F is antiderivative of f if $F'(x) = f(x)$.
So f is an antiderivative of f'.
Likewise f' is an antiderivative of f''.

$f''(x)$	$f'(x)$	$f(x)$	$F(x)$
Positive	Increasing	Concave Up	
Negative	Decreasing	Concave Down	
	Positive	Increasing	Concave Up
	Negative	Decreasing	Concave Down
		Positive	Increasing
		Negative	Decreasing

Left to right these relationships are true 100% of the time.

Right to left there are minor exceptions to the rules.

1. At isolated points over intervals where a function is continually decreasing or continually increasing, the function's first derivative could be zero or undefined.

2. Over an interval where a function is continually concave up or continually concave down, its first derivative could have discontinuity. Pretty much the only time this would happen is when a math instructor wanted to illustrate the reality of this possibility.

f''	f'	f
+	↗	∪
−	↘	∩

Answer each question on this page in reference to the function $y = f(x)$ shown in Figure 10. You may simply indicate the point(s) or intervals(s) in your response to questions 1-5.

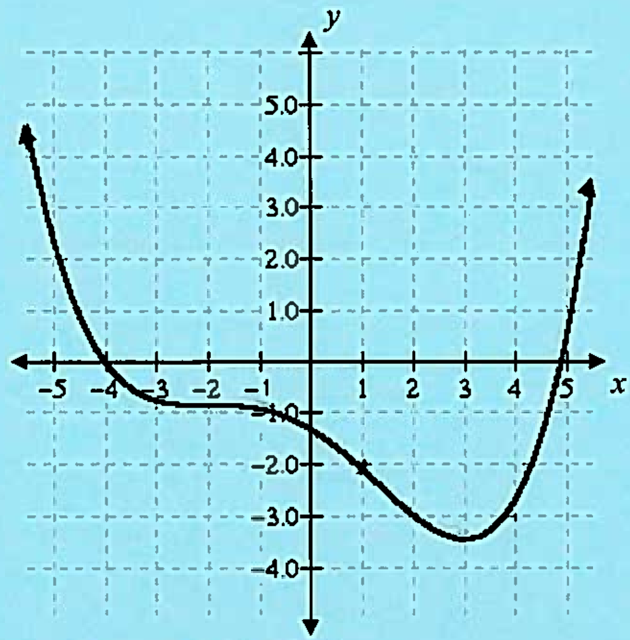


Figure 10: $y = f(x)$

1. At what value(s) of x does $f'(x) = 0$?

   -2 and 3

2. Over what interval(s) is $f'(x)$ positive?

   $(3, \infty)$

3. Over what interval(s) is $f'(x)$ negative?

   $(-\infty, -2) \cup (-2, 3)$

4. Over what interval(s) is $f'(x)$ increasing?

   $(-\infty, -2) \cup (1, \infty)$

5. Over what interval(s) is $f'(x)$ decreasing?

   $(-2, 1)$

6. Write a complete sentence that explains how you visually established the answer to question 1.

   $f'(x) = 0$ at nice smooth local maximums or local minimums and anywhere the graph flattens out to give a horizontal tangent line.

7. Write a complete sentence that explains how you visually established the answer to question 3.

   $f'(x) < 0$ where f is decreasing and the slope of the tangent line to f is negative.

8. Write a complete sentence that explains how you visually established the answer to question 4.

   f' is increasing when f is concave up.

Figure 11 shows the tangent line to $y = f(x) = \dfrac{2x}{x+1}$ at $(-2, 4)$.

