If $x^2 + y^2 = 20$, find $\left.\frac{dy}{dx}\right|_{(x,y)=(2,4)}$ by implicit differentiation.

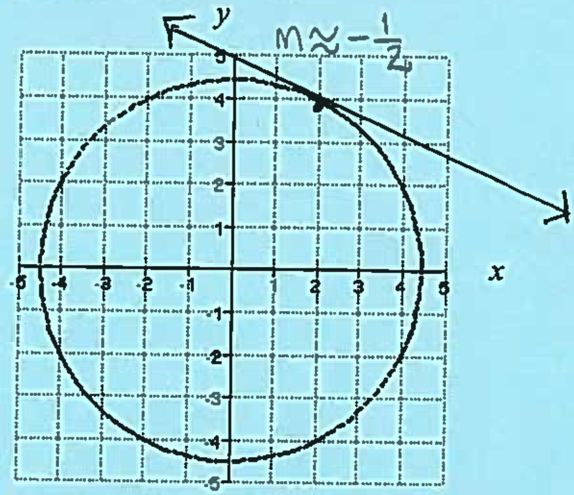


Figure 1: $x^2 + y^2 = 20$

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(20)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$2x + 2y \cdot \frac{d}{dx}(y) = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Treat y as a function of x and applying the chain rule

Outside function
squaring function

inside function
y

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,4)} = -\frac{2}{4}$$

$$= -\frac{1}{2}$$

Find $\frac{dy}{dx}$ by implicit differentiation given that $xy+\cos(y)=y$. Remember to treat y as a function of x and use the chain rule. For example: $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$

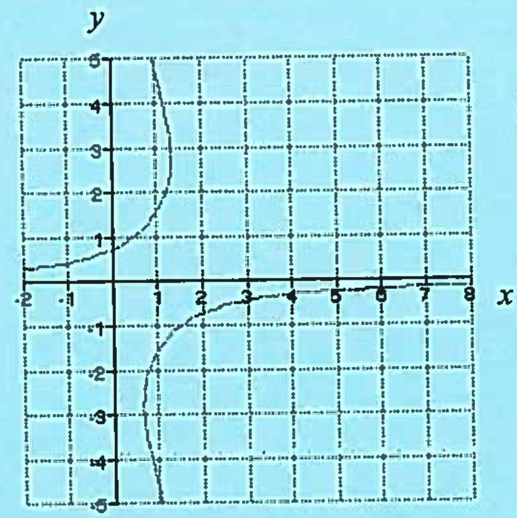


Figure 2: $xy+\cos(y)=y$

$\cos(y)$

Treat y as a function of x

outside function cosine

inside function y

$$\frac{d}{dx}(xy+\cos(y))=\frac{d}{dx}(y)$$

$$\frac{d}{dx}(xy)+\frac{d}{dx}(\cos(y))=\frac{dy}{dx}$$

$$\frac{d}{dx}(x)y+x\frac{d}{dx}(y)-\sin(y)\frac{dy}{dx}=\frac{dy}{dx}$$

$$y+x\frac{dy}{dx}-\sin(y)\frac{dy}{dx}=\frac{dy}{dx}$$

$$y=\frac{dy}{dx}-x\frac{dy}{dx}+\sin(y)\frac{dy}{dx}$$

$$y=\frac{dy}{dx}(1-x+\sin(y))$$

$$\frac{y}{1-x+\sin(y)}=\frac{dy}{dx}$$

We have formulas to differentiate things like x^3 and 3^x, but how do you differentiate x^x? You will use **logarithmic differentiation** to differentiate x^x in exercise 7.3.1 in the lab manual. Let's look at the process of logarithmic differentiation for the function $y = x^{e^x}$.

$$y = x^{e^x}$$

$$\ln(y) = \ln(x^{e^x})$$ Take the natural log of both sides.

$$\ln(y) = e^x \ln(x)$$ Apply properties of logarithms.

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(e^x \ln(x))$$ Use implicit differentiation

