Some Definitions, Tests, and Theorems from Section 4.2 and 4.3

Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

A **critical number** of a function f is a number c **in the domain of f** such that either $f'(c) = 0$ or $f'(c)$ does not exist.

→ local maximum or local minimum

If f has a local extremum at c, then c is a critical number of f.

Increasing/Decreasing Test

1. If $f'(x) > 0$ on an interval, then f is increasing on that interval.
2. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test Suppose that c is a critical number of a continuous function f.

1. If f' changes from positive to negative at c, then f has a local maximum at c.
2. If f' changes from negative to positive at c, then f has a local minimum at c.
3. If f' does not change sign at c, (that is, f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

A function (or its graph) is called **concave upward** on an interval I if f' is an increasing function on I. It is called **concave downward** on I if f' is decreasing on I.

Concavity Test

1. If $f''(x) > 0$ for all x in an interval I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in an interval I, then the graph of f is concave downward on I.

EXAMPLE: Find the critical numbers of $g(x) = \dfrac{\sqrt[5]{(x-4)^3}}{x+2}$.

$x \neq -2$

The domain of g is $(-\infty, -2) \cup (-2, \infty)$.

Note: If the index, 5 in this case, were even we would need exclude values from the domain that would make the radicand negative.

EXAMPLE (continued): Find the critical numbers of $g(x) = \frac{\sqrt[5]{(x-4)^3}}{x+2}$.

$$g'(x) = \frac{(x+2)\frac{d}{dx}\left((x-4)^{3/5}\right) - (x-4)^{3/5}\frac{d}{dx}(x+2)}{(x+2)^2}$$

$$= \frac{(x+2)\cdot\frac{3}{5}(x-4)^{-2/5}\cdot\frac{d}{dx}(x-4) - (x-4)^{3/5}\cdot 1}{(x+2)^2}$$

$$= \frac{\frac{3}{5}(x+2)(x-4)^{-2/5} - (x-4)^{3/5}}{(x+2)^2}\cdot\frac{5(x-4)^{2/5}}{5(x-4)^{2/5}}$$

$$= \frac{3(x+2)(x-4)^0 - 5(x-4)^1}{5(x+2)^2(x-4)^{2/5}}$$

$$= \frac{3(x+2) - 5(x-4)}{5(x+2)^2(x-4)^{2/5}}$$

$$= \frac{3x+6-5x+20}{5(x+2)^2(x-4)^{2/5}}$$

$$= \frac{-2x+26}{5(x+2)^2(x-4)^{2/5}}$$

$$= \frac{-2(x-13)}{5(x+2)^2\sqrt[5]{(x-4)^2}}$$

$g'(x) = 0$ if $x = 13$.

$g'(x)$ is undefined if $x = -2$ or $x = 4$.

The critical numbers of g are 4 and 13.

Note: -2 is not a critical number of g since it is not in the domain of g.

EXAMPLE: Use the first derivative test to find where the local maximum(s) and/or local minimum(s) of $g(x)=\dfrac{\sqrt[5]{(x-4)^3}}{x+2}$ occur. Find the exact coordinates of the local extrema.

Since the only critical numbers of g are 4 and 13, these are the only two x-values at which local extrema can occur.

Table 1: $g'(x)=\dfrac{-2(x-13)}{5(x+2)^2(x-4)^{2/5}}$

Interval	$g'(x)$	Graphical behavior of $g(x)$
$(-2,4)$	positive	increasing
$(4,13)$	positive	increasing
$(13,\infty)$	negative	decreasing

Since the sign of $g'(x)$ changes from positive to negative at $x=13$, the First Derivative Test tells us that there is a local maximum on the graph of $y=g(x)$ at $x=13$.

$$g(13)=\frac{\sqrt[5]{(13-4)^3}}{13+2}$$
$$=\frac{9^{3/5}}{15}$$

The exact coordinates of the local maximum are $\left(13,\frac{9^{3/5}}{15}\right)$.

[Note: If we were going to graph $y=g(x)$, we also would have included the interval $(-\infty,-2)$ in Table 1. The graphical behavior of $y=g(x)$ can change at a discontinuity.]

EXAMPLE: Find the critical numbers of $f(x) = \dfrac{\ln(x)}{x^2}$.