- Use the graph determine $f'(-2)$.
- Except at $x = -1$, what must always be true about the function $f'(x)$? How do you know?
- Find the formula for $f'(x)$.
- Verify your first two answers. ← See page 17

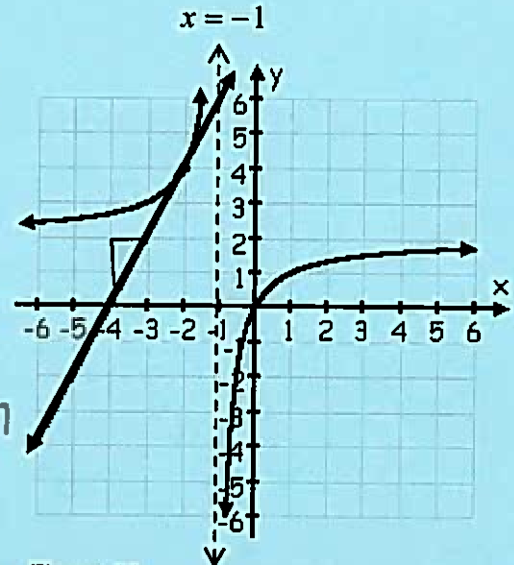


Figure 11

Tangent line to $y = \dfrac{2x}{x+1}$ at $(-2, 4)$.

$$f'(-2) = \frac{\text{rise}}{\text{run}} = \frac{2}{1} = 2$$

Except at $x=-1$, $f'(x) > 0$ because the slopes of the tangent lines to f are positive except at $x=-1$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{\frac{2(x+h)}{x+h+1} - \frac{2x}{x+1}}{h} \cdot \frac{(x+h+1)(x+1)}{(x+h+1)(x+1)} \right]$$

$$= \lim_{h \to 0} \frac{\frac{2(x+h)}{\cancel{x+h+1}} \cdot \cancel{(x+h+1)}(x+1) - \frac{2x}{\cancel{x+1}} \cdot (x+h+1)\cancel{(x+1)}}{h(x+h+1)(x+1)}$$

$$= \lim_{h \to 0} \frac{2(x+h)(x+1) - 2x(x+h+1)}{h(x+h+1)(x+1)}$$

$$= \lim_{h \to 0} \frac{2[\cancel{x^2} \cancel{+x} \cancel{+hx} + h \cancel{-x^2} \cancel{-xh} \cancel{-x}]}{h(x+h+1)(x+1)}$$

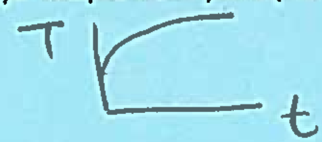
$$= \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}(x+h+1)(x+1)}$$

$$= \lim_{h \to 0} \frac{2}{(x+h+1)(x+1)} = \frac{2}{(x+0+1)(x+1)} = \frac{2}{(x+1)^2}$$

Interpreting First Derivative Values

What needs to be communicated when interpreting an instantaneous rate of change for a non-linear function?

- When?
  You must convey the value of the input variable at the instant you are describing the rate of change.
- What?
  You must communicate whether the output variable is increasing or decreasing at that instant. (Also, unless the input variable is "passage of time", you almost always must specify the entity with respect to which the output is changing.)
- How quickly?
  This part of the description almost always begins with the words "...at a rate of..." Since you have already specified whether the function is increasing or decreasing, the value you state here is always positive. The units on the rate are always "output units per input units."



$T = f(t)$

Example 1

The temperature, T, in degrees Fahrenheit, of a cold yam placed in a hot oven is given by $T = f(t)$, where t is the time in minutes since the yam was put in the oven.

a. What is the sign of $f'(t)$? Why?

$f'(t)$ should be positive because f is increasing.

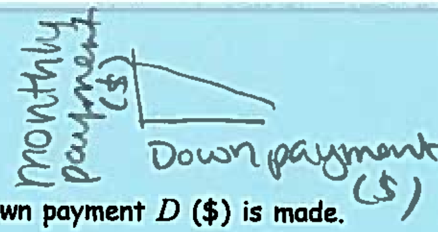
b. What are the units of $f'(20)$? What is the practical meaning of the statement $f'(20) = 2$?