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(e^x)\ln(x) + e^x \frac{d}{dx}(\ln(x))$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = e^x \ln(x) + e^x \cdot \frac{1}{x}$$

$$y\left(\frac{1}{y} \cdot \frac{dy}{dx}\right) = \left(e^x \ln(x) + \frac{e^x}{x}\right)y$$

$$\frac{dy}{dx} = \left(e^x \ln(x) + \frac{e^x}{x}\right)x^{e^x}$$ Factor or distribute

$$= \left(\ln(x) + \frac{1}{x}\right)e^x x^{e^x}$$

Use logarithmic differentiation to differentiate $y = x\sqrt[3]{1+x^2}$

$$\ln(y) = \ln\left(x\sqrt[3]{1+x^2}\right)$$

$$\ln(y) = \ln(x) + \ln\left(\sqrt[3]{1+x^2}\right)$$

$$\ln(y) = \ln(x) + \ln\left((1+x^2)^{1/3}\right)$$

$$\ln(y) = \ln(x) + \frac{1}{3}\ln(1+x^2)$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left(\ln(x) + \frac{1}{3}\ln(1+x^2)\right)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x} + \frac{1}{3}\cdot\frac{1}{1+x^2}\cdot\frac{d}{dx}(1+x^2)$$

$$\frac{1}{x\sqrt[3]{1+x^2}}\cdot\frac{dy}{dx} = \frac{1}{x} + \frac{1}{3}\cdot\frac{1}{1+x^2}\cdot 2x$$

$$\frac{1}{x\sqrt[3]{1+x^2}}\cdot\frac{dy}{dx} = \frac{1}{x} + \frac{2x}{3(1+x^2)}$$

$$\frac{dy}{dx} = x\sqrt[3]{1+x^2}\left(\frac{1}{x} + \frac{2x}{3(1+x^2)}\right)$$

$$\frac{dy}{dx} = \sqrt[3]{1+x^2} + \frac{2x^2\sqrt[3]{1+x^2}}{3(1+x^2)}$$

$$\frac{dy}{dx} = \sqrt[3]{1+x^2} + \frac{2x^2}{3\sqrt[3]{(1+x^2)^2}}$$

$$\frac{\sqrt[3]{u}}{u} = \frac{u^{1/3}}{u} = u^{1/3-1} = u^{-2/3} = \frac{1}{u^{2/3}} = \frac{1}{\sqrt[3]{u^2}}$$

Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 m/s. How fast is the area of the spill increasing when the radius of the spill is 60 m?

Let r represent the radius (in m) of the oil spill at time t (in seconds).

Let A represent the area (in m^2) of the oil spill at time t.

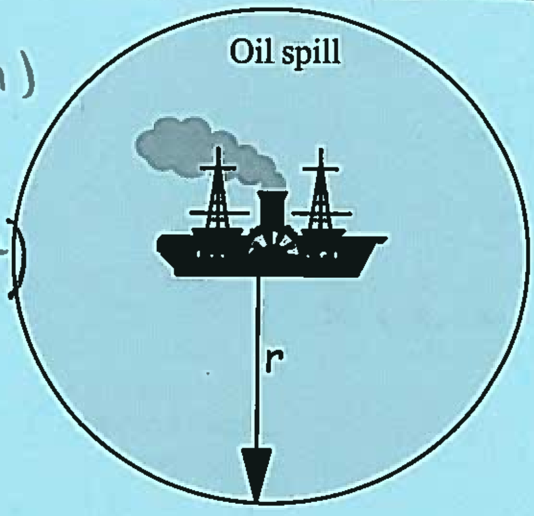


Figure 3: Variable Diagram

Given $\frac{dr}{dt} = 2\frac{m}{s}$, find $\left.\frac{dA}{dt}\right|_{r=60}$.

From Geometry we know $A = \pi r^2$.

So $\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$

$$\frac{dA}{dt} = \pi \frac{d}{dt}(r^2)$$

$$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\therefore \left.\frac{dA}{dt}\right|_{r=60} = 2\pi(60m)\left(2\frac{m}{s}\right)$$

$$= 240\pi \frac{m^2}{s}$$

The area of the spill is increasing at a rate of $240\pi \frac{m^2}{s}$ when the radius of the spill is 60m.

Related Rates Algorithm

The following algorithm - when executed correctly - will lead to success with related rates problems.

1. Draw a picture of the described situation.
2. Label any piece of the picture **for which you are given the rate of change** with a **variable name.**
3. Label the piece of the picture **for which you are asked to find the rate of change** with a **variable name.**
4. Label any piece that does not change with respect to time and for which you are given the **constant value** with that **constant value.**
5. Explicitly define your variables.
6. Your related rates equation contains variables and rates. Somewhere in the problem you are given values for each of the variables at a certain point in time and you are given values for each of the rates except the one you are trying to find. Make a list stating each of the values and rates. If you assign a value of zero to any rate, that piece of the picture **should not have been assigned a variable name.** If there are two or more pieces of the picture for which you were not told the rate of change **you have labeled a piece of the picture that should not have been labeled!** (See footnote).
7. Find the related rates equation **without introducing any additional variables.**
8. Plug the information from step 6 into your rate equation and solve for the unknown rate. You may need a "snapshot" in time diagram to find values of one or more of the variables at the point in time you are interested in.
9. State your conclusion using a complete sentence and proper units.

