9) \\ &= 36 \end{aligned}$$

11 is a solution

**Example:** Write each sentence as an equation. Define your variable. Use different variables. Avoid the variable  $x$ .

- a. The product of 5 and a number is 35.

Let  $t$  represent the number.

$$5t = 35$$

- b. <sup>-4</sup>Four less than 6 times a number is 12.

Let  $A$  represent the number.

$$6A - 4 = 12$$

**Caution!** Commas make a difference. In English, sentences and phrases can take on different meanings depending on the way words are grouped with commas. The same goes for mathematics.

Let's eat, grandma.

Let's eat grandma. ☹

The product of 3 and a number increased by 2 is 18:  $3(x+2) = 18$

The product of 3 and a number, increased by 2, is 18:  $3x + 2 = 18$

A **formula** is an equation that expresses a relationship between two or more variables. A useful formula converts Fahrenheit temperatures to Celsius.  $C = \frac{5}{9}(F - 32)$

**Example:** Water freezes at  $32^{\circ}\text{F}$ . What is the equivalent Celsius temperature?

$$\begin{aligned} C &= \frac{5}{9}(F - 32) \\ &= \frac{5}{9}(32 - 32) \\ &= \frac{5}{9}(0) \\ &= 0 \end{aligned}$$

$0^{\circ}\text{C}$  is equivalent to  $32^{\circ}\text{F}$ .

## Section 1.2 Fractions in Algebra

The **natural numbers** are the numbers that we use for counting. 1, 2, 3, 4, 5, ...

A **mixed number** consists of the sum of a natural number and a fraction, expressed without the use of an addition sign.

$2\frac{3}{5}$  The natural number is 2. The fraction is  $\frac{3}{5}$ .  $2\frac{3}{5}$  means  $2 + \frac{3}{5}$

An **improper fraction** is a fraction whose numerator is greater than its denominator.  $\frac{13}{5}$

**Example:** Convert  $4\frac{3}{7}$  to an improper fraction.

$$4 \cdot 7 + 3$$

$$4\frac{3}{7} = \frac{31}{7}$$

**Example:** Convert  $\frac{43}{5}$  to a mixed number.

$$\frac{43}{5} = 8\frac{3}{5}$$

$$\begin{array}{r} 8 \\ 5 \overline{)43} \\ \underline{40} \\ 3 \end{array}$$

To **factor** a number means to write it as two or more natural numbers being multiplied. For example, 18 can be factored as  $3 \cdot 6$ . In the statement  $3 \cdot 6 = 18$ , 3 and 6 are called the **factors** and 18 is called the **product**.

A **prime number** is a natural number greater than 1 that has only itself and 1 as factors. The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29.

A **composite number** is a natural number greater than 1 that is not a prime number. Every composite can be expressed as the product of prime numbers.

**Example:** Find the prime factorization of 88.

$$\begin{aligned} 88 &= 2 \cdot 2 \cdot 2 \cdot 11 \\ &= 2^3 \cdot 11 \end{aligned}$$

$$\begin{array}{c} 88 \\ \swarrow \searrow \\ 2 \quad 44 \\ \quad \swarrow \searrow \\ \quad 2 \quad 22 \\ \quad \quad \swarrow \searrow \\ \quad \quad 2 \quad 11 \end{array}$$

**Example:** Reduce  $\frac{30}{100}$  by finding the prime factorizations of the numerator and the denominator.

$$\begin{aligned} \frac{30}{100} &= \frac{2 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 5 \cdot 5} \\ &= \frac{2}{2} \cdot \frac{3}{2} \cdot \frac{5}{5} \cdot \frac{1}{5} \quad \text{optional step} \\ &= 1 \cdot \frac{3}{2} \cdot 1 \cdot \frac{1}{5} \quad \text{optional step} \\ &= \frac{3 \cdot 1}{2 \cdot 5} \\ &= \frac{3}{10} \end{aligned}$$

$$\begin{array}{c} 30 \\ \wedge \\ 2 \quad 15 \\ \quad \wedge \\ \quad 3 \quad 5 \end{array}$$

$$\begin{array}{c} 100 \\ \wedge \\ 2 \quad 50 \\ \quad \wedge \\ \quad 2 \quad 25 \\ \quad \quad \wedge \\ \quad \quad 5 \quad 5 \end{array}$$

**Example:** Multiply  $\frac{3}{4} \cdot \frac{2}{7}$ . Reduce, if possible.

$$\begin{aligned} \frac{3}{4} \cdot \frac{2}{7} &= \frac{6}{28} \\ &= \frac{\cancel{2} \cdot 3}{2 \cdot 14} \\ &= \frac{3}{14} \end{aligned}$$

$$\begin{aligned} \frac{3}{4} \cdot \frac{2}{7} &= \frac{\cancel{3} \cdot \cancel{2}}{2 \cdot \cancel{2} \cdot 7} \\ &= \frac{3}{14} \end{aligned}$$