The unit of $f'(20)$ is $\frac{°F}{min}$. The function value $f'(20)=2$ that 20 minutes after the yam was put in the oven the temperature of the yam was increasing at a rate of $2 \frac{°F}{min}$.

c. What is the practical meaning of $f^{-1}(200) = 5$

The function value $f^{-1}(200)=5$ means that the temperature of the yam was 200°F 5 minutes after that yam was placed in the oven.

constants < price of car / interest rate



Example 2

$M = f(D)$ is the monthly payment (\$) on a specific car loan if a down payment D (\$) is made. What is the meaning of $f'(2500) = -0.06$? unit $\frac{\$}{\$}$

The function value $f'(2500) = -0.06$ means that if the down payment is \$2500 the monthly is decreasing, with respect to the down payment, at a rate of 0.06 $\frac{\$}{\$}$.

Side note: If the down payment increased from \$2500 to \$2501 the monthly payment would decrease by about 6¢.

Leibniz Notation

Another way to write $f'(a)$ is $\left.\frac{df}{dx}\right|_{x=a}$.

If $y = f(x)$, another way to write $f'(x)$ is $\frac{dy}{dx}$.

The derivative of f with respect to x such that x=a.

often read "df, dx such that x=a.

Example 3

When making his aerobics tapes, Richard Simmons heart rate increases as the speed of the disco tune increases. $H = f(M)$ models this relationship where H is Rick's heart rate (bpm) and M is the speed of the tune (bpm). What is the practical meaning of the value $f'(110) = 0.15$?

heart rate
$\frac{\text{bpm}}{\text{bpm}}$
speed of the tune

The function $f'(110) = 0.15$ means that when the speed of the tune is 110 bpm, Rick's heart rate is increasing, with respect to the speed of the tune, at a rate of 0.15 $\frac{\text{bpm}}{\text{bpm}}$.

Example: If the tangent line to $y = k(x)$ at $(2,5)$ passes through the point $(1,-3)$, find $k(2)$ and $k'(2)$.

$k(2) = 5$ $\quad$ $k'(2) = \frac{-3-5}{1-2} = \frac{-8}{-1} = 8$

Example: Sketch the graph of a function f for which $\cancel{f(0)} = 3$, $f'(0) = -2$, $f'(2) = 0$, and $f'(4) = 1$

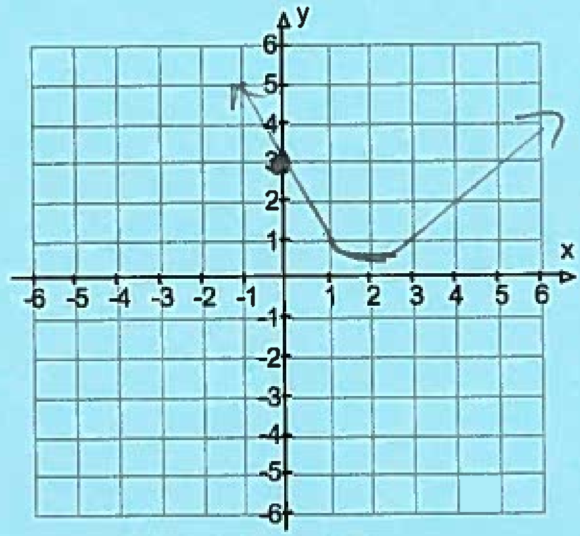


Figure 12: $y = f(x)$

Example: Sketch the graph of a function g for which $g(\cancel{-4}) = -5$, $g'(-4) = 3$, $g'(-2) = 0$, $g'(0) = 3$, and $g'(3)$ is undefined.