The domain of f is $(0,\infty)$ because $\ln(x)$ is undefined for $x \le 0$.

$$f'(x) = \frac{\frac{d}{dx}(\ln(x))x^2 - \ln(x)\frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{\frac{1}{x} \cdot x^2 - \ln(x) \cdot 2x}{x^4}$$ Remember you multiply exponents when you have a power to a power.

$$= \frac{x - 2x\ln(x)}{x^4}$$

$$= \frac{x(1 - 2\ln(x))}{x^4}$$

$$= \frac{1 - 2\ln(x)}{x^3}$$

$f'(0)$ is undefined and $f'(x)$ is undefined for $x \le 0$.

$f'(x) = 0$ if $1 - 2\ln(x) = 0 \Rightarrow \ln(x) = \frac{1}{2}$

$\Rightarrow x = e^{1/2}$

Since the domain of f is $(0,\infty)$, the only critical number of f is $e^{1/2}$.

EXAMPLE: Find the critical numbers of $g(\theta) = \cos^2(\theta)$ on $[0, 2\pi)$.

$$g'(\theta) = 2\cos(\theta)\tfrac{d}{d\theta}(\cos(\theta))$$
$$= -2\cos(\theta)\sin(\theta)$$

$g'(\theta)$ is never undefined
$g'(\theta) = 0$ if $\cos(\theta) = 0$ or $\sin(\theta) = 0$
$\cos(\theta) = 0$ on $[0, 2\pi)$ if $\theta = \frac{\pi}{2}$ or if $\theta = \frac{3\pi}{2}$
$\sin(\theta) = 0$ on $[0, 2\pi)$ if $\theta = 0$ or if $\theta = \pi$
The critical numbers of g on $[0, 2\pi)$ are 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$.

EXAMPLE: Suppose that $h'(t) = t - 2 + \frac{1}{t}$. Where is $h(t)$ increasing? decreasing?

$$h'(t) = t - 2 + \frac{1}{t}$$
$$= t\cdot\frac{t}{t} - 2\cdot\frac{t}{t} + \frac{1}{t}$$
$$= \frac{t^2 - 2t + 1}{t}$$
$$= \frac{(t-1)(t-1)}{t}$$

So, $h'(0)$ is undefined and $h'(t) = 0$ if $t - 1 = 0 \Rightarrow t = 1$.

Table 2: $h'(t) = \frac{(t-1)^2}{t}$

Interval	Sign of $h'(t)$	Graphical behavior of $h(t)$
$(-\infty, 0)$	Negative	Decreasing
$(0,1)$	Positive	Increasing
$(1,\infty)$	Positive	Increasing

So h is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$.

EXAMPLE: Where is $h(t)$ concave up? concave down? Remember $h'(t) = t - 2 + \frac{1}{t}$.

$$\begin{aligned} h''(t) &= \frac{2(t-1)t-(t-1)^2}{t^2} \\ &= \frac{(t-1)\big(2t-(t-1)\big)}{t^2} \\ &= \frac{(t-1)(t+1)}{t^2} \end{aligned}$$

Since $h''(0)$ is undefined and $h''(-1) = h''(1) = 0$, h could change concavity at -1, 0, or 1.

Table 3: $h''(t) = \frac{(t-1)(t+1)}{t^2}$

Interval	Sign of $h''(t)$	Graphical behavior of $h(t)$
$(-\infty,-1)$	Positive	Concave up
$(-1,0)$	Negative	Concave down
$(0,1)$	Negative	Concave down
$(1,\infty)$	Positive	Concave up

So h is concave up on $(-\infty,-1)$ and $(1,\infty)$ and concave down on $(-1,0)$ and $(0,1)$.

A few more examples:

1. Find the Critical Numbers for $k(t) = \dfrac{\sqrt[3]{t-1}}{t-2}$.

$$k'(t) = \frac{\frac{1}{3}(t-1)^{-2/3}(t-2) - (t-1)^{1/3}(1)}{(t-2)^2}$$

$$= \frac{\dfrac{(t-2)}{3(t-1)^{2/3}} - \dfrac{(t-1)^{1/3}}{1} \cdot \dfrac{3(t-1)^{2/3}}{3(t-1)^{2/3}}}{\dfrac{(t-2)^2}{1}}$$

$$= \frac{(t-2) - 3(t-1)}{3(t-1)^{2/3}} \cdot \frac{1}{(t-2)^2}$$

$$= \frac{1-2t}{3(t-2)^2(t-1)^{2/3}}$$

The domain of k is $(-\infty, 2) \cup (2, \infty)$.

