Leibniz Notation

In chapter 2 we used the symbols $f'(x)$ and $f''(x)$ as names for the first and second derivatives (with respect to x) of the function $f(x)$. These are mighty fine names indeed, but the notation is extremely limited in applicability. The limitations, specifically, are:

- The notation can be used only when working with functions and, specifically, functions stated using function notation.
- The notation is limited to applications where the derivative is taken with respect to the independent variable stated in the function.
- $f'(x)$ is the *name* of the first derivative function, it is not a call to differentiate $f(x)$.

Leibniz notation is used to overcome all of these limitations.

The source of all Leibniz notation is the symbol "$\frac{d}{dx}$" (or "$\frac{d}{dt}$" or "$\frac{d}{d\theta}$," etc.). The three example symbols are read, respectively, as "the derivative with respect to x," "the derivative with respect to t," and "the derivative with respect to θ."

Each of these symbols/phrases **is an incomplete phrase** in the same way that the phrase "the square root" is an incomplete phrase. Just as it would never make sense to write "$\sqrt{} = 5$" it is equally nonsensical to write "$\frac{d}{dx} = \cos(x)$."

"$\sqrt{} = 5$" ??? The square root of *what* equals five???

"$\frac{d}{dx} = \cos(x)$" ??? The derivative with respect to x of *what* equals $\cos(x)$???

Just as you must specify the object of which you are finding the square root you must also specify the object that you are differentiating with respect to x.

Several examples of Leibniz Notation appear in Table 1.

Table 1: Leibniz Notation Examples

Object to be Differentiated	Differentiation to be Performed	Result of Differentiation	Mathematical Notation
$\sin(x)$	Differentiate with respect to x	$\cos(x)$	$\frac{d}{dx}(\sin(x)) = \cos(x)$
A function named $f(x)$	Differentiate with respect to x	A function named $f'(x)$	$\frac{d}{dx}(f(x)) = f'(x)$
A function of x named y	Differentiate with respect to x	A function named $\frac{dy}{dx}$	$\frac{d}{dx}(y) = \frac{dy}{dx}$
A function of t named x	Differentiate with respect to t	A function named $\frac{dx}{dt}$	$\frac{d}{dt}(x) = \frac{dx}{dt}$
A function named $f'(u)$	Differentiate with respect to u	A function named $f''(u)$	$\frac{d}{du}(f'(u)) = f''(u)$
A function of t named $\frac{dy}{dt}$	Differentiate with respect to t	A function named $\frac{d^2y}{dt^2}$	$\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$
The function value $f(4)$	Differentiate with respect to x	0	$\frac{d}{dx}(f(4)) = 0$

Important notes about Leibniz notation:

1. $f'(4) \neq \frac{d}{dx}(f(4))$
2. $f'(4) = \frac{d}{dx}(f(x))\Big|_{x=4}$ Leibniz notation is cumbersome in this case. Prime notation is preferable.
3. Leibniz notation can be a call to differentiate $\frac{d}{dx}$ or the name of a derivative function $\frac{dy}{dx}$.
4. $(\sin(x))'$ is not acceptable notation. You must write $\frac{d}{dx}(\sin(x))$.
5. We use prime notation to name functions given in function notation. So, the derivative of $f(x)$ is $f'(x)$.
6. We use Leibniz notation to name functions that are not given in function notation. So, the derivative of a function y is $\frac{dy}{dx}$.

When children are first introduced to the concept of addition, the focus is on conceptual understanding. Consequently, a novice in arithmetic would probably approach the problem "$3+2$" in somewhat the following manner:

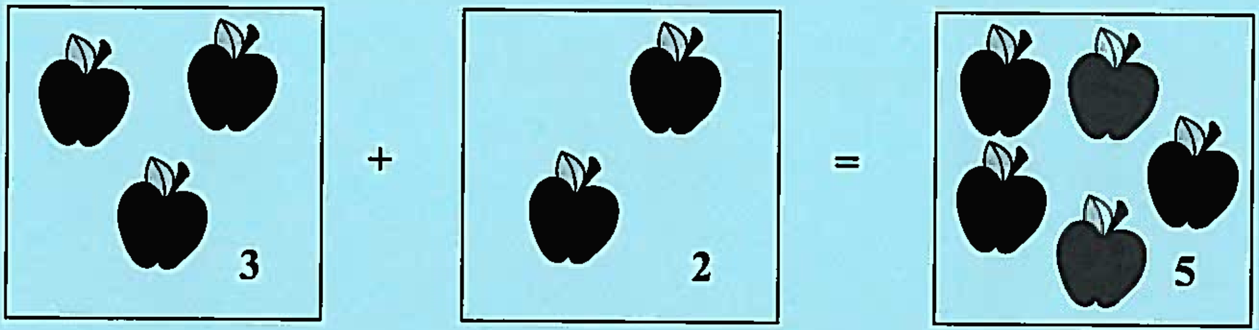

Figure A: A is for Apple!