At 3

cusp
or
discontinuity
or
vertical tangent

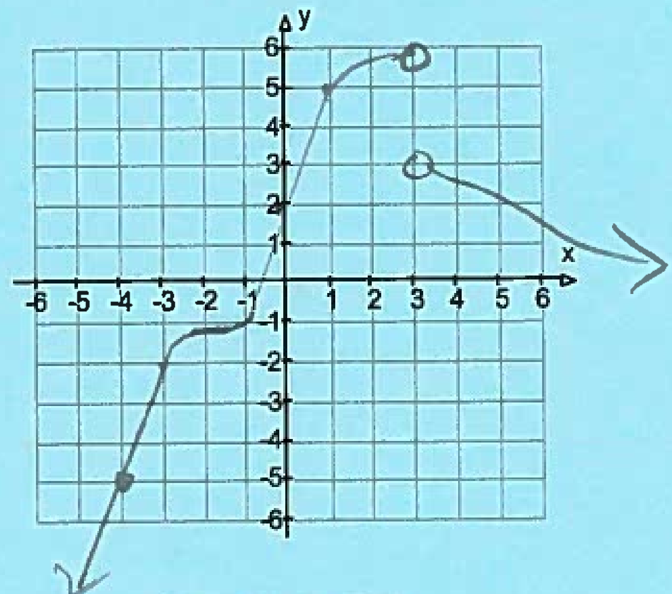
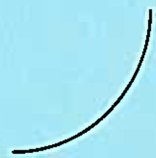
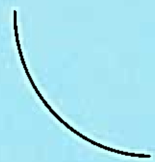
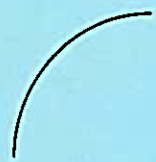
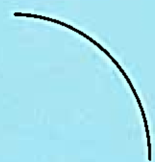


Figure 13: $y = g(x)$

Drawing Functions based upon Derivative Sign Information

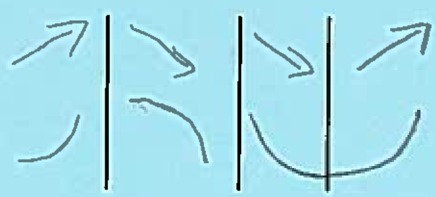
Table 1: 4 Basic Curve Shapes and the attendant Derivative Signs

	$f'(x)>0$	$f'(x)<0$
$f''(x)>0$		
$f''(x)<0$		

<u>Example</u>

Sketch onto Figure 1 a continuous curve, $y=f(x)$, that has the following properties.

- $f(0)=1$
- $f'(x)>0$ over $(-\infty,-2)$ and $(3,\infty)$
- $f'(x)<0$ over $(-2,1)$ and $(1,3)$
- $f''(x)>0$ over $(-\infty,-2)$ and $(1,\infty)$
- $f''(x)<0$ over $(-2,1)$



There must be a cusp at $x=-2$.

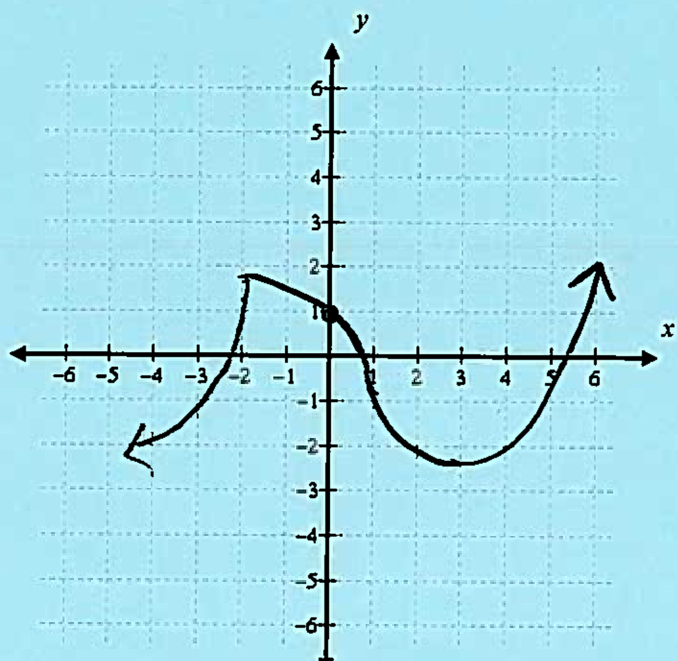
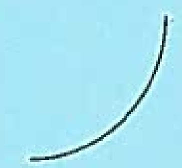
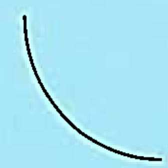
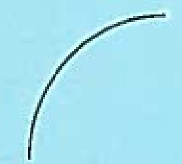
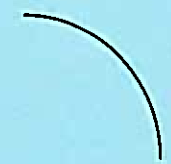


