Last week your homework time was spent finding derivative formulas for polynomial, trigonometric, and exponential functions **and** sums, differences, products, and quotients of these functions. This week, composite functions are thrown into the mix.

Suppose that f is some function of x and $h(x) = f(x^2)$. Suppose we wanted to know the rate of change in the function $h(x)$ at the point where $x = 3$. There are two things that need to be taken into consideration:

- What is the rate of change in the function $u = x^2$ when $x = 3$?
- What is the rate of change in the function $y = f(u)$ when $u = 9$?

The derivative of the outside function evaluated at the inside function • derivative of inside function

It turns out that the derivative formula for functions of the form $h(x) = f(x^2)$ is $h'(x) = f'(x^2) \cdot 2x$. For example:

- $\frac{d}{dx}(\sin(x^2)) = \cos(x^2) \cdot 2x$
- $\frac{d}{dx}\left(e^{x^2}\right) = e^{x^2} \cdot 2x$

The "generic" chain rule can be stated (at least) two different ways:

- If $h(x) = f(g(x))$, then $h'(x) = f'(g(x)) \cdot g'(x)$
- If y is a "function of u" and u is a "function of x", then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

The order in which derivative rules are applied is the reverse of the traditional order of operations. Table 1 illustrates this for a few functions

Table 1: Which Derivative Rule do I use first?

Function Formula	Final Algebraic Operation	First Derivative Rule to Apply
$\sin(x^2 \cdot \cos(x^2))$	$\sin(u)$	$\frac{d}{dx}(\sin(u)) = \cos(u)\frac{d}{dx}(u)$
$\sin(x^2) \cdot \cos(x^2)$	Multiplication	Product Rule
$\frac{\cos(x) + \sin(x)}{x^3 - 4x^2}$	Division	Quotient Rule
$\frac{\cos(x)}{x^3} - \frac{\sin(x)}{4x^2}$	Subtraction	$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$

Examples to be worked in Class

1. Find the derivative of each of the following functions.

a. $y=\sin\left(4x^3\right)$

b. $f(\theta)=\cos\left(4\theta^3\right)$

c. $w=e^{4x^3}$

d. $g(\alpha)=\tan(5\alpha)$

e. $h(x)=\tan\left(5xe^x\right)$

f. $k(x)=5x\tan\left(e^x\right)$

g. $y=\tan\left(e^{5t}\right)$

h. $m(x)=x\ln x$

i. $y=\ln\left(t^2+1\right)$

j. $y=(\ln(a))^4$

k. $n=\ln\left(r^4\right)$

l. $m=4\ln(b)$

m. $f(x)=\ln\left(\dfrac{3x^2}{x^5-2x}\right)$

n. $h(t)=\cot(\cos(5t))$

o. $r=3^t$

p. $k(x)=5^{x^2}$

q. $y=(2-x)^4(3+x)^7$

1

ⓐ $y = \sin(4x^3)$

$\frac{dy}{dx} = \cos(4x^3) \cdot \frac{d}{dx}(4x^3)$

$= \cos(4x^3) \cdot 12x^2$

$= 12x^2 \cos(4x^3)$

The order in which the factors in our answer should appear: coefficient, power functions, exponential functions, trig functions

ⓑ $f(\theta) = \cos(4\theta^3)$

$f'(\theta) = -\sin(4\theta^3) \cdot \frac{d}{d\theta}(4\theta^3)$

$= -\sin(4\theta^3) \cdot 12\theta^2$

$= -12\theta^2 \sin(4\theta^3)$

ⓒ $w = e^{4x^3}$

$\frac{dw}{dx} = e^{4x^3} \cdot \frac{d}{dx}(4x^3)$

$= 12x^2 e^{4x^3}$

ⓓ $g(\alpha) = \tan(5\alpha)$

$g'(\alpha) = \sec^2(5\alpha) \cdot \frac{d}{d\alpha}(5\alpha)$

$= 5\sec^2(5\alpha)$

(e) $h(x) = \tan(5xe^x)$

$$h'(x) = \sec^2(5xe^x) \cdot \frac{d}{dx}(5xe^x)$$

$$= \sec^2(5xe^x)\left[\frac{d}{dx}(5x) \cdot e^x + 5x \cdot \frac{d}{dx}(e^x)\right]$$

$$= \sec^2(5xe^x)\left[5e^x + 5xe^x\right]$$

$$= \sec^2(5xe^x) \cdot 5e^x(1+x)$$

$$= 5e^x \sec^2(5xe^x)(1+x)$$ completely factored

$$= 5e^x \sec^2(5xe^x) + 5xe^x \sec^2(5xe^x)$$ expanded

Either the completely factored form or the expanded form is fine for this type of problem.