<u>footnote</u>
Occasionally a "scratch work" picture must be drawn to eliminate an unwanted variable. This occurs when you are using "textbook" formulas like volume of a cone ($V = \frac{\pi}{3}r^2 h$). If you are told to find the rate of change in volume given the rate of change in height you will need to use additional information in the problem to eliminate r from the volume equation.

Common Errors!!

- Error 1 - labeling pieces that change over time with a constant value
- Error 2 - labeling pieces that are changing but which are irrelevant to the problem. Variable names are assigned to pieces that are changing **if and only if** you are given their rate of change or you are explicitly asked to find their rate of change.
- Error 3- assigning positive derivative values to pieces that are *decreasing* over time

A 13-ft ladder is leaning against a wall. If the top of the ladder slips down the wall at a rate of 2 ft/sec, how fast will the foot of the ladder be moving away from the wall when the top is 5 ft above the ground?

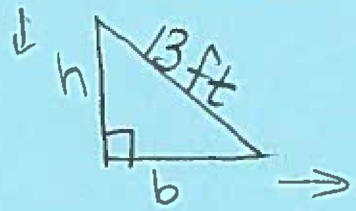


Figure 4: Variable diagram

Let h represent the distance (in feet) from the top of the ladder to the ground at time t (in seconds).

Let b represent the distance (in feet) from the foot of the ladder to the wall at time t.

Given $\frac{dh}{dt} = -2\,\frac{ft}{sec}$, find $\left.\frac{db}{dt}\right|_{h=5}$.

The Pythagorean Theorem tells us $h^2 + b^2 = 13^2$.

So $\frac{d}{dt}(h^2+b^2) = \frac{d}{dt}(13^2)$

$$2h\frac{dh}{dt} + 2b\frac{db}{dt} = 0$$

$$2b\frac{db}{dt} = -2h\frac{dh}{dt}$$

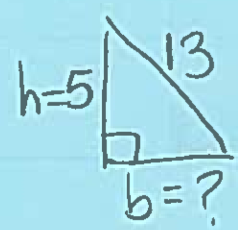
$$\frac{db}{dt} = \frac{-2h\frac{dh}{dt}}{2b}$$

$$\frac{db}{dt} = -\frac{h}{b}\cdot\frac{dh}{dt}$$

$$\therefore \left.\frac{db}{dt}\right|_{h=5} = -\frac{5\cancel{ft}}{12\cancel{ft}}\left(-2\,\frac{ft}{sec}\right)$$

$$= \frac{5}{6}\,\frac{ft}{sec}$$

Figure 4b: snapshot in time when $h=5$



Pythagorean triples
3, 4, 5
5, 12, 13

$b=12$ (you can use $h^2+b^2=13^2$ to find b)

The foot of the ladder will be moving away from the wall at a rate of $\frac{5}{6}\,\frac{ft}{sec}$ when the top is 5 ft above ground.

If the rocket shown in Figure 5 is rising vertically at 880 ft/sec when it is 4000 ft up, how fast must the camera elevation angle change at that instant to keep the rocket in sight?

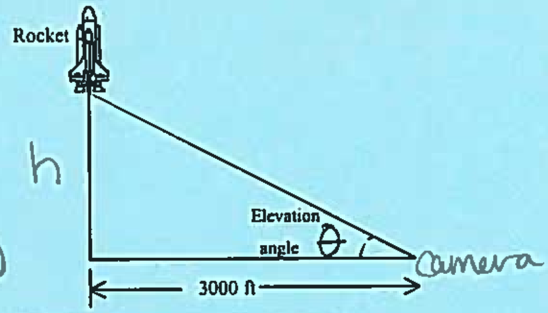


Figure 5: Variable Diagram

Let h represent the height (in feet) of the rocket at time t (in seconds) and θ represent the camera elevation angle (in radians) at time t.

Given $\left.\frac{dh}{dt}\right|_{h=4000} = 880\,\frac{ft}{sec}$, find $\left.\frac{d\theta}{dt}\right|_{h=4000}$.

From trig we know that $\tan(\theta) = \frac{h}{3000}$.