Figure 1: $y=f(x)$

Drawing Functions based upon Derivative Sign Information

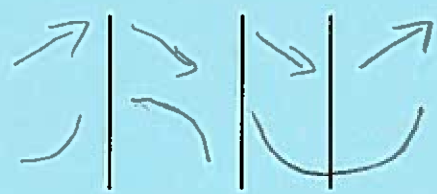
Table 1: 4 Basic Curve Shapes and the attendant Derivative Signs

	$f'(x)>0$	$f'(x)<0$
$f''(x)>0$		
$f''(x)<0$		

Example

Sketch onto Figure 1 a continuous curve, $y=f(x)$, that has the following properties.

- $f(0)=1$
- $f'(x)>0$ over $(-\infty,-2)$ and $(3,\infty)$
- $f'(x)<0$ over $(-2,1)$ and $(1,3)$
- $f''(x)>0$ over $(-\infty,-2)$ and $(1,\infty)$
- $f''(x)<0$ over $(-2,1)$



There must be a cusp at $x=-2$.

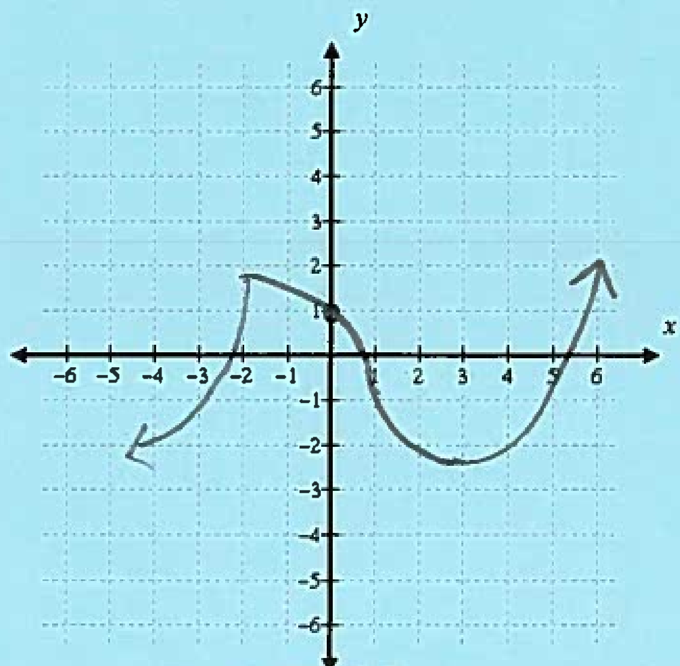
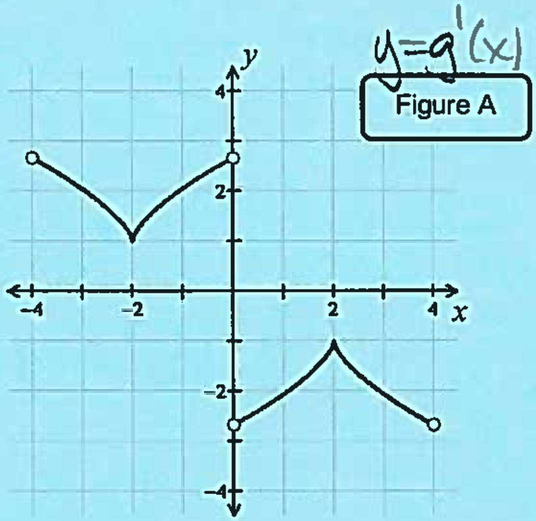
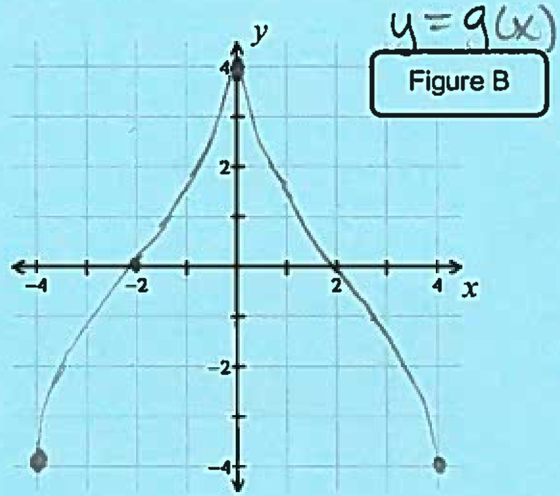