While exercises like those implied by Figure A are necessary in order for the child to develop a conceptual understanding of arithmetic, they are not instructive from a utilitarian perspective; i.e., one does not want to have to draw pictures of apples every time one wants to perform simple addition. Nor, for that matter, does one want to be dependent upon technology to determine the sum of two numbers (at least not "$3+2$"!) Fingers and toes aside, the most effective strategy for finding the sum of two numbers is to memorize the sums for each possible pair of single digit positive integers, memorize a few rules, and learn how to apply the rules for every different conceivable type of sum.

Similarly, your introduction to derivatives was a conceptual introduction. Part of this introduction was to find derivative formulas using the expression $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

From a utilitarian perspective, however, evaluating the limit expression above becomes redundant at best (e.g. every time you use the expression to find the derivative formula for $f(x)=x^2$ the result is $f'(x)=2x$) and cumbersome, if not impossible, at worst. (You try evaluating the limit necessary to find the derivative formula for $g(x)=x^2\sin(x^6)$!)

Just as memorization is the most effective strategy for day to day addition, memorization is the most effective strategy for finding derivative formulas. Your derivative formula sheet lists several rules you *need to memorize.*

When performing arithmetic, there are some facts to memorize ($3\times 4=12$) and some *procedures* to master (borrowing, carrying, etc.) Likewise, with derivative rules some are simply facts to remember (e.g. $\dfrac{d}{dx}(\sin(u))=\cos(u)\cdot\dfrac{d}{dx}(u)$) and some are procedures that must be mastered (e.g. the quotient rule).

In your head rules

The Sum/Difference Rule allows us to differentiate functions terms by term.

$$\frac{d}{dx}\left(3\sin(x)+4\sqrt{x}-5\tan^{-1}(x)\right)=\frac{d}{dx}(3\sin(x))+\frac{d}{dx}(4\sqrt{x})-\frac{d}{dx}\left(5\tan^{-1}(x)\right)$$

The Constant Factor Rule allows us to factor constant factors from the derivative formula.

$$\begin{aligned}\frac{d}{dx}\left(3\sin(x)+4\sqrt{x}-5\tan^{-1}(x)\right)&=\frac{d}{dx}(3\sin(x))+\frac{d}{dx}(4\sqrt{x})-\frac{d}{dx}\left(5\tan^{-1}(x)\right)\\&=3\frac{d}{dx}(\sin(x))+4\frac{d}{dx}(\sqrt{x})-5\frac{d}{dx}\left(\tan^{-1}(x)\right)\end{aligned}$$

Both of the rules are generally done in one's head.

Examples

Find $f'(x)$ if $f(x)=2\sin(x)-5\cos(x)$

Think ⟶ $f'(x)=2\frac{d}{dx}(\sin(x))-5\frac{d}{dx}(\cos(x))$

Write ⟶ $f'(x)=2\cos(x)-5(-\sin(x)$

$$=2\cos(x)+5\sin(x)$$

Find $\frac{dy}{dt}$ if $y=11\sqrt{t}+4\sqrt[3]{t^7}-5t+7+\frac{8}{t^9}$

$$\frac{dy}{dt}=11\frac{d}{dt}(\sqrt{t})+4\frac{d}{dt}\left(t^{7/3}\right)-\frac{d}{dt}(5t)+\frac{d}{dt}(7)+8\frac{d}{dt}(t^{-9}$$

$$=11\cdot\frac{1}{2\sqrt{t}}+4\cdot\frac{7}{3}t^{4/3}-5+0+8(-9t^{-10})$$

$$=\frac{11}{2\sqrt{t}}+\frac{28}{3}\sqrt[3]{t^4}-5-\frac{72}{t^{10}}$$

Show your work rules

In the long run (like after test 3) your goal should be to also perform the Product Rule, Quotient Rule, and Chain Rule in your head. In the short term, however you need to explicitly write out each of these steps. My motivation for this requirement is two-fold. First, it will help you memorize the rules. Second, it will help you master Leibniz Notation. You have additional motivation in that you will not receive full credit for these type problems unless you explicitly write out each application of these rules.