(f) $k(x) = 5x\tan(e^x)$

$$k'(x) = \frac{d}{dx}(5x)\tan(e^x) + 5x\frac{d}{dx}(\tan(e^x))$$

$$= 5\tan(e^x) + 5x\sec^2(e^x) \cdot \frac{d}{dx}(e^x)$$

$$= 5\tan(e^x) + 5x\sec^2(e^x) \cdot e^x$$

$$= 5\tan(e^x) + 5xe^x\sec^2(e^x)$$

$$= 5(\tan(e^x) + xe^x\sec^2(e^x))$$

either answer is fine

g. $y = \tan(e^{5t})$

$$\frac{dy}{dt} = \sec^2(e^{5t}) \cdot \frac{d}{dt}(e^{5t})$$

$$= \sec^2(e^{5t}) \cdot e^{5t} \cdot \frac{d}{dt}(5t)$$

$$= 5e^{5t} \sec^2(e^{5t})$$

h. $m(x) = x \ln(x)$

$$m'(x) = \frac{d}{dx}(x) \ln(x) + x \frac{d}{dx}(\ln(x))$$

$$= \ln(x) + x \cdot \frac{1}{x}$$

$$= \ln(x) + 1$$

i. $y = \ln(t^2+1)$

$$\frac{dy}{dt} = \frac{1}{t^2+1} \cdot \frac{d}{dt}(t^2+1)$$

$$= \frac{1}{t^2+1} \cdot 2t$$

$$= \frac{2t}{t^2+1}$$

j. $y = (\ln(a))^4$

$\frac{dy}{da} = 4(\ln(a))^3 \cdot \frac{d}{da}(\ln(a))$

$= 4(\ln(a))^3 \cdot \frac{1}{a}$

$= \frac{4(\ln(a))^3}{a}$

k. $n = \ln(r^4)$

$\frac{dn}{dr} = \frac{1}{r^4} \cdot \frac{d}{dr}(r^4)$

$= \frac{1}{r^4} \cdot 4r^3$

$= \frac{4r^3}{r^4}$

$= \frac{4}{r}$

l. $m = 4\ln(b)$

$\frac{dm}{db} = 4 \cdot \frac{1}{b}$

$= \frac{4}{b}$

Caution!

What does $\ln a^3$ mean?

Ambiguous

$\ln(a^3)$ or $(\ln(a))^3$?

What does $\sin(a)^3$ mean?

Ambiguous

$(\sin(a))^3$ or $\sin(a^3)$?

$(\sin(a))^3 = \sin^3(a)$

$\ln(r^4) = 4\ln(r)$

Properties of Logarithms

① $\log_b(uv) = \log_b(u) + \log_b(v)$

② $\log_b\left(\frac{u}{v}\right) = \log_b(u) - \log_b(v)$

③ $\log_b(u^p) = p\log_b(u)$

(m) $f(x) = \ln\left(\frac{3x^2}{x^5-2x}\right)$

$= \ln(3x^2) - \ln(x^5-2x)$

$= \ln(3) + \ln(x^2) - \ln(x(x^4-2))$

$= \ln(3) + 2\ln(x) - [\ln(x) + \ln(x^4-2)]$

$= \ln(3) + 2\ln(x) - \ln(x) - \ln(x^4-2)$

$= \ln(3) + \ln(x) - \ln(x^4-2)$

$f'(x) = 0 + \frac{1}{x} - \frac{1}{x^4-2} \cdot \frac{d}{dx}(x^4-2)$

$= \frac{1}{x} - \frac{1}{x^4-2} \cdot 4x^3$

$= \frac{1}{x} - \frac{4x^3}{x^4-2}$ ← okay to leave your answer in this form

$= \frac{1}{x} \cdot \frac{x^4-2}{x^4-2} - \frac{4x^3}{x^4-2} \cdot \frac{x}{x}$ Textbook answers usually have common denominators