Figure 1: $y=f(x)$

Sketching an antiderivative

The graph of a first derivative, $y = g'(x)$, is shown in Figure A. Sketch onto Figure B the **continuous** function $y = g(x)$ given that $g(-4) = -4$ and $g(0) = 4$.





On the interval $(-4,-2)$, g' is decreasing, so g is concave down

On the interval $(-4,-2)$, g' is positive, so g is increasing

On the interval $(-2,0)$, g' is increasing, so g is concave up

On the interval $(-2,0)$, g' is positive, so g is increasing

On the interval $(0,2)$, g' is increasing, so g is concave up

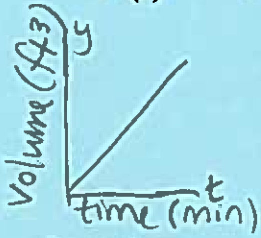
On the interval $(0,2)$, g' is negative, so g is decreasing

On the interval $(2,4)$, g' is decreasing, so g is concave down

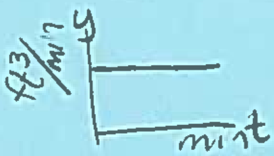
On the interval $(2,4)$, g' is negative, so g is decreasing linear

Water flows *at a constant rate* into a large conical tank (pointy end down ☺). Let $V(t)$ (in ft^3) and $H(t)$ (in ft) be, respectively, the volume of water in the tank and the height of the water in the tank t minutes after the water begins to flow. Suppose that five minutes after the water begins to flow the tank is one quarter full. For each of the following expressions, state the unit on the expression and state whether the value is positive, negative, or zero. **Explain!**

$V'(5)$ $V''(5)$ $H'(5)$ $H''(5)$

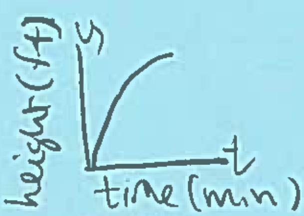


$y=V(t)$



$y=V'(t)$

$y=V''(t)$



$y=H(t)$

The unit of $V'(5)$ *is* $\frac{ft^3}{min}$.

The sign of $V'(5)$ *is* positive because the volume is increasing.

The unit of $V''(5)$ *is* $\frac{ft^3/min}{min}$.

The sign of $V''(5)$ *is* zero (no sign!) since the water flows at a constant rate so V is linear.

The unit of $H'(5)$ *is* $\frac{ft}{min}$.

The sign of $H'(5)$ *is* positive because the height is increasing.

The unit of $H''(5)$ *is* $\frac{ft/min}{min}$.

The sign of $H''(5)$ *is* negative because the height function function is concave down since the rate at which the height increases is decreasing.