So $\frac{d}{dt}(\tan(\theta)) = \frac{d}{dt}\left(\frac{h}{3000}\right)$

$\sec^2(\theta)\cdot\frac{d\theta}{dt} = \frac{1}{3000}\cdot\frac{dh}{dt}$

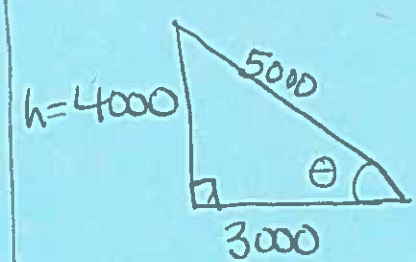
$\frac{d\theta}{dt} = \frac{1}{\sec^2(\theta)}\cdot\frac{1}{3000}\cdot\frac{dh}{dt}$

$\frac{d\theta}{dt} = \frac{\cos^2(\theta)}{3000}\cdot\frac{dh}{dt}$

$\left.\frac{d\theta}{dt}\right|_{h=4000} = \frac{(3/5)^2}{3000\,\cancel{ft}}\cdot 880\frac{ft}{sec}$

$= 0.1056\,\frac{rad}{sec}$

Figure 5a: snapshot in time - when $h=4000$



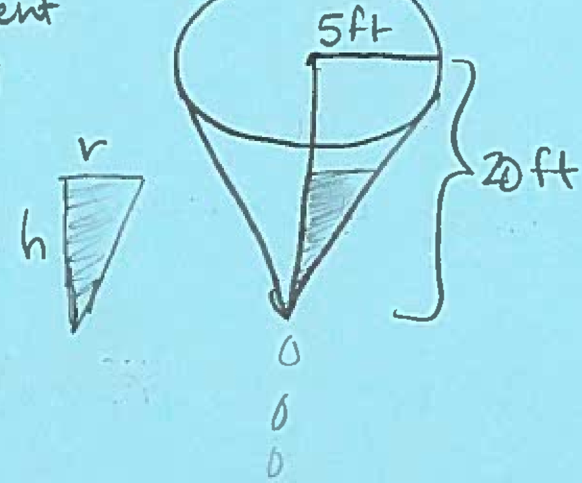
I can find θ or $\cos(\theta)$ when $h=4000$

$\cos(\theta) = \frac{adj}{hyp} = \frac{3000}{5000} = \frac{3}{5}$ or $\tan(\theta) = \frac{4000}{3000} = \frac{4}{3}$, $\theta = \tan^{-1}\left(\frac{4}{3}\right)$

The camera elevation angle must increase at a rate of $0.1056\,\frac{rad}{sec}$ in order to keep the rocket in sight when it is 4000 ft above the launching pad.

A tank filled with water is in the shape of an inverted cone 20 feet high with a circular base (on top) whose radius is 5 feet. Water is running out the bottom of the tank at the constant rate of 2 ft^3/min. How fast is the water level falling when the water is 8 feet deep?

Let V (in ft^3), h (in ft), r (in ft) represent the respective volume, height, and radius of water remaining in the tank at time t (in min).



Given $\frac{dV}{dt} = -2\frac{ft^3}{min}$, find $\left.\frac{dh}{dt}\right|_{h=8}$.

From geometry we know $V = \frac{\pi}{3}r^2h$.

We need to eliminate r from this formula.

Similar triangles tell us $\frac{5}{20} = \frac{r}{h} \Rightarrow \frac{h}{4} = r$

Substituting $\frac{h}{4}$ for r in the volume formula gives us

$$V = \frac{\pi}{3}\left(\frac{h}{4}\right)^2 \cdot h$$

$$V = \frac{\pi}{48}h^3$$

So $\frac{dV}{dt} = \frac{d}{dt}\left(\frac{\pi}{48}h^3\right)$

$$\frac{dV}{dt} = \frac{\pi}{48} \cdot 3h^2\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{16}h^2\frac{dh}{dt}$$

$$\frac{16}{\pi h^2} \cdot \frac{dV}{dt} = \frac{dh}{dt}$$

$$\left.\frac{dh}{dt}\right|_{h=8} = \frac{16}{\pi(8ft)^2}\left(-2\frac{ft^3}{min}\right)$$

$$= -\frac{\cancel{8} \cdot 2 \cdot 2}{\cancel{8} \cdot 8\pi}\frac{ft}{min}$$

$$= -\frac{1}{2\pi}\frac{ft}{min}$$

The water level is falling at a rate of $\frac{1}{2\pi}\frac{ft}{min}$ when the water is 8 ft deep.

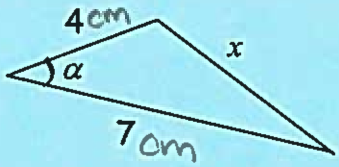


Figure 7: find related rates equation

In Figure 7 x represents the length of the indicated side (cm) and α represents the measurement of the indicated angle (rad); both variables are changing with respect to time, t, measured in minutes. The lengths of the other two sides are fixed at 4 cm and 7 cm. α is increasing at a constant rate of 4 deg/min. Find the rate at which x is changing at the instant $\alpha = 60^o$. Make sure that you clearly communicate whether x is increasing or decreasing.