**Example:** Multiply  $4 \cdot \frac{3}{7}$ . Reduce, if possible.

$$\begin{aligned} 4 \cdot \frac{3}{7} &= \frac{4}{1} \cdot \frac{3}{7} \\ &= \frac{12}{7} \end{aligned}$$

**Example:** Multiply  $\left(4\frac{2}{3}\right)\left(2\frac{1}{2}\right)$ . Reduce, if possible.

$$\begin{aligned}\left(4\frac{2}{3}\right)\left(2\frac{1}{2}\right) &= \frac{14}{3}\left(\frac{5}{2}\right) \\ &= \frac{\cancel{2} \cdot 7 \cdot 5}{3 \cdot \cancel{2}} \\ &= \frac{35}{3}\end{aligned}$$

**Example:** Divide  $\frac{3}{4} \div \frac{2}{7}$ . Reduce, if possible.

$$\begin{aligned}\frac{3}{4} \div \frac{2}{7} &= \frac{3}{4} \cdot \frac{7}{2} \\ &= \frac{21}{8}\end{aligned}$$

**Example:** Divide  $2\frac{3}{4} \div 5$ . Reduce, if possible.

$$\begin{aligned} 2\frac{3}{4} \div 5 &= \frac{11}{4} \div \frac{5}{1} \\ &= \frac{11}{4} \cdot \frac{1}{5} \\ &= \frac{11}{20} \end{aligned}$$

**Example:** Divide  $3\frac{1}{4} \div 1\frac{1}{2}$ . Reduce, if possible.

$$\begin{aligned} 3\frac{1}{4} \div 1\frac{1}{2} &= \frac{13}{4} \div \frac{3}{2} \\ &= \frac{13}{4} \cdot \frac{2}{3} \\ &= \frac{13 \cdot 2}{2 \cdot 2 \cdot 3} \\ &= \frac{13}{6} \end{aligned}$$

**Example:** Subtract  $\frac{7}{12} - \frac{3}{12}$ . Reduce, if possible.

$$\begin{aligned} \frac{7}{12} - \frac{3}{12} &= \frac{7-3}{12} \\ &= \frac{4}{12} \\ &= \frac{\cancel{4} \cdot 1}{\cancel{4} \cdot 3} \\ &= \frac{1}{3} \end{aligned}$$

**Example:** Add  $\frac{3}{4} + \frac{1}{6}$ . Reduce, if possible.

$$\begin{aligned}\frac{3}{4} + \frac{1}{6} &= \frac{3}{4} \cdot \frac{3}{3} + \frac{1}{6} \cdot \frac{2}{2} \\ &= \frac{9}{12} + \frac{2}{12} \\ &= \frac{11}{12}\end{aligned}$$

**Example:** Subtract  $3\frac{2}{3} - 2\frac{1}{2}$ . Reduce, if possible.

$$\begin{aligned}3\frac{2}{3} - 2\frac{1}{2} &= \frac{11}{3} - \frac{5}{2} \\ &= \frac{11}{3} \cdot \frac{2}{2} - \frac{5}{2} \cdot \frac{3}{3} \\ &= \frac{22}{6} - \frac{15}{6} \\ &= \frac{7}{6}\end{aligned}$$

**Example:** Determine if 4 is a solution to  $\frac{1}{2}(a-2)+3 = \frac{3}{8}(3a-4)$ .

$$\begin{aligned}\frac{1}{2}(a-2)+3 &= \frac{1}{2}(4-2)+3 \\ &= \frac{1}{2}(2)+3 \\ &= 1+3 \\ &= 4\end{aligned}$$

$$\begin{aligned}\frac{3}{8}(3a-4) &= \frac{3}{8}(3 \cdot 4 - 4) \\ &= \frac{3}{8}(12-4) \\ &= \frac{3}{8}(8) \\ &= \frac{3}{\cancel{8}}(\frac{\cancel{8}}{1}) \\ &= 3\end{aligned}$$

4 is not a solution.

**Example:** We keep our thermostat set at 68°F. Use the formula  $C = \frac{5}{9}(F-32)$  to find the equivalent temperature on the Celsius scale.

$$\begin{aligned}C &= \frac{5}{9}(F-32) \\ &= \frac{5}{9}(68-32) \\ &= \frac{5}{9}(36) \\ &= \frac{5}{9}\left(\frac{36}{1}\right) \\ &= \frac{5 \cdot \cancel{9} \cdot 4}{\cancel{9} \cdot 1} \\ &= 20\end{aligned}$$

68°F is equivalent to 20°C.