Each of the following sentences is true if one of the words/phrases in Table 1 is inserted into the blank. Find the proper word/phrase for each of the blanks. **Read each sentence carefully!!**

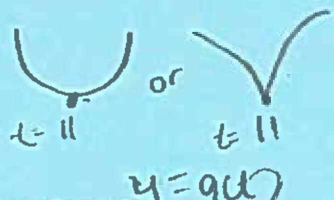
Table 1: Blank Options

Increasing	Decreasing
Concave Up	Concave Down
Positive	Negative
Zero	"Zero or UD" (This is *one* choice)
"Positive, Zero or UD" (This is *one* choice)	"Negative, Zero or UD" (This is *one* choice)
Horizontal	Vertical

Note: UD is short for undefined

- If $f'(x)$ is negative at every point over $(2,5)$, then $f(x)$ is decreasing over the entire interval $(2,5)$.
- If $f''(x)$ is positive at every point over $(2,5)$, then $f'(x)$ is increasing over the entire interval $(2,5)$.
- If $g'(t)$ is increasing over the entire interval $(2,5)$, then $g(t)$ is concave up over the entire interval $(2,5)$.
- If $\overbrace{\text{the slope of } g(t)}^{g'(t)}$ is increasing over the entire interval $(2,5)$, then $g(t)$ is concave up over the entire interval $(2,5)$.
- If the $\overbrace{\text{slope of } g(t)}^{g'(t)}$ is positive over the entire interval $(2,5)$, then $g(t)$ is increasing over the entire interval $(2,5)$.
- If $g(t)$ is continuous and has a local minimum at the point where $t = 11$,

  then $g'(t)$ is positive immediately to the right of $t = 11$.



- g is concave up

  If the slope of $g(t)$ is increasing over the entire interval $(2,5)$, then at any point along the interval $(2,5)$, $g''(t)$ is positive, zero, or UD

$f(x) = x^4$ Always concave up

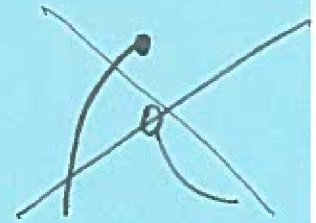
$f'(x) = 4x^3$

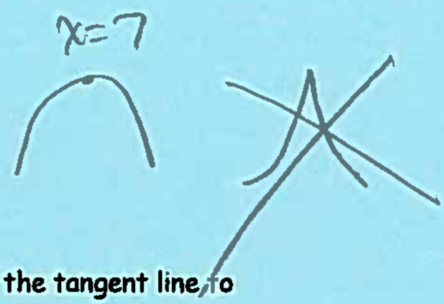
$f''(x) = 12x^2$ $f''(0) = 0$

Table 1: Blank Options

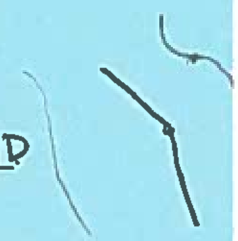
Increasing	Decreasing
Concave Up	Concave Down
Positive	Negative
Zero	"Zero or UD" (This is *one* choice)
"Positive, Zero or UD" (This is *one* choice)	"Negative, Zero or UD" (This is *one* choice)
Horizontal	Vertical

Note: UD is short for undefined





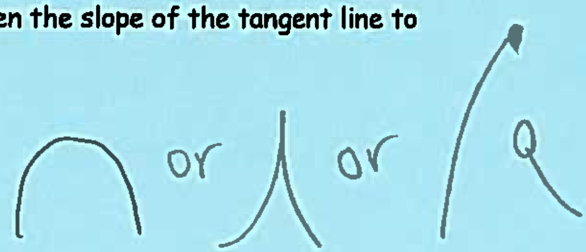
- If $f(x)$ is differentiable and has a local maximum point at $x=7$ then the tangent line to $y=f(x)$ at the point $x=7$ is horizontal.



- If $f(x)$ is decreasing over the entire interval $(-4,4)$ then $f'(2)$ is negative, zero or UD
- If $g'(t)$ is concave up over the entire interval $(2,5)$, then $g''(t)$ is increasing over the entire interval $(2,5)$.