A critical number of a function f is a number c **in the domain of f** such that either $f'(c) = 0$ or $f'(c)$ does not exist.

The critical numbers of $k(t) = \dfrac{\sqrt[3]{t-1}}{t-2}$ are $\dfrac{1}{2}$ and 1. (Note that 2 is not a critical number because it is not in the domain of k; $k$cannot have a local extreme value at a point where it has **no** value! :-O)

Note: If we were making an increasing/decreasing table for $k(t)$, however, we would have to include 2 as an endpoint where the graphical behavior *might* change.

This ⇧ note would not need to be included in your write-up. ☺

This last example shows what we will be doing next week (graphing!).

2. For the function $k(x) = \dfrac{e^x}{\sqrt{x}}$,
   1. Make a table showing the intervals where the function increases and the intervals where the function decreases.
   2. Make a table showing the intervals where the function is concave up and the intervals where the function is concave down.
   3. Find the horizontal and vertical intercepts.
   4. Draw an accurate graph of the function clearly labeling all relevant points.

We should start by stating that the domain of the function $k(x) = \dfrac{e^x}{\sqrt{x}}$ is $(0, \infty)$.

$$k'(x) = \frac{e^x \cdot \sqrt{x} - e^x \cdot \frac{1}{2\sqrt{x}}}{\left(\sqrt{x}\right)^2}$$

$$= \frac{\frac{e^x}{1} \cdot \frac{2\sqrt{x}}{2\sqrt{x}} - \frac{e^x}{2\sqrt{x}}}{\frac{x}{1}}$$

$$= \frac{2xe^x - e^x}{2\sqrt{x}} \cdot \frac{1}{x}$$

$$= \frac{e^x(2x-1)}{2x\sqrt{x}}$$

The critical number for k is the solution to the equation $e^x(2x-1)=0$. Since e^x never equals zero, the only critical number is $\frac{1}{2}$. It should be noted that the only values where $k'(x)$ is undefined are out of the domain of k and, consequently, are not critical numbers.

Table 7: $k'(x) = \frac{e^x(2x-1)}{2x\sqrt{x}}$

Interval	sign of $k'(x)$	$k(x)$ behavior
$\left(0, \frac{1}{2}\right)$	negative	decreasing
$\left(\frac{1}{2}, \infty\right)$	positive	increasing

$$k''(x) = \frac{\left(e^x(2x-1) + e^x \cdot 2\right) \cdot 2x^{3/2} - e^x(2x-1) \cdot 3x^{1/2}}{\left(2x^{3/2}\right)^2}$$

$$= \frac{(2x-1+2) \cdot 2e^x \cdot x\sqrt{x} - (2x-1) \cdot 3e^x\sqrt{x}}{4x^3}$$

$$= \frac{\left((2x+1) \cdot 2x - (2x-1) \cdot 3\right) \cdot e^x\sqrt{x}}{4x^3}$$

$$= \frac{\left(4x^2 - 4x + 3\right)e^x}{4x^{5/2}}$$

A quick application of the quadratic formula reveals that $k''(x)$ has no real zeros. Since $k''(x)$ is positive for all positive x (the domain of k), the curve $y = k(x)$ is always concave up.

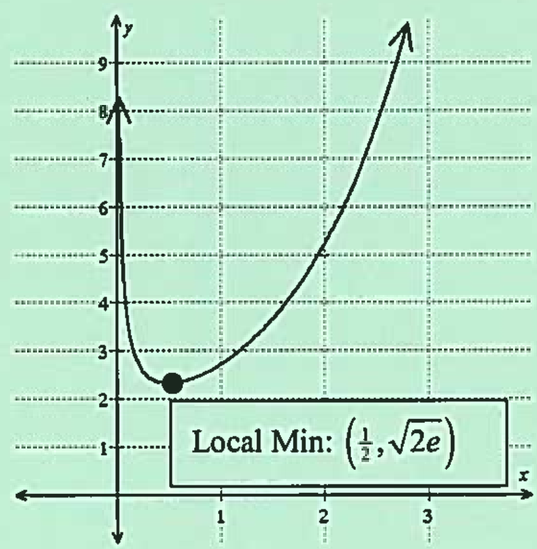

Figure 3: $y = \frac{e^x}{\sqrt{x}}$