Given $\frac{d\alpha}{dt} = 4\,\frac{\text{deg}}{\text{min}} \cdot \frac{\pi \text{ rad}}{180 \text{deg}}$

$= \frac{\pi}{45}\,\frac{\text{rad}}{\text{min}}$, find $\left.\frac{dx}{dt}\right|_{\alpha = 60^\circ = \frac{\pi}{3}}$

The Law of Cosines: $x^2 = 4^2 + 7^2 - 2(4)(7)\cos(\alpha)$

$x^2 = 4^2 + 7^2 - 56\cos(\alpha)$

So $\frac{d}{dt}(x^2) = \frac{d}{dt}\left(4^2 + 7^2 - 56\cos(\alpha)\right)$

$2x\,\frac{dx}{dt} = -56\left(-\sin(\alpha)\,\frac{d\alpha}{dt}\right)$

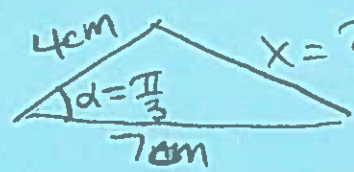
$\frac{dx}{dt} = \frac{56\sin(\alpha)}{2x} \cdot \frac{d\alpha}{dt}$

$\left.\frac{dx}{dt}\right|_{\alpha = \frac{\pi}{3}} = \frac{56\,\text{cm}^2 \sin\left(\frac{\pi}{3}\right)}{2\sqrt{37}\text{ cm}} \cdot \frac{\pi}{45}\,\frac{\text{rad}}{\text{min}}$

$= \frac{56\pi\left(\frac{\sqrt{3}}{2}\right)}{90\sqrt{37}}\,\frac{\text{cm}}{\text{min}}$

$\approx 0.278\,\frac{\text{cm}}{\text{min}}$

Figure 7a: Snapshot in time - when $\alpha = \frac{\pi}{3}$



$x = \sqrt{4^2 + 7^2 - 56\cos\left(\frac{\pi}{3}\right)}$

$= \sqrt{16 + 49 - 56\left(\frac{1}{2}\right)}$

$= \sqrt{65 - 28}$

$= \sqrt{37}$

The side labeled x is increasing at a rate of about $0.278\,\frac{\text{cm}}{\text{min}}$ when $\alpha = 60^\circ$.

Example: A light is at the top of a 16-foot pole. A boy 5 feet tall walks away from the pole at a rate of 4 feet per second. At what rate is the length of his shadow increasing?

Solution: We are given that the pole is 16 feet tall, that the boy is 5 feet tall, and that he walks at a rate of 4 feet per second.

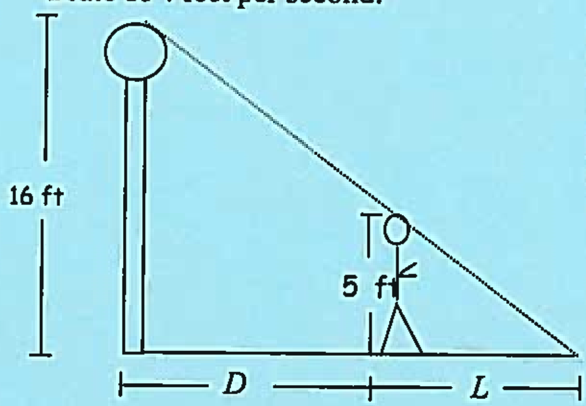


Figure 8: Variable Diagram

Let L represent the length (in feet) of the boy's shadow at time t (in seconds)

Let D represent the distance (in feet) from the base of the pole to the boy at time t, as shown in Figure 8.

Given $\frac{dD}{dt} = 4 \frac{ft}{sec}$, find $\frac{dL}{dt}$.

Using similar triangles, we know $\frac{5}{L} = \frac{16}{D+L}$

$$5(D+L) = 16L$$
$$5D + 5L = 16L$$
$$5D = 11L$$

So $\frac{d}{dt}(5D) = \frac{d}{dt}(11L)$

$$5\frac{dD}{dt} = 11\frac{dL}{dt}$$

$$\frac{5}{11}\frac{dD}{dt} = \frac{dL}{dt}$$

$$\frac{dL}{dt} = \frac{5}{11}\left(4\frac{ft}{sec}\right)$$
$$= \frac{20}{11}\frac{ft}{sec}$$

The length of the boy's shadow is increasing at a rate of $\frac{20}{11}\frac{ft}{sec}$.