$g''(t)$

- If the slope of $g'(t)$ is negative over the entire interval $(-4,4)$, then $g''(2)$ is negative.
- If $g(x)$ is concave up over the entire interval $(-4,4)$, then $g''(2)$ is positive, zero or UD
- If $g(t)$ has a local maximum at the point where $t=1$, then the slope of the tangent line to $y=g(t)$ is zero or UD at $t=1$.



The functions f and g in the following two examples are both differentiable on $(-\infty,\infty)$ and concave up on $(-\infty,\infty)$.

Example 1: $f(t)=\begin{cases}-(t-1)^3-3 & \text{for } t\le 1\\ (t-1)^2-3 & \text{for } t>1\end{cases}$

Figure A: $y=f(t)$

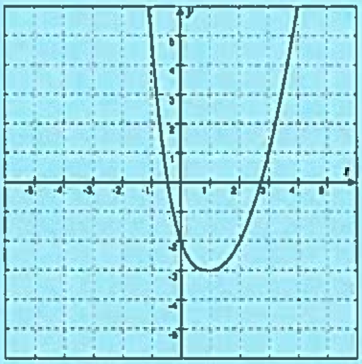


Figure B: $y=f'(t)$

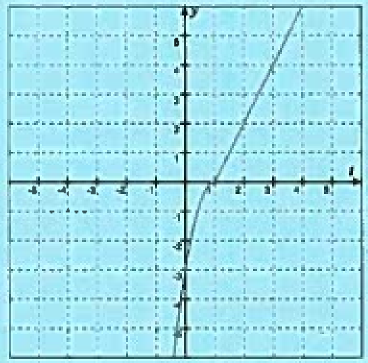
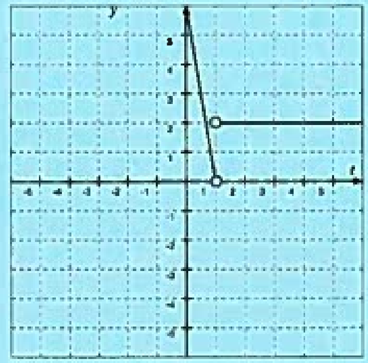


Figure C: $y=f''(t)$



This is an example of a concave up function whose second derivative at 1 is undefined.

Example 2: $g(t)=\frac{1}{4}(t-3)^4+2t-5$

Figure D: $y=g(t)$

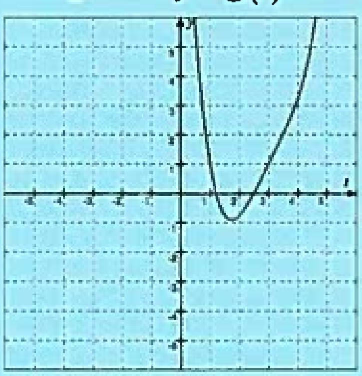


Figure E: $y=g'(t)$

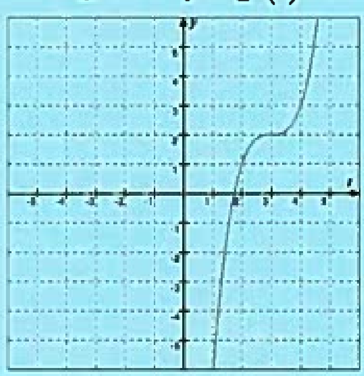
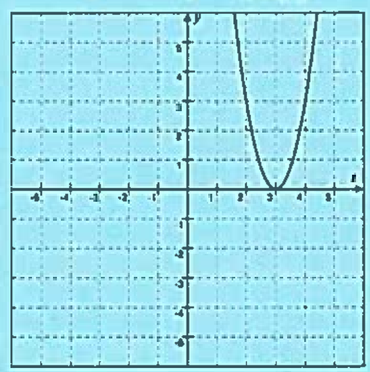


Figure F: $y=g''(t)$



This is an example of a concave up function whose second derivative at 3 is zero.

Note: In these examples, both f' and g' increasing functions, making f and g, respectively, concave up. Remember, even if f' and g' are increasing, it does not guarantee that f'' and g'' are positive.