$= \frac{x^4-2-4x^4}{x(x^4-2)}$

$= \frac{-3x^4-2}{x(x^4-2)}$

(n) $h(t) = \cot(\cos(5t))$

$$h'(t) = -\csc^2(\cos(5t)) \cdot \frac{d}{dt}(\cos(5t))$$

$$= -\csc^2(\cos(5t))\left(-\sin(5t) \cdot \frac{d}{dt}(5t)\right)$$

$$= -\csc^2(\cos(5t))(-\sin(5t) \cdot 5)$$

$$= 5\sin(5t)\csc^2(\cos(5t))$$

1 (o) $r = 3^t$

$\frac{dr}{dt} = \ln(3) \cdot 3^t$

(p) $k(x) = 5^{x^2}$

$k'(x) = \ln(5) \cdot 5^{x^2} \cdot \frac{d}{dx}(x^2)$

$= \ln(5) 5^{x^2} \cdot 2x$

$= 2\ln(5) x \cdot 5^{x^2}$

(q) $y = (2-x)^4 (3+x)^7$

$\frac{dy}{dx} = \frac{d}{dx}\left[(2-x)^4\right](3+x)^7 + (2-x)^4 \frac{d}{dx}\left[(3+x)^7\right]$

$= 4(2-x)^3 \cdot \frac{d}{dx}(2-x)(3+x)^7 + (2-x)^4 \cdot 7(3+x)^6 \cdot \frac{d}{dx}(3+$

$= 4(2-x)^3(-1)(3+x)^7 + (2-x)^4 \cdot 7(3+x)^6 \cdot 1$

$= (2-x)^3(3+x)^6\left[-4(3+x) + 7(2-x)\right]$

$= (2-x)^3(3+x)^6(-12 - 4x + 14 - 7x)$

$= (2-x)^3(3+x)^6(2 - 11x)$

Think $C(p(t))$
composite function

2. Let $C=\sqrt{p}$ be the estimated CO_2 concentration (in ppm) in the atmosphere of an American city of population p (in millions of people). Assume that this function has been roughly valid for the past 30 years.

The population of the Portland metropolitan area (in millions of people) can be somewhat accurately modeled by the function $p=1.528\left(1.0242^t\right)$ where t is the number of years since July 1, 1990.

→ years since 7/1/1990

a. What is the practical meaning of the function value $p(8.5)\approx 1.87$? 1/1/99

The function value $p(8.5)\approx 1.87$ means that on 1-1-99 the population of the Portland metro area was about 1.87 million.

b. What is the practical meaning of the function value $C(1.87)\approx 1.37$?

The function value $C(1.87)\approx 1.37$ means when the population of an American city is 1.87 million people, the CO_2 concentration in the atmosphere of that city is about 1.37 ppm.

c. What is the practical meaning of the function value $\left.\frac{dp}{dt}\right|_{t=8.5}\approx 0.04$? → unit? $\frac{\text{million people}}{\text{year}}$

The function value $\left.\frac{dp}{dt}\right|_{t=8.5}\approx 0.04$ means that on 1-1-99 the population of the Portland metro area was increasing at rate of about 0.04 $\frac{\text{million people}}{\text{year}}$ ($40{,}000\ \frac{\text{people}}{\text{year}}$).

d. What is the practical meaning of the function value $\left.\frac{dC}{dp}\right|_{p=1.87}\approx 0.37$? → unit? $\frac{\text{ppm}}{\text{million people}}$

The function value $\left.\frac{dC}{dp}\right|_{p\approx 1.87}\approx 0.37$ means that when the population of an American city is about 1.87 million people the CO_2 concentration is increasing, with respect to the population, at a rate of about 0.37 $\frac{\text{ppm}}{\text{million people}}$.

e. What is the value and practical meaning of $\left.\frac{dC}{dt}\right|_{t=8.5}$? $\frac{dC}{dt}=\frac{dC}{dp}\cdot\frac{dp}{dt}$

$$\left.\frac{dC}{dt}\right|_{t=8.5}\approx\left.\frac{dC}{dp}\right|_{p\approx 1.87}\cdot\left.\frac{dp}{dt}\right|_{t=8.5}$$
$$\approx 0.37(0.04)$$
$$=0.0148\ \frac{\text{ppm}}{\text{year}}$$

The function value $\left.\frac{dC}{dt}\right|_{t=8.5}\approx 0.0148$ means that on 1-1-99 the CO_2 concentration in the Portland metro area was increasing at a rate of about 0.0148 $\frac{\text{ppm}}{\text{year}}$.

3. When a ball is thrown straight up, it goes up into the air, reaches a maximum altitude, then comes back down. A person stands at the edge of a building and throws a ball straight upward from a height of 48 ft with an initial velocity of 32 ft/sec. The height above the ground of a ball (in feet) t seconds after it is thrown up is given by $h(t) = -16t^2 + 32t + 48$. Find the velocity, v, and acceleration, a, of the ball at time t. When is the velocity negative and when is it positive? When is the acceleration negative and when is it positive? When is the ball speeding up? When is the ball slowing down?