Examples

Find the derivative of $z = 2x^4 \tan(x)$

$$\frac{dz}{dx} = \frac{d}{dx}(2x^4)\tan(x) + 2x^4\frac{d}{dx}(\tan(x))$$

$$= 2\cdot 4x^3\tan(x) + 2x^4\sec^2(x)$$

$$= 8x^3\tan(x) + 2x^4\sec^2(x)$$

$$= 2x^3(4\tan(x) + x\sec^2(x))$$

optional factoring step

Find the derivative of $q(t) = \frac{5t^2}{1+t^4}$

$$q'(t) = \frac{(1+t^4)\frac{d}{dt}(5t^2) - 5t^2\frac{d}{dt}(1+t^4)}{(1+t^4)^2}$$

$$= \frac{(1+t^4)10t - 5t^2(0+4t^3)}{(1+t^4)^2}$$

$$= \frac{10t + 10t^5 - 20t^5}{(1+t^4)^2}$$

$$= \frac{10t - 10t^5}{(1+t^4)^2}$$

$$= \frac{10t(1-t^4)}{(1+t^4)^2}$$

$$= \frac{10t(1+t^2)(1-t^2)}{(1+t^4)^2}$$

$$= \frac{10t(1+t^2)(1+t)(1-t)}{(1+t^4)^2}$$

Find the derivative of $p(t) = \frac{1+t^4}{5t^2}$

$$p(t) = \frac{1}{5t^2} + \frac{t^4}{5t^2}$$

$$= \frac{1}{5}t^{-2} + \frac{t^2}{5}$$

$$p'(t) = \frac{1}{5}(-2t^{-3}) + \frac{1}{5} \cdot 2t$$

$$= -\frac{2}{5t^3} + \frac{2t}{5} \cdot \frac{t^3}{t^3}$$

$$= \frac{-2 + 2t^4}{5t^3}$$

Find the first derivative formula for the function $T = \frac{\sqrt{t}\cos(t)}{3-\sqrt{t}}$

Solution:

$$\frac{dT}{dt} = \frac{d}{dt}\left(\frac{\sqrt{t}\cos(t)}{3-\sqrt{t}}\right)$$

$$= \frac{\frac{d}{dt}\left(\sqrt{t}\cos(t)\right)\left(3-\sqrt{t}\right) - \sqrt{t}\cos(t)\frac{d}{dt}\left(3-\sqrt{t}\right)}{\left(3-\sqrt{t}\right)^2} \qquad \text{Quotient rule}$$

$$= \frac{\overbrace{\left[\frac{d}{dt}\left(\sqrt{t}\right)\cos(t) + \sqrt{t}\frac{d}{dt}(\cos(t))\right]}\left(3-\sqrt{t}\right) - \sqrt{t}\cos(t)\left(-\frac{1}{2\sqrt{t}}\right)}{\left(3-\sqrt{t}\right)^2} \qquad \text{product rule}$$

$$= \frac{\left[\frac{1}{2\sqrt{t}}\cos(t) + \sqrt{t}(-\sin(t))\right]\left(3-\sqrt{t}\right) + \frac{\cos(t)}{2}}{\left(3-\sqrt{t}\right)^2}$$

$$= \frac{\frac{3\cos(t)}{2\sqrt{t}} - \frac{\cos(t)}{2} - 3\sqrt{t}\sin(t) + t\sin(t) + \frac{\cos(t)}{2}}{\left(3-\sqrt{t}\right)^2}$$

$$= \frac{\frac{3\cos(t)}{2\sqrt{t}} - 3\sqrt{t}\sin(t) + t\sin(t)}{\left(3-\sqrt{t}\right)^2} \cdot \frac{2\sqrt{t}}{2\sqrt{t}}$$

$$= \frac{3\cos(t) - 6t\sin(t) + 2t^{3/2}\sin(t)}{2\sqrt{t}\left(3-\sqrt{t}\right)^2}$$

$$= \frac{3\cos(t) - 6t\sin(t) + 2\sqrt{t^3}\sin(t)}{2\sqrt{t}\left(3-\sqrt{t}\right)^2}$$