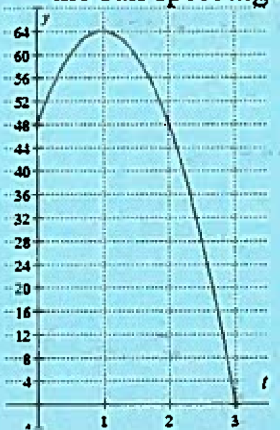
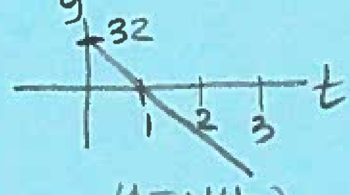


Figure 1: $y=h(t)$

$v(t) = h'(t)$

$= -32t + 32$

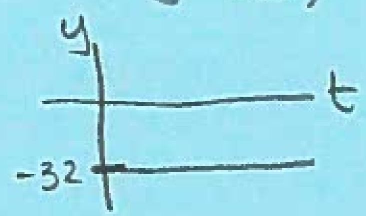


$y = v(t)$

$a(t) = v'(t)$

$= h''(t)$

$= -32$



$y = a(t)$

The velocity is negative on the time interval (1,3) and positive (0,1).

The acceleration is always negative.

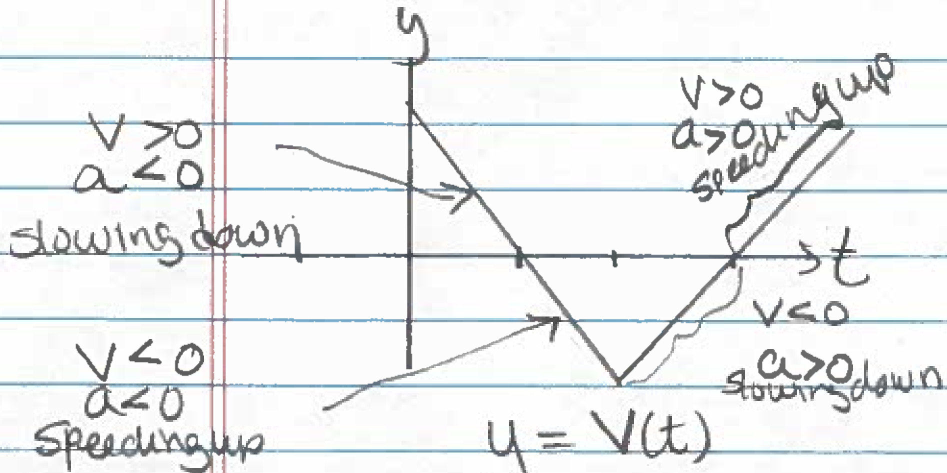
The ball is speeding up on the time interval (1,3) as it is heading back to the ground when the velocity and acceleration are both negative.

The ball is slowing down on the time interval (0,1) as it heads up to its maximum height when the velocity is positive but the acceleration is negative.

sign of v	sign of a	speeding up or ~~slowing down~~
+	+	speeding up
+	−	slowing down
−	+	slowing down
−	−	speeding up

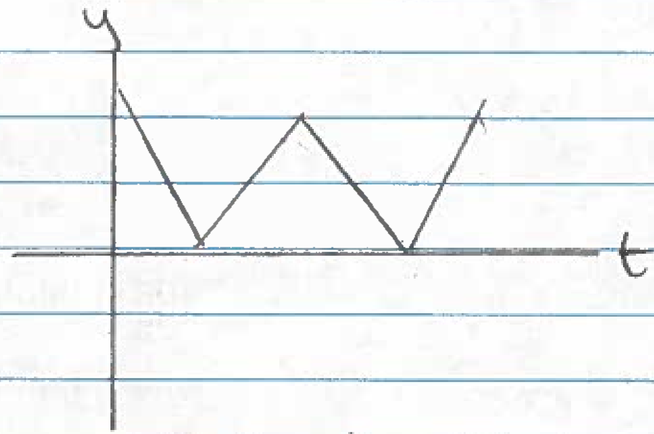
same signs ⇒ speeding up

opposite signs ⇒ slowing down



$y = V(t)$

$|velocity| = speed$



Speed

4. Andy Ant is running back and forth on a twig. Figure 2 shows Andy's position t seconds into his frenzy. Positive positions indicate that Andy is to the right of the midpoint along the twig and negative positions indicate that Andy is to the left of the midpoint along the twig.

Was Andy slowing down or speeding up at $t = 5$? **Justify your conclusion!**

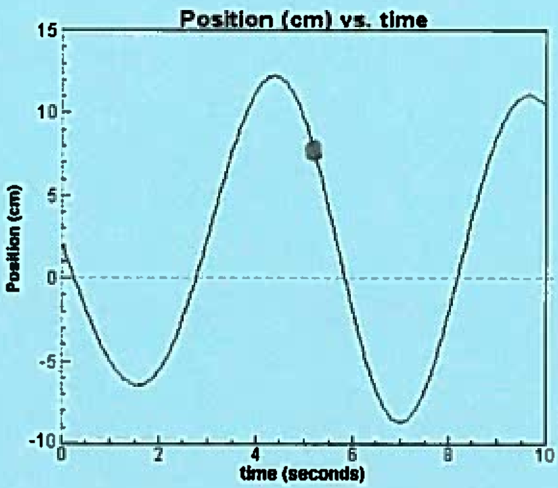


Figure 2: Andy's Position Curve

$V(5) < 0$ since the slope of the tangent line to the position function is negative at $t = 5$.
$a(5)$ is probably negative at $t=5$ since the position curve is concave down at $t=5$.
Andy was speeding up 5seconds into his frenzy since his velocity and acceleration have the same sign.

5. Andy Ant is running back and forth on the same twig on a different day. Figure 3 shows Andy's ***velocity*** t seconds into his frenzy. Positive velocities indicate that Andy is moving right along the twig and negative velocities indicate that Andy is moving left along the twig.

Was Andy slowing down or speeding up at $t = 5$? **Justify your conclusion!**

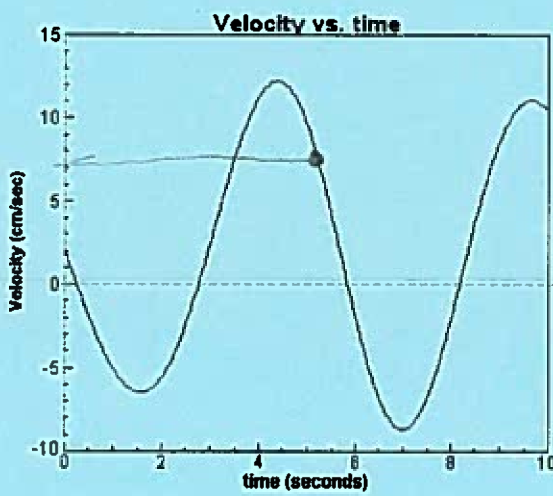


Figure 3: Andy's Velocity Curve

$V(5) > 0$ according to Figure 3.
$a(5) < 0$ since the slope of the tangent line to the velocity curve is negative at $t=5$.
Andy Ant was slowing down 5 seconds into his frenzy since his velocity and acceleration have opposite signs.

Find $k'(0)$ if $k(x)=\sqrt{p(x)}$ using the function $p(x)$ shown in Figure 4.

$$k'(x) = \frac{1}{2\sqrt{p(x)}} \cdot \frac{d}{dx}(p(x))$$

$$= \frac{p'(x)}{2\sqrt{p(x)}}$$

$$\text{So } k'(0) = \frac{p'(0)}{2\sqrt{p(0)}}$$

$$= \frac{1/2}{2\sqrt{4}}$$

$$= \frac{1/2}{2 \cdot 2}$$

$$= \frac{1/2}{4}$$

$$= \frac{1}{2} \cdot \frac{1}{4}$$

$$= \frac{1}{8}$$

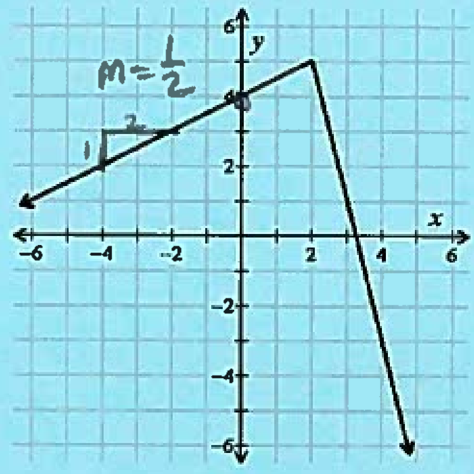


Figure 4: $y = p(x)$

Chain Rule Challenges (complete solutions are posted on my website)

Differentiate the following functions.

1. $w(x)=\sin\left(\tan\left(\sqrt{1+\cos(x)}\right)\right)$

2. $y=\sin\left(\sqrt{e^{2x^3+5x-4}}\right)$

3. $G(x)=[x\sin(2x)+(\tan(x^7))^4]^5$

4. $F(x)=\dfrac{\cos(2x+3)}{e^{x^3-5x+4}}$